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Reparametrization invariance and partial re-summations of the heavy quark expansion

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Abstract: We extend existing work on reparametrization invariance (RPI) of the heavy-quark expansion (HQE). RPI implies relations between different orders of the HQE. We discuss the total rates of inclusive processes and obtain results which have a manifest RPI and can be expressed through matrix elements of operators and states defined in full QCD. This approach leads to a partial re-summation of higher-order terms in the heavy-quark expansion. It has the advantage that for reparametrization-invariant observables it shows that the number of independent non-perturbative parameters can be reduced.

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Contents

1 Introduction

The Heavy Quark Expansion (HQE) has paved the road to a QCD based calculation of inclusive decays of heavy-flavour hadrons and thus has become an indispensable tool in precision flavour physics [1]. In particular, the extraction of the CKM parameters from inclusive semileptonic processes as well as the search for physics beyond the standard model relies heavily on the HQE, which has been pushed to higher orders in the perturbative as well as in the non-perturbative sector to achieve the highest possible accuracy, e.g. in the extraction of V_{cb} [2].

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The HQE can be set up in slightly different ways. The "cleanest" way is to extract the complete mass dependence from the matrix elements that define the HQE parameters. This is achieved by expressing them in terms of static heavy quark fields defined in Heavy Quark Effective Theory (HQET) and hadron states in the infinite mass limit, such that all matrix elements become mass independent. However, the price to be paid is that non-local matrix elements appear which contain the mass dependence of the states in full QCD [3].

This can be avoided by setting up the OPE with fields and states defined in full QCD [4]. To this end, one removes the large part, i.e. the part related to the heavy-quark mass m, of the heavy quark momentum by a phase factor multiplying the fields and by taking matrix elements with states defined in full QCD. In this formulation, the HQE is expressed in terms of local matrix elements only; however, these matrix elements still depend on the heavy-quark mass and the systematics of the expansion in inverse powers of m are less obvious. In particular, the consideration of higher orders requires a careful analysis. Another disadvantage of this approach is that the HQE parameters will depend of the quark mass making them non-universal.

Even when the HQE is defined in this second way, we have to introduce a time-like vector v which has to be chosen in such a way that the "residual" momentum $k = p_Q - mv$ of the heavy quark is a small quantity of order Λ_{QCD}. Although the OPE leading to the HQE is defined as an operator relation, the choice of v anticipates that we aim at computing certain matrix elements; in the case of inclusive decays these will eventually be the forward matrix elements of hadronic states with the hadron momentum p_H; hence a "natural" choice for the velocity vector is $v = p_H/m_H$.

However, the choice of v is not unique and to a large extend arbitrary, and thus the final result for physical quantities cannot depend on v. This Reparametrization Invariance (RPI), i.e. the invariance of physical quantities under small changes of v, has been noticed already long ago in [5] and has subsequently been exploited in [6–9]. Its relation to the Lorentz invariance of full QCD has been investigated in detail in [10, 11].

RPI implies relations between different orders of the HQE. Truncating the HQE at some order $1/m^n$ leaves us thus with a violation of RPI of order $1/m^{n+1}$. This has been studied in some detail for the Lagrangian and the coefficients of its $1/m$ expansion. In turn, RPI fixes the coefficients of towers of operators with different dimensions, which has been noticed in [7, 8] for the Lagrangian and for inclusive rates. While for inclusive *total* rates one arrives at mainly the same conclusions as for the Lagrangian, RPI also constrains the coefficients in the HQE for inclusive *differential* rates, which has been discussed in [7, 8] for low orders in the $1/m$ expansion.

In this paper, we show that for the HQE for the total rates the RPI relations can be used to explicitly perform a re-summation of the towers of operators, in such a way that the result can be written in a fully RPI fashion. Such a re-summation has also been discussed in [7, 8] by introducing specific fields which are RPI, in terms of which reparametrization invariant operator combinations can be formed. In the present paper we proceed one step further and make RPI manifest by expressing the result in terms of matrix elements of operators and states defined in full QCD. This approach has the advantage that, for total rates, the reduction of the number of independent parameters is made explicit. We note that the reduction of independent parameters happens only for total rates, or — more precisely — for Lorentz invariant quantities. In case of differential rates, which are not reparametrization invariant, the relations between different orders on the $1/m_Q$ expansion depend on the kinematics, and thus the number of independent parameters is not reduced.

In the next section we give our definition of the reparametrization transformations and apply this in section 3 to a scalar toy model of QCD, discussing the total rates. In section 4 we consider real QCD, i.e including the spin of the quarks, again focusing on the total rates.

2 Reparametrization transformation

We start from the equation of motion for the heavy quark field Q, which is the Dirac equation

$$(i\not{D} - m)Q(x) = 0 \tag{2.1}$$

where $D = \partial - ig_s A$ is the QCD covariant derivative with the gluon field A. In order to set up the HQE, we introduce a time-like vector v, which we use to split the momentum p_Q of the heavy quark according to $p_Q = mv + k$. This is achieved by a redefinition of the heavy quark field Q according to

$$Q(x) = \exp(-im(v \cdot x))Q_v(x)\,, \tag{2.2}$$

which implies

$$(iD_\mu)Q(x) = \exp(-im(v \cdot x))(iD_\mu + mv_\mu)Q_v(x)\,, \tag{2.3}$$

corresponding to the decomposition of the heavy-quark momentum with $k \sim iD$.

Note that the field Q_v is still the field in full QCD; its equations of motion can be derived from (2.1) and read

$$Q_v = \not{v}Q_v + \frac{i\not{D}}{m}Q_v \tag{2.4}$$

$$(ivD)Q_v = -\frac{1}{2m}(i\not{D})(i\not{D})Q_v = -\frac{1}{2m}(iD)^2Q_v - \frac{1}{2m}(\sigma \cdot G)Q_v \tag{2.5}$$

where

$$(\sigma \cdot G) \equiv (-i\sigma_{\mu\nu})(iD^\mu)(iD^\nu)\,, \qquad \gamma_\mu\gamma_\nu = g_{\mu\nu} + (-i\sigma_{\mu\nu})\,. \tag{2.6}$$

The reparametrization (RP) transformation δ_{RP} corresponding to an infinitesimal change $v_\mu \longrightarrow v_\mu + \delta v_\mu$ is thus

$$\delta_{\mathrm{RP}}\, v_\mu = \delta v_\mu \qquad \text{with} \qquad v \cdot \delta v = 0 \tag{2.7}$$

$$\delta_{\mathrm{RP}}\, iD_\mu = -m\delta v_\mu \tag{2.8}$$

$$\delta_{\mathrm{RP}}\, Q_v(x) = im(x \cdot \delta v)Q_v(x) \quad \text{in particular} \quad \delta_{\mathrm{RP}}\, Q_v(0) = 0\,. \tag{2.9}$$

Note that (2.8) actually follows from (2.9).

For the scalar quarks, which we discuss as a toy model, the equation of motion is

$$\left[(iD)^2 - m^2\right]\phi = 0\,, \tag{2.10}$$

where ϕ is the field of the scalar quark. The redefinition of the field is analogous to (2.2)

$$\phi(x) = \frac{1}{\sqrt{2m}} \exp[-im(vx)]\phi_v(x)\,, \tag{2.11}$$

where we have introduced a normalization factor. Note that due to this factor, the field ϕ_v has a different mass dimension compared to ϕ: $\dim[\phi_v] = 3/2$, while $\dim[\phi] = 1$. This leads to

$$\left[(iD_\mu + mv_\mu)^2 - m^2\right]\phi_v = 2m(ivD)\phi_v + (iD)^2\phi_v = 0 \quad \text{or} \quad (ivD)\phi_v = -\frac{1}{2m}(iD)^2\phi_v\,. \tag{2.12}$$

The additional factor $\sqrt{2m}$ in (2.11) serves to have the proper normalization of the static Lagrangian

$$\mathcal{L} = \phi^\dagger[(iD)^2 - m^2]\phi = \phi_v^\dagger(ivD)\phi_v + \cdots\,. \tag{2.13}$$

In the scalar case the RP transformation $\delta_{\rm RP}$ remains the same with the obvious replacement $Q_v \to \phi_v$.

3 Toy model: scalar quarks

Before considering full QCD, it is instructive to consider scalar quarks, which avoids the unnecessary complications induced by the quark spin. To be explicit, we define a decay of such a scalar quark into a lighter scalar quark and a particle without QCD interactions, mimicking a semileptonic decay of a real heavy quark.

3.1 Reparametrization invariance

We consider the decay of the scalar quark into two lighter scalars ψ and ℓ where only one of the two decay products (ψ) is a color triplet. Thus we consider an effective Hamiltonian of the form

$$H_{\rm eff} = g(\phi^\dagger\psi)\ell\,. \tag{3.1}$$

With this Hamiltonian we can write the total and the differential inclusive rates. We start from the operator

$$R(q) = \int d^4x\, e^{iqx}\, T\left[(\phi^\dagger(x)\psi(x))\,(\psi^\dagger(0)\phi(0))\right], \tag{3.2}$$

where q is the momentum transfer to the (color-neutral) ℓ particle. Clearly this expression is independent of v and thus RPI.

Inserting the rescaling (2.11) we obtain

$$R(S) = \int d^4x\, \frac{1}{2m} e^{-im(S\cdot x)}\, T\left[(\phi_v^\dagger(x)\psi(x))\,(\psi^\dagger(0)\phi_v(0))\right], \tag{3.3}$$

where

$$S = v - \frac{q}{m}\,. \tag{3.4}$$

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This relation is the starting point of an OPE, leading to a series in inverse powers of m. This OPE takes the form

$$R(S) = \sum_{n,i} C_i^{(n)}(S) \mathcal{O}_i^{(n)} \,, \tag{3.5}$$

where the $\mathcal{O}_i^{(n)}$ are local operators of dimension $n+3$ and the coefficients $C_i^{(n)}$ depend on S and are of order $1/m^{n+3}$, assuming dimensionless R. At tree level, the operators $\mathcal{O}_i^{(n)}$ can be written in terms of the fields ϕ_v and a chain of covariant derivatives. We get

$$R(S) = \sum_{n=0}^{\infty} C^{(n)}_{\mu_1\cdots\mu_n}(S) \; \phi_v^\dagger (iD^{\mu_1} \cdots iD^{\mu_n}) \phi_v \,, \tag{3.6}$$

with $\phi_v = \phi_v(0)$. Note that the rate is obtained by taking the discontinuity of R; since the rate has to be real, the operators appearing in the OPE have to be hermitian, so the forward matrix elements are real. This has consequences for the coefficients $C^{(n)}_{\mu_1\cdots\mu_n}(S)$ which we will exploit later on.

