The structure of generic anomalous dimensions and no- π theorem for massless propagators

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ABSTRACT: Extending an argument of [1] for the case of 5-loop massless propagators we prove a host of new exact model-independent relations between contributions proportional to odd and even zetas in generic $\overline{\text{MS}}$ anomalous dimensions as well as in generic massless correlators. In particular, we find a new remarkable connection between coefficients in front of ζ_3 and ζ_4 in the 4-loop and 5-loop contributions to the QCD β -function respectively. It leads to a natural explanation of a simple mechanics behind mysterious cancellations of the π -dependent terms in one-scale Renormalization Group (RG) invariant Euclidean quantities recently discovered in [2]. We give a proof of this no- π theorem for a general case of (not necessarily scheme-independent) one-scale massless correlators. All π -dependent terms in the **six-loop** coefficient of an anomalous dimension (or a β -function) are shown to be explicitly expressible in terms of lower order coefficients for a general one-charge theory. For the case of a scalar $O(n)$ ϕ^4 theory all our predictions for π -dependent terms in 6-loop anomalous dimensions are in full agreement with recent results of [3–5].

KEYWORDS: Perturbative QCD, Renormalization Group

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1 Introduction

The seminal calculation of the Adler function at order α_s^3 [6] demonstrated for the first time a mysterious complete cancellation of all contributions proportional to ζ_4 (abounding in separate diagrams) while odd zetas terms (that is those proportional to ζ_3 and ζ_5) survived and appeared in the final result. Literally the authors of [6] wrote: “*We would like to stress the cancellations of ζ_4 in the final results for $R(s)$. It is very interesting to find the origin of the cancellation of ζ_4 in the physical quantity.*”

Since then it has been noted many times that *all* one scale physical quantities are indeed free from even zetas at order α_s^4 (like corrections to the Bjorken (polarized) DIS sum rule) and *some* of them — like the Adler function — even at next, in fact, five-loop, $\alpha_e\alpha_s^4$ order [7]. By one-scale physical quantities we mean here scale invariant massless correlators (or their proper combinations) in the Euclidean region.¹

¹We do not consider well known terms proportional to various powers of π which are routinely generated during the procedure of analytical continuation to the Minkowskian (negative) values of the momentum transfer Q^2 .

The appearance of ζ_4 in a one-scale physical quantity has been demonstrated in [8] for the case of the 5-loop scalar correlator.

Very recently M. Jamin and R. Miravittas have discovered that after a transition to a new so-called C-scheme [9] all terms proportional to even zetas (ζ_4 and ζ_6 in the cases under consideration) do disappear in the 5-loop scalar correlator as well as in the 5-loop gluon correlator [2] (both enter the hadronic decays of the Higgs boson [10, 11]). They also suggested that the absence of even zetas after transition to the C-scheme is an universal feature of *all* $\mathcal{O}(\alpha_s^5)$ physical quantities² — “no- π^2 conjecture”.

Later many more confirmations of the conjecture have been discussed in [12–14].

The most interesting feature of the conjecture is a direct universal connection between ζ_4 contribution to physical quantities at order α_s^5 and the ζ_4 -term in the QCD β -function which appears first at order α_s^5 [15–17]. To really appreciate the mystery behind these cancellations we want to remind the reader of the following simple facts:

1. a bare physical (massless!) quantity depends on the bare coupling constant, α_s^B ;
2. its renormalization is done with the replacement $\alpha_s^B = \mu^{2\epsilon} Z_a \alpha_s$;
3. the charge Renormalization Constant (RC) Z_a depends on the five-loop coefficient in the β -function — β_5 — starting from the *fifth* order, α_s^5 ;
4. as a result the renormalized physical quantity starts to “feel” β_5 only at the astonishingly large *sixth* order in α_s ;
5. for the case of the scalar correlator the contribution of order α_s^6 corresponds to the fabulously large 7-loop level.

In the present work we suggest a natural explanation of the mystery above as well as a rigorous proof of the no- π^2 conjecture (at least for the 5-loop level), by demonstrating that the ζ_4 term in the β -function is, in fact, not independent but must meet a simple factorization formula:

$$\beta_5^{\zeta_4} = \frac{9}{8} \beta_1 \beta_4^{\zeta_3}, \quad (1.1)$$

where the β -function is defined as:

$$\mu^2 \frac{d}{d\mu^2} a(\mu) = a \beta(a) = a \sum_{i \geq 1} \beta_i a^i, \quad a = \frac{\alpha_s}{4\pi} = \frac{g_s^2}{16\pi^2} \quad (1.2)$$

and the upper-script ζ_i means

$$F^{\zeta_i} = \lim_{\zeta_i \rightarrow 0} \frac{\partial}{\partial \zeta_i} F \quad \text{and (for future reference)} \quad F^{\zeta_i \zeta_j} = \lim_{\zeta_i \rightarrow 0} \frac{\partial}{\partial \zeta_i} F^{\zeta_j}. \quad (1.3)$$

The validity of the factorization formula will be proven *without* any loop calculation and (almost) without any information on the β -function. The proof is based solely on the

²Please, pay attention that the counting of powers of α_s is definition depended, see more details in the next section.

general considerations about the structure of the 4-loop massless propagators discussed in section 7 of [1] and the fact that the 3-loop QCD β -function is free from any zetas [18, 19].

Note that factorization in (1.1) is not trivial even for QCD with the SU(3) gauge group (the coefficient β_4 was first computed in [20] and confirmed in [21]):

$$\begin{array}{cc} \beta_1 & (\partial/\partial\zeta_3)\beta_4 \\ \frac{\partial}{\partial\zeta_4}\beta_5 = \frac{9}{8}\left(\frac{2}{3}n_f - 11\right)\left(-\frac{6472}{81}n_f^2 + \frac{6508}{27}n_f - 3564\right) \end{array}$$

while for a general case it takes the form:

$$\begin{array}{cc} \beta_1 & (\partial/\partial\zeta_3)\beta_4 \\ \frac{\partial}{\partial\zeta_4}\beta_5 = \frac{9}{8}\left(\frac{4}{3}n_f T_F - \frac{11}{3}C_A\right)\left(\frac{44}{9}C_A^4 - \frac{136}{3}C_A^3 n_f T_F \right. \\ \quad + \frac{656}{9}C_A^2 C_F n_f T_F - \frac{224}{9}C_A^2 n_f^2 T_F^2 - \frac{352}{9}C_A C_F^2 n_f T_F \\ \quad - \frac{448}{9}C_A C_F n_f^2 T_F^2 + \frac{704}{9}C_F^2 n_f^2 T_F^2 - \frac{704}{3}\frac{d_A^{abcd} d_A^{abcd}}{N_A} \\ \quad \left. + \frac{1664}{3}\frac{d_F^{abcd} d_A^{abcd}}{N_A} n_f - \frac{512}{3}\frac{d_F^{abcd} d_F^{abcd}}{N_A} n_f^2\right). \end{array} \quad (1.4)$$

We will derive a variety of exact relations similar to (1.1) valid for arbitrary $\overline{\text{MS}}$ [22, 23] anomalous dimensions (ADs). We also will find new identities which relate contributions proportional to even zetas in (not necessarily physical) massless correlators and terms proportional to odd zetas in the corresponding ADs.

In addition we suggest a simple proof of the no- π theorem valid for any number of loops and every massless correlator (including those with non-zero AD) provided the participating Feynman integrals meet a simple condition to be specified later. The condition is shown to be fulfilled for all currently known examples of the theorem. We also construct explicit expressions for the π -dependent contributions to 6-loop ADs and β -function valid for a generic one-charge theory.

If not otherwise stated we will assume the so-called G -scheme for renormalization [24]. The scheme is natural for massless propagators. All ADs, β -functions and Z -factors are identical in $\overline{\text{MS}}$ - and G -schemes. For (finite) renormalized functions there exists a simple conversion rule. Namely, in order to switch from an G -renormalized quantity to the one in the $\overline{\text{MS}}$ -scheme one should make the following replacement in the former: $\ln \mu^2 \rightarrow \ln \mu^2 + 2$ (μ is the renormalization scale, the limit of $\epsilon \rightarrow 0$ is understood).

2 Setup

Let F be any p -function, that is a Green function or a 2-point correlator (or even some combination thereof), expressible in terms of massless propagator-like Feynman integrals (to be named p -integrals below). As an example one could have in mind any of the 11

objects computed and renormalized at the 4-loop level in [25]. F may not necessarily be a gauge invariant object, if there is a gauge dependence we will always assume the Landau gauge fixing condition below. With such convention at hand we can and will effectively ignore the gauge parameter dependence (if any) in all objects under consideration here. The (renormalized) p-function $F \equiv F_R$ can be naturally represented as

$$F_n(a, \ell_\mu) = 1 + \sum_{1 \leq i \leq n}^{0 \leq j \leq i} g_{i,j} \ell_\mu^j a^i, \quad (2.1)$$

where $a = \frac{\alpha_s(\mu)}{4\pi}$, $\ell_\mu = \ln \frac{\mu^2}{Q^2}$ and Q is an (Euclidean) external momentum. The integer n stands for the (maximal) power of α_s appearing in the Feynman diagrams contributing to F_n . The F without n will stand as a shortcut for a formal series F_∞ . In terms of bare quantities F is written as

$$F = Z F_B(a_B, \ell_\mu), \quad Z = 1 + \sum_{i \geq 1}^{1 \leq j \leq i} Z_{i,j} \frac{a^i}{\epsilon^j}, \quad (2.2)$$

with the bare coupling constant and the corresponding RC being

$$a_B = \mu^{2\epsilon} Z_a a, \quad Z_a = 1 + \sum_{i \geq 1}^{1 \leq j \leq i} (Z_a)_{i,j} \frac{a^i}{\epsilon^j}. \quad (2.3)$$

The evolution equation for F reads:

$$\left(\frac{\partial}{\partial \ell_\mu} + \beta a \frac{\partial}{\partial a} \right) F = \gamma F, \quad (2.4)$$

with the AD

$$\gamma(a) = \sum_{i \geq 1} \gamma_i a^i, \quad \gamma_i = -i Z_{i,1}. \quad (2.5)$$

The coefficients of the β -function β_i are related to Z_a in the standard way:

$$\beta_i = i (Z_a)_{i,1}. \quad (2.6)$$

Note that if the AD γ happens to be equal β for a p-function F (an explicit example will be considered later) then the latter should be renormalized with $Z \equiv (Z_a)^{-1}$ (due to different signs in the above expressions for γ_i and β_i).

