Charges of monopole operators in $\widehat{ADE}$ Chern-Simons quiver gauge theories

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ABSTRACT: We compute R-charges of the BPS-monopole operators in $\mathcal{N} = 3$ $\widehat{ADE}$ Chern-Simons quiver gauge theories, along the lines of the work of Benna, Klebanov and Klose in [1]. These theories have a weakly coupled UV completion in terms of $\mathcal{N} = 3$ supersymmetric Chern-Simons Yang-Mills theories. In the UV limit the monopole operators are well approximated by classical solutions. We construct classical BPS and anti-BPS monopole solutions to these theories which preserve $\frac{1}{3}$ supersymmetry all along the RG flow. We compute the $SU(2)_R$ charges in these backgrounds and show that the smallest possible value of quantised $SU(2)_R$ charge is zero in each quiver theory.

KEYWORDS: Solitons Monopoles and Instantons, AdS-CFT Correspondence, Chern-Simons Theories, M-Theory

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1 Introduction

Study of $\text{AdS}_4/\text{CFT}_3$ correspondence [2] received a lot of interest after the discovery of an $\mathcal{N} = 6$ superconformal Chern-Simons (CS) matter theory which describes the world volume theory of multiple M2 branes [16](also see [3]–[7]) in the low energy limit. This theory, widely known as ABJM theory, has gauge group $\text{U}(N)_k \times \text{U}(N)_{-k}$ where k is the CS level. In the matter sector there are four complex scalar fields in $(N, \bar{N})$ representation and their complex conjugate fields in the $(\bar{N}, N)$ representation along with their fermionic partners. ABJM theory can be obtained as IR limit of a brane construction [17, 18] which preserves $\mathcal{N} = 3$ supersymmetry. The field content in the brane construction is similar to ABJM theory but the gauge fields and their superpartners become dynamical in the high energy regime. One of the important objects in ABJM theory is the monopole operator which is crucial for the supersymmetry enhancement from $\mathcal{N} = 6$ to $\mathcal{N} = 8$ for special values of the CS levels $k = 1, 2$.

Monopole operator was first studied in the context of QED and supersymmetric QED in [8, 9] where it is defined as a vortex creating operator with unit vortex charge. The vortex charge is the conserved charge of the current $J^\mu = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho}$, which exists in any three dimensional gauge theory and conserved by virtue of Bianchi identity. Monopole operator can also be thought of as 't Hooft operator which is a topological disorder operator and naturally arises as dual to an “order” operator in a topological quantum field theory [13]. Disorder operators can not be expressed as polynomials of basic field variables in the Lagrangian. Therefore they are defined by specifying the singularities of the classical fields in the theory and performing the path integral with a boundary condition that the fields take those specified configurations at the point of singularity. For example, a $\text{U}(1)$ Dirac monopole in $\mathbb{R}^3$ is defined by specifying the classical gauge field configuration,

$$\vec{A} = \frac{q}{2r \sin \theta} (\pm 1 - \cos \theta) \hat{e}_\varphi \quad (1.1)$$

where $\hat{e}_\varphi$ is the unit vector in spherical polar co-ordinate system (r, θ, φ) , the upper and lower sign is for northern and southern hemisphere respectively, q is the magnetic charge. Observe that the gauge field has a real singularity at $r = 0$. Therefore the statement that there exists such a monopole operator in a theory implies that we are inserting a singular gauge field at a point in space-time. Insertion of such a monopole operator at a point p amounts to integrating over the gauge fields which have a singularity at $x = p$ such that the magnetic flux through a 2-sphere surrounding $x = p$ is q . In a $\text{U}(N)$ gauge theory a monopole operator is obtained by defining a homomorphism,

$$\rho : \text{U}(1) \mapsto \text{U}(N). \quad (1.2)$$

One such mapping takes a $\text{U}(1)$ element $e^{i\alpha}$ to $\text{diag}(e^{iq_1\alpha}, e^{iq_2\alpha}, \dots, e^{iq_N\alpha}) \in \text{U}(N)$. An algebra element gets mapped to $H = \text{diag}(q_1, q_2, \dots, q_N) \in \mathfrak{u}(N)$. It is shown by Goddard, Nuyts and Olive in [14] that when H is written as a linear combination of Cartan generators then $(q_1, q_2, \dots, q_N)$ are the weights of the dual of $\text{U}(N)$. The monopole operator will transform in the $\text{U}(N)$ representation with highest weight state labelled by $(q_1, q_2, \dots, q_N)$ [15].

Monopole operators play crucial role in establishing various non-perturbative dualities in three dimensional quantum field theories [11, 12] and are often useful in condensed matter systems [34–36]. In AdS/CFT studies it has been found that monopole operators with zero conformal dimensions are the ones important for matching the spectra with supergravity. Such monopole operators in ABJM theory which are singlets under the global symmetries were first studied by Benna, Klebanov and Klose (BKK) in [1]. Since ABJM theory is strongly coupled for small values of k , which is the only coupling in the theory, it is difficult to compute the conformal dimension of the monopole operators. To overcome this difficulty BKK introduces a method which goes as follows:

- A small coupling g is introduced through Yang-Mills deformation of the action which provides a weakly coupled UV completion of the theory. The UV completion is an $\mathcal{N} = 3$ CS Yang-Mills theory.¹
- To retain $\mathcal{N} = 3$ supersymmetry one has to add dynamical fields in the adjoint representation. Thus the R-symmetry group of the theory is $SU(2)$.
- In the UV limit the monopole operators are described by classical BPS (anti-BPS) monopole solutions of the CS-Yang-Mills theory.
- Since the final aim is to compute the spectrum of conformal dimension of monopole operator which is valid at the IR fixed point, they compute a quantity in the UV which is related to the conformal dimension and does not change along the renormalisation group (RG) flow. This quantity is the quantised $SU(2)_R$ charge of the monopole operator. Due to the non-abelian nature of the R-symmetry group the quantised charges will take discrete values and will not be changed under the continuous RG flow in the parameter g .
- At the conformal fixed point these charges are related to the conformal dimensions of the BPS monopole operators by state operator correspondence.

Monopole operators in ABJM theory have been extensively studied over many years now. In the light of the important role played by monopole operators in ABJM theory we study monopole operators in a wide class of $\widehat{ADE}$ CS quiver gauge theories. ABJM theory is the low energy world volume description of M2 branes probing a transverse toric hyperKähler manifold which has singularities of the form $\mathbb{C}^4/\mathbb{Z}_k$. Therefore it is natural to construct theories which can arise as world volume theories of multiple M2 branes probing other hyperKähler singularities. In [19] an $\mathcal{N} = 3$ superconformal quiver CS matter theory was constructed which is a world volume theory of M2 branes placed at singularities in the transverse eight dimensional hyper-Kähler manifold and are dual by the AdS/CFT conjecture [24–26] to M-theory on AdS_4 times the seven dimensional tri-Sasakian manifold whose cone is the eight dimensional hyper-Kähler manifold. The corresponding quiver of the field theory is a Dynkin diagram of $\hat{A}$ algebra. It is known that one can always

¹There are other UV completions that have been studied in the literature can be found in [21, 22], where a smaller amount of supersymmetry is preserved along the RG flow.

construct a three dimensional $\mathcal{N} = 3$ superconformal CS matter theory whose field content can be summarised by a $\widehat{ADE}$ quiver diagram [30].

In this note we find classical monopole solutions and compute R-charges of the BPS monopole operators in $\mathcal{N} = 3$ CS quiver gauge theories with $\widehat{A}_{n-1}$, $\widehat{D}_n$ and $\widehat{E}_6$ quiver diagrams along the lines of the work of BKK. The rest of the paper is organised as follows. In section 2 we construct $\mathcal{N} = 3$ action of $\widehat{ADE}$ theories and write down supersymmetry variation equations. In section 3 we obtain classical BPS and anti-BPS monopole solutions in these theories. In section 4 we compute the quantum corrections to the $U(1)_R$ charges and in section 5 computation of $SU(2)_R$ charges is presented. Appendix A contains all notations and conventions used for calculations. Appendix B contains the superfield expressions and explicit expression of the component action. In appendix C we present the computation to check supersymmetry in the $\widehat{D}$ case. In appendix D we give a small example of monopole solution in $\widehat{D}_4$.

2 Action of Yang-Mills deformed $\mathcal{N} = 3$ Chern-Simons theory

One way to obtain a three dimensional CS matter theory with $\mathcal{N} = 3$ supersymmetry is to first construct a theory with $\mathcal{N} = 4$ supersymmetry without a CS term. An $\mathcal{N} = 4$ theory in three dimension is a dimensional reduction of $\mathcal{N} = 2$ theory in four space-time dimension which has the following supersymmetric multiplets,

- (i) $\mathcal{N} = 2$ gauge/vector multiplet = $(\mathcal{N} = 1 \text{ gauge } \mathcal{V}) \oplus (\mathcal{N} = 1 \text{ chiral } \Phi)$ multiplet in adjoint rep. of the gauge group of the theory.
- (ii) $\mathcal{N} = 2$ hypermultiplet = $(\mathcal{N} = 1 \text{ chiral } \mathcal{Z} \text{ in rep. } R_i) \oplus (\mathcal{N} = 1 \text{ chiral } \mathcal{W} \text{ in rep. } R_i^*)$. where, R_i can be fundamental or bi-fundamental under the gauge group of the theory.

The above multiplet is used to construct the $\mathcal{N} = 3$ CS matter theory in three dimensions after dimensional reduction. The component expansions of all the superfields are given in appendix B. Keeping in mind that in three dimension the R-symmetry group of $\mathcal{N} = 3$ theory is $SO(3)$, which is isomorphic to $SU(2)$ at the algebra level, the on-shell component fields in the superfield expansion are arranged in R-symmetry representations in BKK as follows.

The non-auxiliary scalars in the vector multiplet are arranged as,²

$$\phi_{b(j)}^a = (\phi_{(j)})_i (\sigma_i)_b^a = \begin{pmatrix} -\sigma_{(j)} & \phi_{(j)}^\dagger \\ \phi_{(j)} & \sigma_{(j)} \end{pmatrix}, \quad (2.1)$$

where, the lower/upper index is the row/column index, $i = 1, 2, 3$ is $so(3)$ vector index. $\phi_{b(j)}^a$ forms a 3-dimensional representation of $SU(2)$ algebra. Fermions in the vector multiplet are written in terms of a 2×2 matrix,

$$\lambda_{(j)}^{ab} = \begin{pmatrix} \chi_{\sigma(j)} e^{-i\pi/4} & \chi_{\phi(j)}^\dagger e^{-i\pi/4} \\ \chi_{\phi(j)} e^{+i\pi/4} & -\chi_{\sigma(j)}^\dagger e^{+i\pi/4} \end{pmatrix}. \quad (2.2)$$

²The subscript (j) was not present in BKK and will be explained shortly.

Therefore $\lambda_{(j)}^{ab}$ transforms in the reducible representation $\mathbf{2} \times \mathbf{2} = \mathbf{3} + \mathbf{1}$ of $\text{SU}(2)_R$. The $\text{SU}(2)_R$ indices are raised and lowered by the $\text{SU}(2)$ metric ϵ^{ab} with, $\epsilon^{12} = \epsilon_{21} = +1$ and the following relations hold.

$$\begin{aligned} (\lambda_{(j)}^{ab})^* &= -\lambda_{ab(j)} = -\epsilon_{ac}\epsilon_{bd}\lambda_{(j)}^{cd}, \\ (\phi_{b(j)}^a)^* &= \phi_{a(j)}^b = \epsilon_{ac}\epsilon^{bd}\phi_{d(j)}^c. \end{aligned} \quad (2.3)$$

The bifundamental matter fields are written as $\text{SU}(2)_R$ doublets as follows,

$$X_{(j)}^a = \begin{pmatrix} Z_{(j)} \\ W_{(j)}^\dagger \end{pmatrix}, \quad X_{a(j)}^\dagger = \begin{pmatrix} Z_{(j)}^\dagger \\ W_{(j)} \end{pmatrix}, \quad (2.4)$$

and

$$\xi_{(j)}^a = \begin{pmatrix} \omega_{(j)}^\dagger e^{i\pi/4} \\ \zeta_{(j)} e^{-i\pi/4} \end{pmatrix}, \quad \xi_{a(j)}^\dagger = \begin{pmatrix} \omega_{(j)} e^{-i\pi/4} \\ \zeta_{(j)}^\dagger e^{i\pi/4} \end{pmatrix}. \quad (2.5)$$

The component action and the supersymmetry variations will be written in terms of the above R-symmetry representations.

2.1 $\hat{A}$ -type quiver

Before writing down the superspace action for quiver gauge theories let us first set up the notation to express the above field content via a quiver diagram. Figure 1 is a Dynkin diagram of affine A-algebra and can be used to represent the $\mathcal{N} = 3$ field content discussed above. Each of the circles (nodes) are associated with a gauge group factor $\text{U}(N_{(j)})$, a CS level $k_{(j)}$ and contains a gauge multiplet $(\mathcal{V}_{(j)}, \Phi_{(j)})$. The subscript in parentheses (j) is used to label the nodes and the edges in the quiver which runs from 1 to n . The (j) -th edge is the one that joins the (j) -th node to the $(j+1)$ -th node. The arrows represent the bi-fundamental hypermultiplets $\mathcal{Z}_{(j)}$ and $\mathcal{W}_{(j)}$, in the representation $(\mathbf{N}_{(j)}, \bar{\mathbf{N}}_{(j+1)})$ and $(\bar{\mathbf{N}}_{(j)}, \mathbf{N}_{(j+1)})$ respectively. The gauge group of the theory is $\text{U}(N_{(1)}) \times \text{U}(N_{(2)}) \times \cdots \times \text{U}(N_{(n)})$. The CS levels satisfy $\sum_{(j)=1}^n \tilde{n}_{(j)} k_{(j)} = 0$ [30], where $\tilde{n}_{(j)}$ is the co-mark of the (j) -th node which is 1 for all (j) in this case, making $\sum_{(j)=1}^n k_{(j)} = 0$.

The above described field content can be used to write down an $\mathcal{N} = 3$ superconformal CS matter theory. Such a theory was constructed by Jafferis and Tomasiello in [19].

The superspace action of Yang-Mills deformed CS theory consists of following five parts,

$$\mathcal{S} = \mathcal{S}_{\text{CS}} + \mathcal{S}_{\text{YM}} + \mathcal{S}_{\text{adj}} + \mathcal{S}_{\text{mat}} + \mathcal{S}_{\text{pot}}. \quad (2.6)$$

The first three parts involve only vector multiplet fields and the last two parts involve the hypermultiplet fields together with their minimal couplings to the vector multiplet fields. The CS term is,

$$\mathcal{S}_{\text{CS}} = -\frac{i}{8\pi} \int d^3x d^4\theta \int_0^1 ds \sum_{(j)=1}^n \text{tr} \left[k_{(j)} \mathcal{V}_{(j)} \bar{D}^\alpha (e^{s\mathcal{V}_{(j)}} D_\alpha e^{-s\mathcal{V}_{(j)}}) \right]. \quad (2.7)$$

Figure 1. $\hat{A}_{n-1}$ quiver diagram.

The Yang-Mills term is,

$$\mathcal{S}_{\text{YM}} = \frac{1}{4g^2} \int d^3x d^2\theta \sum_{(j)=1}^n \text{tr} \left[\mathcal{U}_{(j)}^\alpha \mathcal{U}_{\alpha(j)} \right] \quad (2.8)$$

where, g is a coupling of mass dimension $\frac{1}{2}$ which is responsible for the RG flow and $\mathcal{U}_{\alpha(j)} = \frac{1}{4} \bar{D}^2 e^{\mathcal{V}_{(j)}} D_\alpha e^{-\mathcal{V}_{(j)}}$ is the super field strength.

The kinetic terms of the adjoint scalar and fermionic fields arise from,

$$\mathcal{S}_{\text{adj}} = \frac{1}{g^2} \int d^3x d^4\theta \sum_{(j)=1}^n \text{tr} \left[-\bar{\Phi}_{(j)} e^{-\mathcal{V}_{(j)}} \Phi_{(j)} e^{\mathcal{V}_{(j)}} \right]. \quad (2.9)$$

After introducing the dimensionful coupling g the theory is not conformal any more. At the IR fixed point $g \rightarrow \infty$ which sets $\mathcal{S}_{\text{YM}}$ and $\mathcal{S}_{\text{adj}}$ to zero thus making the gauge fields and the adjoint fields non-dynamical. We can then integrate out the $\Phi_{(j)}$'s and recover $\mathcal{N} = 3$ superconformal theory.

For the bifundamental matter fields we have the following minimally coupled action,

$$\mathcal{S}_{\text{mat}} = \int d^3x d^4\theta \sum_{(j)=1}^n \text{tr} \left[-\bar{\mathcal{Z}}_{(j)} e^{-\mathcal{V}_{(j)}} \mathcal{Z}_{(j)} e^{\mathcal{V}_{(j+1)}} - \bar{\mathcal{W}}_{(j)} e^{-\mathcal{V}_{(j+1)}} \mathcal{W}_{(j)} e^{\mathcal{V}_{(j)}} \right]. \quad (2.10)$$

The last part of the action is a superpotential term, which is,

$$\mathcal{S}_{\text{pot}} = \int d^3x d^2\theta \sum_{(j)=1}^n W_{(j)} - \int d^3x d^2\bar{\theta} \sum_{(j)=1}^n \bar{W}_{(j)} \quad (2.11)$$

where,

$$\begin{aligned} W_{(j)} &= \text{tr}(\Phi_{(j)} \mathcal{Z}_{(j)} \mathcal{W}_{(j)} - \Phi_{(j)} \mathcal{W}_{(j-1)} \mathcal{Z}_{(j-1)}) + \frac{k_{(j)}}{8\pi} \text{tr}(\Phi_{(j)} \Phi_{(j)}), \\ \bar{W}_{(j)} &= \text{tr}(\bar{\Phi}_{(j)} \bar{\mathcal{W}}_{(j)} \bar{\mathcal{Z}}_{(j)} - \bar{\Phi}_{(j)} \bar{\mathcal{Z}}_{(j-1)} \bar{\mathcal{W}}_{(j-1)}) + \frac{k_{(j)}}{8\pi} \text{tr}(\bar{\Phi}_{(j)} \bar{\Phi}_{(j)}). \end{aligned} \quad (2.12)$$

We can write the gauge transformations of the fields under which the action is invariant as,

$$\begin{aligned}\Phi_{(j)} &\longrightarrow e^{i\Lambda_{(j)}} \Phi_{(j)} e^{-i\Lambda_{(j)}}, & e^{\mathcal{V}_{(j)}} &\longrightarrow e^{i\Lambda_{(j)}} e^{\mathcal{V}_{(j)}} e^{-i\Lambda_{(j)}^\dagger}, \\ \mathcal{Z}_{(j)} &\longrightarrow e^{i\Lambda_{(j)}} \mathcal{Z}_{(j)} e^{-i\Lambda_{(j+1)}}, & \mathcal{W}_{(j)} &\longrightarrow e^{i\Lambda_{(j+1)}} \mathcal{W}_{(j)} e^{-i\Lambda_{(j)}}\end{aligned}\quad (2.13)$$

where, $\Lambda_{(j)} \in \text{U}(N_{(j)})$ is a chiral superfield.

We do Grassmann integration and integrate out the auxiliary fields to get the component action. Since the R-symmetry group $\text{SU}(2)$ is preserved all along the RG flow it is desirable to write the component action as follows where the $\text{SU}(2)_R$ symmetry is manifest.

The component action on $\mathbb{R}^{1,2}$ with signature $(-++)$, written in terms of the above $\text{SU}(2)_R$ multiplets is,

$$\begin{aligned}\mathcal{S}_{\text{kin}} = \int d^3x \sum_{(j)=1}^n \text{tr} \left[-\frac{1}{2g^2} F_{(j)}^{\mu\nu} F_{\mu\nu(j)} + \kappa_{(j)} \epsilon^{\mu\nu\lambda} \left(A_{\mu(j)} \partial_\nu A_{\lambda(j)} + \frac{2i}{3} A_{\mu(j)} A_{\nu(j)} A_{\lambda(j)} \right) \right. \\ \left. - \frac{1}{2g^2} \mathcal{D}_\mu \phi_{b(j)}^a \mathcal{D}^\mu \phi_{a(j)}^b - \frac{1}{2} \kappa_{(j)}^2 g^2 \phi_{b(j)}^a \phi_{a(j)}^b - \frac{i}{2g^2} \lambda_{(j)}^{ab} \not{D} \lambda_{ab(j)} - \frac{i\kappa_{(j)}}{2} \lambda_{(j)}^{ab} \lambda_{ba(j)} \right. \\ \left. - \mathcal{D}_\mu X_{(j)}^\dagger \mathcal{D}^\mu X_{(j)} + i\xi_{(j)}^\dagger \not{D} \xi_{(j)} \right]\end{aligned}\quad (2.14)$$

$$\begin{aligned}\mathcal{S}_{\text{int}} = \int d^3x \sum_{(j)=1}^n \text{tr} \left[-\kappa_{(j)} g^2 X_{a(j)}^\dagger \phi_{b(j)}^a X_{(j)}^b + \kappa_{(j)} g^2 X_{(j-1)}^a \phi_{a(j)}^b X_{b(j-1)}^\dagger \right. \\ \left. - i\xi_{a(j)}^\dagger \phi_{b(j)}^a \xi_{(j)}^b + i\xi_{(j-1)}^a \phi_{a(j)}^b \xi_{b(j-1)}^\dagger + \epsilon_{ac} \lambda_{(j)}^{cb} X_{(j)}^a \xi_{b(j)}^\dagger - \epsilon^{ac} \lambda_{cb(j)} \xi_{(j)}^b X_{a(j)}^\dagger \right. \\ \left. - \epsilon_{ac} \lambda_{(j)}^{cb} \xi_{b(j-1)}^\dagger X_{(j-1)}^a + \epsilon^{ac} \lambda_{cb(j)} X_{a(j-1)}^\dagger \xi_{b(j-1)}^b - \frac{\kappa_{(j)}}{6} \phi_{b(j)}^a [\phi_{c(j)}^b, \phi_{a(j)}^c] \right. \\ \left. - \frac{i}{2g^2} \lambda_{ab(j)} [\phi_{c(j)}^b, \lambda_{(j)}^{ac}] - \frac{g^2}{4} (X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j)} \sigma_i X_{(j)}^\dagger) - \frac{g^2}{4} (X_{(j)}^\dagger \sigma_i X_{(j)}) (X_{(j)}^\dagger \sigma_i X_{(j)}) \right. \\ \left. + \frac{g^2}{2} (X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j-1)}^\dagger \sigma_i X_{(j-1)}) - \frac{1}{2} (X_{(j)} X_{(j)}^\dagger) \phi_{b(j)}^a \phi_{a(j)}^b + X_{a(j)}^\dagger \phi_{c(j)}^b X_{(j)}^a \phi_{b(j+1)}^c \right. \\ \left. - \frac{1}{2} (X_{(j-1)}^\dagger X_{(j-1)}) \phi_{b(j)}^a \phi_{a(j)}^b + \frac{1}{8g^2} [\phi_{b(j)}^a, \phi_{d(j)}^c] [\phi_{a(j)}^b, \phi_{c(j)}^d] \right]\end{aligned}\quad (2.15)$$

where $\kappa_{(j)} \equiv \frac{k_{(j)}}{4\pi}$, $X_{(j)} \sigma_i X_{(j)}^\dagger \equiv X_{(j)}^a (\sigma_i)_a{}^b X_{b(j)}^\dagger$ and $X_{(j)}^\dagger \sigma_i X_{(j)} \equiv X_{a(j)}^\dagger (\sigma_i)^a{}_b X_{(j)}^b$. The $(\sigma_i)_a{}^b = \sigma_i$ are the usual Pauli matrices and the $(\sigma_i)^a{}_b = \sigma_i^T$ are the transpose of the Pauli matrices. The various gauge covariant derivatives above are

$$\begin{aligned}F_{\mu\nu(j)} &= \partial_\mu A_{\nu(j)} - \partial_\nu A_{\mu(j)} + i[A_{\mu(j)}, A_{\nu(j)}], & \mathcal{D}_\mu \phi_{b(j)}^a &= \partial_\mu \phi_{b(j)}^a + i[A_{\mu(j)}, \phi_{b(j)}^a], \\ \mathcal{D}_\mu \lambda_{(j)}^{ab} &= \partial_\mu \lambda_{(j)}^{ab} + i[A_{\mu(j)}, \lambda_{(j)}^{ab}], \\ \mathcal{D}_\mu X_{(j)}^b &= \partial_\mu X_{(j)}^b + iA_{\mu(j)} X_{(j)}^b - iX_{(j)}^b A_{\mu(j+1)}, & \mathcal{D}_\mu X_{b(j)}^\dagger &= \partial_\mu X_{b(j)}^\dagger + iA_{\mu(j+1)} X_{b(j)}^\dagger - iX_{b(j)}^\dagger A_{\mu(j)}, \\ \mathcal{D}_\mu \xi_{(j)}^b &= \partial_\mu \xi_{(j)}^b + iA_{\mu(j)} \xi_{(j)}^b - i\xi_{(j)}^b A_{\mu(j+1)}, & \mathcal{D}_\mu \xi_{b(j)}^\dagger &= \partial_\mu \xi_{b(j)}^\dagger + iA_{\mu(j+1)} \xi_{b(j)}^\dagger - i\xi_{b(j)}^\dagger A_{\mu(j)}.\end{aligned}$$

The full on-shell action in terms of the component fields are given in B.1. It can be checked that when the above action is unpacked following the $\text{SU}(2)_R$ symmetry notation introduced in 2 reproduces the expressions written in B.1. In (2.15), the term $(X_{(j-1)}^\dagger \sigma_i X_{(j-1)})$ should

Figure 2. $\hat{D}_n$ quiver diagram.

be taken as $(X_{(n)}^\dagger \sigma_i X_{(n)})$ for $(j) = 1$ and for $(j) = n$, $\phi_{b(j+1)}^a$ should be taken to be $\phi_{b(1)}^a$. The supersymmetry transformation parameter ε in an $\mathcal{N} = 3$ theory is in the **3** of $SU(2)_R$: $\varepsilon_{ab} = \varepsilon_i (\sigma_i)_{ab}$. The supersymmetry transformations of the non-auxiliary component fields that leave the action (2.14)+(2.15) invariant are

$$\begin{aligned} \delta A_{\mu(j)} &= -\frac{i}{2} \varepsilon_{ab} \gamma_\mu \lambda_{(j)}^{ab}, \\ \delta \lambda_{(j)}^{ab} &= \frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \gamma_\lambda \varepsilon^{ab} - i \not{D} \phi_{c(j)}^b \varepsilon^{ac} + \frac{i}{2} [\phi_{c(j)}^b, \phi_{d(j)}^c] \varepsilon^{ad} + i \kappa_{(j)} g^2 \phi_{c(j)}^b \varepsilon^{ac} \\ &\quad + i g^2 (X_{(j)}^a X_{c(j)}^\dagger \varepsilon^{cb} - \varepsilon^{bc} X_{c(j-1)}^\dagger X_{(j-1)}^a) - \frac{i g^2}{2} (X_{(j)} X_{(j)}^\dagger - X_{(j-1)}^\dagger X_{(j-1)}) \varepsilon^{ab}, \\ \delta \phi_{b(j)}^a &= -\varepsilon_{cb} \lambda_{(j)}^{ca} + \frac{1}{2} \delta_b^a \varepsilon_{cd} \lambda_{(j)}^{cd}, \end{aligned} \quad (2.16)$$

$$\begin{aligned} \delta X_{(j)}^a &= -i \varepsilon_b^a \xi_{(j)}^b, & \delta \xi_{(j)}^a &= \not{D} X_{(j)}^b \varepsilon_b^a + \phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_b^c \phi_{b(j+1)}^a, \\ \delta X_{a(j)}^\dagger &= -i \xi_{b(j)}^\dagger \varepsilon_a^b, & \delta \xi_{a(j)}^\dagger &= \not{D} X_{b(j)}^\dagger \varepsilon_a^b - \phi_{a(j+1)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b. \end{aligned} \quad (2.17)$$

2.2 $\hat{D}$ -type quiver

Figure 2 is a Dynkin diagram of $\hat{D}_n$ algebra. The main differences in $\hat{D}$ quiver diagram from the $\hat{A}$ -type quiver are the external nodes $(j) = 1, 2, 3, 4$ which have only one edge attached to them and the nodes labelled by (5) and $(n+1)$ which have three edges attached to them. The hypermultiplet corresponding to an external edge say (1) is $(\mathcal{Z}_{(1)}, \mathcal{W}_{(1)})$, which are in the gauge representations $(\mathbf{N}_{(1)}, \bar{\mathbf{N}}_{(5)})$ and $(\bar{\mathbf{N}}_{(1)}, \mathbf{N}_{(5)})$ respectively. The representations of the hypermultiplets for other external edges can be written similarly. Rest of the quiver, i.e from $(j) = 6, \dots, n$ is similar to the previous case. The CS levels satisfy $\sum_{(j)} \tilde{n}_{(j)} k_{(j)} = 0$. $\tilde{n}_{(j)} = 1$ for $(j) = 1, 2, 3, 4$ and $\tilde{n}_{(j)} = 2$ for $(j) = 5, 6, \dots, n+1$ which implies,

$$k_{(1)} + k_{(2)} + k_{(3)} + k_{(4)} + 2(k_{(5)} + \dots + k_{(n+1)}) = 0. \quad (2.18)$$

Such quiver gauge theory is dual to M-theory on $AdS_4 \times Y$, where, Y is the base of the hyperKähler cone $\mathbb{H}^{2n-8} // U(1)^n \times SU(2)^{n-3}$ [31].

The superspace action for $\mathcal{S}_{CS}$, $\mathcal{S}_{YM}$ and $\mathcal{S}_{adj}$ remains same except now the limit of summation runs from $(j) = 1$ to $n+1$. The rest of the action is as follows: the minimally

coupled action in the matter sector is as follows,

$$\begin{aligned}
 \mathcal{S}_{\text{mat}} = \int d^3x d^4\theta \, \text{tr} \left[\sum_{(j)=1}^2 \left(-\bar{\mathcal{Z}}_{(j)} e^{-\mathcal{V}_{(j)}} \mathcal{Z}_{(j)} e^{\mathcal{V}_{(5)}} - \bar{\mathcal{W}}_{(j)} e^{-\mathcal{V}_{(5)}} \mathcal{W}_{(j)} e^{\mathcal{V}_{(j)}} \right) \right. \\
 + \sum_{(j)=3}^4 \left(-\bar{\mathcal{Z}}_{(j)} e^{-\mathcal{V}_{(j)}} \mathcal{Z}_{(j)} e^{\mathcal{V}_{(n+1)}} - \bar{\mathcal{W}}_{(j)} e^{-\mathcal{V}_{(n+1)}} \mathcal{W}_{(j)} e^{\mathcal{V}_{(j)}} \right) \\
 \left. + \sum_{(j)=5}^n \left(-\bar{\mathcal{Z}}_{(j)} e^{-\mathcal{V}_{(j)}} \mathcal{Z}_{(j)} e^{\mathcal{V}_{(j+1)}} - \bar{\mathcal{W}}_{(j)} e^{-\mathcal{V}_{(j+1)}} \mathcal{W}_{(j)} e^{\mathcal{V}_{(j)}} \right) \right] \quad (2.19)
 \end{aligned}$$

and

$$\mathcal{S}_{\text{pot}} = \int d^3x d^2\theta W_{(j)} - \int d^3x d^2\bar{\theta} \bar{W}_{(j)}$$

with,

$$\begin{aligned}
 W_{(j)} = \text{tr} \left[\sum_{(j)=1}^n \Phi_{(j)} \mathcal{Z}_{(j)} \mathcal{W}_{(j)} + \sum_{(j)=1}^{n+1} \frac{k_{(j)}}{8\pi} [\Phi_{(j)} \Phi_{(j)}] \right. \\
 \left. - \sum_{(j)=1}^2 \Phi_{(5)} \mathcal{W}_{(j)} \mathcal{Z}_{(j)} - \sum_{(j)=3}^4 \Phi_{(n+1)} \mathcal{W}_{(j)} \mathcal{Z}_{(j)} - \sum_{(j)=5}^n \Phi_{(j+1)} \mathcal{W}_{(j)} \mathcal{Z}_{(j)} \right], \\
 \bar{W}_{(j)} = \text{tr} \left[\sum_{(j)=1}^n \bar{\Phi}_{(j)} \bar{\mathcal{W}}_{(j)} \bar{\mathcal{Z}}_{(j)} + \sum_{(j)=1}^{n+1} \frac{k_{(j)}}{8\pi} [\bar{\Phi}_{(j)} \bar{\Phi}_{(j)}] \right. \\
 \left. - \sum_{(j)=1}^2 \bar{\Phi}_{(5)} \bar{\mathcal{Z}}_{(j)} \bar{\mathcal{W}}_{(j)} - \sum_{(j)=3}^4 \bar{\Phi}_{(n+1)} \bar{\mathcal{Z}}_{(j)} \bar{\mathcal{W}}_{(j)} - \sum_{(j)=5}^n \bar{\Phi}_{(j+1)} \bar{\mathcal{Z}}_{(j)} \bar{\mathcal{W}}_{(j)} \right]. \quad (2.20)
 \end{aligned}$$