The relation (3.5) is RPI as long as the full sum is taken into account. The key observation is that the RP transformation relates subsequent orders in the $1/m$ expansion. In fact, we have

$$\begin{aligned}
0 &= \delta_{\rm RP} R(S) \qquad (3.7)\\
&= \sum_{n=0}^{\infty} \left[\delta_{\rm RP} C^{(n)}_{\mu_1\cdots\mu_n}\right] \phi_v^\dagger (iD^{\mu_1} \cdots iD^{\mu_n}) \phi_v + \sum_{n=0}^{\infty} C^{(n)}_{\mu_1\cdots\mu_n} \left[\delta_{\rm RP} \phi_v^\dagger (iD^{\mu_1} \cdots iD^{\mu_n}) \phi_v\right] \\
&= \sum_{n=0}^{\infty} \left[\delta_{\rm RP} C^{(n)}_{\mu_1\cdots\mu_n}\right] \phi_v^\dagger (iD^{\mu_1} \cdots iD^{\mu_n}) \phi_v \\
&\quad - m \sum_{n=0}^{\infty} C^{(n)}_{\mu_1\cdots\mu_n} \Big[\delta v^{\mu_1} \, \phi_v^\dagger (iD^{\mu_2}) \cdots (iD^{\mu_n}) \phi_v + \delta v^{\mu_2} \, \phi_v^\dagger (iD^{\mu_1})(iD^{\mu_3}) \cdots (iD^{\mu_n}) \phi_v \\
&\qquad\qquad \cdots + \delta v^{\mu_n} \, \phi_v^\dagger (iD^{\mu_1}) \cdots (iD^{\mu_{n-1}}) \phi_v \Big] \,.
\end{aligned}$$

In order to achieve the cancellation between the different orders in the OPE, the coefficients have to satisfy the relation

$$\delta_{\rm RP} C^{(n)}_{\mu_1\cdots\mu_n}(S) = m \, \delta v^\alpha \left(C^{(n+1)}_{\alpha\mu_1\cdots\mu_n}(S) + C^{(n+1)}_{\mu_1\alpha\mu_2\cdots\mu_n}(S) + \cdots + C^{(n+1)}_{\mu_1\cdots\mu_n\alpha}(S) \right) , \tag{3.8}$$

where we have $\delta_{\rm RP} S = \delta v$ from (3.4) and thus

$$\frac{\partial}{\partial S^\alpha} C^{(n)}_{\mu_1\cdots\mu_n}(S) = m \left(C^{(n+1)}_{\alpha\mu_1\cdots\mu_n}(S) + C^{(n+1)}_{\mu_1\alpha\mu_2\cdots\mu_n}(S) + \cdots + C^{(n+1)}_{\mu_1\cdots\mu_n\alpha}(S) \right) . \tag{3.9}$$

This is a remarkable relation, relating subsequent orders in the $1/m$ expansion. It is universal for any differential rate and is valid (with obvious small modifications) also when perturbative radiative corrections are included, we discuss this in more detail below. We note that this relation is a generalization of the relation given in [8] for the differential rates.

In fact, using the tensor decomposition of the first few terms

$$C^{(0)}(S) = a^{(0)}(S^2) \tag{3.10}$$

$$C^{(1)}_\mu(S) = a^{(1)}(S^2)S_\mu \tag{3.11}$$

$$C^{(2)}_{\mu\nu}(S) = a^{(2)}(S^2)g_{\mu\nu} + b^{(2)}(S^2)S_\mu S_\nu \tag{3.12}$$

yields the RPI relations

$$\frac{\partial}{\partial S^\alpha}C^{(0)}(S) = \frac{\partial}{\partial S^\alpha}a^{(0)} = 2(a^{(0)})' S_\alpha = m\, C^{(1)}_\alpha(S) = m\, a^{(1)} S_\alpha \tag{3.13}$$

$$\frac{\partial}{\partial S^\alpha}C^{(1)}_\mu(S) = 2(a^{(1)})' S_\alpha S_\mu + a^{(1)} g_{\mu\alpha} = m(C^{(2)}_{\mu\alpha} + C^{(2)}_{\alpha\mu}) = 2m(b^{(2)} S_\mu S_\alpha + a^{(2)} g_{\mu\alpha})\,. \tag{3.14}$$

3.2 Total rate

Upon integration over the particle ℓ we get the OPE for the total rate;

$$R = \sum_{n=0}^{\infty} c^{(n)}_{\mu_1\cdots\mu_n}(v)\ \phi_v^\dagger (iD^{\mu_1}\cdots iD^{\mu_n})\phi_v\,, \tag{3.15}$$

where the coefficients $c^{(n)}$ now depend only on v. The tensor decomposition for the $c^{(n)}$ has the same structure as (3.10)–(3.12); however now we have to replace S by v. Since $v^2 = 1$, the coefficients $a^{(i)}$ are in this case just numbers, and hence all the derivatives appearing in (3.13) vanish.

We shall explicitly consider terms up to fourth order, which read

$$\begin{aligned} R = {} & c^{(0)}\phi_v^\dagger\phi_v + c^{(1)}_\mu\, \phi_v^\dagger(iD^\mu)\phi_v + c^{(2)}_{\mu\nu}\, \phi_v^\dagger(iD^\mu)(iD^\nu)\phi_v \\ & + c^{(3)}_{\mu\alpha\nu}\, \phi_v^\dagger(iD^\mu)(iD^\alpha)(iD^\nu)\phi_v + c^{(4)}_{\mu\alpha\beta\nu}\, \phi_v^\dagger(iD^\mu)(iD^\alpha)(iD^\beta)(iD^\nu)\phi_v + \cdots \end{aligned} \tag{3.16}$$

and discuss, how this finally generalizes to arbitrary order.

Decomposing the $c^{(n)}$ into linear combinations with all possible tensor structures, and taking into account that only hermitian operators can appear, gives

$$c^{(0)}(v) = a^{(0)} \tag{3.17}$$

$$c^{(1)}_\mu(v) = a^{(1)}v_\mu \tag{3.18}$$

$$c^{(2)}_{\mu\nu}(v) = a^{(2)}g_{\mu\nu} + b^{(2)}v_\mu v_\nu \tag{3.19}$$

$$c^{(3)}_{\mu\alpha\nu}(v) = x_1^{(3)}v_\alpha g_{\mu\nu} + x_2^{(3)}\left[v_\nu g_{\mu\alpha} + v_\mu g_{\nu\alpha}\right] + x_3^{(3)}v_\mu v_\alpha v_\nu \tag{3.20}$$

$$\begin{aligned} c^{(4)}_{\mu\alpha\beta\nu}(v) = {} & y_1^{(4)}g_{\mu\nu}g_{\alpha\beta} + y_2^{(4)}g_{\mu\alpha}g_{\nu\beta} + y_3^{(4)}g_{\mu\beta}g_{\nu\alpha} \\ & + z_1^{(4)}v_\alpha v_\beta g_{\mu\nu} + z_2^{(4)}v_\mu v_\nu g_{\alpha\beta} \\ & + z_3^{(4)}\left[v_\mu v_\alpha g_{\beta\nu} + v_\nu v_\beta g_{\mu\alpha}\right] + z_4^{(4)}\left[v_\mu v_\beta g_{\alpha\nu} + v_\nu v_\alpha g_{\beta\mu}\right] \\ & + w^{(4)}v_\mu v_\alpha v_\beta v_\nu \end{aligned} \tag{3.21}$$

Each of the coefficients $a^{(0)}\cdots w^{(4)}$ corresponds to a linearly independent operator, and RPI will imply relations between the coefficients $a^{(0)}\cdots w^{(4)}$.

The corresponding total rate is then obtained via the optical theorem by taking a forward matrix element of R with the initial state $|H(p_H)\rangle$

$$2m_H\Gamma = \langle R\rangle \equiv \langle H(p_H)|R|H(p_H)\rangle\,. \tag{3.22}$$

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3.2.1 The leading order term

Applying the RP transformation to the leading term gives

$$\delta_{\rm RP} c^{(0)}(v) = 0 \tag{3.23}$$

which leads to the RPI result for the leading term

$$\Gamma = \frac{1}{2m_H}\langle R\rangle = a^{(0)}\frac{1}{2m_H}\langle \phi_v^\dagger \phi_v\rangle\,, \tag{3.24}$$

in terms of a single matrix element. As we show in the appendix, it is given by

$$\langle \phi_v^\dagger \phi_v\rangle = 2m_H\mu_3 = 2m_H\left(1-\frac{\mu_\pi^2}{2m^2}\right)\,,$$

where the last relation is exact to any order in $1/m$ for our definition of μ_π^2.

Before continuing to higher orders, we note that the above result in fact depends on m in a nontrivial way. The parameter μ_3 as well as the parameter μ_π both depend on m; however, in the limit $m\to\infty$ we obtain $\mu_3 = 1$ as expected. As we shall see below, the higher-order terms are such that the result is RPI, which becomes manifest by expressing the leading order result in terms of operators and matrix elements in full QCD, i.e. with no reference to the arbitrary velocity vector v. We claim, that this constitutes an improvement of the HQE, since the corresponding higher-order terms that will appear in the HQE are now implicitly re-summed in the parameter μ_3. We shall return to this when we have discussed the higher orders of the HQE.