Currently our ability to compute p-functions in QCD is limited to 4 loops and the corresponding ADs to 5 loops (see, e.g. [26]). If γ is not vanishing then there are two ways to construct a scale-invariant version of F .

Let us consider them in turn. The first one is to construct an object:

$$\hat{F}^{si}(a, \ell_\mu)_{n+1} = (a)^{\frac{-\gamma_1}{\beta_1}} \exp \left\{ - \int_0^a \frac{dx}{x} \left[\frac{\gamma(x)}{\beta(x)} - \frac{\gamma_1}{\beta_1} \right] \right\} F_n(a, \ell_\mu) \quad (2.7)$$

$$= (a)^{\frac{-\gamma_1}{\beta_1}} \left(1 + \sum_{1 \leq i \leq n}^{0 \leq j \leq i} \hat{d}_{i,j} a^i \ell_\mu^j \right), \quad (2.8)$$

where $a = a(\mu)$. Note that $\hat{F}_{n+1}^{si}$ is built from F_n and the $(n+1)$ -loop AD γ .

We will be interested in even zetas appearing in the coefficients $\hat{d}_i \equiv \hat{d}_{i0}$. It is a well-know fact (see discussion in section 6) that even zeta may appear only starting from 3 loops for correlators and from 4 loops for ADs. In addition β_4 does not depend on ζ_4 (this is a peculiar property of β_4 in QCD). An account of this directly leads to the following representations for the (potentially) π -dependent contributions to $\hat{d}_3$ and $\hat{d}_4$:

$$\hat{d}_3 \stackrel{\pi}{=} -\frac{1}{3\beta_1}(\gamma_4 - 3\beta_1 g_3), \quad (2.9)$$

$$\hat{d}_4 \stackrel{\pi}{=} -\frac{1}{4\beta_1} \left(\gamma_5 - \frac{\gamma_1 \beta_5}{\beta_1} - \frac{\gamma_4 \beta_2}{\beta_1} + \frac{4\gamma_4 g_1}{3} - 4\beta_1 g_4 \right), \quad (2.10)$$

where the sign $\stackrel{\pi}{=}$ means equality modulo π -independent terms. Note that in eq. (2.10) we have used the fact (proven in [1] and discussed later³) that $\hat{d}_3 \stackrel{\pi}{=} 0$.

Unlike the 3-loop level, the 4-loop coefficient $\hat{d}_4$ does provide no-zero contribution proportional to ζ_4 in every available example with $\gamma \neq 0$, but terms with ζ_6 and $\zeta_3 \zeta_4$ never show up. It is of interest that the second combination which does appear in many master integrals [1, 27] never contributes to *all* known 4-loop renormalized correlators (not necessarily scale-invariant) and 5-loop ADs. The reason for it will be clarified in section 6.

The remarkable observation first made in [2] can, in our setup, be precisely formulated as follows:

$$\hat{d}_4 \stackrel{\pi}{=} \frac{\gamma_1 \beta_5}{3\beta_1^2} \quad (2.11)$$

or, in an explicit form,

$$\hat{d}_4 - \frac{\gamma_1 \beta_5}{3\beta_1^2} \stackrel{\pi}{=} -\frac{1}{4\beta_1} \left(\frac{\gamma_1 \beta_5}{3\beta_1} + \gamma_5 - \frac{\gamma_4 \beta_2}{\beta_1} + \frac{4\gamma_4 g_1}{3} - 4\beta_1 g_4 \right) \stackrel{\pi}{=} 0. \quad (2.12)$$

We will prove (2.12) in section 3.

If we change the renormalization scheme as follows:

$$a = \bar{a} (1 + c_1 \bar{a} + c_2 \bar{a}^2 + c_3 \bar{a}^3 + c_4 \bar{a}^4), \quad (2.13)$$

with c_1, c_2 and c_3 all free from even zetas and with

$$c_4 \stackrel{\pi}{=} \frac{1}{3} \frac{\beta_5}{\beta_1} \stackrel{\pi}{=} \frac{3}{8} \beta_4^{\zeta_3} \zeta_4, \quad (2.14)$$

then the function $\hat{F}_5^{si}(\bar{a}, \ell_\mu)$ as well the (5-loop) β -function $\bar{\beta}(\bar{a})$ both loose any dependence on even zetas. We will call the class of renormalization schemes for which

$$\bar{\beta}(\bar{a}) \stackrel{\pi}{=} \mathcal{O}(\bar{a}^{L+1}) \quad \text{and} \quad c_1 \stackrel{\pi}{=} 0 \quad (2.15)$$

as π -independent to L loops schemes. Note that any two π -independent (to L loops) schemes are related by transformation (2.13) with all coefficients $c_1 \dots c_{L-1}$ being π -free.

³See eq. (3.3).

An example of such a scheme has been recently considered in [9] under the name “C-scheme”. In this scheme the β -function is π -independent in all orders. In particular,

$$c_4 = \frac{1}{3} \frac{\beta_5}{\beta_1}. \quad (2.16)$$

The second way to construct a scale-invariant object from F is also well-known.⁴

$$F_{n+1}^{si}(a, \ell_\mu) = \frac{\partial}{\partial \ell_\mu} (\ln F)_{n+1} \equiv \left(\frac{(\gamma(a) - \beta(a)a \frac{\partial}{\partial a}) F_n}{F_n} \right)_{n+1}. \quad (2.17)$$

Note that $F_{n+1}^{si}(a, \ell_\mu)$ starts from the first power of the coupling constant a and is formally composed from $\mathcal{O}(\alpha_s^{n+1})$ Feynman diagrams. In the same time it can be completely restored from F_n and the $(n+1)$ -loop AD γ . This dual nature of F_{n+1}^{si} plays a vital role in all our considerations in the next two sections.

It is convenient to write

$$F_{n+1}^{si}(a, \ell_\mu) = a \left(\gamma_1 + \sum_{1 \leq i \leq n} d_{i,j} \ell_\mu^j a^i \right). \quad (2.18)$$

The coefficients $d_i = d_{i0}$ meet, obviously, the equations (parallel to (2.9) and (2.10)):

$$d_3 \stackrel{\pi}{=} \gamma_4 - 3\beta_1 g_3, \quad (2.19)$$

$$d_4 \stackrel{\pi}{=} \gamma_5 - \frac{\gamma_4 \beta_2}{\beta_1} + \frac{4\gamma_4 g_1}{3} - 4\beta_1 g_4. \quad (2.20)$$

Note, that up to a common factor $\hat{d}_3$ is equal to d_3 ; the same is true for the pair $\hat{d}_4$ and d_4 except for a term proportional to β_5 which *does not appear at all* in d_n if $n \leq 4$. A transition to a π -independent renormalization scheme according to (2.13) and (2.14) leads to

$$F_5^{si}(\bar{a}, \ell_\mu) = a \left(\gamma_1 + \sum_{1 \leq i \leq 4} \bar{d}_{i,j} \ell_\mu^j \bar{a}^i \right), \quad (2.21)$$

with

$$\bar{d}_2 \stackrel{\pi}{=} d_2, \quad \bar{d}_3 \stackrel{\pi}{=} d_3, \quad \bar{d}_4 \stackrel{\pi}{=} d_4 + \frac{\gamma_1 \beta_5}{3\beta_1} \stackrel{\pi}{=} \frac{\gamma_1 \beta_5}{3\beta_1} + \gamma_5 - \frac{\gamma_4 \beta_2}{\beta_1} + \frac{4\gamma_4 g_1}{3} - 4\beta_1 g_4. \quad (2.22)$$

Eqs. (2.12) and (2.22) explicitly demonstrate that if, for a given p-function F_4 , its scale invariant version $F_5^{si}(a, \ell_\mu)$ is π -free in a π -independent renormalization scheme then the function $\hat{F}_5^{si}(a, \ell_\mu)$ is also π -free, and vice versa.

Our main aim is to really understand and, consequently, to prove the equality (which happens to be true in all known so far examples)

$$d_4 + \frac{1}{3} \frac{\gamma_1 \beta_5}{\beta_1} \stackrel{\pi}{=} 0. \quad (2.23)$$

⁴The trick was first applied for constructing a π -free version of a 2-point correlator in [28]. The same quantity as defined in (2.17) was discussed in [29] under the name of the “Renormalization Scheme invariant anomalous dimension”. In DIS similar objects are called “physical anomalous dimensions”, see, e.g. [12].

The equality is naturally separated into two pieces

$$d_4 + \frac{1}{3} \frac{\gamma_1 \beta_5^{\zeta_4}}{\beta_1} \zeta_4 \stackrel{\pi^4}{=} 0, \quad (2.24)$$

$$d_4 \stackrel{\pi^6}{=} 0. \quad (2.25)$$

Assuming the factorization formula (1.1) (which will be derived in section 5), we will prove in what follows that

$$d_4 + \frac{3}{8} \gamma_1 \beta_4^{\zeta_3} \zeta_4 \stackrel{\pi^4}{=} 0, \quad (2.26)$$

$$d_4 \stackrel{\pi^6}{=} 0. \quad (2.27)$$

3 Interplay between ζ_3 , ζ_4 : generic case

In general representation (2.17) is very suitable to discuss the interplay of various transcendental contributions to F , β and γ . This is because, unlike (2.8), $F_n^{\text{si}}(a, \ell_\mu)$ is a combination of a finite number of Feynman diagrams with a very simple renormalization mode: $F^{\text{si}}(a, \ell_\mu) = F_B^{\text{si}}(a_B = Z_a \mu^{2\epsilon} a, \ell_\mu)$ (as we discussed already in the Introduction).

The (bare) function $F_{5,B}^{\text{si}}$ is composed from 1,2,3,4 and 5-loop bare p-integrals.⁵ We will call a (bare) L -loop p-integral $F(Q^2, \epsilon)$ π -safe if the π -dependence of its pole in ϵ and constant part can be completely absorbed into the properly defined “hatted” odd zetas. The first observation of a non-trivial class of π -safe p-integrals — all 3-loop ones — was made in [30]. An extension of the observation on the class of all 4-loop p-integrals was performed in [1]. Here it was shown that, given an arbitrary 4-loop p-integral, its pole in ϵ and constant part depend on even zetas *only* via the following combinations:

$$\hat{\zeta}_3 := \zeta_3 + \frac{3\epsilon}{2} \zeta_4 - \frac{5\epsilon^3}{2} \zeta_6, \quad \hat{\zeta}_5 := \zeta_5 + \frac{5\epsilon}{2} \zeta_6 \quad \text{and} \quad \hat{\zeta}_7 := \zeta_7. \quad (3.1)$$

Note that the π -safeness of, say, all L -loop p-integrals ensures that for any L' -loop p-integral F with $L' < L$ the L -loop combination $\frac{1}{\epsilon^{L-L'}} F$ is a π -safe one with the same choice of the hatted zetas.