This gives the following component action after doing the Grassmann integral and eliminating the auxiliary fields. The interaction part of the action is,

$$\begin{aligned}
 \mathcal{S}_{\text{int}} = \int d^3x \, \text{tr} \left(\sum_{(j)=1}^n \left[-i\xi_{a(j)}^\dagger \phi_{b(j)}^a \xi_{(j)}^b + \epsilon_{ac} \lambda_{(j)}^{cb} X_{(j)}^a \xi_{b(j)}^\dagger - \epsilon^{ac} \lambda_{cb(j)} \xi_{(j)}^b X_{a(j)}^\dagger \right. \right. \\
 - g^2 \kappa_{(j)} X_{a(j)}^\dagger \phi_{b(j)}^a X_{(j)}^b - \frac{1}{2} X_{(j)} X_{(j)}^\dagger \phi_{b(j)}^a \phi_{a(j)}^b - \frac{g^2}{4} (X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j)} \sigma_i X_{(j)}^\dagger) \\
 \left. - \frac{g^2}{4} (X_{(j)}^\dagger \sigma_i X_{(j)}) (X_{(j)}^\dagger \sigma_i X_{(j)}) \right] + \sum_{(j)=1}^2 \left[-\epsilon_{ac} \lambda_{(5)}^{cb} \xi_{b(j)}^\dagger X_{(j)}^a + \epsilon^{ac} \lambda_{cb(5)} X_{a(j)}^\dagger \xi_{(j)}^b + i\xi_{(j)}^a \phi_{a(5)}^b \xi_{b(j)}^\dagger \right. \\
 + -\frac{1}{2} (X_{(j)}^\dagger X_{(j)}) \phi_{b(5)}^a \phi_{a(5)}^b + X_{(j)a}^\dagger \phi_{c(j)}^a X_{(j)}^c \phi_{b(5)}^c + \kappa_{(5)} g^2 X_{(j)}^a \phi_{a(5)}^b X_{b(j)}^\dagger \\
 \left. + \frac{g^2}{2} (X_{(5)} \sigma_i X_{(5)}^\dagger) (X_{(j)}^\dagger \sigma_i X_{(j)}) \right] + \sum_{(j)=3}^4 \left[-\epsilon_{ac} \lambda_{(n+1)}^{cb} \xi_{b(j)}^\dagger X_{(j)}^a + \epsilon^{ac} \lambda_{cb(n+1)} X_{a(j)}^\dagger \xi_{(j)}^b \right. \\
 + i\xi_{(j)}^a \phi_{a(n+1)}^b \xi_{b(j)}^\dagger - \frac{1}{2} (X_{(j)}^\dagger X_{(j)}) \phi_{b(n+1)}^a \phi_{a(n+1)}^b + X_{(j)a}^\dagger \phi_{c(j)}^a X_{(j)}^c \phi_{b(n+1)}^c \\
 \left. + \kappa_{(n+1)} g^2 X_{(j)}^a \phi_{a(n+1)}^b X_{b(j)}^\dagger - \frac{g^2}{2} (X_{(j)}^\dagger \sigma_i X_{(j)}) (X_{(n)}^\dagger \sigma_i X_{(n)}) \right] + \sum_{(j)=5}^n \left[-\epsilon_{ac} \lambda_{(j+1)}^{cb} \xi_{b(j)}^\dagger X_{(j)}^a \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \epsilon^{ac} \lambda_{cb(j+1)} X_{a(j)}^\dagger \xi_{(j)}^b + i \xi_{(j)}^a \phi_{a(j+1)}^b \xi_{b(j)}^\dagger + g^2 \kappa_{(j+1)} X_{(j)}^a \phi_{a(j+1)}^b X_{b(j)}^\dagger - \frac{1}{2} (X_{(j)}^\dagger X_{(j)}) \phi_{b(j+1)}^a \phi_{a(j+1)}^b \\
 & + X_{(j)a}^\dagger \phi_{c(j)}^b X_{(j)}^a \phi_{b(j+1)}^c \Big] + \sum_{(j)=1}^{n+1} \left[\frac{1}{8g^2} [\phi_{b(j)}^a, \phi_{d(j)}^c] [\phi_{a(j)}^b, \phi_{c(j)}^d] - \frac{\kappa_{(j)}}{6} \phi_{b(j)}^a [\phi_{c(j)}^b \phi_{a(j)}^c] \right. \\
 & \left. - \frac{i}{2g^2} \lambda_{ab(j)} [\phi_{c(j)}^b, \lambda^{ac}_{(j)}] \right] + \frac{g^2}{2} \sum_{(j)=6}^n (X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j-1)}^\dagger \sigma_i X_{(j-1)}) \\
 & - \frac{g^2}{2} (X_{(1)}^\dagger \sigma_i X_{(1)}) (X_{(2)}^\dagger \sigma_i X_{(2)}) - \frac{g^2}{2} (X_{(3)}^\dagger \sigma_i X_{(3)}) (X_{(4)}^\dagger \sigma_i X_{(4)}) \Big). \tag{2.21}
 \end{aligned}$$

The kinetic part of the action is,

$$\begin{aligned}
 \mathcal{S}_{\text{kin}} = \int d^3x \text{ tr} \Bigg[\sum_{(j)=1}^{n+1} \Big(- \frac{1}{2g^2} F_{(j)}^{\mu\nu} F_{\mu\nu(j)} + \kappa_{(j)} \epsilon^{\mu\nu\lambda} (A_{\mu(j)} \partial_\nu A_{\lambda(j)} + \frac{2i}{3} A_{\mu(j)} A_{\nu(j)} A_{\lambda(j)}) \\
 - \frac{1}{2g^2} \mathcal{D}_\mu \phi_{b(j)}^a \mathcal{D}^\mu \phi_{a(j)}^b - \frac{i}{2g^2} \lambda_{(j)}^{ab} \not{D} \lambda_{ab(j)} - \frac{i\kappa_{(j)}}{2} \lambda_{(j)}^{ab} \lambda_{ba(j)} - \frac{1}{2} \kappa_{(j)}^2 g^2 \phi_{b(j)}^a \phi_{a(j)}^b \Big) \\
 + \text{tr} \sum_{(j)=1}^n \Big(- \mathcal{D}_\mu X_{(j)}^\dagger \mathcal{D}^\mu X_{(j)} + i \xi_{(j)}^\dagger \not{D} \xi_{(j)} \Big) \Big] \tag{2.22}
 \end{aligned}$$

where the covariant derivatives of the bi-fundamental fields are

$$\begin{aligned}
 (j) = 1, 2: \quad \mathcal{D}_\mu X_{(j)}^b &= \partial_\mu X_{(j)}^b + i A_{\mu(j)} X_{(j)}^b - i X_{(j)}^b A_{\mu(5)}, \quad \mathcal{D}_\mu \xi_{(j)}^b = \partial_\mu \xi_{(j)}^b + i A_{\mu(j)} \xi_{(j)}^b - i \xi_{(j)}^b A_{\mu(5)}, \\
 (j) = 3, 4: \quad \mathcal{D}_\mu X_{(j)}^b &= \partial_\mu X_{(j)}^b + i A_{\mu(j)} X_{(j)}^b - i X_{(j)}^b A_{\mu(n+1)}, \quad \mathcal{D}_\mu \xi_{(j)}^b = \partial_\mu \xi_{(j)}^b + i A_{\mu(j)} \xi_{(j)}^b - i \xi_{(j)}^b A_{\mu(n+1)}.
 \end{aligned}$$

Rest of the covariant derivatives are same as in $\hat{A}$ case. The supersymmetry transformations of the vector multiplet fields are

$$\begin{aligned}
 \delta A_{\mu(j)} &= -\frac{i}{2} \epsilon_{ab} \gamma_\mu \lambda_{(j)}^{ab}, \quad \delta \phi_{b(j)}^a = -\epsilon_{cb} \lambda_{(j)}^{ca} + \frac{1}{2} \delta_b^a \epsilon_{cd} \lambda_{(j)}^{cd}, \quad (j) = 1, \dots, n+1 \\
 \delta \lambda_{(j)}^{ab} &= \frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \gamma_\lambda \epsilon^{ab} - i \not{D} \phi_{c(j)}^b \epsilon^{ac} + \frac{i}{2} [\phi_{c(j)}^b, \phi_{d(j)}^c] \epsilon^{ad} + i \kappa_{(j)} g^2 \phi_{c(j)}^b \epsilon^{ac} \\
 &+ i g^2 (X_{(j)}^a X_{c(j)}^\dagger \epsilon^{cb} - \epsilon^{bc} X_{c(j-1)}^\dagger X_{(j-1)}^a) + \frac{i g^2}{2} ((X^\dagger X)_{(j-1)} - (X X^\dagger)_{(j)}) \epsilon^{ab}, \quad (j) = 6, \dots, n \\
 \delta \lambda_{(5)}^{ab} &= \frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(5)} \gamma_\lambda \epsilon^{ab} - i \not{D} \phi_{c(5)}^b \epsilon^{ac} + \frac{i}{2} [\phi_{c(5)}^b, \phi_{d(5)}^c] \epsilon^{ad} + i \kappa_{(5)} g^2 \phi_{c(5)}^b \epsilon^{ac} \\
 &+ i g^2 \left(X_{(5)}^a X_{c(5)}^\dagger \epsilon^{cb} - \epsilon^{bc} \sum_{(j)=1}^2 X_{c(j)}^\dagger X_{(j)}^a \right) + \frac{i g^2}{2} \left(\sum_{(j)=1}^2 (X^\dagger X)_{(j)} - (X X^\dagger)_{(5)} \right) \epsilon^{ab}, \\
 \delta \lambda_{(n+1)}^{ab} &= \frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(n+1)} \gamma_\lambda \epsilon^{ab} - i \not{D} \phi_{c(n+1)}^b \epsilon^{ac} + \frac{i}{2} [\phi_{c(n+1)}^b, \phi_{d(n+1)}^c] \epsilon^{ad} + i \kappa_{(n+1)} g^2 \phi_{c(n+1)}^b \epsilon^{ac} \\
 &- i g^2 \epsilon^{bc} \sum_{(j)=3,4,n} X_{c(j)}^\dagger X_{(j)}^a + \frac{i g^2}{2} \sum_{(j)=3,4,n} (X^\dagger X)_{(j)} \epsilon^{ab}, \\
 \delta \lambda_{(j)}^{ab} &= \frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \gamma_\lambda \epsilon^{ab} - i \not{D} \phi_{c(j)}^b \epsilon^{ac} + \frac{i}{2} [\phi_{c(j)}^b, \phi_{d(j)}^c] \epsilon^{ad} + i \kappa_{(j)} g^2 \phi_{c(j)}^b \epsilon^{ac} \\
 &+ i g^2 X_{(j)}^a X_{c(j)}^\dagger \epsilon^{cb} - \frac{i g^2}{2} (X X^\dagger)_{(j)} \epsilon^{ab} \quad (j) = 1, 2, 3, 4. \tag{2.23}
 \end{aligned}$$

Note: the reason that the transformations of $\lambda_{(n+1)}^{ab}$ and $\lambda_{(5)}^{ab}$ are different in spite of their symmetry in the quiver diagram is because of our convention of labelling the $X_{(j)}$'s. There

Figure 3. $\widehat{E}_6$ quiver diagram.

is no edge labelled by $(n+1)$. Therefore the analogues of the terms $g^2 i X_{(5)}^a X_{c(5)}^\dagger \varepsilon^{cb}$ and $\frac{ig^2}{2} (\sum_{(j)=1}^2 - (XX^\dagger)_{(5)}) \varepsilon^{ab}$ in $\delta \lambda_{(5)}^{ab}$ are not present in $\delta \lambda_{(n+1)}^{ab}$. One other way to check this is, the term $g^2 i \sum_{(j)=3,4,n} X_{(j)}^a X_{c(j)}^\dagger \varepsilon^{cb}$ is a matrix of size 3×3 while the L.H.S $\delta \lambda_{(n+1)}^{ab}$, is an $(n+1) \times (n+1)$ matrix.

The supersymmetry transformations of the hypermultiplet fields are

$$\begin{aligned}
 (j) = 1, \dots, n : \quad & \delta X_{(j)}^a = -i \varepsilon_b^a \xi_{(j)}^b, \quad \delta X_{a(j)}^\dagger = -i \xi_{b(j)}^\dagger \varepsilon_a^b, \\
 (j) = 1, 2 : \quad & \delta \xi_{(j)}^a = \not{D} X_{(j)}^b \varepsilon_b^a + \phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(5)}^a, \\
 & \delta \xi_{a(j)}^\dagger = \not{D} X_{b(j)}^\dagger \varepsilon_a^b - \phi_{a(5)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b, \\
 (j) = 3, 4 : \quad & \delta \xi_{(j)}^a = \not{D} X_{(j)}^b \varepsilon_b^a + \phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(n+1)}^a, \\
 & \delta \xi_{a(j)}^\dagger = \not{D} X_{b(j)}^\dagger \varepsilon_a^b - \phi_{a(n+1)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b, \\
 (j) = 5, 6, \dots, n : \quad & \delta \xi_{(j)}^a = \not{D} X_{(j)}^b \varepsilon_b^a + \phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(j+1)}^a, \\
 & \delta \xi_{a(j)}^\dagger = \not{D} X_{b(j)}^\dagger \varepsilon_a^b - \phi_{a(j+1)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b. \tag{2.24}
 \end{aligned}$$

2.3 $\widehat{E}_6$ quiver

The $\widehat{E}_6$ quiver is similar to $\widehat{D}_n$ quiver diagram after adding some extra nodes and mapping 7-th node of the former to $n+1$ -th node of the latter. The quiver diagram is given in figure 3. Therefore all the notions can be generalised easily from the $\widehat{D}_n$ case. The comarks in this case are, $\tilde{n}_{(j)} = 1$ for $(j) = 1, 3, 5$, $\tilde{n}_{(j)} = 2$ for $(j) = 2, 4, 6$ and $\tilde{n}_{(j)} = 3$ for $(j) = 7$ which implies that the CS levels satisfy,

$$\kappa_{(1)} + \kappa_{(3)} + \kappa_{(5)} + 2(\kappa_{(2)} + \kappa_{(4)} + \kappa_{(6)}) + 3\kappa_{(7)} = 0. \tag{2.25}$$

The first three parts of the action (2.6) remains same as except the summation limit changes from 1 to 7. We write down the last two parts of the action below.

$$\begin{aligned}
 \mathcal{S}_{\text{mat}} = \int d^3x d^4\theta \text{tr} \left[\sum_{(j)=2,4,6} \left(-\bar{Z}_{(j)} e^{-\nu_{(j)}} Z_{(j)} e^{\nu_{(7)}} - \bar{W}_{(j)} e^{-\nu_{(7)}} W_{(j)} e^{\nu_{(j)}} \right) \right. \\
 \left. + \sum_{(j)=1,3,5} \left(-\bar{Z}_{(j)} e^{-\nu_{(j)}} Z_{(j)} e^{\nu_{(j+1)}} - \bar{W}_{(j)} e^{-\nu_{(j+1)}} W_{(j)} e^{\nu_{(j)}} \right) \right] \tag{2.26}
 \end{aligned}$$

and

$$\mathcal{S}_{\text{pot}} = \int d^3x d^2\theta W_{(j)} - \int d^3x d^2\bar{\theta} \bar{W}_{(j)}$$

with,

$$\begin{aligned} W_{(j)} = \text{tr} & \left[\sum_{(j)=1}^6 \Phi_{(j)} \mathcal{Z}_{(j)} \mathcal{W}_{(j)} + \sum_{(j)=1}^7 \frac{k_{(j)}}{8\pi} [\Phi_{(j)} \Phi_{(j)}] \right. \\ & \left. - \sum_{(j)=2,4,6} \Phi_{(7)} \mathcal{W}_{(j)} \mathcal{Z}_{(j)} - \sum_{(j)=1,3,5} \Phi_{(j+1)} \mathcal{W}_{(j)} \mathcal{Z}_{(j)} \right]. \end{aligned} \quad (2.27)$$

We write down the interaction part of the action since the kinetic part is straight forward from the previous cases.

$$\begin{aligned} \mathcal{S}_{\text{int}} = \int d^3x \text{tr} & \left(\sum_{(j)=1}^6 \left[-i\xi_{a(j)}^\dagger \phi_{b(j)}^a \xi_{(j)}^b + \epsilon_{ac} \lambda^{cb}_{(j)} X_{(j)}^a \xi_{b(j)}^\dagger - \epsilon^{ac} \lambda_{cb(j)} \xi_{(j)}^b X_{a(j)}^\dagger - g^2 \kappa_{(j)} X_{a(j)}^\dagger \phi_{b(j)}^a X_{(j)}^b \right. \right. \\ & \left. - \frac{1}{2} X_{(j)} X_{(j)}^\dagger \phi_{b(j)}^a \phi_{a(j)}^b - \frac{g^2}{4} (X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j)} \sigma_i X_{(j)}^\dagger) - \frac{g^2}{4} (X_{(j)}^\dagger \sigma_i X_{(j)}) (X_{(j)}^\dagger \sigma_i X_{(j)}) \right] \\ & + \sum_{(j)=2,4,6} \left[-\epsilon_{ac} \lambda_{(7)}^{cb} \xi_{b(j)}^\dagger X_{(j)}^a + \epsilon^{ac} \lambda_{cb(7)} X_{a(j)}^\dagger \xi_{(j)}^b + i\xi_{(j)}^a \phi_{a(7)}^b \xi_{b(j)}^\dagger - \frac{1}{2} (X_{(j)}^\dagger X_{(j)}) \phi_{b(7)}^a \phi_{a(7)}^b \right. \\ & + X_{a(j)}^\dagger \phi_{c(j)}^b X_{(j)}^a \phi_{b(7)}^c + \kappa_{(7)} g^2 X_{(j)}^a \phi_{a(7)}^b X_{b(j)}^\dagger - \epsilon_{ac} \lambda_{(j)}^{cb} \xi_{b(j-1)}^\dagger X_{(j-1)}^a + \epsilon^{ac} \lambda_{cb(j)} X_{a(j-1)}^\dagger \xi_{(j-1)}^b \\ & + i\xi_{(j-1)}^a \phi_{a(j)}^b \xi_{b(j-1)}^\dagger + g^2 \kappa_{(j)} X_{(j-1)}^a \phi_{a(j)}^b X_{b(j-1)}^\dagger - \frac{1}{2} (X_{(j-1)}^\dagger X_{(j-1)}) \phi_{b(j)}^a \phi_{a(j)}^b \\ & \left. + \frac{g^2}{2} (X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j-1)}^\dagger \sigma_i X_{(j-1)}) \right] + \sum_{(j)=1,3,5} X_{a(j)}^\dagger \phi_{c(j)}^b X_{(j)}^a \phi_{b(j+1)}^c \\ & + \sum_{(j)=1}^7 \left(\frac{1}{8g^2} [\phi_{b(j)}^a, \phi_{d(j)}^c] [\phi_{a(j)}^b, \phi_{c(j)}^d] - \frac{\kappa_{(j)}}{6} \phi_{b(j)}^a [\phi_{c(j)}^b \phi_{a(j)}^c] - \frac{i}{2g^2} \lambda_{ab(j)} [\phi_{c(j)}^b, \lambda^{ac}_{(j)}] \right) \\ & \left. - \frac{g^2}{2} (X_{(2)}^\dagger \sigma_i X_{(2)}) (X_{(4)}^\dagger \sigma_i X_{(4)}) - \sum_{(j)=2,4} \frac{g^2}{2} (X_{(j)}^\dagger \sigma_i X_{(j)}) (X_{(6)}^\dagger \sigma_i X_{(6)}) \right) \end{aligned} \quad (2.28)$$

The supersymmetry variations of $A_{\mu(j)}$, $\phi_{b(j)}^a$, $X_{(j)}^a$, $X_{a(j)}^\dagger$ are similar to the previous cases. The rest of the supersymmetry variations are

$$\begin{aligned} \delta\lambda_{(j)}^{ab} &= \frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \gamma_\lambda \epsilon^{ab} - i\mathcal{D} \phi_{c(j)}^b \epsilon^{ac} + \frac{i}{2} [\phi_{c(j)}^b, \phi_{d(j)}^c] \epsilon^{ad} + i\kappa_{(j)} g^2 \phi_{c(j)}^b \epsilon^{ac} \\ &+ ig^2 (X_{(j)}^a X_{c(j)}^\dagger \epsilon^{cb} - \epsilon^{bc} X_{c(j-1)}^\dagger X_{(j-1)}^a) + \frac{ig^2}{2} ((X^\dagger X)_{(j-1)} - (X X^\dagger)_{(j)}) \epsilon^{ab} \\ &\quad (j) = 2, 4, 6, \\ \delta\lambda_{(7)}^{ab} &= \frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(7)} \gamma_\lambda \epsilon^{ab} - i\mathcal{D} \phi_{c(7)}^b \epsilon^{ac} + \frac{i}{2} [\phi_{c(7)}^b, \phi_{d(7)}^c] \epsilon^{ad} + i\kappa_{(7)} g^2 \phi_{c(7)}^b \epsilon^{ac} \\ &- ig^2 \epsilon^{bc} \sum_{(j)=2,4,6} X_{c(j)}^\dagger X_{(j)}^a + \frac{ig^2}{2} \sum_{(j)=2,4,6} (X^\dagger X)_{(j)} \epsilon^{ab}, \\ \delta\lambda_{(j)}^{ab} &= \frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \gamma_\lambda \epsilon^{ab} - i\mathcal{D} \phi_{c(j)}^b \epsilon^{ac} + \frac{i}{2} [\phi_{c(j)}^b, \phi_{d(j)}^c] \epsilon^{ad} + i\kappa_{(j)} g^2 \phi_{c(j)}^b \epsilon^{ac} \\ &+ ig^2 X_{(j)}^a X_{c(j)}^\dagger \epsilon^{cb} - \frac{ig^2}{2} (X X^\dagger)_{(j)} \epsilon^{ab} \quad (j) = 1, 3, 5, \end{aligned} \quad (2.29)$$

$$\begin{aligned}
 (j) = 2, 4, 6 : \quad & \delta \xi_{(j)}^a = \not{D} X_{(j)}^b \varepsilon_b^a + \phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(7)}^a, \\
 & \delta \xi_{a(j)}^\dagger = \not{D} X_{b(j)}^\dagger \varepsilon_a^b - \phi_{a(7)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b, \\
 (j) = 1, 3, 5 \quad & \delta \xi_{(j)}^a = \not{D} X_{(j)}^b \varepsilon_b^a + \phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(j+1)}^a, \\
 & \delta \xi_{a(j)}^\dagger = \not{D} X_{b(j)}^\dagger \varepsilon_a^b - \phi_{a(j+1)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b.
 \end{aligned} \tag{2.30}$$

2.4 Action and supersymmetry variations on $\mathbb{R} \times S^2$

Our main goal is to compute the scaling dimensions of the BPS monopole operators in the theory which is related to its charges by operator state correspondence in a radially quantized system. Following the method of BKK, to obtain the action on $\mathbb{R} \times S^2$ from the action on $\mathbb{R}^{1,2}$ one has to carry out the following steps. First we do a Wick rotation to go from $\mathbb{R}^{1,2}$ to $\mathbb{R}^3$ in the following way,

$$x^0 \longrightarrow -ix^3, \quad A^0 \longrightarrow -iA^3, \quad \gamma^0 \longrightarrow -i\gamma^3. \tag{2.31}$$

The next step is to do a co-ordinate transformation from (x^1, x^2, x^3) to the spherical polar co-ordinates (r, θ, φ) . Under this co-ordinate transformation all the terms in the action except the measure is invariant since they are written in a co-ordinate independent form. Then we translate the action to $\mathbb{R} \times S^2$ by introducing a radial variable τ by the following relation,

$$r = e^\tau \tag{2.32}$$

where, $\tau \in \mathbb{R}$ runs from $-\infty$ to $+\infty$, is the Euclidean time. Now one has to perform a Weyl rescaling of the metric by a factor of $e^{-2\tau}$ so that the theory lands on $\mathbb{R} \times S^2$ described by (τ, θ, φ) co-ordinate system. The metric after Weyl rescaling is, $g_{mn} = \text{diag}(1, 1, \sin^2 \theta)$. $m, n = \tau, \theta, \varphi$ are used to denote the space-time indices on $\mathbb{R} \times S^2$. One has to also rescale the $\mathbb{R}^3$ fields to obtain $\mathbb{R} \times S^2$ fields in the following way,

$$\mathcal{X}_{\mathbb{R}^3} = e^{-\dim(\mathcal{X})\tau} \mathcal{X}_{\mathbb{R} \times S^2} \tag{2.33}$$

where, $\mathcal{X}$ is any generic field. The coupling g is also rescaled by,

$$g \longrightarrow e^{-\frac{\tau}{2}} \tilde{g}. \tag{2.34}$$

$\tilde{g}$ is the Yang-Mills coupling on $\mathbb{R} \times S^2$ (for the component fields we do not use any tilde). The vector multiplet fermions remains auxiliary in the IR and drop out from the theory. But since they were dynamical in the UV and can effect the charges of the monopole operators by quantum fluctuation one has to rescale them before doing the Weyl rescaling in the following way,

$$\lambda_{ab(j)} \longrightarrow g \lambda_{ab(j)}. \tag{2.35}$$

2.4.1 $\hat{A}$ -type quiver

Carrying out all the steps described above we convert the action on $\mathbb{R}^{1,2}$ to $\mathbb{R} \times S^2$. The kinetic part of the action is,

$$\begin{aligned} \mathcal{S}_{\text{kin}}^E = \int d\tau d\Omega \sum_{(j)=1}^n \text{tr} \left(\frac{1}{2\tilde{g}^2} F_{(j)}^{mn} F_{mn(j)} - i\kappa_{(j)} \epsilon^{mnk} \left(A_{m(j)} \partial_n A_{k(j)} + \frac{2i}{3} A_{m(j)} A_{n(j)} A_{k(j)} \right) \right. \\ \left. + \mathcal{D}_m X_{(j)}^\dagger \mathcal{D}^m X_{(j)} + \frac{1}{4} X_{(j)}^\dagger X_{(j)} - i\xi_{(j)}^\dagger \not{D} \xi_{(j)} + \frac{1}{2\tilde{g}^2} \mathcal{D}_m \phi_{b(j)}^a \mathcal{D}^m \phi_{a(j)}^b + \frac{1}{2} \kappa_{(j)}^2 \tilde{g}^2 \phi_{b(j)}^a \phi_{a(j)}^b \right. \\ \left. + \frac{i}{2} \lambda_{ab(j)}^\dagger \not{D} \lambda_{ab(j)} + \frac{i\kappa_{(j)} \tilde{g}^2}{2} \lambda_{(j)}^{ab} \lambda_{ba(j)} \right). \end{aligned} \quad (2.36)$$

The interaction part is,

$$\begin{aligned} \mathcal{S}_{\text{int}}^E = \int d\tau d\Omega \sum_{(j)=1}^n \text{tr} \left(\kappa_{(j)} \tilde{g}^2 X_{a(j)}^\dagger \phi_{b(j)}^a X_{(j)}^b - \kappa_{(j)} \tilde{g}^2 X_{a(j-1)}^\dagger \phi_{b(j)}^a X_{b(j-1)}^\dagger + i\xi_{a(j)}^\dagger \phi_{b(j)}^a \xi_{(j)}^b \right. \\ \left. - i\xi_{a(j-1)}^\dagger \phi_{b(j-1)}^a \xi_{b(j-1)}^\dagger - \tilde{g} \epsilon_{ac} \lambda_{(j)}^{cb} X_{(j)}^a \xi_{b(j)}^\dagger + \tilde{g} \epsilon^{ac} \lambda_{cb(j)} \xi_{(j)}^b X_{a(j)}^\dagger + \tilde{g} \epsilon_{ac} \lambda_{(j)}^{cb} \xi_{b(j-1)}^\dagger X_{a(j-1)}^a \right. \\ \left. - \tilde{g} \epsilon^{ac} \lambda_{cb(j)} X_{a(j-1)}^\dagger \xi_{(j-1)}^b - \frac{\kappa_{(j)}}{6} \phi_{b(j)}^a [\phi_{c(j)}^b, \phi_{a(j)}^c] + \frac{i}{2} \lambda_{ab(j)} [\phi_{c(j)}^b, \lambda^{ac}_{(j)}] \right. \\ \left. + \frac{\tilde{g}^2}{4} (X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j)} \sigma_i X_{(j)}^\dagger) + \frac{\tilde{g}^2}{4} (X_{(j)}^\dagger \sigma_i X_{(j)}) (X_{(j)}^\dagger \sigma_i X_{(j)}) + \frac{1}{2} (X_{(j)} X_{(j)}^\dagger) \phi_{b(j)}^a \phi_{a(j)}^b \right. \\ \left. - \frac{\tilde{g}^2}{2} (X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j-1)}^\dagger \sigma_i X_{(j-1)}) + \frac{1}{2} (X_{(j-1)}^\dagger X_{(j-1)}) \phi_{b(j)}^a \phi_{a(j)}^b - X_{a(j)}^\dagger \phi_{c(j)}^b X_{(j)}^a \phi_{b(j+1)}^c \right. \\ \left. - \frac{1}{8\tilde{g}^2} [\phi_{b(j)}^a, \phi_{d(j)}^c] [\phi_{a(j)}^b, \phi_{c(j)}^d] \right). \end{aligned} \quad (2.37)$$

The supersymmetry variations obtained for the action on $\mathbb{R}^{1,2}$ can also be converted to $\mathbb{R} \times S^2$ by following same steps as stated above. The supersymmetry variation for the action on $\mathbb{R} \times S^2$ with the rescaled variation parameter $\tilde{\varepsilon}_{ab} = e^{-\frac{\tau}{2}} \varepsilon_{ab}$ are as follows,

$$\begin{aligned} \delta A_{m(j)} = -\frac{i\tilde{g}}{2} \varepsilon_{ab} \gamma_m \lambda_{(j)}^{ab}, \quad \delta \phi_{b(j)}^a = -\tilde{g} \varepsilon_{cb} \lambda_{(j)}^{ca} + \frac{\tilde{g}}{2} \delta_b^a \varepsilon_{cd} \lambda_{(j)}^{cd}, \\ \delta \lambda_{(j)}^{ab} = \frac{i}{2\tilde{g}} \epsilon^{mnk} F_{mn(j)} \gamma_k \varepsilon^{ab} - \frac{i}{\tilde{g}} \not{D}(\phi_{c(j)}^b) \varepsilon^{ac} - \frac{2i}{3\tilde{g}} \phi_{c(j)}^b \not{D} \varepsilon^{ac} + \frac{i}{2\tilde{g}} [\phi_{c(j)}^b, \phi_{d(j)}^c] \varepsilon^{ad} \\ + i\kappa_{(j)} \tilde{g} \phi_{c(j)}^b \varepsilon^{ac} + i\tilde{g} (X_{(j)}^a X_{c(j)}^\dagger \varepsilon^{cb} - \varepsilon^{bc} X_{c(j-1)}^\dagger X_{(j-1)}^a) \\ + \frac{i\tilde{g}}{2} ((X^\dagger X)_{(j-1)} - (X X^\dagger)_{(j)}) \varepsilon^{ab}, \end{aligned} \quad (2.38)$$

$$\begin{aligned} \delta X_{(j)}^a = -i\varepsilon_b^a \xi_{(j)}^b, \quad \delta \xi_{(j)}^a = \not{D} X_{(j)}^b \varepsilon_b^a + \frac{1}{3} X_{(j)}^b \not{D} \varepsilon_b^a + \phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(j+1)}^a, \\ \delta X_{a(j)}^\dagger = -i\xi_{b(j)}^\dagger \varepsilon_b^a, \quad \delta \xi_{a(j)}^\dagger = \not{D} X_{b(j)}^\dagger \varepsilon_a^b + \frac{1}{3} X_{b(j)}^\dagger \not{D} \varepsilon_a^b - \phi_{a(j+1)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b. \end{aligned} \quad (2.39)$$

2.4.2 $\hat{D}$ -type quiver

The kinetic part remains similar to the previous case. We write down the interaction part of the action on $\mathbb{R} \times S^2$,

$$\begin{aligned}
 \mathcal{S}_{\text{int}}^E = & \int d\tau d\Omega \operatorname{tr} \left(\sum_{(j)=1}^n \left(\kappa_{(j)} \tilde{g}^2 X_{(j)a}^\dagger \phi_{b(j)}^a X_{(j)}^b + i \xi_{a(j)}^\dagger \phi_{b(j)}^a \xi_{(j)}^b - \tilde{g} \epsilon_{ac} \lambda_{(j)}^{cb} X_{(j)}^a \xi_{b(j)}^\dagger \right. \right. \\
 & + \tilde{g} \epsilon^{ac} \lambda_{cb(j)} \xi_{(j)}^b X_{a(j)}^\dagger + \frac{1}{2} X_{(j)} X_{(j)}^\dagger \phi_{b(j)}^a \phi_{a(j)}^b + \frac{\tilde{g}^2}{4} (X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j)} \sigma_i X_{(j)}^\dagger) \\
 & + \frac{\tilde{g}^2}{4} (X_{(j)}^\dagger \sigma_i X_{(j)}) (X_{(j)}^\dagger \sigma_i X_{(j)}) \Big) + \sum_{(j)=1}^2 \left(\tilde{g} \epsilon_{ac} \lambda_{(5)}^{cb} \xi_{b(j)}^\dagger X_{(j)}^a - \tilde{g} \epsilon^{ac} \lambda_{cb(5)} X_{a(j)}^\dagger \xi_{(j)}^b \right. \\
 & - i \xi_{(j)}^a \phi_{a(5)}^b \xi_{b(j)}^\dagger + \frac{1}{2} (X_{(j)}^\dagger X_{(j)}) \phi_{b(5)}^a \phi_{a(5)}^b - X_{(j)a}^\dagger \phi_{c(j)}^b X_{(j)}^a \phi_{b(5)}^c - \kappa_{(5)} \tilde{g}^2 X_{(j)}^a \phi_{a(5)}^b X_{b(j)}^\dagger \\
 & - \frac{\tilde{g}^2}{2} (X_{5} \sigma_i X_{5}^\dagger) (X_{(j)}^\dagger \sigma_i X_{(j)}) \Big) + \sum_{(j)=3}^4 \left(\tilde{g} \epsilon_{ac} \lambda_{(n+1)}^{cb} \xi_{b(j)}^\dagger X_{(j)}^a - \tilde{g} \epsilon^{ac} \lambda_{cb(n+1)} X_{a(j)}^\dagger \xi_{(j)}^b \right. \\
 & - i \xi_{(j)}^a \phi_{a(n+1)}^b \xi_{b(j)}^\dagger + \frac{1}{2} (X_{(j)}^\dagger X_{(j)}) \phi_{b(n+1)}^a \phi_{a(n+1)}^b - X_{(j)a}^\dagger \phi_{c(j)}^b X_{(j)}^a \phi_{b(n+1)}^c \\
 & - \kappa_{(n+1)} \tilde{g}^2 X_{(j)}^a \phi_{a(n+1)}^b X_{b(j)}^\dagger + \frac{\tilde{g}^2}{2} (X_{(j)}^\dagger \sigma_i X_{(j)}) (X_{(n)}^\dagger \sigma_i X_{(n)}) \Big) + \sum_{(j)=5}^n \left(\tilde{g} \epsilon_{ac} \lambda_{(j+1)}^{cb} \xi_{b(j)}^\dagger X_{(j)}^a \right. \\
 & - \tilde{g} \epsilon^{ac} \lambda_{cb(j+1)} X_{a(j)}^\dagger \xi_{(j)}^b - i \xi_{(j)}^a \phi_{a(j+1)}^b \xi_{b(j)}^\dagger - \tilde{g}^2 \kappa_{(j+1)} X_{(j)}^a \phi_{a(j+1)}^b X_{b(j)}^\dagger - X_{(j)a}^\dagger \phi_{c(j)}^b X_{(j)}^a \phi_{b(j+1)}^c \\
 & + \frac{1}{2} (X_{(j)}^\dagger X_{(j)}) \phi_{b(j+1)}^a \phi_{a(j+1)}^b \Big) + \sum_{(j)=1}^{n+1} \left(-\frac{1}{8\tilde{g}^2} [\phi_{b(j)}^a, \phi_{d(j)}^c] [\phi_{a(j)}^b, \phi_{c(j)}^d] - \frac{\kappa_{(j)}}{6} \phi_{b(j)}^a [\phi_{c(j)}^b, \phi_{a(j)}^c] \right. \\
 & + \frac{i}{2} \lambda_{ab(j)} [\phi_{c(j)}^b, \lambda_{ac}^{ac}] \Big) - \frac{\tilde{g}^2}{2} \sum_{(j)=6}^n (X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j-1)}^\dagger \sigma_i X_{(j-1)}) \\
 & + \frac{\tilde{g}^2}{2} (X_{(1)}^\dagger \sigma_i X_{(1)}) (X_{(2)}^\dagger \sigma_i X_{(2)}) + \frac{\tilde{g}^2}{2} (X_{(3)}^\dagger \sigma_i X_{(3)}) (X_{(4)}^\dagger \sigma_i X_{(4)}) \Big). \tag{2.40}
 \end{aligned}$$

