Generating new $N = 2$ small black holes

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ABSTRACT: We use the exact symmetries of the $N = 2$ STU model of Sen and Vafa to classify BPS orbits in this theory. Subsequently, we construct examples of small BPS black holes in this model by solving the associated attractor equations, and use duality symmetries as a solution generating technique to construct a two-center bound state of small BPS black holes in asymptotically flat space-time.

KEYWORDS: Black Holes in String Theory, Supersymmetry and Duality

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1 Introduction

The search for a statistical understanding of black hole entropy as well as attempts to extend the solution space of black hole backgrounds in various theories of gravity has yielded rich insights into the mathematical structures that govern the non-perturbative physics in these theories. In particular, the former endeavour has uncovered modular forms that underlie the microstate organisation of certain classes of supersymmetric dyonic black holes in specific four-dimensional $N = 4$ and $N = 8$ string theories [1–19]. Especially in the $N = 4$ cases, where the counting formulae for a class of dyonic black holes can be written down exactly in terms of Siegel modular forms, studying the solution space of decalent black hole backgrounds restricted to 2-centered small black hole pairs and the lines of marginal stability across which they decay allows one to extract non-perturbative information encoded in the zeros of these modular forms. In both the quest to write down exact counting formulae as well as study the solution space of small black holes, the underlying duality symmetries of the theory not only provide useful checks on allowed solutions and modular forms, but also act as solution generating techniques in the low energy gravity theories obtained from the string theories under consideration. So far, both types of aforementioned analyses have proved elusive in the $N = 2$ case until very recently, when an approximate counting formula, in terms of Siegel modular forms, Jacobi and quasi-modular forms was written down [20] for supersymmetric dyonic black holes in the $N = 2$ STU model of Sen and Vafa (example D of [21]). This is a model that possesses exact duality symmetries. These have been used recently [22] to obtain exact results for the function F that encodes the Wilsonian action describing the coupling of vector multiplets to supergravity in the presence of an infinite set of gravitational coupling functions $\omega^{(n+1)}(S, T, U)$ ($n \geq 0$). The function F is one of the ingredients that enter in the definition of the quantum entropy function [23–25] for BPS black holes in $N = 2$ theories. In the $N = 2$ Sen-Vafa STU model, it is this quantum entropy function for large dyonic

BPS black holes that was rewritten in [20] as an approximate counting formula, in terms of modular forms.

In this note, we will attempt to harness the exact symmetries of the model to investigate the solution space of 2-centered BPS small black holes. We will begin with a classification of BPS orbits using the $\Gamma_0(2)$ duality symmetry of this $N = 2$ model, following the analysis based on $SL(2)$ given in [26]. We will then turn to the construction of small BPS black holes in this $N = 2$ model. This is done by solving the associated attractor equations. To this end, we begin by working in a regime where the gravitational coupling functions $\omega^{(n+1)}(S, T, U)$ simplify, namely, we take two of the moduli S, T, U of the model to be small, while the third modulus is taken to be large. In this regime, the small BPS black holes [27–29] that we obtain carry more than two charges, in agreement with [30]. Subsequently, with the help of duality symmetries, we construct a two-center bound state of small BPS black holes in asymptotically flat space-time, by taking the asymptotic moduli to equal the attractor values of a dyonic BPS black hole carrying the same amount of charge as the combined two-center configuration.

2 Classification of orbits

In $N = 2$ supergravity theories in four dimensions, the couplings of Abelian vector multiplets to supergravity are encoded in a holomorphic function $F(Y, \Upsilon)$ that characterizes the Wilsonian effective action [31]. The function $F(Y, \Upsilon)$ can be decomposed as $F(Y, \Upsilon) = F^{(0)}(Y) + 2i\Omega(Y, \Upsilon)$, where Υ denotes the (rescaled) lowest component of the square of the Weyl superfield, and where $Y^I \in \mathbb{C}$ denote complex scalar fields that reside in the off-shell vector multiplets of the underlying supergravity theory ($I = 0, 1, \dots, n$). These theories admit BPS black holes, which are extremal dyonic black holes carrying electric/magnetic charges (q_I, p^I) .

In this note, we will consider small BPS black holes in the $N = 2$ STU model of Sen and Vafa (example D of [21]). For this model, $I = 0, 1, 2, 3$. We introduce projective coordinates

$$S = -i \frac{Y^1}{Y^0}, \quad T = -i \frac{Y^2}{Y^0}, \quad U = -i \frac{Y^3}{Y^0}. \quad (2.1)$$

This model possesses duality symmetries, namely $\Gamma_0(2)_S \times \Gamma_0(2)_T \times \Gamma_0(2)_U$ symmetry as well as triality symmetry under exchange of S, T, U . Here, $\Gamma_0(2)$ denotes the subgroup

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{2} \right\}, \quad (2.2)$$

where $*$ can take any value in $\mathbb{Z}$.

We may assemble the eight electric/magnetic charges (q_I, p^I) into two vectors,

$$\begin{aligned} Q_e^T &= (q_0, -p^1, q_2, q_3), \\ Q_m^T &= (q_1, p^0, p^3, p^2). \end{aligned} \quad (2.3)$$

Then, under $\Gamma_0(2)_S$ -transformations, these two vectors transform as [30],

$$\begin{aligned} Q_e^T &\rightarrow a Q_e^T - b Q_m^T, \\ Q_m^T &\rightarrow d Q_m^T - c Q_e^T, \end{aligned} \quad (2.4)$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2). \quad (2.5)$$

Each of the four-component vectors Q_e, Q_m corresponds to a point in a four-dimensional lattice made out of two two-dimensional hyperbolic lattices, each with hyperbolic metric h ,

$$h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.6)$$

Introducing the four-dimensional metric $\eta = h \oplus h$, we define the charge bilinears

$$\begin{aligned} 2n &= Q_e^T \eta Q_e = 2(-q_0 p^1 + q_2 q_3), \\ 2m &= Q_m^T \eta Q_m = 2(p^0 q_1 + p^2 p^3), \\ l &= Q_e^T \eta Q_m = q_0 p^0 - q_1 p^1 + q_2 p^2 + q_3 p^3. \end{aligned} \quad (2.7)$$

These bilinears transform as a triplet under $\Gamma_0(2)_S$ [30],

$$\begin{pmatrix} n \\ m \\ l \end{pmatrix} \rightarrow \begin{pmatrix} a^2 & b^2 & -ab \\ c^2 & d^2 & -cd \\ -2ac & -2bd & ad+bc \end{pmatrix} \begin{pmatrix} n \\ m \\ l \end{pmatrix}, \quad (2.8)$$

and the $\Gamma_0(2)$ invariant norm of this vector is $\Delta \equiv 4nm - l^2$.

