Spin(7)-manifolds as generalized connected sums and 3d $\mathcal{N} = 1$ theories

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ABSTRACT: M-theory on compact eight-manifolds with Spin(7)-holonomy is a framework for geometric engineering of 3d $\mathcal{N} = 1$ gauge theories coupled to gravity. We propose a new construction of such Spin(7)-manifolds, based on a generalized connected sum, where the building blocks are a Calabi-Yau four-fold and a G_2 -holonomy manifold times a circle, respectively, which both asymptote to a Calabi-Yau three-fold times a cylinder. The generalized connected sum construction is first exemplified for Joyce orbifolds, and is then used to construct examples of new compact manifolds with Spin(7)-holonomy. In instances when there is a K3-fibration of the Spin(7)-manifold, we test the spectra using duality to heterotic on a T^3 -fibered G_2 -holonomy manifold, which are shown to be precisely the recently discovered twisted-connected sum constructions.

KEYWORDS: Differential and Algebraic Geometry, String Duality, M-Theory

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1 Introduction

Geometric engineering of supersymmetric gauge theories is fairly well-understood in theories with at least four real supercharges. Most prominently in recent years, F-theory has established a precise dictionary between geometric data (and fluxes) and 6d and 4d theories with $\mathcal{N} = 1$ supersymmetry, and dually, M-theory of course has a well-established dictionary between 3d and 5d theories and Calabi-Yau geometries. For 4d $\mathcal{N} = 1$ the realization in terms of M-theory compactifications on G_2 -holonomy manifolds is already much less well understood, in particular due to the scarcity of compact geometries of this type. Even less is known about M-theory on Spin(7)-holonomy compactifications, which yield 3d $\mathcal{N} = 1$ theories.

The goal of this paper is to provide a new construction of compact Spin(7)-holonomy manifolds, which have an interesting field theoretic counterpart in 3d. This is motivated on the one hand to build large sets of examples of 3d $\mathcal{N} = 1$ vacua in M-theory/string theory, and we relate these constructions in certain instances to heterotic string theory on G_2 -holonomy manifolds. A second motivation is to study Spin(7) compactification in the context of M/F-duality and uplifting these to supersymmetry breaking vacua in F-theory [1] based on the observations that 3d $\mathcal{N} = 1$ supersymmetry can be related to the circle-reduction of a not necessarily supersymmetric theory in 4d [2, 3].

The construction that we propose, and refer to as *generalized connected sum* (GCS) construction of Spin(7)-manifolds is motivated by a recent development for compact G_2 -holonomy manifolds in Mathematics, where a large class of compact G_2 -holonomy manifolds were constructed using a *twisted connected sum* (TCS) construction [4–6]. This construction relies on the decomposition of the G_2 -holonomy manifold in terms of two asymptotically cylindrical (acyl) Calabi-Yau three-folds — for a sketch see figure 1. Following these mathematical developments, there has been a resurgence in interest in the string and M-theory compactifications, which for TCS-manifolds have been studied in [7–14]. For a review of M-theory on G_2 and Spin(7)-holonomy manifolds related to earlier results in the '90s and early '00s see [15].

One of the nice features of TCS G_2 -manifolds is that the connected sum gives rise to a field theoretic decomposition in terms higher supersymmetric subsectors [11, 13]: the asymptotic regions that are acyl Calabi-Yau three-folds (times a circle), give rise to 4d $\mathcal{N} = 2$ subsectors, whereas the asymptotic neck region where the two building blocks are glued together is by itself K3 times a cylinder times S^1 , and corresponds to a 4d $\mathcal{N} = 4$ subsector, which is present in the decoupling limit (infinite neck limit). The theory breaks to $\mathcal{N} = 1$, when the asymptotic region is of finite size and the states of the $\mathcal{N} = 4$ vector multiplet become massive, leaving only an $\mathcal{N} = 1$ massless vector. This observation may be used to study non-perturbative corrections, e.g. M2-brane instantons [16], which has been initiated for TCS-manifolds in [7, 8, 14].

For Spin(7)-manifolds the type of constructions have thus far has been rather limited. There are (to our knowledge) two constructions of compact manifolds with Spin(7)-holonomy due to Joyce: either in terms of a Joyce orbifold T^8/Γ [17] or a quotient by an anti-holomorphic involution of a Calabi-Yau four-fold (CY₄) [18]. There are variations of

Figure 1. The twisted connected sum (TCS) construction for G_2 -manifolds: the two acyl Calabi-Yau three-fold building blocks are $X_{\pm} = Z_{\pm} \setminus S_{\pm}^0$, where $S_{\pm}$ denotes the K3-fibers of each building block. Unlike the Spin(7) case, here the gluing is done with a twist (or a Donaldson matching): a hyper-Kähler rotation (HKR) on the K3-fibers, and an exchange of the circles as shown above. The asymptotic region where the gluing is performed is $K3 \times S^1 \times S^1 \times I$. Topologically the resulting G_2 -manifold is K3-fibered over an S^3 .

these constructions using CY_4 with Spin(7)-necks in [19]. In both constructions, the quotient has singularities, which get resolved and shown to give rise to an eight-manifold with Spin(7)-holonomy. Although it is relatively straight forward to construct string and M-theory compactifications on the Joyce orbifolds, understanding the second class of constructions is far less straight-forward. Recent attempts were made, in the context of M/F-theory duality for Spin(7)-manifolds [1] in [20, 21]. One of the challenges in this is determining the 3d $\mathcal{N} = 1$ theory, which is crucially related to the singular loci of the Calabi-Yau quotient, and not only the sector that is 3d $\mathcal{N} = 2$.

In the present paper we provide a different approach to constructing Spin(7)-manifolds, which is closer in spirit to the TCS construction. We will motivate a generalized connected sum (GCS)-construction where the building blocks are an open asymptotically cylindrical (acyl) Calabi-Yau four-fold (CY_4) and a open asymptotically cylindrical (acyl) G_2 -holonomy manifold (times S^1), which asymptote to a Calabi-Yau three-fold (times a cylinder/line). This is shown in figure 2. Explicitly, we construct the acyl CY_4 as

$$CY_3 \hookrightarrow CY_4 \rightarrow \mathbb{C}, \quad (1.1)$$

and the $G_2 = (CY_3 \times \mathbb{R})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as a reflection on the circle and anti-holomorphic involution on the CY_3 . In case the anti-holomorphic involution acts freely, these are so-called ‘barely G_2 -manifolds’ [16], which have holonomy group $SU(3) \rtimes \mathbb{Z}_2$.

Field theoretically we will show that each of the building blocks will give rise to a 3d $\mathcal{N} = 2$ subsector, and the asymptotic region $CY_3 \times$ cylinder results in the limit of an infinite neck region in a 3d $\mathcal{N} = 4$ sector. At finite distance, some of the modes of the vector multiplet of 3d $\mathcal{N} = 4$ become massive and the theory is broken to 3d $\mathcal{N} = 1$.

Figure 2. The generalized connected sum (GCS) construction for Spin(7)-manifolds proposed in this paper: the left hand building block is an the acyl Calabi-Yau four-fold building block. We require this to be asymptotically approach a Calabi-Yau three-fold times a cylinder. One way to realize this is in terms of a CY_3 fibration over an open $\mathbb{P}^1$. The right hand building block is a circle times a G_2 -manifold, which asymptotes to a $CY_3 \times I$ times a circle. The simplest way to realize this is in terms of $G_2 = (CY_3 \times \mathbb{R})/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts as an anti-holomorphic involution. The asymptotic region is $CY_3 \times \text{cylinder}$.