3.2.2 First and second order terms

The first order terms already give insights into the structure of the HQE. Applying the RP transformation (3.9) to $c^{(1)}$ we get a relation between the first and second order terms

$$\delta_{\rm RP} c^{(1)}_\mu(v) = a^{(1)}\delta v_\mu = m\delta v^\alpha\left(c^{(2)}_{\alpha\mu}+c^{(2)}_{\mu\alpha}\right) = 2m\delta v_\mu a^{(2)}\,, \tag{3.25}$$

which implies $a^{(1)} = 2m\,a^{(2)}$, while the coefficient $b^{(2)}$ remains unconstrained. Inserting this relation into (3.6) yields

$$a^{(1)}\,\phi_v^\dagger(ivD)\phi_v + a^{(1)}\,\frac{1}{2m}\phi_v^\dagger(iD)^2\phi_v = a^{(1)}\,\phi_v^\dagger\left((ivD)+\frac{1}{2m}(iD)^2\right)\phi_v\,. \tag{3.26}$$

Note that this particular combination is invariant under reparametrization:

$$\delta_{\rm RP}\left((ivD)+\frac{1}{2m}(iD)^2\right) = 0\,; \tag{3.27}$$

furthermore, its contribution vanishes when acting on the field ϕ_v by the equation of motion.

In the case at hand it means that we may drop the (ivD) terms as soon as this operator acts directly on the field ϕ_v, since RPI ensures that at the next order a corresponding term with the proper coefficient will appear, which will combine with this term to an *exactly* vanishing result. Below we show explicitly that this cancellation also appears for higher-order terms.

In addition, we re-derive the well-known result that there is no term of linear order in $1/m$ in the HQE; this holds true also for the higher-order terms hidden in μ_3, since according to (A.7) the first correction to μ_3 is $\mathcal{O}(1/m^2)$.

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3.2.3 Second and third order terms

At second order, we obtain

$$\begin{aligned}\delta_{\mathrm{RP}} c^{(2)}_{\mu\nu}(v) &= b^{(2)}(\delta v_\mu\, v_\nu + v_\mu\, \delta v_\nu) \\ &= m\delta v^\alpha \left(c^{(3)}_{\mu\nu\alpha} + c^{(3)}_{\mu\alpha\nu} + c^{(3)}_{\alpha\mu\nu}\right) \\ &= m(x_1^{(3)} + 2x_2^{(3)})[\delta v_\mu\, v_\nu + \delta v_\nu\, v_\mu]\end{aligned} \tag{3.28}$$

which implies

$$m(x_1^{(3)} + 2x_2^{(3)}) = b^{(2)}\,. \tag{3.29}$$

The parameterization in (3.21) of $c^{(3)}$ into the various tensor structures corresponds to a choice of the operator basis, such that

$$R^{(3)} = x_1^{(3)} \phi_v^\dagger (iD^\mu)(ivD)(iD_\mu)\phi_v + x_2^{(3)} \phi_v^\dagger \left\{(iD)^2\,,\,(ivD)\right\}\phi_v + x_3^{(3)} \phi_v^\dagger (ivD)^3 \phi_v\,. \tag{3.30}$$

We may solve (3.29) for $x_2^{(3)}$ and insert this into (3.30) to obtain

$$R^{(3)} = \frac{x_1^{(3)}}{2} \phi_v^\dagger \left[(iD_\mu)\,,\,[(ivD)\,,\,(iD^\mu)]\right]\phi_v + \frac{b}{2m}\phi_v^\dagger \left\{(iD)^2\,,\,(ivD)\right\}\phi_v + x_3^{(3)}\phi_v^\dagger (ivD)^3\phi_v\,.$$

This relation suggests a change of the operator basis. The first operator generates the well-known Darwin term ρ_D and does not relate back to the lower orders in $1/m$. RPI fixes the coefficient of the second operator; as discussed in the last section this term combines with the terms of the second order into

$$\begin{aligned} b^{(2)} &\left(\phi_v^\dagger (ivD)^2\phi_v + \frac{1}{2m}\phi_v^\dagger \left\{(ivD)\,,\,(iD)^2\right\}\phi_v\right) \\ &= b^{(2)}\,\phi_v^\dagger \left((ivD) + \frac{1}{2m}(iD)^2\right)^2 \phi_v + \mathcal{O}(1/m^2)\,,\end{aligned} \tag{3.31}$$

where the higher-order term is also generated properly, as we show below. The coefficient $c^{(4)}$ of this term can also be related directly to $c^{(2)}$ via the second-order transformation $(\delta_{\mathrm{RP}})^2$.

Thus we find that there is no new term generated at order $1/m^2$. The expected $1/m^2$ kinetic energy parameter μ_π^2 is contained in the leading order term μ_3. It is well known, that RPI relates the coefficients of the leading term with the one of μ_π^2; here we suggest to consider this as the RPI completion of the leading order, now written in terms of μ_3. As we shall see below, μ_3 will absorb terms at higher orders in the same way as μ_π.

At order $1/m^3$, we find only a single new term which generates the Darwin term with a coefficient that is not constrained by RPI; likewise, the coefficient $x_3^{(3)}$ remains unconstrained. However, we shall see below that RPI will again ensure that this term gets completed such that the equation of motion can be applied to make it vanish.

3.2.4 Third and fourth order terms

Applying the RPI relation (3.9) to $c^{(3)}$ yields the relation

$$\begin{aligned}\delta_{\rm RP}c^{(3)}_{\mu\alpha\nu}(v) &= x_1^{(3)}\delta v_\alpha g_{\mu\nu} + x_2^{(3)}\left[\delta v_\nu g_{\mu\alpha} + \delta v_\mu g_{\nu\alpha}\right] + x_3^{(3)}\left[\delta v_\mu\, v_\alpha v_\nu + \delta v_\alpha\, v_\mu v_\nu + \delta v_\nu\, v_\alpha v_\mu\right] \\ &= 2m\left(y_1^{(4)} + y_3^{(4)}\right)\delta v_\alpha g_{\mu\nu} + m\left(2y_2^{(4)} + y_1^{(4)} + y_3^{(4)}\right)\left[\delta v_\mu g_{\alpha\nu} + \delta v_\nu g_{\alpha\mu}\right] \\ &\quad + 2m\left(z_2^{(4)} + z_4^{(4)}\right)\delta v_\alpha\, v_\mu v_\nu + m\left(2z_3^{(4)} + z_1^{(4)} + z_4^{(4)}\right) v_\alpha\left[\delta v_\mu\, v_\nu + \delta v_\nu\, v_\mu\right].\end{aligned} \tag{3.32}$$

Comparing the different tensor structures we obtain the relations

$$x_1^{(3)} = 2m\left(y_1^{(4)} + y_3^{(4)}\right), \tag{3.33}$$

$$x_2^{(3)} = \frac{b^{(2)}}{2m} - \frac{x_1^{(3)}}{2} = m\left(2y_2^{(4)} + y_1^{(4)} + y_3^{(4)}\right), \tag{3.34}$$

$$x_3^{(3)} = 2m\left(z_2^{(4)} + z_4^{(4)}\right) = m\left(2z_3^{(4)} + z_1^{(4)} + z_4^{(4)}\right). \tag{3.35}$$

There are two contributions to $R^{(4)}$ which are given by

$$R_1^{(4)} = y_1^{(1)}O_1^{(4)} + y_2^{(1)}O_2^{(4)} + y_3^{(1)}O_3^{(4)} \tag{3.36}$$

$$R_2^{(4)} = z_1^{(4)}P_1^{(4)} + z_2^{(4)}P_2^{(4)} + z_3^{(4)}P_3^{(4)} + z_4^{(4)}P_4^{(4)} \tag{3.37}$$

with the basis operators

$$\begin{aligned} O_1^{(4)} &= y_1^{(1)}\phi_v^\dagger(iD_\mu)(iD)^2(iD^\mu)\phi_v \\ O_2^{(4)} &= \phi_v^\dagger((iD)^2)^2\phi_v \\ O_3^{(1)} &= \phi_v^\dagger(iD_\mu)(iD_\nu)(iD^\mu)(iD^\nu)\phi_v \\ P_1^{(4)} &= \phi_v^\dagger(iD_\mu)(ivD)^2(iD^\mu)\phi_v \\ P_2^{(4)} &= \phi_v^\dagger(ivD)(iD)^2(ivD)\phi_v \\ P_3^{(4)} &= \phi_v^\dagger\left\{(ivD)^2\,,\,(iD)^2\right\}\phi_v \\ P_4^{(4)} &= \phi_v^\dagger\left[(ivD)(iD_\mu)(ivD)(iD^\mu) + (iD_\mu)(ivD)(iD^\mu)(ivD)\right]\phi_v \end{aligned}$$

Solving the relations (3.33), (3.34) for $y_1^{(4)}$ and $y_2^{(4)}$ and inserting this into $R_1^{(4)}$ yields

$$\begin{aligned} R_1^{(4)} &= \frac{b^{(2)}}{4m^2}O_2^{(4)} + \frac{x_1^{(3)}}{4m}\phi_v^\dagger\left[(iD_\mu)\,,\,\left[(iD)^2\,,\,(iD^\mu)\right]\right]\phi_v \\ &\quad + \frac{y_3^{(4)}}{2}\phi_v^\dagger[iD_\mu\,,\,iD_\nu][iD^\mu\,,\,iD^\nu]\phi_v\,. \end{aligned} \tag{3.38}$$

The first term is the expected completion of the $(ivD)^2$ in (3.31), while the second term is the RPI completion of the Darwin term,

$$\phi_v^\dagger\left[(iD_\mu)\,,\,\left[(ivD)\,,\,(iD^\mu)\right]\right]\phi_v \to \phi_v^\dagger\left[(iD_\mu)\,,\,\left[\left(ivD + \frac{1}{2m}(iD)^2\right)\,,\,(iD^\mu)\right]\right]\phi_v\,. \tag{3.39}$$

Finally, only the coefficient of the third term remains unconstrained leading to a genuinely new contribution to $R_1^{(4)}$.