We will proceed, assuming for the moment that all p-integrals contributing to F_5^{si} are π -safe. The assumption is discussed and eventually proved in section 7 (at least for the particular classes of p-integrals which we encounter in the present paper).

Any renormalized p-function $F_{5,R}$ can, by definition, be presented as

$$F_{5,R} = F_{5,B} + \sum_i Z_i p_i, \quad (3.2)$$

where Z_i are some UV counterterms and p_i are p-integrals. Let us assume for a moment that all Z_i in (3.2) are π -free and rewrite all (including those contributing to F_n^B) p-integrals

⁵In some cases (correlators of composite operators) also 6-loop p-integrals might contribute (see a discussion in section 7).

in (3.2) in terms of hatted zetas. Then the r.h.s. of (3.2) will depend on hatted odd zetas only (in the limit of $\epsilon \rightarrow 0$). The finiteness of $F_{5,R}$ would ensure then the absence of any π -dependent terms in it in the $\epsilon \rightarrow 0$ limit. Clearly, if some of Z_i do contain odd and/or even zetas then there should exist some constraints relating contributions with odd and even zetas to the p-function and depending on a precise pattern of hiding π -dependent terms inside hatted odd zetas. These constraints will be the main tool in our considerations.

Let us consider first F_4^{si} . It is renormalized with Z_a in three loops which is free from any zetas for QCD. As a result, F_4^{si} should be free from even zetas which means the fulfillment of the identity

$$d_3 \stackrel{\pi}{=} \gamma_4 - 3\beta_1 g_3 \stackrel{\pi}{=} 0. \quad (3.3)$$

Now let us turn to the renormalization of the very function F_3 . Its renormalization mode includes Z_a (in 2 loops, free from any zetas) and Z_γ (in 3 loops). Note that $(Z_\gamma)_3$ depends generically on the combination ζ_3/ϵ . By definition of the AD

$$(Z_\gamma)_{3,1}^{\zeta_3} = -\frac{1}{3}\gamma_3^{\zeta_3}. \quad (3.4)$$

Let us write the renormalized function F_3 as follows⁶

$$F_3 = (Z_\gamma)_{3,1}^{\zeta_3} \zeta_3 \frac{a^3}{\epsilon} + \sum_i z_i(\epsilon) p_i a^3 + \text{terms of order } a^2 \text{ and lower}, \quad (3.5)$$

where p_i are some bare p-integrals and the coefficients z_i are some polynomials in $1/\epsilon$ (the constant term is allowed!) with *rational* coefficients. Assume now that the p-integrals in the above sum are expressed via hatted zetas. The coefficient in front of $\hat{\zeta}_3$ should then be equal

$$-(Z_\gamma)_{3,1}^{\zeta_3} \frac{a^3}{\epsilon} \equiv \gamma_3^{\zeta_3} \frac{a^3}{3\epsilon}.$$

Expressing back $\hat{\zeta}_3$ via normal zetas we arrive at the identity:

$$g_3^{\zeta_4} \equiv \frac{1}{2}\gamma_3^{\zeta_3}. \quad (3.6)$$

Now a look on (3.3) immediately leads to relations valid for any AD γ

$$\gamma_4^{\zeta_4} = \frac{3}{2}\beta_1\gamma_3^{\zeta_3}. \quad (3.7)$$

The fact that $\beta_3 \stackrel{\pi}{=} 0$ is a particular feature of QCD. The above treatment can be easily extended for a general case of $\beta_3^{\zeta_3} \neq 0$. The corresponding generalizations of eqs. (3.3) and (3.7) read

$$d_3 \stackrel{\pi}{=} \gamma_4 - 3\beta_1 g_3 \stackrel{\pi}{=} -\frac{1}{2}\beta_3^{\zeta_3} \gamma_1 \zeta_4, \quad (3.8)$$

$$\gamma_4^{\zeta_4} = \frac{3}{2}\beta_1\gamma_3^{\zeta_3} - \frac{1}{2}\beta_3^{\zeta_3} \gamma_1. \quad (3.9)$$

For a particular case of $\gamma \equiv \beta$ we arrive at

$$\beta_4^{\zeta_4} = \beta_1\beta_3^{\zeta_3}. \quad (3.10)$$

⁶For simplicity we set $\mu = 1$ in all formulas below.

4 Interplay between ζ_5 , ζ_6 : QCD case

Let us now add one more loop and consider F_5^{si} . It is renormalized with the RC Z_a (taken in the 4-loop approximation):

$$F_5^{\text{si}} = \gamma_1 (Z_a)_{4,1}^{\zeta_3} \left(\hat{\zeta}_3 - \frac{3\epsilon}{2} \zeta_4 \right) \frac{a^5}{\epsilon} + \underbrace{\sum_{i,1 \leq j \leq 5} z_{i,j}(\epsilon) \hat{p}_{i,j} a^j}_{\text{underlined}} + \mathcal{O}(\epsilon^0), \quad (4.1)$$

where $\hat{p}_{i,j}$ stands for p -integrals expressed in terms of hatted (odd!) zetas and $z_{i,j}(\epsilon)$ are completely rational polynomials in $1/\epsilon$. The finiteness of (4.1) means that the term proportional to $\hat{\zeta}_5$ in the underlined sum in (4.1) should have a finite coefficient. Thus, the whole p -function $F_5^{\text{si}}(a, \ell_\mu)$ should be free from ζ_6 in the limit $\epsilon \rightarrow 0$. This could be precisely written as $d_4 \stackrel{\pi^6}{=} 0$ or, equivalently,

$$\gamma_5 - \frac{\gamma_4 \beta_2}{\beta_1} + \frac{4\gamma_4 g_1}{3} - 4\beta_1 g_4 \stackrel{\pi^6}{=} \gamma_5 - 4\beta_1 g_4 \stackrel{\pi^6}{=} 0, \quad (4.2)$$

where we have used the fact that $\gamma_4 \stackrel{\pi^6}{=} 0$ (see table 1). On the other hand, the pole part of the coefficient in front of $\hat{\zeta}_3$ in the underlined sum must be equal to $-\gamma_1 (Z_a)_{4,1}^{\zeta_3}/\epsilon$. This directly leads to the identity

$$d_4 \stackrel{\pi}{=} -\gamma_1 \frac{3}{8} \beta_4^{\zeta_3} \zeta_4. \quad (4.3)$$

Thus, we have proved the no- π theorem: for every (QCD!) p -function F_4 the corresponding scheme-invariant function F_5^{si} is π -independent. We will rederive it for generic functions F_{n+1}^{si} in more systematic way in section 9.

Let us write now the renormalized version of F_4 in a form explicitly showing all possible zetas coming from RCs. The form reads (an upper index $n\ell$ in $\hat{p}_i^{n\ell}$ means that the p -integral p_i has n loops)

$$F_4 \stackrel{\pi^6}{=} a^4 \left[Z_{3,1}^{\zeta_3} \frac{\zeta_3}{\epsilon} \sum_i \hat{p}_i^{1\ell} + Z_{4,2}^{\zeta_3} \frac{\zeta_3}{\epsilon^2} + Z_{4,1}^{\zeta_3} \frac{\zeta_3}{\epsilon} + Z_{4,1}^{\zeta_4} \frac{\zeta_4}{\epsilon} + Z_{4,1}^{\zeta_5} \frac{\zeta_5}{\epsilon} + \sum_i \hat{p}_i^{4\ell} \right], \quad (4.4)$$

where we have discarded all terms obviously independent of ζ_6 . The finiteness of coefficient in front of ζ_5 in (4.4) leads to the equality

$$\left(\sum_i \hat{p}_i^{4\ell} \right)^{\zeta_5} = -Z_{4,1}^{\zeta_5}/\epsilon. \quad (4.5)$$

Thus,

$$g_4 \stackrel{\pi^6}{=} -\frac{5}{2} Z_{4,1}^{\zeta_5} \zeta_6 \stackrel{\pi^6}{=} \frac{5}{8} \gamma_4^{\zeta_5} \zeta_6 \quad (4.6)$$

and, finally, after the use of (4.2)

$$\gamma_5^{\zeta_6} = \frac{5}{2} \gamma_4^{\zeta_5} \beta_1. \quad (4.7)$$

All derivations of this section have been done for the QCD case, that is assuming $\beta_3^{\zeta_3} = 0$ and $\beta_4^{\zeta_5} = 0$. A generalization to a generic one-charge theory can be also done along the same lines. But we postpone this to sections 9, 10 and 11 where we will suggest an universal treatment.

5 The connection between ζ_3 and ζ_4 terms in the QCD β -function

As is well-known the use of the Landau gauge implies that

$$Z_a = (Z^{\text{gh}})^{-2} (Z^{\text{gl}})^{-1} \quad \text{and} \quad \beta = -2\gamma^{\text{gh}} - \gamma^{\text{gl}}, \quad (5.1)$$

where γ^{gh} and γ^{gl} stand for the ghost and gluon wave function ADs respectively [31, 32].

Let us consider the following combination of the ghost and gluon self-energies

$$F(a, \ell_\mu) = (1 + \Pi^{\text{gh}})^2 (1 + \Pi^{\text{gl}}) = (Z_a)^{-1} (1 + \Pi_B^{\text{gh}})^2 (1 + \Pi_B^{\text{gl}}). \quad (5.2)$$

This is a p-function with the evolution equation

$$\left(\frac{\partial}{\partial L} + \beta a \frac{\partial}{\partial a} \right) F = \beta F. \quad (5.3)$$

The β -function coefficient β_3 is a rational number while β_4 does depend on ζ_3 . This is the only specific information on the β -function which is needed for our analysis of the transcendental structure of F_4 and the coefficients β_4 and β_5 below. Indeed, the following is true:

Lemma. If F is defined by eq. (5.2) and $\beta_3^{\zeta_3} = \beta_4^{\zeta_5} = 0$ then

1. The coefficients $g_i \stackrel{\pi}{=} 0$ if $i \leq 3$.
2. The coefficient $\beta_4 \stackrel{\pi}{=} 0$.
3. The coefficients $(Z_a)_{4,j}$ are rational for $j \geq 2$.
4. The coefficient $g_4 \stackrel{\pi^4}{=} \frac{3}{8} \beta_4^{\zeta_3} \zeta_4$.
5. The coefficient $\beta_5^{\zeta_6} = 0$.

