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The all-loop conjecture for integrands of reggeon amplitudes in $\mathcal{N} = 4$ SYM

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Abstract: In this paper we present the all-loop conjecture for integrands of Wilson line form factors, also known as reggeon amplitudes, in $\mathcal{N} = 4$ SYM. In particular we present a new gluing operation in momentum twistor space used to obtain reggeon tree-level amplitudes and loop integrands starting from corresponding expressions for on-shell amplitudes. The introduced gluing procedure is used to derive the BCFW recursions both for tree-level reggeon amplitudes and their loop integrands. In addition we provide predictions for the reggeon loop integrands written in the basis of local integrals. As a check of the correctness of the gluing operation at loop level we derive the expression for LO BFKL kernel in $\mathcal{N} = 4$ SYM.

Keywords: Scattering Amplitudes, Extended Supersymmetry, Perturbative QCD

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Contents

1 Introduction

In the last two decades the tremendous progress in understanding of the structure of S-matrix (amplitudes) in gauge theories in various dimensions has been achieved. The most prominent examples of such progress are various results for the amplitudes in $\mathcal{N}=4$ SYM theory. See for review [1, 2]. These results were near to impossible to obtain without plethora of new ideas and approaches to the perturbative computations in gauge theories. These new ideas and approaches mostly exploit analytical properties of amplitudes rather then rely on standard textbook Feynman diagram technique.

It is important to note that these analyticity based approaches appear to be effective not only for computations of the amplitudes but for form factors and correlation functions of local and non-local operators in $\mathcal{N}=4$ SYM and other gauge theories as well [3–12]. So for many important results for the amplitudes in $\mathcal{N}=4$ SYM their analogs for the form factors and correlation functions were found [3, 4, 6, 8–20]. First of all, new variables such as helicity spinors and momentum twistors appear also useful for the description of the form factors and correlation functions [3, 4, 10]. At tree level various recurrence relations (BCFW, CSW e t.c.) were constructed for the form factors of some local [3, 4, 6, 7, 21] as well as non-local [22] operators and various closed solutions for such recurrence relations were obtained [3, 4, 6, 7, 10–12, 19, 20, 23]. Ultimately for the form factors of operators from stress tensor supermultiplet [8] as well as for Wilson line operators [19] the representation in terms of integral over Grassmannian was discovered.[1] Also in the case of the Wilson line operators such representation was generalized to the form factors with arbitrary number of Wilson line operator insertions as well as correlation functions [20]. Dual description for such objects in terms of twistor string theories was investigated and in this context different CHY like representations for form factors were obtained [14, 15, 24]. In addition, unconventional (compared to standard textbooks) geometrical interpretation for the correlation functions of stress tensor supermultiplet operators was conjectured [25] similar to the "Amplituhedron" [26–30] for the amplitudes. At loop level various other results were obtained for the form factors at high orders of perturbation theory and/or number of external particles [3, 13, 16, 31–41] and connection between form factors and integrable systems [8, 42] was discussed. Interesting results [43, 44] also should be mentioned.

The ultimate goal for such investigations, similar to the amplitude case, would be the evaluation, in some closed form, of all factors and correlation functions off all possible operators in $\mathcal{N} = 4$ SYM at arbitrary value of coupling constant.

In this note we are going to continue to work in this direction and consider the possibility of constructing recurrence relations for the loop integrands of the Wilson line form factors in $\mathcal{N} = 4$ SYM theory.

Wilson lines are non-local gauge invariant operators and are interesting objects not only from pure theoretical but also from phenomenological point of view. They appear, for example, in the description of reggeon amplitudes in the framework of Lipatov's effective QCD lagrangian[2] [22, 46–56], within the context of k_T or high-energy factorization [57–60] as well as in the study of processes at multi-Regge kinematics. The Wilson line operators play the role of sources for the reggeized gluons, while their form factors are directly related to amplitudes with reggeized gluons in such framework. The results in this field to a large extent originate from long lasting efforts of St.Peterburg and Novosibirsk groups in the investigation of asymptotic behavior of QFT scattering amplitudes at high energies (Regge limit), which can be tracked back in time to early works [61] of Gribov. These results, in particular, include resummation of leading high energy logarithms $(\alpha_s \ln s)^n$ to all orders in strong coupling constant (LLA resummation) in QCD, which eventually resulted in the discovery of Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [62–66] governing the LLA high energy asymptotic behavior of QCD scattering amplitudes. Today BFKL equation is known at next-to-leading-logarithmic-approximation (NLLA) [67–69]. The current article can be also considered as an effort in this direction, namely towards NNLLA BFKL in the context of $\mathcal{N} = 4$ SYM. More accurately, the results of this article can be considered as a solution to the problem of the reduction of individual Feynman diagrams to a set of

[1] So in this sense these objects at tree level are known for arbitrary number of external legs.

[2] We also want to mention recent work [45], where Wilson lines arise in the process of off-shell analytic continuation of light-front quantized Yang-Mills action.

master integrals in BFKL computations. As in general amplitudes multiloop calculations, in BFKL calculations there are basically two steps in getting the final result. The first one is the reduction of contributing individual Feynman diagrams to a finite set of so called master integrals and the second one is the evaluation of these master integrals themselves. In this paper we discuss only the first part of this problem, which is the easiest one.

This article is organized as follows. In section 2 we remind the reader the definition of the Wilson line form factors and correlation functions as well as give the definition of so called gluing operator $\hat{A}_{i-1i}$ considered in our previous papers [24]. This operator allows one to convert the on-shell amplitudes into the Wilson line form factors and will be heavily used throughout the paper. In section 3 we discuss how the BCFW recurrence relations for the Wilson line form factors are constructed with the use of helicity spinor variables used to describe kinematical data. After that we show how the mentioned BCFW recursion can be derived from the BCFW recursion for on-shell amplitudes by means of application of the gluing operators. Section 4 contains the derivation of the gluing operator in the case when kinematical data are encoded by momentum twistor variables. In section 5 we remind the reader the necessary facts about BCFW recursion for integrands of the on-shell amplitudes in momentum twistor space. After that we show how applying the gluing operator one can formulate similar recurrence relation for the Wilson line form factor integrands as well. We also show how one can directly transform the integrands of on-shell amplitudes into the integrands of the Wilson line form factors on the examples of local on-shell integrands. After that we perform simple but interesting self consistency check of our considerations. Namely starting from our results for tree and one loop level Wilson line form factors we correctly reproduce LO BFKL kernel. Appendix A contains the derivation of the Grassmannian integral representation for the reggeon amplitudes starting from corresponding representation for the on-shell amplitudes.

2 Form factors of Wilson lines and gluing operation

2.1 Form factors of Wilson lines operators

To describe the form factors of Wilson line operators we will use the definition in [22]:

$$\mathcal{W}_p^c(k) = \int d^4x e^{ix\cdot k} \mathrm{Tr}\left\{\frac{1}{\pi g} t^c\, \mathcal{P}\exp\left[\frac{ig}{\sqrt{2}}\int_{-\infty}^{\infty} ds\, p\cdot A_b(x+sp)t^b\right]\right\}, \tag{2.1}$$

where t^c is $\mathrm{SU}(N_c)$ generator,[3] k ($k^2 \neq 0$) is the off-shell reggeized gluon momentum and p is its direction or polarization vector, such that $p^2 = 0$ and $p \cdot k = 0$. The polarization vector and momentum of the reggeized gluon are related to each other through the so called k_T-decomposition of the latter:

$$k^\mu = xp^\mu + k_T^\mu, \quad x \in [0,1]. \tag{2.2}$$

[3]The color generators are normalized as $\mathrm{Tr}(t^a t^b) = \delta^{ab}$.

It is convenient to parametrize such decomposition by an auxiliary light-cone four-vector q^μ, so that

$$k_T^\mu(q) = k^\mu - x(q)p^\mu \quad \text{with} \quad x(q) = \frac{q\cdot k}{q\cdot p} \;\text{ and }\; q^2 = 0. \tag{2.3}$$

Noting that the transverse momentum k_T^μ is orthogonal to both p^μ and q^μ vectors, we may write down the latter in the basis of two "polarization" vectors[4] [46]:

$$k_T^\mu(q) = -\frac{\kappa}{2}\frac{\langle p|\gamma^\mu|q]}{[pq]} - \frac{\kappa^*}{2}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle} \quad \text{with} \quad \kappa = \frac{\langle q|\not{k}|p]}{\langle qp\rangle},\; \kappa^* = \frac{\langle p|\not{k}|q]}{[pq]}. \tag{2.4}$$

It is easy to check, that $k^2 = -\kappa\kappa^*$ and both κ and κ^* variables are independent of auxiliary four-vector q^μ [46]. Also, it turns out convenient to use spinor helicity decomposition of the light-cone four-vector q as $q = |\xi\rangle[\xi|$. $\mathcal{W}_p^c(k)$ is non-local gauge invariant operator and plays the role of source for the reggeized gluon [22, 70], so the form factors of such operators, or off-shell gauge invariant scattering amplitudes in our terminology, are closely related to the reggeon scattering amplitudes, and we will use words off-shell gauge invariant scattering amplitudes, reggeon amplitudes and Wilson line form factors hereafter as synonyms.

Both usual and color ordered reggeon amplitudes with n reggeized and m usual on-shell gluons could be then written in terms of the form factors with multiple Wilson line insertions as [22]:

$$A^*_{m+n}\left(1^\pm,\ldots,m^\pm,g^*_{m+1},\ldots,g^*_{n+m}\right) = \langle\{k_i,\epsilon_i,c_i\}_{i=1}^m|\prod_{j=1}^n \mathcal{W}_{p_{m+j}}^{c_{m+j}}(k_{m+j})|0\rangle\,. \tag{2.5}$$

Here asterisk denotes an off-shell gluon and p, k, c are its direction, momentum and color index. Next $\langle\{k_i,\epsilon_i,c_i\}_{i=1}^m| = \bigotimes_{i=1}^m\langle k_i,\varepsilon_i,c_i|$ and $\langle k_i,\varepsilon_i,c_i|$ denotes an on-shell gluon state with momentum k_i, polarization vector ε_i^- or ε_i^+ and color index c_i, p_j is the direction of the j'th ($j = 1,\ldots,n$) off-shell gluon and k_j is its off-shell momentum. To simplify things, here we are dealing with color ordered amplitudes only. The usual amplitudes are then obtained using their color decomposition, see [19, 71]. For example, the color ordered amplitude with one reggeon and two on-shell gluons with opposite helicity at tree level is given by the following expression:

$$A^*_{2+1}(1^-,2^+,g_3^*) = \frac{\delta^4(\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 + k_3)}{\kappa_3^*}\frac{\langle p_3\,1\rangle^4}{\langle p_3\,1\rangle\langle 1\,2\rangle\langle 2\,p_3\rangle}. \tag{2.6}$$

When dealing with $\mathcal{N}=4$ SYM we may also consider other on-shell states from $\mathcal{N}=4$ supermultiplet. The easiest way to do it is to consider color ordered superamplitudes defined on $\mathcal{N}=4$ on-shell momentum superspace [72, 73]:

$$A^*_{m+n}\left(\Omega_1,\ldots,\Omega_m,g^*_{m+1},\ldots,g^*_{n+m}\right) = \langle\Omega_1\ldots\Omega_m|\prod_{j=1}^n \mathcal{W}_{p_{m+j}}(k_{m+j})|0\rangle, \tag{2.7}$$

[4] Here we used the helicity spinor [1] decomposition of light-like four-vectors p and q. We will also sometime abuse spinor helicity formalism notations and write $\langle q|\gamma^\mu|p]/2 \equiv |p]\langle q|$, $\lambda_q \equiv \langle q|$ and $\tilde{\lambda}_q \equiv [p|$.

where $\langle\Omega_1\Omega_2\ldots\Omega_m| \equiv \bigotimes_{i=1}^m \langle 0|\Omega_i$ and Ω_i $(i=1,\ldots,m)$ denotes an $\mathcal{N}=4$ on-shell chiral superfield [73]:

$$\Omega = g^+ + \tilde{\eta}_A\psi^A + \frac{1}{2!}\tilde{\eta}_A\tilde{\eta}_B\phi^{AB} + \frac{1}{3!}\tilde{\eta}_A\tilde{\eta}_B\tilde{\eta}_C\epsilon^{ABCD}\bar{\psi}_D + \frac{1}{4!}\tilde{\eta}_A\tilde{\eta}_B\tilde{\eta}_C\tilde{\eta}_D\epsilon^{ABCD}g^- . \tag{2.8}$$

Here, g^+, g^- are creation/annihilation operators of gluons with $+1$ and -1 helicities, ψ^A, $\bar{\psi}_A$ stand for creation/annihilation operators of four Weyl spinors with negative helicity $-1/2$ and four Weyl spinors with positive helicity correspondingly, while ϕ^{AB} denote creation/annihilation operators for six scalars (anti-symmetric in the $\mathrm{SU}(4)_R$ R-symmetry indices AB). The $A^*_{m+n}\left(\Omega_1,\ldots,g^*_{n+m}\right)$ superamplitude is then the function of the following kinematic[5] and Grassmann variables

$$A^*_{k,m+n}\left(\Omega_1,\ldots,g^*_{m+n}\right) = A^*_{k,m+n}\left(\{\lambda_i,\tilde{\lambda}_i,\tilde{\eta}_i\}_{i=1}^m;\{k_i,\lambda_{p,i},\tilde{\lambda}_{p,i}\}_{i=m+1}^{m+n}\right). \tag{2.9}$$

and encodes in addition to the amplitudes with gluons also amplitudes with other on-shell states similar to the case of usual on-shell superamplitudes [1]. Here, additional index[6] k in $A^*_{k,m+n}$ denotes the total degree of $A^*_{k,m+n}$ in Grassmann variables η_i, which is given by $4k-4n$. For example the supersymmetrised (in on-shell states) version of (2.6) is given by:

$$\begin{aligned} A^*_{2,2+1}(\Omega_1,\Omega_2,g^*_3) &= \prod_{A=1}^4 \frac{\partial}{\partial\tilde{\eta}^A_{p_3}}\left[\frac{\delta^4(\lambda_1\tilde{\lambda}_1+\lambda_2\tilde{\lambda}_2+k_3)}{\kappa^*_3}\frac{\delta^8(\lambda_{p_3}\tilde{\eta}_{p_3}+\lambda_1\tilde{\eta}_1+\lambda_2\tilde{\eta}_2)}{\langle p_3\,1\rangle\langle 1\,2\rangle\langle 2\,p_3\rangle}\right] \\ &= \frac{\delta^4(\lambda_1\tilde{\lambda}_1+\lambda_2\tilde{\lambda}_2+k_3)}{\kappa^*_3}\frac{\delta^4\left(\tilde{\eta}_1\langle p_3\,1\rangle+\tilde{\eta}_2\langle p_3\,2\rangle\right)}{\langle p_3\,1\rangle\langle 1\,2\rangle\langle 2\,p_3\rangle}. \end{aligned} \tag{2.10}$$

Here we have $k=2$, $m=2$ and $n=1$. We also for simplicity will often drop $\partial^4/\partial\tilde{\eta}^4_{p_i}$ projectors in further considerations.

2.2 Gluing operator: transforming on-shell amplitudes into Wilson line form factors

In [24, 74] it was conjectured that one can compute the form factors of Wilson line operators by means of the four dimensional ambitwistor string theory [75]. In an addition to the standard vertex operators $\mathcal{V}$ and $\widetilde{\mathcal{V}}$, which describe Ω_i on-shell states in $\mathcal{N}=4$ SYM field theory, one can introduce, so called, generalised vertex operators $\mathcal{V}^{\text{gen.}}$ [24]:

$$\mathcal{V}^{\text{gen.}}_j \sim \int A^*_{2,2+1}(\Omega_j,\Omega_{j+1},g^*)\prod_{i=j,j+1}\mathcal{V}_i\,\frac{d^2\lambda_i d^2\tilde{\lambda}_i}{\mathrm{Vol}[\mathrm{GL}(1)]}\,d^4\tilde{\eta}_i. \tag{2.11}$$

Then it was conjectured that the following relation holds at least at tree level:

$$A^*_{k,m+n}\left(\Omega_1,\ldots,g^*_{m+n}\right) = \langle\mathcal{V}_1,\ldots\mathcal{V}_m\mathcal{V}^{\text{gen.}}_{m+1},\ldots,\mathcal{V}^{\text{gen.}}_{m+n}\rangle_{\text{worldsheet fields}}. \tag{2.12}$$

Here $\langle\ldots\rangle$ means average with respect to string worldsheet fields. This conjecture was successfully verified at the level of Grassmannian integral representations for the whole

[5] We used helicity spinor decomposition of on-shell particles momenta.
[6] We hope there will be no confusion with momentum labels.

tree level S-matrix [24, 74] and on several particular examples [24] with fixed number of external states. Effectively the evaluation of the string theory correlation function in (2.12) can be reduced to the action of some integral operator $\hat{A}$ on the on-shell amplitudes. In the case of one Wilson line operator insertion the relation between on-shell amplitude and the Wilson line form factor looks like:

$$A^*_{n+1} = \hat{A}_{n+1,n+2}[A_{n+2}]\,, \tag{2.13}$$

where A_{n+2} is the usual on-shell superamplitude with $n+2$ on-shell external states and the gluing integral operator $\hat{A}_{n+1,n+2}$ acts on the kinematical variables associated with the states Ω_{n+1} and Ω_{n+2}.

The action of $\hat{A}_{n+1,n+2}$ on any function f of variables $\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}_{i=1}^{n+2}$ is formally given by

$$\hat{A}_{n+1,n+2}[f] \equiv \int \prod_{i=n+1}^{n+2} \frac{d^2\lambda_i d^2\tilde{\lambda}_i d^4\tilde{\eta}_i}{\text{Vol}[\text{GL}(1)]} A^*_{2,2+1}(g^*, \Omega_{n+1}, \Omega_{n+2}) \times f\left(\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}_{i=1}^{n+2}\right). \tag{2.14}$$

This expression can be simplified. Performing integration over $\tilde{\lambda}_{n+1}$, $\tilde{\lambda}_{n+2}$, $\tilde{\eta}_{n+1}$ and $\tilde{\eta}_{n+2}$ variables [24] in (2.14) we get

$$\hat{A}_{n+1,n+2}[f] = \frac{\langle p_{n+1}\xi_{n+1}\rangle}{\kappa^*_{n+1}} \int \frac{d\beta_1}{\beta_1} \wedge \frac{d\beta_2}{\beta_2} \frac{1}{\beta_1^2\beta_2}\, f\left(\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}_{i=1}^{n+2}\right)\Big|_*, \tag{2.15}$$

where $\big|_*$ denotes substitutions $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}_{i=n+1}^{n+2} \mapsto \{\lambda_i(\beta), \tilde{\lambda}_i(\beta), \tilde{\eta}_i(\beta)\}_{i=n+1}^{n+2}$ with

$$\begin{aligned}
&\lambda_{n+1}(\beta) = \underline{\underline{\lambda}}_{n+1} + \beta_2\underline{\underline{\lambda}}_{n+2}\,, && \tilde{\lambda}_{n+1}(\beta) = \beta_1\underline{\underline{\tilde{\lambda}}}_{n+1} + \frac{(1+\beta_1)}{\beta_2}\underline{\underline{\tilde{\lambda}}}_{n+2}\,, && \tilde{\eta}_{n+1}(\beta) = -\beta_1\underline{\underline{\tilde{\eta}}}_{n+1}\,,\\
&\lambda_{n+2}(\beta) = \underline{\underline{\lambda}}_{n+2} + \frac{(1+\beta_1)}{\beta_1\beta_2}\underline{\underline{\lambda}}_{n+1}\,, && \tilde{\lambda}_{n+2}(\beta) = -\beta_1\underline{\underline{\tilde{\lambda}}}_{n+2} - \beta_1\beta_2\underline{\underline{\tilde{\lambda}}}_{n+1}\,, && \tilde{\eta}_{n+2}(\beta) = \beta_1\beta_2\underline{\underline{\tilde{\eta}}}_{n+1}\,,
\end{aligned} \tag{2.16}$$

and

$$\underline{\underline{\lambda}}_{n+1} = \lambda_p,\ \underline{\underline{\tilde{\lambda}}}_{n+1} = \frac{\langle\xi|k}{\langle\xi p\rangle},\ \underline{\underline{\tilde{\eta}}}_n = \tilde{\eta}_p;\ \ \underline{\underline{\lambda}}_{n+2} = \lambda_\xi,\ \underline{\underline{\tilde{\lambda}}}_{n+2} = \frac{\langle p|k}{\langle\xi p\rangle},\ \underline{\underline{\tilde{\eta}}}_{n+2} = 0. \tag{2.17}$$

All other variables left unshifted.