Supersymmetry variations on $\mathbb{R} \times S^2$ of the fermions in the vector multiplet are,

$$(j) = 6, \dots, n:$$

$$\begin{aligned}
 \delta \lambda_{(j)}^{ab} = & \frac{i}{2\tilde{g}} \epsilon^{mnk} F_{mn(j)} \gamma_k \varepsilon^{ab} - \frac{i}{\tilde{g}} \not{D} \phi_{c(j)}^b \varepsilon^{ac} - \frac{2i}{3\tilde{g}} \phi_{c(j)}^b \nabla \varepsilon^{ac} + \frac{i}{2\tilde{g}} [\phi_{c(j)}^b, \phi_{d(j)}^c] \varepsilon^{ad} \\
 & + i \kappa_{(j)} \tilde{g} \phi_{c(j)}^b \varepsilon^{ac} + i \tilde{g} (X_{(j)}^a X_{c(j)}^\dagger \varepsilon^{cb} - \varepsilon^{bc} X_{c(j-1)}^\dagger X_{(j-1)}^a) \\
 & + \frac{i\tilde{g}}{2} ((X^\dagger X)_{(j-1)} - (X X^\dagger)_{(j)}) \varepsilon^{ab}, \tag{2.41}
 \end{aligned}$$

$$\begin{aligned}
 \delta \lambda_{(5)}^{ab} = & \frac{i}{2\tilde{g}} \epsilon^{mnk} F_{mn(5)} \gamma_k \varepsilon^{ab} - \frac{i}{\tilde{g}} \not{D} \phi_{c(5)}^b \varepsilon^{ac} - \frac{2i}{3\tilde{g}} \phi_{c(5)}^b \nabla \varepsilon^{ac} + \frac{i}{2\tilde{g}} [\phi_{c(5)}^b, \phi_{d(5)}^c] \varepsilon^{ad} + i \kappa_{(5)} \tilde{g} \phi_{c(5)}^b \varepsilon^{ac} \\
 & + i \tilde{g} \left(X_{(5)}^a X_{c(5)}^\dagger \varepsilon^{cb} - \varepsilon^{bc} \sum_{(j)=1}^2 X_{c(j)}^\dagger X_{(j)}^a \right) + \frac{i\tilde{g}}{2} \left(\sum_{(j)=1}^2 (X^\dagger X)_{(j)} - (X X^\dagger)_{(5)} \right) \varepsilon^{ab}, \tag{2.42}
 \end{aligned}$$

$$\begin{aligned} \delta\lambda_{(n+1)}^{ab} = & \frac{i}{2\tilde{g}}\epsilon^{mnk}F_{mn(n+1)}\gamma_k\varepsilon^{ab} - \frac{i}{\tilde{g}}\not{D}\phi_{c(n+1)}^b\varepsilon^{ac} - \frac{2i}{3\tilde{g}}\phi_{c(n+1)}^b\nabla\varepsilon^{ac} + \frac{i}{2\tilde{g}}[\phi_{c(n+1)}^b, \phi_{d(n+1)}^c]\varepsilon^{ad} \\ & + i\kappa_{(n+1)}\tilde{g}\phi_{c(n+1)}^b\varepsilon^{ac} - i\tilde{g}\varepsilon^{bc}\sum_{(j)=3,4,n}X_c^\dagger X_{(j)}^a + \frac{i\tilde{g}}{2}\sum_{(j)=3,4,n}(X^\dagger X)_{(j)}\varepsilon^{ab}, \end{aligned} \quad (2.43)$$

$(j) = 1, 2, 3, 4:$

$$\begin{aligned} \delta\lambda_{(j)}^{ab} = & \frac{i}{2\tilde{g}}\epsilon^{mnk}F_{mn(j)}\gamma_k\varepsilon^{ab} - \frac{i}{\tilde{g}}\not{D}\phi_{c(j)}^b\varepsilon^{ac} - \frac{2i}{3\tilde{g}}\phi_{c(j)}^b\nabla\varepsilon^{ac} + \frac{i}{2\tilde{g}}[\phi_{c(j)}^b, \phi_{d(j)}^c]\varepsilon^{ad} \\ & + i\kappa_{(j)}\tilde{g}\phi_{c(j)}^b\varepsilon^{ac} + i\tilde{g}X_{(j)}^a X_c^\dagger \varepsilon^{cb} - \frac{i\tilde{g}}{2}(XX^\dagger)_{(j)}\varepsilon^{ab}. \end{aligned} \quad (2.44)$$

Variations of the hypermultiplet fermions are

$$\begin{aligned} (j) = 1, 2: \quad & \delta\xi_{(j)}^a = \not{D}X_{(j)}^b\varepsilon_b^a + \frac{1}{3}X_{(j)}^b\nabla\varepsilon_b^a + \phi_{b(j)}^a\varepsilon_c^b X_{(j)}^c - X_{(j)}^c\varepsilon_c^b\phi_{b(5)}^a, \\ & \delta\xi_{a(j)}^\dagger = \not{D}X_{b(j)}^\dagger\varepsilon_a^b + \frac{1}{3}X_{b(j)}^\dagger\nabla\varepsilon_a^b - \phi_{b(5)}^a\varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger\varepsilon_b^c\phi_{a(j)}^b, \end{aligned} \quad (2.45)$$

$$\begin{aligned} (j) = 3, 4: \quad & \delta\xi_{(j)}^a = \not{D}X_{(j)}^b\varepsilon_b^a + \frac{1}{3}X_{(j)}^b\nabla\varepsilon_b^a + \phi_{b(j)}^a\varepsilon_c^b X_{(j)}^c - X_{(j)}^c\varepsilon_c^b\phi_{b(n+1)}^a, \\ & \delta\xi_{a(j)}^\dagger = \not{D}X_{b(j)}^\dagger\varepsilon_a^b + \frac{1}{3}X_{b(j)}^\dagger\nabla\varepsilon_a^b - \phi_{b(n+1)}^a\varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger\varepsilon_b^c\phi_{a(j)}^b, \end{aligned} \quad (2.46)$$

$$\begin{aligned} (j) = 5, 6, \dots, n: \quad & \delta\xi_{(j)}^a = \not{D}X_{(j)}^b\varepsilon_b^a + \frac{1}{3}X_{(j)}^b\nabla\varepsilon_b^a + \phi_{b(j)}^a\varepsilon_c^b X_{(j)}^c - X_{(j)}^c\varepsilon_c^b\phi_{b(j+1)}^a, \\ & \delta\xi_{a(j)}^\dagger = \not{D}X_{b(j)}^\dagger\varepsilon_a^b + \frac{1}{3}X_{b(j)}^\dagger\nabla\varepsilon_a^b - \phi_{b(j+1)}^a\varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger\varepsilon_b^c\phi_{a(j)}^b \end{aligned} \quad (2.47)$$

2.4.3 $\hat{E}_6$ quiver

The $\hat{E}_6$ action on $\mathbb{R} \times S^2$ is calculated following the same steps. We do not write the expression of the action here since it is straightforward from the $\hat{D}$ -type quiver. Let us write down the supersymmetry variations which are needed to find the monopole solutions later.

$$\begin{aligned} \delta\lambda_{(j)}^{ab} = & \frac{i}{2\tilde{g}}\epsilon^{mnk}F_{mn(j)}\gamma_k\varepsilon^{ab} - \frac{i}{\tilde{g}}\not{D}\phi_{c(j)}^b\varepsilon^{ac} - \frac{2i}{3\tilde{g}}\phi_{c(j)}^b\nabla\varepsilon^{ac} + \frac{i}{2\tilde{g}}[\phi_{c(j)}^b, \phi_{d(j)}^c]\varepsilon^{ad} \\ & + i\kappa_{(j)}\tilde{g}\phi_{c(j)}^b\varepsilon^{ac} + i\tilde{g}(X_{(j)}^a X_c^\dagger \varepsilon^{cb} - \varepsilon^{bc} X_{c(j-1)}^\dagger X_{(j-1)}^a) \\ & + \frac{i\tilde{g}}{2}((X^\dagger X)_{(j-1)} - (XX^\dagger)_{(j)})\varepsilon^{ab} \quad (j) = 2, 4, 6, \\ \delta\lambda_{(7)}^{ab} = & \frac{i}{2\tilde{g}}\epsilon^{mnk}F_{mn(7)}\gamma_k\varepsilon^{ab} - \frac{i}{\tilde{g}}\not{D}\phi_{c(7)}^b\varepsilon^{ac} - \frac{2i}{3\tilde{g}}\phi_{c(7)}^b\nabla\varepsilon^{ac} + \frac{i}{2\tilde{g}}[\phi_{c(7)}^b, \phi_{d(7)}^c]\varepsilon^{ad} \\ & + i\kappa_{(7)}\tilde{g}\phi_{c(7)}^b\varepsilon^{ac} - i\tilde{g}\varepsilon^{bc}\sum_{(j)=2,4,6}X_c^\dagger X_{(j)}^a + \frac{i\tilde{g}}{2}\sum_{(j)=2,4,6}(X^\dagger X)_{(j)}\varepsilon^{ab}, \\ \delta\lambda_{(j)}^{ab} = & \frac{i}{2\tilde{g}}\epsilon^{mnk}F_{mn(j)}\gamma_k\varepsilon^{ab} - \frac{i}{\tilde{g}}\not{D}\phi_{c(j)}^b\varepsilon^{ac} - \frac{2i}{3\tilde{g}}\phi_{c(j)}^b\nabla\varepsilon^{ac} + \frac{i}{2\tilde{g}}[\phi_{c(j)}^b, \phi_{d(j)}^c]\varepsilon^{ad} + i\kappa_{(j)}\tilde{g}\phi_{c(j)}^b\varepsilon^{ac} \\ & + i\tilde{g}X_{(j)}^a X_c^\dagger \varepsilon^{cb} - \frac{i\tilde{g}}{2}(XX^\dagger)_{(j)}\varepsilon^{ab} \quad (j) = 1, 3, 5, \end{aligned} \quad (2.48)$$

$$\begin{aligned} (j) = 2, 4, 6: \quad & \delta\xi_{(j)}^a = \not{D}X_{(j)}^b\varepsilon_b^a + \frac{1}{3}X_{(j)}^b\nabla\varepsilon_b^a + \phi_{b(j)}^a\varepsilon_c^b X_{(j)}^c - X_{(j)}^c\varepsilon_c^b\phi_{b(7)}^a, \\ & \delta\xi_{a(j)}^\dagger = \not{D}X_{b(j)}^\dagger\varepsilon_a^b + \frac{1}{3}X_{b(j)}^\dagger\nabla\varepsilon_a^b - \phi_{b(7)}^a\varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger\varepsilon_b^c\phi_{a(j)}^b, \end{aligned} \quad (2.49)$$

$$\begin{aligned}
 (j) = 1, 3, 5: \quad \delta \xi_{(j)}^a &= \not{D} X_{(j)}^b \varepsilon_b^a + \frac{1}{3} X_{(j)}^b \not{\nabla} \varepsilon_b^a + \phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(j+1)}^a, \\
 \delta \xi_{a(j)}^\dagger &= \not{D} X_{b(j)}^\dagger \varepsilon_a^b + \frac{1}{3} X_{b(j)}^\dagger \not{\nabla} \varepsilon_a^b - \phi_{a(j+1)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_c^b \phi_{a(j)}^b.
 \end{aligned} \tag{2.50}$$

3 Classical monopole solution

3.1 $\widehat{\mathbf{A}}$ -type quiver

In this section we compute the classical bosonic BPS and anti-BPS monopole solution to the theories under consideration. To find a monopole solution we start with the following Dirac monopole solution on $\mathbb{R} \times S^2$,

$$A_{(j)} = \frac{H}{2} (\pm 1 - \cos \theta) d\varphi \tag{3.1}$$

where, $H = \text{diag}(q_1, q_2, \dots, q_N)$ and $q_i \in \mathbb{Z}$ are the magnetic charges. The upper sign is for the northern hemisphere and lower sign is for the southern one. The dual of the field strength of the above gauge field,

$$\epsilon^{\theta\varphi\tau} F_{\theta\varphi} = \frac{H}{2} d\tau \tag{3.2}$$

is a constant. If there exists a BPS monopole solution with such a gauge potential (3.1), then corresponding fermionic variations $\delta \lambda_{(j)}^{ab}$, $\delta \xi_{(j)}^a$, $\delta \xi_{a(j)}^\dagger$ should be equal to zero for a non trivial supersymmetry variation parameter. We can see from (2.38) that $\delta \lambda_{(j)}^{ab}$ contains terms of order $\tilde{g}$ and $\tilde{g}^{-1}$. The goal of finding solutions all along the flow means we have to set them separately to zero. In our following calculation we are also assuming that the background fields (A, ϕ) are in the Cartan subalgebra of the gauge group factors, which makes all commutator vanish.

$\delta \lambda_{(j)}^{ab}$ at order $\frac{1}{\tilde{g}}$. Equating order $\tilde{g}^{-1}$ terms of (2.38) to zero we get,

$$\frac{i}{2g} \epsilon^{mnk} F_{mn(j)} \gamma_k \varepsilon^{ab} - \frac{i}{g} \not{D} \phi_{c(j)}^b \varepsilon^{ac} - \frac{2i}{3g} \phi_{c(j)}^b \not{\nabla} \varepsilon^{ac} = 0. \tag{3.3}$$

Recalling that $\delta \lambda_{(j)}^{ab}$ is in the reducible representation $\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{3}$ of $\text{SU}(2)_R$ we isolate the $\mathbf{1}$ part i.e. the $\text{SU}(2)_R$ trace, by computing $\delta \lambda_{(j)}^{ab} \epsilon_{ba}$. The first term is zero since $\varepsilon_a^a = 0$. Therefore only the trace of the second term contributes to the $\text{SU}(2)_R$ trace of $\delta \lambda_{(j)}^{ab}$ at order $\frac{1}{\tilde{g}}$. Since, $\epsilon^{mnk} F_{mn}$ is a constant, we can have a simple situation where ϕ 's are also constant. Then the second term of the above equation is zero. The third term in (3.3), after using the Killing spinor equation on $\mathbb{R} \times S^2$,

$$\nabla_m \varepsilon = -\frac{1}{2} \gamma_m \gamma^\tau \varepsilon \tag{3.4}$$

gives,

$$\phi_{i(j)} \varepsilon_i = 0 \tag{3.5}$$

i.e, the supersymmetry variation parameter is orthogonal to the background scalar. We can solve (3.3) by choosing,

$$\varepsilon_3 = 0, \quad \phi_{i(j)} \sim \delta_{i3}. \quad (3.6)$$

Then we isolate the **3** part by computing $\delta\lambda_{(j)}^{ab}(\sigma_i)_{ba}$ and we get

$$\frac{i}{2}\epsilon^{mnk}F_{mn(j)}\gamma_k\varepsilon_i + \epsilon_{ijk}\phi_{j(j)}\gamma^\tau\varepsilon_k = 0, \quad (3.7)$$

which is easily solvable in the following two ways:

$$(i) \quad \phi_{3(j)} = -\frac{H}{2}, \quad \varepsilon_1 - i\varepsilon_2 = 0, \quad (3.8)$$

$$(ii) \quad \phi_{3(j)} = \frac{H}{2}, \quad \varepsilon_1 + i\varepsilon_2 = 0. \quad (3.9)$$

In the first case, the preserved supersymmetry is $\varepsilon_1 + i\varepsilon_2$ and we get the BPS solution and in the second case, the preserved supersymmetry is $\varepsilon_1 - i\varepsilon_2$ and we get the anti-BPS solution. In both cases, since one of the three supersymmetry parameters are preserved by the solution we have $\frac{1}{3}$ -BPS solutions. Now let us examine the order $\tilde{g}$ terms in $\delta\lambda_{(j)}^{ab}$.

$\delta\lambda_{(j)}^{ab}$ at order $\tilde{g}$. We obtain from (2.38)

$$\kappa_{(j)}\phi_{c(j)}^b\varepsilon^{ac} + (X_{(j)}^a X_{c(j)}^\dagger \varepsilon^{cb} - \varepsilon^{bc} X_{c(j-1)}^\dagger X_{(j-1)}^a) - \frac{1}{2}(X_{(j)} X_{(j)}^\dagger - X_{(j-1)}^\dagger X_{(j-1)})\varepsilon^{ab} = 0. \quad (3.10)$$

We analyse the above as before by considering the **1** and the **3** parts separately. The $SU(2)_R$ trace of the first term is $\kappa_{(j)}\varepsilon_i\phi_{i(j)}$ which vanishes on using (3.5) and the third term is traceless. The second term gives

$$\begin{aligned} (i) \quad \eta = 1 \quad & (-W_{(j)}^\dagger Z_{(j)}^\dagger + Z_{(j-1)}^\dagger W_{(j-1)}^\dagger)(\varepsilon_1 + i\varepsilon_2) = 0, \\ (ii) \quad \eta = -1 \quad & (-Z_{(j)} W_{(j)} + W_{(j-1)} Z_{(j-1)})(\varepsilon_1 - i\varepsilon_2) = 0. \end{aligned} \quad (3.11)$$

By isolating the **3** part of (3.10) one obtains the following equation,

$$2\kappa_{(j)}\phi_{i(j)} = -X_{(j)}\sigma_i X_{(j)}^\dagger + X_{(j-1)}^\dagger\sigma_i X_{(j-1)} \quad (3.12)$$

by which we see that the hypermultiplet scalars are already constrained by the CS levels and the magnetic charges in addition to other constraints that are yet to come from analysing the hypermultiplet fermions variation equations.

Now, we analyse the hypermultiplet fermion variation equations. We rewrite $\delta\xi_{(j)}^a$ from (2.39) below,

$$\delta\xi_{(j)}^a = \not{D}X_{(j)}^b\varepsilon_b^a + \frac{1}{3}X_{(j)}^b\nabla\varepsilon_b^a + \phi_{b(j)}^a\varepsilon_c^b X_{(j)}^c - X_{(j)}^c\varepsilon_c^b\phi_{b(j)}^a.$$

The last two terms cancel³ because in the special case we are working in, all the $\phi_{(j)}$'s are equal. Using Killing- spinor equation we get,

$$\delta\xi_{(j)}^a = \left(\gamma^m\mathcal{D}_m X_{(j)}^b - \frac{1}{2}X_{(j)}^b\gamma^\tau\right)\varepsilon_b^a.$$

³There won't be a straightforward cancellation in the more general case where the magnetic charges are different at each node, $A_{(j)} \sim H_{(j)}$, $\phi_{i(j)} = -\eta\frac{H_{(j)}}{2}\delta_{i3}$.

Figure 4. $\hat{A}$ quiver diagram with three gauge groups.

Writing two components of the above for $a = 1, 2$ we get,

$$\delta\xi_{(j)}^1 = (\varepsilon_1 + i\varepsilon_2) \left(\gamma^m \mathcal{D}_m X_{(j)}^2 - \frac{1}{2} X_{(j)}^2 \gamma^\tau \right) \quad \text{for } a = 1 \quad (3.13)$$

and

$$\delta\xi_{(j)}^2 = (\varepsilon_1 - i\varepsilon_2) \left(\gamma^m \mathcal{D}_m X_{(j)}^1 - \frac{1}{2} X_{(j)}^1 \gamma^\tau \right) \quad \text{for } a = 2. \quad (3.14)$$

Demanding $\delta\xi_{(j)}^a = 0$, fixes the functional dependence of $X_{(j)}^a$ to be, $X_{(j)}^1 \sim \exp(-\eta\tau/2)$ and $X_{(j)}^2 \sim \exp(\eta\tau/2)$. Then we can make all $\delta\xi_{(j)}^a$'s zero either by the functional dependence or by the condition derived above $\varepsilon_1 - i\eta\varepsilon_2 = 0$. The solution chosen for the X 's also satisfy their classical equation of motion. The only other equation that remains to be satisfied by the so-far obtained background are the equations of motion for the gauge fields, which is

$$\kappa_{(j)} \epsilon^{mnp} F_{np(j)} = X_{(j)} \mathcal{D}^m X_{(j)}^\dagger - \mathcal{D}^m X_{(j)} \cdot X_{(j)}^\dagger - \mathcal{D}^m X_{(j-1)}^\dagger \cdot X_{(j-1)} + X_{(j-1)}^\dagger \mathcal{D}^m X_{(j-1)}. \quad (3.15)$$

To summarise, we have obtained the background gauge and adjoint scalar fields,

$$A_{(j)} = \frac{H}{2} (\pm 1 - \cos \theta) d\varphi, \quad \phi_{i(j)} = -\eta \frac{H}{2} \delta_{i3} \quad (3.16)$$

with $\varepsilon_1 + i\eta\varepsilon_2$ being the preserved supersymmetry. And we have several constraints on the hypermultiplet scalar fields: (i) equation (3.11), (ii) equation (3.12) (iii) the τ dependence that follow from the hypermultiplet fermion variations and (iv) the equation (3.15). We will analyse these constraints and find solutions first for the three node quiver and with the experience gained thus, we can then generalize for a generic n -node quiver.

3.1.1 Example of a three node case

In the previous section we studied how to choose the fields in the vector multiplet to make the fermionic variations (both vector and hypermultiplet) zero. In this section we will consider a simple example of a three node case given in figure 4 and see how to completely fix the hypermultiplet scalar fields to obtain full monopole solution. For that we need to solve,

$$\phi_{l(j)} = -\frac{1}{2\kappa_{(j)}} \left(X_{(j)}(\sigma_l) X_{(j)}^\dagger - X_{(j-1)}^\dagger(\sigma_l) X_{(j-1)} \right) \quad (3.17)$$

where, $l = 1, 2, 3$. (3.17) when expanded in terms of the component fields gives the following set of equations,

$$\begin{aligned}
 l = 1 : \quad & \phi_{1(1)} \longrightarrow Z_{(1)} W_{(1)} + W_{(1)}^\dagger Z_{(1)}^\dagger - Z_{(3)}^\dagger W_{(3)}^\dagger - W_{(3)} Z_{(3)} = 0, \\
 & \phi_{1(2)} \longrightarrow Z_{(2)} W_{(2)} + W_{(2)}^\dagger Z_{(2)}^\dagger - Z_{(1)}^\dagger W_{(1)}^\dagger - W_{(1)} Z_{(1)} = 0, \\
 & \phi_{1(3)} \longrightarrow Z_{(3)} W_{(3)} + W_{(3)}^\dagger Z_{(3)}^\dagger - Z_{(2)}^\dagger W_{(2)}^\dagger - W_{(2)} Z_{(2)} = 0,
 \end{aligned} \tag{3.18}$$

$$\begin{aligned}
 l = 2 : \quad & \phi_{2(1)} \longrightarrow W_{(1)}^\dagger Z_{(1)}^\dagger - Z_{(1)} W_{(1)} - Z_{(3)}^\dagger W_{(3)}^\dagger + W_{(3)} Z_{(3)} = 0, \\
 & \phi_{2(2)} \longrightarrow W_{(2)}^\dagger Z_{(2)}^\dagger - Z_{(2)} W_{(2)} - Z_{(1)}^\dagger W_{(1)}^\dagger + W_{(1)} Z_{(1)} = 0, \\
 & \phi_{2(3)} \longrightarrow W_{(3)}^\dagger Z_{(3)}^\dagger - Z_{(3)} W_{(3)} - Z_{(2)}^\dagger W_{(2)}^\dagger + W_{(2)} Z_{(2)} = 0.
 \end{aligned} \tag{3.19}$$

$$\begin{aligned}
 l = 3 : \quad & \phi_{3(1)} \longrightarrow Z_{(1)} Z_1^\dagger - W_{(1)}^\dagger W_{(1)} - Z_{(3)}^\dagger Z_{(3)} + W_{(3)} W_{(3)}^\dagger = \eta H \kappa_{(1)}, \\
 & \phi_{3(2)} \longrightarrow Z_{(2)} Z_2^\dagger - W_{(2)}^\dagger W_{(2)} - Z_{(1)}^\dagger Z_{(1)} + W_{(1)} W_{(1)}^\dagger = \eta H \kappa_{(2)}, \\
 & \phi_{3(3)} \longrightarrow Z_{(3)} Z_3^\dagger - W_{(3)}^\dagger W_{(3)} - Z_{(2)}^\dagger Z_{(2)} + W_{(2)} W_{(2)}^\dagger = \eta H \kappa_{(3)}.
 \end{aligned} \tag{3.20}$$

For a simple case, there can be the following three possibilities which solve (3.18) and (3.19).

- (i) $Z_{(1)} = Z_{(2)} = Z_{(3)} = 0$
- (ii) $W_{(1)} = W_{(2)} = W_{(3)} = 0$
- (iii) $W_{(1)} = Z_{(2)} = W_{(3)} = 0$ or $Z_{(1)} = W_{(2)} = Z_{(3)} = 0$.

Now we need to solve (3.20) for the three cases written above. Since the CS levels satisfy,

$$\kappa_{(1)} + \kappa_{(2)} + \kappa_{(3)} = 0$$

either one or two of them have to be negative. For a positive semi-definite H , we can solve several constraints on the hypermultiplet scalars in the following ways. These solutions for each case are consistent with the equation of motion of the gauge fields (3.15). For example,

(i) $Z_{(1)} = Z_{(2)} = Z_{(3)} = 0$

BPS $\eta = +1$. (3.20) reduces to,

$$\begin{aligned}
 -W_{(1)}^\dagger W_{(1)} + W_{(3)} W_{(3)}^\dagger &= H \kappa_{(1)}, & -W_{(2)}^\dagger W_{(2)} + W_{(1)} W_{(1)}^\dagger &= H \kappa_{(2)}, \\
 -W_{(3)}^\dagger W_{(3)} + W_{(2)} W_{(2)}^\dagger &= H \kappa_{(3)}
 \end{aligned} \tag{3.21}$$

which are solved by,

$$\begin{aligned}
 W_{(1)} &= \sqrt{A} e^{-\tau/2}, & W_{(1)}^\dagger &= \sqrt{A} e^{\tau/2}, \\
 W_{(2)} &= \sqrt{A - H \kappa_{(2)}} e^{-\tau/2}, & W_{(2)}^\dagger &= \sqrt{A - H \kappa_{(2)}} e^{\tau/2}, \\
 W_{(3)} &= \sqrt{A + H \kappa_{(1)}} e^{-\tau/2}, & W_{(3)}^\dagger &= \sqrt{A + H \kappa_{(1)}} e^{\tau/2}.
 \end{aligned} \tag{3.22}$$

The above equations are satisfied for any diagonal matrix A with entries ≥ 0 assuming, $\kappa_{(1)}, \kappa_{(3)} > 0$ and $\kappa_{(2)} < 0$.

anti-BPS $\eta = -1$. Similarly the anti-BPS case is solved by,

$$\begin{aligned} W_{(1)} &= \sqrt{A} e^{\tau/2}, & W_{(1)}^\dagger &= \sqrt{A} e^{-\tau/2}, \\ W_{(2)} &= \sqrt{A + H\kappa_{(2)}} e^{\tau/2}, & W_{(2)}^\dagger &= \sqrt{A + H\kappa_{(2)}} e^{-\tau/2}, \\ W_{(3)} &= \sqrt{A - H\kappa_{(1)}} e^{\tau/2}, & W_{(3)}^\dagger &= \sqrt{A - H\kappa_{(1)}} e^{-\tau/2} \end{aligned} \quad (3.23)$$

where,

$$A_\alpha \geq q_\alpha |\kappa_{(2)}|, \quad A_\alpha \geq q_\alpha \kappa_{(1)}.$$

A_α and q_α denote the α -th entry of the matrix A and H respectively. Similarly the rest of the cases can be solved.

3.1.2 n node case

Now we can move on to more general case with gauge group $U(N) \times U(N) \times \dots \times U(N)$ and apply similar steps as above to obtain the full monopole solution in this theory. From (3.18) and (3.19) we can see that,

$$Z_{(j)} W_{(j)} = Z_{(j-1)} W_{(j-1)}. \quad (3.24)$$

Because all $Z_{(j)}$'s and $W_{(j)}$'s are diagonal, if we take d_α to be the α -th entry of the matrix $Z_{(j)} W_{(j)}$, then the following is true

$$Z_{(j)\alpha} W_{(j)\alpha} = Z_{(j-1)\alpha} W_{(j-1)\alpha} = d_\alpha (= \text{constant}). \quad (3.25)$$

Also from (3.20) we can write,

$$|Z_{(j)}|_\alpha^2 - |W_{(j)}|_\alpha^2 = |Z_{(j-1)}|_\alpha^2 - |W_{(j-1)}|_\alpha^2 + \eta q_\alpha \kappa_{(j)}.$$

Using the above equation again and again we get,

$$|Z_{(j)}|_\alpha^2 - |W_{(j)}|_\alpha^2 = |Z_{(1)}|_\alpha^2 - |W_{(1)}|_\alpha^2 + \eta q_\alpha K_{(j)} \quad (3.26)$$

where,

$$K_{(j)} = \sum_{(j)=2}^{(j)} \kappa_{(j)}. \quad (3.27)$$

Let $|Z_{(j)}|_\alpha^2 = u_{(j)}$, $|W_{(j)}|_\alpha^2 = v_{(j)}$. Using (3.25) and substituting in (3.26) we get,

$$u_{(j)}^2 u_{(1)} + u_{(j)} (-u_{(1)}^2 + |d_\alpha|^2 - \eta q_\alpha K_{(j)} u_{(1)}) - |d_\alpha|^2 u_{(1)} = 0. \quad (3.28)$$

Above is a quadratic equation in $u_{(j)}$ with the following solution,

$$u_{(j)} = \frac{u_{(1)}^2 - d_\alpha^2 + \eta q_\alpha K_{(j)} u_{(1)} \pm \sqrt{(u_{(1)}^2 - |d_\alpha|^2 + \eta q_\alpha K_{(j)} u_{(1)})^2 - 4d_\alpha^2 u_{(1)}^2}}{2u_{(1)}}. \quad (3.29)$$

These seem to exist for any value of $\kappa_{(j)}$. $u_{(j)}$'s form a moduli space of solutions classified by $Z_{(1)\alpha}, d_\alpha, \arg(Z_{(j)\alpha})$. The above is a general solution.

Let us examine the case where we choose $d_\alpha = 0$. Then from (3.25) either $v_{(1)}$ or $u_{(1)}$ is zero. Also from (3.26) we get,

$$u_{(j)} - v_{(j)} = u_{(1)} - v_{(1)} + \eta q K_{(j)}. \quad (3.30)$$

Now, due to the constraint that $\kappa_{(1)} + \kappa_{(2)} + \dots + \kappa_{(n)} = 0$, some of the $\kappa_{(j)}$'s are positive and the rest are negative. In other words there will be some positive $K_{(j)}$'s, let us call that set $[K_+]$ and rest of them will be negative which are denoted by $[K_-]$. Now let us solve case by case as we did in the three node example.

CASE 1: $v_{(1)} = v_{(2)} = \dots = v_{(n)} = 0$. From (3.30) we get,

$$u_{(j)} = u_{(1)} + \eta q_\alpha K_{(j)}.$$

We study BPS and anti-BPS case separately as follows.

(i) **BPS $\eta = 1$.** Since $u_{(1)}$ is positive, positivity of the L.H.S. implies solutions exist for $K_{(j)} > 0$. For negative $K_{(j)}$'s we get,

$$u_{(1)} - q[|K_-|] > 0 \implies u_{(1)} > q_\alpha \max([|K_-|]). \quad (3.31)$$

(ii) **anti-BPS $\eta = -1$.** There is no restriction on $u_{(1)}$ for $[K_-]$. But for $[K_+]$ we get,

$$u_{(1)} - q[K_+] > 0 \implies u_{(1)} > q \max([K_+]). \quad (3.32)$$

CASE 2: $u_{(1)} = u_{(2)} = \dots = u_{(n)} = 0$. From (3.30) it follows that,

$$v_{(j)} = v_{(1)} - \eta q_\alpha K_{(j)}.$$

(i) **BPS $\eta = 1$.** Positivity of L.H.S. implies solutions exist for $K_{(j)} < 0$, while for positive $K_{(j)}$'s one obtains,

$$v_{(1)} - q_\alpha[K_+] > 0 \implies v_{(1)} > q_\alpha \max([K_+]). \quad (3.33)$$

(ii) **anti-BPS $\eta = -1$.** There is no restriction on $v_{(1)}$ for $[K_+]$. But for $[K_-]$ we get,

$$v_{(1)} - q_\alpha[|K_-|] > 0 \implies v_{(1)} > q_\alpha \max([|K_-|]). \quad (3.34)$$

CASE 3: $v_{(1)} = 0$, some $u_{(j)} = 0$ and complementary $v_{(j)} = 0$. Let us first consider the nodes where $u_{(j)} = 0$. For this case there will be a subset of $[K]$ for which $u_{(j)} = 0$ and K 's are positive or negative. Let us denote the first case by $[K_{u_+}]$ and the second by $[K_{u_-}]$. Similar analysis for the rest of the nodes will bring two more subsets of $[K]$ i.e $[K_{v_+}]$ and $[K_{v_-}]$. Hence we can write,

$$[K] = [K_+] \cup [K_-]$$

and

$$[K_+] = [K_{u_+}] \cup [K_{v_+}].$$

Similarly the following relation holds,

$$[K_-] = [K_{u_-}] \cup [K_{v_-}].$$

Now, let us look at the solutions. For the nodes where $v_{(j)} = 0$, from (3.30) we find,

$$u_{(j)} = u_{(1)} + \eta q_\alpha K_{(j)}.$$

(i) **BPS $\eta = 1$.** We have,

$$u_{(j)} = u_{(1)} + q_\alpha K_{(j)}.$$

Now this $K_{(j)}$ will either belong to $[K_{v_+}]$ or $[K_{v_-}]$. For $[K_{v_+}]$ solutions exist without any restriction on $u_{(1)}$, while for $[K_{v_-}]$ we find,

$$u_{(1)} - q[|K_{v_-}|] > 0 \implies u_{(1)} > q \max(|K_{v_-}|). \quad (3.35)$$

(ii) **anti-BPS $\eta = -1$.** There is no restriction on $u_{(1)}$ for $[K_{v_-}]$. But for $[K_{v_+}]$ we get,

$$u_{(1)} - q_\alpha [K_{v_+}] > 0 \implies u_{(1)} > q_\alpha \max([K_{v_+}]). \quad (3.36)$$

For the nodes where $u_{(j)} = 0$, we have,

$$v_{(j)} = -(u_{(1)} + \eta q_\alpha K_{(j)}).$$

(i) **BPS $\eta = 1$.**

$$v_{(j)} = -(u_{(1)} + q_\alpha K_{(j)}).$$

For the set $[K_{u_+}]$ there is no solution. But for $[K_{u_-}]$ the following holds

$$u_{(1)} - q_\alpha [|K_{u_-}|] < 0 \implies u_{(1)} < q_\alpha \min(|K_{u_-}|). \quad (3.37)$$

(ii) **anti-BPS $\eta = -1$.**

$$v_{(j)} = -(u_{(1)} - q_\alpha K_{(j)}).$$

There is no solution for $[K_{u_-}]$, while for $[K_{u_+}]$ we get,

$$u_{(1)} - q_\alpha [K_{u_+}] < 0 \implies u_{(1)} < q_\alpha \min([K_{u_+}]). \quad (3.38)$$

The above four cases exhaust all the possibilities. To summarize,

BPS solutions

$$q_\alpha \cdot \min(|K_{u_-}|) > u_{(1)} > q_\alpha \cdot \max(|K_{v_-}|) \quad (3.39)$$

anti-BPS solutions

$$q_\alpha \cdot \min([K_{u_+}]) > u_{(1)} > q_\alpha \cdot \max([K_{v_+}]). \quad (3.40)$$

Therefore we can conclude that we have obtained full set of monopole solutions in the CS Yang-Mills $\widehat{A}$ quiver theory with $\mathcal{N} = 3$ supersymmetry and with gauge group $U(N) \times U(N) \times \dots U(N)$.