The transcendental contributions to β_4 and β_5 are constrained by the following relation:

$$\beta_5^{\zeta_4} = \frac{9}{8} \beta_1 \beta_4^{\zeta_3}. \quad (5.4)$$

Proof.

1. Indeed, the p-function F_3 is renormalized with the 3-loop RC Z_a and, thus, can not depend on even zetas (see the discussion after (3.2) and section 7.1 of [1]).
2. This is a direct consequence of the fact that $\beta_4 \equiv \gamma_4$ and $\gamma_4 \stackrel{\pi}{=} 3\beta_1 g_3$.
3. This is an obvious consequence of the well-known fact that higher poles in a RC Z at loop order $(L+1)$ are completely fixed by simple pole terms of Z and Z_a with loop order not exceeding L (so-called 't Hooft's constraints [22]). The constraints read

$$a \frac{\partial Z_{*,k+1}}{\partial a} = \left[\beta a \frac{\partial}{\partial a} - \gamma \right] Z_{*,k}, \quad (5.5)$$

with $Z_{*,k} = \sum_{i \geq k} Z_{ik} a^i$.

4. The renormalized function F_4 can be presented as:

$$F_4 = -(Z_a)_{4,1}^{\zeta_3} \zeta_3 \frac{a^4}{\epsilon} + \sum_i z_i(\epsilon) p_i a^4 + \text{terms of order } a^3 \text{ and lower} . \quad (5.6)$$

Literally repeating considerations after eq. (3.5) we arrive at:

$$g_4 \stackrel{\pi}{=} (Z_a)_{4,1}^{\zeta_3} \frac{3}{2} \zeta_4 \equiv \frac{3}{8} \beta_4^{\zeta_3} \zeta_4 . \quad (5.7)$$

5. Now we consider the p-function $F_{5,B}^{\text{si}}$. By construction, its renormalized version can be written as

$$F_5^{\text{si}}(a, \ell_\mu) = F_{5,B}^{\text{si}}(a_B, \ell_\mu), \quad a_B = \mu^{2\epsilon} (Z_a)_4 a , \quad (5.8)$$

where we have explicitly indicated that higher than 4-loop contributions to RC Z_a have no effect on renormalization of $F_{5,B}^{\text{si}}$ (as the latter starts from $\beta_1 a_B$). As a result we have a representation

$$F_5^{\text{si}} = (Z_a)_{4,1}^{\zeta_3} \zeta_3 \frac{a^5}{\epsilon} \beta_1 + \sum_i z_i(\epsilon) p_i a^5 + \text{terms of order } a^4 \text{ and lower} , \quad (5.9)$$

where all z_i are polynomial in $1/\epsilon$ with *rational* coefficients and p_i are some p-integrals with loop number less or equal 5. Assuming that all p-integrals in (5.9) are rewritten in term of hatted zetas and requiring finiteness of F_5^{si} at $\epsilon \rightarrow 0$ we immediately obtain:

$$d_4 \stackrel{\pi}{=} -\frac{3}{8} \beta_1 \beta_4^{\zeta_3} \zeta_4 , \quad (5.10)$$

where we have used eq. (3.1) and the identity $(Z_a)_{4,1} = \frac{1}{4} \beta_4$. On the other hand, from (2.20) and first two statements of the Lemma we have

$$d_4 \stackrel{\pi}{=} \beta_5 - \frac{\beta_4 \beta_2}{\beta_1} + \frac{4 \beta_4 g_1}{3} - 4 \beta_1 g_4 \stackrel{\pi}{=} \beta_5 - 4 \beta_1 g_4 . \quad (5.11)$$

Finally, by combining eqs. (5.7), (5.10) and (5.11) we arrive at (5.4).

Thus, the puzzle discussed in the Introduction is solved: the surviving ζ_4 in (generic) p-functions $\hat{F}_5^{\text{si}}$ and in F_5^{si} are indeed removed by the ζ_4 piece of the 5-loop coefficient β_5 which, however, is generated (and completely fixed) by β_1 and the ζ_3 term in the 4-loop β -function.

6 The $\pi - \zeta - \epsilon$ structure of p-integrals and renormalization constants

It is instructive to look into the structure of the results obtained in [13, 25] for 11 4-loop QCD p-functions and corresponding 5-loop ADs as displayed by table 1. Is it possible to understand the regularities shown in table 1? Sure.

L	p-integrals	L+1	Z
0	rational	1	rational/ ϵ
1	rational/ ϵ	2	rational/ ϵ^2
2	ζ_3	3	ζ_3/ϵ
3	$\zeta_3/\epsilon, \zeta_4, \zeta_5$	4	$\zeta_3/\epsilon^2, \{\zeta_4, \zeta_5\}/\epsilon$
4	$\zeta_3/\epsilon^2, \{\zeta_4, \zeta_5\}/\epsilon, \zeta_3^2, \zeta_6, \zeta_7$	5	$\zeta_3/\epsilon^3, \{\zeta_4, \zeta_5\}/\epsilon^2, \{\zeta_3^2, \zeta_6, \zeta_7, \}/\epsilon$

Table 1. The structure of p-integrals (expanded in ϵ up to and including the constant ϵ^0 part) and RCs in dependence on the loop number L . The inverse power of ϵ stands for the *maximal* one in generic case; in particular cases it might be less.

1. The rationality⁷ of any 1-loop p-function is essentially made by hand by the choice of the G-scheme. 1-loop RC are, in fact, rational in any minimal scheme be it MS-, $\overline{\text{MS}}$ - or G-scheme. The irrational structure of the p-functions at 2,3 and 4 loop level directly follows from explicit results for the corresponding master integrals as found in [1].

The fact that the set of zetas appearing in $(L+1)$ RCs is the same as the one displayed by L -loop p -integrals directly follows from the following theorem proven in [33].

Theorem 1. *Any $(L+1)$ -loop UV counterterm for any Feynman integral may be expressed in terms of pole and finite parts of some appropriately constructed L -loop p -integrals.*

The maximal power of $1/\epsilon$ with which a given zeta may appear in a L -loop contribution to a renormalization constant Z directly comes from the 't Hooft constraint (5.5).

Note that points 1. and 2. are also covered by remarks 1. and 2. on page 13 of [34].

2. An attentive reader has probably noticed that table 1 does not include a particular combination of the transcendental weight 7, namely, $\zeta_3\zeta_4$. There is a nice explanation of the fact. Indeed, let us take an arbitrary 4-loop p-integral F . If we consider it as a function of the irrational constants $\zeta_3, \dots, \zeta_7$, then the following identity holds (see eq. (3.1))

$$F(\zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7) = F(\hat{\zeta}_3, 0, \hat{\zeta}_5, 0, \hat{\zeta}_7) + \mathcal{O}(\epsilon). \quad (6.1)$$

If a particular combination $\zeta_3\zeta_4$ could appear in the finite part of the function

$$F(\zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7)$$

then its hatted version would, obviously, contain a term proportional to $\hat{\zeta}_3^2/\epsilon$. This would imply that the constant part of a 3-loop p-integral contained ζ_3^2 which is in contradiction to the previous point.

⁷In the sense of absence any irrational constants.

7 π -free representation for 5- and 6-loop p-integrals

In sections 3, 4, and 5 we have assumed that all encountered p-integrals are π -safe. In fact, for the treatment of these 3 sections to be valid the proven existence of π -free form for 4-loop p-integrals is just enough. Indeed, all currently known p -functions F_4 can be separated into 2 classes.

I. The lowest order contribution — the 1 in (2.1) — corresponds to a tree-level (that is loopless) diagram. The class includes QCD propagators and vertex functions (the latter in the propagator-like kinematical regime). Here the only contributions to the scale-invariant p-function F_5^{si} with loop number equal to 5 could come from the derivative $\frac{\partial}{\partial \ell_\mu} F_5$. This derivative annihilates the constant part of the corresponding 5-loop p-integrals and, thus, does depend only on their *pole* parts.

The Adler function in 5-loops also belongs to class I in spite of the fact that its lowest order term corresponds to a 1-loop diagram. This is because by construction (i) the function is scale-invariant and (ii) it is completely determined by the non-constant part of the polarization operator.

Thus, the no- π theorem for class I requires the existence of the π -free form only for the pole part of an arbitrary 5-loop p-integral (Property I).

II. Any correlator $\Pi(a, Q^2)$ of 2 composite operators with non-zero ADs. Here the starting object $F(a, Q^2)$ is constructed by a repeated application of the operator $\frac{\partial}{\partial Q^2}$ to Π in order to arrive to the superficially convergent p-function F . During this procedure the bare p-integrals are effectively multiplied by ϵ . The lowest order contribution to F starts (at least) from the one-loop term. Current technology allow to compute F_4 and the corresponding AD,⁸ γ , up to and including 5 loops. As a result the p-function F_5^{si} is contributed by 6-loop diagrams! However, a little meditation shows that only $1/\epsilon^2$ and higher poles in these integrals could really contribute to F_5^{si} .

Thus, the no- π theorem for class II requires the existence of the π -free form only for arbitrary 6-loop p-integrals multiplied by ϵ^2 (Property II).

Both Properties I and II are direct consequences of the following

Theorem 2. *Let us assume that a π -free form exists for all L -loop p-integrals. Then*

1. *any $(L + 1)$ -loop p-integral times ϵ is π -safe.*
2. *any $(L + 2)$ -loop p-integral times ϵ^2 is π -safe.*

Proof.

1. Let us define the $\widehat{G}$ -scheme by pretending that hatted zetas do not depend on ϵ . This means that all p-integrals are assumed to be expressed in term of the hatted zetas and that

⁸Which is, obviously, equal to the sum of the ADs of the both composite operators.

the extraction of the pole part of a p-integral is defined as:

$$\hat{K} \left(\mathcal{P}(\epsilon) \prod_j \hat{\zeta}_j \right) := \left(\sum_{i \leq 0} \mathcal{P}_i \epsilon^j \right) \prod_j \hat{\zeta}_j, \quad (7.1)$$

with $\mathcal{P}(\epsilon) = \sum_i \epsilon^i \mathcal{P}_i$ being a polynomial in ϵ with rational coefficients. The corresponding coupling constant will be denoted as $\hat{a}$.