We motivate this construction in section 3 by considering a simple Spin(7) Joyce orbifold, T^8/Γ , which is known to have a related G_2 Joyce orbifold, where the precise connection is through M-theory on K3 to Heterotic on T^3 duality applied to this toroidal setting [22]. It is known that such Joyce orbifolds have a TCS-decomposition [12], which we apply to the G_2 Joyce orbifold, and subsequently uplift the decomposition to the Spin(7) Joyce orbifold. The TCS-building blocks then map precisely to an open CY_4 and $G_2 \times S^1$, respectively. This motivates our general construction of GCS Spin(7)-manifolds, which will be given in section 4.

One may ask whether these constructions in fact globally fit together to (resolutions of) CY_4 quotients, like in Joyce’s constructions in [18]. Whenever the acyl G_2 manifold is realized as (the resolution of) a quotient of $X_3 \times \mathbb{R}$, this is indeed the case and we can write our GCS Spin(7)-manifolds globally as resolutions of quotients of CY_4 by an anti-holomorphic involution. Here, one starts with a Calabi-Yau four-fold fibered by a Calabi-Yau three-fold CY_3 as in (1.1), and acts on it with an anti-holomorphic involution on the CY_3 -fiber combined with an action on the base $\mathbb{P}^1$ given by $z_i \leftrightarrow -\bar{z}_j$, $i \neq j$, where z_i are the homogeneous coordinates. The fixed locus of this on the base is a circle and the quotient space is half- CY_4 , which is a CY_3 -fold fibered over a disc, and in the vicinity of the boundary circle we obtain $(CY_3 \times \mathbb{R})/\mathbb{Z}_2$ which is a (barely) G_2 -manifold, times an S^1 . Pulling the base half sphere apart, gives the decomposition into the GCS building blocks. This is sketched in figure 3. The key difference to the examples and constructions in [18] is however that the singularities are either absent or occur over a locus of real dimension at least 1. In Joyce’s construction via CY_4/σ the singularities are only point-like and get resolved using an ALE-space. It seems likely that the analog to this procedure in our construction is gluing in the open, acyl G_2 -manifold.

Figure 3. GCS-construction of Spin(7)-manifolds, from the point of view of a Calabi-Yau four-fold quotient by an anti-holomorphic involution. The CY_4 is as in (1.1) and the picture shows the geometry after the quotient. Away from the boundary circle, which is the fixed locus of the involution, we have a CY_3 fibered over a disc, which is locally a CY_4 . Above the boundary circle (dark green) the quotient acts non-trivially on the $CY_3 \times$ interval, shown in light green, which results in a G_2 . Pulling the base apart, results in the GCS-decomposition of the resulting Spin(7)-manifold.

The paper is structured as follows: we begin with some background material on Spin(7)-holonomy and M-theory compactifications to 3d $\mathcal{N} = 1$ in section 2. In section 3 we motivate our construction by first considering a Joyce orbifold example. Section 4 contains our proposal for the general construction of GCS Spin(7)-manifolds. We also comment on how this construction relates to and differs from other known setups and provide an example of a new Spin(7)-manifold. For some examples of GCS constructions there is a K3-fibration of the Spin(7)-manifold, and we utilize this to apply M-theory/heterotic duality and construct the dual heterotic on G_2 compactifications in section 5, and match the spectra of the theories. We conclude with a discussion and outlook in section 6. For reference we include a brief summary of the TCS-construction of G_2 -manifolds in appendix A.

2 Spin(7)-holonomy, M-theory and 3d $\mathcal{N} = 1$ theories

As a preparation for this work and to introduce some notation, this section reviews some basic aspects of manifolds with holonomy group Spin(7), see [23] for a detailed discussion.

2.1 Spin(7)-holonomy

A compact orientable 8-dimensional manifold Z which has a Ricci-flat metric g with holonomy group contained in Spin(7) supports a closed, self-dual four-form $\Psi = *_g \Psi$, which can be expressed in local coordinates (in which g is the euclidean metric) as

$$\begin{aligned} \Psi = & dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368} - dx_{1458} - dx_{1467} \\ & - dx_{2358} - dx_{2367} - dx_{2457} + dx_{2468} + dx_{3456} + dx_{3478} + dx_{5678} . \end{aligned} \quad (2.1)$$

Here, we use dx_{ijkl} as a shorthand for $dx_i \wedge dx_j \wedge dx_k \wedge dx_l$. This four-form is in the stabilizer of the action of the holonomy group Spin(7). Such a form defines a Spin(7) structure, which is called torsion free if Ψ is closed and self-dual.

Having this structure in place does not necessarily mean that the holonomy group is exactly $\text{Spin}(7)$ and, as long as Z is simply connected, we may discriminate between different cases by computing the $\hat{A}$ genus, which tells us the number of covariantly constant spinors:

$\hat{A}$	$\text{hol}(g)$
1	$\text{Spin}(7)$
2	$\text{SU}(4)$
3	$\text{Sp}(2)$
4	$\text{SU}(2) \times \text{SU}(2)$

(2.2)

All of these manifolds have $b^1(Z) = 0$ and the remaining independent Betti numbers are related to the $\hat{A}$ genus by

$$24\hat{A} = -1 - b^2 + b^3 + b_+^4 - 2b_-^4, \quad (2.3)$$

where $b_\pm^4$ are the dimensions of the (anti)self-dual subspaces of $H^4(Z, \mathbb{Q})$. As we are interested in compactifications of M-theory which preserve $\mathcal{N} = 1$ supersymmetry in three dimensions, we will only be interested in the case $\hat{A} = 1$.

For eight-manifolds Z with holonomy contained in $\text{Spin}(7)$ and $\hat{A}(Z) = 1$, a necessary and sufficient condition for the holonomy group to be all of $\text{Spin}(7)$ is that Z is simply connected, $\pi_1(Z) = 0$ [17, 23]. This still allows for cases with non-trivial subgroups of $\text{Spin}(7)$, and we will see examples of such spaces later on. We will refer to manifolds Z with $\hat{A}(Z) = 1$ and a metric g with $\text{hol}(g) \subseteq \text{Spin}(7)$ as barely $\text{Spin}(7)$ -manifolds.

The dimension of the moduli space of Ricci-flat metrics on a $\text{Spin}(7)$ -manifold is given by $b_-^4 + 1$. Together with $\hat{A} = 1$, this number is already determined by the Euler characteristic and the two Betti numbers b_2 and b_3 by using (2.3)

$$b_-^4 = \frac{\chi}{3} - 9 - b_2 + b_3. \quad (2.4)$$

Calibrated submanifolds of $\text{Spin}(7)$ -manifolds must be of real dimension four and are called Cayley submanifolds. The dimension of the moduli space of such a Cayley submanifold N is [23, 24]

$$m_N = \tau(N) - \frac{1}{2}\chi(N) - \frac{1}{2}N \cdot N, \quad (2.5)$$

where $\tau(N)$ is the Hirzebruch signature and $\chi(N)$ is the Euler characteristic. Note that this expression evaluates to

$$m_N = 4 - \frac{1}{2}N \cdot N \quad (2.6)$$

for a Cayley submanifold which is a K3 surface, where $\tau(N) = 16$ and $\chi(N) = 24$. It hence seems sensible to assume there exist $\text{Spin}(7)$ -manifolds fibered by K3 surfaces over a four-dimensional base. Using duality to heterotic strings, it is precisely the existence of such fibrations which we will conjecture and exploit. In contrast, note that for a Cayley submanifold with the topology of a four-dimensional torus T^4 , the same computation gives

$$m_N = -\frac{1}{2}N \cdot N \quad (2.7)$$

so that $\text{Spin}(7)$ -manifolds cannot possibly be fibered by calibrated four-tori.