Figure 5. $\widehat{D}_n$ quiver diagram with gauge group $U(2N)^{n-3} \times U(N)^4$.

3.2 $\widehat{D}$ -type quiver

We consider the gauge group of the theory to be $U(2N)^{n-3} \times U(N)^4$ [29–31] as they have gravity duals which are the $AdS_4 \times M_7$ vacua of M-theory, where M_7 is a tri-Sasakian manifold. Also this class of theories have been considered in the context of $\widehat{ADE}$ matrix models since they have a nice large N limit. The quiver diagram with the ranks of the gauge groups are given in figure 5. Since the ranks of all gauge groups are not equal like the previous case, we need to choose two different ansatzes for the gauge fields as following,

$$\begin{aligned} A_{(j)} &= \frac{H^{(1)}}{2} (\pm 1 - \cos \theta) d\varphi \quad \text{for } (j) = 1, 2, 3, 4, \\ A_{(j)} &= \frac{H^{(2)}}{2} (\pm 1 - \cos \theta) d\varphi \quad \text{for } (j) = 5, \dots, n+1, \end{aligned} \quad (3.41)$$

where, $H^{(1)} = \text{diag}(q_1, q_2, \dots, q_N)$, $H^{(2)} = \text{diag}(q'_1, q'_2, \dots, q'_{2N})$. The superscript in H is the co-mark of the associated gauge group.

The logic here is same as the $\widehat{A}$ quiver case, i.e we need to make all the supersymmetry variations of the theory zero for such gauge ansatz with a non trivial variation parameter. Since the order $\tilde{g}^{-1}$ terms in $\delta\lambda_{(j)}^{ab}$ is same for $(j) = 1$ to $n+1$, the analysis is same as before and we get analogues of (3.5) and (3.8), which are as follows,

$$\begin{aligned} \phi_{i(j)} \varepsilon_i &= 0, \quad \varepsilon_3 = 0, \quad \varepsilon_1 - i\eta \varepsilon_2 = 0, \\ \phi_{i(j)} &= -\eta \frac{H^{(1)}}{2} \delta_{i3} \quad \text{for } (j) = 1, 2, 3, 4, \\ \phi_{i(j)} &= -\eta \frac{H^{(2)}}{2} \delta_{i3} \quad \text{for } (j) = 5, \dots, n+1. \end{aligned} \quad (3.42)$$

The difference occurs in order $\tilde{g}$ terms in $\delta\lambda_{(j)}^{ab}$. Let us analyse them from (2.41)–(2.44).

Making $\delta\lambda_{(j)}^{ab}|_{\tilde{g}} = 0$, for $(j) = 1, \dots, 4$, we find,

$$\phi_{i(j)} = -\frac{1}{2\kappa_{(j)}} \left(X_{(j)} \sigma_i X_{(j)}^\dagger \right) \quad (j) = 1, \dots, 4. \quad (3.43)$$

Similarly, $\delta\lambda_{(5)}^{ab}|_{\tilde{g}} = 0$ gives,

$$\phi_{i(5)} = -\frac{1}{2\kappa_{(5)}} \left(X_{(5)} \sigma_i X_{(5)}^\dagger - \sum_{(j)=1}^2 X_{(j)}^\dagger \sigma_i X_{(j)} \right). \quad (3.44)$$

$\delta\lambda_{(j)}^{ab}|_{\tilde{g}} = 0$, for $(j) = 6, \dots, n$ implies,

$$\phi_{i(j)} = -\frac{1}{2\kappa_{(j)}} \left(X_{(j)} \sigma_i X_{(j)}^\dagger - X_{(j-1)}^\dagger \sigma_i X_{(j-1)} \right) \quad (j) = 6, \dots, n. \quad (3.45)$$

$\delta\lambda_{(n+1)}^{ab}|_{\tilde{g}} = 0$ gives,

$$\phi_{i(n+1)} = \frac{1}{2\kappa_{(n+1)}} \sum_{(j)=3,4,n} X_{(j)}^\dagger \sigma_i X_{(j)}. \quad (3.46)$$

At this stage we have obtained solutions for $A_{(j)}$ and $\phi_{i(j)}$ explicitly. But to call it a monopole solution we need to check if these solutions make all $\delta\xi_{(j)}^a$'s and its complex conjugates zero. Let us verify it. For $(j) = 1, 2$ we have,

$$\delta\xi_{(j)}^a = \not{D} X_{(j)}^b \varepsilon_b^a + \frac{1}{3} X_{(j)}^b \nabla \varepsilon_b^a + \phi_{b(j+1)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(5)}^a.$$

Using Killing spinor equation (3.4) above, we get for $a = 1$,

$$\delta\xi_{(j)}^1 = \left(\not{D} X_{(j)}^2 - \frac{1}{2} X_{(j)}^2 \gamma^\tau - \eta \frac{H^{(1)}}{2} X_{(j)}^2 + \eta X_{(j)}^2 \frac{H^{(2)}}{2} \right) (\varepsilon_1 + i\varepsilon_2).$$

In the anti-BPS case $(\varepsilon_1 + i\varepsilon_2) = 0$. Hence the above supersymmetry variation vanishes. But in the BPS case the following has to be zero,

$$\left(\not{D} X_{(j)}^2 - \frac{1}{2} X_{(j)}^2 \gamma^\tau - \frac{H^{(1)}}{2} X_{(j)}^2 + X_{(j)}^2 \frac{H^{(2)}}{2} \right) = 0.$$

To cancel the first two terms we fix the functional dependence of $X_{(j)}^2 \sim e^{\eta\tau/2}$. To cancel the last two terms for the simplest case, one can fix the diagonal entries of $H^{(1)}$ and $H^{(2)}$ to be all equal, say q (of course there can be non-trivial choices). Therefore we can write

$$\begin{aligned} H^{(1)} &= \text{diag}(q_1, q_2, \dots, q_N), \quad q_i = q, \quad i = 1, \dots, N, \\ H^{(2)} &= \text{diag}(q_1, q_2, \dots, q_{2N}), \quad q_i = q, \quad i = 1, \dots, 2N. \end{aligned} \quad (3.47)$$

By taking the above ansatz one can check that all the supersymmetry variations $\delta\xi_{(j)}^a$ and their complex conjugates $\delta\xi_{a(j)}^\dagger$ will vanish and we take the functional dependence of the hypermultiplet scalars to be $X_{(j)}^1 \sim e^{-\eta\tau/2}$, $X_{1(j)}^\dagger \sim e^{\eta\tau/2}$, $X_{2(j)}^\dagger \sim e^{-\eta\tau/2}$.

Now we can generalise the same steps used in the $\hat{A}$ case to solve for the hypermultiplet scalars. Let us briefly go through the solutions. For the external nodes,

$(j) = 1, \dots, 4$: we get the following set of equations

$$Z_{(j)} W_{(j)} + W_{(j)}^\dagger Z_{(j)}^\dagger = 0, \quad Z_{(j)} W_{(j)} - W_{(j)}^\dagger Z_{(j)}^\dagger = 0, \quad Z_{(j)} Z_{(j)}^\dagger - W_{(j)}^\dagger W_{(j)} = \eta H^{(1)} \kappa_{(j)}. \quad (3.48)$$

First two equations imply that $Z_{(j)} W_{(j)} = 0$. Therefore we have the following two cases.

Case 1: $Z_{(j)} = 0$

$$W_{(j)}^\dagger W_{(j)} = -\eta H^{(1)} \kappa_{(j)}. \quad (3.49)$$

For a positive semi-definite solution we get more constraints on the CS levels associated to the external nodes of the quiver, i.e $\eta \kappa_{(j)} \leq 0$.

Case 2: $W_{(j)} = 0$

$$Z_{(j)} Z_{(j)}^\dagger = \eta H^{(1)} \kappa_{(j)}. \quad (3.50)$$

For a positive semi-definite solution we get, $\eta \kappa_{(j)} \geq 0$.

For $(j) = 5$: we get,

$$\begin{aligned} Z_{(5)} W_{(5)} + W_{(5)}^\dagger Z_{(5)}^\dagger - \sum_{(j)=1}^2 (Z_{(j)}^\dagger W_{(j)}^\dagger + W_{(j)} Z_{(j)}) &= 0, \\ -i Z_{(5)} W_{(5)} + i W_{(5)}^\dagger Z_{(5)}^\dagger - \sum_{(j)=1}^2 (i Z_{(j)}^\dagger W_{(j)}^\dagger - i W_{(j)} Z_{(j)}) &= 0, \\ Z_{(5)} Z_{(5)}^\dagger - W_{(5)}^\dagger W_{(5)} - \sum_{(j)=1}^2 (Z_{(j)}^\dagger Z_{(j)} - W_{(j)} W_{(j)}^\dagger) &= \eta \kappa_{(5)} H^{(2)}. \end{aligned} \quad (3.51)$$

Using the solutions from the $(j) = 1, \dots, 4$ case, the last summed over terms vanish and the remaining can be solved exactly as before.

For $(j) = 6, \dots, n$: this analysis is similar to the $\hat{A}$ case.

For $(j) = n + 1$: this case is same as $(j) = 5$.

The only remaining thing that one has to keep in mind that X and $X^\dagger$ should satisfy the equations of motions of the gauge fields, which are

$$\begin{aligned} \kappa_{(j)} \epsilon^{mnp} F_{np(j)} &= X_{(j)} \mathcal{D}^m X_{(j)}^\dagger - \mathcal{D}^m X_{(j)} \cdot X_{(j)}^\dagger \quad (j) = 1, 2, 3, 4, \\ \kappa_{(5)} \epsilon^{mnp} F_{np(5)} &= \sum_{(j)=1}^2 (X_{(j)}^\dagger \mathcal{D}^m X_{(j)} - \mathcal{D}^m X_{(j)}^\dagger \cdot X_{(j)}) + (X_{(5)} \mathcal{D}^m X_{(5)}^\dagger - \mathcal{D}^m X_{(5)} \cdot X_{(5)}^\dagger), \\ \kappa_{(n+1)} \epsilon^{mnp} F_{np(n+1)} &= \sum_{(j)=3,4,n} (X_{(j)}^\dagger \mathcal{D}^m X_{(j)} - \mathcal{D}^m X_{(j)}^\dagger \cdot X_{(j)}), \\ \kappa_{(j)} \epsilon^{mnp} F_{np(j)} &= X_{(j)} \mathcal{D}^m X_{(j)}^\dagger - \mathcal{D}^m X_{(j)} \cdot X_{(j)}^\dagger - \mathcal{D}^m X_{(j-1)}^\dagger \cdot X_{(j-1)} + X_{(j-1)}^\dagger \mathcal{D}^m X_{(j-1)} \\ &\quad (j) = 6, \dots, n. \end{aligned} \quad (3.52)$$

3.3 $\hat{E}_6$ quiver

Finding the monopole solution for $\hat{E}_6$ is straight forward from the $\hat{D}$ case. The theory we analyse has gauge group $U(2N)^3 \times U(N)^3 \times U(3N)$ as given in figure 6. In this case we

Figure 6. $\hat{E}_6$ quiver diagram with gauge group $U(2N)^3 \times U(N)^3 \times U(3N)$.

have to start with three different gauge potentials on $\mathbb{R} \times S^2$, since there are three distinct co-marks associated to the nodes, such as,

$$\begin{aligned} A_{(j)} &= \frac{H^{(1)}}{2} (\pm 1 - \cos \theta) d\varphi \quad \text{for } (j) = 1, 3, 5, \\ A_{(j)} &= \frac{H^{(2)}}{2} (\pm 1 - \cos \theta) d\varphi \quad \text{for } (j) = 2, 4, 6, \\ A_{(7)} &= \frac{H^{(3)}}{2} (\pm 1 - \cos \theta) d\varphi \quad \text{for } (j) = 7, \end{aligned} \quad (3.53)$$

where, $H^{(1)} = \text{diag}(q_1, q_2, \dots, q_N)$, $H^{(2)} = \text{diag}(q'_1, q'_2, \dots, q'_{2N})$, $H^{(3)} = \text{diag}(q''_1, q''_2, \dots, q''_{3N})$.

Equating order $\tilde{g}^{-1}$ of $\delta\lambda_{(j)}^{ab}$ one chooses the background scalars to be,

$$\begin{aligned} \phi_{i(j)} &= -\eta \frac{H^{(1)}}{2} \delta_{i3} \quad \text{for } (j) = 1, 3, 5, \\ \phi_{i(j)} &= -\eta \frac{H^{(2)}}{2} \delta_{i3} \quad \text{for } (j) = 2, 4, 6, \\ \phi_{i(7)} &= -\eta \frac{H^{(3)}}{2} \delta_{i3} \quad \text{for } (j) = 7. \end{aligned} \quad (3.54)$$

By doing a similar analysis as in the $\hat{D}$ case, we get the following relations which relates the hypermultiplet scalars to the vector multiplet scalars.

$$\phi_{i(j)} = -\frac{1}{2\kappa_{(j)}} \left(X_{(j)} \sigma_i X_{(j)}^\dagger \right) \quad \text{for } (j) = 1, 3, 5, \quad (3.55)$$

$$\phi_{i(j)} = -\frac{1}{2\kappa_{(j)}} \left(X_{(j)} \sigma_i X_{(j)}^\dagger - X_{(j-1)}^\dagger \sigma_i X_{(j-1)} \right) \quad \text{for } (j) = 2, 4, 6, \quad (3.56)$$

$$\phi_{i(7)} = \frac{1}{2\kappa_{(7)}} \sum_{(j)=2,4,6} X_{(j)}^\dagger \sigma_i X_{(j)}. \quad (3.57)$$

To make the variations of the hypermultiplet fermions vanish one obtains,

$$\begin{aligned} H^{(1)} &= \text{diag}(q_1, \dots, q_N) \quad q_i = q \quad i = 1, \dots, N, \\ H^{(2)} &= \text{diag}(q_1, \dots, q_{2N}) \quad q_i = q \quad i = 1, \dots, 2N, \\ H^{(3)} &= \text{diag}(q_1, \dots, q_{3N}) \quad q_i = q \quad i = 1, \dots, 3N, \end{aligned} \quad (3.58)$$

while, the functional dependence of the X 's remain same as before. Now it is straight forward to solve (3.55), (3.56), (3.57) from the $\widehat{D}$ case.

4 $U(1)_R$ charge

The quantity that is preserved by the static background monopole solution in (3.16) (and their $\widehat{DE}$ analogs) is the $U(1)_R$ charge. This charge is not exact since it is abelian and can receive quantum correction under RG flow. In this section we compute the quantum corrections to the $U(1)_R$ charge, following method of BKK, who do it by computing the normal ordering constant of the $U(1)_R$ charge operator. The $U(1)_R$ charge for ABJM theory is computed in BKK as a consistency check of the results found in [27, 28]. BKK start this computation by considering a simple toy model and then generalising it to ABJM theory. Let us consider a single fermion $\psi(\tau, \Omega)$ on $\mathbb{R} \times S^2$ in presence of a $U(1)$ monopole with charge q which is kept at the centre of S^2 . The equation of motion obeyed by ψ in this system is,

$$\not{D}\psi + \frac{\eta}{2}q\psi = 0, \quad \eta = \pm 1. \quad (4.1)$$

The Dirac operator, $\not{D} = \gamma^\tau \partial_\tau + \not{D}_S$, where $\not{D}_S$ is the operator on S^2 , contains a $U(1)$ monopole solution with charge q . The associated conserved current is,

$$j^\mu = -i\psi^\dagger \gamma^\mu \psi$$

which has the following conserved charge,

$$Q = -i \int d\Omega \psi^\dagger \gamma^\tau \psi. \quad (4.2)$$

Our goal is to find the normal ordering constant of (4.2). To solve for ψ from (4.1) we use the machinery of monopole spinor harmonics [20]. The explicit expressions of monopole spinor harmonics and their properties are given in appendix C of BKK. Monopole spinor harmonics are eigenfunctions of $\not{D}_S$ and forms a basis on S^2 in the presence of a monopole. Therefore we can expand $\psi(\tau, \Omega)$ in the monopole spinor harmonics basis as follows,

$$\psi(\tau, \Omega) = \sum_m \psi_m(\tau) \Upsilon_{qm}^0(\Omega) + \sum_{jm\varepsilon} \psi_{jm}^\varepsilon(\tau) \Upsilon_{qjm}^\varepsilon(\Omega) \quad (4.3)$$

with $\varepsilon = \pm 1$. The time part gets separated because the monopole solution we are considering does not have a τ dependence. j is the total angular momentum quantum number, taking values

$$\begin{aligned} \text{for } q \neq 0 : \quad & j = \frac{|q|-1}{2}, \frac{|q|+1}{2}, \frac{|q|+3}{2}, \dots \quad m = -j, -j+1, \dots, j, \\ \text{for } q = 0 : \quad & j = \frac{|q|+1}{2}, \frac{|q|+3}{2}, \dots \quad m = -j, -j+1, \dots, j. \end{aligned} \quad (4.4)$$

Eigenvalue equations of $\not{D}_S$ are,

$$\not{D}_S \Upsilon_{qm}^0 = 0 \quad \text{for } j = \frac{|q|-1}{2}, q \neq 0, \quad (4.5)$$

$$\not{D}_S \Upsilon_{qjm}^\pm = i\Delta_{jq}^\pm \Upsilon_{qjm}^\pm \quad \text{for } j = \frac{|q|+1}{2}, \frac{|q|+3}{2}, \dots \quad (4.6)$$

where, $\Delta_{jq}^\pm = \pm \frac{1}{2} \sqrt{(2j+1)^2 - q^2}$. Υ_{qm}^0 is called the zero mode since it has zero eigenvalue. For the zero mode, $j = \frac{|q|-1}{2}$ whose multiplicity is $m = |q|$. Now putting the expansion of ψ into (4.1) one obtains the following first order differential equations,

$$\dot{\psi}_m = -\eta \frac{|q|}{2} \psi_m, \quad \frac{d\psi_{jm}^+}{d\tau} = \left(-i\Delta^- - \frac{q}{2}\right) \psi_{jm}^-, \quad \frac{d\psi_{jm}^-}{d\tau} = \left(-i\Delta^+ - \frac{q}{2}\right) \psi_{jm}^+. \quad (4.7)$$

Solving the equation of motion for $j = \frac{|q|-1}{2}$. We obtain,

$$\psi_m(\tau) = A e^{-\eta \frac{|q|}{2} \tau}, \quad A \text{ is an integration constant.} \quad (4.8)$$

This says that the zero modes have energy $E_j = \eta \frac{|q|}{2}$.

Solving the equation of motion for $j = \frac{|q|-1}{2} + p, p \neq 0, \eta = 1$. Dividing the last two equations of (4.7) one obtains ψ_{jm}^- in terms of ψ_{jm}^+ as follows,

$$\psi_{jm}^- = \pm \sqrt{\frac{i\Delta^+ + \frac{q}{2}}{i\Delta^- + \frac{q}{2}}} \psi_{jm}^+ + C, \quad C \text{ is an integration constant.}$$

Substituting this back in equation of motion we get,

$$\psi_{jm}^-(\tau) = (P e^{E_j \tau} + Q e^{-E_j \tau})$$

and

$$\psi_{jm}^+(\tau) = (R e^{E_j \tau} + S e^{-E_j \tau})$$

where, P, Q, R, S are integration constants and fixed by using normalisation of ψ and canonical anti-commutation relations of the operators and $E_j = j + \frac{1}{2}$ is the energy of the corresponding state. We quote the final solution from BKK below

$$\begin{aligned} \psi &= \sum_m \left[c_m u^0 e^{-\frac{|q|}{2} \tau} + d_m^\dagger v^0 e^{\frac{|q|}{2} \tau} \right] \Upsilon_m^0 + \sum_{jm\epsilon} \left[c_{jm} u_j^\epsilon e^{-E_j \tau} + d_{jm}^\dagger v_j^\epsilon e^{E_j \tau} \right] \Upsilon_{jm}^\epsilon, \\ \psi^\dagger &= \sum_m \left[c_m^\dagger u^0 e^{\frac{|q|}{2} \tau} + d_m v^0 e^{-\frac{|q|}{2} \tau} \right] \Upsilon_m^{0\dagger} + \sum_{jm\epsilon} \left[c_{jm}^\dagger u_j^{\epsilon\dagger} e^{E_j \tau} + d_{jm} v_j^{\epsilon\dagger} e^{-E_j \tau} \right] \Upsilon_{jm}^{\epsilon\dagger}. \end{aligned} \quad (4.9)$$

(Note: in the expression of $\psi^\dagger$ we have reversed the sign of τ since we are working with Euclidean time.) The wave-functions for BPS case is,

$$u^0 = 1, \quad v^0 = 0, \quad u_j^+ = v_j^+ = \frac{1}{\sqrt{2}}, \quad u_j^- = -v_j^- = \frac{1}{\sqrt{2}} \left(\frac{q}{2j+1} + i \sqrt{1 - \left(\frac{q}{2j+1} \right)^2} \right) \quad (4.10)$$

and in the anti-BPS case is,

$$u^0 = 0, \quad v^0 = 1, \quad u_j^+ = -v_j^+ = -\frac{1}{\sqrt{2}} \left(\frac{q}{2j+1} + i \sqrt{1 - \left(\frac{q}{2j+1} \right)^2} \right), \quad u_j^- = -v_j^- = \frac{1}{\sqrt{2}}. \quad (4.11)$$

Now, the $U(1)_R$ charge computed by using point splitting regularisation [9] has the following expression,

$$Q(\beta) = -\frac{i}{2} \int d\Omega \left[\psi^\dagger \left(\tau + \frac{\beta}{2} \right) \gamma^\tau \psi \left(\tau - \frac{\beta}{2} \right) - \psi \left(\tau + \frac{\beta}{2} \right) \gamma^\tau \psi^\dagger \left(\tau - \frac{\beta}{2} \right) \right] \quad (4.12)$$

where $\beta > 0$. In the end we take the limit $\beta \rightarrow 0$. Now substituting (4.9) above and using properties of monopole spinor harmonics and canonical anti-commutation relations we get,

$$Q(\beta) = \frac{1}{2} \sum_m \left[c_m^\dagger c_m u^{0\dagger} u^0 (e^{\frac{|q|}{2}\beta} + e^{-\frac{|q|}{2}\beta}) - e^{-\frac{|q|}{2}\beta} u^{0\dagger} u^0 + v^{0\dagger} v^0 e^{-\frac{|q|}{2}\beta} - d_m^\dagger d_m v^{0\dagger} v^0 (e^{-\frac{|q|}{2}\beta} + e^{\frac{|q|}{2}\beta}) \right] \\ + \frac{1}{2} \sum_{jm\varepsilon} \left[c_{jm}^\dagger c_{jm} u_j^{\varepsilon\dagger} u_j^\varepsilon (e^{E_j\beta} + e^{-E_j\beta}) - u_j^{\varepsilon\dagger} u_j^\varepsilon e^{-E_j\beta} + v_j^{\varepsilon\dagger} v_j^\varepsilon e^{-E_j\beta} - d_{jm}^\dagger d_{jm} v_j^{\varepsilon\dagger} v_j^\varepsilon (e^{-E_j\beta} + e^{E_j\beta}) \right]$$

whose normal ordered piece at $\beta = 0$ is,

$$Q_1(\beta = 0) = \sum_m \left(c_m^\dagger c_m u^{0\dagger} u^0 - d_m^\dagger d_m v^{0\dagger} v^0 \right) + \sum_{jm\varepsilon} \left[c_{jm}^\dagger c_{jm} u_j^{\varepsilon\dagger} u_j^\varepsilon - d_{jm}^\dagger d_{jm} v_j^{\varepsilon\dagger} v_j^\varepsilon \right] \quad (4.13)$$

with a normal ordering constant,

$$Q_0(\beta) = -\frac{1}{2} \sum_{jm\varepsilon} \left[u_j^{\varepsilon\dagger} u_j^\varepsilon - v_j^{\varepsilon\dagger} v_j^\varepsilon \right] e^{-\beta E_j} \quad (4.14)$$

where in the last sum the zero mode with $j = \frac{|q|-1}{2}$ is also included. Observing that $\sum_\varepsilon u_j^{\varepsilon\dagger} u_j^\varepsilon = 1$ for every positive energy state ($\sim e^{-E_j\tau}$) and $\sum_\varepsilon v_j^{\varepsilon\dagger} v_j^\varepsilon = 1$ for every negative energy state ($\sim e^{E_j\tau}$), we can write,

$$Q_0(\beta) = -\frac{1}{2} \sum_{\text{states}} \text{sign}(E) e^{-\beta|E|}. \quad (4.15)$$

The above quantity would be zero when we have a symmetric spectrum with respect to $E = 0$. But after turning on the scalar fields one finds that the energy corresponding to the zero mode is, $-\frac{|q|}{2}$ for BPS states and $\frac{|q|}{2}$ for anti-BPS states, i.e the zero mode energy spectrum is not symmetric⁴ But for non zero modes both positive and negative energy states are present for a fixed value of j . Therefore the normal ordering constant,

$$Q_0 = -\eta \frac{|q|}{2} \quad (4.16)$$

is non zero only in the case of zero modes. The factor $|q|$ arises because of the sum over zero modes, which has multiplicity $|q|$. Bosonic fields do not contribute to the $U(1)_R$ charge because their spectrum is symmetric [9].

4.1 Application to $\widehat{A}$ -type quiver

Now we can apply the above method to $\mathcal{N} = 3$ Yang-Mills deformed CS $\widehat{ADE}$ theories. We first compute the $SU(2)_R$ current and then extract the $U(1)_R$ part from that. Under infinitesimal $SU(2)$ transformation fundamental and anti-fundamental $SU(2)_R$ indices transform as,

$$\delta \mathcal{X}^a = i\varepsilon_b^a \mathcal{X}^b, \quad \delta \mathcal{X}_a = -i\varepsilon_a^b \mathcal{X}_b^\dagger, \quad (4.17)$$

⁴The energy spectrum plot can be found in BKK.

where $\mathcal{X}$ represents a generic field. Hence the fields in the vector multiplet transform as,

$$\delta\lambda_{(j)}^{ab} = i\varepsilon_c^a \lambda_{(j)}^{cb} + i\varepsilon_c^b \lambda_{(j)}^{ac}, \quad \delta\phi_{b(j)}^a = i\varepsilon_c^a \phi_{b(j)}^c - i\varepsilon_b^c \phi_{c(j)}^a. \quad (4.18)$$

The fields in the hypermultiplet transform as,

$$\begin{aligned} \delta X_{(j)}^a &= i\varepsilon_b^a X_{(j)}^b, & \delta X_{a(j)}^\dagger &= -i\varepsilon_a^b X_{b(j)}^\dagger, \\ \delta \xi_{(j)}^a &= i\varepsilon_b^a \xi_{(j)}^b, & \delta \xi_{a(j)}^\dagger &= -i\varepsilon_a^b \xi_{b(j)}^\dagger. \end{aligned} \quad (4.19)$$

The conserved $SU(2)_R$ current is,

$$\begin{aligned} (J^\mu)_a^b &= \sum_{(j)=1}^n \text{tr} \left[iX_{(j)a}^\dagger \mathcal{D}_\mu X_{(j)}^b - i\mathcal{D}_\mu X_{(j)a}^\dagger X_{(j)}^b - \xi_{a(j)}^\dagger \gamma^\mu \xi_{(j)}^b + \frac{i}{g^2} \mathcal{D}_\mu \phi_{a(j)}^c \phi_{c(j)}^b \right. \\ &\quad \left. - \frac{i}{g^2} \mathcal{D}_\mu \phi_{c(j)}^b \phi_{a(j)}^c + \frac{1}{2g^2} \lambda_{ac(j)} \gamma^\mu \lambda_{bc(j)} + \frac{1}{2g^2} \lambda_{ca(j)} \gamma^\mu \lambda_{cb(j)} \right] \end{aligned} \quad (4.20)$$

where we have used $J^\mu = (J^\mu)_a^b \varepsilon_b^a$. The $U(1)_R$ component of this current is obtained by contracting $(J^\mu)_a^b$ with $\varepsilon_b^a = (\sigma_3)_b^a$ which gives

$$J^\mu = \sum_{(j)=1}^n \text{tr} \left[-\frac{1}{2} \zeta_{(j)}^\dagger \gamma^\mu \zeta_{(j)} - \frac{1}{2} \omega_{(j)}^\dagger \gamma^\mu \omega_{(j)} + \chi_{\sigma(j)}^\dagger \gamma^\mu \chi_{\sigma(j)} \right]. \quad (4.21)$$

Now the expression of charge in (4.2) has become,

$$Q = -i \int d\Omega \sum_{(j)=1}^n \text{tr} \left[-\frac{1}{2} \zeta_{(j)}^\dagger \gamma^\tau \zeta_{(j)} - \frac{1}{2} \omega_{(j)}^\dagger \gamma^\tau \omega_{(j)} + \chi_{\sigma(j)}^\dagger \gamma^\tau \chi_{\sigma(j)} \right]. \quad (4.22)$$

The equations of motion of the fermions in the far UV limit $\tilde{g} \rightarrow 0$ are,

$$\begin{aligned} \mathcal{D}\zeta_{(j)} + \frac{\eta}{2}[H, \zeta_{(j)}] &= 0, & \mathcal{D}\chi_{\sigma(j)} + \frac{\eta}{2}[H, \chi_{\sigma(j)}] &= 0, \\ \mathcal{D}\omega_{(j)} + \frac{\eta}{2}[H, \omega_{(j)}] &= 0, & \mathcal{D}\chi_{\phi(j)} + \frac{\eta}{2}[H, \chi_{\phi(j)}] &= 0. \end{aligned} \quad (4.23)$$

Now, the brilliant observation of BKK which enables us to use the abelian result of previous section in a non-abelian model is the following. For any mr -th entry of an $N \times N$ matrix ψ (m, r are gauge indices) one can write,

$$[H, \psi]_{mr} = H_{mp} \psi_{pr} - \psi_{mp} H_{pr} = q_m \delta_{mp} \psi_{pr} - \psi_{mp} q_p \delta_{pr} = (q_m - q_r) \psi_{mr} \quad (4.24)$$

which happens because of the diagonal nature of H . Therefore we can treat each component in (4.23) separately, for example,

$$\mathcal{D}\zeta_{(j)mr} + \frac{\eta}{2} q_{mr} \zeta_{(j)mr} = 0$$

where, $q_{mr} \equiv q_m - q_r$. Comparing this with (4.1) one can see that now the effective monopole charge is $q_m - q_r$. Using the result (4.16) from the previous section and adding

contributions from hyper and vector multiplet fermions, we get,

$$Q_R^{mon} = \sum_{(j)}^n \sum_{m,r=1}^N \left[-\frac{1}{2} \cdot \left(-\eta \frac{|q_{mr}|}{2} \right) - \frac{1}{2} \cdot \left(-\eta \frac{|q_{mr}|}{2} \right) + 1 \cdot \left(-\eta \frac{|q_{mr}|}{2} \right) \right] = 0. \quad (4.25)$$

Therefore $U(1)_R$ charges of the monopoles of $\mathcal{N} = 3$ $\widehat{A}$ -type quiver gauge theory is zero. This result is as anticipated as in BKK who obtained, $Q_R^{mon} = \eta \left(\frac{N_f}{2} - 1 \right) \sum_{m,r=1}^N |q_m - q_r|$, where N_f is the number of hypermultiplets between two nodes. Therefore for $N_f = 2$ which is ABJM we get $Q_R^{mon} = 0$. Our result indeed matches with ABJM (i.e $n = 2$) as it should.

4.2 $\widehat{D}$ -type quiver

This case is almost similar to the previous case except for the external edges. The $U(1)_R$ component of the Noether current is,

$$J^\mu = \text{tr} \sum_{(j)=1}^n \left[-\frac{1}{2} \zeta_{(j)}^\dagger \gamma^\mu \zeta_{(j)} - \frac{1}{2} \omega_{(j)}^\dagger \gamma^\mu \omega_{(j)} \right] + \text{tr} \sum_{(j)=1}^{n+1} \chi_{\sigma(j)}^\dagger \gamma^\mu \chi_{\sigma(j)}. \quad (4.26)$$

The equation of motions in the far UV limit of the hypermultiplet fermions are,

$$\begin{aligned} (j) = 5, \dots, n: \quad \mathcal{D}\zeta_{(j)} + \frac{\eta}{2} [H^{(2)}, \zeta_{(j)}] &= 0, & (j) = 1, \dots, 4: \quad \mathcal{D}\zeta_{(j)} + \frac{\eta}{2} (H^{(1)}\zeta_{(j)} - \zeta_{(j)}H^{(2)}) &= 0, \\ \mathcal{D}\omega_{(j)} + \frac{\eta}{2} [H^{(2)}, \omega_{(j)}] &= 0, & \mathcal{D}\omega_{(j)} + \frac{\eta}{2} (H^{(1)}\omega_{(j)} - \omega_{(j)}H^{(2)}) &= 0 \end{aligned} \quad (4.27)$$

and the equations of motions of the vector multiplet fermions are,

$$\begin{aligned} (j) = 5, \dots, n+1 \quad \mathcal{D}\chi_\sigma + \frac{\eta}{2} [H^{(2)}, \chi_{\sigma(j)}] &= 0, & (j) = 1, \dots, 4 \quad \mathcal{D}\chi_{\sigma(j)} + \frac{\eta}{2} [H^{(1)}, \chi_{\sigma(j)}] &= 0, \\ \mathcal{D}\chi_{\phi(j)} + \frac{\eta}{2} [H^{(2)}, \chi_{\phi(j)}] &= 0, & \mathcal{D}\chi_{\phi(j)} + \frac{\eta}{2} [H^{(1)}, \chi_{\phi(j)}] &= 0. \end{aligned} \quad (4.28)$$

It is straight forward to generalise (4.24) for the internal edges, i.e $(j) = 5, \dots, n$ except that now we have $q_m = q_r$. This implies that the fermions associated to the internal edges do not interact with the monopole hence do not contribute to the $U(1)_R$ charge. For the external edges we have,

$$\begin{aligned} (H^{(1)}\zeta_{(j)} - \zeta_{(j)}H^{(2)})_{r\hat{s}} &= H_{rm}^{(1)}\zeta_{(j)m\hat{s}} - \zeta_{(j)r\hat{p}}H_{\hat{p}\hat{s}}^{(2)} = q\delta_{rm}\zeta_{(j)m\hat{s}} - q\zeta_{(j)r\hat{p}}\delta_{\hat{p}\hat{s}} \\ &= q\zeta_{(j)r\hat{s}} - q\zeta_{(j)r\hat{s}} = 0, \end{aligned} \quad (4.29)$$

where the hatted gauge indices imply that $\zeta_{(1)}$ is an $N \times 2N$ matrix and so on. Therefore the hypermultiplet fermions associated with the external edges also do not contribute to the charge. This happens because of our choice of the gauge ansatz in (3.47). Therefore we find that the $U(1)_R$ charges of the monopole operators in the $\widehat{D}$ case do not receive any quantum corrections. The similar steps can be carried out in the $\widehat{E}_6$ case and we find that the $U(1)_R$ charges of the monopole operators do not receive any quantum corrections.