Given a generic charge vector Q_m , we now show that Q_m can be brought into one of the following inequivalent forms by means of sequences of $\Gamma_0(2)$ -transformations:

Proposition. Let $Q_m^T = (q_1, p^0, p^3, p^2) \in \mathbb{Z}^4$ be a charge vector with $\gcd(q_1, p^0, p^3, p^2) = 1$. Then, it can be brought into one of the following inequivalent forms by $\Gamma_0(2)$ -transformations,

$$\begin{aligned} Q_m^T &= (1, m, 0, 0), & m \in \mathbb{Z}. \\ Q_m^T &= (m, 1, 0, 0), & m \in \mathbb{Z}. \\ Q_m^T &= (0, 0, m, 1), & m \in \mathbb{Z}. \\ Q_m^T &= (0, 0, 1, m), & m \in \mathbb{Z}. \end{aligned} \quad (2.9)$$

Proof. Given an integral column vector, $V = \begin{pmatrix} a \\ b \end{pmatrix}$, with non-vanishing entries a and b , we write it as $V = g_V \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, where (α, β) are coprime, and where $g_V = \gcd(|a|, |b|)$ denotes the

greatest common divisor of $|a|$ and $|b|$. We then focus on the vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. First, consider the case when β is even, in which case α is odd. Then, by a sequence of T , T^{-1} , S and S^{-1} transformations, this vector can be brought to the form $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Here, T and S denote the two generators of $\Gamma_0(2)$, given by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad , \quad S = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} . \quad (2.10)$$

The $\Gamma_0(2)$ -transformation that brings the vector into the form $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ must be of the form

$$\begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad (2.11)$$

where the integers (γ, δ) are coprime and satisfy $\alpha\delta - \beta\gamma = 1$, with δ odd and β even. The fact that (γ, δ) are coprime is a consequence of the fact that the determinant of any $\text{SL}(2, \mathbb{Z})$ -matrix is one. The entries of the second row of the $\Gamma_0(2)$ -matrix have to be equal to $(-\beta, \alpha)$ (up to an overall minus sign). The reason is as follows. Suppose that the entries of the second row are given by coprime integers $(-\tilde{\beta}, \tilde{\alpha})$, with $\tilde{\beta}$ even (due to the $\Gamma_0(2)$ nature of the matrix), such that $\tilde{\beta}\alpha = \tilde{\alpha}\beta$. Decompose α into prime factors, $\alpha = p_1 \dots p_n$. Then, none of these prime numbers can divide β , since (α, β) are coprime. Hence, they must divide $\tilde{\alpha}$, and hence $\tilde{\alpha} = m\alpha$, with $m \in \mathbb{Z}$. It follows that $\tilde{\beta} = m\beta$. Since $(\tilde{\alpha}, \tilde{\beta})$ are coprime, it follows that $m = \pm 1$.

Next, consider the case when β is odd, in which case $\alpha = 0 \pmod{2}$. Then, by a sequence of T , T^{-1} , S and S^{-1} transformations, this vector can be brought to the form $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The $\Gamma_0(2)$ -transformation that brings the vector into this form must be of the form

$$\begin{pmatrix} \beta & -\alpha \\ \delta & -\gamma \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad (2.12)$$

where the integers (γ, δ) are coprime and satisfy $\alpha\delta - \beta\gamma = 1$, with $\delta = 0 \pmod{2}$ and γ odd. This follows by using a similar reasoning as above.

Hence, the vector V can be brought into the form

$$V = g_V \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad V = g_V \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.13)$$

by means of a $\Gamma_0(2)$ -transformation. This result will be used in what follows next.

Now, associate to the vector $Q_m^T = (q_1, p^0, p^3, p^2)$ the matrix

$$Q \equiv \begin{pmatrix} q^1 & -p^3 \\ p^2 & p^0 \end{pmatrix} , \quad (2.14)$$

whose determinant equals the charge bilinear m , $\det Q = m$. Let us denote by $\gcd(Q)$ the greatest common divisor of the four charges (q^1, p^3, p^2, p^0) . We now factor out $\gcd(Q)$ from the matrix Q , so that in the following, Q will denote the matrix (2.14) with $\gcd(Q) = 1$.

Now we operate with matrices $A \in \Gamma_0(2)_U$, $B \in \Gamma_0(2)_T$ on Q as follows, $Q \mapsto AQB^T$, to bring Q either into diagonal form or into anti-diagonal form. Note that $\gcd(Q) = 1$ is preserved by this operation. Here, the matrix A takes the form (2.5), with the entries (b, c) replaced by $(-b, -c)$, while the matrix B is of the form (2.5). We proceed to explain this construction. In the first step, we use the result given above to bring the first column of the matrix Q into the form

$$Q = \begin{pmatrix} g_V & -\tilde{p}^3 \\ 0 & \tilde{p}^0 \end{pmatrix} \quad (2.15)$$

when $\tilde{p}^2 = 0 \pmod{2}$, and into the form

$$Q = \begin{pmatrix} 0 & -\tilde{p}^3 \\ g_V & \tilde{p}^0 \end{pmatrix} \quad (2.16)$$

when $\tilde{p}^2$ is odd.

Let us first consider the case (2.15). We operate on it from the left with the $\Gamma_0(2)$ -matrix

$$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad k = \prod_i p_i, \quad p_i \in \{p | p \text{ is prime}, p | g_V, p \nmid |\tilde{p}^3|\}, \quad (2.17)$$

to obtain

$$\begin{pmatrix} g_V & -\tilde{p}^3 + k\tilde{p}^0 \\ 0 & \tilde{p}^0 \end{pmatrix}, \quad (2.18)$$

where we note that the entries of the first row are coprime, for the following reason. Let p denote a prime factor of g_V . Then, it either divides $\tilde{p}^3$, or it does not. If it divides $\tilde{p}^3$, it cannot divide k , and it also cannot divide $\tilde{p}^0$, because $\gcd(Q) = 1$, by assumption. If it does not divide $\tilde{p}^3$, it divides k . Hence, the two entries of the first row are coprime.