2.2 M-theory on Spin(7)-manifolds

M-theory on a Spin(7)-manifold gives rise to 3d $\mathcal{N} = 1$ supersymmetric theory and the spectrum is encoded in the topological data of the Spin(7)-manifold as follows:

#	3d $\mathcal{N} = 1$ Multiplet	11d Origin
$b_3(Z)$	Scalar	$C_3 \supset \sum_{i=1}^{b_3} \rho_i^{(3)} \varphi^i$
$b_2(Z)$	Vector	$C_3 \supset \sum_{i=1}^{b_2} \omega_\alpha^{(2)} v_\mu^\alpha$
$b_4^-(Z)$	Scalar	$\delta g_{\mu\nu} \supset \sum_{I=1}^{b_4^-} (\xi_I^{(4)})_{\mu\rho\sigma\tau} \Phi_\nu^{\rho\sigma\tau} \delta\varphi^I$
1	Scalar	Volume modulus

(2.8)

where $\omega^{(2)}, \rho^{(3)}$ are a basis of harmonic two and three-forms, and $\xi_I^{(4)}$ are a basis of harmonic anti-self-dual four-forms. For 3d $\mathcal{N} = 1$ the scalar multiplet has only a real scalar as its bosonic component and the vector multiplet is just a 3d vector. As we can dualize the 3d vectors to real scalars, compactifications of M-theory on Spin(7)-manifolds hence give rise to

$$n_s = b_-^4 + 1 + b^2 + b^3 = \frac{\chi(Z)}{3} - 8 + 2b^3 \quad (2.9)$$

massless real scalars at the classical level. Anomaly cancellation furthermore requires the introduction of

$$N_{M2} = \frac{1}{24} \chi(Z) - \frac{1}{2} \int_Z G_4 \wedge G_4 \quad (2.10)$$

space-time filling M2-branes [25, 26]. As such M2-branes can freely move on Z , they each contribute a further 8 real degrees of freedom.

In the absence of G_4 -flux, the effective action for a smooth Spin(7)-manifold is a 3d $\mathcal{N} = 1$ field theory with b_2 abelian vectors, and scalars with the following kinetic terms

$$\mathcal{G}_{IJ} \sim \int_Z \xi_I^{(4)} \wedge \xi_J^{(4)}, \quad \mathcal{M}_{\alpha\beta} \sim \int_Z \omega_\alpha^{(2)} \wedge \star \omega_\beta^{(2)}. \quad (2.11)$$

The theory for a smooth Spin(7)-manifold is an abelian gauge theory. With singularities this can enhance to non-abelian gauge symmetries. With G_4 -flux, additional Chern-Simons terms and scalar potential are generated — for an in depth discussion of the effective theory see [20, 27–33]. In this paper we will not consider fluxes but focus on the geometric constructions. Of course it is interesting to consider this in the future and study the effects of these on supersymmetry breaking and potential obstructions to dualities, both M/F-duality [34], where in particular 4d Poincaré invariance could be broken, as well as obstructions to M-theory/weakly-coupled heterotic duality [35, 36].

3 A Spin(7) Joyce orbifold as a generalized connected sum

3.1 Setup and motivation

Our goal is to construct new classes of Spin(7)-manifolds — which we have motivated from various points of view in the introduction. The construction which we will end up with is inspired by combining two ideas:

1. The recent construction of G_2 -holonomy manifolds as twisted connected sums (TCS), with each building block an asymptotically cylindrical Calabi-Yau three-fold. We review this construction in appendix A.
2. M-theory on K3/heterotic on T^3 duality.

Combining these ideas will lead us to consider a generalized connected sum (GCS) construction, where — as we will show — the building blocks are acyl CY four-folds and G_2 -manifolds, respectively. To motivate this we start with a well-known construction by Joyce of both G_2 and Spin(7)-manifolds and a well-known duality, between M-theory and heterotic strings, which will be discussed in more detail in section 5.2, between

$$\text{M-theory on K3} \longleftrightarrow \text{Heterotic on } T^3 \quad (3.1)$$

This duality in 7d is based on the agreement of the moduli spaces of the two compactifications

$$\mathcal{M}_{\text{het}/T^3} = \mathcal{M}_{M/K3} = [\text{SO}(3, \mathbb{Z}) \times \text{SO}(10, \mathbb{Z})] \backslash \text{SO}(3, 19) / [\text{SO}(3) \times \text{SO}(10)] \times \mathbb{R}^+, \quad (3.2)$$

which is both the Narain moduli space of heterotic string theory on T^3 and the moduli space of Einstein metrics on K3 [37] (for a recent exposition in the context of fiber-wise application see [12]). The string coupling on the heterotic side is matched with the volume modulus of the K3. We can fiber this over a four-manifold M_4 in such a way that there is a duality of 3d $\mathcal{N} = 1$ theories

$$\text{M-theory on Spin(7)-manifold } Z_8 \longleftrightarrow \text{Heterotic on } G_2\text{-manifold } J_7 \quad (3.3)$$

where Z_8 is K3-fibered over M_4 and J_7 has T^3 -fibers. This duality has been tested in the case when both manifolds are Joyce orbifolds in [22]. More generally, fiberwise application of this duality will lead us to a correspondence between two realizations of a 3d $\mathcal{N} = 1$ theory, in terms of a Spin(7)-compactification of M-theory, and a T^3 -fibered G_2 -holonomy manifold.¹ It is this setup which will motivate our construction of GCS Spin(7)-manifolds: it is the analog of the TCS-decomposition for G_2 -manifolds on the heterotic side, mapped to M-theory using the duality. In this way we obtain a dual pair of connected sums:

$$\text{M-theory on GCS Spin(7)-manifold} \longleftrightarrow \text{Heterotic on TCS } G_2\text{-manifold} \quad (3.4)$$

As a warmup we now show how this works for a simple G_2 Joyce orbifold, which has a TCS-decomposition and determine what this decomposition corresponds to on the M-theory side. This will give a first hint as to what the general connected sum construction will be for Spin(7)-manifolds.

¹The existence of T^3 -fibrations of G_2 manifolds has also been conjectured in the context of mirror symmetry for G_2 manifolds in [13].

3.2 Joyce orbifolds: Spin(7)- and G_2 -holonomy

We will start the construction of a Spin(7)-manifold Z_8 by considering a Joyce orbifold T^8/Γ , where $\Gamma = \mathbb{Z}_2^4$, where each generator of the order two subgroups acts as follows [23]:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
α	−	−	−	−	+	+	+	+
β	+	+	+	+	−	−	−	−
γ	$\frac{1}{2}-$	$\frac{1}{2}-$	+	+	$\frac{1}{2}-$	$\frac{1}{2}-$	+	+
δ	−	+	$\frac{1}{2}-$	+	$\frac{1}{2}-$	+	$\frac{1}{2}-$	+

(3.5)

The entries $\frac{1}{2}-$ denote $x \rightarrow -x + \frac{1}{2}$. The singular sets are locally given by

$$\begin{aligned}
 S_\alpha &: 4 \times \mathbb{C}^2/\mathbb{Z}_2 \times T^4/\{\pm 1\} \cong \mathbb{C}^2/\mathbb{Z}_2 \times K3 \\
 S_\beta &: 4 \times \mathbb{C}^2/\mathbb{Z}_2 \times T^4/\{\pm 1\} \cong \mathbb{C}^2/\mathbb{Z}_2 \times K3 \\
 S_\gamma &: 2 \times \mathbb{C}^2/\mathbb{Z}_2 \times T^4 \\
 S_\delta &: 2 \times \mathbb{C}^2/\mathbb{Z}_2 \times T^4,
 \end{aligned}
 \tag{3.6}$$

as well as 64 fixed points of $\alpha\beta$, the singularities of which are modelled on $\mathbb{C}^4/\mathbb{Z}_2$. It was shown by Joyce that the resolution of the orbifold in this way yields a Spin(7)-holonomy manifold.