5 $SU(2)_R$ charges of the monopole operators

In this section we compute the quantised $SU(2)_R$ charges of the monopole operators which is the main goal of this note. In the previous sections we have shown that the Yang-Mills deformations to the superconformal CS theories under $\widehat{ADE}$ classification preserve $\mathcal{N} = 3$ supersymmetry. Also we have explicitly solved for the hypermultiplet scalars, which implies that monopole solution exists for more than two gauge groups. Therefore it is legitimate to use BKK method, in $\widehat{ADE}$ quiver theories as well to compute the $SU(2)_R$ charges. We report in this note that, the smallest possible representation of the $SU(2)_R$ charge is zero. This result was anticipated in [1] for the $\widehat{A}$ -type quiver and here it is verified by explicit calculations.

Let us first briefly describe the method used in BKK which is collective co-ordinate quantisation method to obtain the $SU(2)_R$ charge of the BPS background. The first step in obtaining the charges is to generalise the BPS background in (3.16) (and their analogs for $\widehat{DE}$) to arbitrary $SU(2)_R$ orientation as follows,

$$\phi_{i(j)} = -\frac{H}{2} n_i \quad (5.1)$$

where, n_i is the unit vector on two sphere $SU(2)_R/U(1)_R$. It can be checked that these are the bosonic zero modes by computing the equation of motion for $\phi_{(j)}$. In the previous section $\phi_{i(j)}$ was chosen in the **3** direction with $n_i = \delta_{i3}$. Therefore the conserved quantity with that background was $U(1)_R$ charge. Now, to specify the collective co-ordinates one makes the unit vector n_i time dependent in the following way,

$$\phi_{i(j)} = -\frac{H}{2} n_i(\tau) \quad (5.2)$$

$n_i(\tau)$ is the collective co-ordinate of the BPS background. Now the global $SU(2)_R$ symmetry can act on this background whose action is to rotate $n_i(\tau)$ on the two sphere. Since the collective co-ordinate is interacting with the fermions of the theory the motion is not free. The effect of these interactions are obtained by calculating the effective action of the collective co-ordinate, i.e by integrating out the fermions. From the effective Lagrangian we compute the conserved quantity by Noether's procedure. The conserved quantity will be the angular momentum since $SU(2)_R$ acts as rotational symmetry. After computing the angular momentum we compute its quantized values which are the $SU(2)_R$ charges of the BPS-monopole operators in $\widehat{ADE}$ theories.

5.1 Quantum effective action of the collective co-ordinate

In this section we present the details for obtaining the effective action of the collective co-ordinates. Following BKK we first consider a simple model with one fermion in an abelian gauge theory. Then we will generalise the results of the former to Yang-Mills deformed CS quiver theories.

Let us consider a fermion $\psi^a(\tau)$, in the fundamental representation of $SU(2)$, on $\mathbb{R} \times S^2$ with the action,

$$\mathcal{S} = \int d\tau d\Omega \left(-i\psi_a^\dagger \not{D} \psi^a - \frac{iq}{2} n_i(\tau) \psi_a^\dagger (\sigma_i)^a_b \psi^b \right) \quad (5.3)$$

where the operator,

$$\mathcal{D} = \gamma^\tau \partial_\tau + \mathcal{D}_S = \gamma^\tau \partial_\tau + \gamma^\theta \partial_\theta + \gamma^\varphi \nabla_\varphi + \gamma^\varphi A_\varphi$$

on $\mathbb{R} \times S^2$ contains the abelian generalisation of the monopole background (3.16) with monopole charge q . Now to compute the effective action, we expand $\psi(\tau, \Omega)$ in monopole spinor harmonics basis as (4.3). The full action after substituting (4.3) in (5.3) and using properties monopole harmonics we obtain,

$$\begin{aligned} \mathcal{S} = & \sum_m \int d\tau \left(-i \psi_{am}^\dagger \partial_\tau \psi_m^a - \frac{iq}{2} \text{sign}(q) n_i(\tau) \psi_{am}^\dagger (\sigma_i)_b^a \cdot \psi_m^b(\tau) \right) \\ & + \sum_{jm\epsilon} \int d\tau \left(-i \psi_{ajm}^{\epsilon\dagger} \partial_\tau \psi_{jm}^{a\epsilon} + \Delta_{jq}^\epsilon \psi_{ajm}^{-\epsilon\dagger} \psi_{jm}^{a\epsilon} - \frac{iq}{2} n_i(\tau) \psi_{ajm}^{-\epsilon\dagger} (\sigma_i)_b^a \cdot \psi_{jm}^{b\epsilon} \right). \end{aligned} \quad (5.4)$$

The orthogonality property of monopole harmonics ensures that modes with different (jm) values and zero modes do not couple to each other. Therefore, the effective action can be computed easily for each (jm) separately.

The effective action for this system is,

$$\begin{aligned} e^{-\Gamma(\vec{n})} &= \int [d\psi_m^\dagger] [d\psi_m] e^{-\mathcal{S}[\psi_m, \psi_m^\dagger]} \int [d\psi_{jm}^\dagger] [d\psi_{jm}] e^{-\mathcal{S}[\psi_{jm}, \psi_{jm}^\dagger]} \\ &= \det \left(i \delta_a^b \partial_\tau - \frac{iq}{2} n_i(\tau) (\sigma_i)_a^b \right) \\ &\quad \cdot \det \begin{pmatrix} i \delta_a^b \partial_\tau & -\Delta^- - \frac{iq}{2} n_i(\tau) (\sigma_i)_a^b \\ -\Delta^+ - \frac{iq}{2} n_i(\tau) (\sigma_i)_a^b & i \delta_a^b \partial_\tau \end{pmatrix} \\ \implies \Gamma(\vec{n}) &= -\ln \det \left(i \delta_a^b \partial_\tau - \frac{iq}{2} n_i(\tau) (\sigma_i)_a^b \right) \\ &\quad - \ln \det \begin{pmatrix} i \delta_a^b \partial_\tau & -\Delta^- - \frac{iq}{2} n_i(\tau) (\sigma_i)_a^b \\ -\Delta^+ - \frac{iq}{2} n_i(\tau) (\sigma_i)_a^b & i \delta_a^b \partial_\tau \end{pmatrix}. \end{aligned} \quad (5.5)$$

To evaluate this functional determinant we first write the general form of the effective action using derivative expansion in $n_i(\tau)$ as follows,

$$\Gamma(\vec{n}) = \int d\tau \left(-V_{\text{eff}}(\vec{n}) + i \dot{n}_i A_i(\vec{n}) + \frac{1}{2} \dot{n}_i \dot{n}_j B_{ij}(\vec{n}) + \dots \right). \quad (5.6)$$

Now we will expand both (5.5) and (5.6) and compare them to find $A_i(\vec{n})$. We are keeping terms upto first order derivative in n_i , because the higher derivative terms in the above expansion will be dominated by a term proportional to $\frac{1}{\tilde{g}^2} \dot{n}_i^2$, which comes from the bosonic kinetic term in the action, in the far UV limit $\tilde{g} \rightarrow 0$.

Now, to expand the effective action we write,

$$n_i(\tau) = \dot{n}_i + \tilde{n}_i(\tau) \quad (5.7)$$

where, $\dot{n}_i$ is a constant, satisfying $\dot{n}^2 = 1$ and $\tilde{n}_i(\tau)$ is a small fluctuation. Expanding (5.6) around $\dot{n}_i$ we get,

$$\Gamma(\vec{n}) = \int d\tau \left(-V_{\text{eff}}(\vec{n}) - \tilde{n}_i \partial_i V_{\text{eff}}(\vec{n}) - \frac{1}{2} \tilde{n}_i \tilde{n}_j \partial_i \partial_j V_{\text{eff}}(\vec{n}) + i \dot{n}_i A_i(\vec{n}) + i \dot{n}_i \tilde{n}_j \partial_j A_i(\vec{n}) \right. \\ \left. + \frac{1}{2} \dot{n}_i \dot{n}_j B_{ij}(\vec{n}) + \dots \right) \quad (5.8)$$

The above says that to determine A_i we need to look at the terms with two powers of $\tilde{n}_i$ with one derivative. Such term is denoted by $\Gamma_{(2,1)}(\vec{n})$. We present the final result, details of which can be found in BKK.

$$\Gamma_{(2,1)}(\vec{n}) = -\frac{i}{4} \int d\tau \epsilon_{ijk} \dot{n}_i \tilde{n}_j \dot{n}_k \frac{1}{|\vec{n}|^3} \\ + \int d\tau \int \frac{d\omega}{2\pi} \frac{2i\epsilon_{ijk} \dot{m}_i \tilde{m}_j \dot{m}_k (\Delta^+ + \Delta^-) (2\omega^2 + \Delta^{+2} + \Delta^{-2} + 2\dot{m}^2) \omega}{(\omega^4 + 2(\Delta^+ \Delta^- - \dot{m}^2) \omega^2 + \Delta^{+2} \Delta^{-2} + (\Delta^{+2} + \Delta^{-2}) \dot{m}^2 + \dot{m}^4)^2} \quad (5.9)$$

where, $\frac{q}{2} \dot{n}_i := \dot{m}_i$, $\frac{q}{2} \tilde{n}_i := \tilde{m}_i$, ω is the energy. The contribution to the effective action from non-zero modes cancels because of the fact that $\Delta^+ = -\Delta^-$. Comparing the above with, (5.8) we get,

$$\partial_i A_j(\vec{n}) - \partial_j A_i(\vec{n}) = \frac{|q|}{2} \epsilon_{ijk} \frac{\vec{n}_k}{|\vec{n}|^3}. \quad (5.10)$$

The factor $|q|$ occurs because of the sum over zero modes which has multiplicity $m = 2j + 1 = |q|$.

5.2 Application to $\hat{A}$ -type quiver

Now we are ready to generalise the previous result to our case. The relevant part of the action in (2.36) and (2.37) for computing effective action is,

$$\mathcal{S} = \int d\tau d\Omega \sum_{(j)=1}^n \text{tr} \left(-i \xi_{(j)}^\dagger \mathcal{P} \xi_{(j)} + \frac{i}{2} \lambda^{ab}_{(j)} \mathcal{P} \lambda_{ab(j)} - \frac{i}{2} n_i(\tau) \xi_{a(j)}^\dagger (\sigma_i)^a_b [H, \xi_{(j)}^b] \right. \\ \left. - \frac{i}{4} n_i(\tau) \lambda_{ab(j)} (\sigma_i)_c^b [H, \lambda_{(j)}^{ac}] \right). \quad (5.11)$$

Now, to take care of the gauge indices we apply the same trick as (4.24) and rewrite the action with gauge indices as follows,

$$\mathcal{S} = \int d\tau d\Omega \sum_{(j)=1}^n \sum_{m,r=1}^N \left(-i \xi_{(j)rm}^\dagger \mathcal{P} \xi_{(j)mr} - \frac{i}{2} n_i(\tau) (q_m - q_r) \xi_{a(j)rm}^\dagger (\sigma_i)^a_b \xi_{(j)mr}^b \right. \\ \left. - i (\lambda_{1a(j)})_{rm} \mathcal{P} (\lambda_{(j)}^{1a})_{mr} + \frac{i}{2} n_i(\tau) (q_m - q_r) (\lambda_{1a(j)}^\dagger)_{rm} (\sigma_i)^a_b (\lambda_{(j)}^{1b})_{mr} \right). \quad (5.12)$$

To write the vector-multiplet fermion action in the above form so that we can use the result from the abelian case, we have defined the following similarly as BKK,

$$\lambda_{11} = -\lambda_{11}^\dagger, \lambda_{22} = \lambda^{11}, \lambda_{21} = -\lambda^{12}, \lambda^{21} = \lambda_{12}^\dagger, \lambda_{12} = -\lambda_{12}^\dagger, \lambda^{22} = -\lambda_{11}^\dagger.$$

The above action is similar to (5.3) except for the vector multiplet where the sign of interaction term is changed. Now we can treat each matrix elements of the fields as abelian fields and use the result from previous section to compute the effective action. The only modification we have to make is to put a negative sign in the final result for vector multiplet fermions. We obtain,

$$\begin{aligned} \partial_i A_j(\vec{n}) - \partial_j A_i(\vec{n})|_{\text{hyper}} + \partial_i A_j(\vec{n}) - \partial_j A_i(\vec{n})|_{\text{vector}} &= n \sum_{m,r=1}^N \left(\frac{|q_m - q_r|}{2} - \frac{|q_m - q_r|}{2} \right) \epsilon_{ijk} \frac{n_k}{|\vec{n}|^3} \\ &= 0 \end{aligned} \quad (5.13)$$

which means the induced monopole charge, in the $SU(2)_R$ moduli space, due to fermionic interaction is zero. This happens because of the field configuration of $\hat{A}$ -quiver. Quoting the result of BKK who obtained,

$$\partial_i A_j(\vec{n}) - \partial_j A_i(\vec{n})|_{\text{hyper}} + \partial_i A_j(\vec{n}) - \partial_j A_i(\vec{n})|_{\text{vector}} = (N_f - 2) \sum_{m,r=1}^N \left(\frac{|q_m - q_r|}{2} \right) \epsilon_{ijk} \frac{n_k}{|\vec{n}|^3} \quad (5.14)$$

which is zero for ABJM where $N_f = 2$ and hence our result is consistent.

Adding all contributions from bosons and fermions the effective action takes the following form,

$$\Gamma(\vec{n}) = \int d\tau \left(\frac{1}{2} M \dot{n}_i^2 + \lambda(n_i^2 - 1) \right), \quad (5.15)$$

where,

$$M = \frac{n}{2\tilde{g}^2} \text{tr}(H^2) = \frac{n}{2g^2} e^{-2\tau} \text{tr} H^2. \quad (5.16)$$

The above action can be thought of as that of a free particle of mass M moving on a unit sphere due to the presence of Lagrange multiplier (last term) in (5.15). The conserved angular momentum is,

$$L = iM\vec{n} \times \dot{\vec{n}} \quad (5.17)$$

whose quantized values are $l = 0, 1, 2, \dots$. Now, to find the conformal dimension of the monopole operators we solve the Schrödinger equation and read off the scaling dimension from the wave function by using state operator correspondence at IR. It can be checked that to get the correct behaviour of the wave function one has to include second order correction ($\Gamma_{2,2}(\vec{n})$) in the effective action. By doing so we recover the correct exponential behaviour of the wave function and read off the conformal dimension of the monopole operator by using state operator correspondence. This also verifies that the lowest energy state $l = 0$ is BPS in our case and hence the lowest possible value of the conformal dimension is zero.

5.3 Application to $\hat{D}$ -type quiver

Now it is easy to see that in the $\hat{D}$ case everything from the previous section follows for the internal edges. For the external edges also one can generalise the results as follows.

The relevant part of the action for charge computation is,

$$\begin{aligned}
 \mathcal{S} = \int d\tau d\Omega \operatorname{tr} & \left(\sum_{(j)=1}^{n+1} \frac{i}{2} \lambda_{(j)}^{ab} \mathcal{P} \lambda_{ab(j)} + \sum_{(j)=1}^{n+1} \frac{i}{2} \lambda_{ab(j)} [\phi_{c(j)}^b, \lambda_{(j)}^{ac}] \right. \\
 & + \sum_{(j)=3}^4 \left[-i \xi_{(j)}^\dagger \mathcal{P} \xi_{(j)} + i \xi_{a(j)}^\dagger \phi_{b(j)}^a \xi_{(j)}^b - i \xi_{a(j)}^a \phi_{a(n+1)}^b \xi_{b(j)}^\dagger \right] \\
 & + \sum_{(j)=1}^2 \left[-i \xi_{(j)}^\dagger \mathcal{P} \xi_{(j)} - i \xi_{(j)}^a \phi_{a(5)}^b \xi_{b(j)}^\dagger + i \xi_{a(j)}^\dagger \phi_{b(j)}^a \xi_{(j)}^b \right] \\
 & \left. + \sum_{(j)=5}^n \left[-i \xi_{(j)}^\dagger \mathcal{P} \xi_{(j)} - i \xi_{(j)}^a \phi_{a(j+1)}^b \xi_{b(j)}^\dagger + i \xi_{a(j)}^\dagger \phi_{b(j)}^a \xi_{(j)}^b \right] \right). \quad (5.18)
 \end{aligned}$$

The last summation is the same as in $\hat{A}$ case. Rest of the interaction terms for the hypermultiplets after substituting the monopole solution for the gauge fields look as follows,

$$\begin{aligned}
 & \int d\tau d\Omega \operatorname{tr} \left(\sum_{(j)=1}^4 \left[-\frac{i}{2} n_i(\tau) \xi_{a(j)}^\dagger H^{(1)}(\sigma_i)^a_b \xi_{(j)}^b + \frac{i}{2} n_i(\tau) \xi_{a(j)}^\dagger (\sigma_i)^a_b \xi_{(j)}^b H^{(2)} \right] \right) \\
 & = \sum_{(j)=1}^4 \sum_{s=1}^N \sum_{\hat{r}=1}^{2N} \left(-\frac{iq}{2} n_i(\tau) \xi_{a(j)\hat{r}s}^\dagger (\sigma_i)^a_b \xi_{(j)s\hat{r}}^b + \frac{iq}{2} n_i(\tau) \xi_{a(j)\hat{r}s}^\dagger (\sigma_i)^a_b \xi_{(j)s\hat{r}}^b \right) = 0. \quad (5.19)
 \end{aligned}$$

Therefore the external hypermultiplet fermions do not contribute to the charge. By doing same manipulations as above it is easy to see that the hypermultiplets associated to the internal edges also do not contribute to the charge. This can be anticipated from the $\hat{A}$ case if we set $q_m = q_r$. Similarly the action for the vector multiplet fermions after substituting the BPS background is,

$$\begin{aligned}
 \mathcal{S} = \int d\tau d\Omega \operatorname{tr} & \left(\sum_{(j)=1}^4 -i \lambda_{1a(j)} \mathcal{P} \lambda_{(j)}^{1a} + \frac{i}{2} n_i(\tau) \lambda_{1a(j)}^\dagger (\sigma_i)^a_b [H^{(1)}, \lambda_{(j)}^{1b}] \right. \\
 & \left. + \sum_{(j)=5}^{n+1} -i \lambda_{1a(j)} \mathcal{P} \lambda_{(j)}^{1a} + \frac{i}{2} n_i(\tau) \lambda_{1a(j)}^\dagger (\sigma_i)^a_b [H^{(2)}, \lambda_{(j)}^{1b}] \right). \quad (5.20)
 \end{aligned}$$

Now the commutator in above expression becomes,

$$[H^{(2)}, \lambda_{(j)}^{1b}]_{rs} = H_{rp}^{(2)} \lambda_{(j)ps}^{1b} - \lambda_{(j)rp}^{1b} H_{ps}^{(2)} = q \delta_{rp} \lambda_{(j)ps}^{1b} - q \lambda_{(j)rp}^{1b} \delta_{ps} = q \lambda_{(j)rs}^{1b} - q \lambda_{(j)rs}^{1b} = 0$$

which implies that they do not contribute to the $SU(2)_R$ charge. Therefore we obtain,

$$\partial_i A_j(\vec{n}) - \partial_j A_i(\vec{n})|_{\text{hyper}} + \partial_i A_j(\vec{n}) - \partial_j A_i(\vec{n})|_{\text{vector}} = 0. \quad (5.21)$$

The same logic applies in the $\hat{E}_6$ case as well and we get same result as above.

6 Discussion

To summarise our results,

- In this note we have constructed actions and supersymmetry variations of three dimensional $\mathcal{N} = 3$ Yang-Mills deformed CS quiver gauge theories with $\hat{A}_{n-1}, \hat{D}_n, \hat{E}_6$ quiver diagrams. These theories flow to a conformal fixed point in the IR via RG flow.
- We have obtained $\frac{1}{3}$ BPS and anti-BPS monopole solutions in the $\hat{A}_{n-1}$ quiver case with equal ranks of all gauge groups.
- We have obtained $\frac{1}{3}$ BPS and anti-BPS monopole solutions in the $\hat{D}_n$ quiver theory where we choose the gauge group to be $U(N)^4 \times U(2N)^{n-3}$ which has been studied extensively in the context of matrix models and have a dual M-theory description. Similarly in the $\hat{E}_6$ case we find BPS and anti-BPS monopole solution with a gauge group $U(N)^3 \times U(2N)^3 \times U(3N)$.
- We find that the quantum corrections to the $U(1)_R$ charges are zero for each quiver. In the $\hat{A}$ case this happens because of the field content of the theory, i.e contribution to the $U(1)_R$ charge coming from hypermultiplet fermions precisely cancel the contribution from vector multiplet fermions. In the $\widehat{DE}$ case this happens because of our choice of gauge ansatz which is equally charged under all $U(1)$ factors of the gauge groups.
- We find that the lowest possible value of quantised $SU(2)_R$ charge for each quiver theory is zero. In the $\hat{A}$ case, it is observed by calculating the path integral for the adjoint fermions that they give negative contribution to the R-charge and therefore cancelling the positive contribution from hypermultiplet fermions and making net quantum correction zero. The contribution from the adjoint fermions wouldn't have been captured in the IR theory, as they are not dynamical in the IR. In the $\widehat{DE}$ case we find $SU(2)_R$ charges to be zero for the same reasons explained in the previous point.

Our result is similar to ABJM theory which has monopole operator of zero conformal dimension. In fact these monopole operators are the ones needed to match the spectrum of ABJM theory with dual gravity theory.

Some of the interesting questions that maybe worth exploring are as follows:

- It is well known that in ABJM theory there is a supersymmetry enhancement from $\mathcal{N} = 6$ to $\mathcal{N} = 8$ for $k = 1, 2$. In this phenomena monopole operators played an important role. It will be interesting to check if there is any supersymmetry enhancement in $\widehat{ADE}$ theories as well.
- To the best of our knowledge the brane construction for exceptional quivers is not well understood [32]. One can try to see if the results derived here are useful for brane engineering [33].

- In interacting CFT's, monopole operators are usually studied via state operator correspondence, where the monopole operators become states on $\mathbb{R} \times S^2$ (considering three dimensional theories), which provides quantized flux through S^2 due to a monopole kept at the centre of S^2 [8]. Then one quantizes the theory in the monopole background and find several quantities like scaling dimensions, superconformal index. A recent study on monopole operators in CS matter theories in [23] proposes a prescription to describe the monopole operators as local operators directly on $\mathbb{R}^3$. In an $\mathcal{N} = 2$ abelian SQED with single charged chiral multiplet and a CS term with level k , it can be done by giving a singular profile to the bosonic and fermionic fields in the theory along with the singular gauge field at the insertion point, keeping in mind that they should be consistent with the equations of motion and Gauss law constraints. They find $\frac{1}{4}$ -BPS monopoles on $\mathbb{R}^3$ and compute their dimensions. As suggested in [23] that this method is applicable to continuous deformation of ABJM theory as it won't affect the discrete global charges of the monopole operators in the theory. Therefore it would be nice to apply this method in $\widehat{ADE}$ theories as a consistency check.

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A Notations and conventions

We use similar conventions as BKK. $\alpha, \beta = 1, 2$ are spinor indices raised and lowered from the left, $\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta$ and $\psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta$, with $\epsilon^{12} = -\epsilon_{12} = 1$. $a, b, c, d = 1, 2$ are R-symmetry indices which are raised and lowered by $SU(2)_R$ metric ϵ^{ab} , with $\epsilon^{12} = \epsilon_{21} = +1$. To contract the spinor indices we use NW-SE convention. Inner products on superspace in three dimension $\theta^2 = \theta^\alpha \theta_\alpha, \theta \bar{\theta} = \theta^\alpha \bar{\theta}_\alpha, \bar{\theta}^2 = \bar{\theta}^\alpha \bar{\theta}_\alpha, \theta \gamma^\mu \bar{\theta} = \theta^\alpha (\gamma^\mu)_\alpha^\beta \bar{\theta}_\beta$. For doing the superspace integral we use,

$$\theta^\alpha \theta^\beta = -\frac{1}{2} \epsilon^{\alpha\beta} \theta^2, \quad \bar{\theta}^\alpha \bar{\theta}^\beta = -\frac{1}{2} \epsilon^{\alpha\beta} \bar{\theta}^2. \quad (\text{A.1})$$

The super covariant derivatives in x basis has the following expression,

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i(\gamma^\mu)_\alpha^\beta \bar{\theta}_\beta \frac{\partial}{\partial x^\mu}, \quad \bar{D}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} + i(\gamma^\mu)_\alpha^\beta \theta_\beta \frac{\partial}{\partial x^\mu}. \quad (\text{A.2})$$

The Fierz identities are,

$$\begin{aligned}
 (\psi_1 \psi_2)(\psi_3 \psi_4) &= -\frac{1}{2}(\psi_1 \psi_4)(\psi_3 \psi_2) - \frac{1}{2}(\psi_1 \gamma^\mu \psi_4)(\psi_3 \gamma_\mu \psi_2), \\
 (\psi_1 \psi_2)(\psi_3 \gamma^\mu \psi_4) &= -\frac{1}{2}(\psi_1 \gamma^\mu \psi_4)(\psi_3 \psi_2) - \frac{1}{2}(\psi_1 \psi_4)(\psi_3 \gamma^\mu \psi_2) - \frac{1}{2}\epsilon^{\mu\nu\rho}(\psi_1 \gamma_\nu \psi_4)(\psi_3 \gamma_\rho \psi_2), \\
 (\psi_1 \gamma^\mu \psi_2)(\psi_3 \gamma^\nu \psi_4) &= -\frac{1}{2}g^{\mu\nu}(\psi_1 \psi_4)(\psi_3 \psi_2) + \frac{1}{2}g^{\mu\nu}(\psi_1 \gamma^\rho \psi_4)(\psi_3 \gamma_\rho \psi_2) - (\psi_1 \gamma^{(\mu} \psi_4)(\psi_3 \gamma^{\nu)} \psi_2) \\
 &\quad - \frac{1}{2}\epsilon^{\mu\nu\rho}[(\psi_1 \gamma_\rho \psi_4)(\psi_3 \psi_2) - (\psi_1 \psi_4)(\psi_3 \gamma_\rho \psi_2)].
 \end{aligned} \tag{A.3}$$

On $\mathbb{R}^{1,2}$. We list here the conventions used in the Minkowski space $\mathbb{R}^{1,2}$ with metric $ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2$. $\mu, \nu = 0, 1, 2$ to denote space-time indices. Choices of γ matrices are $(\gamma^\mu)_\alpha^\beta = (i\sigma^2, \sigma^1, \sigma^3)$ which satisfy $\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu\rho} \gamma_\rho$. Notice that $(\gamma^\mu)_{\alpha\beta} = (-\mathbb{1}, -\sigma^3, \sigma^1)$ is symmetric. The Killing spinor equation is,

$$\mathcal{D}_\mu \varepsilon = 0. \tag{A.4}$$

On $\mathbb{R} \times S^2$. The metric on $\mathbb{R} \times S^2$ is $ds^2 = g_{mn} dx^m dx^n = d\tau^2 + (d\theta^2 + \sin^2 \theta d\varphi^2)$. $k, m, n = 1, 2, 3$ are space time (Euclidean) indices. Choices for gamma matrices are as follows. In the tangent frame $(\gamma^A)_\alpha^\beta = (-\sigma^2, \sigma^1, \sigma^3)$, which satisfy $\gamma^A \gamma^B = \delta^{AB} + i\epsilon^{ABC} \gamma^C$, where $A = 1, 2, 3$ are flat indices in the tangent frame. The gamma matrices $\gamma^m = e_A^m \gamma^A$ satisfy the Clifford algebra $\gamma^m \gamma^n + \gamma^n \gamma^m = 2g^{mn}$, where e_A^m are the vierbeins, taking values $e_1^\theta = 1, e_2^\varphi = \frac{1}{\sin \theta}$. The covariant derivative of a spinor ψ is,

$$\begin{aligned}
 \nabla_m \psi &= (\partial_m + \omega_m) \psi \quad \text{where } \omega_m \text{ is the spin connection,} \\
 \omega_m &= \frac{1}{4} \omega_{mAB} \gamma^{AB}, \quad \gamma^{AB} = \frac{1}{2} [\gamma^A, \gamma^B]
 \end{aligned} \tag{A.5}$$

where, $\omega_{\varphi 21} = -\omega_{\varphi 12} = \cos \theta$. The covariant derivatives of a vector $\mathcal{A}^n$ is,

$$\nabla_m \mathcal{A}^n = \partial_m \mathcal{A}^n + \Gamma_{mp}^n \mathcal{A}^p \tag{A.6}$$

where the non zero components of the Christoffel connection are $\Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta$, $\Gamma_{\theta\varphi}^\varphi = \frac{\cos \theta}{\sin \theta}$.

B Component action computation

We use dimensionally reduced $\mathcal{N} = 2$ multiplet of four dimension which are again written in terms of $\mathcal{N} = 1$ superfields. The component expansion of the $\mathcal{N} = 2$ superfields are given as follows. In the gauge multiplet we have a vector superfield,

$$\mathcal{V}_{(j)} = 2i \theta \bar{\theta} \sigma_{(j)}(x) - 2\theta \gamma^m \bar{\theta} A_{m(j)}(x) + \sqrt{2}i \theta^2 \bar{\theta} \chi_{\sigma(j)}^\dagger(x) - \sqrt{2}i \bar{\theta}^2 \theta \chi_{\sigma(j)}(x) + \theta^2 \bar{\theta}^2 D_{(j)}(x)$$

and an adjoint chiral superfield,

$$\begin{aligned}
 \Phi_{(j)} &= \phi_{(j)}(x_L) + \sqrt{2} \theta \chi_{\phi(j)}(x_L) + \theta^2 F_{\phi(j)}(x_L), \\
 \bar{\Phi}_{(j)} &= \phi_{(j)}^\dagger(x_R) - \sqrt{2} \bar{\theta} \chi_{\phi(j)}^\dagger(x_R) - \bar{\theta}^2 F_{\phi(j)}^\dagger(x_R).
 \end{aligned}$$

In the hypermultiplet we have two bifundamental chiral superfields,

$$\begin{aligned}\mathcal{Z}_{(j)} &= Z_{(j)}(x_L) + \sqrt{2}\theta\zeta_{(j)}(x_L) + \theta^2 F_i(x_L), & \bar{\mathcal{Z}}_{(j)} &= Z_{(j)}^\dagger(x_R) - \sqrt{2}\bar{\theta}\zeta_{(j)}^\dagger(x_R) - \bar{\theta}^2 F_i^\dagger(x_R), \\ \mathcal{W}_{(j)} &= W_{(j)}(x_L) + \sqrt{2}\theta\omega_{(j)}(x_L) + \theta^2 G_{(j)}(x_L), & \bar{\mathcal{W}}_{(j)} &= W_{(j)}^\dagger(x_R) - \sqrt{2}\bar{\theta}\omega_{(j)}^\dagger(x_R) - \bar{\theta}^2 G_{(j)}^\dagger(x_R)\end{aligned}$$

where, $x_L^m = x^m - i\theta^\alpha(\gamma^m)_\alpha^\beta \bar{\theta}_\beta$ and $x_R^m = x^m + i\theta^\alpha(\gamma^m)_\alpha^\beta \bar{\theta}_\beta$.