The integration with respect to $\beta_{1,2}$ will be understood as a residue form [76] and will be evaluated by means of the composite residue in points $res_{\beta_2=0} \circ res_{\beta_1=-1}$. For example, one can obtain [24] the Wilson line form factor $A^*_{3,3+1}(1^-, 2^+, 3^-, g_4^*)$ from 5 point NMHV on-shell amplitude $A^*_{3,5}(1^-, 2^+, 3^-, 4^-, 5^+)$:

$$A^*_{3,3+1}(1^-, 2^+, 3^-, g_4^*) = \hat{A}_{45}[A^*_{3,5}(1^-, 2^+, 3^-, 4^-, 5^+)], \tag{2.18}$$

where [19, 46]

$$A^*_{3,3+1}(1^-, 2^+, 3^-, g_4^*) = \delta^4\left(\sum_{i=1}^{3} \lambda_i\tilde{\lambda}_i + k_4\right) \frac{1}{\kappa_4} \frac{[2p_4]^4}{[12][23][3p_4][p_41]}. \tag{2.19}$$

Several Wilson line operator insertions correspond to the consecutive action of several gluing operators. For example $A^*_{3,0+3}(g^*_1, g^*_2, g^*_3)$ can be obtained [24] from 6 point NMHV amplitude $A_{3,6}(1^-2^+3^-4^+5^-6^+)$:

$$A^*_{3,0+3}(g^*_1, g^*_2, g^*_3) = (\hat{A}_{12} \circ \hat{A}_{34} \circ \hat{A}_{56})[A_{3,6}(1^-2^+3^-4^+5^-6^+)], \tag{2.20}$$

where $A^*_{3,0+3}$ is given by ($\mathbb{P}'$ is the permutation operator which shifts all spinor and momenta labels by $+1$ mod 3.):

$$\begin{aligned} A^*_{3,0+3}(g^*_1, g^*_2, g^*_3) &= \delta^4(k_1 + k_2 + k_3) \left(1 + \mathbb{P}' + \mathbb{P}'^2\right) f, \\ f &= \frac{\langle p_1 p_2\rangle^3 [p_2 p_3]^3}{\kappa_3 \kappa^*_1 \langle p_2|k_1|p_3] \langle p_1|k_3|p_2] \langle p_2|k_1|p_2]}. \end{aligned} \tag{2.21}$$

There is another way of representing the action of gluing operator. One can note that (2.14) is in fact equivalent to the action of a pair of consecutive BCFW bridge operators, in terminology of [77], on the f function weighted with an inverse soft factor. Namely, if one [77] defines $[i, j\rangle$ BCFW shift operator as $Br(i, j)$ (see figure 1) which acts on the function f of the arguments $\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}_{i=1}^{n+2}$, $1 \leq i, j \leq n+2$ according to:

$$\begin{aligned} &Br(i, i+1) \left[f\left(\ldots, \lambda_i, \tilde{\lambda}_i \tilde{\eta}_i, \ldots, \lambda_j, \tilde{\lambda}_j, \tilde{\eta}_j, \ldots\right)\right] = \int \frac{d\alpha}{\alpha} f\left(\ldots, \lambda_i, \hat{\tilde{\lambda}}_i, \hat{\tilde{\eta}}_i, \ldots, \hat{\lambda}_j, \tilde{\lambda}_j, \tilde{\eta}_j, \ldots\right) \\ &= \int \frac{d\alpha}{\alpha} f\left(\ldots, \lambda_i, \tilde{\lambda}_i - \alpha\tilde{\lambda}_j, \tilde{\eta}_i - \alpha\tilde{\eta}_j, \ldots, \lambda_j + \alpha\lambda_i, \tilde{\lambda}_j, \tilde{\eta}_j, \ldots\right), \end{aligned} \tag{2.22}$$

then one can see that the following relation holds:

$$\hat{A}_{n+1,n+2}[\ f\ ] = Br(n+1, n+2) \circ Br(n+2, n+1) \left[S^{-1}(1, n+2, n+1)\ f\ \right], \tag{2.23}$$

where

$$S(1, n+2, n+1) = \frac{\kappa^*_{n+1} \langle 1 n+1\rangle}{\langle 1 n+2\rangle \langle n+2 n+1\rangle}, \tag{2.24}$$

and function the f depends on $\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}_{i=1}^{n+2}$ arguments.

Note also that since $Br(i, j)$ operators act naturally on on-shell diagrams [77] one can easily consider the action of $\hat{A}_{n+1,n+2}$ operator on the top-cell diagram corresponding to the $A_{k,n+2}$ tree level on-shell amplitude. The top-cell for $A_{k,n+2}$ in its turn can be represented as the integral over Grassmannian L^k_{n+2} [77] (here let's ignore integration contour for a moment):

$$L^k_{n+2} = \int \frac{d^{k\times n+2} C}{\mathrm{Vol}[\mathrm{GL}(k)]} \frac{\delta^{k\times 2}(C \cdot \tilde{\lambda}) \delta^{k\times 4}(C \cdot \tilde{\eta}) \delta^{(n+2-k)\times 2}(C^\perp \cdot \lambda)}{(1 \cdots k)(2 \cdots k+1) \cdots (n+2 \cdots k-1)}. \tag{2.25}$$

Then one can see that the following relation also holds:

$$\hat{A}_{n+1,n+2}\left[L^k_{n+2}\right] = \Omega^k_{n+1}\,, \tag{2.26}$$

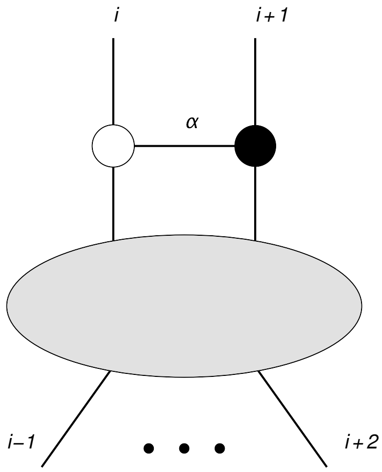


Figure 1. The action of $Br(i, i+1)$ operator on the on-shell diagram. White blob is $\overline{\text{MHV}}_3$ amplitude, black one — MHV_3.

where Ω^k_{n+2} is the Grassmannian integral representation for the off-shell amplitude $A^*_{k,n+1}$, with the Wilson line insertion positioned after the on-shell state with number n [19], if the appropriate integration contour is chosen for Ω^k_{n+2}:[7]

$$\Omega^k_{n+2} = \int \frac{d^{k\times(n+2)}C'}{\text{Vol}[GL(k)]} Reg. \frac{\delta^{k\times 2}\left(C' \cdot \underline{\underline{\tilde{\lambda}}}\right)\delta^{k\times 4}\left(C' \cdot \underline{\underline{\tilde{\eta}}}\right)\delta^{(n+2-k)\times 2}\left(C'^{\perp} \cdot \underline{\underline{\lambda}}\right)}{(1\cdots k)\cdots(n+1\cdots k-2)(n+2\,1\cdots k-1)}, \tag{2.27}$$

with

$$Reg. = \frac{\langle \xi_{n+1} p_{n+1}\rangle}{\kappa^*_{n+1}} \frac{(n+2\,1\cdots k-1)}{(n+1\,1\cdots k-1)}, \tag{2.28}$$

and

$$\begin{aligned}
&\underline{\underline{\lambda}}_i = \lambda_i, && i = 1,\ldots n, && \underline{\underline{\lambda}}_{n+1} = \lambda_{p_{n+1}}, && \underline{\underline{\lambda}}_{n+2} = \xi_{n+1} \\
&\underline{\underline{\tilde{\lambda}}}_i = \tilde{\lambda}_i, && i = 1,\ldots n, && \underline{\underline{\tilde{\lambda}}}_{n+1} = \frac{\langle \xi_{n+1}|k_{n+1}}{\langle \xi_{n+1}\, p_{n+1}\rangle}, && \underline{\underline{\tilde{\lambda}}}_{n+2} = -\frac{\langle p_{n+1}|k_{n+1}}{\langle \xi_{n+1}\, p_{n+1}\rangle}, \\
&\underline{\underline{\tilde{\eta}}}_i = \tilde{\eta}_i, && i = 1,\ldots n, && \underline{\underline{\tilde{\eta}}}_{n+1} = \tilde{\eta}_{p_{n+1}}, && \underline{\underline{\tilde{\eta}}}_{n+2} = 0.
\end{aligned} \tag{2.29}$$

The action of several $\hat{A}_{i,i+1}$ operators can be considered among the same lines and the result reproduces Grassmannian representation of the form factors with multiple Wilson line operator insertion obtained in [20].

At the end of this section let us make the following comment. Both on-shell and off-shell amplitudes (Wilson line form factors) can be represented by means of the BCFW recursion relations. But due to different analytical properties (Wilson line form factors will have additional type of poles corresponding to the Wilson line propagators [46]) the recursion for on-shell and off-shell amplitudes looks rather different. However, from the examples similar to ones considered above (namely (2.18) and (2.18)) one can note that

[7] One can think of this as alternative derivation of the results of appendix A of [24]. See also appendix of the current article for notation explanation.

not only gluing operator maps on-shell amplitudes to off-shell ones but one can choose representation for the on-shell amplitude in terms of the BCFW recursion in such a way that each BCFW term from on-shell amplitude will be mapped one-to-one to the terms from the BCFW recursion for the off-shell amplitudes. So a natural question to ask is whether it is possible to *derive* the BCFW recursion for the Wilson line form factors from the BCFW recursion for the on-shell amplitudes. We will address this question in the next section.

3 BCFW recursion for Wilson line form factors

3.1 Off-shell BCFW from analyticity

First let us remind the reader the main results of [46] and comment on supersymmetric extension of the off-shell BCFW recursion. The off-shell BCFW recursion for the reggeon amplitudes with an arbitrary number of off-shell reggeized gluons was worked out in [46]. Similar to the BCFW recursion [78, 79] for the on-shell amplitudes it is based on the observation, that a contour integral of an analytical function f vanishing at infinity equals to zero, that is

$$\oint \frac{dz}{2\pi i}\frac{f(z)}{z} = 0 \tag{3.1}$$

and the integration contour expands to infinity. Taking the above integral by residues we get

$$f(0) = -\sum_i \frac{\mathrm{res}_i f(z)}{z_i}, \tag{3.2}$$

where the sum is over all poles of f and $\mathrm{res}_i f(z)$ is a residue of f at pole z_i. Using this, one can relate the off-shell amplitude to the sum over contributions of its factorisation channels, which in turn can be represented as the off-shell amplitudes with smaller number of external states. In the original on-shell BCFW recursion the z-dependence of scattering amplitude is obtained by a z-dependent shift of particle's momenta. Similarly, the off-shell gluon BCFW recursion of [46] is formulated using a shift of momenta for two external gluons i and j with a vector

$$e^\mu = \frac{1}{2}\langle p_i|\gamma^\mu|p_j], \qquad p_i\cdot e = p_j\cdot e = e\cdot e = 0, \tag{3.3}$$

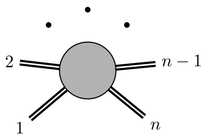
so that

$$\hat{k}_i^\mu(z) \equiv k_i^\mu + ze^\mu = x_i(p_j)p_i^\mu - \frac{\kappa_i - [p_ip_j]z}{2}\frac{\langle p_i|\gamma^\mu|p_j]}{[p_ip_j]} - \frac{\kappa_i^*}{2}\frac{\langle p_j|\gamma^\mu|p_i]}{\langle p_jp_i\rangle}, \tag{3.4}$$

$$\hat{k}_j^\mu(z) \equiv k_j^\mu - ze^\mu = x_j(p_i)p_j^\mu - \frac{\kappa_j}{2}\frac{\langle p_j|\gamma^\mu|p_i]}{[p_jp_i]} - \frac{\kappa_j^* + \langle p_ip_j\rangle z}{2}\frac{\langle p_i|\gamma^\mu|p_j]}{\langle p_ip_j\rangle}. \tag{3.5}$$

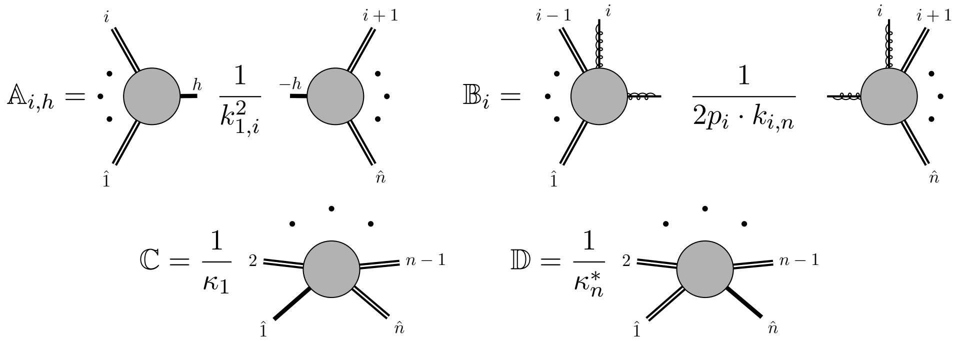
This shift does not violate momentum conservation and we still have $p_i\cdot\hat{k}_i(z) = 0$ and $p_j\cdot\hat{k}_j(z) = 0$. We would like to note, that the overall effect of shifting momenta is that the values of κ_i and κ_j^* shift, while κ_i^* and κ_j stay unshifted. In the on-shell limit the above shift corresponds to the usual $[i,j\rangle$ BCFW shift. Note also, that we could have chosen another shift vector $e^\mu = \frac{1}{2}\langle p_j|\gamma^\mu|p_i]$ and shift κ_i^* and κ_j instead. The off-shell amplitudes we consider in this paper do also have a correct large z ($z \to \infty$) behavior [46], so that we should not worry about boundary terms at infinity.

The sum over the poles (3.2) for z-dependent off-shell gluon scattering amplitude is given by the following graphical representation[8] [46]:



$$= \sum_{i=2}^{n-2} \sum_h \mathbb{A}_{i,h} + \sum_{i=2}^{n-1} \mathbb{B}_i + \mathbb{C} + \mathbb{D}, \tag{3.6}$$

where



(3.7)

$k_{i,j}^\mu \equiv k_i^\mu + k_{i+1}^\mu + \cdots + k_j^\mu$ and h is an internal on-shell gluon helicity or a summation index over all on-shell states in the Nair on-shell supermultiplet in the supersymmetric case discussed later. Here and below we use the convention that double lines may stand both for off-shell and on-shell gluons. The coil crossed with a line correspond to the off-shell gluons (Wilson line operator insertion). The thick solid lines stand for on-shell particles. The off-shell coil lines can be bent apart to form a single eikonal quark lines [22, 46]. According to this k_j^μ in $k_{i,j}^\mu$ can be either off-shell or on-shell depending on the context.

Let's now discuss each type of the terms encountered in (3.6) in more details. The $\mathbb{A}_{i,h}$ terms are usual on-shell BCFW terms, which correspond to the z-poles at which denominator of internal gluon (and also fermion or scalar) propagator $\hat{k}_{1,i}^2(z)$ vanishes:

$$\hat{k}_{1,i}^2(z) = 0. \tag{3.8}$$

This is standard BCFW on-shell condition for physical states of $\mathcal{N} = 4$ SYM supermultiplet.

The $\mathbb{B}_i$ term is a new one and is unique to the BCFW recursion for the off-shell amplitudes. It originates from the situation when the denominators of eikonal propagators coming from Wilson line expansion vanish, that is

$$p_i \cdot \hat{k}_{i,n}(z) = 0 \tag{3.9}$$

[8] We are considering the color ordered scattering amplitudes and without loss of generality may use shift of two adjacent legs 1 and n.

and p_i^μ is the direction of the Wilson line associated with the off-shell gluon. It is important to understand that condition $p_i \cdot \hat{k}_{i,n}(z) = 0$ fixes only the *direction* of momentum flowing through the Wilson line $\hat{k}_{i,n}$. The off-shell momenta $\hat{k}_L = k_{i-1} + \ldots + \hat{k}_1$ and $\hat{k}_R = k_i + \ldots + \hat{k}_n$, which belongs to the off-shell amplitudes in eq. (3.7), term $\mathbb{B}_i$, are different ($\hat{k}_L = -\hat{k}_R$), but satisfy the same condition $p_i \cdot \hat{k}_{L/R} = 0$. We also want to stress that this term is present only if i labels an off-shell external gluon. In addition, let us note that κ_i^* factors in the pair of off-shell amplitudes, which contribute to this term, are given explicitly by the following expressions:

$$\kappa_{i,L}^* = \frac{\langle p_i|\hat{k}_1 + \ldots + k_{i-1}|q_i]}{[p_i q_i]}, \quad \kappa_{i,R}^* = \frac{\langle p_i|\hat{k}_n + \ldots + k_{i+1}|q_i]}{[p_i q_i]}, \tag{3.10}$$

where $\kappa_{i,L}^*$ and $\kappa_{i,R}^*$ belong to the off-shell amplitudes positioned to the left and to the right in eq. (3.7) for the term $\mathbb{B}_i$.

The $\mathbb{C}$ term is only present if the gluon number 1 is off-shell. It is also unique to the BCFW recursion for the off-shell amplitudes. It appears due to vanishing of the external momentum square

$$\hat{k}_1^2(z) = 0. \tag{3.11}$$

Similarly, the $\mathbb{D}$ term is due to vanishing of the external momentum square $\hat{k}_n^2(z)$. It turns out that both these contributions could be calculated in terms of the same BCFW term with the off-shell gluons 1 or n exchanged for the on-shell ones. The helicity of the on-shell gluons depends on the type of the term ($\mathbb{C}$ or $\mathbb{D}$) and the shift vector e^μ ($\frac{1}{2}\langle p_i|\gamma^\mu|p_j]$ or $\frac{1}{2}\langle p_j|\gamma^\mu|p_i]$) used. We refer the reader to [46] for further details and examples.

The use of shifts involving only on-shell legs also allows one to perform the supersymmetrization of the off-shell BCFW recursion introduced in [46]. Indeed, it is easy to see, that the supersymmetric shifts of momenta and corresponding Grassmann variables are given by the on-shell BCFW $[i,j\rangle$ super-shifts:[9]

$$|\hat{i}] = |1] + z|j], \qquad |\hat{j}\rangle = |j\rangle - z|i\rangle, \qquad \hat{\eta}_A^i = \eta_A^i + z\eta_A^j. \tag{3.12}$$

No other spinors or Grassmann variables shift.

3.2 Off-shell BCFW from gluing operation

The aim of this section is to derive the off-shell recursion relations described above from the BCFW recursion for the on-shell amplitudes by means of the gluing operator. Before proceeding with general derivation let us consider a simple example first: we will take the BCFW recursion for the on-shell 6-point NMHV amplitude $A_{3,6}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$ and transform it into the three point off-shell amplitude $A_{0+3}^*(g_1^*, g_2^*, g_3^*)$ considered in [46]. This off-shell amplitude in its turn also can be obtained from the off-shell BCFW recursion, when external momenta 1 and 3 are shifted. Contributions corresponding to this shift are given in figure 2, and the sum of these three terms is given by (2.21).

[9]These shifts respect both momentum and supermomentum conservation.

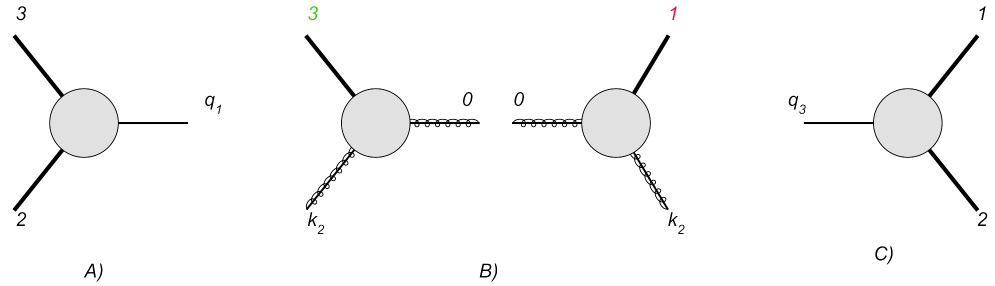


Figure 2. The off-shell BCFW contributions representing $A^*_{0+3}(g^*_1, g^*_2, g^*_3)$ for shift of the k_1 and k_3 external momenta. The first and the third terms labeled A) and C) are type $\mathbb{C}$ and $\mathbb{D}$ contributions, while the second, labeled B), term is type $\mathbb{B}$ contribution of the off-shell BCFW recursion (3.6).

