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3d field theory, plane partitions and triple Macdonald polynomials

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ABSTRACT: We argue that MacMahon representation of Ding-Iohara-Miki (DIM) algebra spanned by plane partitions is closely related to the Hilbert space of a 3d field theory. Using affine matrix model we propose a generalization of Bethe equations associated to DIM algebra with solutions also labelled by plane partitions. In a certain limit we identify the eigenstates of the Bethe system as new *triple* Macdonald polynomials depending on an infinite number of families of time variables. We interpret these results as first hints of the existence of an integrable 3d field theory, in which DIM algebra plays the same role as affine algebras in 2d WZNW models.

KEYWORDS: Conformal and W Symmetry, Integrable Field Theories, Matrix Models, Bethe Ansatz

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1 Introduction

Nekrasov functions [1–3] obtained by applying the localization technique to supersymmetric gauge theories in four, five and six dimensions effectively sum up instanton contributions to their partition functions. Instanton configurations fixed under toric action of $U(1)_{t_1, t_2}^2$ are labelled by tuples of Young diagrams, which technically arise as ideals in the ring of polynomials in two variables, or as residues of the LMNS integral [4–7]. Young diagrams naturally remind one of the basis in the Hilbert space of a 2d field theory. It turns out that this connection can indeed be made precise leading to the famous AGT correspondence [8–10]. Each term in the instanton sum over partitions corresponds to a particular state in the Hilbert space (Verma module) of a 2d CFT. The states featuring in the correspondence are eigenstates of a certain family of CFT integrals of motion [11–13] and are naturally labelled by tuples of Young diagrams. The Hilbert space of the 2d theory is a representation of W_N -algebra, and is identified with the equivariant cohomology of the gauge theory instanton moduli space.

In this paper we develop the basics of a similar, though not yet as precise, identification for *plane* partitions, i.e. 3d Young diagrams. To this end in section 2 we study 3d free scalar field theory and identify (chiral half of) its Hilbert space as the space of plane partitions. We then consider in section 3, instead of the LMNS integral, the affine matrix model, which poles are labelled by plane partitions. We derive the quantum spectral curve for this model and, taking its Nekrasov-Shatashvili (NS) limit [14], obtain certain Baxter TQ

equation. A particular case of this Baxter equation already appeared in the literature¹ [16–22] as an equation determining the eigenstates of the integrals of motion associated to the Ding-Iohara-Miki algebra $U_{t_1, t_2}(\widehat{\mathfrak{gl}}_1)$. In this restricted case the solutions were labelled by K -tuples of Young diagrams, so that the eigenstates spanned a tensor product of K Fock representations (K $2d$ fields). However, the slight generalization of the Baxter equation obtained from the affine matrix model has solutions labelled by K -tuples of $3d$ Young diagrams. The states corresponding to these solutions, therefore, span the tensor product of what is called the *MacMahon representations* of $U_{t_1, t_2}(\widehat{\mathfrak{gl}}_1)$. It is natural to associate this representation with the states of a $3d$ field.

Finally, in section 4 we give a concrete construction of a basis in the MacMahon representation of DIM as a subrepresentation in an infinite tensor product of Fock representations. The basis, which we call the family of *triple* Macdonald polynomials, is formed by the eigenstates of the Cartan subalgebra of DIM algebra. This basis can also be obtained in a certain limit of DIM Baxter equation mentioned above. It is tempting to speculate that this construction implies the existence of an *integrable* $3d$ field theory, though we are currently very far from giving its concrete description. Ideologically, this connection might be justified by noticing that loop algebras (affine, Virasoro or W_N) naturally appear in $2d$ CFT, while DIM algebra is essentially a *double* loop algebra, and thus should be associated to a $3d$ theory.

We use the standard notation for Young diagrams $Y = [Y_1, Y_2, \dots, Y_m]$ with $Y_1 \geq Y_2 \geq \dots \geq Y_m$ throughout the paper. Plane partition $\pi = [\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(n)}]$ is built from the “non-increasing” sequence of Young diagrams $\pi^{(1)} \supset \pi^{(2)} \supset \dots \supset \pi^{(n)}$.

2 $3d$ massless free scalar and $3d$ partitions

In this section we demonstrate how the Hilbert space of the simplest possible $3d$ field theory, the massless free scalar, is related to the space of plane partitions. We start by recalling the prototypical example of $2d$ free scalar, whose Hilbert space factorizes into two chiral parts, each of which is a Fock space $\mathcal{F}$ with basis labelled by ordinary partitions (Young diagrams). We then analyze the $3d$ setup using similar arguments. In particular, we introduce an analogue of the holomorphic factorization and observe that the Hilbert space of the free scalar in $3d$ (after certain reduction) factorizes into a product of two *MacMahon* spaces spanned by *plane* partitions.

Our considerations are very elementary and the free scalar is only a toy model for more interesting $3d$ field theories (see e.g. the recent work [23]). Still, we believe that the general structure of the Hilbert space will remain the same as in the free case, especially if one considers *integrable* $3d$ models.

2.1 Warm-up: $2d$ free scalar

One way to quantize a classical model is to quantize the space of solutions to its classical equations of motion. One can also interpret it as the quantization of the space of initial

¹These Bethe equations follow from the general algebro-geometric framework proposed in [15].

data, since each solution is uniquely determined by the initial conditions.² In the mechanical systems the quantization amounts to replacing the coordinates on the space of solutions (e.g. initial coordinates and momenta) with non-commuting operators. In the field theory this procedure is the same in principle, though the space under consideration is infinite-dimensional, which may require more subtle treatment. However, for the free fields, no technical problems arise and the quantization can be carried out straightforwardly.

We will work in the Euclidean signature and the *radial* coordinate will play the role of time. The classical equations of motion for a $2d$ massless free scalar read

$$(\partial_x^2 + \partial_y^2)\phi(x, y) = 0 \quad (2.1)$$

Changing to the radial coordinates (r, φ) we get

$$[(r\partial_r)^2 + \partial_\varphi^2]\phi(r, \varphi) = 0 \quad (2.2)$$

One can find a basis of solutions by separating the variables, i.e. $\phi(r, \varphi) = f(r)g(\varphi)$. From the fact that $g(\varphi)$ needs to be single-valued we get

$$g(\varphi) = e^{im\varphi}, \quad m \in \mathbb{Z}. \quad (2.3)$$

The equation for $f(r)$ is then

$$(r\partial_r)^2 f(r) = m^2 f(r), \quad (2.4)$$

and thus

$$f(r) = r^{\pm m}. \quad (2.5)$$

At this point we need to specify on which part of the $2d$ plane the theory lives. We assume it to be a large disk D_2 around $r = 0$. Therefore, we can only consider *positive* powers of r , and thus $f(r) = r^{|m|}$. We ignore the *zero mode* solutions $f(r) = 1$ and $f(r) = \ln r$ corresponding to $m = 0$. One can assume that the logarithmic mode is killed by the boundary conditions on ∂D_2 , while the constant mode could be discarded is one additionally requires $f(r)$ to vanish at $r = 0$.³ We will observe a similar situation when we come to the $3d$ case: part of the solutions will need to be discarded. One can also notice that the space of solutions which we have thrown away is exactly the space of solutions to the free scalar equations of motion *in one dimension less*. Indeed the functions 1 and $\ln r$ correspond to initial position and momentum of a *particle on a line*. In string theory this coordinate and momentum correspond to the motion of the string center of mass, while other solutions represent the harmonics.

Regular solutions to the free field equations of motion are thus labelled by $m \in \mathbb{Z} \setminus \{0\}$:

$$\phi_m(r, \varphi) = r^{|m|} e^{im\varphi}. \quad (2.6)$$

²We assume that the Cauchy problem for the classical model is well posed.

³This requirement may seem superficial. To see that it is natural one can consider the “inverted” quantization scheme in which only the modes vanishing at $r = \infty$ are kept. In this case both the constant and logarithmic modes are discarded.

To quantize the space of solutions we need to introduce the coordinates. Those are the coefficients a_{-n} and $\bar{a}_{-n}$ in the expansion of a general field $\phi(r, \varphi)$ in the basis (2.6):

$$\phi(r, \varphi) = \sum_{m \geq 1} \frac{1}{m} a_{-m} r^m e^{im\varphi} + \sum_{m \geq 1} \frac{1}{m} \bar{a}_{-m} r^m e^{-im\varphi} \quad (2.7)$$

They constitute in fact only half of the coordinates — the so-called positive-frequency parts. The other half corresponds to the solutions which are singular at $r = 0$, but are instead regular at $r = \infty$. If we introduce the *complex structure* on the $2d$ plane, so that the complex coordinates read $z = re^{i\varphi}$, $\bar{z} = re^{-i\varphi}$, then the coordinates a_{-n} and $\bar{a}_{-n}$ correspond to *holomorphic* and *antiholomorphic* fields respectively.