For $R_2^{(4)}$, we solve the relations (3.35) for $z_2^{(4)}$ and $z_3^{(4)}$ and change to a more convenient base. We find

$$R_2^{(4)} = \frac{x_3^{(3)}}{2m}\left[P_2^{(4)} + P_3^{(4)}\right] - z_1^{(4)}\phi_v^\dagger\left[(ivD)\,,\,(iD_\mu)\right]\left[(ivD)\,,\,(iD^\mu)\right]\phi_v \qquad (3.40)$$
$$- \frac{u^{(4)}}{2}\phi_v^\dagger\Big\{(ivD)\,,\,\left[(iD^\mu)\,,\,\left[(iD_\mu)\,,\,(ivD)\right]\right]\Big\}\phi_v\,,$$

with $u^{(4)} = z_1^{(4)} + z_4^{(4)}$. The first term is part of the RPI completion of the $(ivD)^3$ term appearing in the third order. The remaining terms are not constrained by RPI, the second term can be interpreted as the square of the chromo-electric field, while the last term will vanish by the equations of motion.

In summary, we find that at tree-level, the total rate for scalar-quark QCD up to the order $1/m^4$ can be written in terms of four parameters only. We define these parameters as

$$\langle\phi_v^\dagger\phi_v\rangle = 2m_H\,\mu_3 \qquad (3.41)$$

$$\frac{1}{2}\left\langle\phi_v^\dagger\left[(iD_\mu)\,,\,\left[\left(ivD + \frac{1}{2m}(iD)^2\right)\,,\,(iD^\mu)\right]\right]\phi_v\right\rangle = 2m_H\,\rho_D^3 \qquad (3.42)$$

$$\langle\phi_v^\dagger[iD_\mu\,,\,iD_\nu][iD^\mu\,,\,iD^\nu]\phi_v\rangle = 2m_H\,r_G^4 \qquad (3.43)$$

$$\langle\phi_v^\dagger\left[(ivD)\,,\,(iD_\mu)\right]\left[(ivD)\,,\,(iD^\mu)\right]\phi_v\rangle = 2m_H\,r_E^4 \qquad (3.44)$$

In particular, we note the absence of operators such as $[(iD)^2]^2$. This can be understood as a consequence of Lorentz invariance. The argument becomes particularly simple, if we ignore the presence of gluons and evaluate the forward matrix element of R between free quark states. This matrix element will be a Lorentz invariant quantity and hence will depend on the square of the quark momentum p. Inserting $p = mv + k$ yields $p^2 = m^2 + 2m(vk) + k^2$, and by the equation of motion we find $2m(vk) + k^2 = 0$ ensuring that $p^2 = m^2$. Hence all terms but the leading one vanish by the equation of motion, i.e. all operators involving k^2 and (vk) will appear only in the particular combination dictated by the equation of motion.

At this point it is also convenient to compare the above formulation with the one where the covariant derivative is split into a spatial and a time derivative, according to

$$iD_\mu = v_\mu(ivD) + D_\mu^\perp\,. \qquad (3.45)$$

While this splitting is very useful in different contexts, it is not useful for the present investigation. In fact, rewriting the HQE in terms of operators involving (ivD) and $iD^\perp$ will again re-arrange the terms, without giving additional insights.

3.2.5 Re-summation and relation to full QCD: scalar toy model

The parameters we found up to order $1/m^4$ depend on the mass of the heavy quark in a nontrivial way and imply a re-summation of higher orders of the HQE in such a way that

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the final result is actually RPI. This fact can be made manifest by rewriting the matrix elements in terms of QCD states and operators.

While the states are already the ones of full QCD, we still need to un-do the phase redefinition of the quark fields. For the leading term μ_3 this obvious, since we have

$$\langle \phi_v^\dagger \phi_v \rangle = 2m \, \langle \phi^\dagger \phi \rangle \,,$$

and thus μ_3 is a matrix element defined in full QCD.

The next term is the Darwin term ρ_D, for which we use the relation

$$e^{-imvx} \left(ivD + \frac{1}{2m}(iD)^2 \right) = \frac{1}{2m}((iD)^2 - m^2) e^{-imvx} \,. \tag{3.46}$$

Since the mass term does not contribute in the commutator, we find

$$\phi_v^\dagger \left[(iD_\mu) \,, \left[\left(ivD + \frac{1}{2m}(iD)^2 \right) \,, (iD^\mu) \right] \right] \phi_v = \phi^\dagger \left[(iD_\mu) \,, \left[(iD)^2 \,, (iD^\mu) \right] \right] \phi \,. \tag{3.47}$$

In fact, the Darwin term is related to the chromo-electric field $E \sim [iD_\mu \,, ivD]$ which is a quantity defined in a specific frame. RPI ensures that

$$\phi_v^\dagger \ldots [iD_\mu \,, ivD] \ldots \phi_v \to \phi_v^\dagger \ldots \left[iD_\mu \,, \left(ivD + \frac{1}{2m}(iD)^2 \right) \right] \ldots \phi_v, \tag{3.48}$$

where the ellipses denote any combination of derivatives or other operators involving the light degrees of freedom. Replacing the field ϕ_v by ϕ yields

$$\begin{aligned} &\phi_v^\dagger \ldots \left[iD_\mu \,, \left(ivD + \frac{1}{2m}(iD)^2 \right) \right] \ldots \phi_v \\ &\quad = \phi^\dagger \ldots \left[iD_\mu \,, (iD)^2 \right] \ldots \phi = \phi^\dagger \ldots \{ iD_\alpha \,, [iD_\mu \,, iD^\alpha] \} \ldots \phi \,. \end{aligned} \tag{3.49}$$

Taking a matrix element of this operator with momentum eigenstates shows that this indeed becomes the chromo-electric field in the rest frame of the this state.

In a similar way the remaining terms can be re-expressed in terms of full QCD and become

$$\langle \phi_v^\dagger [iD_\mu, iD_\nu][iD^\mu, iD^\nu] \phi_v \rangle = 2m \langle \phi^\dagger [iD_\mu, iD_\nu][iD^\mu, iD^\nu] \phi \rangle \,, \tag{3.50}$$

$$\langle \phi_v^\dagger \left[(ivD), (iD_\mu) \right] \left[(ivD), (iD^\mu) \right] \phi_v \rangle = 2m \langle \phi^\dagger \left[(iD)^2, (iD_\mu) \right] \left[(iD)^2, (iD^\mu) \right] \phi \rangle. \tag{3.51}$$

Note that the power counting is now much less obvious, since the power in $1/m$ is no longer simply related to the number of derivatives appearing in the operators. Nevertheless, the leading term in a $1/m$ expansion of the matrix elements always reproduces the proper static limit, and the higher order terms are arranged such that the final result is RPI.

We may compare our approach to the approach using RPI covariant fields as discussed in [7, 11]. In contrast to the approach in [7, 11], we propose to write the results in terms of matrix elements of full QCD operators with states defined in full QCD. As stated above, this renders the power counting more complicated: it can be seen from e.g. the Darwin term (3.47) that the dimension of the operator is no longer related to the power in $1/m$.

Note that the framework of HQE allows us to fully map this to QCD operators, which we consider advantageous. However, in a theory, where the matching to QCD is less trivial (such as various effective theories used in the context of chiral effective theories in hadron and nuclear physics) RPI will also yield constraints, since the underlying theory has to be Lorentz invariant.

3.2.6 Generalization to arbitrary orders

From the above arguments it becomes clear that one may systematically access higher orders by an iterative process. Starting from a suitable tensor decomposition of the coefficients $c^{(n)}$ and $c^{(n+1)}$ one makes use of (3.9) to obtain relations between the coefficients of the tensor decomposition of $c^{(n)}$ and $c^{(n+1)}$. Taking into account the information obtained from lower orders m, $m \leq n$ one can determine the elements of the operator basis which are constrained by RPI and the ones which emerge as new parameters. However, genuinely new matrix elements are only the ones where no (ivD) factor appears next to the field ϕ_v, since such a contribution will vanish exactly once it is properly combined with higher orders.

Finally, in order to make the invariance under reparametrization manifest, one always can rewrite the operators and covariant derivatives in terms of full QCD operators, which are without any reference to the velocity vector.

4 Real quarks

Taking into account the quark spin does not change the general idea, the discussion becomes only a bit more tedious. We start with eq. (3.5) in real QCD

$$R(S) = \sum_{n=0}^{\infty} C^{(n)}_{\mu_1\cdots\mu_n}(S) \otimes \bar{Q}_v (iD_{\mu_1}\cdots iD_{\mu_n}) Q_v \,. \tag{4.1}$$

Here $\otimes$ is a short hand for the Dirac structure:

$$\begin{aligned} R(S) &= \sum_{n=0}^{\infty} \left[C^{(n)}_{\mu_1\cdots\mu_n}(S) \right]_{\alpha\beta} \bar{Q}_{v,\alpha} (iD_{\mu_1}\cdots iD_{\mu_n}) Q_{v,\beta} \\ &= \sum_{n=0}^{\infty} \sum_{\Gamma} C^{(n,\Gamma)}_{\mu_1\cdots\mu_n} \bar{Q}_v (iD_{\mu_1}\cdots iD_{\mu_n}) \Gamma Q_v \,, \end{aligned} \tag{4.2}$$

where the sum over Γ runs over the basis of the 16 Dirac matrices $1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5, i\gamma_\mu\gamma_5$ and

$$C^{(n,\Gamma)}_{\mu_1\cdots\mu_n} = \frac{1}{4} \mathrm{Tr}[\Gamma \, C^{(n)}_{\mu_1\cdots\mu_n}] \,.$$

Applying the RP transformation (2.7), (2.8), (2.9), we arrive at the RPI relation

$$\delta_{\mathrm{RP}} C^{(n)}_{\mu_1\cdots\mu_n} = m\,\delta v^\alpha \left(C^{(n+1)}_{\alpha\mu_1\cdots\mu_n} + C^{(n+1)}_{\mu_1\alpha\mu_2\cdots\mu_n} + \cdots + C^{(n+1)}_{\mu_1\cdots\mu_n\alpha} \right) \quad n = 0,1,2,\ldots \tag{4.3}$$

The difference with (3.9) is that the coefficients are now Dirac-matrix valued.