B.1 $\hat{A}$ -type quiver

The CS part of the action takes the following form,

$$\begin{aligned}\mathcal{S}_{\text{CS}} &= \int d^3x \sum_{(j)=1}^n \text{tr} \left(-2\kappa_{(j)}\sigma_{(j)}D_{(j)} + \kappa_{(j)}\epsilon^{\mu\nu\lambda} \left(A_{\mu(j)}\partial_\nu A_{\lambda(j)} + \frac{2i}{3}A_{\mu(j)}A_{\nu(j)}A_{\lambda(j)} \right) \right. \\ &\quad \left. + \frac{i\kappa_{(j)}}{2}\chi_{\sigma(j)}\chi_{\sigma(j)}^\dagger + \frac{i\kappa_{(j)}}{2}\chi_{\sigma(j)}^\dagger\chi_{\sigma(j)} \right)\end{aligned}\quad (\text{B.1})$$

where, $\kappa_{(j)} = \frac{k_{(j)}}{4\pi}$. The Yang-Mills part is,

$$\begin{aligned}\mathcal{S}_{\text{YM}} &= \int d^3x \text{tr} \sum_{(j)=1}^n \left(-\frac{1}{2g^2}F_{\mu\nu(j)}F_{(j)}^{\mu\nu} + \frac{i}{g^2}\chi_{\sigma(j)}\not{D}\chi_{\sigma(j)}^\dagger - \frac{1}{g^2}(\mathcal{D}_m\sigma_{(j)})(\mathcal{D}^m\sigma_{(j)}) + \frac{1}{g^2}D_{(j)}^2 \right. \\ &\quad \left. + \frac{i}{g^2}\chi_{\sigma(j)}[\sigma_{(j)},\chi_{\sigma(j)}^\dagger] \right).\end{aligned}\quad (\text{B.2})$$

The component expression of $\mathcal{S}_{\text{adj}}$ is,

$$\begin{aligned}\mathcal{S}_{\text{adj}} &= \int d^3x \text{tr} \sum_{(j)=1}^n \left(-\frac{1}{g^2}(\mathcal{D}_m\phi_{(j)}^\dagger)(\mathcal{D}^m\phi_{(j)}) + \frac{i}{g^2}\chi_{\phi(j)}\not{D}\chi_{\phi(j)}^\dagger - \frac{i}{g^2}\phi_{(j)}^\dagger[\chi_{\sigma(j)},\chi_{\phi(j)}] \right. \\ &\quad + \frac{i}{g^2}\chi_{\phi(j)}^\dagger[\sigma_{(j)},\chi_{\phi(j)}] + \frac{i}{g^2}\chi_{\phi(j)}^\dagger[\chi_{\sigma(j)}^\dagger,\phi_{(j)}] + \frac{1}{g^2}[\sigma_{(j)},\phi_{(j)}][\sigma_{(j)},\phi_{(j)}^\dagger] - \frac{1}{g^2}\phi_{(j)}^\dagger\phi_{(j)}D_{(j)} \\ &\quad \left. + \frac{1}{g^2}\phi_{(j)}^\dagger D_{(j)}\phi_{(j)} + \frac{1}{g^2}F_{\phi(j)}^\dagger F_{\phi(j)} \right).\end{aligned}\quad (\text{B.3})$$

The matter action is,

$$\begin{aligned}\mathcal{S}_{\text{mat}} &= \int d^3x \sum_{(j)=1}^n \text{tr} \left(-\mathcal{D}_m Z_{(j)} \cdot \mathcal{D}^m Z_{(j)}^\dagger + i\zeta_{(j)}^\dagger \not{D}\zeta_{(j)} - Z_{(j)}^\dagger Z_{(j)} D_{(j+1)} + Z_{(j)}^\dagger D_{(j)} Z_{(j)} \right. \\ &\quad - iZ_{(j)}^\dagger (\zeta_{(j)}\chi_{\sigma(j+1)} - \chi_{\sigma(j)}\zeta_{(j)}) - i\zeta_{(j)}^\dagger (Z_{(j)}\chi_{\sigma(j+1)}^\dagger - \chi_{\sigma(j)}^\dagger Z_{(j)}) \\ &\quad - i\zeta_{(j)}^\dagger (\sigma_{(j)}\zeta_{(j)} - \zeta_{(j)}\sigma_{(j+1)}) - Z_{(j)}^\dagger Z_{(j)}\sigma_{(j+1)}^2 + 2Z_{(j)}^\dagger \sigma_{(j)} Z_{(j)}\sigma_{(j+1)} \\ &\quad - Z_{(j)}^\dagger \sigma_{(j)}^2 Z_{(j)} + F_{(j)}^\dagger F_{(j)} - \mathcal{D}_m W_{(j)} \cdot \mathcal{D}^m W_{(j)}^\dagger + i\omega_{(j)}\not{D}\omega_{(j)}^\dagger - W_{(j)}^\dagger W_{(j)} D_{(j)} \\ &\quad + W_{(j)}^\dagger D_{(j+1)} W_{(j)} - iW_{(j)}^\dagger (\omega_{(j)}\chi_{\sigma(j)} - \chi_{\sigma(j+1)}\omega_{(j)}) - i\omega_{(j)}^\dagger (W_{(j)}\chi_{\sigma(j)}^\dagger - \chi_{\sigma(j+1)}^\dagger W_{(j)}) \\ &\quad + i\omega_{(j)}^\dagger (\omega_{(j)}\sigma_{(j)} - \sigma_{(j+1)}\omega_{(j)}) - W_{(j)}^\dagger W_{(j)}\sigma_{(j)}^2 + 2W_{(j)}^\dagger \sigma_{(j+1)} W_{(j)}\sigma_{(j)} \\ &\quad \left. - W_{(j)}^\dagger \sigma_{(j+1)}^2 W_{(j)} + G_{(j)}^\dagger G_{(j)} \right).\end{aligned}\quad (\text{B.4})$$

The component action for $\mathcal{S}_{\text{pot}}$ is,

$$\begin{aligned}
 \mathcal{S}_{\text{pot}} = & \int d^3x \text{tr} \left(\phi_{(j)} Z_{(j)} G_{(j)} - \phi_{(j)} \zeta_{(j)} \omega_{(j)} + \phi_{(j)} F_{(j)} W_{(j)} - \chi_{\phi_{(j)}} Z_{(j)} \omega_{(j)} \right. \\
 & - \chi_{\phi_{(j)}} \zeta_{(j)} W_{(j)} + F_{\phi_{(j)}} Z_{(j)} W_{(j)} - \phi_{(j)} W_{(j-1)} F_{(j-1)} + \phi_{(j)} \omega_{(j-1)} \zeta_{(j-1)} \\
 & - \phi_{(j)} G_{(j-1)} Z_{(j-1)} + \chi_{\phi_{(j)}} \omega_{(j-1)} Z_{(j-1)} + \chi_{\phi_{(j)}} W_{(j-1)} \zeta_{(j-1)} - F_{\phi_{(j)}} W_{(j-1)} Z_{(j-1)} \\
 & + \phi_{(j)}^\dagger W_{(j)}^\dagger F_{(j)}^\dagger + \phi_{(j)}^\dagger \omega_{(j)}^\dagger \zeta_{(j)}^\dagger + \phi_{(j)}^\dagger G_{(j)}^\dagger Z_{(j)}^\dagger + \chi_{\phi_{(j)}}^\dagger W_{(j)}^\dagger \zeta_{(j)}^\dagger + \chi_{\phi_{(j)}}^\dagger \omega_{(j)}^\dagger Z_{(j)}^\dagger \\
 & + F_{\phi_{(j)}}^\dagger W_{(j)}^\dagger Z_{(j)}^\dagger - \phi_{(j)}^\dagger Z_{(j-1)}^\dagger G_{(j-1)}^\dagger - \phi_{(j)}^\dagger \omega_{(j-1)}^\dagger \zeta_{(j-1)}^\dagger - \phi_{(j)}^\dagger F_{(j-1)}^\dagger W_{(j-1)}^\dagger \\
 & \left. - \chi_{\phi_{(j)}}^\dagger Z_{(j-1)}^\dagger \omega_{(j)}^\dagger - \chi_{\phi_{(j)}}^\dagger \zeta_{(j-1)}^\dagger W_{(j-1)}^\dagger - F_{\phi_{(j)}}^\dagger Z_{(j-1)}^\dagger W_{(j-1)}^\dagger \right) \\
 & + \int d^3x \frac{\kappa_{(j)}}{2} \text{tr} \left(2\phi_{(j)} F_{\phi_{(j)}} - \chi_{\phi_{(j)}} \chi_{\phi_{(j)}} + 2\phi_{(j)}^\dagger F_{\phi_{(j)}}^\dagger + \chi_{\phi_{(j)}}^\dagger \chi_{\phi_{(j)}}^\dagger \right). \quad (\text{B.5})
 \end{aligned}$$

Auxiliary fields are eliminated by using their equation of motions which are

$$\begin{aligned}
 D_{(j)} &= \frac{g^2}{2} \left(2\kappa_{(j)} \sigma_{(j)} - \frac{1}{g^2} [\phi_{(j)}, \phi_{(j)}^\dagger] - Z_{(j)} Z_{(j)}^\dagger + W_{(j)}^\dagger W_{(j)} + Z_{(j-1)}^\dagger Z_{(j-1)} - W_{(j-1)} W_{(j-1)}^\dagger \right), \\
 F_{\phi_{(j)}} &= -g^2 (W_{(j)}^\dagger Z_{(j)}^\dagger - Z_{(j-1)}^\dagger W_{(j-1)}^\dagger + \kappa_{(j)} \phi_{(j)}^\dagger), \\
 F_{(j)} &= -(\phi_{(j)}^\dagger W_{(j)}^\dagger - W_{(j)}^\dagger \phi_{(j+1)}^\dagger), \\
 G_{(j)} &= -(Z_{(j)}^\dagger \phi_{(j)}^\dagger - \phi_{(j+1)}^\dagger Z_{(j)}^\dagger), \\
 F_{\phi_{(j)}}^\dagger &= -g^2 (Z_{(j)} W_{(j)} - W_{(j-1)} Z_{(j-1)} + \kappa_{(j)} \phi_{(j)}), \\
 F_{(j)}^\dagger &= -(W_{(j)} \phi_{(j)} - \phi_{(j+1)} W_{(j)}), \\
 G_{(j)}^\dagger &= -(\phi_{(j)} Z_{(j)} - Z_{(j)} \phi_{(j+1)}). \quad (\text{B.6})
 \end{aligned}$$

After eliminating the auxiliary fields one obtains,

$$\begin{aligned}
 (\mathcal{L}_{\text{aux}}) &= -\frac{g^2}{2} \left((Z_{(j)} Z_{(j)}^\dagger)^2 + (W_{(j)} W_{(j)}^\dagger)^2 + Z_{(j)} Z_{(j)}^\dagger W_{(j)}^\dagger W_{(j)} + Z_{(j)}^\dagger Z_{(j)} W_{(j)} W_{(j)}^\dagger \right) \\
 &+ \frac{g^2}{2} (Z_{(j)} Z_{(j)}^\dagger Z_{(j-1)}^\dagger Z_{(j-1)} - Z_{(j)} Z_{(j)}^\dagger W_{(j-1)}^\dagger W_{(j-1)} - W_{(j)}^\dagger W_{(j)} Z_{(j-1)}^\dagger Z_{(j-1)} \\
 &+ W_{(j)}^\dagger W_{(j)} W_{(j-1)}^\dagger W_{(j-1)} + g^2 W_{(j)}^\dagger Z_{(j)}^\dagger W_{(j-1)} Z_{(j)} + g^2 Z_{(j-1)}^\dagger W_{(j-1)}^\dagger Z_{(j)} W_{(j)} \\
 &- \frac{1}{4g^2} [\phi_{(j)}, \phi_{(j)}^\dagger]^2 + \kappa_{(j)} \sigma_{(j)} [\phi_{(j)}, \phi_{(j)}^\dagger] - \kappa_{(j)}^2 g^2 \sigma_{(j)}^2 + \kappa_{(j)} g^2 \sigma_{(j)} (Z_{(j)} Z_{(j)}^\dagger - Z_{(j-1)}^\dagger Z_{(j-1)} \\
 &+ W_{(j-1)} W_{(j-1)}^\dagger - W_{(j)}^\dagger W_{(j)} + \kappa_{(j)} g^2 \phi_{(j)} (Z_{(j-1)}^\dagger W_{(j-1)}^\dagger - W_{(j)}^\dagger Z_{(j)}^\dagger) \\
 &+ \kappa_{(j)} g^2 \phi_{(j)}^\dagger (W_{(j-1)} Z_{(j-1)} - Z_{(j)} W_{(j)}) - \kappa_{(j)}^2 g^2 \phi_{(j)}^\dagger \phi_{(j)} \\
 &- \frac{1}{2} (\phi_{(j)} \phi_{(j)}^\dagger + \phi_{(j)}^\dagger \phi_{(j)}) (Z_{(j)} Z_{(j)}^\dagger + W_{(j)}^\dagger W_{(j)} + Z_{(j-1)}^\dagger Z_{(j-1)} + W_{(j-1)} W_{(j-1)}^\dagger). \quad (\text{B.7})
 \end{aligned}$$

B.2 $\widehat{D}$ -type quiver

The Euler-Lagrange equations of motion of the auxiliary fields are,

$$\begin{aligned}
 D_{(j)} &= \frac{g^2}{2} \left(2\kappa_{(j)}\sigma_{(j)} - \frac{1}{g^2}[\phi_{(j)}, \phi_{(j)}^\dagger] - Z_{(j)}Z_{(j)}^\dagger + W_{(j)}^\dagger W_{(j)} \right) \quad (j) = 1, 2, 3, 4, \\
 D_{(5)} &= \frac{g^2}{2} \left(2\kappa_{(5)}\sigma_{(5)} - \frac{1}{g^2}[\phi_{(5)}, \phi_{(5)}^\dagger] + \sum_{(j)=1}^2 (Z_{(j)}^\dagger Z_{(j)} - W_{(j)} W_{(j)}^\dagger) - Z_{(5)}Z_{(5)}^\dagger + W_{(5)}^\dagger W_{(5)} \right), \\
 D_{(n+1)} &= \frac{g^2}{2} \left(2\kappa_{(n+1)}\sigma_{(n+1)} - \frac{1}{g^2}[\phi_{(n+1)}, \phi_{(n+1)}^\dagger] + \sum_{(j)=3,4,n} (Z_{(j)}^\dagger Z_{(j)} - W_{(j)} W_{(j)}^\dagger) \right), \\
 D_{(j)} &= \frac{g^2}{2} \left(2\kappa_{(j)}\sigma_{(j)} - \frac{1}{g^2}[\phi_{(j)}, \phi_{(j)}^\dagger] + Z_{(j-1)}^\dagger Z_{(j-1)} - W_{(j-1)} W_{(j-1)}^\dagger - Z_{(j)}Z_{(j)}^\dagger + W_{(j)}^\dagger W_{(j)} \right) \\
 &\quad (j) = 6, \dots, n, \\
 F_{\phi_{(j)}} &= -g^2 (W_{(j)}^\dagger Z_{(j)}^\dagger + \kappa_{(j)}\phi_{(j)}^\dagger) \quad (j) = 1, 2, 3, 4, \\
 F_{\phi_{(5)}} &= -g^2 (W_{(5)}^\dagger Z_{(5)}^\dagger + \kappa_{(5)}\phi_{(5)}^\dagger - Z_{(1)}^\dagger W_{(1)}^\dagger - Z_{(2)}^\dagger W_{(2)}^\dagger), \\
 F_{\phi_{(n+1)}} &= -g^2 \left(\kappa_{(n+1)}\phi_{(n+1)}^\dagger - \sum_{(j)=3,4,n} Z_{(j)}^\dagger W_{(j)}^\dagger \right), \\
 F_{\phi_{(j)}} &= -g^2 (\kappa_{(j)}\phi_{(j)}^\dagger + W_{(j)}^\dagger Z_{(j)}^\dagger - Z_{(j-1)}^\dagger W_{(j-1)}^\dagger) \quad (j) = 6, \dots, n, \tag{B.8}
 \end{aligned}$$

$$\begin{aligned}
 F_{(j)} &= -(\phi_{(j)}^\dagger W_{(j)}^\dagger - W_{(j)}^\dagger \phi_{(5)}^\dagger) \quad (j) = 1, 2, \\
 F_{(j)} &= -(\phi_{(j)}^\dagger W_{(j)}^\dagger - W_{(j)}^\dagger \phi_{(n+1)}^\dagger) \quad (j) = 3, 4, \\
 F_{(j)} &= -(\phi_{(j)}^\dagger W_{(j)}^\dagger - W_{(j)}^\dagger \phi_{(j+1)}^\dagger) \quad (j) = 5, \dots, n, \\
 G_{(j)} &= -(Z_{(j)}^\dagger \phi_{(j)}^\dagger - \phi_{(5)}^\dagger Z_{(j)}^\dagger) \quad (j) = 1, 2, \\
 G_{(j)} &= -(Z_{(j)}^\dagger \phi_{(j)}^\dagger - \phi_{(n+1)}^\dagger Z_{(j)}^\dagger) \quad (j) = 3, 4, \\
 G_{(j)} &= -(Z_{(j)}^\dagger \phi_{(j)}^\dagger - \phi_{(j+1)}^\dagger Z_{(j)}^\dagger) \quad (j) = 5, \dots, n. \tag{B.9}
 \end{aligned}$$

We present the component action for only the $\mathcal{L}_{\text{pot}}$ and $\mathcal{L}_{\text{mat}}$ since rest of the part is straightforward from the previous section.

$$\begin{aligned}
 \mathcal{L}_{\text{pot}} &= \text{tr} \left(\sum_{(j)=1}^n \left[\phi_{(j)} Z_{(j)} G_{(j)} - \phi_{(j)} \zeta_{(j)} \omega_{(j)} + \phi_{(j)} F_{(j)} W_{(j)} - \chi_{\phi_{(j)}} Z_{(j)} \omega_{(j)} - \chi_{\phi_{(j)}} \zeta_{(j)} W_{(j)} + F_{\phi_{(j)}} Z_{(j)} W_{(j)} \right. \right. \\
 &\quad \left. \left. + \phi_{(j)}^\dagger W_{(j)}^\dagger F_{(j)}^\dagger + \phi_{(j)}^\dagger \omega_{(j)}^\dagger \zeta_{(j)}^\dagger + \phi_{(j)}^\dagger G_{(j)}^\dagger Z_{(j)}^\dagger + \chi_{\phi_{(j)}}^\dagger W_{(j)}^\dagger \zeta_{(j)}^\dagger + \chi_{\phi_{(j)}}^\dagger \omega_{(j)}^\dagger Z_{(j)}^\dagger + F_{\phi_{(j)}}^\dagger W_{(j)}^\dagger Z_{(j)}^\dagger \right] \right. \\
 &\quad \left. + \sum_{(j)=1}^2 \left[-\phi_{(5)} W_{(j)} F_{(j)} + \phi_{(5)} \omega_{(j)} \zeta_{(j)} - \phi_{(5)} G_{(j)} Z_{(j)} + \chi_{\phi_{(5)}} \omega_{(j)} Z_{(j)} + \chi_{\phi_{(5)}} W_{(j)} \zeta_{(j)} - F_{\phi_{(5)}} W_{(j)} Z_{(j)} \right. \right. \\
 &\quad \left. \left. - \phi_{(5)}^\dagger Z_{(j)}^\dagger G_{(j)}^\dagger - \phi_{(5)}^\dagger \zeta_{(j)}^\dagger \omega_{(j)}^\dagger - \phi_{(5)}^\dagger F_{(j)}^\dagger W_{(j)}^\dagger - \chi_{\phi_{(5)}}^\dagger Z_{(j)}^\dagger \omega_{(j)}^\dagger - \chi_{\phi_{(5)}}^\dagger \zeta_{(j)}^\dagger W_{(j)}^\dagger - F_{\phi_{(5)}}^\dagger Z_{(j)}^\dagger W_{(j)}^\dagger \right] \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{(j)=3}^4 \left[-\phi_{(n+1)} W_{(j)} F_{(j)} + \phi_{(n+1)} \omega_{(j)} \zeta_{(j)} - \phi_{(n+1)} G_{(j)} Z_{(j)} + \chi_{\phi_{(n+1)}} \omega_{(j)} Z_{(j)} + \chi_{\phi_{(n+1)}} W_{(j)} \zeta_{(j)} \right. \\
 & - F_{\phi_{(n+1)}} W_{(j)} Z_{(j)} - \phi_{(n+1)}^\dagger Z_{(j)}^\dagger G_{(j)}^\dagger - \phi_{(n+1)}^\dagger \omega_{(j)}^\dagger \zeta_{(j)}^\dagger - \phi_{(n+1)}^\dagger F_{(j)}^\dagger W_{(j)}^\dagger - \chi_{\phi_{(n+1)}^\dagger} Z_{(j)}^\dagger \omega_{(j)}^\dagger \\
 & \left. - \chi_{\phi_{(n+1)}^\dagger} \zeta_{(j)}^\dagger W_{(j)}^\dagger - F_{\phi_{(n+1)}^\dagger} Z_{(j)}^\dagger W_{(j)}^\dagger \right] + \sum_{(j)=5}^n \left[-\phi_{(j+1)} W_{(j)} F_{(j)} + \phi_{(j+1)} \omega_{(j)} \zeta_{(j)} - \phi_{(j+1)} G_{(j)} Z_{(j)} \right. \\
 & + \chi_{\phi_{(j+1)}} \omega_{(j)} Z_{(j)} + \chi_{\phi_{(j+1)}} W_{(j)} \zeta_{(j)} - F_{\phi_{(j+1)}} W_{(j)} Z_{(j)} - \phi_{(j+1)}^\dagger Z_{(j)}^\dagger G_{(j)}^\dagger - \phi_{(j+1)}^\dagger \omega_{(j)}^\dagger \zeta_{(j)}^\dagger \\
 & \left. - \phi_{(j+1)}^\dagger F_{(j)}^\dagger W_{(j)}^\dagger - \chi_{\phi_{(j+1)}^\dagger} Z_{(j)}^\dagger \omega_{(j)}^\dagger - \chi_{\phi_{(j+1)}^\dagger} \zeta_{(j)}^\dagger W_{(j)}^\dagger - F_{\phi_{(j+1)}^\dagger} Z_{(j)}^\dagger W_{(j)}^\dagger \right] \\
 & + \int d^3x \sum_{(j)=1}^{n+1} \frac{\kappa_{(j)}}{2} \text{tr} \left[2\phi_{(j)} F_{\phi_{(j)}} - \chi_{\phi_{(j)}} \chi_{\phi_{(j)}} + 2\phi_{(j)}^\dagger F_{\phi_{(j)}^\dagger} + \chi_{\phi_{(j)}^\dagger} \chi_{\phi_{(j)}^\dagger} \right] \Bigg), \tag{B.10}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{L}_{\text{mat}} = & \text{tr} \left(\sum_{(j)=1}^2 \left[-\mathcal{D}_m Z_{(j)} \mathcal{D}^m Z_{(j)}^\dagger + i\zeta_{(j)}^\dagger \not{D} \zeta_{(j)} - Z_{(j)}^\dagger Z_{(j)} D_{(5)} + Z_{(j)}^\dagger D_{(j)} Z_{(j)} - iZ_{(j)}^\dagger (\zeta_{(j)} \chi_{\sigma(5)} \right. \right. \\
 & - \chi_{\sigma(5)} \zeta_{(j)}) - i\zeta_{(j)}^\dagger (Z_{(j)} \chi_{\sigma(5)}^\dagger - \chi_{\sigma(5)}^\dagger Z_{(j)}) - i\zeta_{(j)}^\dagger (\sigma_{(j)} \zeta_{(j)} - \zeta_{(j)} \sigma_{(5)}) - Z_{(j)}^\dagger Z_{(j)} \sigma_{(5)}^2 \\
 & + 2Z_{(j)}^\dagger \sigma_{(j)} Z_{(j)} \sigma_5 - Z_{(j)}^\dagger \sigma_{(j)}^2 Z_{(j)} + F_{(j)}^\dagger F_{(j)} \Big] - (\mathcal{D}_m W_{(j)}) (\mathcal{D}^m W_{(j)}^\dagger) + i\omega_{(j)} \not{D} \omega_{(j)}^\dagger \\
 & - W_{(j)}^\dagger W_{(j)} D_{(j)} + W_{(j)}^\dagger D_{(5)} W_{(j)} + i\omega_{(j)}^\dagger (\omega_{(j)} \sigma_{(j)} - \sigma_{(5)} \omega_{(j)}) - iW_{(j)}^\dagger (\omega_{(j)} \chi_{\sigma(5)} - \chi_{\sigma(5)} \omega_{(j)}) \\
 & \left. - i\omega_{(j)}^\dagger (W_{(j)} \chi_{\sigma(5)}^\dagger - \chi_{\sigma(5)}^\dagger W_{(j)}) - W_{(j)}^\dagger W_{(j)} \sigma_{(j)}^2 + 2W_{(j)}^\dagger \sigma_{(5)} W_{(j)} \sigma_{(j)} - W_{(j)}^\dagger \sigma_{(5)}^2 W_{(j)} + G_{(j)}^\dagger G_{(j)} \right] \\
 & + \sum_{(j)=3}^4 \left[-\mathcal{D}_m Z_{(j)} \mathcal{D}^m Z_{(j)}^\dagger + i\zeta_{(j)}^\dagger \not{D} \zeta_{(j)} - Z_{(j)}^\dagger Z_{(j)} D_{(n+1)} + Z_{(j)}^\dagger D_{(j)} Z_{(j)} \right. \\
 & - iZ_{(j)}^\dagger (\zeta_{(j)} \chi_{\sigma(n+1)} - \chi_{\sigma(n+1)} \zeta_{(j)}) - i\zeta_{(j)}^\dagger (Z_{(j)} \chi_{\sigma(n+1)}^\dagger - \chi_{\sigma(n+1)}^\dagger Z_{(j)}) - i\zeta_{(j)}^\dagger (\sigma_{(j)} \zeta_{(j)} - \zeta_{(j)} \sigma_{(n+1)}) \\
 & - Z_{(j)}^\dagger Z_{(j)} \sigma_{(n+1)}^2 + 2Z_{(j)}^\dagger \sigma_{(j)} Z_{(j)} \sigma_{(n+1)} - Z_{(j)}^\dagger \sigma_{(j)}^2 Z_{(j)} + F_{(j)}^\dagger F_{(j)} \Big] - (\mathcal{D}_m W_{(j)}) (\mathcal{D}^m W_{(j)}^\dagger) \\
 & + i\omega_{(j)} \not{D} \omega_{(j)}^\dagger - W_{(j)}^\dagger W_{(j)} D_{(j)} + W_{(j)}^\dagger D_{(n+1)} W_{(j)} + i\omega_{(j)}^\dagger (\omega_{(j)} \sigma_{(j)} - \sigma_{(n+1)} \omega_{(j)}) \\
 & - iW_{(j)}^\dagger (\omega_{(j)} \chi_{\sigma(n+1)} - \chi_{\sigma(n+1)} \omega_{(j)}) - i\omega_{(j)}^\dagger (W_{(j)} \chi_{\sigma(n+1)}^\dagger - \chi_{\sigma(n+1)}^\dagger W_{(j)}) - W_{(j)}^\dagger W_{(j)} \sigma_{(j)}^2 \\
 & + 2W_{(j)}^\dagger \sigma_{(n+1)} W_{(j)} \sigma_{(j)} - W_{(j)}^\dagger \sigma_{(n+1)}^2 W_{(j)} + G_{(j)}^\dagger G_{(j)} \Big] + \sum_{(j)=5}^n \left[-\mathcal{D}_m Z_{(j)} \mathcal{D}^m Z_{(j)}^\dagger \right. \\
 & + i\zeta_{(j)}^\dagger \not{D} \zeta_{(j)} - Z_{(j)}^\dagger Z_{(j)} D_{(j)+1} + Z_{(j)}^\dagger D_{(j)} Z_{(j)} - iZ_{(j)}^\dagger (\zeta_{(j)} \chi_{\sigma(j+1)} - \chi_{\sigma(j+1)} \zeta_{(j)}) \\
 & - i\zeta_{(j)}^\dagger (Z_{(j)} \chi_{\sigma(j+1)}^\dagger - \chi_{\sigma(j+1)}^\dagger Z_{(j)}) - i\zeta_{(j)}^\dagger (\sigma_{(j)} \zeta_{(j)} - \zeta_{(j)} \sigma_{(j+1)}) - Z_{(j)}^\dagger Z_{(j)} \sigma_{(j+1)}^2 \\
 & + 2Z_{(j)}^\dagger \sigma_{(j)} Z_{(j)} \sigma_{(j+1)} - Z_{(j)}^\dagger \sigma_{(j)}^2 Z_{(j)} + F_{(j)}^\dagger F_{(j)} \Big] - (\mathcal{D}_m W_{(j)}) (\mathcal{D}^m W_{(j)}^\dagger) + i\omega_{(j)} \not{D} \omega_{(j)}^\dagger \\
 & - W_{(j)}^\dagger W_{(j)} D_{(j)} + W_{(j)}^\dagger D_{(j+1)} W_{(j)} + i\omega_{(j)}^\dagger (\omega_{(j)} \sigma_{(j)} - \sigma_{(j+1)} \omega_{(j)}) \\
 & - iW_{(j)}^\dagger (\omega_{(j)} \chi_{\sigma(j+1)} - \chi_{\sigma(j+1)} \omega_{(j)}) - i\omega_{(j)}^\dagger (W_{(j)} \chi_{\sigma(j+1)}^\dagger - \chi_{\sigma(j+1)}^\dagger W_{(j)}) - W_{(j)}^\dagger W_{(j)} \sigma_{(j)}^2 \\
 & \left. + 2W_{(j)}^\dagger \sigma_{(j+1)} W_{(j)} \sigma_{(j)} - W_{(j)}^\dagger \sigma_{(j+1)}^2 W_{(j)} + G_{(j)}^\dagger G_{(j)} \right] \Bigg). \tag{B.11}
 \end{aligned}$$

B.3 $\widehat{E}_6$ quiver

The component action in this case is similar to the $\widehat{D}_n$ case. The auxiliary fields are eliminated using the following equations,

$$\begin{aligned}
 D_{(j)} &= \frac{g^2}{2} \left(2\kappa_{(j)}\sigma_{(j)} - \frac{1}{g^2} [\phi_{(j)}, \phi_{(j)}^\dagger] - Z_{(j)}Z_{(j)}^\dagger + W_{(j)}^\dagger W_{(j)} \right) & (j) = 1, 3, 5, \\
 D_{(7)} &= \frac{g^2}{2} \left(2\kappa_{(7)}\sigma_{(7)} - \frac{1}{g^2} [\phi_{(7)}, \phi_{(7)}^\dagger] + \sum_{(j)=2,4,6} (Z_{(j)}^\dagger Z_{(j)} - W_{(j)}W_{(j)}^\dagger) \right), \\
 D_{(j)} &= \frac{g^2}{2} \left(2\kappa_{(j)}\sigma_{(j)} - \frac{1}{g^2} [\phi_{(j)}, \phi_{(j)}^\dagger] + Z_{(j-1)}^\dagger Z_{(j-1)} - W_{(j-1)}^\dagger W_{(j-1)} - Z_{(j)}Z_{(j)}^\dagger + W_{(j)}^\dagger W_{(j)} \right) & (j) = 2, 4, 6, \\
 F_{\phi_{(j)}} &= -g^2 (W_{(j)}^\dagger Z_{(j)}^\dagger + \kappa_{(j)}\phi_{(j)}^\dagger) & (j) = 1, 3, 5, \\
 F_{\phi_{(7)}} &= -g^2 \left(\kappa_{(7)}\phi_{(7)}^\dagger - \sum_{(j)=2,4,6} Z_{(j)}^\dagger W_{(j)}^\dagger \right), \\
 F_{\phi_{(j)}} &= -g^2 (\kappa_{(j)}\phi_{(j)}^\dagger + W_{(j)}^\dagger Z_{(j)}^\dagger - Z_{(j-1)}^\dagger W_{(j-1)}^\dagger) & (j) = 2, 4, 6, \\
 F_{(j)} &= -(\phi_{(j)}^\dagger W_{(j)}^\dagger - W_{(j)}^\dagger \phi_{(7)}^\dagger) & (j) = 2, 4, 6, \\
 F_{(j)} &= -(\phi_{(j)}^\dagger W_{(j)}^\dagger - W_{(j)}^\dagger \phi_{(j+1)}^\dagger) & (j) = 1, 3, 5, \\
 G_{(j)} &= -(Z_{(j)}^\dagger \phi_{(j)}^\dagger - \phi_{(7)}^\dagger Z_{(j)}^\dagger) & (j) = 2, 4, 6, \\
 G_{(j)} &= -(Z_{(j)}^\dagger \phi_{(j)}^\dagger - \phi_{(j+1)}^\dagger Z_{(j)}^\dagger) & (j) = 1, 3, 5.
 \end{aligned} \tag{B.12}$$

C Checking supersymmetry variation of $\widehat{D}_n$

We present some steps of checking the supersymmetry variation of $\widehat{D}_n$ action. We vary all the fields in (2.21) and (2.22) simultaneously and substitute (2.23) and (2.24). The terms which combine and cancel are labelled by same alphabet. We do not write tr in front of every term, which implies the trace of gauge indices.