The operators corresponding to the coordinates a_{-n} and $\bar{a}_{-n}$ commute between themselves and are the *creation* operators. They can be used to build all states of the model out of the vacuum state $|\emptyset\rangle$. After the choice of complex structure is made, the Hilbert space $\mathcal{H}_{\text{tot}}^{2d}$ of the model, therefore, splits into a tensor product $\mathcal{H}^{2d} \otimes \bar{\mathcal{H}}^{2d}$ of holomorphic and antiholomorphic states:

$$\mathcal{H}^{2d} = \bigoplus_{m_1 \geq m_2 \geq \dots \geq m_k} \mathbb{C} a_{-m_1} a_{-m_2} \dots a_{-m_k} |\emptyset\rangle \quad (2.8)$$

$$\bar{\mathcal{H}}^{2d} = \bigoplus_{m_1 \geq m_2 \geq \dots \geq m_k} \mathbb{C} \bar{a}_{-m_1} \bar{a}_{-m_2} \dots \bar{a}_{-m_k} |\emptyset\rangle \quad (2.9)$$

We can introduce the *grading* L_0 on $\mathcal{H}$ by assigning degree m to a_{-m} . This choice of the degree corresponds to the action of the dilatation operator $z\partial_z$ on the solutions. Similar grading $\bar{L}_0 \simeq \bar{z}\partial_{\bar{z}}$ acts on $\bar{\mathcal{H}}^{2d}$.

The states of $\mathcal{H}^{2d}$ are labelled by Young diagrams $\{m_1, m_2, \dots, m_k\}$ (notice that the sequence is non-increasing). The partition function of the model can be written as a product of two Dedekind eta-functions:

$$\begin{aligned} Z'_{2d}(q, \bar{q}) &= \text{Tr}_{\mathcal{H}_{\text{tot}}^{2d}} \left(q^{L_0} \bar{q}^{\bar{L}_0} \right) = \text{Tr}_{\mathcal{H}^{2d}} \left(q^{L_0} \right) \text{Tr}_{\bar{\mathcal{H}}^{2d}} \left(\bar{q}^{\bar{L}_0} \right) \\ &= \prod_{m \geq 1} \frac{1}{1 - q^m} \frac{1}{1 - \bar{q}^m} \\ &= \sum_{Y \in 2d \text{ YD}} q^{|Y|} \sum_{W \in 2d \text{ YD}} \bar{q}^{|W|}. \end{aligned} \quad (2.10)$$

where Y and W are $2d$ Young diagrams and $|Y|$ denotes the total number of boxes in Y . The coordinates of the zero modes, that we have thrown away earlier (hence a prime in Z'_{2d}) would have contributed to the partition function by an overall factor of the form q^P , which can be thought of as the partition function of a particle with a given momentum, or as the contribution of the center of mass coordinate.

Let us recapitulate the result in the $2d$ case. The solutions of the equations of motion can be split into two parts: those regular at $r = 0$ and those regular at $r = \infty$. The first subspace corresponds to the creation operators. There is also a small subspace of zero modes, which needs to be discarded (or accounted for separately). After introduction

of the complex structure, creation operators further split into holomorphic and antiholomorphic subsets, each of which produces the Hilbert space with basis labelled by Young diagrams. The partition function is the product of two generating functions for the number of Young diagrams.

2.2 3d free scalar

We will now consider the 3d free scalar and analyze its quantization in the same spirit as the 2d case in the previous section. The equation of motion reads

$$(\partial_x^2 + \partial_y^2 + \partial_t^2)\phi(x, y, t) = 0 \quad (2.11)$$

or in the spherical coordinates:

$$[(r\partial_r)^2 + r\partial_r + \Delta_{S^2}]\phi(r, \theta, \varphi) = 0 \quad (2.12)$$

where Δ_{S^2} is the Laplacian operator on a unit sphere. Separating the variables we get $\phi(r, \theta, \varphi) = f(r)g(\theta, \varphi)$, where $g(\theta, \varphi)$ should be an eigenfunction of Δ_{S^2} . It is thus given by the spherical harmonics:

$$g(\theta, \varphi) = Y_{j,m}(\theta, \varphi), \quad j \in \mathbb{Z}_{\geq 0}, \quad -j \leq m \leq j, \quad (2.13)$$

$$\Delta_{S^2} Y_{j,m}(\theta, \varphi) = -j(j+1)Y_{j,m}(\theta, \varphi). \quad (2.14)$$

The corresponding function $f(r)$ satisfies

$$r\partial_r(r\partial_r + 1)f(r) = j(j+1)f(r), \quad (2.15)$$

and thus

$$f(r) = r^j \quad \text{or} \quad r^{-1-j} \quad (2.16)$$

Only the first choice is regular at $r = 0$, so the solutions are given by

$$\phi_{j,m}(r, \theta, \varphi) = r^j Y_{j,m}(\theta, \varphi), \quad j \in \mathbb{Z}_{\geq 0}, \quad -j \leq m \leq j. \quad (2.17)$$

Similarly to the 2d case we would like to discard certain solutions and split the remaining ones using some analogue of the complex structure.⁴ The natural choice of the solutions to be discarded are those with $m = 0$. One way to justify this choice is to require the solutions to vanish at the t axis. This condition can be enforced on the distant boundary of the ball B_3 . There is only one discarded solution for any given j , so they can be thought of as corresponding to a 2d chiral scalar. Thus, the discarded solutions follow the pattern established in 2d: they correspond to the states of the field in one dimension *less* than the original model.

We split the remaining solutions into those with positive and negative m calling them, by gross abuse of the terminology, holomorphic and antiholomorphic respectively. This

⁴The relevant mathematical structure here is probably the *contact* structure represented by the standard one-form $\lambda = dt + xdy - ydx = dt - \frac{i}{2}(\bar{z}dz - z d\bar{z})$ in $\mathbb{R}^3$. Notice that $d\lambda = 2dx \wedge dy = idz \wedge d\bar{z}$ represents the complex (or symplectic) structure on the constant t slices of $\mathbb{R}^3$.

choice of splitting may be thought of as an additional geometric structure, which the model is supplemented with (see footnote 4). Just as in the $2d$ case, the equations of motion *themselves* do not imply any choice of splitting. However, in $2d$ an additional choice of complex structure allows one to define holomorphic and antiholomorphic modes. We would like to view the splitting of $3d$ modes in a similar fashion.

The coordinates on the space of solutions are the coefficients in the expansion:

$$\phi(r, \theta, \varphi) = \sum_{j \geq 1} \sum_{m=1}^j a_{-(j,m)} r^j Y_{j,m}(\theta, \varphi) + \sum_{j \geq 1} \sum_{m=1}^j \bar{a}_{-(j,m)} r^j Y_{j,-m}(\theta, \varphi) \quad (2.18)$$

Let us mention that similar results for the quantization of the free scalar field in $3d$ can be obtained by the standard canonical quantization. To see this, we have to consider the Minkowski version of the free field model. More specifically, let the radial direction r become the time direction, while the space is spanned by (θ, φ) , so that the sphere space-time is locally $\mathbb{R}_r \times S^2$. In accordance with the general prescription we find the momentum corresponding to the field ϕ to be $\pi = \partial_r \phi$. The canonical Poisson brackets are evaluated for a given time slice, e.g. $r = 1$:

$$\{\phi(\theta, \varphi), \pi(\theta', \varphi')\} = \delta^{(2)}(\theta - \theta', \varphi - \varphi'), \quad (2.19)$$

$$\{\phi(\theta, \varphi), \phi(\theta', \varphi')\} = \{\pi(\theta, \varphi), \pi(\theta', \varphi')\} = 0, \quad (2.20)$$

where $\delta^{(2)}(\theta - \theta', \varphi - \varphi') = \frac{1}{\sin(\theta - \theta')} \delta(\theta - \theta') \delta(\varphi - \varphi')$ is the Delta function on S^2 . The commutation relations (2.19), (2.20) are more conveniently written by expanding the field and its conjugate momentum in the complete basis of spherical harmonics $Y_{j,m}(\theta, \varphi)$ on the sphere S^2 :

$$\phi(\theta, \varphi) = \sum_{j \geq 0} \sum_{m=-j}^j \phi_{j,m} Y_{j,m}(\theta, \varphi), \quad (2.21)$$

$$\pi(\theta, \varphi) = \sum_{j \geq 0} \sum_{m=-j}^j \pi_{j,m} Y_{j,m}(\theta, \varphi), \quad (2.22)$$