From this, the Betti numbers are computed as follows. First note that the only even classes in $H^\bullet(T^8)$ under the $\mathbb{Z}_2^4$ (besides $H^0(T^8)$ and $H^8(T^8)$) are given by 14 classes in $H^4(T^8)$. Resolving the 8 singularities of the form $\mathbb{C}^2/\mathbb{Z}_2 \times T^4/\{\pm 1\}$ gives

	$\mathbb{C}^2/\mathbb{Z}_2 \times T^4/\{\pm 1\}$	$\widetilde{\mathbb{C}^2/\mathbb{Z}_2 \times T^4/\{\pm 1\}}$
b^2	6	7
b^3	0	0
b^4	1	7

(3.7)

whereas the 4 singularities of the form $\mathbb{C}^2/\mathbb{Z}_2 \times T^4$ yield

	$\mathbb{C}^2/\mathbb{Z}_2 \times T^4$	$\widetilde{\mathbb{C}^2/\mathbb{Z}_2 \times T^4}$
b^2	6	7
b^3	4	8
b^4	1	7

(3.8)

Hence each of these 12 singularities contributes $b^2 = 1$ and $b^4 = 6$ and only the second 4 contribute a non-zero $b^3 = 4$. Finally, the resolution of the 64 singularities of the form $\mathbb{C}^4/\mathbb{Z}_2^2$ contribute $b^4 = 1$ and nothing else. Altogether we can compute

$$\begin{aligned}
 b^2 &= 0 + 8 \cdot 1 + 4 \cdot 1 + 0 = 12 \\
 b^3 &= 0 + 4 \cdot 4 + 0 = 16 \\
 b^4 &= 14 + 8 \cdot 6 + 4 \cdot 6 + 64 = 150.
 \end{aligned}
 \tag{3.9}$$

The key observation to make is that this $\text{Spin}(7)$ -manifold gives rise naturally to a G_2 -manifold as follows: consider the T^7 given by $x_1, x_2, x_3, x_5, x_6, x_7, x_8$, and act with $\mathbb{Z}_2^\beta \oplus \mathbb{Z}_2^\gamma \oplus \mathbb{Z}_2^\delta$. This is a Joyce construction of G_2 -manifolds by orbifolds as already observed in [22].

3.3 TCS-decomposition of the Joyce G_2 -manifold

Our goal is to first identify the TCS description of this Joyce G_2 -orbifold and to then lift this to a connected sum description, GCS, for the $\text{Spin}(7)$ -manifold. First note that $T^7/\langle\beta, \gamma, \delta\rangle$ is fibered over an interval

$$x_6 : (\beta, \gamma, \delta) = \left(-, -\frac{1}{2}, +\right) \quad (3.10)$$

which identifies this as $x_6 \sim -x_6 \sim -x_6 + \frac{1}{2}$, and thus $x_6 \in I = [0, 1/4]$. The generator δ is the stabilizer of x_6 and so

$$T^7/\langle\delta\rangle = T_{(2,6,8)}^3 \times T_{(1,3,5,7)}^4/\mathbb{Z}_2^\delta = T^3 \times \text{K3}. \quad (3.11)$$

The notation $T_{(2,6,8)}^3$ indicates the three-torus along the coordinates x_2, x_6, x_8 . We now need to identify the action of β and γ on this space. Again, there will be two components, which we denote by X_0 and $X_{1/4}$, located at $x_6 = 0, 1/4$, which are given by

$$\begin{aligned} X_0 : \quad & S_{(8)}^1 \times \left(T_{(1,3,5,7)}^4/\mathbb{Z}_2^\delta \times S_{(2)}^1 \times I_{x_6}\right) / \mathbb{Z}_2^\beta \\ X_{1/4} : \quad & S_{(2)}^1 \times \left(T_{(1,3,5,7)}^4/\mathbb{Z}_2^\delta \times S_{(8)}^1 \times I_{x_6}\right) / \mathbb{Z}_2^\gamma. \end{aligned} \quad (3.12)$$

Each of these halves is an open, K3-fibered CY three-fold. To see this introduce the coordinates

$$X_0 : \quad \begin{cases} z_1^0 = x_1 + ix_3 \\ z_2^0 = x_5 + ix_7 \\ z_3^0 = x_6 + ix_8 \end{cases} \quad X_{1/4} : \quad \begin{cases} z_1^{1/4} = x_7 + ix_3 \\ z_2^{1/4} = x_5 + ix_1 \\ z_3^{1/4} = x_6 + ix_2 \end{cases} \quad (3.13)$$

The K3s are along z_1, z_2 , where δ acts non-trivially. The remaining coordinates combine with the interval coordinate x_6 . At $x_6 = 0(1/4)$ the $x_8(x_2)$ circle pinches. A sketch is shown in figure 4. The holomorphic three-form is invariant under these actions, making these open CY_3 .

3.4 Uplift to GCS-decomposition of the $\text{Spin}(7)$ -manifold

Adding back the circle along the x_4 coordinate, as well as the additional orbifold generator α , α acts trivially on x_6 , so that the fibration over the interval remains intact, but $\alpha : z_1^0 \rightarrow -z_1^0$, so that the $\Omega^{(3,0)}$ form is not invariant any longer. Let us now again define two building blocks along the x_6 interval, close to either boundary. The interesting observation is that the two halves behave quite differently: first consider $x_6 = 0$. Define another complex coordinate with the action of α

$$z_4^0 \equiv x_2 + ix_4 \xrightarrow{\alpha} -z_4^0. \quad (3.14)$$

Figure 4. The GCS decomposition of the Spin(7) Joyce orbifold: the building blocks are Z_0 and $Z_{1/4}$, which are an open Calabi-Yau four-fold and an open G_2 -manifold times $S^1_{(8)}$. These are glued together along the x_6 coordinate. Both asymptote to X_s times a cylinder, where X_s is the Schoen Calabi-Yau three-fold. In this particular example each building block contains a K3, which is also present in the neck region of the Schoen Calabi-Yau.

Together with the action of α on z_i^0 , we see that $\Omega_0^{(4,0)} = dz_1^0 \wedge dz_2^0 \wedge dz_3^0 \wedge dz_4^0$ is invariant, and we obtain a building block which is an open Calabi-Yau four-fold

$$Z_0 : \quad \left(X_0 \times S^1_{(2)} \times S^1_{(4)} \right) / \mathbb{Z}_2^\alpha. \quad (3.15)$$

At the other end of the x_6 interval we have

$$Z_{1/4} : \quad \left(X_{1/4} \times S^1_{(8)} \times S^1_{(4)} \right) / \mathbb{Z}_2^\alpha = S^1_{(8)} \times M_7, \quad (3.16)$$

where M_7 will be shown to be a G_2 holonomy manifold. The action of α is

$$\alpha : \quad \begin{cases} \Omega_{1/4}^{(3,0)} \rightarrow \overline{\Omega_{1/4}^{(3,0)}} \\ x_4 \rightarrow -x_4. \end{cases} \quad (3.17)$$

From this we can define the G_2 -form

$$\Phi_3 = dx_4 \wedge \omega^{(1,1)} + \text{Re } \Omega^{(3,0)}, \quad * \Phi_3 = \frac{1}{2} \omega^{(1,1)} \wedge \omega^{(1,1)} - dx_4 \wedge \text{Im } \Omega^{(3,0)}, \quad (3.18)$$

which is invariant, as follows from the action of α and the coordinates $z^{1/4}$.