Next, we operate on (2.18) from the right to bring the first row into canonical form, i.e. either into the form $(1 \ 0)$ or into the form $(0 \ 1)$, depending on whether $-\tilde{p}^3 + k\tilde{p}^0$ is even or odd. This is achieved by using the transpose of the matrix given in either (2.11) or (2.12). Therefore, if $-\tilde{p}^3 + k\tilde{p}^0$ is even, we obtain

$$\begin{pmatrix} 1 & 0 \\ r & s \end{pmatrix}, \quad (2.19)$$

with some integer entries r and s , whereas if $-\tilde{p}^3 + k\tilde{p}^0$ is odd, we obtain

$$\begin{pmatrix} 0 & 1 \\ \tilde{r} & \tilde{s} \end{pmatrix}, \quad (2.20)$$

with some integer entries $\tilde{r}$ and $\tilde{s}$. Let us first consider the case (2.19). If $r = 0 \pmod{2}$, we operate on the matrix (2.19) from the left with

$$\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}, \quad (2.21)$$

to bring the charge matrix into the diagonal form

$$\begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad (2.22)$$

with $s = m = \det(Q)$, which corresponds to a charge vector $Q_m^T = (1, m, 0, 0)$. If r is odd, we operate on the matrix (2.19) from the left with the $\Gamma_0(2)$ matrix

$$\begin{pmatrix} 1 & 0 \\ -(r-1) & 1 \end{pmatrix}, \quad (2.23)$$

to obtain

$$\begin{pmatrix} 1 & 0 \\ 1 & s \end{pmatrix}. \quad (2.24)$$

If s is even, we operate on this matrix from the right with the matrix

$$\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}, \quad (2.25)$$

to obtain

$$\begin{pmatrix} 1 & -s \\ 1 & 0 \end{pmatrix}. \quad (2.26)$$

Finally, operating from the left with the matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad (2.27)$$

we obtain for the charge matrix

$$\begin{pmatrix} 0 & -s \\ 1 & 0 \end{pmatrix}, \quad (2.28)$$

with $s = m = \det(Q)$, which corresponds to a charge vector $Q_m^T = (0, 0, m, 1)$. If, on the other hand, s is odd, we operate on (2.24) with

$$\begin{pmatrix} 1 & 0 \\ s-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad (2.29)$$

with $s = m = \det(Q)$, which corresponds to a charge vector $Q_m^T = (1, m, 0, 0)$.

Next, let us consider the case (2.20). If $\tilde{s} = 0 \pmod 2$, we operate on (2.20) from the left with the $\Gamma_0(2)$ matrix

$$\begin{pmatrix} 1 & 0 \\ -\tilde{s} & 1 \end{pmatrix}, \quad (2.30)$$

to obtain

$$\begin{pmatrix} 0 & 1 \\ \tilde{r} & 0 \end{pmatrix}, \quad (2.31)$$

with $\tilde{r} = -m = -\det(Q)$, which corresponds to a charge vector $Q_m^T = (0, 0, -1, -m)$. By operating with the generator $-\mathbb{I}$ on (2.31) we bring this charge vector into the form $Q_m^T = (0, 0, 1, m)$. If $\tilde{s}$ is odd, we operate on (2.20) from the left with the $\Gamma_0(2)$ matrix

$$\begin{pmatrix} 1 & 0 \\ -(\tilde{s}-1) & 1 \end{pmatrix}, \quad (2.32)$$

to obtain

$$\begin{pmatrix} 0 & 1 \\ \tilde{r} & 1 \end{pmatrix}. \quad (2.33)$$

Now we operate from the right with

$$\begin{pmatrix} 1 & 0 \\ -\tilde{r} & 1 \end{pmatrix} \quad (2.34)$$

to obtain

$$\begin{pmatrix} -\tilde{r} & 1 \\ 0 & 1 \end{pmatrix}, \quad (2.35)$$

and subsequently from the left with

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad (2.36)$$

to obtain

$$\begin{pmatrix} -\tilde{r} & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.37)$$

with $\tilde{r} = -m = -\det(Q)$, which corresponds to a charge vector $Q_m^T = (m, 1, 0, 0)$.

Finally, let us consider the case when $\tilde{p}^2$ is odd. Then, we act on the left of (2.16) with the matrix (2.17), to obtain

$$\begin{pmatrix} kg_V & -\tilde{p}^3 + k\tilde{p}^0 \\ g_V & \tilde{p}^0 \end{pmatrix}. \quad (2.38)$$

The elements in the first line of (2.38) are coprime, for the reason given below (2.18), and therefore, following the same steps as before, we conclude that the case of odd $\tilde{p}^2$ reduces to the case of even $\tilde{p}^2$.

Hence, we have shown that when $\gcd(q_1, p^0, p^3, p^2) = 1$, we can bring Q_m into one of the following four inequivalent forms,

$$\begin{aligned} Q_m^T &= (1, m, 0, 0) , & m \in \mathbb{Z} . \\ Q_m^T &= (m, 1, 0, 0) , & m \in \mathbb{Z} . \\ Q_m^T &= (0, 0, m, 1) , & m \in \mathbb{Z} . \\ Q_m^T &= (0, 0, 1, m) , & m \in \mathbb{Z} . \end{aligned} \tag{2.39}$$

□

3 Small BPS black holes in the STU-model

BPS black holes in $N = 2$ supergravity theories carry electric/magnetic charges (q_I, p^I) .¹ These charges give rise to three charge bilinears (n, m, l) , which for the STU-model are given by (2.7). Then, one introduces the quartic charge combination $\Delta \equiv 4nm - l^2$, to define small BPS black holes, as follows.

Definition. A small BPS black hole is a BPS black hole carrying electric/magnetic charges (q_I, p^I) such that the charge combination $\Delta = 4nm - l^2$ vanishes, and such that its macroscopic (Wald) entropy $\mathcal{S}$ is, for large charges, given by $\mathcal{S} \propto \sqrt{Q^2}$. Here Q^2 denotes a linear combination of products of two charges.

Remark: At the two-derivative level, a BPS black hole has a non-vanishing horizon area proportional to $\sqrt{\Delta}$, with $\Delta > 0$. Therefore, a small BPS black hole does not exist as a solution to the equations of motion of the underlying $N = 2$ supergravity theory at the two-derivative level. For a small BPS black hole to exist, higher-curvature corrections need to be taken into account.² These are encoded in $\Omega(Y, \Upsilon)$.