C.1 Variation of the kinetic part

We now explicitly show the term by term variation of the kinetic part of the action.

$$\begin{aligned}
 (1) \quad & -\frac{1}{g^2} \sum_{(j)=1}^{n+1} \text{tr}(\delta F_{\mu\nu})_{(j)} F_{(j)}^{\mu\nu} = \sum_{(j)=1}^{n+1} \underbrace{\frac{i}{g^2} \varepsilon_{ab} \gamma_\mu \lambda_{(j)}^{ab} \mathcal{D}_\nu F_{(j)}^{\mu\nu}}_B \\
 (2) \quad & \sum_{(j)=1}^{n+1} \kappa_{(j)} \epsilon^{\mu\nu\lambda} \delta \left(A_{\mu(j)} \partial_\nu A_{\lambda(j)} + \frac{2i}{3} A_{\mu(j)} A_{\nu(j)} A_{\lambda(j)} \right) = \sum_{(j)=1}^{n+1} \underbrace{-\frac{i\kappa_{(j)}}{2} \epsilon^{\mu\nu\lambda} \text{tr} \left(\varepsilon_{ab} \gamma_\mu \lambda^{ab} F_{\nu\lambda(j)} \right)}_C
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & \sum_{(j)=1}^n -\delta(\mathcal{D}_\mu X_{(j)}^\dagger, \mathcal{D}^\mu X_{(j)}) \\
 &= \sum_{(j)=1}^n \underbrace{(-i\xi_{b(j)}^\dagger \varepsilon_b^a) \cdot \mathcal{D}_\mu \mathcal{D}^\mu X_{(j)}}_D + \sum_{(j)=1}^n \underbrace{(\frac{1}{2}\varepsilon_{ab}\gamma_\mu \lambda_{(j)}^{ab}) \cdot (\mathcal{D}^\mu X_{(j)} \cdot X_{(j)}^\dagger - X_{(j)} \mathcal{D}^\mu X_{(j)}^\dagger)}_E \\
 &+ \sum_{(j)=1}^n \underbrace{(-i\varepsilon_b^a \xi_{(j)}^b) \cdot \mathcal{D}_\mu \mathcal{D}^\mu X_{(j)}^\dagger}_{D^*} + \sum_{(j)=1}^2 \underbrace{(\frac{1}{2}\varepsilon_{ab}\gamma_\mu \lambda_{(5)}^{ab}) \cdot (\mathcal{D}^\mu X_{(j)}^\dagger \cdot X_{(j)} - X_{(j)}^\dagger \mathcal{D}^\mu X_{(j)})}_{E_5} \\
 &+ \sum_{(j)=3}^4 \underbrace{(\frac{1}{2}\varepsilon_{ab}\gamma_\mu \lambda_{(n+1)}^{ab}) \cdot (\mathcal{D}^\mu X_{(j)}^\dagger \cdot X_{(j)} - X_{(j)}^\dagger \mathcal{D}^\mu X_{(j)})}_{E_{n+1}} + \sum_{(j)=5}^n \underbrace{(\frac{1}{2}\varepsilon_{ab}\gamma_\mu \lambda_{(j+1)}^{ab}) \cdot (\mathcal{D}^\mu X_{(j)}^\dagger \cdot X_{(j)} - X_{(j)}^\dagger \mathcal{D}^\mu X_{(j)})}_F \\
 &\underbrace{-X_{(j)}^\dagger \mathcal{D}^\mu X_{(j)}}_F
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & \sum_{(j)=1}^n i\delta(\xi_{(j)}^\dagger, \mathcal{P}\xi_{(j)}) \\
 &= \sum_{(j)=5}^n \left(\underbrace{i(-\phi_{a(j+1)}^b \varepsilon_b^c X_{c(j)}^\dagger) \mathcal{P}\xi_{(j)} + i(-X_{(j)}^c \varepsilon_c^b \phi_{b(j+1)}^a) \cdot \mathcal{P}\xi_{(j)}^\dagger}_{G} + \underbrace{\xi_{(j)}^\dagger \cdot \gamma^\mu \xi_{(j)} (-\frac{i}{2}\varepsilon_{ab}\gamma_\mu \lambda_{(j+1)}^{ab})}_I \right) \\
 &+ \sum_{(j)=1}^2 \left(\underbrace{i(-\phi_{a(5)}^b \varepsilon_b^c X_{c(j)}^\dagger) \mathcal{P}\xi_{(j)} + i(-X_{(j)}^c \varepsilon_c^b \phi_{b(5)}^a) \cdot \mathcal{P}\xi_{(j)}^\dagger}_{G_5} + \underbrace{\xi_{(j)}^\dagger \cdot \gamma^\mu \xi_{(j)} (-\frac{i}{2}\varepsilon_{ab}\gamma_\mu \lambda_{(5)}^{ab})}_I \right) \\
 &+ \sum_{(j)=3}^4 \left(\underbrace{i(-\phi_{a(n+1)}^b \varepsilon_b^c X_{c(j)}^\dagger) \mathcal{P}\xi_{(j)} + i(-X_{(j)}^c \varepsilon_c^b \phi_{b(n+1)}^a) \cdot \mathcal{P}\xi_{(j)}^\dagger}_{G_{n+1}} + \underbrace{\xi_{(j)}^\dagger \cdot \gamma^\mu \xi_{(j)} (-\frac{i}{2}\varepsilon_{ab}\gamma_\mu \lambda_{(n+1)}^{ab})}_I \right) \\
 &+ \sum_{(j)=1}^n \left(\underbrace{-i\varepsilon_a^b \mathcal{P}X_{b(j)}^\dagger \mathcal{P}\xi_{(j)}}_{D^*} - \underbrace{i\varepsilon_b^a \mathcal{P}X_{(j)}^b \mathcal{P}\xi_{(j)}^\dagger}_D \right. \\
 &\left. - \underbrace{\xi_{(j)}^\dagger \cdot \gamma^\mu (-\frac{i}{2}\varepsilon_{ab}\gamma_\mu \lambda_{(j)}^{ab}) \cdot \xi_{(j)}}_I + \underbrace{iX_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b \mathcal{P}\xi_{(j)} + i\phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c \cdot \mathcal{P}\xi_{(j)}^\dagger}_H \right)
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & \sum_{(j)=1}^{n+1} -\frac{1}{2g^2} \delta(\mathcal{D}_\mu \phi_{b(j)}^a) \mathcal{D}^\mu \phi_{a(j)}^b - \frac{1}{2g^2} \mathcal{D}_\mu \phi_{b(j)}^a \delta(\mathcal{D}^\mu \phi_{a(j)}^b) \\
 &= \underbrace{\frac{1}{g^2} \varepsilon_{cb} \mathcal{D}_\mu \lambda_{(j)}^{ca} \mathcal{D}^\mu \phi_{a(j)}^b}_J - \frac{1}{2g^2} \varepsilon_{cd} \mathcal{D}_\mu \lambda_{(j)}^{cd} \mathcal{D}^\mu \phi_{a(j)}^a \xrightarrow{0} - \underbrace{\frac{1}{2g^2} \varepsilon_{cd} \gamma_\mu [\lambda_{(j)}^{cd}, \phi_{b(j)}^a] \mathcal{D}^\mu \phi_{a(j)}^b}_K
 \end{aligned}$$

$$(6) \quad \sum_{(j)=1}^{n+1} -\frac{1}{2} \kappa_{(j)}^2 g^2 \delta(\phi_{b(j)}^a \phi_{a(j)}^b) = \underbrace{\kappa_{(j)}^2 g^2 \varepsilon_{cb} \lambda_{(j)}^{ca} \phi_{a(j)}^b}_L$$

$$\begin{aligned}
 (7) \quad & \sum_{(j)=1}^{n+1} -\frac{i}{2g^2} \delta(\lambda_{(j)}^{ab}) \mathcal{P} \lambda_{ab(j)} \\
 &= -\frac{i}{g^2} \sum_{(j)=1}^n \underbrace{\left(ig^2 X_{(j)}^a X_{c(j)}^\dagger \varepsilon^{cb} - \frac{ig^2}{2} (X X^\dagger)_{(j)} \varepsilon^{ab} \right)}_E \mathcal{P} \lambda_{ab(j)} \\
 &\quad - \underbrace{\frac{i}{g^2} \left(-\varepsilon^{bc} g^2 i \sum_{(j)=1}^2 X_{c(j)}^\dagger X_{(j)}^a + \frac{ig^2}{2} \sum_{(j)=1}^2 (X^\dagger X)_{(j)} \varepsilon^{ab} \right)}_{E_5} \mathcal{P} \lambda_{ab(5)} \\
 &\quad - \underbrace{\frac{i}{g^2} \sum_{(j)=6}^n \left(-g^2 i \varepsilon^{bc} X_{c(j-1)}^\dagger X_{(j-1)}^a + \frac{ig^2}{2} (X^\dagger X)_{(j-1)} \varepsilon^{ab} \right)}_E \mathcal{P} \lambda_{ab(j)} \\
 &\quad - \underbrace{\frac{i}{g^2} \left(-ig^2 \varepsilon^{bc} \sum_{(j)=3,4,n} X_{c(j)}^\dagger X_{(j)}^a + \frac{ig^2}{2} \sum_{(j)=3,4,n} (X^\dagger X)_{(j)} \varepsilon^{ab} \right)}_{E_{n+1}} \mathcal{P} \lambda_{ab(n+1)} \\
 &\quad + \sum_{(j)=1}^{n+1} \left(\underbrace{\frac{1}{2g^2} \lambda_{(j)}^{ab} \gamma^\mu \left[-\frac{i}{2} \varepsilon_{ab} \gamma_\mu \lambda_{(j)}^{ab} \cdot \lambda_{ab(j)} \right]}_M + \underbrace{\frac{i}{2g^2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \varepsilon^{ab} \gamma_\lambda \mathcal{P} \lambda_{ab(j)}}_B + \underbrace{\frac{1}{g^2} \varepsilon^{ac} \mathcal{P} \phi_{c(j)}^b \mathcal{P} \lambda_{ab(j)}}_J \right. \\
 &\quad \left. + \underbrace{\frac{1}{2g^2} [\phi_{c(j)}^b, \phi_{d(j)}^c] \varepsilon^{ad} \mathcal{P} \lambda_{ab(j)}}_K + \underbrace{\kappa_{(j)} \phi_{c(j)}^b \varepsilon^{ac} \mathcal{P} \lambda_{ab(j)}}_N \right) \\
 (8) \quad & - \sum_{(j)=1}^{n+1} \frac{\kappa_{(j)}}{2} i \delta \left(\lambda_{(j)}^{ab} \lambda_{ba(j)} \right) \\
 &= -i \sum_{(j)=1}^n \kappa_{(j)} \underbrace{\left(+ig^2 X_{(j)}^a X_{c(j)}^\dagger \varepsilon^{cb} - \frac{ig^2}{2} (X X^\dagger)_{(j)} \varepsilon^{ab} \right)}_O \lambda_{ba(j)} \\
 &\quad - i \kappa_{(5)} \underbrace{\left(-g^2 i \varepsilon^{bc} \sum_{(j)=1}^2 X_{c(j)}^\dagger X_{(j)}^a + \frac{ig^2}{2} \sum_{(j)=1}^2 (X^\dagger X)_{(j)} \varepsilon^{ab} \right)}_O \lambda_{ba(5)} \\
 &\quad - i \sum_{(j)=6}^n \kappa_{(j)} \underbrace{\left(-ig^2 \varepsilon^{bc} X_{c(j-1)}^\dagger X_{(j-1)}^a + \frac{ig^2}{2} (X^\dagger X)_{(j-1)} \varepsilon^{ab} \right)}_O \lambda_{ba(j)} \\
 &\quad - i \kappa_{(n+1)} \underbrace{\left(-ig^2 \varepsilon^{bc} \sum_{(j)=3,4,n} X_{c(j)}^\dagger X_{(j)}^a + \frac{ig^2}{2} \sum_{(j)=3,4,n} (X^\dagger X)_{(j)} \varepsilon^{ab} \right)}_O \lambda_{ba(n+1)} \\
 &\quad - i \sum_{(j)=1}^{n+1} \kappa_{(j)} \left(\underbrace{\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \gamma_\lambda \varepsilon^{ab}}_C - \underbrace{i \mathcal{P} \phi_{c(j)}^b \varepsilon^{ac}}_N + \underbrace{\frac{i}{2} [\phi_{c(j)}^b, \phi_{d(j)}^c] \varepsilon^{ad}}_P + \underbrace{\kappa_{(j)} g^2 i \phi_{c(j)}^b \varepsilon^{ac}}_L \right) \lambda_{ba(j)}
 \end{aligned}$$

C.2 Variation of the interaction part

We now explicitly show the term by term variation of the interaction part of the action.

$$\begin{aligned}
 (9) & -i \sum_{(j)=1}^n \delta(\xi_{a(j)}^\dagger \phi_{b(j)}^a \xi_{(j)}^b) \\
 & = -i \sum_{(j)=1}^2 \delta \xi_{a(j)}^\dagger \cdot \phi_{b(j)}^a \xi_{(j)}^b - i \sum_{(j)=3}^4 \delta \xi_{a(j)}^\dagger \cdot \phi_{b(j)}^a \xi_{(j)}^b - i \sum_{(j)=5}^n \delta \xi_{a(j)}^\dagger \cdot \phi_{b(j)}^a \xi_{(j)}^b \\
 & \quad - i \sum_{(j)=1}^n \xi_{a(j)}^\dagger \cdot \delta \phi_{b(j)}^a \cdot \xi_{(j)}^b - i \sum_{(j)=1}^2 \xi_{a(j)}^\dagger \phi_{b(j)}^a \delta \xi_{(j)}^b - i \sum_{(j)=3}^4 \xi_{a(j)}^\dagger \phi_{b(j)}^a \delta \xi_{(j)}^b \\
 & \quad - i \sum_{(j)=5}^n \xi_{a(j)}^\dagger \phi_{b(j)}^a \delta \xi_{(j)}^b.
 \end{aligned}$$

Substituting transformation equations in each terms we get the following terms,

$$\begin{aligned}
 (9.1) & -i \sum_{(j)=1}^2 \left(\underbrace{(-\varepsilon_a^b \not{D} X_{b(j)}^\dagger)}_H - \underbrace{\phi_{a(5)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b}_Q \right) \cdot \phi_{d(j)}^a \xi_{(j)}^d \\
 & \quad + \xi_{a(j)}^\dagger \phi_{b(j)}^a \left(\underbrace{\not{D} X_{(j)}^c \varepsilon_c^b}_H + \underbrace{\phi_{d(j)}^b \varepsilon_c^d X_{(j)}^c - X_{(j)}^c \varepsilon_c^d \phi_{d(5)}^b}_Q \right)
 \end{aligned}$$

$$\begin{aligned}
 (9.2) & -i \sum_{(j)=3}^4 \left(\underbrace{(-\varepsilon_a^b \not{D} X_{b(j)}^\dagger)}_H - \underbrace{\phi_{a(n+1)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b}_Q \right) \cdot \phi_{m(j)}^a \xi_{(j)}^m \\
 & \quad + \xi_{m(j)}^\dagger \phi_{a(j)}^m \left(\underbrace{\not{D} X_{(j)}^b \varepsilon_b^a}_H + \underbrace{\phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(n+1)}^a}_Q \right)
 \end{aligned}$$

$$\begin{aligned}
 (9.3) & -i \sum_{(j)=5}^n \left(\underbrace{(-\varepsilon_a^b \not{D} X_{b(j)}^\dagger)}_H - \underbrace{\phi_{a(j+1)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b}_Q \right) \cdot \phi_{m(j)}^a \xi_{(j)}^m \\
 & \quad + \xi_{m(j)}^\dagger \phi_{a(j)}^m \left(\underbrace{\not{D} X_{(j)}^b \varepsilon_b^a}_H + \underbrace{\phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(j+1)}^a}_Q \right)
 \end{aligned}$$

$$(9.4) \underbrace{i \sum_{(j)=1}^n \xi_{a(j)}^\dagger \cdot (-\varepsilon_{cb} \lambda_{(j)}^{ca} + \frac{1}{2} \delta_b^a \varepsilon_{cd} \lambda_{(j)}^{cd}) \cdot \xi_{(j)}^b}_I$$

$$\begin{aligned}
 (10) & \sum_{(j)=1}^n \epsilon_{ac} \delta(\lambda_{(j)}^{cb} X_{(j)}^a \xi_{b(j)}^\dagger) \\
 & = \epsilon_{ac} \sum_{(j)=1}^n \left(\underbrace{-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \varepsilon^{cb} \gamma_\lambda}_{D} + \underbrace{i \varepsilon^{cd} \not{D} \phi_{d(j)}^b}_H + \underbrace{\frac{i}{2} [\phi_{m(j)}^b, \phi_{d(j)}^m] \varepsilon^{cd}}_Q + \underbrace{\kappa_{(j)} g^2 i \phi_{d(j)}^b \varepsilon^{cd}}_R \right) \\
 & \quad + \underbrace{i g^2 X_{(j)}^c X_{d(j)}^\dagger \varepsilon^{db} - \frac{i g^2}{2} (X X^\dagger)_{(j)} \varepsilon^{cb}}_S \cdot X_{(j)}^a \xi_{b(j)}^\dagger \\
 & \quad + \underbrace{\epsilon_{ac} \left(-g^2 i \varepsilon^{bd} \sum_{(j)=1}^2 X_{d(j)}^\dagger X_{(j)}^c + \frac{i g^2}{2} \sum_{(j)=1}^2 (X^\dagger X)_{(j)} \varepsilon^{cb} \right)}_S \cdot X_{(5)}^a \xi_{b(5)}^\dagger
 \end{aligned}$$

$$\begin{aligned}
 & + \epsilon_{ac} \sum_{(j)=6}^n \underbrace{\left(-g^2 i \varepsilon^{bd} X_{d(j-1)}^\dagger X_{(j-1)}^c + \frac{ig^2}{2} (X^\dagger X)_{(j-1)} \varepsilon^{cb} \right)}_S \cdot X_{(j)}^a \xi_{b(j)}^\dagger \\
 & + \epsilon_{am} \sum_{(j)=1}^2 \underbrace{\lambda_{(j)}^{mb} X_{(j)}^a (-\phi_{b(5)}^d \varepsilon_d^c X_{c(j)}^\dagger)}_T + \epsilon_{ac} \sum_{(j)=3}^4 \underbrace{\lambda_{(j)}^{cb} X_{(j)}^a (-\phi_{b(n+1)}^m \varepsilon_m^d X_{d(j)}^\dagger)}_T \\
 & + \epsilon_{mn} \sum_{(j)=5}^n \underbrace{\lambda_{(j)}^{na} X_{(j)}^m (-\phi_{a(j+1)}^b \varepsilon_b^c X_{c(j)}^\dagger)}_T + \epsilon_{ac} \sum_{(j)=1}^n \underbrace{\lambda_{(j)}^{cb} \cdot (-i \varepsilon_d^a \xi_{(j)}^d)}_I \cdot \xi_{b(j)}^\dagger \\
 & + \epsilon_{am} \sum_{(j)=1}^n \lambda_{(j)}^{mb} X_{(j)}^a \left(\underbrace{\not{D} X_{c(j)}^\dagger \varepsilon_b^c}_E + \underbrace{X_{c(j)}^\dagger \varepsilon_d^c \phi_{b(j)}^d}_T \right) \\
 (11) \quad & \sum_{(j)=1}^n \delta \left[-\epsilon^{ac} \lambda_{cb(j)} \xi_{(j)}^b X_{a(j)}^\dagger \right] \\
 & = - \sum_{(j)=1}^n \epsilon^{ac} \epsilon_{cm} \epsilon_{bn} \underbrace{\left(-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \varepsilon^{mn} \gamma_\lambda \right)}_{D^*} + \underbrace{i \varepsilon^{md} \not{D} \phi_{d(j)}^n}_H + \underbrace{\frac{i}{2} [\phi_{r(j)}^n, \phi_{d(j)}^r] \varepsilon^{md}}_Q + \underbrace{\kappa_{(j)} g^2 i \phi_{d(j)}^n \varepsilon^{md}}_R \\
 & + \underbrace{ig^2 X_{(j)}^m X_{d(j)}^\dagger \varepsilon^{dn} - \frac{ig^2}{2} (X X^\dagger)_{(j)} \varepsilon^{mn}}_S \cdot \xi_{(j)}^b X_{a(j)}^\dagger \\
 & - \epsilon^{ac} \epsilon_{cm} \epsilon_{bn} \underbrace{\left(-g^2 i \varepsilon^{nd} \sum_{(j)=1}^2 X_{d(j)}^\dagger X_{(j)}^m + \frac{ig^2}{2} \sum_{(j)=1}^2 (X^\dagger X)_{(j)} \varepsilon^{ab} \right)}_S \cdot \xi_{(5)}^b X_{a(5)}^\dagger \\
 & - \epsilon^{ac} \epsilon_{cm} \epsilon_{bn} \sum_{(j)=6}^n \underbrace{\left(+g^2 i (-\varepsilon^{nr} X_{r(j-1)}^\dagger X_{(j-1)}^m) + \frac{ig^2}{2} ((X^\dagger X)_{(j-1)}) \varepsilon^{mn} \right)}_S \cdot \xi_{(j)}^b X_{a(j)}^\dagger \\
 & - \sum_{(j)=1}^2 \underbrace{\epsilon^{ac} \lambda_{cb(j)} \cdot (-X_{(j)}^m \varepsilon_m^d \phi_{d(5)}^b) \cdot X_{a(j)}^\dagger}_{T} - \sum_{(j)=3}^4 \underbrace{\epsilon^{mc} \lambda_{ca(j)} \cdot (-X_{(j)}^c \varepsilon_c^b \phi_{b(n+1)}^a) \cdot X_{m(j)}^\dagger}_T \\
 & - \sum_{(j)=5}^n \underbrace{\epsilon^{mn} \lambda_{na(j)} \cdot (-X_{(j)}^c \varepsilon_c^b \phi_{b(j+1)}^a) \cdot X_{m(j)}^\dagger}_T - \sum_{(j)=1}^n \underbrace{\epsilon^{ac} \lambda_{cb(j)} \xi_{(j)}^b \cdot (-i \xi_{d(j)}^\dagger \varepsilon_a^d)}_I \\
 & - \sum_{(j)=1}^n \epsilon^{ac} \lambda_{cb(j)} \cdot \left(\underbrace{\not{D} X_{(j)}^d \varepsilon_d^b}_E + \underbrace{\phi_{d(j)}^b \varepsilon_m^d X_{(j)}^m}_T \right) \cdot X_{a(j)}^\dagger \\
 (12) \quad & \sum_{(j)=1}^n -\kappa_{(j)} g^2 \delta(X_{a(j)}^\dagger \phi_{b(j)}^a X_{(j)}^b) \\
 & = - \underbrace{\sum_{(j)=1}^n \kappa_{(j)} g^2 (-i \xi_{c(j)}^\dagger \varepsilon_a^c) \cdot \phi_{b(j)}^a X_{(j)}^b}_R - \underbrace{\sum_{(j)=1}^n \kappa_{(j)} g^2 X_{a(j)}^\dagger (-\varepsilon_{pb} \lambda_{(j)}^{pa} + \frac{1}{2} \delta_b^a \varepsilon_{pq} \lambda_{(j)}^{pq}) \cdot X_{(j)}^b}_O \\
 & - \underbrace{\sum_{(j)=1}^n \kappa_{(j)} g^2 X_{a(j)}^\dagger \phi_{b(j)}^a \cdot (-i \varepsilon_c^b \xi_{(j)}^c)}_R
 \end{aligned}$$

$$\begin{aligned}
 (13) \quad & \sum_{(j)=1}^n \delta \left[-\frac{1}{2} X_{(j)} X_{(j)}^\dagger \phi_{b(j)}^a \phi_{a(j)}^b \right] \\
 &= \underbrace{\frac{i}{2} (\varepsilon_c^d \xi_{(j)}^\dagger X_{c(j)}^\dagger) \phi_{b(j)}^a \phi_{a(j)}^b + \frac{i}{2} (X_{(j)} \xi_{d(j)}^\dagger \varepsilon_c^d) \phi_{b(j)}^a \phi_{a(j)}^b}_Q \\
 &\quad - \underbrace{\frac{1}{2} (X_{(j)} X_{(j)}^\dagger) (\varepsilon_{cb} \lambda_{(j)}^{ca} \phi_{a(j)}^b)}_T + \underbrace{\frac{1}{2} \delta_b^a \varepsilon_{cd} \lambda_{(j)}^{cd} \phi_{a(j)}^b}_0 \\
 &\quad - \underbrace{\frac{1}{2} (X_{(j)} X_{(j)}^\dagger) (-\varepsilon_{ca} \phi_{b(j)}^a \lambda_{(j)}^{cb})}_T + \underbrace{\frac{1}{2} \delta_a^b \varepsilon_{cd} \phi_{b(j)}^a \lambda_{(j)}^{cd}}_0 \\
 (14) \quad & \sum_{(j)=1}^n \delta \left[-\frac{g^2}{4} (X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j)} \sigma_i X_{(j)}^\dagger) \right] \\
 &= \frac{ig^2}{2} \sum_{(j)=1}^n \left(\underbrace{(X_{(j)} \sigma_i X_{(j)}^\dagger) \varepsilon_b^a \xi_{(j)}^b \sigma_i X_{(j)}^\dagger + (X_{(j)} \sigma_i X_{(j)}^\dagger) X_{(j)} \sigma_i \xi_{b(j)}^\dagger \varepsilon_a^b}_S \right) \\
 (15) \quad & \frac{g^2}{2} \sum_{(j)=1}^n \delta \left[-(X_{(j)}^\dagger \sigma_i X_{(j)}) (X_{(j)}^\dagger \sigma_i X_{(j)}) \right] \\
 &= \frac{ig^2}{2} \sum_{(j)=1}^n \left(\underbrace{(X_{(j)}^\dagger \sigma_i X_{(j)}) \xi_{b(j)}^\dagger \varepsilon_a^b \sigma_i X_{(j)} + (X_{(j)}^\dagger \sigma_i X_{(j)}) X_{(j)}^\dagger \sigma_i \varepsilon_b^a \xi_{(j)}^b}_S \right) \\
 (16) \quad & \sum_{(j)=1}^2 -\epsilon_{ac} \delta (\lambda_{(5)}^{cb} \xi_{b(j)}^\dagger X_{(j)}^a) \\
 &= -\epsilon_{ac} \left(\underbrace{-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(5)} \varepsilon^{cb} \gamma_\lambda}_D + \underbrace{i \varepsilon^{cd} \not{D} \phi_{d(5)}^b}_{G_5} + \underbrace{\frac{i}{2} [\phi_{p(5)}^b, \phi_{d(5)}^p] \varepsilon^{cd}}_Q + \underbrace{\kappa_{(5)} g^2 i \phi_{d(5)}^b \varepsilon^{cd}}_R \right. \\
 &\quad \left. + g^2 i \underbrace{(X_{(5)}^c X_{d(5)}^\dagger \varepsilon^{db} - \varepsilon^{bd} \sum_{(j)=1}^2 X_{d(j)}^\dagger X_{(j)}^c)}_S + \frac{ig^2}{2} \sum_{(j)=1}^2 ((X^\dagger X)_{(j)} - (X X^\dagger)_{(5)}) \varepsilon^{cb} \right) \cdot \sum_{(j)=1}^2 \xi_{b(j)}^\dagger X_{(j)}^a \\
 &\quad - \epsilon_{ac} \lambda_{(5)}^{cb} \sum_{(j)=1}^2 \left(\underbrace{(\not{D} X_{d(j)}^\dagger \varepsilon_b^d)}_{E_5} - \underbrace{\phi_{b(5)}^d \varepsilon_d^p X_{p(j)}^\dagger + X_{d(j)}^\dagger \varepsilon_p^d \phi_{b(j)}^p}_{T} \cdot X_{(j)}^a - \underbrace{\xi_{b(j)}^\dagger (-i \varepsilon_d^a \xi_{(j)}^d)}_I \right) \\
 (17) \quad & \sum_{(j)=1}^2 \delta \left[+\epsilon^{ac} \lambda_{cb(5)} X_{a(j)}^\dagger \xi_{(j)}^b \right] \\
 &= \epsilon^{ac} \epsilon_{cm} \epsilon_{bn} \left(\underbrace{-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(5)} \varepsilon^{mn} \gamma_\lambda}_{D^*} + \underbrace{i \varepsilon^{md} \not{D} \phi_{d(5)}^n}_{G_5} + \underbrace{\frac{i}{2} [\phi_{r(5)}^n, \phi_{d(5)}^r] \varepsilon^{md}}_Q + \underbrace{\kappa_{(5)} g^2 i \phi_{d(5)}^n \varepsilon^{md}}_R \right. \\
 &\quad \left. + g^2 i \underbrace{(X_{(5)}^m X_{d(5)}^\dagger \varepsilon^{dn} - \varepsilon^{nd} \sum_{(j)=1}^2 X_{d(j)}^\dagger X_{(j)}^m)}_S + \frac{ig^2}{2} \left(\sum_{(j)=1}^2 (X^\dagger X)_{(j)} - (X X^\dagger)_{(5)} \right) \varepsilon^{mn} \right) \cdot \sum_{(j)=1}^2 X_{a(j)}^\dagger \xi_{(j)}^b \\
 &\quad + \epsilon^{ac} \lambda_{cb(5)} \sum_{(j)=1}^2 \left(\underbrace{(-i \xi_{d(j)}^\dagger \varepsilon_a^d) \cdot \xi_{(j)}^b}_I + X_{a(j)}^\dagger \cdot \underbrace{(\not{D} X_{(j)}^c \varepsilon_c^b + \phi_{d(j)}^b \varepsilon_c^d X_{(j)}^c - X_{(j)}^c \varepsilon_c^d \phi_{d(5)}^b)}_{E_5 + T} \right)
 \end{aligned}$$

$$\begin{aligned}
 (18) \quad & \sum_{(j)=1}^2 \delta \left[+i \xi^a_{(j)} \phi^b_{a(5)} \xi^\dagger_{b(j)} \right] \\
 &= i \sum_{(j)=1}^2 \left(\underbrace{\left(-\varepsilon^a_c \not{D} X^c_{(j)} \right)}_{G_5} + \underbrace{\left(\phi^a_{d(j)} \varepsilon^d_c X^c_{(j)} - X^c_{(j)} \varepsilon^d_c \phi^a_{d(5)} \right)}_Q \phi^b_{a(5)} \xi^\dagger_{b(j)} \right. \\
 &\quad \left. + \underbrace{\xi^a_{(j)} \cdot \left(-\varepsilon_{cb} \lambda^{ca}_{(5)} + \frac{1}{2} \delta^a_b \varepsilon_{cd} \lambda^{cd}_{(5)} \right) \cdot \xi^\dagger_{b(j)}}_I + \underbrace{\xi^d_{(j)} \phi^a_{d(5)} \cdot \left(\not{D} X^\dagger_{b(j)} \varepsilon^b_a - \phi^b_{a(5)} \varepsilon^c_b X^\dagger_{c(j)} + X^\dagger_{c(j)} \varepsilon^c_b \phi^b_{a(j)} \right)}_{G_5 + T} \right) \\
 (19) \quad & \sum_{(j)=1}^2 \delta \left[-\frac{1}{2} (X^\dagger_{(j)} X_{(j)}) \phi^a_{b(5)} \phi^b_{a(5)} \right] \\
 &= -\frac{1}{2} \sum_{(j)=1}^2 \left(\underbrace{\left((-i \xi^\dagger_{d(j)} \varepsilon^d_c) X^c_{(j)} \cdot \phi^a_{b(5)} \phi^b_{a(5)} - i X^\dagger_{(j)} \cdot \varepsilon^a_d \xi^d_{(j)} \cdot \phi^a_{b(5)} \phi^b_{a(5)} \right)}_Q \right. \\
 &\quad \left. - \underbrace{X^\dagger_{(j)} X_{(j)} \cdot \varepsilon_{cb} \lambda^{ca}_{(5)} \phi^b_{a(5)} + X^\dagger_{(j)} X_{(j)} \cdot \phi^a_{b(5)} \cdot \varepsilon_{cb} \lambda^{ca}_{(5)}}_T \right) \\
 (20) \quad & \sum_{(j)=1}^2 \delta \left[X^\dagger_{(j)a} \phi^b_{c(j)} X^a_{(j)} \phi^c_{b5} \right] \\
 &= \sum_{(j)=1}^2 \left(\underbrace{\left((-i \xi^\dagger_{d(j)} \varepsilon^d_a) \phi^b_{c(j)} X^a_{(j)} \phi^c_{b5} + X^\dagger_{(j)a} \cdot \left(-\varepsilon_{dc} \lambda^{db}_{(j)} \right) X^a_{(j)} \phi^c_{b(5)} \right)}_Q \right. \\
 &\quad \left. - \underbrace{i X^\dagger_{(j)a} \phi^b_{c(j)} \varepsilon^a_d \xi^d_{(j)} \cdot \phi^c_{b5}}_Q \right. \\
 &\quad \left. + \underbrace{X^\dagger_{(j)a} \phi^b_{c(j)} X^a_{(j)} (-\varepsilon_{db} \lambda^{dc}_{(5)})}_T \right) \\
 (21) \quad & \sum_{(j)=1}^2 \delta \left[\kappa_{(j)} g^2 X^a_{(j)} (\phi^b_a)_5 X^\dagger_{b(j)} \right] \\
 &= \kappa_{(j)} g^2 \sum_{(j)=1}^2 \left(\underbrace{\left(-i \varepsilon^a_b \xi^b_{(j)} \phi^b_{a(5)} X^\dagger_{b(j)} - X^a_{(j)} \varepsilon_{ca} \lambda^{cb}_{(5)} X^\dagger_{b(j)} + \frac{1}{2} X^b_{(j)} \varepsilon_{cd} \lambda^{cd}_{(5)} X^\dagger_{b(j)} + X^a_{(j)} \phi^b_{a(5)} \cdot (-i \xi^\dagger_{d(j)} \varepsilon^d_b) \right)}_R \right) \\
 (22) \quad & \sum_{(j)=1}^2 \delta \left[\frac{g^2}{2} (X_5 \sigma_i X^\dagger_5) (X^\dagger_{(j)} \sigma_i X_{(j)}) \right] \\
 &= \frac{g^2}{2} \sum_{(j)=1}^2 \left(\underbrace{\left((-i \varepsilon^a_b \xi^b_{(5)}) \sigma_i X^\dagger_5 + X_5 \sigma_i \cdot (-i \xi^\dagger_{b(5)} \varepsilon^b_a) \right) \cdot (X^\dagger_{(j)} \sigma_i X_{(j)})}_S \right. \\
 &\quad \left. + \underbrace{(X_5 \sigma_i X^\dagger_5) \cdot \left((-i \xi^\dagger_{b(j)} \varepsilon^b_a) \sigma_i X_{(j)} + X^\dagger_{(j)} \sigma_i (-i \varepsilon^a_b \xi^b_{(j)}) \right)}_S \right) \\
 (23) \quad & \sum_{(j)=3}^4 -\epsilon_{ac} \delta \left(\lambda^{cb}_{(n+1)} \xi^\dagger_{b(j)} X^a_{(j)} \right) \\
 &= \sum_{(j)=3}^4 \left(-\epsilon_{ac} \underbrace{\left(-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(n+1)} \varepsilon^{cb} \gamma_\lambda + i \varepsilon^{cd} \not{D} \phi^b_{d(n+1)} \right)}_D + \underbrace{i \varepsilon^{cd} \not{D} \phi^b_{d(n+1)}}_{G_{n+1}} + \underbrace{\frac{i}{2} [\phi^b_{p(n+1)}, \phi^p_{d(n+1)}] \varepsilon^{cd}}_Q + \underbrace{\kappa_{(n+1)} g^2 i \phi^b_{d(n+1)} \varepsilon^{cd}}_R \right)
 \end{aligned}$$