In summary we obtained a GCS-construction of a Spin(7) Joyce orbifold Z , which has two building blocks, an open CY_4 and an open G_2 , W times a circle, respectively. The

geometry in the middle of the x_6 interval is

$$Q = X_s \times S_{(8)}^1 \times I_{(6)}, \quad (3.19)$$

where the X_s is a CY_3 along the directions 1, 2, 3, 4, 5, 7. Not too surprisingly, this is in fact the Schoen Calabi-Yau three-fold. The action along these coordinates, written in terms of the complex coordinates of Z_0 is

$$\begin{aligned} \alpha : \quad (z_1^0, z_2^0, z_4^0) &\rightarrow (-z_1^0, z_2^0, -z_4^0) \\ \delta : \quad (z_1^0, z_2^0, z_4^0) &\rightarrow \left(-z_1^0 + i\frac{1}{2}, -z_2^0, z_4^0\right). \end{aligned} \quad (3.20)$$

This is precisely the action required to get the Schoen Calabi-Yau as a Joyce orbifold (see [12] for a detailed discussion of that). Inside X_s , there is a K3 surface along z_1^0, z_2^0 . On the other hand the CY_4 Z_0 is fibered by X_s over x_6, x_8 .

Note that we may also think about this $\text{Spin}(7)$ orbifold as a quotient of a Calabi-Yau four-fold X_4 by an anti-holomorphic involution. The action of α, β, δ respects the holomorphic coordinates z_i^0 on T^8 and leaves $\Omega_0^{(4,0)}$ invariant, so that it produces a Calabi-Yau orbifold. The full $\text{Spin}(7)$ orbifold is then formed by acting with γ , which acts as an anti-holomorphic involution. We expect this structure to persist after resolution.

The Joyce orbifold (3.5) has several generalizations, which can be parametrized in terms of the following data:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
α	−	−	−	−	+	+	+	+
β	+	+	+	+	−	−	−	−
γ	c_1-	c_2-	+	+	c_5-	c_6-	+	+
δ	d_1-	+	d_3-	+	d_5-	+	d_7-	+

(3.21)

where the shift vectors can take the following values

$$\begin{aligned} I : \quad \mathbf{c} &= \frac{1}{2}(1, 1, 1, 1), \quad \mathbf{d} = \frac{1}{2}(0, 1, 1, 1) \\ II : \quad \mathbf{c} &= \frac{1}{2}(1, 0, 1, 0), \quad \mathbf{d} = \frac{1}{2}(0, 1, 1, 1) \\ III : \quad \mathbf{c} &= \frac{1}{2}(1, 1, 1, 0), \quad \mathbf{d} = \frac{1}{2}(0, 1, 1, 0) \\ IV : \quad \mathbf{c} &= \frac{1}{2}(1, 0, 1, 0), \quad \mathbf{d} = \frac{1}{2}(0, 1, 1, 0). \end{aligned} \quad (3.22)$$

Case I is the orbifold studied above. Instead of the fibration over the interval (3.10), in this general setting, we consider pulling these orbifolds apart along x_3 , with the interval $x_3 \in [0, 1/4]$. There is an asymptotic middle-region, where the space is $X_s = T_{(1,2,5,6,7,8)}^6 / \langle \beta, \gamma \rangle$, which in fact is again the Schoen Calabi-Yau three-fold for all of these Joyce orbifolds. The asymptotic manifolds at the fixed points $x_3 = 0, \frac{1}{4}$ are again open a Calabi-Yau four-fold and G_2 -manifold M_7 , respectively, given by

$$\begin{aligned} Z_0 : \quad & (I_{(3)} \times S_{(4)}^1 \times X_0) / \mathbb{Z}_2^\alpha \\ Z_{1/4} : \quad & S_{(4)}^1 \times (I_{(3)} \times X_0) / \mathbb{Z}_2^\delta. \end{aligned} \quad (3.23)$$

The analysis will follow very much along the lines of the first example we studied and we will instead now move to generalize this construction beyond Joyce orbifolds.

We have seen that the Joyce orbifold Z_8 (3.5) has a natural decomposition in terms of a GCS-construction, where one building block is an open Calabi-Yau four-fold, and the other is a G_2 -manifold times a circle, where both geometries asymptote to a Calabi-Yau three-fold X_3 times a cylinder. With this example in mind, we now turn to providing a generalization of this construction in the next section.

4 Spin(7)-manifolds as generalized connected sums

4.1 The construction

In the discussion of the last section, we were led to think about Spin(7) Joyce orbifolds as being glued from two eight-manifolds: a non-compact Calabi-Yau four-fold and the product of a non-compact G_2 -manifold with a circle. These two halves are glued along their overlap, which is the product of a Calabi-Yau three-fold with an open cylinder $I \times \mathbb{S}^1$ — see figure 2. We are now going to propose a generalization of this procedure in which both the two building blocks $Z_{\pm}$ and the resulting Spin(7)-manifold Z are no longer (resolutions of) orbifolds. To start this discussion, let us first propose the following definitions:

Definition. *An asymptotically cylindrical (acyl) Calabi-Yau four-fold Z_+ is a non-compact algebraic four-fold, which admits a Ricci-flat metric, is simply connected, and is diffeomorphic (as a real manifold) to the product of a cylinder $I \times \mathbb{S}^1$ and a compact Calabi-Yau three-fold X_3 outside of a compact submanifold κ . The Ricci-flat metric of Z_+ exponentially asymptotes to the Ricci flat metric on the product $I \times \mathbb{S}^1 \times X_3 \cong X_4 \setminus \kappa$.*

Asymptotically Calabi-Yau manifolds were discussed in [38]. Similar to the explicit construction of acyl three-folds in [5, 6], we expect to be able to construct Z_+ by excising a fiber X_3 from a compact four-fold $\tilde{Z}_+$ with $c_1(\tilde{Z}_+) = [X_3]$, which is fibered by Calabi-Yau three-folds. This implies that we can think of producing Z_+ by appropriately cutting a compact, CY₃-fibered Calabi-Yau four-fold in half, or via toric methods as in [9]. Likewise we define:

Definition. *An asymptotically Calabi-Yau (acyl) G_2 -manifold Z_- is a non-compact manifold of G_2 holonomy, which is diffeomorphic to the product of an interval I with a compact Calabi-Yau three-fold X_3 outside of a compact submanifold of Z_- , and the Ricci-flat metric on Z_- exponentially asymptotes to the Ricci-flat metric on the product manifold $I \times X_3$.*

Such manifolds were discussed in [39]. Note that such G_2 -manifolds are easily constructed as (resolutions of) orbifolds $(X_3 \times \mathbb{R}_t)/\mathbb{Z}_2$ with the $\mathbb{Z}_2$ acting as $t \rightarrow -t$ and as an anti-holomorphic involution on the Calabi-Yau three-fold X_3 . The quotient by such involutions can be thought of as a Calabi-Yau three-fold X_3 fibered over an half-open interval which undergoes a degeneration at one end. In particular, it is not surprising if the Ricci-flat metric on such a resolution of $(X_3 \times \mathbb{R})/\mathbb{Z}_2$ asymptotes to the Ricci-flat metric on $X_3 \times \mathbb{R}$ far away from the origin of $\mathbb{R}$. A particularly simple case is given by freely

acting anti-holomorphic involutions. In this case no resolution is required, and the Ricci flat metric on $(X_3 \times \mathbb{R})/\mathbb{Z}_2$ is simply the quotient of the Ricci-flat metric on $X_3 \times \mathbb{R}$. The simplicity of this examples comes at a price, however, as the holonomy group of such acyl G_2 -manifolds is not the full G_2 [39, 40], but only $SU(3) \rtimes \mathbb{Z}_2$, so that we can call them acyl barely G_2 -manifolds. Such acyl barely G_2 -manifolds in particular have a nontrivial fundamental group $\mathbb{Z}_2$.