so that

$$\{\phi_{j,m}, \pi_{j',m'}\} = \delta_{j,j'} \delta_{m,-m'}, \quad (2.23)$$

$$\{\phi_{j,m}, \phi_{j',m'}\} = \{\pi_{j,m}, \pi_{j',m'}\} = 0. \quad (2.24)$$

In Euclidean field theory, the modes $\phi_{j,m}, \pi_{j,m}$ are recast into two sets of solutions to the field equations: those regular at $r = 0$ and those regular at $r = \infty$, as in eq. (2.16). In Minkowski field theory these two sets of modes are usually called the positive and negative frequency modes, because the powers of r become the frequencies after the Euclidean rotation $\ln r \mapsto i \ln r$. The coefficients of these modes become the classical versions of the creation and annihilation operators, which we denote by $\alpha_{j,m}^\dagger$ and $\alpha_{j,m}$ respectively:

$$\phi_{j,m} = \alpha_{j,m}^\dagger + \alpha_{j,-m}, \quad (2.25)$$

$$\pi_{j,m} = \alpha_{j,m}^\dagger - \alpha_{j,-m}, \quad (2.26)$$

$\alpha_{j,m}$ (resp. $\alpha_{j,-m}^\dagger$) corresponds to the coefficient in front of the solution regular at $r = 0$ (resp. $r = \infty$) in eq. (2.16). The canonical Poisson brackets (2.23), (2.24) now dictate the brackets between $\alpha_{j,m}$ and $\alpha_{j,m}^\dagger$:

$$\{\alpha_{j,m}, \alpha_{j',m'}^\dagger\} = \frac{1}{2} \delta_{j,j'} \delta_{m,m'}, \quad (2.27)$$

$$\{\alpha_{j,m}, \alpha_{j',m'}\} = \{\alpha_{j,m}^\dagger, \alpha_{j',m'}^\dagger\} = 0. \quad (2.28)$$

The canonical quantization upgrades the Poisson brackets in eqs. (2.27), (2.28) to commutators, so that $\alpha_{j,m}^\dagger$ and $\alpha_{j,m}$ become creation and annihilation operators of the free field states.

The Hilbert space of the free field is obtained by acting with creation operators $\alpha_{j,m}^\dagger$ on the vacuum vector $|\emptyset\rangle$. The splitting of the modes into “holomorphic” and “antiholomorphic” ones, which we have described above, corresponds to splitting the creation operators into those with $m > 0$ and those with $m < 0$ and throwing away those with $m = 0$. Thus we have the identification between the operators obtained from canonical quantization and $a_{-(j,m)}$, $\bar{a}_{-(j,m)}$ featuring in eq. (2.18))

$$a_{-(j,m)} = \alpha_{j,m}^\dagger, \quad \bar{a}_{-(j,m)} = \alpha_{j,-m}^\dagger, \quad j \geq 1, \quad m = 1, \dots, j. \quad (2.29)$$

The Hilbert space $\mathcal{H}_{\text{tot}}^{3d}$ is (as in section 2.1) a tensor product of two factors $\mathcal{H}^{3d} \otimes \bar{\mathcal{H}}^{3d}$. The space of holomorphic states $\mathcal{H}^{3d}$ is spanned by the vectors created from the vacuum $|\emptyset\rangle$ by the strings of operators $a_{-(j,m)}$:

$$\mathcal{H}^{3d} = \bigoplus_{\substack{j_1 \geq j_2 \geq \dots \geq j_k \\ m_1 \geq m_2 \geq \dots \geq m_k}} \bigoplus_{m_1=1}^{j_1} \dots \bigoplus_{m_k=1}^{j_k} \mathbb{C} a_{-(j_1,m_1)} a_{-(j_2,m_2)} \dots a_{-(j_k,m_k)} |\emptyset\rangle \quad (2.30)$$

$$\bar{\mathcal{H}}^{3d} = \bigoplus_{\substack{j_1 \geq j_2 \geq \dots \geq j_k \\ m_1 \geq m_2 \geq \dots \geq m_k}} \bigoplus_{m_1=1}^{j_1} \dots \bigoplus_{m_k=1}^{j_k} \mathbb{C} \bar{a}_{-(j_1,m_1)} \bar{a}_{-(j_2,m_2)} \dots \bar{a}_{-(j_k,m_k)} |\emptyset\rangle \quad (2.31)$$

The grading on $\mathcal{H}^{3d}$ corresponds to the the total power of r in the solution, so that $a_{-(j,m)}$ has degree j .

Quite remarkably, the states on a given level in $\mathcal{H}^{3d}$ are in one to one correspondence with *plane partitions*:

	Plane partitions	States in $\mathcal{H}$
deg = 0	$\emptyset$	$ \emptyset\rangle$
deg = 1	$[[1]]$	$a_{-(1,1)} \emptyset\rangle$
deg = 2	$[[1, 1]], \quad [[2]], \quad [[1], [1]]$	$a_{-(1,1)}^2 \emptyset\rangle, \quad a_{-(2,1)} \emptyset\rangle, \quad a_{-(2,2)} \emptyset\rangle$
deg = 3	$[[1, 1, 1]], \quad [[2, 1]], \quad [[3]]$ $[[1, 1], [1]], \quad [[2], [1]], \quad [[1], [1], [1]]$	$a_{-(1,1)}^3 \emptyset\rangle, \quad a_{-(2,1)} a_{-(1,1)} \emptyset\rangle, \quad a_{-(3,1)} \emptyset\rangle$ $a_{-(2,2)} a_{-(1,1)} \emptyset\rangle, \quad a_{-(3,2)} \emptyset\rangle, \quad a_{-(3,3)} \emptyset\rangle$

(2.32)

Figure 1. A plane partition, which can be represented by an array $\pi_{i,j} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 \end{pmatrix}$. The diagonal cross section inside π is shown in pale blue, it defines a Young diagram $Y = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$. The arm (resp. leg) lengths corresponding to the boxes on the cross section are shown in red (resp. blue). They form two Young tableaux of shape Y : $P = \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & \end{array}$ and $Q = \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 1 & \end{array}$.

From the table (2.32) one can see that at least on the first three levels the number of plane partitions and the number of states in $\mathcal{H}^{3d}$ coincide. The correspondence is in fact a combinatorial theorem⁵ [26–30]. Let us briefly recall the steps of the proof:

1. Plane partitions are in one-to-one correspondence with pairs (P, Q) of Young tableaux sharing the same shape. Young tableaux of shape Y is a Young diagram Y filled with positive integers so that the numbers in the rows are *non-increasing* from left to right and the numbers in the columns are *decreasing* from top to bottom. An example of Young tableaux of shape $[4, 2, 1]$ is

$$\begin{array}{|c|c|c|c|} \hline 3 & 3 & 1 & 1 \\ \hline 2 & 2 & & \\ \hline 1 & & & \\ \hline \end{array} \quad (2.33)$$

Let π be a plane partition. It can be represented as a $2d$ array of non-negative integers $\pi_{i,j}$, such that

$$\pi_{i,j} \geq \pi_{i,j+1}, \quad \pi_{i,j} \geq \pi_{i+1,j}. \quad (2.34)$$

and $\pi_{i,j} = 0$ for i or j sufficiently large. If we dissect π along the diagonal $i = j$, the cross section is a Young diagram Y defined by the non-increasing sequence of integers $Y_i = \pi_{i,i}$. The tableaux P and Q corresponding to π have the shape Y and are obtained from the “arm” and “leg” lengths corresponding to the boxes lying in the cross section, as shown in figure 1. The defining properties of the Young tableaux are satisfied for P and Q . Indeed, the numbers in the rows of P (resp. Q) are non-increasing, since the corresponding arms (resp. legs) are stacked directly on top of each other, while the entries in a given column correspond to arms and lengths in a given horizontal slice of π , and therefore decrease as one goes further from the vertical coordinate axis.

2. We perform the *inverse* Robertson-Shensted-Knuth (RSK) transformation [26–29] on the pair of tableaux (P, Q) . The RSK correspondence is a *bijection* between pairs of

⁵We thank the referee for pointing out the second reference and M. Bershtein and A. Popolitov for pointing out the third one.