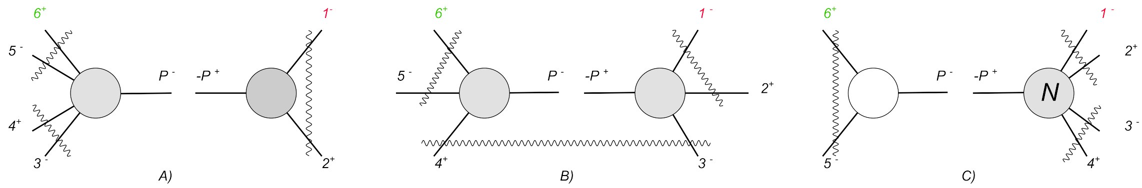


Figure 3. The action of the gluing operators $\hat{A}_{ii+1}$ (represented by wavy line) on $[1^-, 6^+\rangle$ BCFW representation of $A_{3,6}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$ amplitude.

So let's consider $A_{3,6}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$ amplitude represented via the standard BCFW shift $[1^-, 6^+\rangle$. The amplitude is then given, once again, by three terms (see figure 3)

$$A_{3,6}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+) = A + B + C, \tag{3.13}$$

where

$$A = A_{2,5}(3^-, 4^+, 5^-, \hat{6}^+, \hat{P}^+)\frac{1}{q^2_{1,2}}A_{2,3}(\hat{1}^-, 2^+, -\hat{P}^-), \tag{3.14}$$

$$B = A_{2,4}(4^+, 5^-, \hat{6}^+, \hat{P}^-)\frac{1}{q^2_{1,3}}A_{2,4}(\hat{1}^-, 2^+, 3^-, -\hat{P}^+), \tag{3.15}$$

$$C = A_{1,3}(5^-, \hat{6}^+, \hat{P}^+)\frac{1}{q^2_{5,6}}A_{3,5}(\hat{1}^-, 2^+, 3^-, 4^+, -\hat{P}^-). \tag{3.16}$$

Here $q_{a,b} = \sum_{i=a}^{b} q_i$ and q_i denote on-shell particle momenta.

Now we are going to consider the action of our gluing operators on $A_{3,6}$ on-shell amplitude, which will convert all pairs of $i^-, (i+1)^+$ gluons into Wilson line operator insertions (reggeized gluons). As we have discussed in previous sections to do this we have to take into account the action of the following combination of the gluing operations on $A_{3,6}$:

$$A_{0+3}(g^*_1, g^*_2, g^*_3) = \hat{A}_{12} \circ \hat{A}_{34} \circ \hat{A}_{56}[A_{3,6}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)]. \tag{3.17}$$

Let's consider each contribution in details.

We will start with A contribution first:

$$\hat{A}_{12} \circ \hat{A}_{34} \circ \hat{A}_{56}[A] = \hat{A}_{34} \circ \hat{A}_{56}[A_{2,5}(\hat{6}^+, 5^-, 4^+, 3^-, \hat{P}^-)]\hat{A}_{12}\left[\frac{1}{p_{1,2}^2} A_{2,3}(\hat{1}^-, 2^+, -\hat{P}^+)\right]. \tag{3.18}$$

It turns out, that both the value of the BCFW shift parameter z as well as shifted spinors are regular after we made $\big|_*$ substitutions (see (2.15) and (2.2)) corresponding to gluing operations and took limits $\beta_2 \to 0,\ \beta_1 \to -1$. Let us introduce the following notation for the spinors entering k_T-decomposition of momenta of three reggeized gluons, each spinor will be labeled by corresponding gluing operator:

$$\hat{A}_{12} \ \mapsto \ |p_1\rangle,\ |\xi_1\rangle;\ \hat{A}_{34} \ \mapsto \ |p_2\rangle,\ |\xi_2\rangle;\ \hat{A}_{56} \ \mapsto \ |p_3\rangle,\ |\xi_3\rangle. \tag{3.19}$$

The original value of the on-shell z parameter ($z = \frac{[21]}{[62]}$) transformed after the action of $\hat{A}_{12}$ into

$$z = \frac{[21]}{[62]} \mapsto \frac{\kappa_1 \langle \xi_3\, p_3\rangle}{\kappa_3^* [p_3\, p_1]} \tag{3.20}$$

and helicity spinor decomposition of momentum $\hat{P}$ is given now by

$$|\hat{P}\rangle = x(p_3)|p_1\rangle - \frac{\kappa_1^*}{\langle p_3\, p_1\rangle}|p_3\rangle\,, \quad |\hat{P}] = |p_1]\,, \tag{3.21}$$

where we used k_T-decomposition of the first reggeized gluon g_1^* momentum k_1:

$$k_1 = x(p_3)p_1 - \frac{\kappa_1}{[p_1\, p_3]}|p_1\rangle[p_3| - \frac{\kappa_1^*}{\langle p_3\, p_1\rangle}|p_3\rangle[p_1|\,. \tag{3.22}$$

Then, it is easy to see that

$$\hat{A}_{34} \circ \hat{A}_{56}[A_{2,5}(3^-, 4^+, 5^-, \hat{6}^+, \hat{P}^+)] = A^*_{1+2}(g_2^*, \hat{g}_3^*, \hat{P}^+), \tag{3.23}$$

where $\hat{g}_3^*$ denotes reggeized gluon g_3^* with momentum shifted as

$$\hat{k}_3 = k_3 + \frac{\kappa_1}{[p_1\, p_3]}|p_1\rangle[p_3|. \tag{3.24}$$

For the term

$$\hat{A}_{12}\left[\frac{1}{q_{1,2}^2} A_{2,3}(\hat{1}^-, 2^+, -\hat{P}^-)\right] \tag{3.25}$$

we have

$$\hat{A}_{12}\left[\frac{1}{q_{1,2}^2} A_{2,3}(\hat{1}^-, 2^+, -\hat{P}^-)\right] = res_{\beta_1=-1} \circ res_{\beta_2=0}[\omega_A], \tag{3.26}$$

where

$$\omega_A = -\frac{\langle p_1\,\xi_1\rangle}{\kappa_1^*}\left(\frac{1}{k_1^2}\frac{\langle 1\hat{P}\rangle^3}{\langle 12\rangle\langle 2\hat{P}\rangle}\right)\Big|_* \frac{1}{\beta_1^2\beta_2}\frac{d\beta_1\wedge d\beta_2}{\beta_1\beta_2}$$
$$= \frac{1}{\kappa_1}\frac{1}{(1+\beta_1)}\frac{d\beta_1\wedge d\beta_2}{\beta_1\beta_2} + \text{less singular terms.} \tag{3.27}$$

Taking residues and combining everything together we finally get for A term

$$\hat{A}_{12}\circ\hat{A}_{34}\circ\hat{A}_{56}[A] = \frac{1}{\kappa_1}A^*_{1+2}(\hat{g}_3^*, g_2^*, \hat{P}^+), \tag{3.28}$$

which is precisely the $\mathbb{C}$ term from the off-shell BCFW recursion [46] for $A^*_{0+3}(g_1^*, g_2^*, g_3^*)$ reggeon amplitude (see (3.6)). The C term can be analysed similarly. In this case we get

$$\hat{A}_{12}\circ\hat{A}_{34}\circ\hat{A}_{56}[C] = \frac{1}{\kappa_3}A^*_{1+2}(\hat{g}_1^*, g_2^*, \hat{P}^-)\,, \tag{3.29}$$

where helicity decomposition of momentum $\hat{P}$ is given by

$$|\hat{P}] = \left(x(p_1)|p_3] - \frac{\kappa_1}{[p_3\,p_1]}|p_1]\right), \quad |\hat{P}\rangle = |p_3\rangle. \tag{3.30}$$

This is precisely the $\mathbb{D}$ term from the off-shell BCFW recursion [46] for $A^*_{0+3}(g_1^*, g_2^*, g_3^*)$ amplitude.

Now let us turn to B contribution to $A_{3,6}$ (see figure 2). The value of z parameter in this case transforms under the action of the gluing operator as

$$z = \frac{q_{1,3}^2}{\langle 1|q_2+q_3|6]} \mapsto \frac{\langle p_2|k_1+k_2|p_2]}{\langle p_1\,p_2\rangle\,[p_2\,p_3]}. \tag{3.31}$$

Here it is convenient to consider first the action of $\hat{A}_{34}$. In this case the value of $\hat{P}$ momentum is given by

$$\hat{P} = \hat{q}_1 + q_2 + \frac{(1+\beta_1)}{\beta_2}\kappa_2^*\frac{|p_2\rangle[p_2|}{\langle\xi_2\,p_2\rangle} + \text{less singular terms}, \tag{3.32}$$

before residues evaluation. So for the whole B term after $\hat{A}_{34}$ action we have

$$\hat{A}_{34}[B] = \hat{A}_{34}\left[A_{2,4}(4^+,5^-,\hat{6}^+,\hat{P}^-)\frac{1}{q_{1,3}^2}A_{2,4}(\hat{1}^-,2^+,3^-,-\hat{P}^+)\right]$$
$$= res_{\beta_1=-1}\circ res_{\beta_2=0}[\omega_B] \tag{3.33}$$

where

$$\omega_B = \frac{\langle p_2\,\xi_2\rangle}{\kappa_2^*}\left(\frac{\langle 5\,\hat{P}\rangle^4}{\langle 4\,5\rangle\langle 5\,\hat{6}\rangle\langle\hat{6}\,\hat{P}\rangle\langle\hat{P}\,4\rangle}\frac{1}{q_{1,3}^2}\frac{\langle\hat{1}\,3\rangle^4}{\langle\hat{1}\,2\rangle\langle 2\,3\rangle\langle 3\,\hat{P}\rangle\langle\hat{P}\,1\rangle}\right)\Big|_* \frac{1}{\beta_1^2\beta_2}\frac{d\beta_1\wedge d\beta_2}{\beta_1\beta_2}. \tag{3.34}$$

Evaluating corresponding residues we get

$$\hat{A}_{34}[B] = \frac{1}{\hat{\kappa}^*_{2,L}}\frac{\langle 5p_2\rangle^4}{\langle\hat{6}5\rangle\langle 5p_2\rangle\langle p_2\hat{6}\rangle}\times\frac{1}{\langle p_2|k_1+k_2|p_2]}\times\frac{1}{\hat{\kappa}^*_{2,R}}\frac{\langle 1p_2\rangle^4}{\langle 12\rangle\langle 2p_2\rangle\langle p_2 1\rangle}, \tag{3.35}$$

where

$$\hat{\kappa}_{2,L}^{*} = \frac{\langle p_2|q_5 + \hat{q}_6|\xi]}{[p_2\xi]}, \ \hat{\kappa}_{2,R}^{*} = \frac{\langle p_2|q_2 + \hat{q}_1|\xi]}{[p_2\xi]}. \tag{3.36}$$

The action of $\hat{A}_{12} \circ \hat{A}_{56}$ can be evaluated in similar fashion and finally we arrive at

$$\hat{A}_{12} \circ \hat{A}_{56} \circ \hat{A}_{34}[B] = A_{0+2}^{*}(g_2^{*}, \hat{g}_3^{*}) \frac{1}{\langle p_2|k_1|p_2]} A_{0+2}^{*}(\hat{g}_1^{*}, g_2^{*}), \tag{3.37}$$

which is exactly $\mathbb{B}$ term in the off-shell BCFW recursion [46] for $A_{0+3}^{*}(g_1^{*}, g_2^{*}, g_3^{*})$ amplitude.

So we see the pattern here: the contributions $\mathbb{C}$ and $\mathbb{D}$ from the off-shell BCFW recursion for $A_{0+3}^{*}(g_1^{*}, g_2^{*}, g_3^{*})$ amplitude are reproduced from the on-shell BCFW recursion for $A_{3,6}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$ when gluing operator is acting on three point MHV_3 or $\overline{\mathrm{MHV}}_3$ sub-amplitudes (with degenerate kinematics), while $\mathbb{B}$ contribution is reproduced by the action of gluing operator on both sides of the BCFW bridge, see figure 3.

These observations can be immediately generalized to the situation with arbitrary on-shell amplitude. One can obtain the off-shell BCFW recursion for $A_{0+n}^{*}(g_1^{*}, \ldots, g_n^{*})$ with the shift of the off-shell momenta k_1 and k_n from the BCFW recursion for $A_{n,2n}(1^-, 2^+, \ldots, (2n)^+)$ on-shell amplitude represented by $[1^-, (2n)^+\rangle$ shift. Indeed, the terms $\mathbb{C}$ and $\mathbb{D}$ are reproduced when gluing operator acts on MHV_3 or $\overline{\mathrm{MHV}}_3$ on-shell amplitudes. Repeating the steps identical to the previous discussion (see (3.28)) we get:

$$\begin{aligned} &\hat{A}_{2n-1\ 2n} \circ \ldots \circ \hat{A}_{12} \left[A_{n-1,2n-1}\left(3^-, \ldots, \widehat{(2n)}^+, \hat{P}^+\right) \frac{1}{q_{1,2}^2} A_{2,3}(\hat{1}^-, 2^+, -\hat{P}^-) \right] \\ &\quad = \frac{1}{\kappa_1} A_{1+(n-1)}^{*}(g_2^{*}, \ldots, \hat{g}_n^{*}, \hat{P}^+), \end{aligned} \tag{3.38}$$

where

$$|\hat{P}\rangle = \left(x(p_n)|p_1\rangle - \frac{\kappa_1^{*}}{\langle p_n p_1\rangle}|p_n\rangle \right), \quad |\hat{P}] = |p_1]. \tag{3.39}$$

Similarly for $\overline{\mathrm{MHV}}_3$ we have:

$$\begin{aligned} &\hat{A}_{2n-1\ 2n} \circ \ldots \circ \hat{A}_{12} \left[A_{1,3}\left((2n-1)^-, \widehat{(2n)}^+, -\hat{P}^+\right) \frac{1}{q_{2n-1,2n}^2} A_{n,2n-1}(\hat{1}^-, \ldots, \hat{P}^-) \right] \\ &\quad = \frac{1}{\kappa_n^{*}} A_{1+(n-1)}^{*}(\hat{g}_1^{*}, \ldots, g_{n-1}^{*}, \hat{P}^-), \end{aligned} \tag{3.40}$$

with

$$|\hat{P}] = \left(x(p_1)|p_n] - \frac{\kappa_1}{[p_n p_1]}|p_1] \right), \quad |\hat{P}\rangle = |p_n\rangle. \tag{3.41}$$

When gluing operator $\hat{A}_{ii+1}$ acts on legs separated by the BCFW bridge the $\mathbb{B}$ type contribution is reproduced. In this case the on-shell BCFW shift z is replaced by

$$z = \frac{\langle p_i|k_1 + \ldots + k_i|p_i]}{\langle p_1\, p_i\rangle [p_i\, p_n]}, \tag{3.42}$$

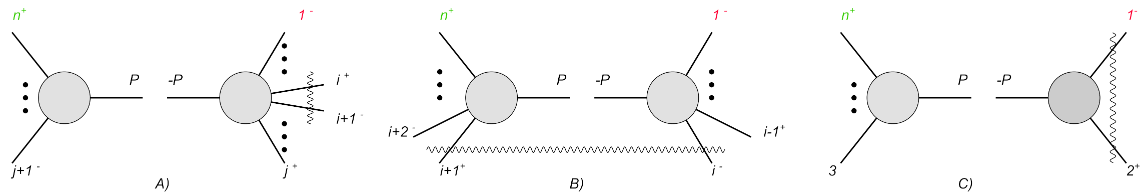


Figure 4. The action of the gluing operators $\hat{A}_{ii+1}$ on the individual BCFW terms of the on-shell $[1^-, n^+\rangle$ recurcion in general case. *A*) diagrams will give $\mathbb{A}$ type terms of the off-shell BCFW recurcion, *B*) type diagrams will give $\mathbb{B}$ type terms, while *C*) diagrams will give $\mathbb{C}$ and $\mathbb{D}$ type terms of the off-shell BCFW recurcion.

and the factor $1/q_{1,i}^2$ is replaced by

$$\frac{1}{\langle p_i|k_1 + \ldots + k_i|p_i]} \tag{3.43}$$

times β_i factors. The explicit proof that such contribution in general case gives us the $\mathbb{B}$ term can be done by induction and can be sketched as follows: one can decompose each A_{k_i,n_i} on-shell amplitude in individual BCFW terms via on-shell diagram representation [77] into combination of MHV_3 and $\overline{\text{MHV}}_3$ vertexes (on-shell diagrams). The action of the gluing operator on such on-shell diagrams was considered in the previous section. After that one have to reassemble $A^*_{2,2+1}$, MHV_3 and $\overline{\text{MHV}}_3$ amplitudes together. As the result one can obtain that:

$$\begin{aligned} \ldots \circ \hat{A}_{ii+1} \circ \ldots &\left[A_{k_1,n_1}\left((i+1)^+, \ldots, \widehat{(2n)}^+, \hat{P}^-\right) \frac{1}{q_{1,i}^2} A_{k_2,n_2}(\hat{1}^-, \ldots, i^-, -\hat{P}^+)\right] \\ &= A^*_{0+i}(\hat{g}_1^*, \ldots, g_i^*) \frac{1}{\langle p_i|k_1 + \ldots + k_i|p_i]} A^*_{0+(n-i)}(g_i^*, \ldots, \hat{g}_n^*). \end{aligned} \tag{3.44}$$

Also, presented above relation is implicitly guaranteed by the Grassmannian integral representation of the on-shell and off-shell amplitudes. It was shown [24, 74] that the latter could be easily related with each other by means of the same gluing operations (see (2.26) and appendix A) which imply similar relation for the individual residues of the top-forms. We see that the gluing operator transforms ordinary $1/P^2$ propagator type poles of the on-shell amplitudes into eikonal ones, when external legs, on which the gluing operator acts, are separated by the BCFW bridge (see figure 4 B). The value of the BCFW shift parameter z is adjusted accordingly to match the off-shell BCFW recursion term $\mathbb{B}$.

All other contributions (see figure 4) reproduce $\mathbb{A}$ type terms from [46], which also can be shown by induction. Indeed it is easy to see that the pole factor remains of the same $1/P^2$ type in this case, which corresponds to the propogators of the on-shell states of $\mathcal{N} = 4$ SYM, and the value of z is adjusted to match $\mathbb{A}$ term of the off-shell BCFW.

The fact that one can transform each individual term in the BCFW recursion for the on-shell amplitudes into terms of the BCFW recursion for the Wilson line form factors using gluing operators $\hat{A}_{i,i+1}$ in fact is not (very)surprising and in some sense trivial. Indeed, as was mentioned before, if the relation (2.26) holds on the level of the Grassmannian integrals (top-cell diagrams) then it likely will hold for the individual residues (boundaries of top-cells) as well.[10] And it is also natural, that in the case of Ω^k_{n+2}, and its generalisations to multiple Wilson line insertions, the residues of Ω^k_{n+2} can be identified with the individual BCFW terms of recursion for the off-shell amplitudes in full analogy with the on-shell case.

However observation that one can transform the BCFW recursion for the on-shell amplitudes into the BCFW recursion for the Wilson line form factors at tree level opens up exiting possibility. It is known that one can formulate the BCFW recursion not only for the tree level on-shell amplitudes in $\mathcal{N}=4$ SYM theory but for the loop integrands as well [77, 80]. *If we can transform on-shell BCFW recursion into off-shell one at tree level, then what about recursion for the loop integrands* ?

The next sections will be dedicated to discussion of this question. Namely we will investigate what will happen if we apply the analog of the gluing operators to the BCFW recursion for $\mathcal{N}=4$ SYM loop integrands of the on-shell amplitudes. Our ultimate goal is to present arguments that using the gluing operator one can transform the integrands of the on-shell amplitudes into the integrands of the Wilson line form factors at any given number of loops and external states.

4 Gluing operation in momentum twistor space

The integrands of the on-shell amplitudes in the planar limit are naturally formulated using momentum super twistors variables, in particular this is the case for theis BCFW recursion representation [80]. So, to proceed with our main goal we need to fprmulate our gluing operation in momentum twistor variables as well.