For the $N = 2$ STU-model of Sen and Vafa [21], the Wilsonian function $F(Y, \Upsilon) = F^{(0)}(Y) + 2i\Omega(Y, \Upsilon)$ takes the form [22],³

$$\begin{aligned} F^{(0)}(Y) &= -\frac{Y^1 Y^2 Y^3}{Y^0} , \\ \Omega(Y, \Upsilon) &= \Upsilon \sum_{n=0}^{\infty} \left(\frac{\Upsilon}{(Y^0)^2} \right)^n \omega^{(n+1)}(S, T, U) . \end{aligned} \tag{3.1}$$

¹The charges (q_I, p^I) used in this paper are those defined in [37].

²For a recent discussion on the subtleties in defining small BPS black holes we refer to [36].

³In [22], the coupling functions $\omega^{(n+1)}(S, T, U)$ were determined by exploiting the duality symmetries of the STU-model. These were implemented by adding to $\Omega(Y, \Upsilon)$ a term $2\Upsilon\gamma \ln Y^0$. Since this term is not part of the Wilsonian effective action, we will not consider it in the following.

The gravitational coupling functions $\omega^{(n+1)}(S, T, U)$ were determined in [22], using the duality symmetries of the model, namely invariance under $\Gamma_0(2)_S \times \Gamma_0(2)_T \times \Gamma_0(2)_U$ transformations and under triality (i.e. exchanges of S, T and U). The coupling functions $\omega^{(n+1)}(S, T, U)$ are all non-vanishing when evaluated at a generic point in the S, T, U -moduli space, and are determined in terms of the first gravitational coupling function $\omega^{(1)}(S, T, U)$,

$$\omega^{(1)}(S, T, U) = \omega(S) + \omega(T) + \omega(U) , \quad (3.2)$$

where [32]

$$\omega(S) = -\frac{\gamma}{2} \ln \vartheta_2^8(S), \quad \gamma = -\frac{1}{256\pi}, \quad \vartheta_2(S) = 2\eta^2(2S)/\eta(S). \quad (3.3)$$

Note that

$$\operatorname{Re} S > 0, \quad \operatorname{Re} T > 0, \quad \operatorname{Re} U > 0, \quad (3.4)$$

in view of the fact that $\vartheta_2(\tau)$ is defined on the complex upper half-plane $\{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$.

The expressions for the coupling functions $\omega^{(n+1)}(S, T, U)$ obtained in [22] are complicated. However, as noted in [22], they do simplify when some of the moduli S, T, U are taken to large or to small values. In the following, we will choose a regime where these couplings simplify, namely the regime where $\operatorname{Re} S \gg 1$ and $\operatorname{Re} T \ll 1, \operatorname{Re} U \ll 1$. We proceed with a discussion of the $\omega^{(n+1)}(S, T, U)$ in this regime.

We begin by discussing the behaviour of $\omega(\tau)$ in the limit of large and of small $\operatorname{Im} \tau$. Using the product representation for $\vartheta_2(\tau)$, valid in the punctured disc $\{q \in \mathbb{C} \setminus \{0\} : |q| < 1\}$,

$$\vartheta_2(\tau) = 2q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n)^2, \quad q = e^{2\pi i \tau}, \quad (3.5)$$

we obtain for large $\operatorname{Im} \tau \gg 1$,

$$\ln \vartheta_2^8(\tau) = 2\pi i \tau + \mathcal{O}(q), \quad (3.6)$$

and hence, setting $S = -i\tau$,

$$\omega(S) = \gamma \pi S + \mathcal{O}(q), \quad (3.7)$$

For small $\operatorname{Im} \tau \ll 1$, we proceed as follows. Using the relations

$$\vartheta_2(\tau) = \sqrt{\frac{i}{\tau}} \vartheta_4\left(-\frac{1}{\tau}\right) \quad (3.8)$$

as well as

$$\vartheta_4(\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1/2})^2, \quad (3.9)$$

we obtain, setting $T = -i\tau$,

$$\omega(T) = 2\gamma \ln T + \mathcal{O}(e^{-2\pi/T}), \quad (3.10)$$

provided $\operatorname{Re}(1/T) \gg 1$. The latter will be enforced in the next subsection by constructing a BPS configuration that satisfies $\operatorname{Im} T = 0$.

In the regime mentioned above, the higher gravitational coupling functions $\omega^{(n+1)}(S, T, U)$, with $n \geq 1$, simplify, and become proportional to

$$\omega^{(n+1)}(S, T, U) \propto \frac{\gamma^{n+1}}{(TU)^n}. \quad (3.11)$$

This is a consequence of the behaviour $\partial_S \omega \sim 1, \partial_T \omega \sim 1/T, \partial_U \omega \sim 1/U$ in this regime, as well as of the explicit form of the $\omega^{(n+1)}(S, T, U)$, with $n \geq 1$, given in [22]. Then, the function $F(Y, \Upsilon)$ takes the approximate form,

$$F(Y, \Upsilon) = -\frac{Y^1 Y^2 Y^3}{Y^0} + 2i\Upsilon \left(\gamma \pi S + 2\gamma \ln T + 2\gamma \ln U + \sum_{n=1}^{\infty} \left(\frac{\Upsilon}{(Y^0)^2} \right)^n \frac{\gamma^{n+1} \alpha_n}{(TU)^n} \right), \quad (3.12)$$

with constants $\alpha_n \in \mathbb{C}$. Here we assume that the constant coefficients α_n are such that the infinite sum in (3.12), viewed as a power series in $z \equiv 1/[(Y^0)^2 TU]$, has a non-vanishing radius of convergence, and that $z = -2$ is in the region of convergence, cf. (3.19). However, we cannot verify this at the present stage, because of lack of knowledge about the precise form of the constant coefficients α_n .