We are now ready to propose our construction of what we will call generalized connected sum (GCS) $Spin(7)$ -manifolds. Take an acyl Calabi-Yau four-fold Z_+ with asymptotic neck region isomorphic to $I_{t_+} \times \mathbb{S}_{\theta_+}^1 \times X_3^+$ for $t_+ = 0 \cdots l$ and a acyl G_2 -manifold Z_- with asymptotic neck region isomorphic to $I_{t_-} \times X_3^-$ for $t_- = 0 \cdots l$. Then Z_+ and $Z_- \times \mathbb{S}_{\theta_-}^1$ can be glued as topological manifolds to a manifold Z by identifying the neck regions $I_{t_{\pm}} \times \mathbb{S}_{\theta_{\pm}}^1 \times X_3^{\pm}$ such that

$$\begin{aligned} t_+ &= l - t_- \\ \theta_+ &= \theta_- \\ \phi : X_3^+ &\cong X_3^- \end{aligned} \tag{4.1}$$

for a biholomorphic map ϕ . If the Ricci-flat metrics on $Z^{\pm}$ asymptote to the Ricci-flat metrics on $X_3^{\pm}$, this takes us close to a Ricci-flat metric on Z . We conjecture that for long enough neck regions (l large enough) there exists a Ricci-flat metric g associated with a torsion free $Spin(7)$ structure on Z , which is found by a small perturbation of the Ricci-flat metrics on Z_+ and $Z_- \times \mathbb{S}^1$.

If $Z_{\pm}$ are both simply connected, it follows from the Seifert-van Kampen theorem that Z is a simply connected eight-manifold. This means that the holonomy group of Z must be equal to $Spin(7)$ (and not a subgroup) if Z has a torsion-free $Spin(7)$ structure.

Before exploring the consequences of our proposed construction further, let us briefly discuss the mathematical work needed to put our proposal on firm ground. The work [38–40] on acyl Calabi-Yau four-folds and acyl G_2 -manifolds and their deformation theory, together with clear criteria when the asymptotic Calabi-Yau three-folds $X_3^{\pm}$ allow a biholomorphic map ϕ , should clarify under which circumstances a gluing can be found for a given pair of such manifolds. In our examples, we can easily find such diffeomorphisms by realizing $X_3^{\pm}$ as hypersurfaces in toric varieties, so that a diffeomorphic pair can be simply constructed by writing down identical algebraic equations. The crucial task is then to show the existence of a Ricci-flat metric with holonomy $Spin(7)$ on the resulting topological manifold Z . By making the neck regions very long, we expect that the torsion introduced when gluing Z_+ and $Z_- \times \mathbb{S}^1$ can be made sufficiently small for a torsion-free $Spin(7)$ structure to exist nearby.

As we shall discuss in more detail in section 4.5, there are instances of GCS $Spin(7)$ -manifolds which can globally be realized as (resolutions) of quotients of Calabi-Yau four-folds. Quotients by free actions, where no resolution is needed, or situations in which resolutions with Ricci-flat metrics can be constructed provide non-trivial examples of our construction in which their existence is proven by other means.

4.2 Calibrating forms

Both the acyl Calabi-Yau four-fold Z_+ and the product of the acyl G_2 -manifold Z_- with a circle have a torsion free $\text{Spin}(7)$ structure. In this section we show how the identification (4.1) between the asymptotic neck regions produces a globally defined $\text{Spin}(7)$ structure.

Let us denote the Kähler form and holomorphic four-form of Z_+ by ω_+ and Ω_+ . The Cayley four-form defining a torsion-free $\text{Spin}(7)$ structure on Z_+ is then given by

$$\Psi_+ = \text{Re } \Omega_+ + \frac{1}{2} \omega_+ \wedge \omega_+. \quad (4.2)$$

In the asymptotic neck region, we can introduce coordinates t_+ and θ_+ and decompose the $\text{SU}(4)$ structure as

$$\begin{aligned} \omega_+ &= \omega_{3,+} + dt_+ \wedge d\theta_+ \\ \Omega_+ &= d\theta_+ \wedge \text{Re } \Omega_{3,+} - dt_+ \wedge \text{Im } \Omega_{3,+} + i(dt_+ \wedge \text{Re } \Omega_{3,+} + d\theta_+ \wedge \text{Im } \Omega_{3,+}), \end{aligned} \quad (4.3)$$

where $\omega_{3,+}$ and $\Omega_{3,+}$ are the Kähler form and holomorphic three-form on X_3^+ . This means that the $\text{Spin}(7)$ structure in the neck region is

$$\Psi_+ = d\theta_+ \wedge \text{Re } \Omega_{3,+} - dt_+ \wedge \text{Im } \Omega_{3,+} + \frac{1}{2} \omega_{3,+} \wedge \omega_{3,+} + d\theta_+ \wedge dt_+ \wedge \omega_{3,+}. \quad (4.4)$$

Similarly, the $\text{Spin}(7)$ structure on $Z_- \times \mathbb{S}_{\theta_-}^1$ is given by

$$\Psi_- = d\theta_- \wedge \varphi_- + * \varphi_- \quad (4.5)$$

in terms of the G_2 structure φ_- on Z_- . In the neck region, this is further decomposed as

$$\begin{aligned} \varphi_- &= -dt_- \wedge \omega_{3,-} + \text{Re } \Omega_{3,-} \\ * \varphi_- &= \frac{1}{2} \omega_{3,-} \wedge \omega_{3,-} + dt_- \wedge \text{Im } \Omega_{3,-} \end{aligned} \quad (4.6)$$

in terms of the Kähler form and holomorphic three-form on X_3^- . We hence find that the $\text{Spin}(7)$ structure on the neck region of $Z_- \times \mathbb{S}^1$ can be written as

$$\Psi_- = -d\theta_- \wedge dt_- \wedge \omega_{3,-} + d\theta_- \wedge \text{Re } \Omega_{3,-} + \frac{1}{2} \omega_{3,-} \wedge \omega_{3,-} + dt_- \wedge \text{Im } \Omega_{3,-}. \quad (4.7)$$

This means that the two $\text{Spin}(7)$ structure $\Psi_{\pm}$ are consistently glued together under a diffeomorphism which identifies the neck regions as

$$\begin{aligned} t_+ &= -t_- \\ \theta_+ &= \theta_- \\ \omega_{3,+} &= \omega_{3,-} \\ \Omega_{3,+} &= \Omega_{3,-} \end{aligned} \quad (4.8)$$

which is nothing but the map (4.1) proposed earlier.

4.3 Topology of GCS Spin(7)-manifolds

The easiest topological number to determine for the GCS Spin(7)-manifold Z is given by the Euler characteristic $\chi(Z)$. As it is additive, we have

$$\chi(Z) = \chi(Z_+) + \chi(Z_- \times \mathbb{S}^1) - \chi(Z_+ \cap (Z_- \times \mathbb{S}^1)) = \chi(Z_+), \quad (4.9)$$

where we have used the fact that the Euler characteristic vanishes for any manifold with an $\mathbb{S}^1$ factor.