To do this, let us recall how the momentum twistor variables are introduced. We start with so called zone super variables or dual super coordinates y_i and ϑ_i which are related with on-shell momenta and its supersymmetric counterpart as [1, 81]:

$$q_i = \lambda_i \tilde{\lambda}_i = y_{i+1} - y_i, \quad \lambda_i \eta_i = \vartheta_{i+1} - \vartheta_i. \tag{4.1}$$

The introduction of dual super coordinates helps to trivialize the conservation of super momentum [1, 81] and figure 5 shows the momentum conservation geometrically for the case of $n=4$ on-shell and one off-shell momenta as an example. There we have a contour in the dual space formed by on-shell particles momenta together with two auxiliary on-shell momenta 5 and 6 used to describe off-shell momentum.

The momentum super twistor variables $\mathcal{Z}_i = (\lambda_i, \mu_i, \eta_i)$ [81] are then defined through the following incidence relations

$$\mu_i = \lambda_i y_i = \lambda_i y_{i+1}, \quad \tilde{\eta}_i = \lambda_i \vartheta_i = \lambda_i \vartheta_{i+1}. \tag{4.2}$$

[10] This can be explicitly seen for some particular case considering integration contours for tree level amplitudes L^3_{n+2} and Ω^3_{n+2}, and probably can be easily generalised for the case of arbitrary number of Wilson line insertions and arbitrary value of k [24]. Here however we avoided considerations of integration contours completely.

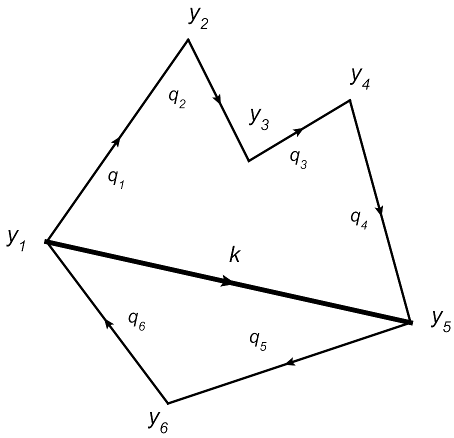


Figure 5. Momenta and dual coordinates in the case of the amplitude with one off-shell and $n = 4$ on-shell legs. In contrast to the case of the on-shell amplitudes, the n on-shell momenta do not add up to zero but to the off-shell gluon momentum k: $q_1 + \ldots + q_4 = k$, which in its turn can be decomposed as a pair of auxiliary on-shell momenta $k = q_5 + q_6$.

The bosonic part of $\mathcal{Z}_i$ will be labeled as $Z_i = (\lambda_i, \mu_i)$. Inverting presented above relations we get

$$\tilde{\lambda} = \mu \cdot Q\,, \quad \tilde{\eta} = \eta \cdot Q\,, \quad Q_{ij} = \frac{\delta_{i-1\,j}\langle i\, i+1\rangle + \delta_{ij}\langle i+1\, i-1\rangle + \delta_{i+1\,j}\langle i-1\, i\rangle}{\langle i-1\, i\rangle\langle i\, i+1\rangle}. \tag{4.3}$$

Here $\tilde{\lambda} \equiv (\tilde{\lambda}_1 \cdots \tilde{\lambda}_n)$, $\tilde{\eta} \equiv (\tilde{\eta}_1 \cdots \tilde{\eta}_n)$ and it is assumed that $\sum_{i=1}^n q_i = 0$. The transition from momentum twistors to helicity spinors could be performed with a formula like:[11]

$$\mu = \tilde{Q} \cdot \tilde{\lambda}\,, \quad \eta = \tilde{Q} \cdot \tilde{\eta}\,, \quad \tilde{Q}_{ij} = \begin{cases} \langle j\, i\rangle & \text{if } 1 < j < i \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

Note, that momentum super twistors trivialize both on-shell condition $q_i^2 = 0$ and mentioned above conservation of super momentum.

To construct the gluing operator acting in momentum twistor space let us recall that the initial gluing operator in helicity spinor variables can be represented as an action of two consecutive BCFW bridges times some regulator[12] factor (2.23). The BCFW bridge operators can also be defined in momentum twistor space using special version of the on-shell diagrams [84]. The action of $[i, i+1\rangle$ the BCFW shift bridge operator $br(\hat{i}, i+1)$ in momentum twistor representation on the function Y of $\{\mathcal{Z}_i\}_{i=1}^n$ variables is given by [80, 84]:

$$Y'(\mathcal{Z}_1, \ldots, \mathcal{Z}_n) = br(\hat{i}, i+1)\,[Y(\mathcal{Z}_1, \ldots, \mathcal{Z}_n)] \equiv \int \frac{dc}{c} Y(\mathcal{Z}_1, \ldots, \hat{\mathcal{Z}}_i, \ldots \mathcal{Z}_n), \tag{4.5}$$

where Y, Y' are both functions of n momentum super twistors variables and $\hat{\mathcal{Z}}_i = \mathcal{Z}_i + c\mathcal{Z}_{i+1}$. We also do not require Y and Y' to be Yangian invariants.

[11]The matrix $\tilde{Q}_{ij}$ is a formal inverse of singular map Q_{ij}, see [82, 83] for details.

[12]We call inverse soft factor (2.24) regulator because it makes soft holomorphic limit with respect to one of the auxiliary on-shell momenta, which encodes off-shell one, regular [19, 46].

As new on-shell diagrams in momentum twistor space are no longer built from ordinary MHV_3 and $\overline{\mathrm{MHV}}_3$ vertexes (amplitudes), then in the definition of the gluing operator we will have, in principle, to change the form of regulator factor. So, to construct the gluing operator in momentum twistors we will consider the following ansatz:

$$\hat{A}_{i-1,i}^{\text{m.twistor}}[\ldots] = N\ br(\hat{i}, i+1) \circ br(\widehat{i+1}, i)\,[M\ \ \ldots] \tag{4.6}$$

with two unknown rational functions of helicity spinors λ_i (first components of momentum twistors) — measure M and normalization coefficient N. To fix N, M functions we require that

$$\hat{A}_{n+1,n+2}^{\text{m.twistor}}\left[\mathcal{L}_{n+2}^{k}\right] = \omega_{n+2}^{k}\,, \tag{4.7}$$

where (k is $\mathrm{N}^{(k-2)}$MHV degree and the use of an appropriate integration contour is assumed):

$$\omega_{n+2}^{k} = \int \frac{d^{(k-2)\times(n+2)}D}{\mathrm{Vol}[\mathrm{GL}(k-2)]}\ Reg.\ \frac{\delta^{4(k-2)|4(k-2)}(D\cdot\mathcal{Z})}{(1\ \ldots\ k-2)\ \ldots\ (n+2\ \ldots\ k-3)}\,,$$
$$Reg. = \frac{1}{1+\frac{\langle p_{n+1}\,\xi_{n+1}\rangle}{\langle p_{n+1}\,1\rangle}\frac{(n+2\ 2\ \ldots\ k-2)}{(1\ \ldots\ k-2)}}\,, \tag{4.8}$$

is the momentum twistor Grassmannian integral representation for the ratio of amplitudes with one Wilson line operator insertion $A^{*}_{k,n+1}/A^{*}_{2,n+1}$ and $\mathcal{L}_{n+2}^{k}$ is the Grassmannian representation for $A_{k,n+2}/A_{k=2,n+2}$ on-shell amplitude ratio:

$$\mathcal{L}_{n+2}^{k} = \int \frac{d^{(k-2)\times(n+2)}D}{\mathrm{Vol}[\mathrm{GL}(k-2)]}\frac{\delta^{4(k-2)|4(k-2)}(D\cdot\mathcal{Z})}{(1\ \ldots\ k-2)\ \ldots\ (n+2\ \ldots\ k-3)}. \tag{4.9}$$

That is our gluing operation should transform the Grassmannian integral representation of on-the shell amplitudes into corresponding Grassmannian integral representation for the off-shell amplitudes. From this requirement we get

$$M = N^{-1} = S(i+1, i, i-1)\,, \tag{4.10}$$

where S is the usual soft factor

$$S(i+1,i,i-1) = \frac{\kappa^{*}_{i-1}\langle i-1\,i+1\rangle}{\langle i\,i+1\rangle\langle i-1\,i\rangle}\,. \tag{4.11}$$

Computation details can be found in appendix A.

So, finally, we have the following expression for the gluing operation in momentum twistor space

$$\hat{A}_{i-1,i}^{\text{m.twistor}}[\ldots] = S(i+1,i,i-1)^{-1}\ br(\hat{i}, i+1)\circ br(\widehat{i+1}, i)\,[S(i+1,i,i-1)\ldots]\,. \tag{4.12}$$

It may be, at first glance, surprising that here in momentum twistor space we used $[i, i+1\rangle$ BCFW shift and not $[i-1, i\rangle$ as for the gluing operation in the helicity spinors representation. In fact, mentioned before the two BCFW shifts are equivalent, see, for example,

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discussion in [1]. It is also assumed that in the construction of dual variables (dual contour) we decompose any off-shell momentum k_i that we encounter in a pair of (complex) on-shell momenta as [8, 19]:

$$k_i = |p_i\rangle \frac{\langle \xi_i | k_i}{\langle p_i \xi_i \rangle} + |\xi_i\rangle \frac{\langle p_i | k_i}{\langle p_i \xi_i \rangle}. \tag{4.13}$$

So we will have a pair of axillary momentum twistor variables Z_i and Z_{i+1} which encode information about off-shell momenta k_i (see figure 5 as example). The same is also true for supersymmetric counterparts of k_i momenta [19].

Now, when we have an explicit definition of the gluing operator $\hat{A}_{ii+1}$ in momentum twistor space, let us proceed with particular applications of it. Hereafter we will drop *m.twistor* subscript to simplify the notations and hope that it will not lead to any confusion. First, consider the ratio

$$\mathcal{P}^{4(k-2)}_{n+2}(\mathcal{Z}_1, \ldots, \mathcal{Z}_{n+2}) = A_{k,n+2}/A_{2,n+2}. \tag{4.14}$$

Applying to it the gluing operation $\hat{A}_{i-1,i}$ we will have (see figure 6):

$$\hat{A}_{i-1,i}\left[\mathcal{P}^{4(k-2)}_{n+2}\right] = S(i+1,i,i-1)^{-1} \int \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2} \frac{\langle i-1\, i+1\rangle + \alpha_1 \langle i-1\, i\rangle + \alpha_1\alpha_2 \langle i-1\, i+1\rangle}{\langle i\, i+1\rangle(\langle i-1\, i\rangle + \alpha_2 \langle i-1\, i+1\rangle)}$$
$$\times \mathcal{P}^{4(k-2)}_{n+2}(\ldots, \mathcal{Z}_i + \alpha_2 \mathcal{Z}_{i+1}, \mathcal{Z}_{i+1} + \alpha_1 \mathcal{Z}_i + \alpha_1\alpha_2 \mathcal{Z}_{i+1} \ldots). \tag{4.15}$$

Taking the residues at $\alpha_1 = 0$, $\alpha_2 = -\frac{\langle i-1\, i\rangle}{\langle i-1\, i+1\rangle}$ we finally get

$$\hat{A}_{i-1,i}\left[\mathcal{P}^{4(k-2)}_{n+2}\right] = \mathcal{P}^{4(k-2)}_{n+2}\left(\ldots, \mathcal{Z}_i - \frac{\langle i-1\, i\rangle}{\langle i-1\, i+1\rangle}\mathcal{Z}_{i+1}, \mathcal{Z}_{i+1}, \ldots\right), \tag{4.16}$$

which should be proportional to $A^*_{k,n+1}$ i.e.

$$\frac{A^*_{k,n+1}}{A^*_{2,n+1}} = \hat{A}_{i-1,i}\left[\mathcal{P}^{4(k-2)}_{n+2}\right], \tag{4.17}$$

where the Wilson line operator insertion is positioned between the on-shell states with numbers $i-2$ and $i+1$. If one has to consider several Wilson line operator insertions then one should apply the gluing operation several times similar to the examples considered in section 3.2. We see that application of the gluing operation in momentum twistor space significantly simplifies compared to the helicity spinor case (2.15) and amounts to just a shift of i-th momentum super twistor:

$$\mathcal{Z}^*_i = \mathcal{Z}_i - \frac{\langle i-1\, i\rangle}{\langle i-1\, i+1\rangle}\mathcal{Z}_{i+1}, \tag{4.18}$$

which is similar to the BCFW shift. This shift, as expected, transforms ordinary $1/P^2_{k,j}$ propagators, which in momentum twistor space are proportional to $1/\langle k-1 k j-1 j\rangle$, into eikonal ones (up to $\langle ij\rangle$ factors) if i belongs to $(k-1,\ k,\ j-1,\ j)$ set. As an example, let's consider the NMHV 6-point amplitude considered in the previous section. The term B in figure 3 before gluing contained propagator pole which in dual variables is given by

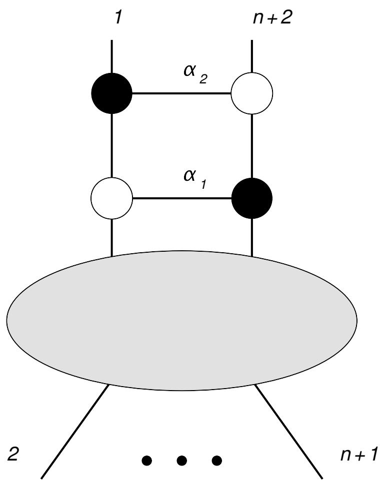

Figure 6. Double BCFW bridge $br(1,\widehat{n+2})\circ br(n+2,\hat{1})$.

$1/x_{14}^2$ with $x_{14}^2 \sim \langle 6134\rangle$. So under the action of $\hat{A}_{34}$ we get $\langle 6134\rangle \mapsto \langle 6134^*\rangle$, which in its turn can be transformed as:

$$\langle 6134^*\rangle = \langle 6134\rangle + \frac{\langle 34\rangle}{\langle 35\rangle}\langle 6135\rangle = \frac{\langle 34\rangle}{\langle 35\rangle}\langle 3|(q_1+q_2+q_3)q_4|5\rangle \sim \langle p_2|q_1+q_2|p_2], \qquad (4.19)$$

where we used explicit expressions for q_3 and q_4 ($k = q_3 + q_4$):

$$q_3 = |p_2\rangle\frac{\langle\xi|k}{\langle p_2\xi\rangle},\ q_4 = |\xi\rangle\frac{\langle p_2|k}{\langle p_2\xi\rangle},\ k|p_2\rangle = \kappa^*|p_2]. \qquad (4.20)$$

Comparing this expression to (3.35) we see that $1/\langle 6134^*\rangle$ is given exactly by the eikonal propagator. This observation can be easily generalized to the arbitrary k,n case. So the whole analysis of section 3.2 can be performed in the momentum twistor space with the identical result, which is accumulated in (4.17), (4.16) relations. We will not repeat it here and will restrain ourselves to the consideration of some particular examples.

Namely, let's reproduce the results for $A^*_{3,4+1}/A^*_{2,4+1}$ and $A^*_{3,2+2}/A^*_{2,2+2}$ off-shell amplitudes using the gluing operator. To do this we start with the ratio $\mathcal{P}_6^4 = A_{3,6}/A_{2,6}$ of the on-shell amplitudes

$$\mathcal{P}_6^4 = [12345] + [13456] + [12356]\,, \qquad (4.21)$$

where as usual five-bracket is given by [1]:

$$[i\ j\ k\ l\ m] = \frac{\delta^4(\langle i\ j\ k\ l\rangle\eta_m + \text{cyclic permutation})}{\langle i\ j\ k\ l\rangle\langle j\ k\ l\ m\rangle\langle k\ l\ m\ i\rangle\langle l\ m\ i\ j\rangle\langle m\ i\ j\ k\rangle} \qquad (4.22)$$

with four-brackets defined as

$$\langle i\ j\ k\ l\rangle = \varepsilon_{ABCD}Z_i^A Z_j^B Z_k^C Z_l^D\,. \qquad (4.23)$$

Then we have[13]

$$\hat{A}_{5,6}\left[\mathcal{P}_6^4\right] = \frac{1}{1+\frac{\langle p_5\xi_5\rangle}{\langle p_5 1\rangle}\frac{\langle 1345\rangle}{\langle 3456\rangle}}[13456] + \frac{1}{1+\frac{\langle p_5\xi_5\rangle}{\langle p_5 1\rangle}\frac{\langle 1235\rangle}{\langle 2356\rangle}}[12356] + [12345],$$
$$\hat{A}_{5,6}\left[\mathcal{P}_6^4\right] = \frac{A^*_{3,4+1}}{A^*_{2,4+1}}\left(\Omega_1,\ldots,\Omega_4,g_5^*\right), \tag{4.24}$$

and[14]

$$\hat{A}_{3,4}\circ\hat{A}_{5,6}\left[\mathcal{P}_6^4\right] = c_{35}[12345] + c_{36}[12356] + c_{46}[13456],$$
$$\hat{A}_{3,4}\circ\hat{A}_{5,6}\left[\mathcal{P}_6^4\right] = \frac{A^*_{3,2+2}}{A^*_{2,2+2}}\left(\Omega_1,\Omega_2,g_3^*,g_4^*\right), \tag{4.25}$$

with

$$c_{35} = \frac{1}{1+\frac{\langle p_3\xi_3\rangle\langle 1235\rangle}{\langle p_3p_4\rangle\langle 1234\rangle}}, \quad c_{36} = \frac{1}{1+\frac{\langle p_4\xi_4\rangle}{\langle p_4 1\rangle}\frac{\langle 1235\rangle}{\langle 2356\rangle}}, \quad c_{46} = \frac{1}{1+\frac{\langle p_3\xi_3\rangle}{\langle p_3p_4\rangle}\frac{\langle 1356\rangle}{\langle 1346\rangle}}\frac{1}{1+\frac{\langle p_4\xi_4\rangle}{\langle p_4 1\rangle}\frac{\langle 1345\rangle}{\langle 3456\rangle}}. \tag{4.26}$$

These results are in complete agreement with previously obtained results from the off-shell BCFW [46] and Grassmannian integral representation [19, 20].

In general the on-shell ratio function $\mathcal{P}_{k,n+2} = \mathcal{P}_{n+2}^{4(k-2)}(\mathcal{Z}_1,\ldots,\mathcal{Z}_{n+2})$ can be found for fixed n and k via the solution of the on-shell BCFW recursion in momentum twistor space [80]:

$$\mathcal{P}_{k,n}(\mathcal{Z}_1,\ldots,\mathcal{Z}_n) = \mathcal{P}_{k,n-1}(\mathcal{Z}_1,\ldots,\mathcal{Z}_{n-1}) \tag{4.27}$$
$$+\sum_{j=2}^{n-2}[j-1,j,n-1,n,1]\mathcal{P}_{k_1,n+2-j}(\mathcal{Z}_{I_j},\mathcal{Z}_j,\mathcal{Z}_{j+1},\ldots,\hat{\mathcal{Z}}_{n_j})\mathcal{P}_{k_2,j}(\mathcal{Z}_{I_j},\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\mathcal{Z}_{j-1}),$$

where[15] $\mathcal{Z}_{n_j} = (n-1,n)\cap(1,j-1,j)$, $\hat{\mathcal{Z}}_{I_j} = (j-1,j)\cap(1,n-1,n)$, $k_1+k_2+1=k$. We will make more comments about the structure of this recursion relation in the next section. From practical point of view the easiest way to compute Wilson line form factor with f on-shell states and m Wilson line operator insertions is to solve (4.27) for $n=f+2m$ and then apply m gluing operators via (4.16) rule.

Now, when we have the definition of the gluing operator $\hat{A}_{ii+1}$ in momentum twistor space and some practice with the tree level answers we are ready to consider loop integrands.

5 Loop integrands

The natural way to define planar loop integrands unambiguously is to use momentum twistors or dual coordinates. The loop integrand $I^L_{k,n}$ for on-shell L-loop amplitude $A^L_{k,n}$

[13] It is assumed that the momentum super twistors $\mathcal{Z}_5$ and $\mathcal{Z}_6$ are sent to corresponding off-shell kinematics related to off-shell momenta of g_5^* reggeized gluon.

[14] We again assume corresponding off-shell kinematics for momentum super twistors $\mathcal{Z}_3$-$\mathcal{Z}_6$ describing reggeized gluons g_3^* and g_4^*.