4.1 Total rate for real quarks

For the total rate, the coefficients still depend only on the velocity v, where now we have to take into account the spinor structure of the coefficients. The first few terms read

$$R = \bar{Q}_v C^{(0)}(v) Q_v + \bar{Q}_v C^{(1)}_\mu(v)(iD^\mu) Q_v + \bar{Q}_v C^{(2)}_{\mu\nu}(v)(iD^\mu)(iD^\nu) Q_v + \cdots \tag{4.4}$$

The coefficients up to $1/m^2$ are

$$C^{(0)}(v) = a_0 + \hat{a}_0 \not{v} , \tag{4.5}$$

$$C^{(1)}_\mu(v) = v_\mu \left(a_1 + \hat{a}_1 \not{v}\right) + \gamma_\mu \left(b_1 + \hat{b}_1 \not{v}\right) , \tag{4.6}$$

$$C^{(2)}_{\mu\nu}(v) = v_\mu v_\nu \left(a_2 + \hat{a}_2 \not{v}\right) + g_{\mu\nu} \left(b_2 + \hat{b}_2 \not{v}\right) + (v_\mu \gamma_\nu + v_\nu \gamma_\mu) \left(d_2 + \hat{d}_2 \not{v}\right) + g_2(-i\sigma_{\mu\nu}) \tag{4.7}$$

where the coefficients $a_1 \ldots g_2$ are only functions of the quark masses and of the strong coupling α_s and we only consider hermitian operators. We have dropped all parity-odd contributions, since we only discuss ground-state mesons.

4.1.1 The leading order

Employing now relation (4.3) to the leading coefficient we get

$$\delta_{\rm RP} C^{(0)} = \hat{a}_0 \; \delta v^\alpha \gamma_\alpha \overset{\rm RPI}{=} m \, \delta v^\alpha \, C^{(1)}_\alpha = m \, \delta v^\alpha \gamma_\alpha \left(b_1 + \hat{b}_1 \not{v}\right) . \tag{4.8}$$

Comparing the Dirac and the tensor structure, we obtain the relations

$$b_1 = \frac{1}{m} \hat{a}_0 \qquad \text{and} \qquad \hat{b}_1 = 0 , \tag{4.9}$$

while a_1 and $\hat{a}_1$ remain unconstrained. Gathering the leading and the first order term yields

$$R = (a_0 + \hat{a}_0) \bar{Q}_v Q_v + a_1 \bar{Q}_v (ivD) Q_v + \hat{a}_1 \bar{Q}_v (ivD) \not{v} Q_v , \tag{4.10}$$

where the two leading coefficients a_0 and $\hat{a}_0$ are related by the equation of motion (2.4). As we shall see, this feature will also be present in higher orders.

The leading term is given by the matrix element μ_3 defined in the appendix. Furthermore, RPI enforces that the contribution proportional to $\hat{a}_0$ involving $\not{v}$ is related to the term with γ_α proportional to b_1. As we shall see below, this will eventually allow us to replace $\not{v} \to 1$, i.e. there will be no contribution with a single γ_α matrix.

4.1.2 First and second order terms

In the next step we apply (4.3) to the first order term to obtain

$$\delta_{\rm RP} C^{(1)}_\mu = \delta v_\mu \left(a_1 + \hat{a}_1 \not{v}\right) + \left(\hat{a}_1 v_\mu + \hat{b}_1 \gamma_\mu\right) \delta\not{v} \tag{4.11}$$

$$\overset{\rm RPI}{=} m \delta v^\alpha \left(C^{(2)}_{\mu\alpha} + C^{(2)}_{\alpha\mu}\right) \tag{4.12}$$

$$= m \delta v^\alpha \left[2 g_{\mu\alpha} \left(b_2 + \hat{b}_2 \not{v}\right) + 2 \gamma_\alpha v_\mu \left(d_2 + \hat{d}_2 \not{v}\right)\right]$$

from which we obtain the relations

$$b_2 = \frac{1}{2m} a_1 \qquad \hat{b}_2 = \frac{1}{2m} \hat{a}_1 \qquad d_2 = \frac{1}{2m} \hat{a}_1 \quad \hat{d}_2 = 0 . \tag{4.13}$$

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Collecting all the (non-zero) terms of up to order $1/m^2$ we get

$$R = (a_0 + \hat{a}_0)\bar{Q}_v Q_v + a_1 \bar{Q}_v \left((ivD) + \frac{1}{2m}(iD)^2 \right) Q_v \\ + \hat{a}_1 \bar{Q}_v \left\{ \left((ivD) + \frac{1}{2m}(iD)^2 \right), \left(\not{v} + \frac{1}{m}(i\not{D}) \right) \right\} Q_v \\ + g_2 \bar{Q}_v (\sigma \cdot G) Q_v + \cdots, \tag{4.14}$$

where the ellipses denote terms of higher order and terms that are total derivatives; the latter do not contribute to the relevant forward matrix elements.

Similar to what happened in the leading order, the terms with $\not{v}$ combine with the corresponding terms at the next order to yield the equation of motion. Eventually this means that these terms may be lumped into the contributions with the unit Dirac matrix. To this end, the Dirac decomposition of the coefficients $C^{(n)}$ can be reduced to the terms with 1 and $\sigma_{\mu\nu}$.

Furthermore, the equation of motion (2.5) now yields

$$\left((ivD) + \frac{1}{2m}(iD)^2 \right) Q_v = -\frac{1}{2m}(\sigma \cdot G) Q_v, \tag{4.15}$$

where the left and the right hand side are both RPI. Finally,

$$R = (a_0 + \hat{a}_0)\, \bar{Q}_v Q_v + \left(g_2 - \frac{a_1 + \hat{a}_1}{2m} \right) \bar{Q}_v (\sigma \cdot G) Q_v + \mathcal{O}(1/m^3). \tag{4.16}$$

This expression is a re-derivation of the known result, that the HQE does not have $1/m$ contributions. Furthermore, up to order $1/m^2$ the HQE contains two non-perturbative parameters μ_3 and μ_G (or equivalently μ_π and μ_G) which we have defined in the appendix.

4.1.3 Second and third order terms

Dropping all terms with $\not{v}$ and single γ matrices, we only consider

$$C^{(2)}_{\mu\nu}(v) = v_\mu v_\nu a_2 + g_{\mu\nu} b_2 + g_2(-i\sigma_{\mu\nu}) \tag{4.17}$$

$$C^{(3)}_{\mu\alpha\nu}(v) = x_1^{(3)} v_\alpha g_{\mu\nu} + x_2^{(3)} (v_\nu g_{\mu\alpha} + v_\mu g_{\nu\alpha}) + x_3^{(3)} v_\mu v_\alpha v_\nu \\ + \xi_1^{(3)} v_\alpha(-i\sigma_{\mu\nu}) + \xi_2^{(3)} (v_\nu(-i\sigma_{\mu\alpha}) + v_\mu(-i\sigma_{\alpha\nu})). \tag{4.18}$$

The spin independent terms (i.e. the ones without a σ matrix) yield the same result as for the scalar case. However, the first term in $C^{(2)}$ will generate a term with $(ivD)^2$ which will combine in the same way as in the scalar case to the RPI combination in (4.15), which now generates a contribution of $1/(4m^2)(\sigma \cdot G)^2$. These terms will appear in the fourth order.

The spin-dependent (denoted by the superscript σ) terms yield

$$\delta_{\mathrm{RP}} C^{(2,\sigma)}_{\mu\nu} = 0 = m\delta v^\alpha \left(C^{(3,\sigma)}_{\mu\nu\alpha} + C^{(3,\sigma)}_{\mu\alpha\nu} + C^{(3,\sigma)}_{\alpha\mu\nu} \right) \\ = m\, \xi_1^{(3)} \delta v^\alpha (\sigma_{\mu\alpha} v_\nu + \sigma_{\alpha\nu} v_\mu). \tag{4.19}$$

From this we conclude that $\xi_1^{(3)} = 0$ and thus we find that in total rates the usual spin-orbit term ρ_{LS} is absent; this has been noticed already in previous papers [12, 13]. This is related to the definition of μ_G in (A.18) in terms of the full covariant derivative instead of the spatial components only. Using the definition of μ_G with spatial components only yields an expression which is not RPI, rather it is related by reparametrization to ρ_{LS} and hence the corresponding combination can be treated as a single parameter, i.e. by our definition of μ_G.

The remaining term in $C^{(3)}$ contains an (ivD) which acts on Q_v. This term will be completed in a reparametrization-invariant way at higher orders, rendering a fourth-order contribution.

Thus we find at order $1/m^3$ only one "genuine" contribution, which will generate the Darwin term ρ_D with the operator structure

$$\bar{Q}_v \left[(iD_\mu)\,,\, [(ivD)\,,\, (iD^\mu)]\right] Q_v\,.$$

4.2 Third and fourth order terms

At the fourth order we obtain the same results as in the scalar case. For the additional spin-dependent terms, we find

$$\begin{aligned} C^{(4\,\sigma g)}_{\mu\alpha\beta\nu} &= \alpha_1^{(4)}(-i\sigma_{\mu\nu})g_{\alpha\beta} + \alpha_2^{(4)}(-i\sigma_{\alpha\beta})g_{\mu\nu} \\ &\quad + \alpha_3^{(4)}\left[(-i\sigma_{\mu\alpha})g_{\beta\nu} + (-i\sigma_{\beta\nu})g_{\mu\alpha}\right] + \alpha_4^{(4)}\left[(-i\sigma_{\mu\beta})g_{\alpha\nu} + (-i\sigma_{\alpha\nu})g_{\mu\beta}\right]\,, \end{aligned} \tag{4.20}$$

corresponding to a linear combination of four hermitian operators:

$$R_1^{(4,\sigma)} = \alpha_1^{(4)}S_1^{(4)} + \alpha_2^{(4)}S_2^{(4)} + \alpha_3^{(4)}S_3^{(4)} + \alpha_4^{(4)}S_4^{(4)}\,, \tag{4.21}$$

with

$$S_1^{(4)} = \bar{Q}_v(iD_\mu)(iD)^2(iD_\nu)(-i\sigma^{\mu\nu})Q_v\,, \tag{4.22}$$

$$S_2^{(4)} = \bar{Q}_v(iD_\alpha)(\sigma\cdot G)(iD^\alpha)Q_v\,, \tag{4.23}$$

$$S_3^{(4)} = \bar{Q}_v\left\{(iD)^2\,,\,(\sigma\cdot G)\right\}Q_v\,, \tag{4.24}$$

$$S_4^{(4)} = \bar{Q}_v\left[(iD_\mu)(iD_\alpha)(iD^\mu)(iD_\beta) + (iD_\alpha)(iD^\mu)(iD_\beta)(iD_\mu)\right](-i\sigma^{\alpha\beta})Q_v\,. \tag{4.25}$$