Young tableaux and generalized permutations, i.e. lexicographically ordered lists of pairs of integers. In this correspondence the first (resp. second) entry in each pair fills the tableaux Q (resp. P). The detailed description of the correspondence and the proof that it is indeed one-to-one is given in [28, 29] (see also [31]). In the example from figure 1 we get:

$$RSK^{-1}\left(\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 1 & \\ \hline \end{array}\right) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix} \quad (2.35)$$

- Any permutation σ of n objects can be written as an $n \times n$ matrix with entry σ_{ij} equal to 1 if $j = \sigma(i)$ and 0 otherwise. Similarly, any generalized permutation Σ can be written as an integer-valued matrix, in which Σ_{ij} is given by the number of pairs $\begin{pmatrix} i \\ j \end{pmatrix}$ in Σ . In our example, eq. (2.35), we get the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (2.36)$$

- To an $n \times n$ matrix of integers Σ_{ij} representing a generalized permutation we associate a state in $\mathcal{H}^{3d}$:

$$\prod_{i,j=1}^n a_{-(i+j-1,j)}^{\Sigma_{ij}} |\emptyset\rangle. \quad (2.37)$$

Evidently, $i + j - 1 \geq j \geq 1$, so the labels of the creation operators are in the range specified by eq. (2.30). In the example from figure 1, the corresponding matrix is (2.36) and the resulting state is

$$a_{-(2,2)} a_{-(2,1)} a_{-(3,1)} |\emptyset\rangle. \quad (2.38)$$

Notice that due to the properties of the RSK correspondence, the level of the state is equal to the total number of boxes in the initial partition, e.g. 7 in eq. (2.38) and figure 1.

Since the RSK correspondence is a bijection, so is our mapping of plane partitions to states in $\mathcal{H}^{3d}$. The $3d$ partition function (with $m = 0$ states removed) is thus equal to the product of two *MacMahon* functions:⁶

$$\begin{aligned} Z'_{3d}(q, \bar{q}) &= \text{Tr}_{\mathcal{H}_{\text{tot}}^{3d}} \left(q^{L_0} \bar{q}^{\bar{L}_0} \right) = \text{Tr}_{\mathcal{H}^{3d}} (q^{L_0}) \text{Tr}_{\bar{\mathcal{H}}^{3d}} (\bar{q}^{\bar{L}_0}) \\ &= \prod_{j \geq 1} \frac{1}{(1 - q^j)^j} \frac{1}{(1 - \bar{q}^j)^j} \\ &= \sum_{\pi \in 3d \text{ YD}} q^{|\pi|} \sum_{\xi \in 3d \text{ YD}} \bar{q}^{|\xi|} \end{aligned} \quad (2.39)$$

Adding back the solutions, which we have discarded corresponds to multiplying the partition function by an additional eta-function. The grading $L_0 + \bar{L}_0$ can in fact be understood

⁶Hints of the connection between $3d$ scalar and MacMahon representation were mentioned in [24].

as a scaling operator acting on the free boson field. The difference $L_0 - \bar{L}_0$ doesn't seem to have such a direct physical interpretation.

What we see is that a particular splitting of the states of the free boson in $3d$ gives the analogue of holomorphic factorization for the $2d$ boson and that the states of the Hilbert space are labelled by (pairs of) plane partitions π .

2.3 MacMahon representation $M_c(u)$ of $W_{1+\infty}$ and DIM algebras

The description of the chiral Hilbert space $\mathcal{H}^{3d}$ spanned by plane partitions π in terms of creation operators $a_{-(j,m)}$ might look a little unconventional. In this section we recall a very similar description of the MacMahon representation of the $W_{1+\infty}$ algebra discovered in [25] (see also the review [24]). In this way we make a connection between the $W_{1+\infty}$ algebra (and as we will see, DIM algebra in general) and the states of the $3d$ free scalar.

DIM algebra can be thought of as a t -deformation of the Lie algebra $W_{1+\infty}$. This Lie algebra is the central extension⁷ of the algebra of difference operators $W_{(n,m)} \simeq q^{-\frac{nm}{2}} z^n q^{mz\partial_z}$:

$$W_{1+\infty} : \quad [W_{n,m}, W_{k,l}] = \left(q^{\frac{mk-nl}{2}} - q^{-\frac{mk-nl}{2}} \right) W_{n+k, m+l} + (nc_1 + mc_2) \delta_{n+k,0} \delta_{m+l,0},$$

$$n, m, k, l \in \mathbb{Z}, \quad (2.40)$$

where c_1 and c_2 are two central charges. One can equivalently understand $W_{1+\infty}$ as the algebra of maps from the quantum torus $(z, q^{z\partial_z})$ to $\mathbb{C}$. Notice that the relations of the algebra are manifestly invariant under the action of the $SL(2, \mathbb{Z})$ automorphism group, which is the mapping class group of the (quantum) torus. In particular, the central charges transform as a doublet under this group. There is also a pair of grading operators (d_1, d_2) which assign the weight (n, m) to the generator $W_{n,m}$.

We are not going to write down the relations for the DIM algebra for $t \neq q$, since they can be easily found in the literature [32, 33]. There are, however, two general remarks. First of all, the t -deformed algebra is still $SL(2, \mathbb{Z})$ -invariant. Secondly, the algebra is symmetric under the permutation of the parameters q , t^{-1} and $\frac{t}{q}$. The remnant of this large symmetry is the symmetry of eq. (2.40) under $q \leftrightarrow q^{-1}$ (one needs to simultaneously invert the sign of $W_{n,m}$).

We will describe the MacMahon representation $M_c(u)$ in the basis corresponding to the differential operators $w_{n,m} \simeq z^n (z\partial_z)^m$, instead of the difference operators $W_{n,m} \simeq z^n q^{mz\partial_z}$ used in eq. (2.40), but the argument is similar in both cases. The difference between two choices of basis is only superficial. Indeed, one can expand the basis generators $W_{n,m} \simeq z^n q^{mz\partial_z}$ in $\ln q$ and get a series in the generators $w_{n,m} \simeq z^n (z\partial_z)^m$. The two formulations of the algebra are completely equivalent, though the structure constants look much simpler in the basis $W_{n,m}$. Notice the difference between the indices in $w_{n,m}$ and $W_{n,m}$: in the first case $m \in \mathbb{Z}_{\geq 0}$ since m labels the term in the Taylor expansion in $\ln q$, while in the second n and m are general integers.

Consider the highest weight representation with central charge $(c_1, c_2) = (c, 0)$. The highest weight state $|v\rangle$, is the eigenstate of $w_{0,m}$ annihilated by $w_{n,m}$ for $n \geq 1$. The states

⁷Our choice of the central extension is equivalent to the more conventional choice leading to the usual central extension in the Virasoro subalgebra.

of the representation are obtained by acting on $|v\rangle$ with the operators $w_{n,m}$ with $n < 0$ and $m \geq 0$.

We further require there to be only a finite number of states on a given level. This is an extremely stringent requirement which fixes the allowed eigenvalues of $w_{0,m}$ almost completely. To get a finite number of states on each level we need to have a lot of null-states. It turns out that the simplest nontrivial representation of this form has the null-subspaces generated by the following states [25]:

$$\text{Null states:} \quad |\chi_n\rangle \simeq z^{-n} \prod_{j=0}^{n-1} (z\partial_z - j)|v\rangle \simeq \partial_z^n |v\rangle, \quad n > 0 \quad (2.41)$$

where we have used the identification between the generators and the differential operators to write the null-states explicitly. Following the pattern of null states, we see that there are exactly n independent generators $w_{-n,m}$ for a given degree $n > 0$: those for which $0 \leq m \leq n-1$. All the other generators produce null-states. This situation is exactly parallel to the chiral free boson Hilbert space $\mathcal{H}^{3d}$, which is generated by $a_{-(j,m)}$ with $1 \leq m \leq j$. Thus, there is a direct equivalence between the holomorphic states of the $3d$ free boson (2.32) and the states of the MacMahon representation $M_c(u)$ of $W_{1+\infty}$:

$W_{1+\infty}$	$3d$ boson	
$w_{-n_1, m_1} w_{-n_2, m_2} \cdots w_{-n_k, m_k} v\rangle$	$a_{-(n_1, m_1+1)} a_{-(n_2, m_2+1)} \cdots a_{-(n_k, m_k+1)} \emptyset\rangle$	
$n_i > 0, \quad 0 \leq m_i \leq n_i - 1$	$n_i > 0, \quad 1 \leq m_i + 1 \leq n_i$	(2.42)

The only essential difference between the two pictures is that the creation and annihilation operators $a_{-(j,m)}$ with non-opposite vectors (j, m) *commute* while the commutator of two generators $W_{n,m}$ is nonzero in general.

Let us look for the analogy to this situation in the $2d$ free boson. Indeed, there are creation operators a_{-m} , which commute for non-opposite m and there are also *Virasoro* generators $L_{-n} \sim \sum_{k \in \mathbb{Z}} :a_{k-n} a_{-k}:$, which form a nontrivial Lie algebra. We can hypothesize that the same phenomenon happens in the $3d$ case and the $W_{1+\infty}$ generators are expressed as *bilinear combinations* of bosonic operators $a_{(j,m)}$:

$$w_{n,m} \stackrel{?}{\sim} \sum_{k,l} :a_{n+k, m+l} a_{-k, -l}: \quad (2.43)$$

However, before applying eq. (2.43) one needs to understand the limits of summation, the precise nature of the normal ordering and possible convergence issues. We will not attempt this task here.