[15] $(i,j)\cap(k,p,m)\equiv\mathcal{Z}_i\langle jkpm\rangle+\mathcal{Z}_j\langle ikpm\rangle$.

in this language is defined as[16]

$$A^{(L)}_{k,n}/A^{(0)}_{2,n} = \int_{reg} \prod_{m=1}^{L} d^4 l_m I^{(L)}_{k,n}(\mathcal{Z}_1,\ldots,\mathcal{Z}_n; l_1,\ldots,l_L)\,, \tag{5.1}$$

where momentum super twistors $\mathcal{Z}_1,\ldots,\mathcal{Z}_n$ describe kinematics of external particles and reg stands for regularization needed by loop integrals. Here $I_{k,n}$ is a rational function of both loop integration and external kinematical variables. Moreover, $I_{k,n}$ is cyclic in external momentum super twistors. It is also assumed that loop integrand is completely symmetrized in loop variables $l_1,\ldots,l_L$. Rewriting the latter in terms of bi-twistors ($l_m \equiv (A_m B_m) \equiv (AB)_m$) the loop integration measure takes the form [80]:

$$d^4 l = \langle AB d^2 A\rangle\langle AB d^2 B\rangle = \frac{d^4 Z_A d^4 Z_B}{\mathrm{Vol}[\mathrm{GL}(2)]}\,, \tag{5.2}$$

where we dropped out factors $\langle\lambda_A\,\lambda_B\rangle = \langle \mathcal{Z}_A\mathcal{Z}_B I_\infty\rangle$ as the integrands in $\mathcal{N}=4$ SYM are always dual conformal invariant. Here I_∞ denotes infinity bi-twistor [81]. The integral over the line (AB) is given by the integrals over the points Z_A, Z_B modulo GL(2) transformations leaving them on the same line.

5.1 BCFW for integrands of Wilson lines form factors and correlation functions

Now let us see what modifications occur to the on-shell integrand BCFW recursion in the off-shell case. The loop-level BCFW for on-shell amplitudes in $\mathcal{N}=4$ SYM was worked out in detail in [80] (see also [85, 86] for situation with less SUSY) and the result for $\hat{\mathcal{Z}}_n = \mathcal{Z}_n + w\mathcal{Z}_{n-1}$ shift reads

$$\begin{aligned} I^{(L)}_{k,n} =& I^{(L)}_{k,n-1}(\mathcal{Z}_1,\ldots,\mathcal{Z}_{n-1}) \\ &+\sum_{j=2}^{n-2}[j-1,j,n-1,n,1] I^{(L_1)}_{k_1,n+2-j}(\mathcal{Z}_{I_j},\mathcal{Z}_j,\mathcal{Z}_{j+1},\ldots,\hat{\mathcal{Z}}_{n_j}) I^{(L_2)}_{k_2,j}(\mathcal{Z}_{I_j},\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\mathcal{Z}_{j-1}) \\ &+\int\frac{d^{4|4}\mathcal{Z}_A d^{4|4}\mathcal{Z}_B}{\mathrm{Vol}[\mathrm{GL}(2)]}\int_{\mathrm{GL}(2)}[A,B,n-1,n,1] I^{(L-1)}_{k+1,n+2}(\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\hat{\mathcal{Z}}_{n_{AB}},\mathcal{Z}_A,\mathcal{Z}_B)\,, \end{aligned} \tag{5.3}$$

where $\hat{\mathcal{Z}}_{n_j} = (n-1,n)\cap(1,j-1,j)$, $\mathcal{Z}_{I_j} = (j-1,j)\cap(1,n-1,n)$, $\hat{\mathcal{Z}}_{n_{AB}} = (n-1,n)\cap(A,B,1)$ and $k_1+k_2+1=k$. The $\int_{\mathrm{GL}(2)}$ integral is defined as follows. First we set $\mathcal{Z}_A\to\mathcal{Z}_A+\alpha\mathcal{Z}_B\equiv\mathcal{Z}'_A$ and $\mathcal{Z}_B\to\mathcal{Z}_B+\beta\mathcal{Z}_A\equiv\mathcal{Z}'_B$, which is equivalent to moving points $\mathcal{Z}_A$ and $\mathcal{Z}_B$ without changing the line they span. Then we calculate composite residue in α, β such that $\langle A',1,n-1,n\rangle\to 0$ and $\langle B',1,n-1,n\rangle\to 0$, what is equivalent to taking points A', B' to lie on the plane $\langle 1,n-1,n\rangle$:

$$\int_{\mathrm{GL}(2)} \equiv \int_{\langle A',1,n-1,n\rangle\to 0} d\alpha \int_{\langle B',1,n-1,n\rangle\to 0} d\beta\,(1-\alpha\beta)^2\,. \tag{5.4}$$

Taking the residue as above is equivalent to setting $\mathcal{Z}'_A,\mathcal{Z}'_B$ to $(A,B)\cap(1,n-1,n)$ and the Jacobian factor $(1-\alpha\beta)^2$ makes poles in α,β simple.

[16] Here by dividing on MHV amplitude we mean that we are factoring out $\langle 12\rangle\ldots\langle n1\rangle$ product and dropping momentum conservation delta function.

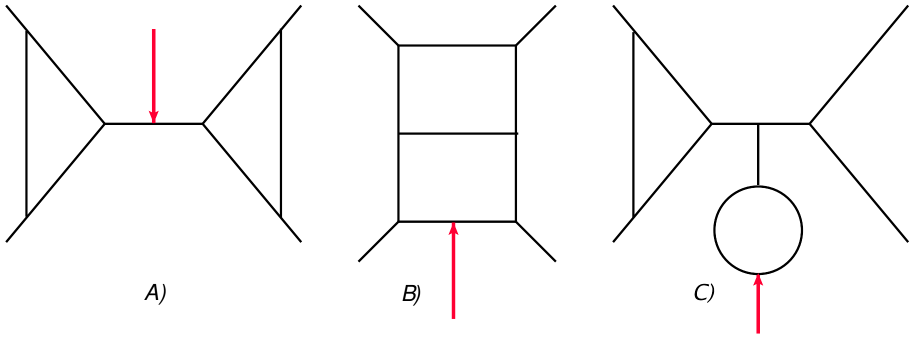

Figure 7. Different possible types of pole contributions to loop BCFW recursion. Here we depicted scalar integrals for two loop $n = 4$ example. Red arrows indicate propagators which we are cutting when evaluating residues. Term C) is actually absent in $\mathcal{N} = 4$ SYM case as well as A).

Next, let us make some comments about the origin of different terms in (5.3). The first two terms, namely

$$I^{(L)}_{k,n-1}(\mathcal{Z}_1,\ldots,\mathcal{Z}_{n-1}) + \sum_{j=2}^{n-2}[j-1,j,n-1,n,1] I^{(L_1)}_{k_1,n+2-j}(\mathcal{Z}_{I_j},\ldots,\hat{\mathcal{Z}}_{n_j}) I^{(L_2)}_{k_2,j}(\mathcal{Z}_{I_j},\ldots,\mathcal{Z}_{j-1}) \tag{5.5}$$

originate from the poles in the BCFW shift parameter w coming from propagators which does not contain loop momentum dependence:

$$\langle i-1 i n-1 \hat{n}(w)\rangle = 0. \tag{5.6}$$

that is from propagators connecting loop integrals, see figure 7 A. These contributions are identical both at tree and loop level. The term containing GL(2) integration

$$\int_{\mathrm{GL}(2)} [A,B,n-1,n,1] I^{(L-1)}_{k+1,n+2}(\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\hat{\mathcal{Z}}_{n_{AB}},\mathcal{Z}_A,\mathcal{Z}_B) \tag{5.7}$$

is present only at the loop level. It originates from the poles in the BCFW shift parameter w coming from propagators containing loop momenta [80], see figure 7 B. At L loop level for n point amplitude the residue at such pole corresponds to the so called forward limit of $L-1$ loop $n+2$ point amplitude. Indeed, if we consider L loop integrand[17] of some amplitude $I^{(L)}_n(\{p_1,\ldots,p_n\},l_1,\ldots,l_L)$, where $\{p_1,\ldots,p_n\}$ are external momenta and consider residue at the pole $1/l_L^2$ corresponding to L'th loop integration we will get (see figure 8)

$$Res_{l_L^2=0}\, I^{(L)}_n \sim I^{(L-1)}_{n+2}(\{p_1,\ldots,p_n,-l_L,l_L\},l_1,\ldots,l_{L-1}). \tag{5.8}$$

In momentum twistor space residue can be evaluated as follows. For simplicity let's consider $L = 1$ example to make formulas more readable. The generalization for general L is trivial. The n-point amplitude integrand is the function of the following variables

[17]Here we assume some specific "appropriate" choice of loop momenta. The corresponding ambiguity in the choice of loop momenta can be removed [80] if one considers dual (or momentum twistor) variables and planar limit, which we are interested in.

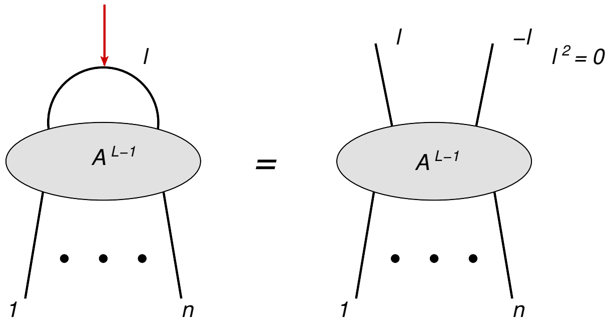

Figure 8. Evaluation of residue at the pole of loop propagator (cut) resulting in the forward limit. Red arrow indicates which propagator we are cutting.

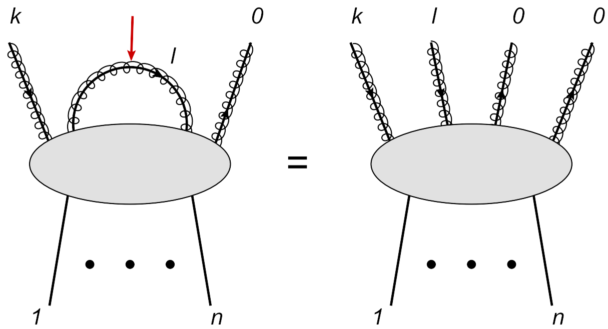

Figure 9. Evaluation of residue in the pole (cut) of eikonla loop propagator. Red arrow indicates which propagator we are cutting.

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$I_n^{(1)}(\{\mathcal{Z}_1 \ldots, \mathcal{Z}_n\}, \mathcal{Z}_A, \mathcal{Z}_B)$. The residue at the point (we consider $\hat{\mathcal{Z}}_n = \mathcal{Z}_n + w\mathcal{Z}_{n-1}$ shift and take residue with respect to w parameter)

$$\langle AB1\hat{n}(w)\rangle = 0 \tag{5.9}$$

is given by:

$$Res_{\langle AB1\hat{n}\rangle=0}\ I_n^{(1)} \sim A_{n+2}^{\text{tree}}(\mathcal{Z}_1, \ldots, \hat{\mathcal{Z}}_n, \hat{\mathcal{Z}}_B, \hat{\mathcal{Z}}_B), \tag{5.10}$$

where

$$\begin{aligned} \hat{\mathcal{Z}}_n &= (n-1, n) \cap (A, B, 1), \\ \hat{\mathcal{Z}}_B &= (A, B) \cap (n-1, n, 1). \end{aligned} \tag{5.11}$$

This is analog of (5.8) in momentum twistor space, see also figure 9 and 10. The first expression for $\hat{\mathcal{Z}}_n$ solves $\langle AB1\hat{n}\rangle = 0$. The second expression for $\hat{\mathcal{Z}}_B$ is the consequence of the first one and the forward limit. See [1] for detailed derivation and discussion. The expression (5.10) in this limit could be obtained from the expression for $A_{n+2}^{\text{tree}}(\mathcal{Z}_1, \ldots, \hat{\mathcal{Z}}_n, \mathcal{Z}_A, \hat{\mathcal{Z}}_B)$ at general kinematics[18] by introducing GL(2) integration with $[A, B, n{-}1, n, 1]$ weight (5.7).

Now let's see how similar to (5.3) the recurrence relation for Wilson line form factors can be constructed. Let's consider integrand $I_{k,n+1}^{*(L)}$ of $A_{k,n+1}^{*(L)}(\Omega_1^*, \ldots, \Omega_n, g_{n+1}^*)$ Wilson line

[18] General in a sense that there are no collinear twistors in contrast to (5.10).

form factor. As one will try to reconstruct it via $\hat{\mathcal{Z}}_i = \mathcal{Z}_i + w\mathcal{Z}_{i-1}$ shift he/she will encounter two types of contributions. The first type will be given by the residues with respect to propagators which does not contain loop momentum dependence. These can be considered along the same lines as in sections 3 and 4. The second type of contribution is the residues with respect to propagator poles with loop momentum dependence. Now in contrast to the case of on-shell amplitudes we have two types of propagator poles. Ordinary $1/l^2$ poles and eikonal ones $1/\langle p|l|p]$. To simplify discussion let's consider one-loop case. Generalization to higher loops can be easily done by induction. The residue evaluation with respect to $1/l^2$ poles is identical to the on-shell amplitudes case and is given by forward limit of tree level Wilson line form factor with $n+2$ on-shell legs (k as usual is off-shell momentum with direction p and $\{q_1, \ldots, q_n\}$ are on-shell momenta):

$$Res_{l_L^2=0}\, I_{n+1}^{*(L=1)} \sim A_{(n+2)+1}^{*(tree)}(\{q_1, \ldots, q_n, -l_L, l_L\}, \{p, k\}). \tag{5.12}$$

The terms which include eikonal propagator pole residue are a little more complicated. Surprisingly, here similar to the on-shell case we also have forward like limit. For example, consider Wilson line form factor at one loop level $I_n^{*(1)}(\{q_1, \ldots, q_n\}, \{k, p\}, l)$. Here once again $\{q_1, \ldots, q_n\}$ are on-shell momenta and k is off-shell momentum with direction p. Using decomposition (4.13) we can decompose our off-shell momentum into pair of on-shell momenta $k' = |p\rangle k|\xi\rangle/\langle p\xi\rangle$, $k'' = |\xi\rangle k|p\rangle/\langle \xi p\rangle$ and formally write this integrand as $I_n^{*(1)}(\{q_1, \ldots, q_n, k'', k'\}, l)$. Considering residue for the pole $1/\langle p|l|p]$ we enforce on loop momentum l condition $\langle p|l|p] = 0$, $l^2 \neq 0$ (see also (3.9) and discussion there). This results in (see figure 9)

$$Res_{\langle p|l|p]=0}\, I_{n+1}^{*(1)} \sim A_{n+2}^{*}(\{q_1, \ldots, q_n, k'', k', l', l''\}), \tag{5.13}$$

where $l' = |p\rangle l|\xi'\rangle/\langle \xi' p\rangle$, $l'' = |\xi'\rangle l|p\rangle/\langle \xi' p\rangle$. Now using the freedom in the choice of $|\xi'\rangle$ one can set l' and k' collinear to each other:

$$(l')^\mu = -(k')^\mu/\langle \xi' p\rangle \tag{5.14}$$

up to scalar factor $\langle \xi' p\rangle$. This resembles the on-shell forward limit kinematics of (5.8) for $n+4$ point off-shell amplitude. So presumably the residue with respect to Wilson line propagators can, in principle, be evaluated in momentum twistor space along the same lines as (5.10) and (5.11). Consideration of this eikonal residue type, however, *can be avoided entirely* if one will choose BCFW shift in such a way that w parameter will not appear in eikonal propagators at all.

To see this let's consider once again one-loop case, that is we take the solution of (5.3) for n external particles $I_{k,n}^{(L=1)}$ and apply gluing operator $\hat{A}_{n-1n}$ to it

$$I_{k,(n-2)+1}^{*(L=1)} = \hat{A}_{n-1n}\left[I_{k,n}^{(L=1)}\right]. \tag{5.15}$$

We will assume that the tree level form factors and on-shell amplitudes are related as

$$\hat{A}_{n-1n}[A_{k,n}(\Omega_1 \ldots, \Omega_n)] = A_{k,(n-2)+1}^{*}(\Omega_1 \ldots, \Omega_{n-2}, g_{n-1}^{*}). \tag{5.16}$$

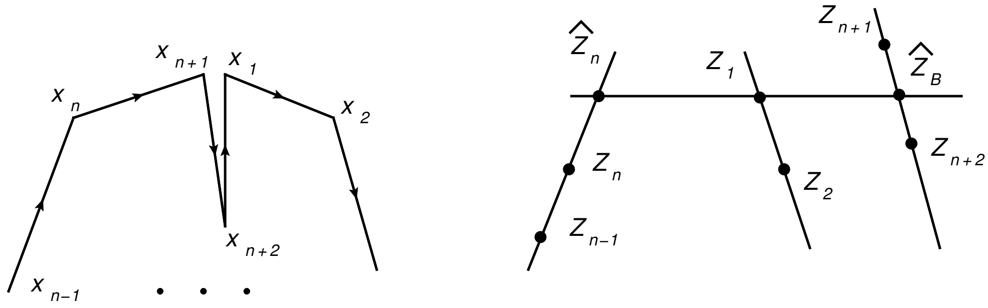

Figure 10. Forward limit in momentum and momentum twistor space. In momentum space we are gluing x_{n+1} with x_1 while keeping x_n fixed in such a way that $x_{1n}^2 = 0$. In momentum twistor space this equivalent to gluing Z_{n+1} and Z_{n+2} with $\hat{Z}_B$. The same is also true for their supersymmetric counterparts.

What we are going to show now is that $I^{*(L=1)}_{k,(n-2)+1}$ will have appropriate factorization properties[19] for one loop Wilson line form factor and that it can be obtained from recurrence relation similar to (5.3), where only poles of (5.12) type will contribute. I.e. there always will be possibility to choose the BCFW shift in such a way that only $1/P^2$ type poles will contribute to recursion.

To show this let us consider all possible BCFW shifts in $I^{*(L=1)}_{k,(n-2)+1}$. But first let us note that in the case under consideration the pair of axillary momentum twistor variables $\mathcal{Z}_n$ and $\mathcal{Z}_{n-1}$ is used to encode information about off-shell momentum k according to (4.13). So, the only possible propagators which contain loop momentum and which will be affected by gluing operator $\hat{A}_{n-1n}$ are given by $\langle ABn-1n\rangle$ and $\langle ABn1\rangle$. More accurately, only $\langle ABn-1n\rangle$ will be transformed into eikonal propagator $\langle ABn-1n^*\rangle$ since $\langle ABn^*1\rangle = \langle ABn1\rangle$. Equivalently one can note that due to the cyclical symmetry the only possible eikonal propagator with loop momentum dependence in $A^{*(L=1)}_{k,(n-2)+1}$ will depend on $Z_A, Z_B, Z_{n-1}, Z_n, Z_1$ momentum twistors.

Now let's return to the shifts. If we shift $\mathcal{Z}_i$ as $\hat{\mathcal{Z}}_i = \mathcal{Z}_i + w\mathcal{Z}_{i-1}$ for $i = 1, \ldots, n-2$, then the shift parameter w will not affect the eikonal propagator and the corresponding residues with respect to w can be evaluated according to (5.12), so that the result will be given by the forward limit of the tree level Wilson line form factor with n on-shell states. These is precisely the desired factorization property. For this form factor we also know that the relation (5.16) holds. So we see that in such cases the gluing operation indeed transforms solutions of (5.3) into Wilson line form factors similar to tree level.

As for the shifts involving $\mathcal{Z}_{n-1}$ and $\mathcal{Z}_n$, we can always choose to shift $\hat{\mathcal{Z}}_n = \mathcal{Z}_n + w\mathcal{Z}_{n-1}$, so that the w parameter drops out of $\langle ABn-1n^*\rangle$ bracket and will remain only in $\langle AB\hat{n}1\rangle$ bracket, which is again not affected by the action of $\hat{A}_{n-1n}$ gluing operator. This gives us

$$Res_{\langle AB\hat{n}1\rangle=0}\ \hat{A}_{n-1n}[I_n^{(1)}] \sim \hat{A}_{n-1n}[A^{\text{tree}}_{n+2}] = A^*_{(n)+1}(\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-1}, \hat{\mathcal{Z}}_n^*, \hat{\mathcal{Z}}_B, \hat{\mathcal{Z}}_B), \tag{5.17}$$

[19]That is the corresponding residue will be given by forward limit of tree level Wilson line form factor with $n+2$ on-shell states.

where

$$\hat{\mathcal{Z}}_n^* = (n-1, n^*) \cap (A, B, 1),$$
$$\hat{\mathcal{Z}}_B = (A, B) \cap (n-1, n, 1),$$
$$\mathcal{Z}_n^* = \mathcal{Z}_n + \frac{\langle p\xi\rangle}{\langle p1\rangle}\mathcal{Z}_1. \tag{5.18}$$

Here we see that $\hat{\mathcal{Z}}_n^*$ solves $\langle AB\hat{n}^*1\rangle = \langle AB\hat{n}1\rangle = 0$ — the same condition as in the case of the on-shell amplitudes. So once again we have appropriate factorization properties and we also see that the gluing operation indeed transforms solutions of (5.3) into the Wilson line form factors.