In the near-horizon region, a BPS black hole solution is determined in terms of the charges carried by the black hole. The field Υ takes a real constant value at the horizon of the BPS black hole, and the horizon values of the scalar fields Y^I supporting the BPS black hole are determined in terms of the charges of the black hole by solving the attractor equations [33]

$$\begin{aligned} Y^I - \bar{Y}^I &= ip^I, \\ F_I - \bar{F}_I &= iq_I, \\ \Upsilon &= -64, \end{aligned} \quad (3.13)$$

where $F_I = \partial F / \partial Y^I$. The semi-classical (Wald) entropy of the BPS black hole takes the form [33]

$$\mathcal{S} = \pi (p^I F_I - q_I Y^I + 4\Upsilon \text{Im } F_{\Upsilon})|_{\text{attractor}}, \quad (3.14)$$

where $F_{\Upsilon} = \partial F / \partial \Upsilon$. We note the useful relation

$$4\pi \Upsilon \gamma = 1. \quad (3.15)$$

We now proceed to solve the attractor equations (3.13) in the regime specified above, with $F(Y, \Upsilon)$ given in (3.12). We first construct a small BPS solution with $p^0 \neq 0$ (and magnetic bilinear $m = 0$). Subsequently, we use the $\Gamma_0(2)_S$ duality symmetry of the model to construct a small BPS black hole with $p^0 = 0$ (and $m \neq 0$).

3.1 Small BPS black holes with $p^0 \neq 0$

We construct small BPS black holes with non-vanishing charges (p^0, p^1, q_2, q_3) . The magnetic bilinear m vanishes.

Small BPS black holes with charges (p^0, q_2, q_3) were considered in [30]. However, they were constructed using a non-holomorphic function $F = F^{(0)} + 2i\Omega$, with a real Ω . This

came about by implementing the duality symmetries of the model through the inclusion of non-holomorphic terms. In the present setup, however, the duality symmetries are implemented by adding a term proportional to $\ln Y^0$, as explained in footnote 3, and hence we will revisit the analysis of small black holes with charges (p^0, q_2, q_3) , also paying attention to the behaviour of the higher gravitational coupling functions $\omega^{(n+1)}(S, T, U)$.

We will work in the regime mentioned above, namely large $\text{Re } S \gg 1$, and small $\text{Re } T \ll 1$, $\text{Re } U \ll 1$. The function F is given in (3.12). Introducing $c_1 = -1/128$, we write F as

$$F = -\frac{Y^1 Y^2 Y^3}{Y^0} - i\Upsilon c_1 \left[i\frac{Y^1}{Y^0} - \frac{2}{\pi} \ln \left(-i\frac{Y^2}{Y^0} \right) - \frac{2}{\pi} \ln \left(-i\frac{Y^3}{Y^0} \right) + \sum_{n=1}^{\infty} \frac{\beta_n}{(Y^2 Y^3)^n} \right], \quad (3.16)$$

where $\beta_n = (-)^{n+1}(\Upsilon\gamma)^n \alpha_n / \pi$. On the attractor $\Upsilon = -64$, the β_n are constant, $\beta_n = 4(-1/4\pi)^{n+1} \alpha_n$, and $\Upsilon c_1 = 1/2 > 0$.

The magnetic attractor equations (3.13) imply that

$$Y^2 = \bar{Y}^2, \quad Y^3 = \bar{Y}^3, \quad (3.17)$$

as well as

$$Y^0 = \frac{1}{2}(\phi^0 + ip^0), \quad Y^1 = \frac{1}{2}(\phi^1 + ip^1). \quad (3.18)$$

Using this, the electric attractor equation $F_1 - \bar{F}_1 = 0$ gives

$$Y^2 Y^3 = \Upsilon c_1. \quad (3.19)$$

Inserting this into the electric attractor equation $F_0 - \bar{F}_0 = 0$, we get

$$\phi^0 = 0, \quad (3.20)$$

and therefore Y^0 is purely imaginary,

$$Y^0 = \frac{ip^0}{2}. \quad (3.21)$$

The electric attractor equation $F_2 - \bar{F}_2 = iq_2$ yields

$$\frac{2\phi^1 \Upsilon c_1}{p^0} + \left(\frac{4\Upsilon c_1}{\pi} + 2 \sum_{n=1}^{\infty} \frac{n \beta_n}{(\Upsilon c_1)^n} \right) = q_2 Y^2, \quad (3.22)$$

where we used (3.19). Similarly, the electric attractor equation $F_3 - \bar{F}_3 = iq_3$ yields

$$\frac{2\phi^1 \Upsilon c_1}{p^0} + \left(\frac{4\Upsilon c_1}{\pi} + 2 \sum_{n=1}^{\infty} \frac{n \beta_n}{(\Upsilon c_1)^n} \right) = q_3 Y^3 |Y^0|^2. \quad (3.23)$$

Comparing (3.22) with (3.23), we infer

$$q_2 Y^2 = q_3 Y^3. \quad (3.24)$$

Multiplying $Y^2 Y^3 = \Upsilon_{c_1}$ with $q_2 q_3$ and using (3.24) yields

$$(q_2 Y^2)^2 = q_2 q_3 \Upsilon_{c_1} > 0. \quad (3.25)$$

Hence we infer

$$q_2 q_3 > 0 \quad (3.26)$$

as well as

$$\begin{aligned} Y^2 &= \pm \frac{\sqrt{q_2 q_3 \Upsilon_{c_1}}}{q_2}, \\ Y^3 &= \pm \frac{\sqrt{q_2 q_3 \Upsilon_{c_1}}}{q_3}. \end{aligned} \quad (3.27)$$

Inserting this into (3.22) or (3.23) determines

$$\phi^1 = \frac{p^0}{2\Upsilon_{c_1}} \left(-\frac{4\Upsilon_{c_1}}{\pi} - 2 \sum_{n=1}^{\infty} \frac{n \beta_n}{(\Upsilon_{c_1})^n} \pm \sqrt{q_2 q_3 \Upsilon_{c_1}} \right). \quad (3.28)$$

Next, we display the expressions for S, T, U ,

$$\begin{aligned} S &= -\frac{1}{2\Upsilon_{c_1}} \left(-\frac{4\Upsilon_{c_1}}{\pi} - 2 \sum_{n=1}^{\infty} \frac{n \beta_n}{(\Upsilon_{c_1})^n} \pm \sqrt{q_2 q_3 \Upsilon_{c_1}} \right) - i \frac{p^1}{p^0}, \\ T &= \mp \frac{2\sqrt{q_2 q_3 \Upsilon_{c_1}}}{q_2 p^0}, \\ U &= \mp \frac{2\sqrt{q_2 q_3 \Upsilon_{c_1}}}{q_3 p^0}. \end{aligned} \quad (3.29)$$

Note that $\text{Im } T = \text{Im } U = 0$, in agreement with what was stated below (3.10).