The reparametrization (4.3) relates these terms to the spin-dependent ones in $C^{(3)}$:

$$\begin{aligned} \delta_{\rm RP}C^{(3,\sigma)}_{\mu\alpha\nu}(v) &= \xi_2^{(3)}\left(\delta v_\nu(-i\sigma_{\mu\alpha}) + \delta v_\mu(-i\sigma_{\alpha\nu})\right) \\ &= 2m\left(\alpha_1^{(4)}+\alpha_4^{(4)}\right)(-i\sigma_{\mu\nu})\delta v_\alpha + m\left(2\alpha_3^{(4)}+\alpha_2^{(4)}+\alpha_4^{(4)}\right)\left[\delta v_\nu(-i\sigma_{\mu\alpha})+\delta v_\mu(-i\sigma_{\alpha\nu})\right] \\ &\quad + m(\alpha_1^{(4)}+\alpha_4^{(4)})\delta v^\beta\left[(-i\sigma_{\mu\beta})g_{\alpha\nu}+(-i\sigma_{\beta\nu})g_{\mu\alpha}\right]Q_v\,. \end{aligned} \tag{4.26}$$

From this relation we obtain the equations

$$m(2\alpha_3^{(4)} + \alpha_2^{(4)} + \alpha_4^{(4)}) = \xi_2^{(3)}\,, \tag{4.27}$$

$$\alpha_1^{(4)} + \alpha_4^{(4)} = 0\,. \tag{4.28}$$

Solving these equations for $\alpha_3^{(4)}$ and inserting this into $R_1^{(4,\sigma)}$ yields

$$R_1^{(4,\sigma)} = \frac{\xi_2^{(3)}}{2m} S_3^{(4)} + \alpha_1^{(4)} \left[S_1^{(4)} + S_2^{(4)} - S_4^{(4)} + \left(\frac{S_3^{(4)}}{2} - S_2^{(4)} \right) \right] - \frac{\alpha_2^{(4)}}{2} \left[S_3^{(4)} - 2S_2^{(4)} \right] \tag{4.29}$$

The first term is the expected completion of terms appearing in the third order

$$\bar{Q}_v \left\{ (ivD) \, , \, (\sigma \cdot G) \right\} Q_v \to \bar{Q}_v \left\{ \left(ivD + \frac{1}{2m} (iD)^2 \right) \, , \, (\sigma \cdot G) \right\} Q_v \, ,$$

while the remaining terms remain unconstrained and have a simple physical interpretation in terms of the operators

$$S_1^{(4)} + S_2^{(4)} - S_4^{(4)} = \bar{Q}_v \left[(iD_\mu) \, , \, (iD_\alpha) \right] \left[(iD_\beta) \, , \, (iD^\mu) \right] (-i\sigma^{\alpha\beta}) Q_v \, , \tag{4.30}$$

$$S_3^{(4)} - 2S_2^{(4)} = \bar{Q}_v \left[(iD_\mu) \, , \, \left[(iD_\mu) \, , \, (\sigma \cdot G) \right] \right] Q_v \, . \tag{4.31}$$

The matrix element of the first operator is related to $\sigma \cdot (G \times G)$, while the second operator is related to $\mathcal{D}^2(\sigma \cdot G)$, where $\mathcal{D}$ is the covariant derivative in the adjoint representation, acting on G.

The second contribution can be parametrized as

$$\begin{aligned} C_{\mu\alpha\beta\nu}^{(4\,\sigma vv)} = {} & \beta_1^{(4)} (-i\sigma_{\mu\nu}) v_\alpha v_\beta + \beta_2^{(4)} (-i\sigma_{\alpha\beta}) v_\mu v_\nu \\ & + \beta_3^{(4)} \left[(-i\sigma_{\mu\alpha}) v_\nu v_\beta + (-i\sigma_{\nu\beta}) v_\mu v_\alpha \right] + \beta_4^{(4)} \left[(-i\sigma_{\nu\alpha}) v_\mu v_\beta + (-i\sigma_{\mu\beta}) v_\nu v_\alpha \right] \, , \end{aligned} \tag{4.32}$$

corresponding to the linear combination of operators

$$R_2^{(4,\sigma)} = \beta_1^{(4)} U_1^{(4)} + \beta_2^{(4)} U_2^{(4)} + \beta_3^{(4)} U_3^{(4)} + \beta_4^{(4)} U_4^{(4)} \, , \tag{4.33}$$

with

$$U_1^{(4)} = \bar{Q}_v (iD_\mu)(ivD)^2 (iD_\nu)(-i\sigma^{\mu\nu}) Q_v \, , \tag{4.34}$$

$$U_2^{(4)} = \bar{Q}_v (ivD)(\sigma \cdot G)(ivD) Q_v \, , \tag{4.35}$$

$$U_3^{(4)} = \bar{Q}_v \left\{ (ivD)^2 \, , \, (\sigma \cdot G) \right\} Q_v \, , \tag{4.36}$$

$$U_4^{(4)} = \bar{Q}_v \left[(ivD)(iD_\alpha)(ivD)(iD_\beta) + (iD_\alpha)(ivD)(iD_\beta)(ivD) \right] (-i\sigma^{\alpha\beta}) Q_v \, . \tag{4.37}$$

Using the reparametrization relation (4.3) we find no terms of this form in $\delta_{\rm RP} C^{(3)}$ and thus

$$0 = m(\beta_1^{(4)} + \beta_4^{(4)}) \delta v^\beta \left[v_\mu v_\alpha (-i\sigma_{\nu\beta}) + v_\nu v_\alpha (-i\sigma_{\beta\nu}) \right] \, , \tag{4.38}$$

from which we conclude

$$\beta_1^{(4)} = -\beta_4^{(4)} \, , \tag{4.39}$$

while all other operator coefficients remain unconstrained. Inserting this into $R_2^{(4,\sigma)}$, we write

$$R_2^{(4,\sigma)} = (\beta_2^{(4)} - \beta_2^{(4)}) U_2^{(4)} + \beta_3^{(4)} U_3^{(4)} + \beta_1^{(4)} \left[U_1^{(4)} + U_2^{(4)} - U_4^{(4)} \right] . \tag{4.40}$$

The operators $U_2^{(4)}$ and $U_3^{(4)}$ have (ivD) factors acting directly on Q_v and thus will contribute only to higher orders, while the only non-vanishing contribution at order $1/m^4$ is

$$U_1^{(4)} + U_2^{(4)} - U_4^{(4)} = \bar{Q}_v \left[D_\mu \,,\, ivD\right] \left[ivD \,,\, iD_\nu\right] (-i\sigma^{\mu\nu}) Q_v \,. \tag{4.41}$$

The matrix element of this operator corresponds to the product $\sigma \cdot (\vec{E} \times \vec{E})$ where $\vec{E}$ is the chromo-electric field.

4.3 Re-summation and relation to full QCD: real quarks

Up to order $1/m^4$, we find in total eight independent parameters at tree level, defined by the matrix elements

$$\langle \bar{Q}_v Q_v \rangle = 2m_H \mu_3 \,, \tag{4.42}$$

$$\langle \bar{Q}_v (iD_\alpha)(iD_\beta)(-i\sigma^{\alpha\beta}) Q_v \rangle = 2m_H d_H \mu_G^2 \,, \tag{4.43}$$

$$\frac{1}{2} \langle \bar{Q}_v \left[(iD_\mu) \,,\, \left[\left(ivD + \frac{1}{2m}(iD)^2 \right) \,,\, (iD^\mu) \right] \right] Q_v \rangle = 2m_H \rho_D^3 \,, \tag{4.44}$$

$$\langle \bar{Q}_v \left[(iD_\mu) \,,\, (iD_\nu) \right] \left[(iD^\mu) \,,\, (iD^\nu) \right] Q_v \rangle = 2m_H r_G^4 \,, \tag{4.45}$$

$$\langle \bar{Q}_v \left[(ivD) \,,\, (iD_\mu) \right] \left[(ivD) \,,\, (iD^\mu) \right] Q_v \rangle = 2m_H r_E^4 \,, \tag{4.46}$$

$$\langle \bar{Q}_v \left[(iD_\mu) \,,\, (iD_\alpha) \right] \left[(iD^\mu) \,,\, (iD_\beta) \right] (-i\sigma^{\alpha\beta}) Q_v \rangle = 2m_H d_H s_B^4 \,, \tag{4.47}$$

$$\langle \bar{Q}_v \left[(ivD) \,,\, (iD_\alpha) \right] \left[(ivD) \,,\, (iD_\beta) \right] (-i\sigma^{\alpha\beta}) Q_v \rangle = 2m_H d_H s_E^4 \,, \tag{4.48}$$

$$\langle \bar{Q}_v \left[iD_\mu \,,\, \left[iD^\mu \,,\, \left[iD_\alpha \,,\, iD_\beta \right] \right] \right] (-i\sigma^{\alpha\beta}) Q_v \rangle = 2m_H d_H s_{qB}^4 \,, \tag{4.49}$$

where $d_H = 1$ for pseudo scalar mesons, $d_H = -1/3$ for vector mesons and $d_H = 0$ for baryons. We note that these operators contain higher orders of $1/m$ in such a way that the result is RPI to all orders. The proper power counting can still be performed, since the contributions appearing at order $1/m^n$ do not contain any pieces of powers $1/m^k$ with $k \leq n$. Thus the result is correct to order $1/m^n$, but is fully RPI.