Equivalently using the same arguments as above one can show that in $A^{*(L=1)}_{k,(n-2)+1}$ in pair $\mathcal{Z}_{n-1}, \mathcal{Z}_n$ one can always choose to shift $\hat{\mathcal{Z}}_n = \mathcal{Z}_n + w\mathcal{Z}_{n-1}$ so that w will drop out from eikonal propagator. That is for all $\hat{\mathcal{Z}}_i$, which describe both on-shell and off-shell momenta, one can choose such shifts that will not affect eikonal propagators with loop momentum dependance and the corresponding recurrence relations will contain only contribution of (5.5) and (5.12) type.

This considerations can be easily generalized by induction to arbitrary loop level and to arbitrary number of gluing operators applied. So we may conclude that application of $\hat{A}_{i-1i}$ to (5.3) will likely result in a valid recursion relation for loop integrands of Wilson line form factors (off-shell amplitudes) similar to tree level case. For example if we chose $i = n$, to match our previous considerations, we will get recurrence relation for the integrand $I^{*(L)}_{k,(n-2)+1}$ of Wilson line form factor when operator is inserted after on-shell state with number $n-2$:

$$\begin{aligned}
I^{*(L)}_{k,(n-2)+1} &= I^{(L)}_{k,n-1}(\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-1}) \\
&+ \sum_{j=2}^{n-2}[j-1, j, n-1, n^*, 1] I^{(L_1)}_{k_1,n+2-j}(\mathcal{Z}_{I_j}, \mathcal{Z}_j, \mathcal{Z}_{j+1}, \ldots, \hat{\mathcal{Z}}^*_{n_j}) I^{(L_2)}_{k_2,j}(\mathcal{Z}_{I_j}, \mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_{j-1}) \\
&+ \int \frac{d^{4|4}\mathcal{Z}_A d^{4|4}\mathcal{Z}_B}{\mathrm{Vol}[\mathrm{GL}(2)]} \int_{\mathrm{GL}(2)} [A, B, n-1, n^*, 1] I^{(L-1)}_{k+1,n+2}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \hat{\mathcal{Z}}^*_{n_{AB}}, \mathcal{Z}_A, \mathcal{Z}_B),
\end{aligned} \tag{5.19}$$

where $\hat{\mathcal{Z}}_{n_j} = (n-1, n^*) \cap (1, j-1, j)$, $\mathcal{Z}_{I_j} = (j-1, j) \cap (1, n-1, n)$, $\hat{\mathcal{Z}}_{n_{AB}} = (n-1, n^*) \cap (A, B, 1)$ and $k_1 + k_2 + 1 = k$. $\mathcal{Z}_n^*$ is given by (5.18). As before, to encode off-shell momenta we use twistor variables with numbers $n-1$ and n. p and ξ are light-cone vectors entering k_T-decomposition of this off-shell momentum k. Spinors $|p\rangle$ and $|\xi\rangle$ are obtained from corresponding vectors.

One can also skip the solution of this new recursion and apply $\hat{A}_{i-1i}$ directly to the solutions of on-shell recursion relation (5.3), that is to the on-shell integrands, similar to the tree level case (4.16). In the next section we will consider such action using local form of integrands instead of non-local form produced directly by BCFW recursion.

At the end of this section we want to make the following note: in general forward limits may not be well defined [80], because *on the level of integrands* one may encounter contributions from tadpoles and bubble type integrals on external on-shell legs (see figure 7 C as

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an example). However, such contributions are absent in $\mathcal{N} = 4$ SYM on-shell amplitudes due to the enhanced SUSY cancellations [1, 80]. Their analogs are also absent for the Wilson line form factors (off-shell reggeon amplitudes) — there are no tadpoles diagrams involving closed Wilson line propagators and bubbles on external Wilson line are also equal to 0 on integrand level (see Feynman rules in [46]).

5.2 Gluing operation and local integrands

Now, following our discussion in the previous subsection we conclude that the integrands for the planar off-shell L-loop amplitudes could be obtained from the corresponding on-shell integrands by means of the same gluing procedure as was used by us at tree level. Namely, for reggeon amplitude with n reggeized gluons (Wilson line operator insertions) and no on-shell states $I^{*(L)}_{k,0+n}(g_1^*,\ldots,g_n^*)$ we should have:

$$\begin{aligned} I^{*(L)}_{k,0+n} &= \hat{A}_{2n-1\,2n}\circ\ldots\circ\hat{A}_{12}\Big[I_{k,2n}\Big] \\ &= I^{(L)}_{k,2n}\left(\mathcal{Z}_1,\mathcal{Z}_2-\frac{\langle 1\,2\rangle}{\langle 1\,3\rangle}\mathcal{Z}_3,\ldots,\mathcal{Z}_{2n-1},\mathcal{Z}_{2n}-\frac{\langle 2n-1\,2n\rangle}{\langle 2n-1 1\rangle}\mathcal{Z}_1\right). \end{aligned} \tag{5.20}$$

Here it is assumed that $I^{*(L)}_{k,0+n}$ is normalized by $A^{*(0)}_{2,0+n}$ similar to the definition of on-shell integrands (5.1). The loop integrands for reggeon amplitudes (Wilson line form factors) with on-shell states can be obtained from (5.20) by removing necessary number of $\hat{A}_{i-1i}$ operators.

The loop integrands produced by the BCFW recursion are non-local in general [80]. However, it is still possible to rewrite the integrands in a manifestly local form.[20] Moreover, one may choose as a basis the set of chiral integrals with unit leading singularities [80, 87]. The leading singularities are generally defined as the residues of a complex, multidimensional integrals of integrands in question over $\mathbb{C}^{4L}$, where L is the loop order. The computation of residues for the integrands expressed in momentum twistors is then ultimately related to the classic Schubert problem in the enumerative geometry of $\mathbb{CP}^3$ [87]. When the residues of integral associated to at least one of its Schubert problems are not the same then the integral is called chiral. In the case when the integral has at most one non-zero residue for the solutions to each Schubert problem then the integral is called completely chiral. If all non-vanishing residues are the same up to a sign then it is possible to normalize them, so that all residues are ± 1 or 0. The integrals with this property are called pure integrals or integrals with unit leading singularities.

The application of the gluing operation to the on-shell integrands written in the local form follows the general rule (5.20). Let's see some particular examples. At one-loop for MHV n-point integrand we have[21] [80, 87]:

$$I^{(1)}_{2,n} = \sum_{i<j}\frac{\langle AB(i-1\,i\,i+1)\cap(j-1\,j\,j+1)\rangle\langle Xij\rangle}{\langle AB\,X\rangle\langle AB\,i-1\,i\rangle\langle AB\,i\,i+1\rangle\langle AB\,j-1\,j\rangle\langle AB\,j\,j+1\rangle}. \tag{5.21}$$

[20] This procedure spoils the Yangian-invariance of each term in the on-shell case however.

[21] $(i-1\,i\,i+1)\cap(j-1\,j\,j+1)\equiv Z_{i-1}Z_i\langle i+1j-1jj+1\rangle+Z_iZ_{i+1}\langle i-1j-1jj+1\rangle+Z_{i-1}Z_{i+1}\langle ij-1jj+1\rangle$.

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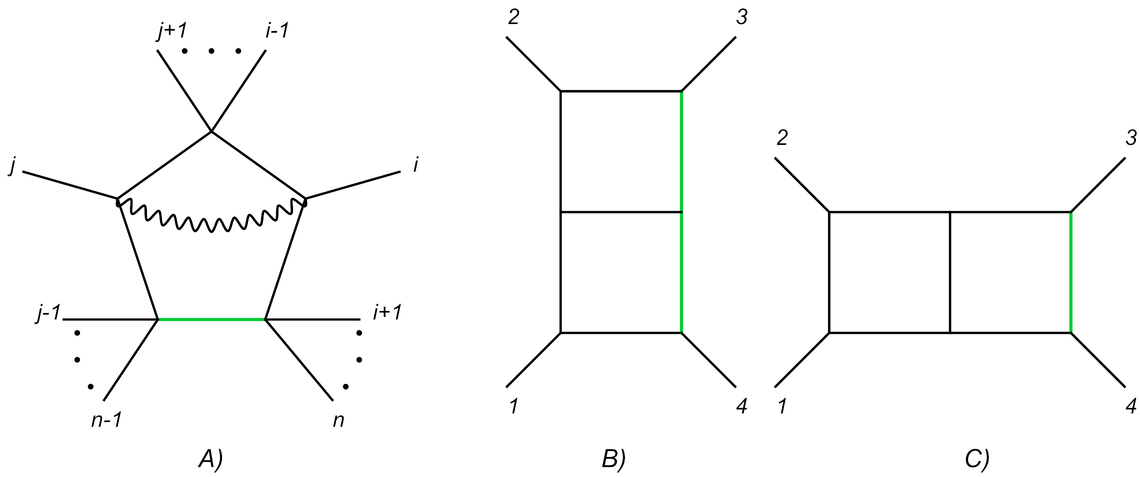

Figure 11. Integrals for $A^{*(1)}_{2,(n-2)+1}(\Omega_1,\ldots,\Omega_{n-2},g^*_{n-1})$ and $A^{*(2)}_{2,2+1}(\Omega_1,\Omega_2,g^*_3)$. Green lines corresponds to eikonal propagators with shifted twistor. Wavy line corresponds to numerator of the form $\langle AB(ij)_W\rangle$, where $(ij)_W=(i-1\,i\,i+1)\cap(j-1\,j\,j+1)$.

This expressions is cyclic invariant and sum in the above expression is independent from X, but contains spurious poles $\langle AB\,X\rangle$ term by term. If we choose $X=(k\,k+1)$ then all poles are manifestly physical but cyclic invariance will be lost. To obtain corresponding expression $I^{*(1)}_{2,(n-2)+1}$ for the amplitude with one off-shell leg in place of two last on-shell legs $A^{*(1)}_{2,(n-2)+1}(\Omega_1,\ldots,\Omega_{n-2},g^*_{n-1})$ we just shift momentum super twistor $\mathcal{Z}_n$. Also it is convenient to choose $X=(n-1n)$:

$$I^{*(1)}_{2,(n-2)+1}=\sum_{i<j}\frac{\langle AB(i-1\,i\,i+1)\cap(j-1\,j\,j+1)\rangle\langle n-1n^*ij\rangle}{\langle AB\,n-1\,n^*\rangle\langle AB\,i-1\,i\rangle\langle AB\,i\,i+1\rangle\langle AB\,j-1\,j\rangle\langle AB\,j\,j+1\rangle}, \tag{5.22}$$

where Z^*_n is given by:

$$Z^*_n=Z_n-\frac{\langle p\xi\rangle}{\langle p1\rangle}Z_1. \tag{5.23}$$

See figure 11 A. Legs $n-1$ and n describe off-shell momentum, so that p and ξ are light-cone vectors entering k_T-decomposition of this momentum k.

Next, taking the expression for the integrand of 2-loop 4-point MHV on-shell amplitude [80, 87]:

$$I^{(2)}_{2,4}=\frac{\langle 2341\rangle\langle 3412\rangle\langle 4123\rangle}{\langle AB41\rangle\langle AB12\rangle\langle AB23\rangle\langle CD23\rangle\langle CD34\rangle\langle CD41\rangle\langle ABCD\rangle}+\text{cyclic, no repeat} \tag{5.24}$$

and applying $\hat{A}_{3,4}$ gluing operation we get for the integrand of $A^{*(2)}_{2,2+1}(\Omega_1,\Omega_2,g^*_3)$ (See figure 11 B and C)

$$\begin{aligned}I^{*(2)}_{2,2+1}=&\frac{\langle 2341\rangle\langle 3412\rangle\langle 4123\rangle}{\langle AB41\rangle\langle AB12\rangle\langle AB23\rangle\langle CD23\rangle\langle CD34^*\rangle\langle CD41\rangle\langle ABCD\rangle}\\&+\frac{\langle 3412\rangle\langle 4123\rangle\langle 1234\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34^*\rangle\langle CD34^*\rangle\langle CD41\rangle\langle CD12\rangle\langle ABCD\rangle},\end{aligned} \tag{5.25}$$

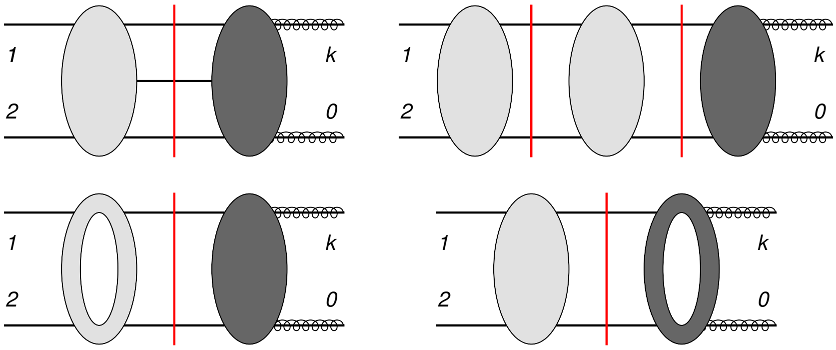

Figure 12. Unitarity cuts for $A^{*(2)}_{2,2+1}(\Omega_1, \Omega_2, g_3^*)$ where only $1/l^2$ propagators have been cut. Vertical red line represents cuts of corresponding propagators. Grey blobs are on-shell amplitudes with $k = 2, 3$. Dark grey blobs are Wilson line form factors with $k = 2, 3$.

where Z_4^* is given by:

$$Z_4^* = Z_4 - \frac{\langle p\xi\rangle}{\langle p1\rangle} Z_1. \tag{5.26}$$

As always we assume off-shell kinematics for legs 3 and 4, so that p and ξ are light-cone vectors entering k_T-decomposition of the off-shell gluon momentum k. Note also that this result is consistent with two and three particle unitarity cuts. See figure 12.

The introduced gluing operation also allows us easily obtained expressions for integrands of off-shell remainder functions starting from their on-shell counterparts. Indeed, starting from integrand for 1-loop on-shell remainder function

$$\mathcal{R}^{(1)}_{k,n} = I^{(1)}_{k,n} - \mathcal{P}^{4(k-2)}_n I^{(1)}_{2,n} \tag{5.27}$$

and applying gluing operation $\hat{A}_{n-1,n}$ we may obtain the expression for off-shell remainder function with one off-shell leg in place of two last on-shell legs

$$\mathcal{R}^{*(1)}_{k,(n-2)+1} = I^{*(1)}_{k,(n-2)+1} - \hat{A}_{n-1,n}\left[\mathcal{P}^{4(k-2)}_n\right] I^{*(1)}_{2,(n-2)+1}\,. \tag{5.28}$$

That is, for example taking integrand for $\mathcal{R}^{*(1)}_{3,6}$ on-shell remainder function written in terms of chiral octagons [87]:

$$\begin{aligned}\mathcal{R}^{*(1)}_{3,6} = &\frac{1}{2}([1,2,3,4,5]+[1,2,3,5,6^*]+[1,2,3,6^*,4])I_8(1,3,4,6^*)+\frac{1}{6}[1,2,3,4,6^*]I^{odd}_8(1,3,4,6^*)\\ &-\frac{1}{6}([1,3,4,5,6^*]-[1,2,3,4,5])I^{odd}_8(1,3,4,5)+\frac{1}{6}([1,2,4,5,6^*]+[1,3,4,5,6^*])I^{odd}_8(1,4,5,6^*)\,,\end{aligned} \tag{5.29}$$

where

$$I^{odd}_8(i,j,k,l) \equiv I_8(i,j,k,l) - I_8(j,k,l,i) \tag{5.30}$$

and (see figure 13)

$$\begin{aligned}I_8(i,j,k,l) = &\frac{\langle ABij\rangle\langle AB(j-1\,j\,j+1)\cap(k-1\,k\,k+1)\rangle}{\langle ABi-1\,i\rangle\langle ABi\,i+1\rangle\langle ABj-1\,j\rangle\langle ABj\,j+1\rangle}\\ &\times\frac{\langle ABkl\rangle\langle AB(l-1\,l\,l+1)\cap(i-1\,i\,i+1)\rangle}{\langle ABk-1\,k\rangle\langle ABk\,k+1\rangle\langle ABl-1\,l\rangle\langle ABll+1\rangle}.\end{aligned} \tag{5.31}$$

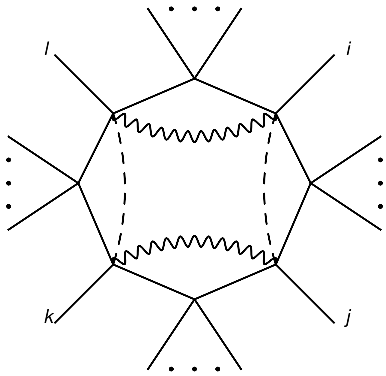

Figure 13. Chiral octagons integral. Dashed line connecting i and j external legs represents numerator of the form $\langle ABij \rangle$.

As before $\mathcal{Z}_6^*$ is defined as

$$\mathcal{Z}_6^* = \mathcal{Z}_6 - \frac{\langle p\xi \rangle}{\langle p1 \rangle} \mathcal{Z}_1 \,, \tag{5.32}$$

and we again assume off-shell kinematics for legs 5 and 6 with p and ξ denoting light-cone vectors entering k_T-decomposition of reggeized gluon momentum.

Now we would like to show one simple but interesting test both for our tree and loop level constructions (4.16), (5.20) and obtain the expression for LO BFKL kernel with gluing operation.

5.3 LO BFKL and gluing operation

Within BFKL approach [62–66] amplitudes of scattering of some quantum states $A+B \to A' + B'$, which can be partons in hadron, hadrons themselves, high energy electrons etc., at large center of mass energy $\sqrt{s}$ and fixed momentum transfer $\sqrt{-t}$, $s \gg |t|$ can be represented as

$$\mathcal{A}_{AB}^{A'B'} = \langle \Phi_{A'A} | \mathrm{e}^{\alpha_s N \ln(s/s_0)\ K_{\mathrm{BFKL}}} | \Phi_{B'B} \rangle \,, \tag{5.33}$$

where the so called impact factors $\langle \Phi_{A'A}|$ and $|\Phi_{B'B}\rangle$ are process dependent functions and describe the transitions $A \to A'$ and $B \to B'$. This scattering, in the mentioned above regime, can be described via interaction with special quasiparticles — so called reggeized gluons. BFKL kernel K_{BFKL} describes the self interaction of these reggeized gluons. s_0 is some process related energy scale. See for example [70] for detailed discussion.

Let us now calculate the LO kernel of BFKL equation in $\mathcal{N}=4$ SYM with the use of our gluing operation. At LO order it is given by two contribution so called real and virtual one. Consider virtual contribution first (also see figure 14 A).