Theorem 1 insures then that all $3, 4, \dots, (L+1)$ counterterms are π -free⁹ in the new scheme. Let now $\langle \Gamma \rangle$ be an arbitrary $(L+1)$ -loop p-integral (corresponding to a Feynman graph Γ). Its $\hat{G}$ -renormalized (and, consequently, finite at $\epsilon \rightarrow 0$) version reads

$$R \langle \Gamma \rangle(Q^2, \mu^2) = \langle \Gamma \rangle(Q^2, \mu^2) + Z_\Gamma + \boxed{\sum_\gamma Z_\gamma \langle \Gamma/\gamma \rangle(Q^2) + \dots}. \quad (7.2)$$

Here Z_γ is the UV Z-factor corresponding to a OPI subgraph γ of Γ , Z_Γ is the UV counterterm for the very Feynman integral $\langle \Gamma \rangle$ and dots stand for contributions with two and more UV subtractions. The finiteness of $R \langle \Gamma \rangle$ implies that

$$\epsilon \langle \Gamma \rangle(Q^2, \mu^2) = -\epsilon \left(Z_\Gamma + \sum_\gamma Z_\gamma \langle \Gamma/\gamma \rangle(Q^2) + \dots \right) + \mathcal{O}(\epsilon). \quad (7.3)$$

Note that the r.h.s. of eq. (7.2) includes only bare p-integrals with loop number not exceeding L as well as π -free UV counterterms. Point 1. is proven.

2. Let us consider (7.2) with $\langle \Gamma \rangle$ being an $(L+2)$ -loop p-integral. Every particular term in the boxed part of eq. (7.2) is a product of some (μ -independent!) Z-factors and a reduced Feynman integral, the latter by construction includes a factor $(\mu^2)^{n\epsilon}$, with n being its loop number. The very p-integral $\langle \Gamma \rangle(Q^2, \mu^2)$ depends on μ via a factor $(\mu^2)^{(L+2)\epsilon}$. Thus, after applying the operator $\mu^2 \frac{\partial}{\partial \mu^2}$ to eq. (7.3) we obtain

$$\epsilon^2 \langle \Gamma \rangle(Q^2, \mu^2) = -\epsilon^2 \left(\sum_\gamma Z_\gamma \left(1 - \frac{L(\gamma)}{L+2} \right) \langle \Gamma/\gamma \rangle(Q^2) + \dots \right) + \mathcal{O}(\epsilon), \quad (7.4)$$

where $L(\gamma)$ is the loop number of the subgraph γ . Let us demonstrate that eq. (7.4) indeed proves point 2. First, in the $\hat{G}$ -scheme every RC Z_γ in its r.h.s. is π -free as its loop number can not exceed $(L+1)$. Second, one can also check that every term $Z_\gamma \epsilon^2 \langle \Gamma/\gamma \rangle$ is π -safe p-integral. For instance, if $L(\gamma) = 1$ then $\langle \Gamma/\gamma \rangle$ is an $(L+1)$ -loop p-integral and $Z_\gamma = C/\epsilon$, with C being an ϵ -independent constant. As a result we find that

$$Z_\gamma \epsilon^2 \langle \Gamma/\gamma \rangle = C \epsilon \langle \Gamma/\gamma \rangle \quad (7.5)$$

is a π -safe combination due to point 1.

⁹We always assume that the dependence of hatted zetas on normal ones is polynomial in ϵ .

8 Adler function at order α_s^5

It has been conjectured in [2] that “a ζ_4 term should arise in the Adler function at order α_s^5 in the $\overline{\text{MS}}$ -scheme, and that this term is expected to disappear in the C -scheme as well”.

Our analysis certainly confirms the conjecture and upgrade it to a firm prediction *provided* that the product $\epsilon F(Q^2)$ is π -safe for every 6-loop p-integral $F(Q^2)$. Due to Theorem 2 the condition is equivalent to the π -safeness of the whole class of 5-loop p-integrals. Fortunately, very recently the last statement has received strong support from explicit evaluation of many 5-loop master p-integrals in [34].

9 All order no- π theorem

In this and next sections we will consider the case of a generic one-charge theory, not necessarily QCD. We will not assume any specific properties of the corresponding β -function and ADs, in particular $\beta_3^{\zeta_3}$ can have non-zero value.

In fact, the $\hat{G}$ -scheme suggests a strong generalization of the no- π theorem proven in section 3. Indeed, the following statement is true.

All order no- π theorem. Let F be any L -loop massless correlator and all L -loop p-integrals form a π -safe class. Then F is π -free in any (massless) renormalization scheme for which corresponding β -function and AD γ are both π -free at least at the level of $L+1$ loops.

Proof. The theorem is obviously true for the $\hat{G}$ -scheme. Let us denote the corresponding coupling constant as $\hat{a}$, (π -free!) RG functions as $\hat{\beta}(\hat{a})$ and $\hat{\gamma}(\hat{a})$, and the very function F as $\hat{F}$. For a different scheme the functions $F(a)$, $\beta(a)$ and $\gamma(a)$ are related via the standard conversion formulas:

$$a = \hat{a} \left(1 + \sum_{1 \leq i \leq L} \hat{c}_i \hat{a}^i \right), \quad (9.1)$$

$$F(a) = \left(1 + \sum_{1 \leq i \leq L} \hat{b}_i \hat{a}^i \right) \hat{F}(\hat{a}), \quad (9.2)$$

$$\beta(a) = \hat{\beta}(\hat{a}) + \left(\sum_{1 \leq i \leq L} i \hat{c}_i \hat{a}^i \right) / \left(1 + \sum_{1 \leq i \leq L} \hat{c}_i \hat{a}^i \right), \quad (9.3)$$

$$\gamma(a) = \hat{\gamma}(\hat{a}) + \hat{\beta}(\hat{a}) \left(\sum_{1 \leq i \leq L} i \hat{b}_i \hat{a}^i \right) / \left(1 + \sum_{1 \leq i \leq L} \hat{b}_i \hat{a}^i \right), \quad (9.4)$$

where it is understood that $\hat{a}$ in eqs. (9.2)–(9.4) is expressed in terms of a with the use of the inverted version of (9.1).

As relation (9.1) connects two π -independent to order $(L+1)$ schemes all coefficients c_i in it should be π -free (see the comment after eq. (2.15)).

Thus, $\hat{\beta}(\hat{a}(a))$ and $\hat{\gamma}(\hat{a}(a))$ are π -free to order $(L+1)$. Finally, the requirement that $\gamma(a) \stackrel{\pi}{=} \mathcal{O}(a^{L+2})$ and eq. (9.4) mean all coefficients b_i with $i = 1, \dots, L$ should also be π -free.

10 Odd and even zetas in RG-functions up to 6 loops

The $\hat{G}$ -scheme has some remarkable features. Indeed, one can see just from its definition that the corresponding “hatted” Green function, ADs and Z -factors can be obtained from the normal (that is computed with the G -scheme) by very simple rules.

- As a first step we make a formal replacement of the coupling constant a by $\hat{a}$ in every G -renormalized Green function, AD and Z -factor we want to transform to the $\hat{G}$ -scheme.
- Renormalized Green function $\hat{F}(\hat{a})$ is obtained from $F(\hat{a})$ by setting to zero *all* even zetas in the latter (both are assumed as taken at $\epsilon = 0$).
- The same rule works for ADs and β -functions.
- If Z is a (G -scheme) renormalization constant then one should not only nullify all even zetas in $Z(\hat{a})$ but also replace every odd zeta term in it with its “hatted” counterpart.

For future reference we write below the connection between hatted and normal zeta for the class of 5-loop p-integrals [34]:

$$\begin{aligned}\hat{\zeta}_3 &:= \zeta_3 + \frac{3\epsilon}{2}\zeta_4 - \frac{5\epsilon^3}{2}\zeta_6 + \frac{21\epsilon^5}{2}\zeta_8, & \hat{\zeta}_5 &:= \zeta_5 + \frac{5\epsilon}{2}\zeta_6 - \frac{35\epsilon^3}{4}\zeta_8, \\ \hat{\zeta}_7 &:= \zeta_7 + \frac{7\epsilon}{2}\zeta_8, & \hat{\varphi} &:= \varphi - 3\epsilon\zeta_4\zeta_5 + \frac{5\epsilon}{2}\zeta_3\zeta_6 \quad \text{and} \quad \hat{\zeta}_9 := \zeta_9,\end{aligned}\tag{10.1}$$

where

$$\varphi := \frac{3}{5}\zeta_{5,3} + \zeta_3\zeta_5 - \frac{29}{20}\zeta_8 = \zeta_{6,2} - \zeta_{3,5} \approx -0.1868414\tag{10.2}$$

and multiple zeta values are defined as¹⁰

$$\zeta_{n_1, n_2} := \sum_{i>j>0} \frac{1}{i^{n_1} j^{n_2}}.\tag{10.3}$$

Once we know which objects appear at 5-loop p-integrals we can construct a second table displaying the structure of 5-loop p-integrals and 6-loop ADs (in constructing the table and in what follows we assume that *all* class of 5-loop p-integrals is π -safe [34]). Once again, some combinations including even zetas, namely, $\zeta_4\zeta_5$ and $\zeta_3\zeta_6$ do not appear in table 2 essentially due to the π -safeness of 5-loop p-integrals as expressed by (10.1).

It is interesting to note that representation (10.1) is in a sense “too deep” for the task of presentation of 5-loop p-integrals in the hatted form. Indeed, a simple inspection of table 2 clearly shows that the hatted form of a 5-loop p-integral can not depend on the last available terms in ϵ -expansions of $\hat{\zeta}_3$, $\hat{\zeta}_5$ and $\hat{\varphi}$ (we do not consider the higher than ϵ^0 terms in the ϵ -expansion of 5-loop p-integrals).

¹⁰Note, that the papers [5, 34] use somewhat different notation for multiple zeta values: $\left[\zeta_{n_1, n_2}\right]^{\text{our}} = \left[\zeta_{n_2, n_1}\right]^{[5, 34]}$. The definition (10.3) is in agreement to the one employed, e.g. in [4, 35–37].