Demanding $\text{Re } S \gg 1$, and recalling that $\Upsilon_{c_1} = 1/2 > 0$, we infer that we have to pick the negative sign in the expression for $\text{Re } S \gg 1$ given in (3.29), so that

$$\begin{aligned} \text{Re } S &= \frac{1}{2\Upsilon_{c_1}} \left(\frac{4\Upsilon_{c_1}}{\pi} + 2 \sum_{n=1}^{\infty} \frac{n \beta_n}{(\Upsilon_{c_1})^n} + \sqrt{q_2 q_3 \Upsilon_{c_1}} \right), \\ \text{Re } T &= \frac{2\sqrt{q_2 q_3 \Upsilon_{c_1}}}{q_2 p^0}, \\ \text{Re } U &= \frac{2\sqrt{q_2 q_3 \Upsilon_{c_1}}}{q_3 p^0}. \end{aligned} \quad (3.30)$$

Demanding $\text{Re } T > 0$, $\text{Re } U > 0$ requires taking

$$q_2 p^0 > 0, \quad q_3 p^0 > 0, \quad (3.31)$$

and hence, using (3.26), we infer that the charges p^0, q_2, q_3 have to be either all positive or all negative. To ensure $\text{Re } S \gg 1$, we require $|q_2| \gg 1, |q_3| \gg 1$. To ensure that $\text{Re } T$ and $\text{Re } U$ are small, we take $|q_2| \sim |q_3|$. We therefore take $|q_2| \gg 1, |q_3| \gg 1, |q_2| \sim |q_3|, |p^0| \gg 1$ to achieve $\text{Re } T \ll 1, \text{Re } U \ll 1$. Note that the solution depends crucially on the

combination $\Upsilon c_1 \neq 0$, and hence only exists due to the gravitational coupling functions encoded in Ω , as expected for a small BPS black hole.

Next, we compute the semi-classical Wald entropy (3.14) of this small BPS black hole, retaining only contributions proportional to charges. First, we note that on the solution,

$$F_1 = 0, \quad (3.32)$$

as well as

$$p^0 F_0 = -\frac{8\Upsilon c_1}{\pi}, \quad (3.33)$$

which we drop, since it is independent of charges. Next, using (3.24) as well as (3.27) (where we have to take the negative sign), we obtain

$$-q_I Y^I = -2q_2 Y^2 = 2\sqrt{q_2 q_3 \Upsilon c_1}. \quad (3.34)$$

And finally, using

$$F_\Upsilon = ic_1 \left[S - \frac{4}{\pi} \ln Y^0 \right] = ic_1 \left[S - \frac{4}{\pi} \ln |p^0| \right], \quad (3.35)$$

which is valid in the approximation that we drop terms independent of charges, we obtain

$$4\Upsilon \text{Im} F_\Upsilon = 4\Upsilon c_1 \left[\text{Re} S - \frac{4}{\pi} \ln |p^0| \right] = \sqrt{\Upsilon c_1} \sqrt{q_2 q_3} - \frac{16\Upsilon c_1}{\pi} \ln |p^0|, \quad (3.36)$$

again up to a charge independent constant. Hence, for large charges, we obtain, with $\Upsilon c_1 = 1/2$,

$$\begin{aligned} \mathcal{S} &= 4\pi \sqrt{\Upsilon c_1} \sqrt{q_2 q_3} - 16\Upsilon c_1 \ln |p^0| \\ &= 2\pi \sqrt{2 q_2 q_3} - 8 \ln |p^0|. \end{aligned} \quad (3.37)$$

In the limit where the charges p^0, q_2, q_3 are taken to be uniformly large, the first term is the leading term, and it equals $A/2$, where A denotes the area of the event horizon of the small BPS black hole. Note that the result for the BPS entropy (3.37) does not depend on the charge p^1 . The term $\ln |p^0|$ represents a correction term, whose coefficient will however change when including corrections to the semi-classical Wald entropy computed by the quantum entropy function [20, 23, 24] (cf. footnote 3), and hence we drop this term in (3.37). Then, the result (3.37) agrees with the one obtained previously in [30].

In the above, we have assumed that the coefficients α_n and β_n are such that the infinite sums in (3.16) and in (3.28) are convergent. As already mentioned below (3.12), we cannot verify this at the present stage. At any rate, these infinite sums give rise to charge independent contributions to the semi-classical entropy, which we discarded.

When dropping all charge independent terms, the solution given in (3.21) and (3.29) takes the form

$$\begin{aligned}
Y^0 &= \frac{ip^0}{2}, \\
S &= \frac{1}{2} \sqrt{\frac{q_2 q_3}{\Upsilon c_1}} - i \frac{p^1}{p^0}, \\
T &= \frac{2\sqrt{q_2 q_3 \Upsilon c_1}}{q_2 p^0}, \\
U &= \frac{2\sqrt{q_2 q_3 \Upsilon c_1}}{q_3 p^0}.
\end{aligned} \tag{3.38}$$

We will use this solution to generate a small BPS black hole solution with $p^0 = 0$ in the next subsection.

3.2 Small BPS black holes with $p^0 = 0$

Now we construct small BPS black holes with $p^0 = 0$ and non-vanishing magnetic bilinear m . These black holes will have to carry non-vanishing charges p^1, p^2, p^3 , to ensure that the real part of the moduli S, T, U is non-vanishing.

In order to obtain a solution with $p^0 = 0$, we start from the solution (3.38) and apply a duality transformation to it, as follows. We act with a $\Gamma_0(2)_S$ transformation on the charges, rewriting explicitly (2.4):

$$\begin{aligned}
p^0 &\rightarrow dp^0 + cp^1, & q_0 &\rightarrow aq_0 - bq_1, \\
p^1 &\rightarrow ap^1 + bp^0, & q_1 &\rightarrow dq_1 - cq_0, \\
p^2 &\rightarrow dp^2 - cq_3, & q_2 &\rightarrow aq_2 - bp^3, \\
p^3 &\rightarrow dp^3 - cq_2, & q_3 &\rightarrow aq_3 - bp^2.
\end{aligned} \tag{3.39}$$

The moduli Y^I transform in a similar manner, by performing the replacement $p^I \rightarrow Y^I$, $q_I \rightarrow F_I$ in (3.39). We apply the following $\Gamma_0(2)_S$ transformation to (3.38),

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{3.40}$$

To ensure that the resulting charge configuration has vanishing charge p^0 , we choose the charge p^1 in (3.38) to have the value $p^1 = p^0/2$. Then, using (3.39), we obtain a new charge configuration $(\tilde{q}_I, \tilde{p}^I)$ with $\tilde{p}^0 = 0$ and with non-vanishing charges $(\tilde{q}_2 = q_2, \tilde{q}_3 = q_3, \tilde{p}^1 = p^0/2, \tilde{p}^2 = 2q_3, \tilde{p}^3 = 2q_2)$. The magnetic charge bilinear m is non-vanishing and given by $m = \tilde{p}^2 \tilde{p}^3 = 4q_2 q_3$.