5.3.1 Virtual part of LO BFKL

To compute virtual contribution to the LO BFKL we need the Regge trajectory. The latter could be conveniently extracted from the one-loop correlation function of two Wilson lines playing the role of sources for reggeized gluons [70]. Namely, we have to compute the following off-shell amplitude:

$$\langle 0|\mathcal{W}_{p_1}(k)\mathcal{W}_{p_2}(-k)|0\rangle = A^*_{2,0+2}(g_1^*,g_2^*) = \hat{A}_{12}\circ\hat{A}_{34}\left[A_{2,4}(1^-,2^+,3^-,4^+)\right] . \tag{5.34}$$

At tree level we have[22]

$$A^{*(0)}_{2,0+2} = \frac{\langle p_1\,\xi_1\rangle}{\kappa_1^*}\frac{\langle p_2\,\xi_2\rangle}{\kappa_2^*}\left(\frac{\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\right)\Bigg|_*\prod_{i=1}^{2}\frac{1}{\beta^2_{1,(i)}\beta_{2,(i)}}\frac{d\beta_{1,(i)}\wedge d\beta_{2,(i)}}{\beta_{1,(i)}\,\beta_{2,(i)}} , \tag{5.35}$$

where $\beta_{1,(1)},\beta_{2,(1)}$ parameters correspond to $\hat{A}_{12}$ gluing operation and those with (2) subscripts to $\hat{A}_{34}$. Evaluating $\Big|_*$ substitutions and taking composite residues at $\beta_{1,(i)}=-1$, $\beta_{2,(i)}=0$ we get

$$A^{*(0)}_{2,0+2} = -\frac{\langle p_1\,p_2\rangle^2}{\kappa_1^*\kappa_2^*} . \tag{5.36}$$

Now we should recall that the Wilson lines were used here to describe scattering of two fast moving particles at high energy.[23] This restricts further our kinematics, so that $p_1\cdot p_2 = s/2$ (s is the usual Mandelstam variable) and momentum transfer between two particles is restricted by two orthogonality conditions $k\cdot p_1 = k\cdot p_2 = 0$. The latter two conditions allow us to write down transverse momentum transfer as

$$k = c_1\lambda_{p_1}\tilde{\lambda}_{p_2} + c_2\lambda_{p_2}\tilde{\lambda}_{p_1} , \tag{5.37}$$

so that $t\equiv k^2 = c_1c_2 s$ and[24]

$$\kappa_1^*\kappa_2^* = \frac{c_2\langle p_1\,p_2\rangle\,[p_1\,p_2]}{[p_1\,p_2]}\frac{c_1\langle p_2\,p_1\rangle\,[p_2\,p_1]}{[p_2\,p_1]} = -c_1c_2\langle p_1\,p_2\rangle^2 = -\frac{t}{s}\langle p_1\,p_2\rangle^2. \tag{5.38}$$

Then for $A^{*(0)}_{2,0+2}$ amplitude we have

$$A^{*(0)}_{2,0+2} = \frac{s}{t} . \tag{5.39}$$

Now let us turn to the integrand of the corresponding one-loop amplitude. The latter is given for $n=4$ by (5.21):

$$I^{*(1)}_{2,0+2} = \frac{\langle 1234\rangle^2}{\langle 12^*AB\rangle\langle 23AB\rangle\langle 34^*AB\rangle\langle 41AB\rangle}, \tag{5.40}$$

with

$$Z_2^* = Z_2 - \frac{\langle p_1\xi_1\rangle}{\langle p_1p_2\rangle}Z_3,\ \ Z_4^* = Z_4 - \frac{\langle p_2\xi_2\rangle}{\langle p_2p_1\rangle}Z_1. \tag{5.41}$$

This expression can be rewritten in spinor helicity variables as:

$$I^{*(1)}_{2,0+2} = \frac{t^2\langle p_1\,p_2\rangle^2}{4\kappa_1^*\kappa_2^*}\frac{1}{l^2(l+k)^2\;l\cdot p_1\;l\cdot p_2} = \frac{st}{4}\frac{1}{l^2(l+k)^2\;l\cdot p_1\;l\cdot p_2} . \tag{5.42}$$

[22]The gluing details are similar to those considered in sections 2 and 3.
[23]For the introduction to corresponding description see [70].
[24]It is convenient here to chose $\xi_1 = p_2$ and $\xi_2 = p_1$.

The same result can also be obtained within helicity spinor picture

$$I^{*(1)}_{2,0+2} = \hat{A}_{12} \circ \hat{A}_{34} \left[I^{(1)}_{2,4}(1^-, 2^+, 3^-, 4^+) \right] , \tag{5.43}$$

where we arrange loop momenta as:

$$I^{(1)}_{2,4}(1^-, 2^+, 3^-, 4^+) = \frac{(q_1+q_2)^2(q_2+q_3)^2}{l^2(l+q_2)^2(l+q_1+q_2)^2(l-q_3)^2} . \tag{5.44}$$

In the expression above q_i, $(i = 1, \ldots, 4)$ are momenta of external gluons and l is loop momentum.

In LO BFKL regime we are interested in leading logarithmic approximation (LLA) to high-energy scattering amplitude. The latter could be obtained using Sudakov decomposition of loop integration momentum and retaining only logarithmic in Mandelstam invariant s contribution. That is

$$l = \alpha p_1 + \beta p_2 + l_\perp, \quad p_i \cdot l_\perp = 0, \quad k \cdot p_i = 0, \tag{5.45}$$

$$d^D l = \frac{s}{2}\, d\alpha\, d\beta\, d^{D-2} l_\perp, \tag{5.46}$$

and we are interested in the following regime (here m is some problem related mass scale):

$$1 \gg \alpha \gg \beta \sim \frac{m^2}{s} \ll 1. \tag{5.47}$$

Then

$$p_2 \cdot l = \alpha s/2 + p_1 \cdot l_\perp = \alpha s/2 \tag{5.48}$$

$$p_1 \cdot l = \beta s/2 + p_2 \cdot l_\perp = \beta s/2 \tag{5.49}$$

$$l^2 = \alpha\beta s/2 - l_\perp^2 \tag{5.50}$$

$$(l+k)^2 = \alpha\beta s/2 - (l_\perp + k_\perp)^2. \tag{5.51}$$

Now taking residue in β at 0 and integrating over α from m^2/s to 1 we get

$$\begin{aligned} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2(l+k)^2(p_1 l)(p_2 l)} &= \frac{s}{2(2\pi)^4} \int \frac{d\alpha}{\alpha s/2} \frac{d\beta}{\beta s/2} \frac{d^{D-2} l_\perp}{[\alpha\beta s/2 - l_\perp^2][\alpha\beta s/2 - (l_\perp + k_\perp)^2]} \\ &= \frac{1}{4\pi^3 s} \log\left(\frac{s}{m^2}\right) \int \frac{d^2 l_\perp}{l_\perp^2 (l_\perp + k_\perp)^2}. \end{aligned} \tag{5.52}$$

So finally we get

$$A^{*(0+1)}_{2,0+2} = A^{*(0)}_{2,0+2} \left\{ 1 - \frac{g^2}{16\pi^3} \log\left(\frac{s}{m^2}\right) \int \frac{k_\perp^2\, d^2 l_\perp}{l_\perp^2 (l_\perp + k_\perp)^2} \right\} . \tag{5.53}$$

This expression tells us that in LLA approximation[25] with account for color factor ($C_A = N$ for SU(N) gauge group) for reggeized gluon propagator we get

$$\langle 0|\mathcal{W}_{p_1}(k)\mathcal{W}_{p_2}(-k)|0\rangle\Big|_{\text{LLA}} \sim \frac{1}{k_\perp^2} \left(\frac{s}{m^2}\right)^{\omega(k_\perp^2)} , \tag{5.54}$$

[25] We should resume logarithmic terms, which exponentiate.

where[26] ($\alpha_s = g^2/(4\pi)$, $t = -k_\perp^2$):

$$\omega(t) = -\alpha_s N \int \frac{d^{2+\varepsilon} l_\perp}{(2\pi)^{2+\varepsilon}} \frac{k_\perp^2}{l_\perp^2 (l_\perp + k_\perp)^2} = -\frac{N\alpha_s}{4\pi} \frac{2(k_\perp^2)^{\varepsilon/2}}{\varepsilon} \tag{5.55}$$

is the famous LO BFKL Regge trajectory, which at LO is the same in QCD and $\mathcal{N}=4$ SYM. See for example [88–90]. Using this result for virtual part of BFKL kernel we can write [70]:

$$-\alpha_s N\, K^V_{\mathrm{BFKL}} = -\frac{1}{2}\delta^{(2)}(k-k')\left(\omega(k_\perp) + \omega(k_\perp - r_\perp)\right). \tag{5.56}$$

Here r is the momentum transfer for $A \to A'$ scattering $r = p_{A'} - p_A$.

In conclusion we would like also to note the following interesting fact. In $\mathcal{N}=4$ SYM the four point on-shell amplitude $A_{2,4}$ has a remarkable property[27] of being Regge exact, i.e. the contribution of the gluon Regge trajectory to the amplitude ($c(t)$ is the gluon impact factor)

$$A_{2,4}(s,t) = c(t)^2 \left(\frac{s}{-t}\right)^{\omega(t)} + \text{subleading terms in } \frac{|t|}{s}, \tag{5.57}$$

coincides with the exact expression for $A_{2,4}(s,t)$ as a function of arbitrary s and t.

5.3.2 Real part of LO BFKL

Now let's consider real contribution. This contribution is given by the integrated product of two, so called, Lipatov's L_μ RRP vertexes [70], see figure 14 B. To compute this contribution we may note, that Lipatov's RRP L_μ vertex (tree level reggeon-reggeon-particle amplitude) is related to reggeon amplitudes $A^*_{2,2+1}(g_1^*, g_2^*, 3^+)$ and $A^*_{3,2+1}(g_1^*, g_2^*, 3^-)$ as

$$L_\mu(k,k') = (k'+k)_\mu + n^-_\mu \left(\frac{k^2}{k'^-} - k^+\right) + n^+_\mu \left(\frac{k'^2}{k^+} - k'^-\right), \tag{5.58}$$

$$\begin{aligned} A^*_{2,2+1}(g_1^*, g_2^*, 3^+) &= \delta^4(k-k'-q_3)\mathbf{A}^*_{2,2+1}(k,-k',-q_3) = \delta^4(k-k'-q_3)\frac{\epsilon_3^{\mu,+} L_\mu(k,k')}{k^2 k'^2}, \\ A^*_{3,2+1}(g_1^*, g_2^*, 3^-) &= \delta^4(k-k'-q_3)\mathbf{A}^*_{3,2+1}(k,-k',-q_3) = \delta^4(k-k'-q_3)\frac{\epsilon_3^{\mu,-} L_\mu(k,k')}{k^2 k'^2}, \end{aligned} \tag{5.59}$$

which in their turns could be obtained with two our gluing operations applied to 5-point on-shell amplitude $A_{2,5}$. Here k, k' are reggeized gluons g_1^* and g_1^* momenta with $k-k'-q_3=0$ and $n^\pm$ are normalized light like directions for reggeized gluons

$$n^- = \frac{2p_1}{\sqrt{s}}, \quad n^+ = \frac{2p_2}{\sqrt{s}}, \ (n^- n^+) = 2, \ (kn^\pm) \equiv k^\pm, \tag{5.60}$$

and $\epsilon_3^\pm$ are polarization vectors of on-shell gluon with momentum $-q_3$. It is assumed that in the definitions of $A^*_{2,2+1}(g_1^*, g_2^*, 3^+)$ and $A^*_{3,2+1}(g_1^*, g_2^*, 3^-)$ amplitudes one has to take in

[26] Here we introduced dimensional regularization of otherwise divergent integral.
[27] See the discussion in [91].

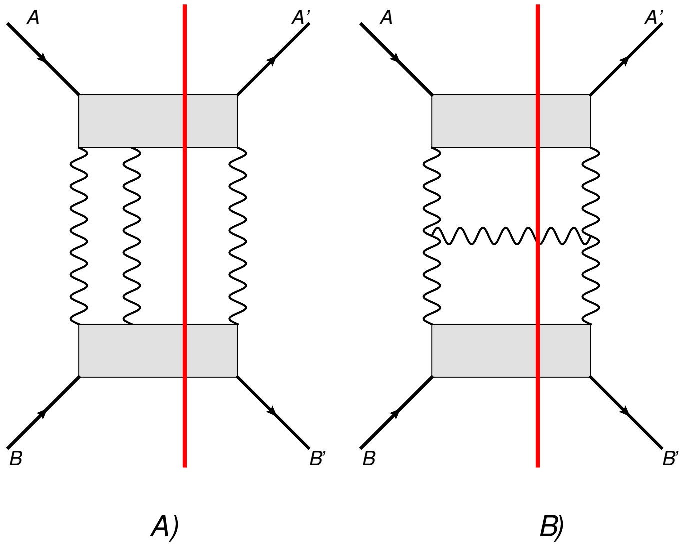

Figure 14. Typical Feynman diagrams contributing to (5.33) i.e. to the BFKL kernel at LO in $\mathcal{N}=4$ SYM. At large center of mass energy $\sqrt{s}$ and fixed momentum transfer $\sqrt{-t}$ the asymptotical behaviour of an amplitude is given by its imaginary part [50, 70]: $\mathcal{A}_{AB}^{A'B'}|_{s\gg 1}\sim \mathrm{Im}[\mathcal{A}_{AB}^{A'B'}]$. Grey squares represents impact factors, wavy lines represents gluon propagators, vertical red line represents cuts of corresponding propagators and impact factors. Diagrams of type A) gives contribution to K_{BFKL}^{V} and their total sum is equivalent to evaluation of (5.53). Diagrams of type B) give contribution to K_{BFKL}^{R} and their total sum is equivalent to evaluation of (5.62).

k_T decomposition of k and k' momenta direction vectors as $p_1=n^-$ and $p_2=n^+$. We have also defined functions $\mathbf{A}_{2,2+1}^{*}$ and $\mathbf{A}_{3,2+1}^{*}$ which are given by corresponding Wilson line form factors stripped from momentum conservation delta functions:

$$\begin{aligned}\mathbf{A}_{2,2+1}^{*}(k_1,k_2,q_3)&=\frac{1}{\kappa_1^{*}\kappa_2^{*}}\frac{\langle n^-n^+\rangle^3}{\langle 3n^+\rangle\langle n^-3\rangle},\\ \mathbf{A}_{3,2+1}^{*}(k_1,k_2,q_3)&=\frac{1}{\kappa_1\kappa_2}\frac{[n^-n^+]^3}{[3n^+][n^-3]}.\end{aligned}\tag{5.61}$$

Performing Sudakov decomposition[28] of reggeized gluon momentum the contribution of real radiation to BFKL kernel takes the form [70]:

$$K_{\mathrm{BFKL}}^{R}(k_\perp,k'_\perp,r)\ln\frac{s}{m^2}=\frac{s}{2}\int d\alpha_k d\beta_{k'}L^{\mu}(k,k')L_{\mu}(r-k,r-k')\frac{\delta(\alpha_k\beta_{k'}s+(k_\perp-k'_\perp)^2)}{k_\perp'^2(r_\perp-k'_\perp)^2}.\tag{5.62}$$

Note that factor $L^{\mu}(k,k')L_{\mu}(r-k,r-k')=g_{\mu\nu}L^{\mu}(k,k')L^{\nu}(r-k,r-k')$ can be rewritten purely in terms of Wilson line form factors. Namely using gauge invariance of $A_{2,2+1}^{*}(g_1^*,g_2^*,3^+)$ and $A_{3,2+1}^{*}(g_1^*,g_2^*,3^-)$ we can replace

$$g_{\mu\nu}\mapsto\sum_{i=\pm}\epsilon_{\mu}^{(i)}\epsilon_{\nu}^{(i)},\tag{5.63}$$

[28]See for example [70] for details.

so that ($r-k\equiv m$ and $r-k'\equiv m'$)

$$\frac{L^{\mu}(k,k')L_{\mu}(m,m')}{k^2k'^2m^2m'^2} = \mathbf{A}^*_{2,2+1}(k,-k',-q_3)\mathbf{A}^*_{2,2+1}(m,-m',-q_3) + \mathbf{A}^*_{3,2+1}(k,-k',-q_3)\mathbf{A}^*_{3,2+1}(m,-m',-q_3). \tag{5.64}$$

Also note that in this case no other particles besides gluons from $\mathcal{N}=4$ SYM supermultiplet give contribution to real radiation. This happens due to the R-charge conservation.

The integral over $\beta_{k'}$ is taken with the help of δ-function, while the integration over α_k is performed over the interval $[\frac{m^2}{s},1]$. This way we get

$$K^R_{\mathrm{BFKL}}(k_\perp,k'_\perp,r) = -\frac{r^2_\perp}{k'^2_\perp(r_\perp-k'_\perp)^2} + \frac{k^2_\perp}{k'^2_\perp(k_\perp-k'_\perp)^2} + \frac{(r_\perp-k_\perp)^2}{(r_\perp-k'_\perp)^2(k_\perp-k'_\perp)^2}. \tag{5.65}$$

Altogether with account for the Regge trajectories contributions we recover LO expression for BFKL kernel $K_{\mathrm{BFKL}} = K^R_{\mathrm{BFKL}} + K^V_{\mathrm{BFKL}}$ [70]:

$$K_{\mathrm{BFKL}}(k_\perp,k'_\perp,r) = -\frac{r^2_\perp}{k'^2_\perp(r_\perp-k'_\perp)^2} + \frac{k^2_\perp}{k'^2_\perp(k_\perp-k'_\perp)^2} + \frac{(r_\perp-k_\perp)^2}{(r_\perp-k'_\perp)^2(k_\perp-k'_\perp)^2} - \frac{1}{2}\delta^{(2)}(k-k')\int\frac{d^2l_\perp}{4\pi^2}\left\{\frac{k^2_\perp}{l^2_\perp(k_\perp-l_\perp)^2} + \frac{(k_\perp-r_\perp)^2}{(l_\perp-r_\perp)^2(k_\perp-l_\perp)^2}\right\}. \tag{5.66}$$

6 Conclusion

In this paper we considered the derivation of the BCFW recurrence relation for the Wilson line form factors and correlation functions (off-shell reggeon amplitudes) both at tree and at integrand level. We have shown that starting from the BCFW recursion for on-shell amplitudes and using so called "gluing operator" one can obtain recursion relations for the Wilson line form factors. The latter is true both at tree and integrand level in helicity spinor and momentum twistor representations. The gluing operation also allows one easily convert known local integrands of the on-shell amplitudes into integrands of the Wilson line form factors. These results are condensed in formulas (4.16), (4.17) and (5.19), (5.20) for tree and loop level correspondingly. We have verified our considerations by reproducing LO BFKL kernel. We also made some predictions regarding the structure of the integrands of Wilson line form factors at higher loops/large number of external states.

As far as we can understand our construction is not limited to the Wilson line operators only. Indeed, using [24] similar gluing operator the form factors of stress tensor operator supermultiplet could be constructed. Also, presumably, analogs of gluing operator for all other type of local operators in $\mathcal{N}=4$ SYM theory should exist. The only real obstacle in this direction is that at the level of integrands for local single trace operators we should account for nonplanar contributions. So the notion of "integrand" in this case is somewhat obscure at first sight. Nevertheless we think that one can still introduce integrands in setup similar to considerations in [92], where nonplanar contributions to the on-shell amplitudes were considered in momentum twistor variables.

We hope that the presented results will be interesting and useful for people both from $\mathcal{N}=4$ SYM "amplitudology" and BFKL/reggeon physics communities.

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A Gluing operation and Grassmannians

Let's see how the use of the gluing operation in momentum twistors could easily reproduce known Grassmannian integral representations for the tree-level off-shell amplitudes [19, 20]. We start with the Grassmannian integral representation for the on-shell amplitudes in momentum twistors:

$$\mathcal{L}_{k,n+2} = \frac{A_{k,n+2}}{A_{2,n+2}} = \int \frac{d^{(k-2)\times(n+2)}D}{\mathrm{Vol}[\mathrm{GL}(k-2)]} \frac{\delta^{4(k-2)|4(k-2)}(D\cdot\mathcal{Z})}{(1\ \ldots\ k-2)\ \ldots\ (n+2\ \ldots\ k-3)}\,, \tag{A.1}$$

here (also similar notations are used in (2.26))

$$\delta^{4(k-2)|4(k-2)}(D\cdot\mathcal{Z}) = \prod_{a=1}^{k-2} \delta^{4|4}\left(\sum_{i=1}^{n+2} D_{ai}\mathcal{Z}_i\right), \tag{A.2}$$

and $(i_1,\ldots,i_{k-2})$ is minor constructed from columns of D matrix with numbers $i_1,\ldots,i_{k-2}$. Applying to this expression the gluing operation $\hat{A}_{n+1,n+2}$ amounts to the following shifts of momentum super twistors:

$$\mathcal{Z}_1 \to (1+\alpha_1\alpha_2)\mathcal{Z}_1 + \alpha_1\mathcal{Z}_{n+2} \equiv \mathcal{Z}'_1\,, \tag{A.3}$$

$$\mathcal{Z}_{n+2} \to \mathcal{Z}_{n+2} + \alpha_2\mathcal{Z}_1 \equiv \mathcal{Z}'_{n+2}\,, \tag{A.4}$$

so that

$$D\cdot\mathcal{Z} \to D'\cdot\mathcal{Z}\,, \tag{A.5}$$

where

$$D'_1 = (1+\alpha_1\alpha_2)D_1 + \alpha_2 D_{n+2}\,, \tag{A.6}$$

$$D'_{n+2} = \alpha_1 D_1 + D_{n+2} \tag{A.7}$$

All other momentum super twistors are unshifted and we have $D'_i = D_i$. The inverse transformation from D's to D''s is then given by

$$D_1 = D'_1 - \alpha_2 D'_{n+2}\,, \tag{A.8}$$

$$D_{n+2} = -\alpha_1 D'_1 + (1+\alpha_1\alpha_2)D'_{n+2}\,. \tag{A.9}$$

With these transformations it is easy to write down transformation rules for minors. For example, we have

$$(1\ldots k-2) \to (1\ldots k-2)' - \alpha_2(n+2\,2\ldots k-2)' \tag{A.10}$$

$$(n-k+5\ldots n+2) \to -\alpha_1(n-k+5\ldots n+1\,1)' + (1+\alpha_1\alpha_2)(n-k+5\ldots n+2)' \tag{A.11}$$

Finally performing transition in the Grassmannian integral from D's to D''s and taking residues at $\alpha_1 = 0$ and $\alpha_2 = -\frac{\langle n+1\, n+2\rangle}{\langle n+1\, 1\rangle} = -\frac{\langle p_{n+1}\, \xi_{n+1}\rangle}{\langle p_{n+1}\, 1\rangle}$ we get

$$\frac{A^*_{k,n+1}}{A^*_{2,n+1}} = \int_\Gamma \frac{d^{(k-2)\times(n+2)}D'}{\mathrm{Vol}[\mathrm{GL}(k-2)]} \frac{1}{1+\frac{\langle p_{n+1}\,\xi_{n+1}\rangle}{\langle p_{n+1}\,1\rangle}\frac{(n+2\,2\,\ldots\,k-2)'}{(1\,\ldots\,k-2)'}} \frac{\delta^{4(k-2)|4(k-2)}(D'\cdot\mathcal{Z})}{(1\,\ldots\,k-2)'\,\ldots\,(n+2\,\ldots\,k-3)'}\,, \tag{A.12}$$

Similarly applying several gluing operations we recover formula [20].