Two copies Z_+, Z'_+ of the acyl Calabi-Yau four-fold Z_+ can be glued to a compact Calabi-Yau four-fold $X_4 = Z_+ \cup Z'_+$ such that $Z_+ \cap Z'_+ = I \times \mathbb{S}^1 \times X_3$ for a Calabi-Yau three-fold X_3 . It follows that $\chi(X_4) = 2\chi(Z_+) = 2\chi(Z)$. As the Euler characteristic of any Calabi-Yau four-fold is divisible by six [41], this implies that $\chi(Z)$ is divisible by 3, so that b_4^- in (2.4) is always an integer.

As we have defined an acyl Calabi-Yau four-fold to be simply connected, but left room for the possibility of an acyl G_2 -manifold to have a non-trivial fundamental group, the Seifert-van Kampen theorem tells us that Z is simply connected if and only if Z_- is simply connected. If Z_- is not simply connected, Z can have a non-trivial fundamental group, which signals that the holonomy group of Z is smaller than Spin(7) [17, 23].

The easiest way to construct acyl G_2 -manifolds is by a free quotient $X_3 \times \mathbb{R}$ by $\mathbb{Z}_2$, in which case the fundamental group of Z_- is not trivial but equal to $\mathbb{Z}_2$. This works as follows. Consider two points identified by the anti-holomorphic involution on X_3 over the origin of $\mathbb{R}$. Any path connecting two such points becomes a closed loop in the quotient and, as the involution acts freely, cannot be homotopic to a point. If X_3 is simply connected, all such loops are homotopic, so that $\pi_1((X_3 \times \mathbb{R})/\mathbb{Z}_2) = \mathbb{Z}_2$, which implies that Z_- only has holonomy group $\text{SU}(3) \rtimes \mathbb{Z}_2$, i.e. is a barely G_2 manifold. On Z , such loops give rise to a non-trivial fundamental group, which is $\mathbb{Z}_2$ as well, so that Z does not have the full holonomy group Spin(7). From the point of view of physics, compactification on such ‘barely’ Spin(7)-manifolds does not give rise to extended supersymmetry as there is still only one covariantly constant spinor. In fact, it is not hard to see that Z has holonomy group $\text{SU}(4) \rtimes \mathbb{Z}_2$ in this case: the holonomy group $\text{SU}(4)$ of Z_+ and the holonomy group $\text{SU}(3) \rtimes \mathbb{Z}_2$ of Z_- share a common $\text{SU}(3)$, the holonomy group of X_3 .

We can compute the cohomology groups of Z in terms of the cohomology groups of $Z_\pm$ and the pull-backs to X_3 by using the Mayer-Vietoris exact sequence, which implies that

$$H^i(Z, \mathbb{Z}) = \ker \gamma^i \oplus \text{coker } \gamma^{i-1}, \quad (4.10)$$

where

$$\gamma^i : H^i(Z_+, \mathbb{Z}) \oplus H^i(Z_- \times \mathbb{S}^1, \mathbb{Z}) \rightarrow H^i(X_3 \times \mathbb{S}^1, \mathbb{Z}) \quad (4.11)$$

can be expressed in terms of the restriction maps

$$\begin{aligned} \beta_+^i &: H^i(Z_+, \mathbb{Z}) \rightarrow H^i(X_3 \times \mathbb{S}^1, \mathbb{Z}) \\ \beta_-^i &: H^i(Z_-, \mathbb{Z}) \rightarrow H^i(X_3, \mathbb{Z}). \end{aligned} \quad (4.12)$$

Note that both β_-^i and β_-^{i-1} feature in γ^i due to the product $\mathbb{S}^1$.

Under a few assumptions, which are met in the examples to be discussed later, we can now explicitly work out the various contributions to $H^\bullet(Z)$. First of all, let us assume that the acyl Calabi-Yau four-fold Z_+ can be constructed from an algebraic four-fold $\tilde{Z}$ which is fibered by Calabi-Yau three-folds by excising a fiber X_3 . Furthermore, let us assume that the images of β_+^2 and β_+^4 are surjective and that $b^3(Z_+) = b^5(Z_+) = 0$.

As noted above, we assume that we can construct Z_- as (a resolution) of the quotient $(X_3 \times \mathbb{R}_t)/\mathbb{Z}_2$ by an anti-holomorphic involution. The resolution must be such that it preserves a metric of holonomy G_2 on Z_- and the property of Z_- being asymptotically cylindrical. We will denote the numbers of even/odd classes of X_3 under this $\mathbb{Z}_2$ by b_e^i and b_o^i , respectively. Finally, the kernels of the restriction maps $\beta_\pm^i$ and their ranks are abbreviated as

$$N_\pm^i \equiv \ker \beta_\pm^i \quad n_\pm^i \equiv |N_\pm^i|. \quad (4.13)$$

With this notation and under the assumptions we have made, (4.10) implies that

$$\begin{aligned} b^1(Z) &= 0 \\ b^2(Z) &= n_+^2 + n_-^2 + b_e^2 \\ b^3(Z) &= n_-^2 + n_-^3 \\ b^4(Z) &= n_-^3 + n_-^4 + n_+^4 + b_o^2 + b_o^3 + b_e^3 + b_e^4 \\ b^5(Z) &= n_-^4 + n_-^5 \\ b^6(Z) &= n_+^6 + n_-^5 + b_o^4 \\ b^7(Z) &= 0. \end{aligned} \quad (4.14)$$

This can be seen as follows. For $H^1(Z)$, the unique class in $H^1(X_3 \times \mathbb{S}^1)$ is in the image of γ^1 and solely originates from $H^1(Z_- \times \mathbb{S}^1)$. Hence $|\ker(\gamma^1)| = 0$, which together with $\text{coker}(\gamma^0) = 0$ implies $b^1(Z) = 0$.

Let us now consider $b^2(Z)$. As the unique class in $H^1(Z_- \times \mathbb{S}^1)$ is in the image of γ^1 , we find that $\text{coker}(\gamma^1) = 0$ so that $H^2(Z) = \ker(\gamma^2)$ follows. In turn, $\ker(\gamma^2)$ has three different contributions:

$$\ker(\gamma^2) = \ker(\beta_+^2) \oplus \ker(\beta_-^2) \oplus [\text{im}(\beta_+^2) \cap \text{im}(\beta_-^2)]. \quad (4.15)$$

As β_+^2 is surjective by assumption, the last term is simply given b_e^2 and we recover the expression in (4.14).

The computation for $b^3(Z)$ is made particularly simple by our assumptions. As $|\text{coker}(\gamma^2)| = 0$, it only receives a contribution from $\ker(\gamma^3)$. As $b^3(Z_+) = 0$, the only contribution to $b^3(Z)$ is from $\ker(\beta_-^2) \oplus \ker(\beta_-^3) = N_-^2 \oplus N_-^3$.

For $b^4(Z)$, all potential contributions are non-trivial. First of all, there is a contribution n_+^4 from $\ker \beta_+^4$, as well as $n_-^3 + n_-^4$ from $\ker \beta_-^3 + \ker \beta_-^4$. Furthermore, there are b_o^2 classes in $\text{coker}(\gamma^3)$, which correspond to three-forms with one leg on the product $\mathbb{S}^1$ of the cylinder region, and b_o^3 classes in $\text{coker}(\gamma^3)$ which correspond to three-forms purely on X_3 . Finally, there is a contribution

$$(\text{im } \beta_+^4) \cap [(\text{im } \beta_-^4) \oplus (\text{im } \beta_-^3)], \quad (4.16)$$

which simply has dimension $b_e^4 + b_e^3$ as β_+^4 is surjective. Together, all of these contributions give the expression for $b^4(Z)$ in (4.14).