L	p-integrals	L+1	Z
5	$\zeta_3/\epsilon^3, \{\zeta_4, \zeta_5\}/\epsilon^2, \{\zeta_3^2, \zeta_6, \zeta_7\}/\epsilon,$ $\zeta_3\zeta_4, \zeta_8, \zeta_3\zeta_5, \varphi, \zeta_3^3, \zeta_9$	6	$\zeta_3/\epsilon^4, \{\zeta_4, \zeta_5\}/\epsilon^3, \{\zeta_3^2, \zeta_6, \zeta_7\}/\epsilon^2,$ $\{\zeta_3\zeta_4, \zeta_8, \zeta_3\zeta_5, \varphi, \zeta_3^3, \zeta_9\}/\epsilon$

Table 2. The structure of 5-loop p-integrals (expanded in ϵ up to and including the constant ϵ^0 part) and 6-loop RCs. The inverse power of ϵ stands for the *maximal* one in generic case; in particular cases it might be less.

Our next aim is to find explicitly the coefficients c_i in the relation between the charge RCs $\hat{Z}_a$ and Z_a in $\hat{G}$ - and G -schemes

$$\hat{a} = a \left(1 + \sum_{1 \leq i \leq L} c_i a^i \right). \quad (10.4)$$

As the bare charge must not depend on the choice of the renormalization scheme the coefficients c_i are fixed by requiring that

$$Z_a a = \hat{Z}_a(\hat{a})\hat{a}, \quad (10.5)$$

where $\hat{a}$ is expressed in terms of a via eq. (10.4). A simple counting of powers of a in (10.5) shows that one can find the coefficients $c_1 \dots c_6$ in terms of $\beta_1 \dots \beta_6$.

For simplicity we start from the case of 4 loops. On general grounds we can write

$$\beta = \beta_1 a + \beta_2 a^2 + (r_3 + \beta_3^{\zeta_3} \zeta_3) a^3 + (r_4 + \beta_4^{\zeta_3} \zeta_3 + \beta_4^{\zeta_4} \zeta_4 + \beta_4^{\zeta_5} \zeta_5) a^4, \quad (10.6)$$

where r_i is β_i with all zetas set to zero. The corresponding RCs Z_a and $\hat{Z}_a$ read:

$$\begin{aligned} Z_a = 1 + \frac{a\beta_1}{\epsilon} + a^2 \left(\frac{1}{2\epsilon} \beta_2 + \frac{1}{\epsilon^2} \beta_1^2 \right) + a^3 \left(\frac{1}{3\epsilon} (r_3 + \beta_3^{\zeta_3} \zeta_3) + \frac{7}{6\epsilon^2} \beta_1 \beta_2 + \frac{1}{\epsilon^3} \beta_1^3 \right) \\ + a^4 \left(\frac{1}{4\epsilon} (r_4 + \beta_4^{\zeta_3} \zeta_3 + \beta_4^{\zeta_4} \zeta_4 + \beta_4^{\zeta_5} \zeta_5) + \frac{1}{\epsilon^2} \left(\frac{5}{6} \beta_1 r_3 + \frac{5}{6} \beta_1 \beta_3^{\zeta_3} \zeta_3 + \frac{3}{8} \beta_2^2 \right) \right. \\ \left. + \frac{23}{12\epsilon^3} \beta_1^2 \beta_2 + \frac{1}{\epsilon^4} \beta_1^4 \right) \end{aligned} \quad (10.7)$$

and

$$\begin{aligned} \hat{Z}_a = 1 + \frac{\hat{a}}{\epsilon} \beta_1 + \hat{a}^2 \left(\frac{1}{2\epsilon} \beta_2 + \frac{1}{\epsilon^2} \beta_1^2 \right) + \hat{a}^3 \left(\frac{1}{3\epsilon} (r_3 + \beta_3^{\zeta_3} \hat{\zeta}_3) + \frac{7}{6\epsilon^2} \beta_1 \beta_2 + \frac{1}{\epsilon^3} \beta_1^3 \right) \\ + \hat{a}^4 \left(\frac{1}{4\epsilon} (r_4 + \beta_4^{\zeta_3} \hat{\zeta}_3 + \beta_4^{\zeta_5} \hat{\zeta}_5) + \frac{1}{\epsilon^2} \left(\frac{5}{6} \beta_1 r_3 + \frac{5}{6} \beta_1 \beta_3^{\zeta_3} \hat{\zeta}_3 + \frac{3}{8} \beta_2^2 \right) \right. \\ \left. + \frac{23}{12\epsilon^3} \beta_1^2 \beta_2 + \frac{1}{\epsilon^4} \beta_1^4 \right). \end{aligned} \quad (10.8)$$

Equation (10.5) can be easily solved with the result

$$\begin{aligned} c_1 = c_2 = 0, \\ c_3 = -\frac{1}{2} \beta_3^{\zeta_3} \zeta_4 + \frac{5\epsilon^2}{6} \beta_3^{\zeta_3} \zeta_6 - \frac{7\epsilon^4}{2} \beta_3^{\zeta_3} \zeta_8, \\ c_4 = \frac{1}{4\epsilon} (\beta_4^{\zeta_4} - \beta_1 \beta_3^{\zeta_3}) \zeta_4 - \frac{3}{8} \beta_4^{\zeta_3} \zeta_4 - \frac{5}{8} \beta_4^{\zeta_5} \zeta_6 \\ + \frac{5\epsilon}{12} \beta_1 \beta_3^{\zeta_3} \zeta_6 + \epsilon^2 \left(\frac{5}{8} \beta_4^{\zeta_3} \zeta_6 + \frac{35}{16} \beta_4^{\zeta_5} \zeta_8 \right) - \frac{7\epsilon^3}{4} \beta_1 \beta_3^{\zeta_3} \zeta_8 - \frac{21\epsilon^4}{8} \beta_4^{\zeta_3} \zeta_8. \end{aligned} \quad (10.9)$$

As the coefficients c_i have to be finite at $\epsilon \rightarrow 0$ we arrive at the connection

$$\beta_4^{\zeta_4} = \beta_1 \beta_3^{\zeta_3}, \quad (10.10)$$

which has been already established in section 3 (see eq. (3.10)). Repeating the same reasonings for 5 and 6 loops we arrive at exact relations which express *all* π -dependent terms in the coefficients β_5 and β_6 in terms of β_1, β_2 and contributions (proportional to odd zetas) to β_3, β_4 and $\beta_3, \beta_4, \beta_5$ correspondingly. The relations read

$$\begin{aligned} \beta_5^{\zeta_4} &= \frac{1}{2} \beta_2 \beta_3^{\zeta_3} + \frac{9}{8} \beta_1 \beta_4^{\zeta_3}, \\ \beta_5^{\zeta_6} &= \frac{15}{8} \beta_1 \beta_4^{\zeta_5}, \\ \beta_5^{\zeta_3 \zeta_4} &= 0, \end{aligned} \quad (10.11)$$

$$\begin{aligned} \beta_6^{\zeta_4} &= \frac{3}{4} \beta_2 \beta_4^{\zeta_3} + \frac{6}{5} \beta_1 \beta_5^{\zeta_3}, \\ \beta_6^{\zeta_6} &= -\beta_1^3 \beta_3^{\zeta_3} + \frac{5}{4} \beta_2 \beta_4^{\zeta_5} + 2 \beta_1 \beta_5^{\zeta_5}, \\ \beta_6^{\zeta_8} &= \frac{14}{5} \beta_1 \beta_5^{\zeta_7}, \\ \beta_6^{\zeta_3 \zeta_4} &= \frac{12}{5} \beta_1 \beta_5^{\zeta_3^2}, \\ \beta_6^{\zeta_3 \zeta_6} &= 0, \\ \beta_6^{\zeta_4 \zeta_5} &= 0. \end{aligned} \quad (10.12)$$

In exactly the same way we derive similar constraints on a generic AD γ . Indeed, let $F_L(a, \ell_\mu)$ is a p-function with evolution eq. (2.4) and $\hat{F}_L(\hat{a}, \ell_\mu)$ is its $\hat{G}$ -scheme counterpart. Both functions should be finite at $\epsilon \rightarrow 0$ and meet a conversion relation:

$$\hat{F} = \left(1 + \sum_{1 \leq i \leq L} b_i a^i \right) F + \mathcal{O}(a^{L+1}) \quad (10.13)$$

or, equivalently,

$$\hat{Z}(\hat{a}) = \left(1 + \sum_{1 \leq i \leq L} b_i a^i \right) Z(a) + \mathcal{O}(a^{L+1}). \quad (10.14)$$

The finiteness of the coefficients b_i leads to the following constraints

$$\gamma_4^{\zeta_4} = \frac{3}{2} \gamma_3^{\zeta_3} \beta_1 - \frac{1}{2} \gamma_1 \beta_3^{\zeta_3}, \quad (10.15)$$

$$\gamma_5^{\zeta_4} = \frac{3}{2} \gamma_4^{\zeta_3} \beta_1 + \frac{3}{2} \gamma_3^{\zeta_3} \beta_2 - \gamma_2 \beta_3^{\zeta_3} - \frac{3}{8} \gamma_1 \beta_4^{\zeta_3},$$

$$\gamma_5^{\zeta_6} = \frac{5}{2} \gamma_4^{\zeta_5} \beta_1 - \frac{5}{8} \gamma_1 \beta_4^{\zeta_5},$$

$$\gamma_5^{\zeta_3 \zeta_4} = 0, \quad (10.16)$$

$$\begin{aligned}
 \gamma_6^{\zeta_4} &= \frac{3}{2}\gamma_5^{\zeta_3}\beta_1 + \frac{3}{2}\gamma_4^{\zeta_3}\beta_2 + \frac{3}{2}\gamma_3^{\zeta_3}r_3 - \frac{3}{2}(\gamma_3 - \gamma_3^{\zeta_3}\zeta_3)\beta_3^{\zeta_3} - \frac{3}{4}\gamma_2\beta_4^{\zeta_3} - \frac{3}{10}\gamma_1\beta_5^{\zeta_3}, \\
 \gamma_6^{\zeta_6} &= \frac{5}{2}\gamma_5^{\zeta_5}\beta_1 - \frac{5}{2}\gamma_3^{\zeta_3}\beta_1^3 + \frac{5}{2}\gamma_4^{\zeta_5}\beta_2 + \frac{3}{2}\gamma_1\beta_1^2\beta_3^{\zeta_3} - \frac{5}{4}\gamma_2\beta_4^{\zeta_5} - \frac{1}{2}\gamma_1\beta_5^{\zeta_5}, \\
 \gamma_6^{\zeta_8} &= \frac{7}{2}\gamma_5^{\zeta_7}\beta_1 - \frac{7}{10}\gamma_1\beta_5^{\zeta_7}, \\
 \gamma_6^{\zeta_3\zeta_4} &= 3\gamma_5^{\zeta_3^2}\beta_1 - \frac{3}{5}\gamma_1\beta_5^{\zeta_3^2}, \\
 \gamma_6^{\zeta_3\zeta_6} &= 0, \\
 \gamma_6^{\zeta_4\zeta_5} &= 0.
 \end{aligned} \tag{10.17}$$

Note, that if $\gamma = \beta$ then relations (10.15)–(10.17) become equivalent to (10.10)–(10.12) correspondingly.