3 Affine matrix models and DIM Bethe equations

In this section we derive the analogue of the quantum spectral curve for affine matrix models. Taking its Nekrasov-Shatashvili limit we obtain Baxter and Bethe equations, which are slight generalizations of those given in [16–22]. Their solutions are labelled by

plane partitions. Following the results of the previous section this hints at a possible connection with an *integrable* 3d field theory.

Affine matrix model is one of the species from the zoo of “network matrix models” [34–37] obtained from refined topological strings [40–45]. Concretely, this model corresponds to the compactified strip geometry. In the geometric engineering terms it gives the 5d $\mathcal{N} = 1$ $U(K)$ gauge theory with an extra adjoint multiplet, or equivalently 6d abelian linear quiver. The matrix model average of a function $f(\vec{x})$ is given by:

$$\langle f(\vec{x}) \rangle_{N,u,\vec{z},\vec{w}} = \frac{1}{Z} \oint_{\mathcal{C}} \prod_{i=1}^N \left\{ \frac{dx_i}{x_i} x_i^u \prod_{a=1}^K \frac{\left(\frac{w_a}{x_i}; q\right)_{\infty}}{\left(\frac{z_a}{x_i}; q\right)_{\infty}} \right\} \prod_{i \neq j}^N \frac{\left(\frac{x_i}{x_j}; q\right)_{\infty} \left(t_1 t_2 \frac{x_i}{x_j}; q\right)_{\infty}}{\left(t_1 \frac{x_i}{x_j}; q\right)_{\infty} \left(t_2 \frac{x_i}{x_j}; q\right)_{\infty}} f(\vec{x}), \quad (3.1)$$

where Z is the integral without the insertion and the contour $\mathcal{C}$ can be chosen to encircle the poles of the integrand. The parameters z_i and w_j can be viewed as Kähler parameters of the strip geometry, or Coulomb moduli of the 5d theory. These poles correspond to K -tuples of plane partitions $\pi^{(a)}$, $a = 1, \dots, K$:

$$x_I = z_a q^{\pi_{i,j}^{(a)} - 1} t_1^{1-i} t_2^{1-j}. \quad (3.2)$$

The partitions $\pi^{(a)}$ have the total *floor area* N , while the total number of boxes can be arbitrary. The residues at the poles transform naturally under arbitrary permutations of t_1 , t_2 and q^{-1} and the simultaneous transpositions of the plane partitions $\pi^{(a)}$ (assuming N to be sufficiently large).

3.1 Quantum spectral curve

We follow the standard technique to obtain the quantum spectral curve (or loop equations, or Ward identities, or qq -characters) for the model and consider an integral of a total difference:⁸

$$0 = \oint_{\mathcal{C}} \prod_{k=1}^N \frac{dx_k}{x_k} \sum_{i=1}^N (1 - q^{x_i \partial_{x_i}}) \left\{ \frac{x_i^3 \prod_{a=1}^K (x_i - z_a)}{x_i - \xi} \times \prod_{j \neq i} \frac{\left(\frac{t_1 t_2}{q} x_i - x_j\right) (t_1^{-1} x_i - x_j) (t_2^{-1} x_i - x_j)}{(x_i - x_j)} \mu(\vec{x}) \right\}, \quad (3.3)$$

where

$$\mu(\vec{x}) = \prod_{i=1}^N \left\{ x_i^u \prod_{a=1}^K \frac{\left(\frac{w_a}{x_i}; q\right)_{\infty}}{\left(\frac{z_a}{x_i}; q\right)_{\infty}} \right\} \prod_{i \neq j}^N \frac{\left(\frac{x_i}{x_j}; q\right)_{\infty} \left(t_1 t_2 \frac{x_i}{x_j}; q\right)_{\infty}}{\left(t_1 \frac{x_i}{x_j}; q\right)_{\infty} \left(t_2 \frac{x_i}{x_j}; q\right)_{\infty}} \quad (3.4)$$

is the matrix model measure. The difference operator acts on the measure as follows:

$$q^{x_i \partial_{x_i}} \mu(\vec{x}) = q^u \prod_{a=1}^K \frac{q x_i - w_a}{q x_i - z_a} \prod_{i \neq j} \frac{(q x_i - x_j) (t_1 x_i - x_j) (t_2 x_i - x_j) \left(\frac{q}{t_1 t_2} x_i - x_j\right)}{(x_i - x_j) (t_1 t_2 x_i - x_j) \left(\frac{q}{t_1} x_i - x_j\right) \left(\frac{q}{t_2} x_i - x_j\right)} \mu(\vec{x}) \quad (3.5)$$

⁸The contour integral in the definition of the model is equivalent to a Jackson integral, for which total differences play the role of total derivatives.

so that half of the factors cancel with the extra factors in eq. (3.3). The identity for the averages following from eq. (3.3) is

$$\left\langle \sum_{i=1}^N \frac{x_i^3 \prod_{a=1}^K (x_i - z_a)}{x_i - \xi} \prod_{j \neq i} \frac{\left(\frac{t_1 t_2}{q} x_i - x_j\right) (t_1^{-1} x_i - x_j) (t_2^{-1} x_i - x_j)}{(x_i - x_j)} - \sum_{i=1}^N q^{u+3} \frac{x_i^3 \prod_{a=1}^K (q x_i - w_a)}{q x_i - \xi} \prod_{j \neq i} \frac{(t_1 x_i - x_j) (t_2 x_i - x_j) \left(\frac{q}{t_1 t_2} x_i - x_j\right)}{(x_i - x_j)} \right\rangle = 0. \quad (3.6)$$

We can use one more standard trick [46, 47] and rewrite the sums under the average as contour integrals over an auxiliary parameter y :

$$\begin{aligned} & \sum_{i=1}^N \frac{x_i^3 \prod_{a=1}^K (x_i - z_a)}{x_i - \xi} \prod_{j \neq i} \frac{\left(\frac{t_1 t_2}{q} x_i - x_j\right) (t_1^{-1} x_i - x_j) (t_2^{-1} x_i - x_j)}{(x_i - x_j)} \\ &= \frac{1}{\left(\frac{t_1 t_2}{q} - 1\right) (t_1^{-1} - 1) (t_2^{-1} - 1)} \\ & \quad \times \oint_{\mathcal{C}_x} dy \frac{\prod_{a=1}^K (y - z_a)}{(y - \xi)} \prod_{j=1}^N \frac{\left(\frac{t_1 t_2}{q} y - x_j\right) (t_1^{-1} y - x_j) (t_2^{-1} y - x_j)}{(y - x_j)} \\ & \sum_{i=1}^N \frac{x_i^3 \prod_{a=1}^K (q x_i - w_a)}{q x_i - \xi} \prod_{j \neq i} \frac{\left(\frac{q}{t_1 t_2} x_i - x_j\right) (t_1 x_i - x_j) (t_2 x_i - x_j)}{(x_i - x_j)} \end{aligned} \quad (3.7)$$

$$\begin{aligned} &= \frac{1}{\left(\frac{q}{t_1 t_2} - 1\right) (t_1 - 1) (t_2 - 1)} \\ & \quad \times \oint_{\mathcal{C}_x} dy \frac{\prod_{a=1}^K (q y - w_a)}{(q y - \xi)} \prod_{j=1}^N \frac{\left(\frac{q}{t_1 t_2} y - x_j\right) (t_1 y - x_j) (t_2 y - x_j)}{(y - x_j)} \end{aligned} \quad (3.8)$$

where the contour $\mathcal{C}_x$ encircles all the points x_j . Deforming the contour we pick up the residues at $y = \xi$ and at $y = 0, \infty$. The former residue gives the expression for the quantum spectral curve while the latter two produce polynomials in ξ with coefficients polynomially depending on x_i .

$$\begin{aligned} K_+(\xi) & \left\langle \frac{Q\left(\frac{t_1 t_2}{q} \xi\right) Q\left(\frac{\xi}{t_1}\right) Q\left(\frac{\xi}{t_2}\right)}{Q(\xi)} \right\rangle \\ & + q^{u+1} K_-(\xi) \left\langle \frac{Q\left(\frac{\xi}{t_1 t_2}\right) Q\left(\frac{t_1}{q} \xi\right) Q\left(\frac{t_2}{q} \xi\right)}{Q\left(\frac{\xi}{q}\right)} \right\rangle = \text{Pol}_{2N+K}(\xi) \end{aligned} \quad (3.9)$$

where

$$Q(\xi) = \prod_{j=1}^N (\xi - x_j), \quad K_+(\xi) = \prod_{a=1}^K (\xi - z_a), \quad K_-(\xi) = \prod_{a=1}^K (\xi - w_a). \quad (3.10)$$

This equation can be viewed as a three-parametric (t_1, t_2 and q) *deformation* of the Baxter equation associated to DIM algebra [20–22]. Notice also that the equation is symmetric under permutation of t_1, t_2 and $\frac{q}{t_1 t_2}$. This symmetry combines with the symmetry of the integrand under the permutation of t_1, t_2 and q^{-1} , which we mentioned in the beginning of this section. Overall, the matrix model is symmetric under permutation of *four parameters*, t_1, t_2, q^{-1} and $\frac{q}{t_1 t_2}$ with their product being equal to one.