The associated moduli $\tilde{Y}^I$ are, to leading order in the charges, given by

$$\begin{aligned}\tilde{Y}^0 &= Y^0 - 2Y^1 = \frac{p^0}{2} \sqrt{\frac{q_2 q_3}{\Upsilon c_1}}, \\ \tilde{Y}^1 &= Y^1 = -\frac{p^0}{4} \sqrt{\frac{q_2 q_3}{\Upsilon c_1}} + \frac{ip^0}{4}, \\ \tilde{Y}^2 &= Y^2 + 2F_3 = iq_3, \\ \tilde{Y}^3 &= Y^3 + 2F_2 = iq_2.\end{aligned}\tag{3.41}$$

In obtaining these leading order expressions, we used $F = F^{(0)} = -Y^1 Y^2 Y^3 / Y^0$, rather than (3.16). These expressions receive subleading corrections in the charges which we have dropped. Since $\Gamma_0(2)_S$ constitutes a symmetry of the model, the configuration (3.41) describes a small BPS black hole solution, to leading order in the charges.

The constraints (3.26), (3.31) on the charges ensure that the transformed fields satisfy $\text{Re } \tilde{S}, \text{Re } \tilde{T}, \text{Re } \tilde{U} > 0$. In addition, using $|q_2|, |q_3|, |p^0| \gg 1, |q_2| \sim |q_3|$ (cf. below (3.31)), we infer from (3.41) that $\text{Re } S, \text{Re } T, \text{Re } U \ll 1$.

4 Two-center bound state of small BPS black holes

Using the small BPS black holes constructed in the previous section, we now construct a two-center bound state of small BPS black holes in asymptotically flat space-time. In doing so, we take the total charge of this configuration to be the one of a large dyonic BPS black hole. In addition, we take the asymptotic values of the moduli Y^I to equal the attractor values of the associated dyonic BPS black hole.

Let us denote the charge configuration of a BPS black hole by

$$\begin{pmatrix} Q_e^T \\ Q_m^T \end{pmatrix} = \begin{pmatrix} q_0 & -p^1 & q_2 & q_3 \\ q_1 & p^0 & p^3 & p^2 \end{pmatrix},\tag{4.1}$$

with Q_e^T, Q_m^T as in (2.3). We consider a two-center configuration of small BPS black holes with charge vectors Γ and $\tilde{\Gamma}$ given by

$$\begin{aligned}\Gamma &= \begin{pmatrix} 0 & -p^1 & q_2 & q_3 \\ 0 & 0 & 2q_2 & 2q_3 \end{pmatrix}, \\ \tilde{\Gamma} &= \begin{pmatrix} 0 & -2\tilde{q}_3 & 0 & \tilde{q}_3 \\ \tilde{q}_1 & 0 & 2\tilde{q}_1 & \tilde{p}^2 \end{pmatrix}.\end{aligned}\tag{4.2}$$

Here, the charge vector Γ describes the small BPS black hole given in (3.41), while the charge vector $\tilde{\Gamma}$ describes another small BPS black hole, obtained from the first one by triality, as follows. Starting from the small BPS black hole with non-vanishing charges (p^0, p^1, q_2, q_3) constructed in subsection 3.1, we construct a small BPS black hole with non-vanishing charges (p^0, p^2, q_1, q_3) by interchanging $I = 1 \leftrightarrow I = 2$. Then, we apply a $\Gamma_0(2)_T$ transformation to this charge configuration. Such a transformation acts as in (3.39) on the

charges, with $I = 1$ and $I = 2$ interchanged. Then, picking the transformation (3.40), we obtain $\tilde{\Gamma}$. Note that Γ and $\tilde{\Gamma}$ describe small BPS black holes whose attractor values S, T, U satisfy $\text{Re } S, \text{Re } T, \text{Re } U \ll 1$. Also, we recall that the charges of these small BPS black holes are constrained, as discussed below (3.31). For concreteness, we take

$$p^1, q_2, q_3, \tilde{q}_1, \tilde{q}_3, \tilde{p}^2 > 0. \quad (4.3)$$

The total charge $\Gamma_{\text{tot}} = \Gamma + \tilde{\Gamma}$ of the two-center configuration is

$$\Gamma_{\text{tot}} = \begin{pmatrix} 0 & -(p^1 + 2\tilde{q}_3) & q_2 & q_3 + \tilde{q}_3 \\ \tilde{q}_1 & 0 & 2(\tilde{q}_1 + q_2) & \tilde{p}^2 + 2q_3 \end{pmatrix}. \quad (4.4)$$

Next, defining the $\Gamma_0(2)_S \times \Gamma_0(2)_T \times \Gamma_0(2)_U$ invariant product

$$\langle \Gamma, \tilde{\Gamma} \rangle = Q_e^T \eta \tilde{Q}_m - \tilde{Q}_e^T \eta Q_m, \quad (4.5)$$

we obtain for the two-center configuration,

$$\langle \Gamma, \tilde{\Gamma} \rangle = q_2 \tilde{p}^2 + 2\tilde{q}_1 q_3 - \tilde{q}_1 p^1 - 2\tilde{q}_3 q_2. \quad (4.6)$$