References

[1] H. Elvang and Y.-t. Huang, *Scattering Amplitudes*, arXiv:1308.1697 [INSPIRE].

[2] S. Weinzierl, *Tales of 1001 Gluons*, Phys. Rept. **676** (2017) 1 [arXiv:1610.05318] [INSPIRE].

[3] A. Brandhuber, B. Spence, G. Travaglini and G. Yang, *Form Factors in $N = 4$ Super Yang-Mills and Periodic Wilson Loops*, JHEP **01** (2011) 134 [arXiv:1011.1899] [INSPIRE].

[4] A. Brandhuber, O. Gurdogan, R. Mooney, G. Travaglini and G. Yang, *Harmony of Super Form Factors*, JHEP **10** (2011) 046 [arXiv:1107.5067] [INSPIRE].

[5] O.T. Engelund and R. Roiban, *Correlation functions of local composite operators from generalized unitarity*, JHEP **03** (2013) 172 [arXiv:1209.0227] [INSPIRE].

[6] L.V. Bork, *On form factors in $\mathcal{N} = 4$ SYM theory and polytopes*, JHEP **12** (2014) 111 [arXiv:1407.5568] [INSPIRE].

[7] B. Penante, B. Spence, G. Travaglini and C. Wen, *On super form factors of half-BPS operators in $N = 4$ super Yang-Mills*, JHEP **04** (2014) 083 [arXiv:1402.1300] [INSPIRE].

[8] R. Frassek, D. Meidinger, D. Nandan and M. Wilhelm, *On-shell diagrams, Graßmannians and integrability for form factors*, JHEP **01** (2016) 182 [arXiv:1506.08192] [INSPIRE].

[9] L.V. Bork and A.I. Onishchenko, *On soft theorems and form factors in $\mathcal{N} = 4$ SYM theory*, JHEP **12** (2015) 030 [arXiv:1506.07551] [INSPIRE].

[10] L. Koster, V. Mitev, M. Staudacher and M. Wilhelm, *Composite Operators in the Twistor Formulation of $N = 4$ Supersymmetric Yang-Mills Theory*, Phys. Rev. Lett. **117** (2016) 011601 [arXiv:1603.04471] [INSPIRE].

[11] L. Koster, V. Mitev, M. Staudacher and M. Wilhelm, *All tree-level MHV form factors in $\mathcal{N} = 4$ SYM from twistor space*, JHEP **06** (2016) 162 [arXiv:1604.00012] [INSPIRE].

[12] L. Koster, V. Mitev, M. Staudacher and M. Wilhelm, *On Form Factors and Correlation Functions in Twistor Space*, JHEP **03** (2017) 131 [arXiv:1611.08599] [INSPIRE].

[13] L.V. Bork, *On NMHV form factors in $N = 4$ SYM theory from generalized unitarity*, JHEP **01** (2013) 049 [arXiv:1203.2596] [INSPIRE].

[14] A. Brandhuber, E. Hughes, R. Panerai, B. Spence and G. Travaglini, *The connected prescription for form factors in twistor space*, JHEP **11** (2016) 143 [arXiv:1608.03277] [INSPIRE].

[15] S. He and Z. Liu, *A note on connected formula for form factors*, *JHEP* **12** (2016) 006 [arXiv:1608.04306] [INSPIRE].

[16] A. Brandhuber, B. Penante, G. Travaglini and C. Wen, *The last of the simple remainders*, *JHEP* **08** (2014) 100 [arXiv:1406.1443] [INSPIRE].

[17] J. Maldacena and A. Zhiboedov, *Form factors at strong coupling via a Y-system*, *JHEP* **11** (2010) 104 [arXiv:1009.1139] [INSPIRE].

[18] Z. Gao and G. Yang, *Y-system for form factors at strong coupling in AdS_5 and with multi-operator insertions in AdS_3*, *JHEP* **06** (2013) 105 [arXiv:1303.2668] [INSPIRE].

[19] L.V. Bork and A.I. Onishchenko, *Wilson lines, Grassmannians and gauge invariant off-shell amplitudes in $\mathcal{N}=4$ SYM*, *JHEP* **04** (2017) 019 [arXiv:1607.02320] [INSPIRE].

[20] L.V. Bork and A.I. Onishchenko, *Grassmannian integral for general gauge invariant off-shell amplitudes in $\mathcal{N}=4$ SYM*, *JHEP* **05** (2017) 040 [arXiv:1610.09693] [INSPIRE].

[21] L.V. Bork, D.I. Kazakov and G.S. Vartanov, *On MHV Form Factors in Superspace for $\mathcal{N}=4$ SYM Theory*, *JHEP* **10** (2011) 133 [arXiv:1107.5551] [INSPIRE].

[22] P. Kotko, *Wilson lines and gauge invariant off-shell amplitudes*, *JHEP* **07** (2014) 128 [arXiv:1403.4824] [INSPIRE].

[23] L.V. Bork and A.I. Onishchenko, *Grassmannians and form factors with $q^2=0$ in $\mathcal{N}=4$ SYM theory*, *JHEP* **12** (2016) 076 [arXiv:1607.00503] [INSPIRE].

[24] L.V. Bork and A.I. Onishchenko, *Four dimensional ambitwistor strings and form factors of local and Wilson line operators*, arXiv:1704.04758 [INSPIRE].

[25] B. Eden, P. Heslop and L. Mason, *The Correlahedron*, *JHEP* **09** (2017) 156 [arXiv:1701.00453] [INSPIRE].

[26] N. Arkani-Hamed and J. Trnka, *The Amplituhedron*, *JHEP* **10** (2014) 030 [arXiv:1312.2007] [INSPIRE].

[27] N. Arkani-Hamed and J. Trnka, *Into the Amplituhedron*, *JHEP* **12** (2014) 182 [arXiv:1312.7878] [INSPIRE].

[28] Y. Bai and S. He, *The Amplituhedron from Momentum Twistor Diagrams*, *JHEP* **02** (2015) 065 [arXiv:1408.2459] [INSPIRE].

[29] N. Arkani-Hamed, Y. Bai and T. Lam, *Positive Geometries and Canonical Forms*, *JHEP* **11** (2017) 039 [arXiv:1703.04541] [INSPIRE].

[30] N. Arkani-Hamed, H. Thomas and J. Trnka, *Unwinding the Amplituhedron in Binary*, *JHEP* **01** (2018) 016 [arXiv:1704.05069] [INSPIRE].

[31] L.V. Bork, D.I. Kazakov and G.S. Vartanov, *On form factors in $N=4$ SYM*, *JHEP* **02** (2011) 063 [arXiv:1011.2440] [INSPIRE].

[32] A. Brandhuber, G. Travaglini and G. Yang, *Analytic two-loop form factors in $N=4$ SYM*, *JHEP* **05** (2012) 082 [arXiv:1201.4170] [INSPIRE].

[33] M. Wilhelm, *Amplitudes, Form Factors and the Dilatation Operator in $\mathcal{N}=4$ SYM Theory*, *JHEP* **02** (2015) 149 [arXiv:1410.6309] [INSPIRE].

[34] D. Nandan, C. Sieg, M. Wilhelm and G. Yang, *Cutting through form factors and cross sections of non-protected operators in $\mathcal{N}=4$ SYM*, *JHEP* **06** (2015) 156 [arXiv:1410.8485] [INSPIRE].

JHEP06(2018)129

[35] F. Loebbert, D. Nandan, C. Sieg, M. Wilhelm and G. Yang, *On-Shell Methods for the Two-Loop Dilatation Operator and Finite Remainders*, *JHEP* **10** (2015) 012 [arXiv:1504.06323] [INSPIRE].

[36] F. Loebbert, C. Sieg, M. Wilhelm and G. Yang, *Two-Loop* SL(2) *Form Factors and Maximal Transcendentality*, *JHEP* **12** (2016) 090 [arXiv:1610.06567] [INSPIRE].

[37] R.H. Boels, B.A. Kniehl, O.V. Tarasov and G. Yang, *Color-kinematic Duality for Form Factors*, *JHEP* **02** (2013) 063 [arXiv:1211.7028] [INSPIRE].

[38] G. Yang, *Color-kinematics duality and Sudakov form factor at five loops for $N = 4$ supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **117** (2016) 271602 [arXiv:1610.02394] [INSPIRE].

[39] T. Gehrmann, J.M. Henn and T. Huber, *The three-loop form factor in $N = 4$ super Yang-Mills*, *JHEP* **03** (2012) 101 [arXiv:1112.4524] [INSPIRE].

[40] R. Boels, B.A. Kniehl and G. Yang, *Master integrals for the four-loop Sudakov form factor*, *Nucl. Phys.* **B 902** (2016) 387 [arXiv:1508.03717] [INSPIRE].

[41] P. Banerjee, P.K. Dhani, M. Mahakhud, V. Ravindran and S. Seth, *Finite remainders of the Konishi at two loops in $\mathcal{N} = 4$ SYM*, *JHEP* **05** (2017) 085 [arXiv:1612.00885] [INSPIRE].

[42] A. Brandhuber, M. Kostacinska, B. Penante, G. Travaglini and D. Young, *The* SU(2|3) *dynamic two-loop form factors*, *JHEP* **08** (2016) 134 [arXiv:1606.08682] [INSPIRE].

[43] R. Huang, Q. Jin and B. Feng, *Form Factor and Boundary Contribution of Amplitude*, *JHEP* **06** (2016) 072 [arXiv:1601.06612] [INSPIRE].

[44] O.T. Engelund, *Lagrangian Insertion in the Light-Like Limit and the Super-Correlators/Super-Amplitudes Duality*, *JHEP* **02** (2016) 030 [arXiv:1502.01934] [INSPIRE].

[45] P. Kotko and A.M. Stasto, *Wilson lines in the MHV action*, *JHEP* **09** (2017) 047 [arXiv:1706.00052] [INSPIRE].

[46] A. van Hameren, *BCFW recursion for off-shell gluons*, *JHEP* **07** (2014) 138 [arXiv:1404.7818] [INSPIRE].

[47] A. van Hameren and M. Serino, *BCFW recursion for TMD parton scattering*, *JHEP* **07** (2015) 010 [arXiv:1504.00315] [INSPIRE].

[48] K. Kutak, A. Hameren and M. Serino, *QCD amplitudes with 2 initial spacelike legs via generalised BCFW recursion*, *JHEP* **02** (2017) 009 [arXiv:1611.04380] [INSPIRE].

[49] L.N. Lipatov, *Gauge invariant effective action for high-energy processes in QCD*, *Nucl. Phys.* **B 452** (1995) 369 [hep-ph/9502308] [INSPIRE].

[50] L.N. Lipatov, *Small x physics in perturbative QCD*, *Phys. Rept.* **286** (1997) 131 [hep-ph/9610276] [INSPIRE].

[51] R. Kirschner, L.N. Lipatov and L. Szymanowski, *Effective action for multi-Regge processes in QCD*, *Nucl. Phys.* **B 425** (1994) 579 [hep-th/9402010] [INSPIRE].

[52] R. Kirschner, L.N. Lipatov and L. Szymanowski, *Symmetry properties of the effective action for high-energy scattering in QCD*, *Phys. Rev.* **D 51** (1995) 838 [hep-th/9403082] [INSPIRE].

[53] A. van Hameren, P. Kotko and K. Kutak, *Multi-gluon helicity amplitudes with one off-shell leg within high energy factorization*, *JHEP* **12** (2012) 029 [arXiv:1207.3332] [INSPIRE].

[54] A. van Hameren, P. Kotko and K. Kutak, *Helicity amplitudes for high-energy scattering*, *JHEP* **01** (2013) 078 [arXiv:1211.0961] [INSPIRE].

JHEP06(2018)129

[55] A. van Hameren, *One-loop amplitudes with an off-shell gluon*, in *17th conference on Elastic and Diffractive Scattering (EDS 17)*, Prague, Czech Republic, June 26–30, 2017 [arXiv:1710.10963] [INSPIRE].

[56] A. van Hameren, *Calculating off-shell one-loop amplitudes for k_T-dependent factorization: a proof of concept*, arXiv:1710.07609 [INSPIRE].

[57] L.V. Gribov, E.M. Levin and M.G. Ryskin, *Semihard Processes in QCD*, Phys. Rept. **100** (1983) 1 [INSPIRE].

[58] S. Catani, M. Ciafaloni and F. Hautmann, *High-energy factorization and small x heavy flavor production*, Nucl. Phys. **B 366** (1991) 135 [INSPIRE].

[59] J.C. Collins and R.K. Ellis, *Heavy quark production in very high-energy hadron collisions*, Nucl. Phys. **B 360** (1991) 3 [INSPIRE].

[60] S. Catani and F. Hautmann, *High-energy factorization and small x deep inelastic scattering beyond leading order*, Nucl. Phys. **B 427** (1994) 475 [hep-ph/9405388] [INSPIRE].

[61] V.N. Gribov, *A Reggeon diagram technique*, *Sov. Phys. JETP* **26** (1968) 414 [INSPIRE].

[62] L.N. Lipatov, *Reggeization of the Vector Meson and the Vacuum Singularity in Nonabelian Gauge Theories*, *Sov. J. Nucl. Phys.* **23** (1976) 338 [INSPIRE].

[63] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, *Multi-Reggeon Processes in the Yang-Mills Theory*, *Sov. Phys. JETP* **44** (1976) 443 [INSPIRE].

[64] V.S. Fadin, E.A. Kuraev and L.N. Lipatov, *On the Pomeranchuk Singularity in Asymptotically Free Theories*, Phys. Lett. **B 60** (1975) 50 [INSPIRE].

[65] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, *The Pomeranchuk Singularity in Nonabelian Gauge Theories*, *Sov. Phys. JETP* **45** (1977) 199 [INSPIRE].

[66] I.I. Balitsky and L.N. Lipatov, *The Pomeranchuk Singularity in Quantum Chromodynamics*, *Sov. J. Nucl. Phys.* **28** (1978) 822 [INSPIRE].

[67] V.S. Fadin and L.N. Lipatov, *BFKL Pomeron in the next-to-leading approximation*, Phys. Lett. **B 429** (1998) 127 [hep-ph/9802290] [INSPIRE].

[68] M. Ciafaloni and G. Camici, *Energy scale(s) and next-to-leading BFKL equation*, Phys. Lett. **B 430** (1998) 349 [hep-ph/9803389] [INSPIRE].

[69] V.S. Fadin, R. Fiore, M.G. Kozlov and A.V. Reznichenko, *Proof of the multi-Regge form of QCD amplitudes with gluon exchanges in the NLA*, Phys. Lett. **B 639** (2006) 74 [hep-ph/0602006] [INSPIRE].

[70] I. Balitsky, *High-energy QCD and Wilson lines*, hep-ph/0101042 [INSPIRE].

[71] L.J. Dixon, *Calculating scattering amplitudes efficiently*, in *QCD and beyond. Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI-95*, Boulder, U.S.A., June 4–30, 1995, pp. 539–584, 1996 [hep-ph/9601359] [INSPIRE].

[72] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Dual superconformal symmetry of scattering amplitudes in $N = 4$ super-Yang-Mills theory*, Nucl. Phys. **B 828** (2010) 317 [arXiv:0807.1095] [INSPIRE].

[73] V.P. Nair, *A Current Algebra for Some Gauge Theory Amplitudes*, Phys. Lett. **B 214** (1988) 215 [INSPIRE].

[74] L.V. Bork and A.I. Onishchenko, *Ambitwistor strings and reggeon amplitudes in $N = 4$ SYM*, Phys. Lett. **B 774** (2017) 403 [arXiv:1704.00611] [INSPIRE].

JHEP06(2018)129

[75] Y. Geyer, A.E. Lipstein and L.J. Mason, *Ambitwistor Strings in Four Dimensions*, *Phys. Rev. Lett.* **113** (2014) 081602 [arXiv:1404.6219] [INSPIRE].

[76] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, *A Duality For The S Matrix*, *JHEP* **03** (2010) 020 [arXiv:0907.5418] [INSPIRE].

[77] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, A.B. Goncharov, A. Postnikov and J. Trnka, *Scattering Amplitudes and the Positive Grassmannian*, Cambridge University Press (2012).

[78] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B 715** (2005) 499 [hep-th/0412308] [INSPIRE].

[79] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [hep-th/0501052] [INSPIRE].

[80] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, S. Caron-Huot and J. Trnka, *The All-Loop Integrand For Scattering Amplitudes in Planar $N = 4$ SYM*, *JHEP* **01** (2011) 041 [arXiv:1008.2958] [INSPIRE].

[81] A. Hodges, *Eliminating spurious poles from gauge-theoretic amplitudes*, *JHEP* **05** (2013) 135 [arXiv:0905.1473] [INSPIRE].

[82] N. Arkani-Hamed, J. Bourjaily, F. Cachazo and J. Trnka, *Local Spacetime Physics from the Grassmannian*, *JHEP* **01** (2011) 108 [arXiv:0912.3249] [INSPIRE].

[83] J.L. Bourjaily, *Efficient Tree-Amplitudes in $N = 4$: Automatic BCFW Recursion in Mathematica*, arXiv:1011.2447 [INSPIRE].

[84] Y. Bai and S. He, *The Amplituhedron from Momentum Twistor Diagrams*, *JHEP* **02** (2015) 065 [arXiv:1408.2459] [INSPIRE].

[85] R.H. Boels, *On BCFW shifts of integrands and integrals*, *JHEP* **11** (2010) 113 [arXiv:1008.3101] [INSPIRE].

[86] R.H. Boels and H. Lüo, *On-shell recursion relations for generic integrands*, arXiv:1610.05283 [INSPIRE].

[87] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo and J. Trnka, *Local Integrals for Planar Scattering Amplitudes*, *JHEP* **06** (2012) 125 [arXiv:1012.6032] [INSPIRE].

[88] A.V. Kotikov and L.N. Lipatov, *NLO corrections to the BFKL equation in QCD and in supersymmetric gauge theories*, *Nucl. Phys.* **B 582** (2000) 19 [hep-ph/0004008] [INSPIRE].

[89] A.V. Kotikov and L.N. Lipatov, *DGLAP and BFKL evolution equations in the $N = 4$ supersymmetric gauge theory*, in *35th Annual Winter School on Nuclear and Particle Physics*, Repino, Russia, February 19–25, 2001 [hep-ph/0112346] [INSPIRE].

[90] A.V. Kotikov and L.N. Lipatov, *DGLAP and BFKL equations in the $N = 4$ supersymmetric gauge theory*, *Nucl. Phys.* **B 661** (2003) 19 [*Erratum ibid.* **B 685** (2004) 405] [hep-ph/0208220] [INSPIRE].

[91] J.M. Drummond, G.P. Korchemsky and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, *Nucl. Phys.* **B 795** (2008) 385 [arXiv:0707.0243] [INSPIRE].

[92] P. Du, G. Chen and Y.-K.E. Cheung, *Permutation relations of generalized Yangian Invariants, unitarity cuts and scattering amplitudes*, *JHEP* **09** (2014) 115 [arXiv:1401.6610] [INSPIRE].

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