3.2 NS limit and Bethe equations

In the NS limit $q \rightarrow 1$ the matrix model integral can be evaluated by saddle point method. The quantum spectral curve equation (3.9) is then understood as an equation determining the saddle points $\vec{x}$. The average signs in eq. (3.9), therefore, disappear and it reduces to the DIM Baxter TQ equation:

$$K_+(\xi) \frac{Q\left(\frac{t_1 t_2}{q} \xi\right) Q\left(\frac{\xi}{t_1}\right) Q\left(\frac{\xi}{t_2}\right)}{Q(\xi)} + q^{u+1} K_-(\xi) \frac{Q\left(\frac{\xi}{t_1 t_2}\right) Q\left(\frac{t_1}{q} \xi\right) Q\left(\frac{t_2}{q} \xi\right)}{Q\left(\frac{\xi}{q}\right)} = \text{Pol}_{2N+K}(\xi). \quad (3.11)$$

where $Q(\xi)$ is a polynomial with roots x_i . The only difference between our equation and the standard one considered in the literature [20–22] is that the roots of the polynomials $K_+(\xi)$ and $K_-(\xi)$ are *a priori* not related to each other. In [20–22], however, one had the condition $K_-(\xi) = K_+(t_1 t_2 \xi)$. In the matrix model language this constraint relating w_a and z_a is interpreted as the *degeneracy condition* for the vertex operators sitting at points z_a — their weights (dimensions) take special, quantized values. As we will see momentarily, lifting of the constraint has dramatic consequences for the structure of the solutions: *with* the constraint they are enumerated (up to permutations) by K -tuples of Young diagrams, while *without* it the solutions are much more numerous and correspond to K -tuples of plane partitions.

Bethe equations following from the Baxter equation (3.11) are:

$$\begin{aligned} K_+(x_i) \prod_{j \neq i} \frac{(t_1 t_2 x_i - x_j) \left(\frac{x_i}{t_1} - x_j\right) \left(\frac{x_i}{t_2} - x_j\right)}{x_i - x_j} \\ = e^\tau K_-(x_i) \prod_{j \neq i} \frac{\left(\frac{x_i}{t_1 t_2} - x_j\right) (t_1 x_i - x_j) (t_2 x_i - x_j)}{x_i - x_j} \end{aligned} \quad (3.12)$$

where $e^\tau = \lim_{q \rightarrow 1} q^{u+1}$ (we assume that u scales as $\frac{1}{\ln q}$ in the NS limit). Let us consider two limits $\tau \rightarrow \pm\infty$ of these equations and see how plane partitions arise as solutions. Our analysis is very similar to that of [16–19], though the end results are different for the reason discussed above.

1. $\tau \rightarrow -\infty$. Equations (3.12) simplify into:

$$K_+(x_i) \prod_{j \neq i} \frac{(t_1 t_2 x_i - x_j) \left(\frac{x_i}{t_1} - x_j\right) \left(\frac{x_i}{t_2} - x_j\right)}{x_i - x_j} = 0. \quad (3.13)$$

Suppose for a moment that $K = 1$, $N = 2$ and one of the variables, say x_1 sits at point z_1 . Then, equation (3.13) for $i = 1$ is already satisfied and we need not consider it anymore. The next variable, x_2 should solve one of the remaining equations. It cannot sit at z_1 because of the denominator $(x_1 - x_2)$, and therefore should be either $t_1 z_1$, $t_2 z_1$ or $(t_1 t_2)^{-1} z_1$. These three solutions correspond to three plane partitions with $N = 2$ boxes.

In general one can see that the solutions are labelled by K -tuples of plane partitions $\pi'^{(a)}$ with total of N boxes. They are given by

$$x_i = z_a t_1^{1-i} t_2^{1-j} (t_1 t_2)^{k-1}, \quad (i, j, k) \in \pi'^{(a)}. \quad (3.14)$$

We use a prime to distinguish partitions $\pi'^{(a)}$ from the partitions $\pi^{(a)}$ from eq. (3.2) labelling the poles of the affine matrix model integrand. In the NS limit the poles condense, so that the size of the saddle point partitions $\pi^{(a)}$ is actually infinite. Notice that N is the *total number of boxes* in $\pi'^{(a)}$, while the partitions $\pi^{(a)}$ have total *floor area* N and arbitrary number of boxes.

2. $\tau \rightarrow +\infty$. This limit is analogous to the previous one:

$$K_-(x_i) \prod_{j \neq i} \frac{\left(\frac{x_i}{t_1 t_2} - x_j\right) (t_1 x_i - x_j) (t_2 x_i - x_j)}{x_i - x_j} = 0. \quad (3.15)$$

The solutions are again labelled by plane partitions, but the roots cluster around the points w_a , not z_a :

$$x_i = w_a t_1^{i-1} t_2^{j-1} (t_1 t_2)^{1-k}, \quad (i, j, k) \in \pi'^{(a)}. \quad (3.16)$$

The solutions for general τ extrapolate between two limits $\tau \rightarrow \pm\infty$, but are still enumerated by plane partitions. We have verified this claim by numerically solving the equations for small N and K . We can then ask what happens in the limit when $w_a \rightarrow t_1 t_2 z_a$? One can notice that in this case plane partitions with height more than one cease to be solutions, because of extra cancellations between the factors in K_+ and K_- . We have also verified this effect numerically. More generally, if $w_a \rightarrow (t_1 t_2)^n z_a$ the solutions are enumerated by plane partitions with height at most n .

The solutions to Bethe equations correspond to the basis of eigenstates of DIM integrals of motion in the corresponding DIM representation. For general K_+ and K_- the representation is a tensor product of MacMahon modules $M_{\frac{w_1}{z_1}}(z_1) \otimes \cdots \otimes M_{\frac{w_K}{z_K}}(z_K)$ with spectral parameters z_a and central charges $(\ln \frac{w_a}{z_a}, 0)$. The basis is given by the K -tuples of plane partitions. For special degenerate value of the central charge equal to $(\ln(t_1 t_2), 0)$ the MacMahon module becomes reducible. After factoring out the invariant subspace one obtains the *Fock* representation with standard basis labelled by Young diagrams. The disappearing solutions of the Bethe equations correspond to the invariant subspaces factored out from the MacMahon modules. For $w_a \rightarrow (t_1 t_2)^n z_a$ the MacMahon module degenerates into a subquotient of tensor product of n Fock modules. Correspondingly, the central charge of the resulting representation is $(n \ln(t_1 t_2), 0)$.

Let us summarize the results of this section. We have derived the quantum spectral curve equation for the affine matrix model and studied the Bethe equations obtained in the NS limit. The solutions of these equations are in general labelled by K -tuples of plane partitions and span the tensor product of MacMahon representations of DIM algebra $U_{t_1, t_2}(\widehat{\mathfrak{gl}}_1)$. The Bethe equations, therefore describe certain (eigen)states of a $3d$ field theory. In the next section we give an explicit construction of the basis of eigenstates in the single MacMahon module.

Let us emphasize that the Bethe eigenstates form a basis in a representation of DIM algebra, i.e. there is an action of DIM algebra on them. The action is described in terms of combinatorics of $3d$ Young diagrams, i.e. the currents generating the DIM algebra act on a Young diagram by either adding or removing a box with some weight depending on the diagram and the box. However, it seems that this action is not easy to see in the matrix model picture (3.1). The toy example of a similar situation is the XXX spin chain where Bethe eigenstates form a special basis in the representation $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ of the Yangian $Y[\mathfrak{sl}_2]$, though the explicit action on them is somewhat complicated. It would be interesting to work out the DIM action on Bethe eigenstates using the Ward identities of the matrix model (3.1).