We have chosen these operators in such a way that they have a clear physical interpretation. We have

$$\mu_G^2 \sim \langle \bar{Q}_v (\vec{\sigma} \cdot \vec{B}) Q_v \rangle \tag{4.50}$$

$$\rho_D^3 \sim \langle \bar{Q}_v (\mathrm{Div} \vec{E}) Q_v \rangle \tag{4.51}$$

$$r_G^4 \sim \langle \bar{Q}_v (\vec{E}^2 - \vec{B}^2) Q_v \rangle \tag{4.52}$$

$$r_E^4 \sim \langle \bar{Q}_v \vec{E}^2 Q_v \rangle \tag{4.53}$$

$$s_B^4 \sim \langle \bar{Q}_v (\vec{B} \times \vec{B}) \cdot \vec{\sigma} Q_v \rangle \tag{4.54}$$

$$s_E^4 \sim \langle \bar{Q}_v (\vec{E} \times \vec{E}) \cdot \vec{\sigma} Q_v \rangle \tag{4.55}$$

$$s_{qB}^4 \sim \langle \bar{Q}_v (\Box \, \vec{\sigma} \cdot \vec{B}) Q_v \rangle \tag{4.56}$$

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We note that all these operators involve at least one gluon field; in the formal limit $g_s \to 0$ all higher dimensional operators vanish and only the leading $\bar{Q}_v Q_v$ remains.

Comparing our results with those in e.g. [12] we notice that the RPI approach yields a smaller number of parameters. This is due to the fact that reparametrization strictly links coefficients of some of the parameters listed in [12] and hence these parameters are not independent. In the RPI approach advertised here these terms are combined in a single parameter.

Finally, to make RPI manifest, we may as well express these operators in terms of full QCD operators. As we have shown explicitly for the case of the Darwin term, any appearance of (ivD) will become completed by a higher-order term in an RPI fashion. In terms of full QCD fields and the corresponding derivatives this means that

$$ivD\, Q_v \to \frac{1}{2m}((iD)^2 - m^2)Q$$

and hence we can write our operators and matrix elements as

$$\langle \bar{Q}Q \rangle = 2m_H \mu_3 \,, \tag{4.57}$$

$$\langle \bar{Q}(iD_\alpha)(iD_\beta)(-i\sigma^{\alpha\beta})Q \rangle = 2m_H d_H \mu_G^2 \,, \tag{4.58}$$

$$\frac{1}{4m}\langle \bar{Q}\left[(iD_\mu)\,,\, \left[(iD)^2\,,\,(iD^\mu)\right]\right] Q \rangle = 2m_H \rho_D^3 \,, \tag{4.59}$$

$$\langle \bar{Q}\left[(iD_\mu)\,,\,(iD_\nu)\right]\left[(iD^\mu)\,,\,(iD^\nu)\right] Q \rangle = 2m_H r_G^4 \,, \tag{4.60}$$

$$\frac{1}{4m^2}\langle \bar{Q}\left[(iD)^2\,,\,(iD_\mu)\right]\left[(iD)^2\,,\,(iD^\mu)\right] Q \rangle = 2m_H r_E^4 \,, \tag{4.61}$$

$$\langle \bar{Q}\left[(iD_\mu)\,,\,(iD_\alpha)\right]\left[(iD^\mu)\,,\,(iD_\beta)\right](-i\sigma^{\alpha\beta})Q \rangle = 2m_H d_H s_B^4 \,, \tag{4.62}$$

$$\frac{1}{4m^2}\langle \bar{Q}\left[(iD)^2\,,\,(iD_\alpha)\right]\left[(iD)^2\,,\,(iD_\beta)\right](-i\sigma^{\alpha\beta})Q \rangle = 2m_H d_H s_E^4 \,, \tag{4.63}$$

$$\langle \bar{Q}\left[iD_\mu\,,\,\left[iD^\mu\,,\,\left[iD_\alpha\,,\,iD_\beta\right]\right]\right](-i\sigma^{\alpha\beta})Q \rangle = 2m_H d_H s_{qB}^4 \,. \tag{4.64}$$

Note that the power counting becomes now more complicated, since the dimension of the operator (i.e. the number of derivatives in the operator) no longer corresponds to the order in the $1/m$ expansion.

5 Beyond tree level

Most of the relations derived in this paper also hold beyond tree level, since RPI must hold also beyond tree level. However, the OPE (4.1) must be generalized to include all possible operators with the relevant dimension at each order. These operators are built from quark fields (light and heavy) and gluon fields as well as from derivatives acting on these fields. We define that all light quark and gluon fields as well as the derivatives acting on these fields are invariant under reparametrization and thus the behavior of any operator under reparametrization is defined. Since the total sum of the OPE is again RPI, the generalization of (4.3) is obvious.

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The operators which appear up to order $1/m^4$ have been written down e.g. in [12] and their RG mixing has been discussed. Up to order $1/m^2$ the full OPE does not have any additional operators beyond the ones we have defined at tree level, which means that our conclusions remain true to all orders in α_s. At order $1/m^3$ new operators appear, which are four quark operator of the form

$$(\bar{Q}_v \Gamma Q_v)(\bar{q}\Gamma q) \text{ and } (\bar{Q}_v \Gamma T^a Q_v)(\bar{q} T^a \Gamma q) \tag{5.1}$$

where T^a are the generators of color SU(3) and Γ is a (v independent) Dirac matrix. These operators are RPI, therefore their coefficients may not be related by RPI to any other coefficient. Likewise, one can directly write these operators in full QCD by simply dropping the subscript v in (5.1).

In some applications it may happen, that the four-quark operators depend on v by a v dependence of Γ (e.g. $\Gamma = \not{v}$). In such a case (4.3) will relate this operator to a term appearing at higher order, a relation that holds at any order in α_s.

Finally, we have to consider the radiative corrections to the color-octet operators. We consider as an illustration r_E^4 in (4.46). Writing the chromo-electric field in its components

$$E^a_\mu T^a = [(ivD)\,,\,(iD_\mu)]\,, \tag{5.2}$$

we see that r_E^4 only contains the (tree level) combination

$$[(ivD)\,,\,(iD_\mu)][(ivD)\,,\,(iD^\mu)] = -E^a_\mu E^{\mu,b}\,\frac{1}{2}\left\{T^a, T^b\right\}, \tag{5.3}$$

where

$$\left\{T^a, T^b\right\} = d^{abc}T^c + \frac{1}{3}\delta^{ab} \tag{5.4}$$

and $d^{abc} = d^{bac}$ the SU(3) symmetric symbol. It has been shown that these two contributions receive different radiative corrections [9, 14]. Therefore, beyond tree level, r_E^4 will generate two independent parameters.

6 Alternative normalization

The normalization of the leading term μ_3 is derived from the conservation of the heavy flavour current in full QCD (see appendix A). However, it is worthwhile to point out that there is also an alternative normalization possible which will relate $\bar{Q}Q$ to the hadron mass.

We start from the energy momentum $\Theta_{\mu\nu}$ tensor of QCD, including the heavy quark. Energy and momentum conservation implies

$$\partial^\mu \Theta_{\mu\nu} = 0\,, \tag{6.1}$$

which gives the normalization

$$\langle H(p)|\Theta_{\mu\nu}|H(p)\rangle = \langle \Theta_{\mu\nu}\rangle = 2p_\mu p_\nu\,. \tag{6.2}$$

Using the expression from QCD for the energy momentum tensor and taking the trace, we get

$$\Theta^{\mu}{}_{\mu} = m\,\bar{Q}Q + \frac{\beta(\alpha_s)}{4\pi} G^{a}_{\mu\nu} G^{\mu\nu,\,a} \tag{6.3}$$

where we have added the well-known contribution of the trace anomaly. Taking the forward matrix element gives

$$\langle \Theta^{\mu}{}_{\mu} \rangle = 2m_H^2 = m\langle \bar{Q}Q \rangle + \frac{\beta(\alpha_s)}{4\pi} \langle G^{a}_{\mu\nu} G^{\mu\nu,\,a} \rangle\,, \tag{6.4}$$

and hence we obtain

$$m\,\mu_3 = m_H - \frac{1}{2m_H} \frac{\beta(\alpha_s)}{4\pi} \langle G^{a}_{\mu\nu} G^{\mu\nu,\,a} \rangle\,. \tag{6.5}$$

We note that inserting (A.19) yields an exact expression for the hadron mass

$$m_H = m - \frac{1}{2m}(\mu_\pi^2 - \mu_G^2) + \frac{1}{2m_H} \frac{\beta(\alpha_s)}{4\pi} \langle G^{a}_{\mu\nu} G^{\mu\nu,\,a} \rangle\,. \tag{6.6}$$

This can be compared to the $1/m$ expansion for the pseudoscalar meson ground state:

$$m_H = m + \bar{\Lambda} + \frac{1}{2m}\left(\lim_{m\to\infty} \mu_\pi^2 - \lim_{m\to\infty} \mu_G^2 \right) + \mathcal{O}(1/m^3)\,, \tag{6.7}$$

from which we conclude that

$$\lim_{m\to\infty} \frac{1}{2m_H} \frac{\beta(\alpha_s)}{4\pi} \langle G^{a}_{\mu\nu} G^{\mu\nu,\,a} \rangle = \bar{\Lambda} = \lim_{m\to\infty} (m_H - m)\,, \tag{6.8}$$

leading to

$$\lim_{m\to\infty} \mu_3 = 1 \tag{6.9}$$

as expected.

However, we may use the RPI formulation to write for a weak decay of a heavy hadron H the relation

$$\Gamma \propto G_F^2 m^5 \mu_3 = G_F^2 m^4 \left(m_H - \frac{1}{2m_H} \frac{\beta(\alpha_s)}{4\pi} \langle G^{a}_{\mu\nu} G^{\mu\nu,\,a} \rangle \right) \tag{6.10}$$

corresponding to the leading order result in the reparametrization-invariant formulation.

7 Conclusion

We have made use of the fact that RPI relates different orders in the HQE to perform partial re-summations. Computing up to a specific order in $1/m$, we combine terms of higher order in such a way that the result becomes RPI. This can be made manifest by writing the resulting matrix elements for the non-perturbative input as matrix elements of operators and states defined in full QCD, which do not have any reference to the velocity vector needed to set up the HQE.

Clearly the RPI improved results calculated to a certain order contain arbitrarily high orders in $1/m$; however, they are still correct only up to the order one has actually calculated, since at each order new terms appear, which are not related by reparametrization

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to terms appearing at lower order. In turn, our approach re-sums all the terms which may be related back to lower-order terms, and thus we expect an improvement.