We have successfully checked all the constraints above for a number of particular examples.

In QCD it was done for the 5-loop β -function and the quark mass AD known from [8, 16, 17, 38] as well as for all vertex and field ADs (taken in the Landau gauge) computed in [13].

At the level of 6 loops we have checked that the (π -dependent contributions to) terms of order $n_f^5\alpha_s^6$ (for the β -functions) as well as terms of order $n_f^5\alpha_s^6$ and of order $n_f^4\alpha_s^6$ (the quark AD) computed in [39–41] are in agreement with relations (10.12) and (10.17) respectively.

For the case of the normal QCD with the SU(3) gauge group our results for π -dependent contributions to the 6-loop coefficients of the β -function and the quark mass AD read (the known terms are boxed)

$$\begin{aligned}
 \beta_6 \stackrel{\pi}{=} & \boxed{\frac{608}{405}n_f^5\zeta_4} + n_f^4\left(\frac{164792}{1215}\zeta_4 - \frac{1840}{27}\zeta_6\right) + n_f^3\left(-\frac{4173428}{405}\zeta_4 + \frac{1800280}{243}\zeta_6\right) \\
 & + n_f^2\left(\frac{68750632}{405}\zeta_4 - \frac{13834700}{81}\zeta_6\right) + n_f\left(-\frac{146487538}{135}\zeta_4 + \frac{40269130}{27}\zeta_6\right) \\
 & + 99(44213\zeta_4 - 64020\zeta_6),
 \end{aligned} \tag{10.18}$$

$$\begin{aligned}
 \gamma_6^m \stackrel{\pi}{=} & \boxed{\frac{320}{243}n_f^5\zeta_4 + n_f^4\left(-\frac{90368}{405}\zeta_4 + \frac{22400}{81}\zeta_6\right)} \\
 & + n_f^3\left(-\frac{92800}{27}\zeta_3\zeta_4 - \frac{2872156}{405}\zeta_4 + \frac{503360}{243}\zeta_6\right) \\
 & + n_f^2\left(\frac{661760}{9}\zeta_3\zeta_4 + \frac{155801234}{405}\zeta_4 - \frac{378577520}{729}\zeta_6 + \frac{12740000}{81}\zeta_8\right) \\
 & + n_f\left(-\frac{1413280}{3}\zeta_3\zeta_4 - \frac{4187656168}{1215}\zeta_4 + \frac{5912758120}{729}\zeta_6 - \frac{96071360}{27}\zeta_8\right) \\
 & + 3194400\zeta_3\zeta_4 + \frac{272688530}{81}\zeta_4 - \frac{6778602160}{243}\zeta_6 + 15889720\zeta_8.
 \end{aligned} \tag{10.19}$$

The results for generic gauge group are rather bulky, they can be found in Mathematica [42] and FORM [43] supplementary files attached to this paper.

All 6-loop RG functions known from [3–5] for the case of the $O(n)$ ϕ^4 model agree with the constraints (provided that the transcendental constant $\zeta_{5,3}$ is replaced by φ according to relation (10.2)).

11 Odd and even zetas in p-functions up to 5 loops

The knowledge of the coefficients c_i and b_i in the relations (10.4) and (10.13) allows to reconstruct the full p-function F from its hatted version $\hat{F}$. Indeed, let

$$F(a) = 1 + \sum_{1 \leq i \leq L} F_i a^i. \quad (11.1)$$

Then

$$\hat{F}(\hat{a}) = 1 + \sum_{1 \leq i \leq L} \hat{F}_i \hat{a}^i \quad \text{with} \quad \hat{F}_i = (F_i)|_{\pi=0}. \quad (11.2)$$

At the limit of $\epsilon \rightarrow 0$ the coefficients c_i and b_i defined in the previous section read (the constraints (10.10)–(10.12) and (10.15)–(10.17) are taken into account; the results can be also found in computer-readable form attached to this paper as supplementary material)

$$\begin{aligned} c_1 &= c_2 = 0, \\ c_3 &= -\frac{1}{2} \beta_3^{\zeta_3} \zeta_4, \\ c_4 &= -\frac{3}{8} \beta_4^{\zeta_3} \zeta_4 - \frac{5}{8} \beta_4^{\zeta_5} \zeta_6, \\ c_5 &= -\frac{3}{10} \beta_5^{\zeta_3} \zeta_4 - \frac{3}{5} \beta_5^{\zeta_3^2} \zeta_3 \zeta_4 + \frac{1}{4} \beta_1^2 \beta_3^{\zeta_3} \zeta_6 - \frac{1}{2} \beta_5^{\zeta_5} \zeta_6 - \frac{7}{10} \beta_5^{\zeta_7} \zeta_8, \end{aligned} \quad (11.3)$$

and

$$\begin{aligned} b_1 &= b_2 = 0, \\ b_3 &= -\frac{1}{2} \gamma_3^{\zeta_3} \zeta_4, \\ b_4 &= -\frac{3}{8} \gamma_4^{\zeta_3} \zeta_4 - \frac{5}{8} \gamma_4^{\zeta_5} \zeta_6, \\ b_5 &= -\frac{3}{10} \gamma_5^{\zeta_3} \zeta_4 - \frac{3}{5} \gamma_5^{\zeta_3^2} \zeta_4 \zeta_3 + \left(-\frac{1}{2} \gamma_5^{\zeta_5} + \frac{1}{2} \beta_1^2 \gamma_3^{\zeta_3} - \frac{1}{4} \beta_3^{\zeta_3} \beta_1 \gamma_1 \right) \zeta_6 - \frac{7}{10} \gamma_5^{\zeta_7} \zeta_8. \end{aligned} \quad (11.4)$$

Finally, we can now restore all π -dependent terms in $F_1 - F_5$ from eqs. (10.13), (11.3) and (11.4) with the result:

$$F_1 = \hat{F}_1, \quad (11.5)$$

$$F_2 = \hat{F}_2, \quad (11.6)$$

$$F_3 = \hat{F}_3 + \frac{1}{2} \gamma_3^{\zeta_3} \zeta_4, \quad (11.7)$$

$$F_4 = \hat{F}_4 + \left(-\frac{1}{2} \hat{F}_1 \beta_3^{\zeta_3} + \frac{3}{8} \gamma_4^{\zeta_3} + \frac{1}{2} \gamma_3^{\zeta_3} \hat{F}_1 \right) \zeta_4 + \frac{5}{8} \gamma_4^{\zeta_5} \zeta_6, \quad (11.8)$$

$$\begin{aligned} F_5 &= \hat{F}_5 + \left(-\beta_3^{\zeta_3} \hat{F}_2 - \frac{3}{8} \hat{F}_1 \beta_4^{\zeta_3} + \frac{3}{10} \gamma_5^{\zeta_3} + \frac{3}{8} \gamma_4^{\zeta_3} \hat{F}_1 + \frac{1}{2} \gamma_3^{\zeta_3} \hat{F}_2 \right) \zeta_4 + \frac{3}{5} \gamma_5^{\zeta_3^2} \zeta_4 \zeta_3 \\ &\quad + \left(\frac{1}{2} \gamma_5^{\zeta_5} + \frac{1}{4} \beta_3^{\zeta_3} \beta_1 \gamma_1 - \frac{5}{8} \hat{F}_1 \beta_4^{\zeta_5} + \frac{5}{8} \hat{F}_1 \gamma_4^{\zeta_5} - \frac{1}{2} \gamma_3^{\zeta_3} \beta_1^2 \right) \zeta_6 + \frac{7}{10} \gamma_5^{\zeta_7} \zeta_8. \end{aligned} \quad (11.9)$$

We have checked that eqs. (11.5)–(11.9) correctly describe the π -dependent contributions to all 11 4-loop p-functions from [25] as well as their 5-loop scale invariant versions constructed according to the definition (2.17) with the use of 5-loop ADs from [13].

12 Discussion and conclusions

We have clarified and proved the no- π theorem for all one-scale RG-invariant Euclidean correlators $\hat{F}_5^{si}$ first suggested in [2]. The theorem is extended to a case of generic Euclidean correlators with arbitrary high loop number.

We have found many new identities relating contributions proportional to odd and even zetas in generic $\overline{\text{MS}}$ ADs and β -functions. The new identities allow to reconstruct all π -dependent terms in a RG function at the level L loops in terms of the same function and the β -function taken at loop level $(L - 1)$ and less with $L = 4, 5$ and 6 . We have explicitly elaborated the corresponding predictions for $L = 4, 5$ and 6 and found full agreement of our results with calculated results where the latter are available.

For the case of the $O(n) \phi^4$ theory there exist since recently results for the corresponding ADs at the 7 loop level [4]. In fact, we have extended our formulas to this case and successfully reproduced all the π -dependent terms in 7-loop ADs and the 6-loop propagator of the scalar field. We are planning to publish a detailed account of these results in the near future.

Finally, the use of the $\hat{G}$ -scheme is not limited to one-charge models or/and to the Landau gauge fixing. It is clear that the derivation of sections 10 and 11 can be straightforwardly extended on a general case of a QFT model with a few coupling constants considered in a generic covariant gauge. More loops are also not a problem *provided* corresponding generalizations of representation (10.1) can be constructed.

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Note added. After completing our work we have been informed by E. Panzer that he independently has found a way to reconstruct π -dependent terms in RG functions of the $O(n) \phi^4$ model.