4 Triple Macdonald polynomials

In this section we build the states corresponding to the solutions of Bethe equations (3.13) for the simplest case of $K = 1$. We model the MacMahon module as a subspace inside the infinite tensor product of Fock modules $\mathcal{F}(u)$ of DIM algebra $U_{t_1, t_2}(\widehat{\mathfrak{gl}}_1)$ with specially adjusted spectral parameters. The infinite tensor product of Fock spaces, each of which is effectively a $2d$ free boson, can be seen as a discretized construction of a $3d$ field. One can view such a discretization as follows. The Fock space $\mathcal{F}(u)$ is the Hilbert space of a $2d$ boson on a circle S^1 (the time direction is assumed to be $\mathbb{R}$). An infinite tensor product of Fock spaces is then the Hilbert space of an infinite family of free $2d$ bosons on a collection of circles. This infinite collection of independent bosons on circles can be seen as an infinite set of modes of a *single* $3d$ boson on a $2d$ space slice e.g. on a torus $S^1 \times S^1$. The modes in this case naturally have two discrete indices, corresponding to the momenta along two circles. As we have argued in section 2.3, in the case of MacMahon representation the relevant $2d$ space slice is S^2 rather than $S^1 \times S^1$. However, this only changes the mode labels, while the overall construction of “ $3d$ from $2d$ ” still goes through, as we show below.

Concretely, we consider the eigenfunctions of the Cartan subalgebra of DIM algebra. The first element of this subalgebra is the zero mode of the current x_0^+ , which corresponds to generator $W_{0,1}$ in the $W_{1+\infty}$ notation. We will diagonalize this element in the tensor product of Fock representations⁹ $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_L)$, which we identify with the space of polynomials in L families of time variables $p_{a,n}$, $a = 1, \dots, L$, $n \geq 1$. The action of x_0^+ on

⁹The definition of the tensor product of representations requires the choice of the coproduct Δ in the DIM algebra. This amounts to choosing a preferred direction in the space of central charge vectors. Here we choose the horizontal direction.

this space is written as follows:

$$x_0^+|_{\mathcal{F}(u_1)\otimes\cdots\otimes\mathcal{F}(u_L)} = \oint_{\mathcal{C}_0} \frac{dz}{z} \sum_{a=1}^L u_a \Lambda_a(z), \quad (4.1)$$

where $\mathcal{C}_0$ is a small contour around zero and

$$\Lambda_a(z) = \exp \left\{ \sum_{n \geq 1} \frac{1-t_2^n}{n} z^n \left[p_{a,n} + (1-(t_1 t_2)^{-n}) \sum_{b=1}^{a-1} p_{b,n} \right] \right\} \exp \left\{ \sum_{n \geq 1} (1-t_1^n) z^{-n} \frac{\partial}{\partial p_{a,n}} \right\}. \quad (4.2)$$

The equation for the eigenfunctions reads

$$\begin{aligned} x_0^+|_{\mathcal{F}(u_1)\otimes\cdots\otimes\mathcal{F}(u_L)} M_{\vec{Y}}^{(t_1, t_2)}(\vec{u}|p_{a,n}) \\ = \sum_{a=1}^L u_a \left[1 - (1-t_1)(1-t_2) \sum_{(i,j) \in Y_a} t_1^{j-1} t_2^{i-1} \right] M_{\vec{Y}}^{(t_1, t_2)}(\vec{u}|p_{a,n}) \end{aligned} \quad (4.3)$$

The eigenfunctions $M_{\vec{Y}}^{(t_1, t_2)}(\vec{u}|p_{a,n})$ are called generalized Macdonald polynomials [46, 48, 49] and depend on L -tuple of Young diagrams $\vec{Y}$ and L -tuple¹⁰ of spectral parameters $\vec{u}$.

For certain special values of spectral parameters called the resonances, i.e. when $u_a = u_{a+1} t_1^m t_2^l$, part of the tensor product of Fock modules completely decouples (i.e. the corresponding states become null-vectors inside the DIM representation) leaving only polynomials corresponding to the *special* L -tuples of Young diagrams. The same phenomenon can be observed when considering the Nekrasov expansion of the conformal block. To get the right answer for *degenerate* values of the intermediate dimensions one should constraint the Young diagrams appearing in the expansion to satisfy the *Burge conditions* [50–52]. We would like to consider here the simplest case of the resonance when $u_a = u(t_1 t_2)^{1-a}$. In this case the surviving L -tuples of Young diagrams form a plane partition $\pi = \{Y_1, \dots, Y_L\}$ of height L .

Another way to see the decoupling of states for spectral parameters in resonance is to consider the R -matrix for the DIM algebra [53–56] acting in the tensor product of Fock spaces $\mathcal{F}(u) \otimes \mathcal{F}(\frac{u}{x})$. The R -matrix in the resonant case becomes singular. Indeed, the eigenvalues of the R -matrix can be read off from its expression in the vertical representation (see [53–56] for details):

$$\mathcal{R}_{\lambda\mu}(x) = (t_1 t_2)^{\frac{|\lambda|+|\mu|}{2}} \frac{G_{\lambda\mu}^{(t_1, t_2^{-2})}(x)}{G_{\lambda\mu}^{(t_1, t_2^{-1})}(t_1 t_2 x)}. \quad (4.4)$$

For $x = t_1 t_2$ the denominator in eq. (4.4) is finite for any Young diagrams λ and μ while the numerator vanishes whenever μ doesn't fit inside λ . Since the operator x_0^+ is an element of DIM algebra, its action on the tensor product commutes with the R -matrix. Thus, using

¹⁰More precisely $M_{\vec{Y}}^{(t_1, t_2)}(\vec{u}|p_{a,n})$ depend only on $(L-1)$ ratios of u_a .

the R -matrix one can project out all the states $|\lambda\rangle\otimes|\mu\rangle$ in the tensor product $\mathcal{F}(u)\otimes\mathcal{F}(\frac{t_1 t_2}{x})$, except those satisfying $\lambda \supset \mu$.

The effect here is similar to the fusion of spins in a spin chain. Take as an example the rational R -matrix $R(x-y) = 1 - \frac{P}{x-y}$ acting in the tensor product of two fundamental evaluation representations $\mathbb{C}^n$ at points x and y (P is the permutation operator). This R -matrix becomes a projector for the resonant values of the spectral parameters $x = y \pm 1$. For these values one can take a projection of the tensor product using $R(\pm 1)$ and, iterating this procedure, build evaluation representations with general spin.

Back to the DIM case: we notice that generalized Macdonald polynomials are *stable* in the following sense. Consider an L -tuple of Young diagrams $\vec{Y}$, in which only first m diagrams are non-empty. Then $M_{\vec{Y}}^{(t_1, t_2)}(\vec{u}|p_{a,n})$ does not depend on L as long as $L \geq m$. In particular, $M_{\vec{Y}}^{(t_1, t_2)}(\vec{u}|p_{a,n})$ in this case depends only on the first m times $p_{a,n}$, $a = 1, \dots, m$ but not on those with $m < a \leq L$. Using this stability property we can take the limit $L \rightarrow \infty$ of the polynomials $M_{\vec{Y}}^{(t_1, t_2)}(u(t_1 t_2)^{1-a}|p_{a,n})$. We denote the resulting polynomials by $M_{\pi}^{(t_1, t_2, (t_1 t_2)^{-1})}(p)$ and call them *triple* Macdonald polynomials. They depend on the plane partition π and an infinite number of families of time variables $p_{a,n}$ $a \geq 1$, though for any concrete π only a finite number of $p_{a,n}$ enters $M_{\pi}^{(t_1, t_2, (t_1 t_2)^{-1})}(p)$.

The eigen-equation for triple Macdonald polynomials reads:

$$\begin{aligned} & (1 - (t_1 t_2)^{-1}) x_0^+ | \otimes_{a \geq 1} \mathcal{F}(u(t_1 t_2)^{1-a}) M_{\pi}^{(t_1, t_2, (t_1 t_2)^{-1})}(p) \\ &= u \left[1 - (1-t_1)(1-t_2) (1-(t_1 t_2)^{-1}) \sum_{(i,j,a) \in \pi} t_1^{j-1} t_2^{i-1} (t_1 t_2)^{1-a} \right] M_{\pi}^{(t_1, t_2, (t_1 t_2)^{-1})}(p) \end{aligned} \quad (4.5)$$

Notice how the eigenvalues in the r.h.s. of eq. (4.5) can be expressed as the sum of the Bethe roots x_i from eq. (3.16). The eigenvalues are invariant with respect to the permutation of t_1, t_2 and $(t_1 t_2)^{-1}$ and the corresponding simultaneous transposition of π . Only part of this symmetry exchanging t_1 and t_2 is manifest in our description, and it would be very interesting to understand the other hidden part.