$$\begin{aligned}
 & \underbrace{-g^2 i \varepsilon^{bd} \sum_{(j)=3,4,n} X_{d(j)}^\dagger X_{(j)}^c + \frac{ig^2}{2} \sum_{(j)=3,4,n} (X^\dagger X)_{(j)} \varepsilon^{cb}}_S \cdot \xi_{b(j)}^\dagger X_{(j)}^a \\
 & - \epsilon_{ac} \lambda_{(n+1)}^{cb} \left(\underbrace{(\not{D} X_{d(j)}^\dagger \varepsilon_b^d - \phi_{b(n+1)}^d \varepsilon_d^p X_{p(j)}^\dagger)}_{E_{n+1}} + \underbrace{X_{d(j)}^\dagger \varepsilon_p^d \phi_{b(j)}^p}_{G_{n+1}} \right) X_{(j)}^a - \underbrace{\epsilon_{ac} \lambda_{(n+1)}^{cb} \xi_{b(j)}^\dagger (-i \varepsilon_d^a \xi_{(j)}^d)}_I \\
 (24) \quad & \sum_{(j)=3}^4 \delta \left[\epsilon^{ac} \lambda_{cb(n+1)} X_{a(j)}^\dagger \xi_{(j)}^b \right] \\
 & = \sum_{(j)=3}^4 \left(\epsilon_{bc} \left(\underbrace{-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(n+1)} \varepsilon^{ac} \gamma_\lambda + i \varepsilon^{ad} \not{D} \phi_{d(n+1)}^c}_{D^*} + \underbrace{\frac{i}{2} [\phi_{p(n+1)}^c, \phi_{d(n+1)}^p] \varepsilon^{ad}}_{G_{n+1}} + \underbrace{\kappa_{(n+1)} g^2 i \phi_{d(n+1)}^c \varepsilon^{ad}}_Q \right) \right. \\
 & \quad \left. - g^2 i \varepsilon^{cd} \sum_{(j)=3,4,n} X_{d(j)}^\dagger X_{(j)}^a + \frac{ig^2}{2} \sum_{(j)=3,4,n} (X^\dagger X)_{(j)} \varepsilon^{cd} \right) \cdot X_{a(j)}^\dagger \xi_{(j)}^b + \underbrace{\epsilon^{ac} \lambda_{cb(n+1)} \cdot (-i \xi_{d(j)}^\dagger \varepsilon_a^d \cdot \xi_{(j)}^b)}_I \\
 & \quad + \epsilon^{ac} \lambda_{cb(n+1)} \cdot X_{a(j)}^\dagger \cdot \left(\underbrace{(\not{D} X_{(j)}^m \varepsilon_m^b)}_{E_{n+1}} + \underbrace{\phi_{m(j)}^b \varepsilon_d^m X_{(j)}^d - X_{(j)}^d \varepsilon_d^m \phi_{m(n+1)}^b}_{T} \right) \\
 (25) \quad & + \sum_{(j)=3}^4 \delta \left[i \xi_{(j)}^a \phi_{a(n+1)}^b \xi_{b(j)}^\dagger \right] \\
 & = i \sum_{(j)=3}^4 \left(\left(\underbrace{(-\varepsilon_b^a \not{D} X_{(j)}^b)}_{G_{n+1}} + \underbrace{\phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(n+1)}^a}_{Q} \right) \phi_{a(n+1)}^m \xi_{m(j)}^\dagger + \underbrace{\xi_{(j)}^a (-\varepsilon_{ca} \lambda_{(n+1)}^{cb})}_I \right. \\
 & \quad \left. + \frac{1}{2} \delta_a^b \varepsilon_{cd} \lambda_{(n+1)}^{cd} \right) \xi_{b(j)}^\dagger + \xi_{(j)}^m \phi_{m(n+1)}^a \left(\underbrace{(\not{D} X_{b(j)}^\dagger \varepsilon_a^b)}_{G_{n+1}} - \underbrace{\phi_{a(n+1)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b}_{Q} \right) \\
 (26) \quad & \sum_{(j)=3}^4 \delta \left[-\frac{1}{2} (X_{(j)}^\dagger X_{(j)}) \phi_{b(n+1)}^a \phi_{a(n+1)}^b \right] \\
 & = -\frac{1}{2} \sum_{(j)=3}^4 \left(\underbrace{(-i \xi_{d(j)}^\dagger \varepsilon_a^d) \cdot X_{(j)} \phi_{b(n+1)}^a \phi_{a(n+1)}^b + X_{(j)}^\dagger \cdot (-i \varepsilon_d^a \xi_{(j)}^d) \phi_{b(n+1)}^a \phi_{a(n+1)}^b}_Q \right. \\
 & \quad \left. + \underbrace{X_{(j)}^\dagger \cdot X_{(j)} (-\varepsilon_{cb} \lambda_{(n+1)}^{ca}) \phi_{a(n+1)}^b + X_{(j)}^\dagger \cdot X_{(j)} \phi_{b(n+1)}^a (-\varepsilon_{ca} \lambda_{(n+1)}^{cb})}_T \right) \\
 (27) \quad & \sum_{(j)=3}^4 \delta \left[+X_{(j)_a}^\dagger \phi_{c(j)}^b X_{(j)}^a \phi_{b(n+1)}^c \right] \\
 & = \sum_{(j)=3}^4 \left(\underbrace{(-i \xi_{d(j)}^\dagger \varepsilon_a^d) \phi_{c(j)}^b X_{(j)}^a \phi_{b(n+1)}^c}_{Q} + \underbrace{X_{(j)_a}^\dagger (-\varepsilon_{dc} \lambda_{(j)}^{db}) X_{(j)}^a \phi_{b(n+1)}^c}_T \right. \\
 & \quad \left. + \underbrace{X_{(j)_a}^\dagger \phi_{c(j)}^b (-i \varepsilon_d^a \xi_{(j)}^d) \cdot \phi_{b(n+1)}^c}_{Q} + \underbrace{X_{(j)_a}^\dagger \phi_{c(j)}^b X_{(j)}^a (-\varepsilon_{db} \lambda_{(n+1)}^{dc})}_T \right) \\
 (28) \quad & \sum_{(j)=3}^4 \delta \left[\kappa_{(j)} g^2 X_{(j)}^a (\phi_{a(n+1)}^b) X_{b(j)}^\dagger \right] \\
 & = \kappa_{(j)} g^2 \sum_{(j)=3}^4 \left(\underbrace{(-i \varepsilon_d^a \xi_{(j)}^d) \phi_{a(n+1)}^b X_{b(j)}^\dagger}_R + \underbrace{X_{(j)}^a \cdot (-\varepsilon_{ca} \lambda_{(n+1)}^{cb} + \frac{1}{2} \delta_a^b \varepsilon_{cd} \lambda_{(n+1)}^{cd}) X_{b(j)}^\dagger}_O \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \underbrace{X_{(j)}^a (\phi_a^b)_{(n+1)} (-i \xi_{d(j)}^\dagger \varepsilon_b^d)}_R \Big) \\
 (29) \quad & \sum_{(j)=3}^4 \delta \left[-\frac{g^2}{2} (X_{(j)}^\dagger \sigma_i X_{(j)}) (X_{(n)}^\dagger \sigma_i X_{(n)}) \right] \\
 & = -\frac{g^2}{2} \sum_{(j)=3}^4 \left(\underbrace{\left((-i \xi_{b(j)}^\dagger \varepsilon_a^b) \sigma_i X_{(j)} + X_{(j)}^\dagger \sigma_i (-i \varepsilon_b^a \xi_{(j)}^b) \right) (X_{(n)}^\dagger \sigma_i X_{(n)})}_S \right. \\
 & \quad \left. + \underbrace{(X_{(j)}^\dagger \sigma_i X_{(j)}) \left((-i \xi_{b(n)}^\dagger \varepsilon_a^b) \sigma_i X_{(n)} + X_{(n)}^\dagger \sigma_i (-i \varepsilon_b^a \xi_{(n)}^b) \right)}_S \right) \\
 (30) \quad & + \sum_{(j)=5}^n -\epsilon_{ac} \delta (\lambda_{(j+1)}^{cb} \xi_{b(j)}^\dagger X_{(j)}^a) \\
 & = -\epsilon_{ac} \sum_{(j)=6}^n \left(\underbrace{\left(-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \varepsilon^{cb} \gamma_\lambda + i \varepsilon^{cd} \not{D} \phi_{d(j)}^b \right)}_D + \underbrace{\frac{i}{2} [\phi_{m(j)}^b, \phi_{d(j)}^m] \varepsilon^{cd}}_Q + \underbrace{\kappa_{(j)} g^2 i \phi_{m(j)}^b \varepsilon^{cm}}_R \right. \\
 & \quad \left. + \underbrace{g^2 i (X_{(j)}^c X_{m(j)}^\dagger \varepsilon^{mb} - \varepsilon^{bd} X_{d(j-1)}^\dagger X_{(j-1)}^c) + \frac{ig^2}{2} ((X^\dagger X)_{(j-1)} - (X X^\dagger)_{(j)}) \varepsilon^{cb}}_S \right) \xi_{b(j-1)}^\dagger X_{(j-1)}^a \\
 & - \epsilon_{ac} \left(\underbrace{\left(-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(n+1)} \varepsilon^{cb} \gamma_\lambda + i \varepsilon^{cd} \not{D} \phi_{d(n+1)}^b \right)}_D + \underbrace{\frac{i}{2} [\phi_{m(n+1)}^b, \phi_{d(n+1)}^m] \varepsilon^{cd}}_Q + \underbrace{\kappa_{(n+1)} g^2 i \phi_{m(j)}^b \varepsilon^{cm}}_R \right. \\
 & \quad \left. + \underbrace{g^2 i \sum_{(j)=3,4,n} (-\varepsilon^{bd} X_{d(j)}^\dagger X_{(j)}^c)}_S + \underbrace{\sum_{(j)=3,4,n} \frac{ig^2}{2} ((X^\dagger X)_{(j-1)}) \varepsilon^{cb}}_S \right) \xi_{bn}^\dagger X_{(n)}^a \\
 & - \epsilon_{ac} \sum_{(j)=6}^{n+1} \left(\underbrace{\lambda_{(j)}^{cb} (\not{D} X_{d(j-1)}^\dagger \varepsilon_b^d)}_F - \underbrace{\phi_{b(j)}^d \varepsilon_d^m X_{m(j-1)}^\dagger}_T \right. \\
 & \quad \left. + \underbrace{X_{d(j-1)}^\dagger \varepsilon_m^d \phi_{b(j-1)}^m X_{(j-1)}^a}_T + \underbrace{\lambda_{(j)}^{cb} \xi_{b(j-1)}^\dagger (-i \varepsilon_a^d \xi_{(j-1)}^d)}_I \right) \\
 (31) \quad & \sum_{(j)=5}^n \delta \left[\epsilon^{ac} \lambda_{cb(j+1)} X_{a(j)}^\dagger \xi_{(j)}^b \right] \\
 & = \epsilon_{bn} \sum_{(j)=6}^{n+1} \left(\underbrace{\left(-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \varepsilon^{an} \gamma_\lambda + i \varepsilon^{ad} \not{D} \phi_{d(j)}^n \right)}_{D^*} + \underbrace{\frac{i}{2} [\phi_{r(j)}^n, \phi_{d(j)}^r] \varepsilon^{ad}}_Q + \underbrace{\kappa_{(j)} g^2 i \phi_{d(j)}^n \varepsilon^{ad}}_R \right) X_{a(j-1)}^\dagger \xi_{(j-1)}^b \\
 & + \epsilon_{bn} \sum_{(j)=6}^n \left[\underbrace{g^2 i (X_{(j)}^a X_{d(j)}^\dagger \varepsilon^{dn} - \varepsilon^{nd} X_{d(j-1)}^\dagger X_{(j-1)}^a) + \frac{ig^2}{2} ((X^\dagger X)_{(j-1)} - (X X^\dagger)_{(j)}) \varepsilon^{an}}_S \right] X_{a(j-1)}^\dagger \xi_{(j-1)}^b \\
 & + \epsilon_{bn} \underbrace{\left(-ig^2 \varepsilon^{nc} \sum_{(j)=3,4,n} X_{c(j)}^\dagger X_{(j)}^a + \frac{ig^2}{2} \sum_{(j)=3,4,n} (X^\dagger X)_{(j)} \varepsilon^{an} \right)}_S X_{a(n)}^\dagger \xi_{(n)}^b \\
 & + \underbrace{\epsilon^{ac} \sum_{(j)=5}^n \left(\lambda_{cb(j+1)} \cdot (-i \xi_{d(j-1)}^\dagger \varepsilon_a^d) \xi_{(j)}^b \right)}_I
 \end{aligned}$$

$$\begin{aligned}
 & + \epsilon^{mn} \sum_{(j)=5}^n \left(\lambda_{na(j+1)} X_{m(j)}^\dagger \cdot \underbrace{(\not{D} X_{(j)}^b \varepsilon_b^a)}_F + \underbrace{\phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(j+1)}^a}_T \right) \\
 (32) \quad & \sum_{(j)=5}^n \delta \left[i \xi_{(j)}^a \phi_{a(j+1)}^b \xi_{b(j)}^\dagger \right] \\
 & = i \sum_{(j)=5}^n \left(\underbrace{(-\varepsilon_b^a \not{D} X_{(j)}^b)}_G + \underbrace{\phi_{b(j)}^a \varepsilon_c^b X_{(j)}^c - X_{(j)}^c \varepsilon_c^b \phi_{b(j+1)}^a}_Q \cdot \phi_{a(j+1)}^b \xi_{b(j)}^\dagger \right. \\
 & \quad + \xi_{(j)}^a \underbrace{(-\varepsilon_{ca} \lambda_{(j+1)}^{cb} + \frac{1}{2} \delta_b^a \varepsilon_{cd} \lambda_{(j+1)}^{cd}) \cdot \xi_{b(j)}^\dagger}_I \\
 & \quad \left. + \xi_{(j)}^m \phi_{m(j+1)}^a \cdot \underbrace{(\not{D} X_{b(j)}^\dagger \varepsilon_a^b - \phi_{a(j+1)}^b \varepsilon_b^c X_{c(j)}^\dagger + X_{c(j)}^\dagger \varepsilon_b^c \phi_{a(j)}^b)}_Q \right) \\
 (33) \quad & \sum_{(j)=5}^n \delta \left[+ g^2 \kappa_{(j+1)} X_{(j)}^a \phi_{a(j+1)}^b X_{b(j)}^\dagger \right] \\
 & = g^2 \kappa_{(j+1)} \sum_{(j)=5}^n \left(\underbrace{(-i \varepsilon_c^a \xi_{(j)}^c) \cdot \phi_{a(j+1)}^b X_{b(j)}^\dagger}_R + \underbrace{X_{(j)}^a (-\varepsilon_{ca} \lambda_{(j+1)}^{cb} + \frac{1}{2} \delta_a^b \varepsilon_{cd} \lambda_{(j+1)}^{cd}) \cdot X_{b(j)}^\dagger}_O \right. \\
 & \quad \left. + \underbrace{X_{(j)}^a \phi_{a(j+1)}^b \cdot (-i \xi_{c(j)}^\dagger \varepsilon_b^c)}_R \right) \\
 (34) \quad & \sum_{(j)=5}^n \delta \left[-\frac{1}{2} (X_{(j)}^\dagger X_{(j)}) (\phi_{b(j+1)}^a) (\phi_{a(j+1)}^b) \right] \\
 & = -\frac{1}{2} \sum_{(j)=5}^n \left(\underbrace{(-i \xi_{b(j)}^\dagger \varepsilon_a^b) X_{(j)} \phi_{b(j+1)}^a \phi_{b(j+1)}^b + X_{(j)}^\dagger (-i \varepsilon_d^a \xi_{(j)}^d) \phi_{b(j+1)}^a \phi_{b(j+1)}^b}_Q \right. \\
 & \quad \left. + \underbrace{(X_{(j)}^\dagger X_{(j)}) (-\varepsilon_{cb} \lambda_{(j+1)}^{ca}) \phi_{a(j+1)}^b + (X_{(j)}^\dagger X_{(j)}) \phi_{b(j+1)}^a (-\varepsilon_{ca} \lambda_{(j+1)}^{cb})}_T \right) \\
 (35) \quad & \sum_{(j)=5}^n \delta \left[X_{(j)a}^\dagger \phi_{c(j)}^b X_{(j)}^a \phi_{b(j+1)}^c \right] \\
 & = \sum_{(j)=5}^n \left(\underbrace{(-i \xi_{b(j)}^\dagger \varepsilon_a^b) \phi_{c(j)}^b X_{(j)}^a \phi_{b(j+1)}^c}_Q + \underbrace{X_{(j)a}^\dagger (-\varepsilon_{dc} \lambda_{(j)}^{db}) X_{(j)}^a \phi_{b(j+1)}^c}_T \right. \\
 & \quad \left. + \underbrace{X_{(j)a}^\dagger \phi_{c(j)}^b (-i \varepsilon_d^a \xi_{(j)}^d) \phi_{b(j+1)}^c}_Q + \underbrace{X_{(j)a}^\dagger \phi_{c(j)}^b X_{(j)}^a (-\varepsilon_{db} \lambda_{(j+1)}^{dc})}_T \right) \\
 (36) \quad & \sum_{(j)=1}^{n+1} \delta \left[\frac{1}{8g^2} [\phi_{b(j)}^a, \phi_{d(j)}^c] [\phi_{a(j)}^b, \phi_{c(j)}^d] \right] \\
 & = -\frac{1}{4g^2} \varepsilon_{pb} [\lambda_{(j)}^{pa}, \phi_{d(j)}^c] [\phi_{a(j)}^b, \phi_{c(j)}^d] - \frac{1}{4g^2} \varepsilon_{pd} [\phi_{b(j)}^a, \lambda_{(j)}^{pc}] [\phi_{a(j)}^b, \phi_{c(j)}^d] \\
 & \quad \underbrace{\hspace{10em}}_U \\
 (37) \quad & \sum_{(j)=1}^{n+1} \delta \left[-\frac{\kappa_{(j)}}{6} \phi_{b(j)}^a [\phi_{c(j)}^b, \phi_{a(j)}^c] \right] = \frac{\kappa_{(j)}}{2} \underbrace{\left(-\varepsilon_{cb} \lambda_{(j)}^{ca} + \frac{1}{2} \delta_b^a \varepsilon_{cd} \lambda_{(j)}^{cd} \right) [\phi_{m(j)}^b, \phi_{a(j)}^m]}_P
 \end{aligned}$$

$$\begin{aligned}
 (38) \quad & \sum_{(j)=1}^{n+1} \delta \left[-\frac{i}{2g^2} \lambda_{ab(j)} [\phi_{c(j)}^b, \lambda_{(j)}^{ac}] \right] \\
 &= -\frac{i}{2g^2} \sum_{(j)=1}^{n+1} \left(\lambda_{ab(j)} [\delta \phi_{c(j)}^b, \lambda_{(j)}^{ac}] \right) - \frac{i}{2g^2} \sum_{(j)=1}^4 \left(\delta \lambda_{ab(j)} [\phi_{c(j)}^b, \lambda_{(j)}^{ac}] + \lambda_{ab(j)} [\phi_{c(j)}^b, \delta \lambda_{(j)}^{ac}] \right) \\
 &\quad - \frac{i}{2g^2} \left(\delta \lambda_{ab(5)} [\phi_{c(5)}^b, \lambda_{(5)}^{ac}] + \lambda_{ab(5)} [\phi_{c(j)}^b, \delta \lambda_{(5)}^{ac}] \right) \\
 &\quad - \frac{i}{2g^2} \sum_{(j)=6}^n \left(\delta \lambda_{ab(j)} [\phi_{c(j)}^b, \lambda_{(j)}^{ac}] + \lambda_{ab(j)} [\phi_{c(j)}^b, \delta \lambda_{(j)}^{ac}] \right) \\
 &\quad - \frac{i}{2g^2} \left(\delta \lambda_{ab(n+1)} [\phi_{c(n+1)}^b, \lambda_{(n+1)}^{ac}] + \lambda_{ab(n+1)} [\phi_{c(n+1)}^b, \delta \lambda_{(n+1)}^{ac}] \right)
 \end{aligned}$$

Terms from the above,

$$(38.1) \quad -\frac{i}{2g^2} \sum_{(j)=1}^{n+1} \left(\lambda_{ab(j)} [\delta \phi_{c(j)}^b, \lambda_{(j)}^{ac}] \right) = -\frac{i}{2g^2} \underbrace{\sum_{(j)=1}^{n+1} \lambda_{ab(j)} \left[-\varepsilon_{dc} \lambda_{(j)}^{db} + \frac{1}{2} \delta_c^b \varepsilon_{mn} \lambda_{(j)}^{mn}, \lambda_{(j)}^{ac} \right]}_M$$

$$\begin{aligned}
 (38.2) \quad & -\frac{i}{2g^2} \sum_{(j)=1}^4 \left(\delta \lambda_{ab(j)} [\phi_{c(j)}^b, \lambda_{(j)}^{ac}] + \lambda_{ab(j+1)} [\phi_{c(j)}^b, \delta \lambda_{(j)}^{ac}] \right) \\
 &= -\frac{i}{2g^2} \varepsilon_{am} \varepsilon_{bn} \sum_{(j)=1}^4 \left(\underbrace{-\frac{1}{2} \varepsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \varepsilon^{mn} \gamma_\lambda}_J + \underbrace{i \varepsilon^{mr} \mathcal{D} \phi_{r(j)}^n}_K + \underbrace{\frac{i}{2} [\phi_{r(j)}^n, \phi_{d(j)}^r] \varepsilon^{md}}_U + \underbrace{\kappa_{(j)} g^2 i \phi_{d(j)}^n \varepsilon^{md}}_P \right. \\
 &\quad \left. + i g^2 \underbrace{X_{(j)}^m X_{d(j)}^\dagger \varepsilon^{dn} - \frac{ig^2}{2} (X X^\dagger)_{(j)} \varepsilon^{mn}}_T \right) [\phi_{c(j)}^b, \lambda_{(j)}^{ac}] \\
 &\quad - \frac{i}{2g^2} \sum_{(j)=1}^4 \lambda_{ab(j)} \left[\phi_{c(j)}^b, \underbrace{\frac{1}{2} \varepsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \gamma_\lambda \varepsilon^{ac}}_J - \underbrace{i \mathcal{D} \phi_{d(j)}^c \varepsilon^{ad}}_K + \underbrace{\frac{i}{2} [\phi_{r(j)}^c, \phi_{d(j)}^r] \varepsilon^{ad}}_U + \underbrace{\kappa_{(j)} g^2 i \phi_{d(j)}^c \varepsilon^{ad}}_P \right. \\
 &\quad \left. + i g^2 \underbrace{X_{(j)}^a X_{d(j)}^\dagger \varepsilon^{dc} - \frac{ig^2}{2} (X X^\dagger)_{(j)} \varepsilon^{ac}}_T \right]
 \end{aligned}$$

$$\begin{aligned}
 (38.3) \quad & -\frac{i}{2g^2} \varepsilon_{am} \varepsilon_{bn} \left(\underbrace{-\frac{1}{2} \varepsilon^{\mu\nu\lambda} F_{\mu\nu(5)} \varepsilon^{mn} \gamma_\lambda}_J + \underbrace{i \varepsilon^{md} \mathcal{D} \phi_{d(5)}^n}_K + \underbrace{\frac{i}{2} [\phi_{r(5)}^n, \phi_{d(5)}^r] \varepsilon^{md}}_U + \underbrace{\kappa_{(5)} g^2 i \phi_{d(5)}^n \varepsilon^{md}}_P \right. \\
 &\quad \left. + g^2 i (X_{(5)}^m X_{d(5)}^\dagger \varepsilon^{dn} - \varepsilon^{nd} \sum_{(j)=1}^2 X_{d(j)}^\dagger X_{(j)}^m) + \frac{ig^2}{2} \left(\sum_{(j)=1}^2 (X^\dagger X)_{(j)} - (X X^\dagger)_{(5)} \right) \varepsilon^{mn} \right) [\phi_{c(5)}^b, \lambda_{(5)}^{ac}] \\
 &\quad - \frac{i}{2g^2} \lambda_{ab(5)} \left[\phi_{c(5)}^b, \underbrace{\frac{1}{2} \varepsilon^{\mu\nu\lambda} F_{\mu\nu(5)} \gamma_\lambda \varepsilon^{ac}}_J - \underbrace{i \mathcal{D} \phi_{d(5)}^c \varepsilon^{ad}}_K + \underbrace{\frac{i}{2} [\phi_{r(5)}^c, \phi_{d(5)}^r] \varepsilon^{ad}}_U + \underbrace{\kappa_{(5)} g^2 i \phi_{d(5)}^c \varepsilon^{ad}}_P \right. \\
 &\quad \left. + g^2 i (X_{(5)}^a X_{d(5)}^\dagger \varepsilon^{dc} - \varepsilon^{cd} \sum_{(j)=1}^2 X_{d(j)}^\dagger X_{(j)}^a) + \frac{ig^2}{2} \left(\sum_{(j)=1}^2 (X^\dagger X)_{(j)} - (X X^\dagger)_{(5)} \right) \varepsilon^{ac} \right]
 \end{aligned}$$

$$\begin{aligned}
 (38.4) \quad & -\frac{i}{2g^2} \sum_{(j)=6}^n \left(\delta\lambda_{ab(j)} [\phi_{c(j)}^b, \lambda_{(j)}^{ac}] + \lambda_{ab(j)} [\phi_{c(j)}^b, \delta\lambda_{(j)}^{ac}] \right) \\
 & = -\frac{i}{2g^2} \epsilon_{am} \epsilon_{bn} \sum_{(j)=6}^n \left(\underbrace{\left(-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \epsilon^{mn} \gamma_\lambda + i \epsilon^{mr} \not{D} \phi_{r(j)}^n \right)}_J + \underbrace{\frac{i}{2} [\phi_{r(j)}^n, \phi_{d(j)}^r] \epsilon^{md}}_U + \underbrace{\kappa_{(j)} g^2 i \phi_{d(j)}^n \epsilon^{md}}_P \right. \\
 & \quad \left. + g^2 i \underbrace{\left(X_{(j)}^m X_{d(j)}^\dagger \epsilon^{dn} - \epsilon^{nd} X_{d(j-1)}^\dagger X_{(j-1)}^m \right)}_T + \frac{ig^2}{2} \left((X^\dagger X)_{(j-1)} - (X X^\dagger)_{(j)} \right) \epsilon^{mn} \right) [\phi_{c(j)}^b, \lambda_{(j)}^{ac}] \\
 & - \frac{i}{2g^2} \sum_{(j)=6}^n \lambda_{ab(j)} \left[\phi_{c(j)}^b, \underbrace{\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(j)} \gamma_\lambda \epsilon^{ac}}_J - \underbrace{i \not{D} \phi_{d(j)}^c \epsilon^{ad}}_K + \underbrace{\frac{i}{2} [\phi_{r(j)}^c, \phi_{d(j)}^r] \epsilon^{ad}}_U + \underbrace{\kappa_{(j)} g^2 i \phi_{d(j)}^c \epsilon^{ad}}_P \right. \\
 & \quad \left. + g^2 i \underbrace{\left(X_{(j)}^a X_{d(j)}^\dagger \epsilon^{dc} - \epsilon^{cd} X_{d(j-1)}^\dagger X_{(j-1)}^a \right)}_T + \frac{ig^2}{2} \left((X^\dagger X)_{(j-1)} - (X X^\dagger)_{(j)} \right) \epsilon^{ac} \right] \\
 (38.5) \quad & -\frac{i}{2g^2} \epsilon_{am} \epsilon_{bn} \left(\underbrace{\left(-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(n+1)} \epsilon^{mn} \gamma_\lambda + i \epsilon^{mc} \not{D} \phi_{c(n+1)}^n \right)}_J + \underbrace{\frac{i}{2} [\phi_{c(n+1)}^n, \phi_{d(n+1)}^c] \epsilon^{md}}_U + \underbrace{\kappa_{(n+1)} g^2 i \phi_{c(n+1)}^n \epsilon^{mc}}_P \right. \\
 & \quad \left. - i g^2 \epsilon^{nc} \underbrace{\sum_{(j)=3,4,n} X_{c(j)}^\dagger X_{(j)}^m}_T + \frac{ig^2}{2} \sum_{(j)=3,4,n} (X^\dagger X)_{(j)} \epsilon^{mn} \right) [\phi_{r(n+1)}^b, \lambda_{(n+1)}^{ar}] \\
 & - \frac{i}{2g^2} \lambda_{ab(n+1)} \left[\phi_{c(n+1)}^b, \underbrace{\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu(n+1)} \gamma_\lambda \epsilon^{ac}}_J - \underbrace{i \not{D} \phi_{d(n+1)}^c \epsilon^{ad}}_K + \underbrace{\frac{i}{2} [\phi_{m(n+1)}^c, \phi_{d(n+1)}^m] \epsilon^{ad}}_U \right. \\
 & \quad \left. + \underbrace{\kappa_{(n+1)} g^2 i \phi_{d(n+1)}^c \epsilon^{ad}}_P - i g^2 \epsilon^{cd} \underbrace{\sum_{(j)=3,4,n} X_{d(j)}^\dagger X_{(j)}^a}_T + \frac{ig^2}{2} \sum_{(j)=3,4,n} (X^\dagger X)_{(j)} \epsilon^{ac} \right] \\
 (39) \quad & \frac{g^2}{2} \sum_{(j)=6}^n \delta \left((X_{(j)} \sigma_i X_{(j)}^\dagger) (X_{(j-1)}^\dagger \sigma_i X_{(j-1)}) \right) \\
 & = -\frac{ig^2}{2} \underbrace{\left(\epsilon_b^a \xi_{(j)}^b \sigma_i X_{(j)}^\dagger (X_{(j-1)}^\dagger \sigma_i X_{(j-1)}) \right)}_S - \frac{ig^2}{2} \underbrace{\left(X_{(j)} \sigma_i \xi_{b(j)}^\dagger \epsilon_a^b (X_{(j-1)}^\dagger \sigma_i X_{(j-1)}) \right)}_S \\
 & \quad - \frac{ig^2}{2} \underbrace{\left((X_{(j)} \sigma_i X_{(j)}^\dagger) \xi_{b(j-1)}^\dagger \epsilon_a^b \sigma_i X_{(j-1)} \right)}_S - \frac{ig^2}{2} \underbrace{\left((X_{(j)} \sigma_i X_{(j)}^\dagger) X_{(j-1)}^\dagger \sigma_i \epsilon_b^a \xi_{(j-1)}^b \right)}_S \\
 (40) \quad & \frac{g^2}{2} \delta \left(- (X_1^\dagger \sigma_i X_1) (X_2^\dagger \sigma_i X_2) \right) \\
 & = -\frac{g^2}{2} \underbrace{\left(((-i \xi_{b_1}^\dagger \epsilon_a^b) \sigma_i X_1 + X_1^\dagger \sigma_i (-i \epsilon_b^a \xi_1^b)) (X_2^\dagger \sigma_i X_2) + (X_1^\dagger \sigma_i X_1) ((-i \xi_{b_2}^\dagger \epsilon_a^b) \sigma_i X_2 + X_2^\dagger \sigma_i (-i \epsilon_b^a \xi_2^b)) \right)}_S \\
 (41) \quad & \frac{g^2}{2} \delta \left(- (X_{(3)}^\dagger \sigma_i X_{(3)}) (X_{(4)}^\dagger \sigma_i X_{(4)}) \right) \\
 & = -\frac{g^2}{2} \left((\delta X_{(3)}^\dagger \sigma_i X_{(3)} + X_{(3)}^\dagger \sigma_i \delta X_{(3)}) (X_{(4)}^\dagger \sigma_i X_{(4)}) + (X_{(3)}^\dagger \sigma_i X_{(3)}) (\delta X_{(4)}^\dagger \sigma_i X_{(4)} + X_{(4)}^\dagger \sigma_i \delta X_{(4)}) \right) \\
 & = -\frac{g^2}{2} \underbrace{\left((-i \xi_{b(3)}^\dagger \epsilon_a^b \sigma_i X_{(3)} - i \epsilon_b^a X_{(3)}^\dagger \sigma_i \xi_{(3)}^b) (X_{(4)}^\dagger \sigma_i X_{(4)}) + (X_{(3)}^\dagger \sigma_i X_{(3)}) ((-i \xi_{b(4)}^\dagger \epsilon_a^b) \sigma_i X_{(4)} + X_{(4)}^\dagger \sigma_i (-i \epsilon_b^a \xi_{(4)}^b)) \right)}_S
 \end{aligned}$$

D Monopole solution of $\hat{D}_4$

In this section we will consider a simple example with gauge group $U(N)^4 \times U(2N)$. For $\hat{D}_4$, 5-th and $n+1$ -th nodes coincide. The constraint on CS levels is,

$$\kappa_{(1)} + \kappa_{(2)} + \kappa_{(3)} + \kappa_{(4)} + 2\kappa_{(5)} = 0. \quad (\text{D.1})$$

Equation (3.43) and (3.44) reduces to,

$$\begin{aligned} \phi_{i(j)} &= -\frac{1}{2\kappa_{(j)}} \left(X_{(j)} \sigma_i X_{(j)}^\dagger \right) \quad (j) = 1, \dots, 4, \\ \phi_{i(5)} &= \frac{1}{2\kappa_{(5)}} \left(\sum_{(j)=1}^4 X_{(j)}^\dagger \sigma_i X_{(j)} \right). \end{aligned} \quad (\text{D.2})$$

Writing (D.2) in terms of the component fields, we get the following set of equations,

For $(j) = 1, \dots, 4$

$$Z_{(j)} W_{(j)} + W_{(j)}^\dagger Z_{(j)}^\dagger = 0, \quad Z_{(j)} W_{(j)} - W_{(j)}^\dagger Z_{(j)}^\dagger = 0, \quad Z_{(j)} Z_p^\dagger - W_{(j)}^\dagger W_{(j)} = \eta H^{(1)} \kappa_{(j)} \quad (\text{D.3})$$

For $(j) = 5$:

$$\begin{aligned} \sum_{(j)=1}^4 (Z_{(j)}^\dagger W_{(j)}^\dagger + W_{(j)} Z_{(j)}) &= 0, \quad \sum_{(j)=1}^4 (Z_{(j)}^\dagger W_{(j)}^\dagger - W_{(j)} Z_{(j)}) = 0, \\ \sum_{(j)=1}^4 \left(Z_{(j)}^\dagger Z_{(j)} - W_{(j)} W_{(j)}^\dagger \right) &= -\eta H^{(2)} \kappa_{(5)}. \end{aligned} \quad (\text{D.4})$$

There can be the following simple possibilities which solve the above equations.

- (i) $Z_{(j)} = 0$, for $(j) = 1, 2, \dots, 4$
- (ii) $W_{(j)} = 0$, for $(j) = 1, 2, \dots, 4$
- (iii) $W_{(1)} = Z_{(2)} = W_{(3)} = Z_{(4)} = 0$ or $Z_{(1)} = W_{(2)} = Z_{(3)} = W_{(4)} = 0$

We need to solve the above three equations for the three cases written above. For positive semi-definite $H^{(1)}, H^{(2)}$, we can solve the several constraints on the hyper multiplet scalars for (anti) BPS solutions in the following ways.

Case (i) $Z_{(j)} = 0, \forall(j)$

BPS $\eta = +1$.

$$-W_{(j)}^\dagger W_{(j)} = H^{(1)} \kappa_{(j)}, \quad \sum_{(j)=1}^4 W_{(j)} W_{(j)}^\dagger = H^{(2)} \kappa_{(5)}$$

which can be solved by choosing,

$$W_{(j)} = A_{(j)} e^{-\tau/2}, \quad W_{(j)}^\dagger = A_{(j)}^\dagger e^{\tau/2},$$

where $A_{(j)}$ is an $2N \times N$ matrix such that,

$$A_{(j)}^\dagger A_{(j)} = -H^{(1)} \kappa_{(j)}, \quad \sum_{(j)=1}^4 A_{(j)} A_{(j)}^\dagger = H^{(2)} \kappa_{(5)}. \quad (\text{D.5})$$

Therefore for a positive semi definite solution $\kappa_{(j)} \leq 0$ for $(j) = 1, \dots, 4$ and $\kappa_{(5)} \geq 0$ which is compatible with (D.1).

anti-BPS $\eta = -1$. Similarly the anti-BPS case is solved by,

$$W_{(j)} = A_{(j)} e^{\tau/2}, \quad W_{(j)}^\dagger = A_{(j)}^\dagger e^{-\tau/2} \quad (\text{D.6})$$

such that, $A_{(j)}^\dagger A_{(j)} = H^{(1)} \kappa_{(j)}$, $\sum_{(j)=1}^4 A_{(j)} A_{(j)}^\dagger = -H^{(2)} \kappa_{(5)}$. Therefore for a positive semi definite solution $\kappa_{(j)} \geq 0$ for $(j) = 1, \dots, 4$ and $\kappa_{(5)} \leq 0$. This is again compatible with (D.1). Solving case (ii) is straight forward now.

Case (iii) $W_{(1)} = Z_{(2)} = W_{(3)} = Z_{(4)} = W_{(5)} = 0$

BPS $\eta = +1$.

$$\begin{aligned} Z_{(1)} Z_{(1)}^\dagger &= H^{(1)} \kappa_{(1)}, & W_{(2)}^\dagger W_{(2)} &= -H^{(1)} \kappa_{(2)}, \\ Z_{(3)} Z_{(3)}^\dagger &= H^{(1)} \kappa_{(3)}, & W_{(4)}^\dagger W_{(4)} &= -H^{(1)} \kappa_{(4)}. \end{aligned} \quad (\text{D.7})$$

For $(j) = 5$

$$\left(Z_{(1)}^\dagger Z_{(1)} - W_{(2)} W_{(2)}^\dagger + Z_{(3)}^\dagger Z_{(3)} - W_{(4)} W_{(4)}^\dagger \right) = -H^{(2)} \kappa_{(5)} \quad (\text{D.8})$$

Solving the above we find

$$\begin{aligned} Z_{(1)} &= A_{(1)} e^{-\tau/2}, & Z_{(1)}^\dagger &= A_{(1)}^\dagger e^{\tau/2}, \\ W_{(2)} &= B_{(2)} e^{-\tau/2}, & W_{(2)}^\dagger &= B_{(2)}^\dagger e^{\tau/2}, \\ Z_{(3)} &= C_{(3)} e^{-\tau/2}, & Z_{(3)}^\dagger &= C_{(3)}^\dagger e^{\tau/2}, \\ W_{(4)} &= D_{(4)} e^{-\tau/2}, & W_{(4)}^\dagger &= D_{(4)}^\dagger e^{\tau/2}, \end{aligned} \quad (\text{D.9})$$

such that, $A_{(1)} A_{(1)}^\dagger = H^{(1)} \kappa_{(1)}$, $B_{(2)}^\dagger B_{(2)} = -H^{(1)} \kappa_{(2)}$, $C_{(3)} C_{(3)}^\dagger = H^{(1)} \kappa_{(3)}$, $D_{(4)}^\dagger D_{(4)} = -H^{(1)} \kappa_{(4)}$, $\left(A_{(1)}^\dagger A_{(1)} - B_{(2)}^\dagger B_{(2)} + C_{(3)}^\dagger C_{(3)} - D_{(4)}^\dagger D_{(4)} \right) = -H^{(2)} \kappa_{(5)}$. For a positive semi-definite solution, $\kappa_{(1)} \geq 0, \kappa_{(2)} \leq 0, \kappa_{(3)} \geq 0, \kappa_{(4)} \leq 0, \kappa_{(5)} \leq 0$. The anti-BPS case can be similarly solved by reversing the signs of the CS levels.

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