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References

- [1] P.A. Baikov and K.G. Chetyrkin, *Four Loop Massless Propagators: An Algebraic Evaluation of All Master Integrals*, *Nucl. Phys. B* **837** (2010) 186 [[arXiv:1004.1153](https://arxiv.org/abs/1004.1153)] [[INSPIRE](#)].
- [2] M. Jamin and R. Miravitllas, *Absence of even-integer ζ -function values in Euclidean physical quantities in QCD*, *Phys. Lett. B* **779** (2018) 452 [[arXiv:1711.00787](https://arxiv.org/abs/1711.00787)] [[INSPIRE](#)].
- [3] D.V. Batkovich, K.G. Chetyrkin and M.V. Kompaniets, *Six loop analytical calculation of the field anomalous dimension and the critical exponent η in $O(n)$ -symmetric ϕ^4 model*, *Nucl. Phys. B* **906** (2016) 147 [[arXiv:1601.01960](https://arxiv.org/abs/1601.01960)] [[INSPIRE](#)].
- [4] O. Schnetz, *Numbers and Functions in Quantum Field Theory*, *Phys. Rev. D* **97** (2018) 085018 [[arXiv:1606.08598](https://arxiv.org/abs/1606.08598)] [[INSPIRE](#)].
- [5] M.V. Kompaniets and E. Panzer, *Minimally subtracted six loop renormalization of $O(n)$ -symmetric ϕ^4 theory and critical exponents*, *Phys. Rev. D* **96** (2017) 036016 [[arXiv:1705.06483](https://arxiv.org/abs/1705.06483)] [[INSPIRE](#)].
- [6] S.G. Gorishnii, A.L. Kataev and S.A. Larin, *The $O(\alpha_s^3)$ -corrections to $\sigma_{\text{tot}}(e^+e^- \rightarrow \text{hadrons})$ and $\Gamma(\tau^- \rightarrow \nu_\tau + \text{hadrons})$ in QCD*, *Phys. Lett. B* **259** (1991) 144 [[INSPIRE](#)].
- [7] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Adler Function, Bjorken Sum Rule and the Crewther Relation to Order α_s^4 in a General Gauge Theory*, *Phys. Rev. Lett.* **104** (2010) 132004 [[arXiv:1001.3606](https://arxiv.org/abs/1001.3606)] [[INSPIRE](#)].
- [8] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Five-loop fermion anomalous dimension for a general gauge group from four-loop massless propagators*, *JHEP* **04** (2017) 119 [[arXiv:1702.01458](https://arxiv.org/abs/1702.01458)] [[INSPIRE](#)].
- [9] D. Boito, M. Jamin and R. Miravitllas, *Scheme Variations of the QCD Coupling and Hadronic τ Decays*, *Phys. Rev. Lett.* **117** (2016) 152001 [[arXiv:1606.06175](https://arxiv.org/abs/1606.06175)] [[INSPIRE](#)].
- [10] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Scalar correlator at $\mathcal{O}(\alpha_s^4)$, Higgs decay into b -quarks and bounds on the light quark masses*, *Phys. Rev. Lett.* **96** (2006) 012003 [[hep-ph/0511063](https://arxiv.org/abs/hep-ph/0511063)] [[INSPIRE](#)].
- [11] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *On Higgs decays to hadrons and the R -ratio at N^4LO* , *JHEP* **08** (2017) 113 [[arXiv:1707.01044](https://arxiv.org/abs/1707.01044)] [[INSPIRE](#)].
- [12] J. Davies and A. Vogt, *Absence of π^2 terms in physical anomalous dimensions in DIS: Verification and resulting predictions*, *Phys. Lett. B* **776** (2018) 189 [[arXiv:1711.05267](https://arxiv.org/abs/1711.05267)] [[INSPIRE](#)].
- [13] K.G. Chetyrkin, G. Falcioni, F. Herzog and J.A.M. Vermaseren, *Five-loop renormalisation of QCD in covariant gauges*, *JHEP* **10** (2017) 179 [[arXiv:1709.08541](https://arxiv.org/abs/1709.08541)] [[INSPIRE](#)].
- [14] B. Ruijl, F. Herzog, T. Ueda, J.A.M. Vermaseren and A. Vogt, *R^* -operation and five-loop calculations*, in proceedings of *13th International Symposium on Radiative Corrections: Application of Quantum Field Theory to Phenomenology (RADCOR2017)*, St. Gilgen, Austria, September 24–29, 2017 [[arXiv:1801.06084](https://arxiv.org/abs/1801.06084)] [[INSPIRE](#)].

- [15] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Five-Loop Running of the QCD coupling constant*, *Phys. Rev. Lett.* **118** (2017) 082002 [[arXiv:1606.08659](#)] [[INSPIRE](#)].
- [16] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *The five-loop β -function of Yang-Mills theory with fermions*, *JHEP* **02** (2017) 090 [[arXiv:1701.01404](#)] [[INSPIRE](#)].
- [17] T. Luthe, A. Maier, P. Marquard and Y. Schröder, *The five-loop β -function for a general gauge group and anomalous dimensions beyond Feynman gauge*, *JHEP* **10** (2017) 166 [[arXiv:1709.07718](#)] [[INSPIRE](#)].
- [18] O.V. Tarasov, A.A. Vladimirov and A.Yu. Zharkov, *The Gell-Mann-Low Function of QCD in the Three Loop Approximation*, *Phys. Lett.* **B 93** (1980) 429 [[INSPIRE](#)].
- [19] S.A. Larin and J.A.M. Vermaseren, *The Three loop QCD β -function and anomalous dimensions*, *Phys. Lett.* **B 303** (1993) 334 [[hep-ph/9302208](#)] [[INSPIRE](#)].
- [20] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, *The Four loop β -function in quantum chromodynamics*, *Phys. Lett.* **B 400** (1997) 379 [[hep-ph/9701390](#)] [[INSPIRE](#)].
- [21] M. Czakon, *The Four-loop QCD β -function and anomalous dimensions*, *Nucl. Phys.* **B 710** (2005) 485 [[hep-ph/0411261](#)] [[INSPIRE](#)].
- [22] G. 't Hooft, *Dimensional regularization and the renormalization group*, *Nucl. Phys.* **B 61** (1973) 455 [[INSPIRE](#)].
- [23] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, *Deep Inelastic Scattering Beyond the Leading Order in Asymptotically Free Gauge Theories*, *Phys. Rev.* **D 18** (1978) 3998 [[INSPIRE](#)].
- [24] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, *New Approach to Evaluation of Multiloop Feynman Integrals: The Gegenbauer Polynomial x Space Technique*, *Nucl. Phys.* **B 174** (1980) 345 [[INSPIRE](#)].
- [25] B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Four-loop QCD propagators and vertices with one vanishing external momentum*, *JHEP* **06** (2017) 040 [[arXiv:1703.08532](#)] [[INSPIRE](#)].
- [26] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Massless Propagators, $R(s)$ and Multiloop QCD*, *Nucl. Part. Phys. Proc.* **261-262** (2015) 3 [[arXiv:1501.06739](#)] [[INSPIRE](#)].
- [27] R.N. Lee, A.V. Smirnov and V.A. Smirnov, *Master Integrals for Four-Loop Massless Propagators up to Transcendentality Weight Twelve*, *Nucl. Phys.* **B 856** (2012) 95 [[arXiv:1108.0732](#)] [[INSPIRE](#)].
- [28] J.A.M. Vermaseren, S.A. Larin and T. van Ritbergen, *The four loop quark mass anomalous dimension and the invariant quark mass*, *Phys. Lett.* **B 405** (1997) 327 [[hep-ph/9703284](#)] [[INSPIRE](#)].
- [29] J. Kubo, S. Sakakibara and P.M. Stevenson, *The Renormalization Scheme Ambiguity and Perturbation Theory Near a Fixed Point*, *Phys. Rev.* **D 29** (1984) 1682 [[INSPIRE](#)].
- [30] D.J. Broadhurst, *Dimensionally continued multiloop gauge theory*, [hep-th/9909185](#) [[INSPIRE](#)].
- [31] J.C. Taylor, *Ward Identities and Charge Renormalization of the Yang-Mills Field*, *Nucl. Phys.* **B 33** (1971) 436 [[INSPIRE](#)].
- [32] A. Blasi, O. Piguet and S.P. Sorella, *Landau gauge and finiteness*, *Nucl. Phys.* **B 356** (1991) 154 [[INSPIRE](#)].

- [33] K.G. Chetyrkin and V.A. Smirnov, *R^* Operation Corrected*, *Phys. Lett. B* **144** (1984) 419 [[INSPIRE](#)].
- [34] A. Georgoudis, V. Gonçalves, E. Panzer and R. Pereira, *Five-loop massless propagator integrals*, [arXiv:1802.00803](#) [[INSPIRE](#)].
- [35] J. Blumlein, D.J. Broadhurst and J.A.M. Vermaseren, *The Multiple Zeta Value Data Mine*, *Comput. Phys. Commun.* **181** (2010) 582 [[arXiv:0907.2557](#)] [[INSPIRE](#)].
- [36] D. Broadhurst, *Multiple Zeta Values and Modular Forms in Quantum Field Theory*, Springer Vienna, Vienna (2013) [[INSPIRE](#)].
- [37] R.N. Lee and K.T. Mingulov, *Introducing SummerTime: a package for high-precision computation of sums appearing in DRA method*, *Comput. Phys. Commun.* **203** (2016) 255 [[arXiv:1507.04256](#)] [[INSPIRE](#)].
- [38] T. Luthe, A. Maier, P. Marquard and Y. Schröder, *Five-loop quark mass and field anomalous dimensions for a general gauge group*, *JHEP* **01** (2017) 081 [[arXiv:1612.05512](#)] [[INSPIRE](#)].
- [39] J.A. Gracey, *The QCD β -function at $\mathcal{O}(1/N_f)$* , *Phys. Lett. B* **373** (1996) 178 [[hep-ph/9602214](#)] [[INSPIRE](#)].
- [40] M. Ciuchini, S.E. Derkachov, J.A. Gracey and A.N. Manashov, *Quark mass anomalous dimension at $\mathcal{O}(1/N_f^2)$ in QCD*, *Phys. Lett. B* **458** (1999) 117 [[hep-ph/9903410](#)] [[INSPIRE](#)].
- [41] M. Ciuchini, S.E. Derkachov, J.A. Gracey and A.N. Manashov, *Computation of quark mass anomalous dimension at $\mathcal{O}(1/N_f^2)$ in quantum chromodynamics*, *Nucl. Phys. B* **579** (2000) 56 [[hep-ph/9912221](#)] [[INSPIRE](#)].
- [42] Wolfram Research Inc., *Mathematica*, version 11.3.
- [43] J.A.M. Vermaseren, *New features of FORM*, [math-ph/0010025](#) [[INSPIRE](#)].