The most elementary property of Macdonald polynomials is that for $t_1 = t_2^{-1}$ they reduce to the product of Schur polynomials:

$$M_{\pi}^{(t_1, t_1^{-1}, 1)}(p) = \prod_{i=1}^{h(\pi)} s_{\pi(i)}(p_{i,n}) \quad (4.6)$$

They also satisfy the inversion relation analogous to that of ordinary Macdonald polynomials:

$$M_{\pi}^{(t_1, t_2, (t_1 t_2)^{-1})} \left(-\frac{1-t_1^n}{1-t_2^{-n}} p_{a,n} \right) = (-1)^{|\pi|} \prod_{i=1}^{h(\pi)} \frac{C_{\pi(i)}}{C'_{\pi(i)}} M_{\pi^T}^{(t_2, t_1, (t_1 t_2)^{-1})}(p) \quad (4.7)$$

where π^T denotes one of the transpositions of π , which acts on the slices of fixed height, and

$$C_Y = \prod_{(i,j) \in Y} (1 - t_1^{Y_i - j + 1} t_2^{i - Y_j^T}), \quad C'_Y = \prod_{(i,j) \in Y} (1 - t_1^{Y_i - j} t_2^{i - Y_j^T - 1}). \quad (4.8)$$

4.1 Examples

Let us give explicit expressions for triple Macdonald polynomials on the first three levels:

$$\begin{aligned}
 M_{[[1]]}^{(t_1, t_2, (t_1 t_2)^{-1})}(p) &= p_{1,1} \\
 M_{[[2]]}^{(t_1, t_2, (t_1 t_2)^{-1})}(p) &= \frac{(t_1 - 1)(t_2 + 1)p_{1,2}}{2(t_1 - t_2)} - \frac{(t_1 + 1)(t_2 - 1)p_{1,1}^2}{2(t_1 - t_2)} \\
 M_{[[1,1]]}^{(t_1, t_2, (t_1 t_2)^{-1})}(p) &= \frac{p_{1,1}^2}{2} - \frac{p_{1,2}}{2} \\
 M_{[[1],[1]]}^{(t_1, t_2, (t_1 t_2)^{-1})}(p) &= -\frac{(t_1 t_2 - 1)(t_2^2 t_1^2 + t_2 t_1^2 + t_2^2 t_1 - t_2 t_1 - 2)p_{1,1}^2}{2t_1 t_2 (t_1^2 t_2 - 1)(t_1 t_2^2 - 1)} \\
 &\quad + \frac{(t_1 - 1)(t_2 + 1)(t_1 t_2 - 1)p_{1,2}}{2(t_1^2 t_2 - 1)(t_1 t_2^2 - 1)} + p_{2,1} p_{1,1} \\
 M_{[[3]]}^{(t_1, t_2, (t_1 t_2)^{-1})}(p) &= \frac{(t_1 + 1)(t_1^2 + t_1 + 1)(t_2 - 1)^2 p_{1,1}^3}{6(t_1 - t_2)(t_1^2 - t_2)} \\
 &\quad - \frac{(t_1 - 1)(t_1^2 + t_1 + 1)(t_2 - 1)(t_2 + 1)p_{1,2} p_{1,1}}{2(t_1 - t_2)(t_1^2 - t_2)} \\
 &\quad + \frac{(t_1 - 1)^2(t_1 + 1)(t_2^2 + t_2 + 1)p_{1,3}}{3(t_1 - t_2)(t_1^2 - t_2)} \\
 M_{[[2,1]]}^{(t_1, t_2, (t_1 t_2)^{-1})}(p) &= -\frac{(t_2 - 1)(t_2 t_1 + 2t_1 + 2t_2 + 1)p_{1,1}^3}{6(t_1 - t_2^2)} + \frac{(t_2 + 1)(t_1 t_2 - 1)p_{1,2} p_{1,1}}{2(t_1 - t_2^2)} \\
 &\quad - \frac{(t_1 - 1)(t_2^2 + t_2 + 1)p_{1,3}}{3(t_1 - t_2^2)} \\
 M_{[[1,1,1]]}^{(t_1, t_2, (t_1 t_2)^{-1})}(p) &= \frac{p_{1,1}^3}{6} - \frac{1}{2} p_{1,2} p_{1,1} + \frac{p_{1,3}}{3} \\
 M_{[[2],[1]]}^{(t_1, t_2, (t_1 t_2)^{-1})}(p) &= \frac{(t_1 + 1)(t_2 - 1)(t_1 t_2 - 1)(t_2^2 t_1^3 + 2t_2 t_1^3 + t_2^2 t_1^2 - t_2 t_1^2 + t_2^2 t_1 - t_2 t_1 - 3)p_{1,1}^3}{6t_1(t_1 - t_2)t_2(t_1^3 t_2 - 1)(t_1 t_2^2 - 1)} \\
 &\quad - \frac{(t_1 + 1)(t_2 - 1)p_{2,1} p_{1,1}^2}{2(t_1 - t_2)} \\
 &\quad - \frac{(t_1 - 1)(t_2 + 1)(t_1 t_2 - 1)(t_2^2 t_1^3 + t_2^2 t_1^2 - t_2 t_1^2 + t_2^2 t_1 - t_2 t_1 - 1)p_{1,2} p_{1,1}}{2t_1(t_1 - t_2)t_2(t_1^3 t_2 - 1)(t_1 t_2^2 - 1)} \\
 &\quad + \frac{(t_1 - 1)^2(t_1 + 1)(t_1 t_2 - 1)(t_2^2 + t_2 + 1)p_{1,3}}{3(t_1 - t_2)(t_1^3 t_2 - 1)(t_1 t_2^2 - 1)} + \frac{(t_1 - 1)(t_2 + 1)p_{1,2} p_{2,1}}{2(t_1 - t_2)} \\
 M_{[[1,1],[1]]}^{(t_1, t_2, (t_1 t_2)^{-1})}(p) &= -\frac{(t_1 t_2 - 1)(t_1^2 t_2^3 + 2t_1 t_2^3 + t_1^2 t_2^2 - t_1 t_2^2 + t_1^2 t_2 - t_1 t_2 - 3)p_{1,1}^3}{6t_1 t_2(t_1^2 t_2 - 1)(t_1 t_2^3 - 1)} \\
 &\quad + \frac{(t_1 t_2 - 1)(t_1^2 t_2^3 + t_1^2 t_2^2 - t_1 t_2^2 + t_1^2 t_2 - t_1 t_2 - 1)p_{1,2} p_{1,1}}{2t_1 t_2(t_1^2 t_2 - 1)(t_1 t_2^3 - 1)} \\
 &\quad - \frac{(t_1 - 1)(t_1 t_2 - 1)(t_2^2 + t_2 + 1)p_{1,3}}{3(t_1^2 t_2 - 1)(t_1 t_2^3 - 1)} + \frac{1}{2} p_{2,1} p_{1,1}^2 - \frac{1}{2} p_{1,2} p_{2,1}
 \end{aligned}$$

5 Conclusions and outlook

We have investigated several situations in which plane partitions naturally arise. Technically, they label the eigenvectors inside the MacMahon representation of DIM algebra

and the corresponding solutions to Bethe equations. We give arguments that the natural framework to work with such representations is not the $2d$ fields, but $3d$ ones, where the (chiral) Hilbert space is spanned by plane partitions.

Let us give directions in which we would like to extend our results in the future.

Matrix models of the type we considered can be understood as Dotsenko-Fateev representations [57, 58] of certain W -algebra conformal blocks. The quantum spectral curve (3.9) corresponds to a particular generator of the W -algebra commuting with a set of screening charges, whose correlator gives the affine matrix model. This W -algebra is associated to a circular quiver with one node [59–61]. The advantage of our quantum spectral curve is that it is a finite expression, whereas the generators given in [59–61] were infinite series. One can notice that the origin of this difference lies in the difference between classes of DIM representations, since the W -algebra provides the quantization of DIM representation ring. The representation corresponding to (3.9) is what is called in [20–22] a module of “finite type”, whereas the generator given in [59–61] is associated to a Fock module, which is not of finite type. It should be possible to write down the relations of the W -algebra associated to DIM algebra *explicitly* in the basis corresponding to *finite* type modules.

Plane partitions naturally arise in the *crystal melting* models of (refined) topological vertex $C_{\lambda\mu\nu}(\mathbf{q}, \mathbf{t})$ [38–45, 62]. Moreover, the matrix model (3.1) actually reduces to the refined crystal melting partition function in the NS limit, i.e. $q \rightarrow 1$ (the remaining parameters t_1, t_2 become $\mathbf{q}$ and $\mathbf{t}^{-1}$). The only missing ingredients are the non-empty asymptotics of the plane partitions featuring in the melting crystal model. We plan to investigate this connection in the future. We would also like to point out one mysterious phenomenon along this direction. Refined topological vertex $C_{\lambda\mu\nu}(\mathbf{q}, \mathbf{t})$ can be identified with the *intertwiner* of Fock representations of DIM algebra [63]. Simultaneously, the Baxter equations obtained from the corresponding matrix model in the NS limit are relations in the *representation ring* of DIM algebra. How to make a direct connection between an intertwiner acting between states of the representations and a relation between the products of representation spaces?

Very recently a connection between *solid* (i.e. $4d$) partitions and gauge theory was put forward in [64]. It would be interesting to understand the relation between our results and those of [64].

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