The two-center configuration represents a two-center bound state, provided the separation distance $|x|$ between the two BPS centers is finite and positive [34, 35]. This distance is given by [34, 35]

$$|x| = \frac{1}{2} \langle \Gamma, \tilde{\Gamma} \rangle \frac{|Z(\Gamma_{\text{tot}})|}{\text{Im}(Z(\Gamma)\bar{Z}(\tilde{\Gamma}))} \Big|_{r=\infty} > 0, \quad (4.7)$$

where $Z = Z(\Gamma)$ denotes the central charge computed from the prepotential function $F^{(0)}$ at spatial infinity,

$$Z(p^I, q_I) = p^I F_I^{(0)} - q_I Y^I. \quad (4.8)$$

Here, the Y^I denote the values at spatial infinity, which are arbitrary. In the following, we choose these values to equal the attractor values computed in the near-horizon region of a large BPS black hole with total charge (4.4). We determine these values by solving the attractor equations (3.13) with $F = F^{(0)}$,

$$\begin{cases} Y^0 = \frac{\phi^0}{2} \\ Y^1 = \frac{\phi^1 + i(p^1 + 2\tilde{q}_3)}{2} \\ Y^2 = \frac{\phi^2 + i(2q_3 + \tilde{p}^2)}{2} \\ Y^3 = \frac{\phi^3 + i(2q_2 + 2\tilde{q}_1)}{2} \end{cases} \quad \begin{cases} \frac{Y^1 Y^2 Y^3}{(Y^0)^2} - \frac{\bar{Y}^1 \bar{Y}^2 \bar{Y}^3}{(\bar{Y}^0)^2} = 0 \\ -\frac{Y^2 Y^3}{(Y^0)} + \frac{\bar{Y}^2 \bar{Y}^3}{(\bar{Y}^0)} = i\tilde{q}_1 \\ -\frac{Y^1 Y^3}{(Y^0)} + \frac{\bar{Y}^1 \bar{Y}^3}{(\bar{Y}^0)} = iq_2 \\ -\frac{Y^1 Y^2}{(Y^0)} + \frac{\bar{Y}^1 \bar{Y}^2}{(\bar{Y}^0)} = i(q_3 + \tilde{q}_3) \end{cases} \quad (4.9)$$

We obtain the attractor values

$$\begin{aligned} \phi^0 &= \pm \frac{4(\tilde{q}_1 + q_2)(\tilde{p}^2 + 2q_3)(p^1 + 2\tilde{q}_3)}{\sqrt{-(p^1)^2 \tilde{q}_1^2 - (\tilde{p}^2)^2 q_2^2 - 4(\tilde{q}_1 q_3 - q_2 \tilde{q}_3)^2 + 4\tilde{p}^2 q_2(\tilde{q}_1 q_3 + 2\tilde{q}_1 q_3 + q_2 \tilde{q}_3) + 2p^1 \tilde{q}_1(2\tilde{q}_1 q_3 + q_2(\tilde{p}^2 + 4q_3 + 2\tilde{q}_3))}}, \\ S &= i \frac{-p^1 \tilde{q}_1 + \tilde{p}^2 q_2 + 2\tilde{q}_1 q_3 + 4q_2 q_3 + 2q_2 \tilde{q}_3}{4(\tilde{q}_1 + q_2)(\tilde{p}^2 + 2q_3)} + \frac{p^1 + 2\tilde{q}_3}{\phi^0}, \\ T &= i \frac{p^1 \tilde{q}_1 - \tilde{p}^2 q_2 + 2\tilde{q}_1 q_3 + 4\tilde{q}_1 \tilde{q}_3 + 2q_2 \tilde{q}_3}{4(\tilde{q}_1 + q_2)(p^1 + 2\tilde{q}_3)} + \frac{\tilde{p}^2 + 2q_3}{\phi^0}, \\ U &= i \frac{p^1 \tilde{q}_1 + \tilde{p}^2 q_2 - 2(\tilde{q}_1 q_3 + 2q_2 \tilde{q}_3)}{2(\tilde{p}^2 + 2q_3)(p^1 + 2\tilde{q}_3)} + \frac{2q_2 + 2q_3}{\phi^0}. \end{aligned} \quad (4.10)$$

This requires to take

$$-(p^1)^2 \tilde{q}_1^2 - (\tilde{p}^2)^2 q_2^2 - 4(\tilde{q}_1 q_3 - q_2 \tilde{q}_3)^2 + 4\tilde{p}^2 q_2 (\tilde{q}_1 q_3 + 2\tilde{q}_1 q_3 + q_2 \tilde{q}_3) + 2p^1 \tilde{q}_1 (2\tilde{q}_1 q_3 + q_2 (\tilde{p}^2 + 4q_3 + 2\tilde{q}_3)) > 0, \quad (4.11)$$

to ensure that the expression for ϕ^0 is well defined. In addition, we demand

$$\begin{aligned} \text{Re}S &= \frac{p^1 + 2\tilde{q}_3}{\phi^0} > 0, \\ \text{Re}T &= \frac{\tilde{p}^2 + 2q_3}{\phi^0} > 0, \\ \text{Re}U &= \frac{2q_2 + 2q_3}{\phi^0} > 0. \end{aligned} \quad (4.12)$$

Next, we evaluate (4.7), using

$$\begin{aligned} Z(\Gamma) &= p^1 F_1 + 2q_3 F_2 + 2q_2 F_3 - q_2 Y^2 - q_3 Y^3, \\ Z(\tilde{\Gamma}) &= 2\tilde{q}_3 F_1 + \tilde{p}^2 F_2 + 2\tilde{q}_1 F_3 - \tilde{q}_1 Y^1 - \tilde{q}_3 Y^3. \end{aligned} \quad (4.13)$$

For simplicity, we choose the charges (4.3) as follows,

$$\tilde{p}^2 = 2p^1 = 2p > 0, \quad q_2 = q_3 = \tilde{q}_1 = \tilde{q}_3 = q > p > 0. \quad (4.14)$$

This ensures that (4.11) is satisfied, and so is (4.12), provided we choose $\phi^0 > 0$ in (4.10). For (4.6) we obtain

$$\langle \Gamma, \tilde{\Gamma} \rangle = pq > 0. \quad (4.15)$$

And finally, we compute

$$\text{Im}(Z(\Gamma)\bar{Z}(\tilde{\Gamma})) = \frac{pq^2}{\phi^0(48q - p)}(3p^3 - 114p^2q - 1780pq^2 + 640q^3), \quad (4.16)$$

which is positive for $q > 3p$. Hence we conclude that for the choice of charges (4.14), with $q > 3p$, the separation (4.7) is finite and positive. Thus, the two-center configuration with total charge (4.4) describes a two-center bound state of small BPS black holes in asymptotically flat space-time, with the asymptotic moduli taken to equal the attractor values of a dyonic BPS black hole with total charge (4.4).

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