In this paper, we have confined our discussion to total rates. In this case, a side effect of RPI is that the number of independent parameters in the HQE is reduced compared to earlier analyses. While the relations implied by RPI have been found some time ago, this has never been used to explicitly reduce the number of independent parameters at $\mathcal{O}(1/m^4)$.

The "genuine" terms at higher orders, which do not relate back to lower orders by reparametrization, are all due to the presence of gluons or additional light quarks. The matching calculation to compute the OPE coefficients for the total rate is conveniently done using free quark and gluon states. In our approach the leading operator $\bar{Q}Q$ is the only contribution which appears in the matching using only the two quarks and no gluons; all higher-order terms require at least one gluon in the matching calculation. Therefore, all the matrix elements which have a zero-gluon matrix element are contained in the matrix element of $\bar{Q}Q$.

The relations obtained from RPI have been formulated for the general case, i.e. also for differential rates. However, for differential rates the RPI relation imply differential equations for the coefficients of the HQE; a detailed analysis for the differential case is beyond the scope of this work and will be exploited in future work.

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A Matrix elements

A.1 Normalization for the scalar case

Here we collect the expression for the relevant forward matrix elements for the scalar case. The leading matrix element is

$$\langle \phi_v^\dagger \phi_v \rangle = \langle H(p)|\phi_v^\dagger \phi_v|H(p)\rangle = 2m\langle \phi^\dagger \phi \rangle \tag{A.1}$$

where $|H(p)\rangle$ is the hadronic state of full (scalar) QCD. We note that we may re-insert the full QCD operators and define

$$\langle \phi^\dagger \phi \rangle = 4m_H^2 \mu_3 \tag{A.2}$$

with a hadronic parameter μ_3.

By a similar argument as for real quarks we can show that $\mu_3 = 1$ up to terms of order $1/m^2$: we note that the equation of motion (2.10) for ϕ has a conserved current of the form

$$J_\mu = \phi^\dagger (i \overleftrightarrow{D}_\mu)\phi = -i\left((D_\mu\phi)^\dagger\phi - \phi^\dagger(D_\mu\phi)\right) . \tag{A.3}$$

Inserting the rescaled field we get (replacing ϕ by ϕ_v)

$$J_\mu = v_\mu \phi_v^\dagger \phi_v + \frac{1}{2m} \phi_v^\dagger i \overleftrightarrow{D}_\mu \phi_v$$

Taking the forward matrix element of this operator yields

$$\langle J_\mu \rangle = 2p_\mu = 2m_H v_\mu = v_\mu \langle \phi_v^\dagger \phi_v \rangle + \frac{1}{m} \langle \phi_v^\dagger i D_\mu \phi_v \rangle \tag{A.4}$$

where we have made a choice for the velocity v to be $v = p_H/m_H$. Contracting with v^μ we get

$$2m_H = \langle \phi_v^\dagger \phi_v \rangle + \frac{1}{m} \langle \phi_v^\dagger (ivD) \phi_v \rangle = \langle \phi_v^\dagger \phi_v \rangle - \frac{1}{2m^2} \langle \phi_v^\dagger (iD)^2 \phi_v \rangle \tag{A.5}$$

Defining the kinetic- energy parameter μ_π as

$$\langle \phi_v^\dagger (iD)^2 \phi_v \rangle = -2m_H \mu_\pi^2 \tag{A.6}$$

we finally get

$$\mu_3 = 1 - \frac{\mu_\pi^2}{2m^2} \tag{A.7}$$

As discussed in the text, there is no matrix element at order $1/m^2$ in the total rates, beyond μ_π^2 which only appears in the expression for μ_3.

A.2 Real QCD

From the equation of motion (2.4), we find for a Dirac matrix Γ

$$\bar{Q}_v (iD_{\mu_1}) \dots (iD_{\mu_n}) \Gamma Q_v = \bar{Q}_v (iD_{\mu_1}) \dots (iD_{\mu_n}) \Gamma \not{v} Q_v + \frac{1}{m} \bar{Q}_v (iD_{\mu_1}) \dots (iD_{\mu_n}) \Gamma (i\not{D}) Q_v \tag{A.8}$$

$$\bar{Q}_v (iD_{\mu_1}) \dots (iD_{\mu_n}) \Gamma Q_v = \bar{Q}_v (iD_{\mu_1}) \dots (iD_{\mu_n}) \not{v} \Gamma Q_v + \frac{1}{m} \bar{Q}_v (i\not{D}) (iD_{\mu_1}) \dots (iD_{\mu_n}) \Gamma Q_v \\ + \text{total derivative} \tag{A.9}$$

where the total derivative will vanish when a forward matrix element is taken.

We obtain

$$\langle \bar{Q}_v (iD_{\mu_1}) \dots (iD_{\mu_n}) \Gamma Q_v \rangle = \frac{1}{2} \langle \bar{Q}_v (iD_{\mu_1}) \dots (iD_{\mu_n}) \{ \Gamma \, , \not{v} \} Q_v \rangle \\ + \frac{1}{2m} \langle \bar{Q}_v \{ (i\not{D}) \, , (iD_{\mu_1}) \dots (iD_{\mu_n}) \Gamma \} Q_v \, , \rangle \tag{A.10}$$

which yields for $\Gamma = 1$

$$\langle \bar{Q}_v (iD_{\mu_1}) \dots (iD_{\mu_n}) Q_v \rangle = \langle \bar{Q}_v (iD_{\mu_1}) \dots (iD_{\mu_n}) \not{v} Q_v \rangle \\ + \frac{1}{2m} \langle \bar{Q}_v \{ (i\not{D}) \, , (iD_{\mu_1}) \dots (iD_{\mu_n}) \} Q_v \rangle \, , \tag{A.11}$$

and for $\Gamma = \gamma_\alpha$

$$\langle \bar{Q}_v (iD_{\mu_1}) \dots (iD_{\mu_n}) \gamma_\alpha Q_v \rangle = v_\alpha \langle \bar{Q}_v (iD_{\mu_1}) \dots (iD_{\mu_n}) Q_v \rangle \\ + \frac{1}{2m} \langle \bar{Q}_v \{ (i\not{D}) \, , \gamma_\alpha (iD_{\mu_1}) \dots (iD_{\mu_n}) \} Q_v \rangle \, . \tag{A.12}$$

These relations show that all the contributions with a single γ_α can be dropped.

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The leading order matrix element is defined as in the scalar case

$$\langle \bar{Q}_v Q_v \rangle = 2m_H \mu_3 \,. \tag{A.13}$$

Using the relation (A.12) for $n = 0$ we get

$$\langle \bar{Q}_v \gamma_\alpha Q_v \rangle = v_\alpha \langle \bar{Q}_v Q_v \rangle + \frac{1}{m} \langle \bar{Q}_v (iD_\alpha) Q_v \rangle \,. \tag{A.14}$$

Contracting with v^α we obtain

$$\langle \bar{Q}_v \not{v} Q_v \rangle = \langle \bar{Q}_v Q_v \rangle + \frac{1}{m} \langle \bar{Q}_v (ivD) Q_v \rangle = \langle \bar{Q} \not{v} Q \rangle = 2m_H \,, \tag{A.15}$$

where we used the conservation of the b quark current in QCD. Furthermore, we may use (2.5) to obtain

$$\langle \bar{Q}_v Q_v \rangle = 2m_H + \frac{1}{2m^2} \langle \bar{Q}_v (i\not{D})(i\not{D}) Q_v \rangle \,. \tag{A.16}$$

Finally, using the definitions

$$\langle \bar{Q}_v (iD)^2 Q_v \rangle = -2m_H \mu_\pi^2 \,, \tag{A.17}$$

$$\langle \bar{Q}_v (\sigma \cdot G) Q_v \rangle = 2m_H d_H \mu_G^2 \,, \tag{A.18}$$

where $d_H = 1$ for pseudo scalar mesons, $d_H = -1/3$ for vector mesons and $d_H = 0$ for baryons. We find

$$\mu_3 = 1 - \frac{1}{2m^2} (\mu_\pi^2 - d_H \mu_G^2) \,. \tag{A.19}$$

B Example: tree level $B \to X_s \gamma$

As an example, we compute the radiative $b \to s\gamma$ decay at tree level but with higher-order $1/m$ corrections. For illustration, we consider only the contribution from the operator O_7

$$\frac{\lambda}{2} \bar{s} \sigma_{\mu\nu} (1 + \gamma_5) b F^{\mu\nu} \quad \text{with} \quad \lambda = \frac{em}{16\pi^2} |C_7(m) V_{ts} V_{tb}^*| \,. \tag{B.1}$$

We find, considering massless s quarks,

$$T = -2\lambda^2 \, \bar{b}_v \left[\sigma_{\mu\alpha} q^\alpha \left(\frac{1}{\not{S} + i\not{D}} \right) \sigma_{\nu\beta} q^\beta g^{\mu\nu} \frac{1}{q^2} \right] b_v \,, \tag{B.2}$$

where $S = p - q$, and q is the photon momentum. From the expansion of the s quark propagator, we obtain

$$\begin{aligned} \frac{1}{\not{S} + i\not{D}} &= \frac{1}{\not{S}} - \frac{1}{\not{S}} i\not{D} \frac{1}{\not{S}} + \frac{1}{\not{S}} i\not{D} \frac{1}{\not{S}} i\not{D} \frac{1}{\not{S}} + \cdots \\ &= \frac{1}{S^2} \not{S} - \left(\frac{1}{S^2} \right)^2 \not{S} i\not{D} \not{S} + \left(\frac{1}{S^2} \right)^3 \not{S} i\not{D} \not{S} i\not{D} \not{S} + \cdots \,. \end{aligned} \tag{B.3}$$

Performing the loop integration and taking the imaginary part yields for the total rate;

$$\Gamma_{bsg} = \frac{\lambda^2 m^3}{4\pi} \left[\mu_3 - \frac{2}{m^2} \mu_G^2 - \frac{10\rho_D^3}{3m^3} - \frac{1}{3m^4} \left(4r_G^4 + 4r_E^4 + \frac{1}{4} s_{qB}^4 - 2s_E^4 \right) \right] , \tag{B.4}$$

where we indeed see the expected reduction to independent matrix elements. Note that the contribution of s_B^4 is absent in this relation which is accidental.

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