The computation of $b^5(Z)$ again only receives a contribution from $\ker \gamma^5$ as $\text{coker } \gamma^4$ is empty by assumption. Furthermore, γ^5 is only non-trivial on $Z_- \otimes \mathbb{S}^1$, so that $|\ker \gamma^5| = n_-^4 + n_-^5$.

The computation of $b^6(Z)$ is similar to $b^2(Z)$, it receives a term b_o^4 from $\text{coker}(\gamma^5)$, as well as terms n_-^5 and n_+^6 from $\ker(\gamma^6)$. Note that $N_-^6 = 0$ as Z_- is a G_2 manifold.

Finally, $\text{coker}(\gamma^6) = 0$ and $\ker(\gamma^7) = 0$ gives $b^7(Z) = 0$.

The various contributions n_-^i and h_e^i, h_o^i are not all independent and their relations make Poincaré duality for (4.14) manifest. For anti-holomorphic involutions of a Calabi-Yau three-fold X_3 , the Lefschetz fixed-point theorem combined with Poincaré duality implies that $b_o^2 = b_e^4$ and $b_e^2 = b_o^4$. Furthermore, Poincaré duality on the compact G_2 manifold $(X_3 \times \mathbb{S}^1)/\mathbb{Z}_2$ implies that $n_-^2 = n_-^5$ and $n_-^3 = n_-^4$. Together with $n_+^2 = n_+^6$, these relations imply that $b^i(Z) = b^{8-i}(Z)$ in (4.14).

The number of deformations of the Ricci-flat metric is given by $b_-^4 + 1$ for manifolds of Spin(7)-holonomy. Using (2.4), this can be written as

$$b_-^4 + 1 = -8 + \frac{1}{3} (2 - b^2 + b^3 + b^4) , \quad (4.17)$$

which becomes

$$b_-^4 + 1 = -8 + n_-^3 + \frac{1}{3} (2 + n_+^4 - n_+^2 + 2b_o^2 - b_e^2 + b_e^3 + b_o^3) \quad (4.18)$$

in our GCS-construction. The Euler characteristic of Z is given by

$$\chi(Z) = 2 + 2b^2(X_3) + b^3(X_3) + n_+^4 + 2n_+^2 , \quad (4.19)$$

and only depends on the data of Z_+ , as expected.

Even though the topology of the resolutions of Joyce's Spin(7)-orbifolds discussed in section 3 is most conveniently computed otherwise, one can also use their decomposition as GCS Spin(7)-manifolds together with the relations of this sections to find their topology from resolutions of two building blocks $Z_\pm$.

4.4 3d field theory and sectors of enhanced supersymmetry

In compactifications of M-Theory on a smooth Spin(7)-manifold, the low energy effective theory at the classical level gives a 3d $\mathcal{N} = 1$ supergravity theory with $n_v = b^2(Z)$ massless U(1) vector multiplets and $n_r = b^3(Z) + b_-^4(Z) + 1$ massless uncharged real scalar multiplets. As the Spin(7)-manifolds considered here are glued from pieces with holonomy SU(4) and G_2 , respectively, we expect to find subsectors of enhanced supersymmetry in our spectrum, similar to the observations made in [11] regarding TCS G_2 manifolds, to arise from localized forms in the building blocks.

Consider first the multiplets arising from localized forms on $Z_- \times \mathbb{S}^1$. By (4.14), each two-form in N_-^2 gives rise to both a two-form and a three-form on Z , which combine to form the bosonic field content of a 3d $\mathcal{N} = 2$ vector multiplet. Furthermore, each

three-form in N_-^3 gives rise to a real scalar due to its appearance in the formula for $b^3(Z)$ in (4.14). As for compactifications of M-Theory on a G_2 -manifold times a circle, the moduli associated to these three-forms pair up with deformations of the metric to form 3d $\mathcal{N} = 2$ chiral multiplets: this can be seen from (4.18), which shows that each three-form in N_-^3 corresponds to an anti-self-dual four-form, i.e. a deformation of the metric.

Let us now turn to multiplets which arise from localized forms on Z_+ . Each two-form of Z_+ in N_+^2 gives rise to a 3d $\mathcal{N} = 1$ vector multiplet due to its contribution to $b^2(Z)$. As for compactifications of M-Theory on Calabi-Yau four-folds, we expect this degree of freedom to pair up with a real scalar to the bosonic degrees of freedom of a 3d $\mathcal{N} = 2$ vector multiplet. This degree of freedom must originate from an anti-self-dual four-form in N_+^4 . Furthermore, we expect the remaining anti-self-dual four-forms in N_+^4 to appear pairwise, so as to combine into 3d $\mathcal{N} = 2$ chiral multiplets.

To see how this comes about, we need to exploit the fact that Z_+ and its compactification $\tilde{Z}_+$ are Kähler manifolds and carry a Lefschetz $SU(2)$ action. The upshot is that the anti-self-dual four-forms on $\tilde{Z}_+$ are given by the four-forms in $H^{3,1}(\tilde{Z}_+) \oplus H^{1,3}(\tilde{Z}_+)$, together with four-forms of the type $\omega_k \wedge J$, where J is the Kähler form on $\tilde{Z}_+$ and the ω_k are $(1,1)$ forms on $\tilde{Z}_+$ such that $\omega_k \wedge J^3 = 0$. As every two-form in N_+^2 gives a four-form in N_+^4 upon wedging with J , we can hence associate an anti-self-dual four-form in N_+^4 on Z_+ to every element of N_+^2 (modulo the $(1,1)$ forms in the image of β_+^2). The remaining anti-self-dual four-forms in N_+^4 then have to appear pairwise, as they must correspond to $H^{3,1}(\tilde{Z}_+) \oplus H^{1,3}(\tilde{Z}_+)$. As we can think about the forms in N_+^4 as being localized on the acyl Calabi-Yau four-fold Z_+ far away from the gluing region, (anti) self-duality on Z_+ will imply (anti) self-duality of the corresponding forms on Z .

Note that this argument precisely reflects how these degrees of freedom originate in Physics. For two-forms in N_+^2 and self-dual four-forms in N_+^4 , there are deformations of the Ricci-flat metric of Z_+ (Kähler and complex structure deformations), which do not alter the cylindrical region in which Z_+ is glued to $Z_- \times \mathbb{S}^1$. As we expect the Ricci-flat metric of Z to be well approximated by the Ricci-flat metrics of Z_+ far away from the gluing region, these become deformations of the $Spin(7)$ -manifold Z if we stretch the neck region sufficiently long. The metric deformations associated with N_+^2 and N_+^4 , together with the 3d vectors originating from the C -field on N_+^2 , only see the geometry of a Calabi-Yau four-fold and hence give rise to a subsector with 3d $\mathcal{N} = 2$ supersymmetry. As the property of forms being (anti) self-dual is a topological constraint, the counting and identifications we have performed will persist throughout the moduli space of Z , so that the subsectors with enhanced $\mathcal{N} = 2$ supersymmetry in 3d will persist. We have summarized the result of this discussion in table 1.

The remaining moduli of M-Theory on the GCS $Spin(7)$ -manifold Z do not appear in multiplets with enhanced supersymmetry, as they are associated with the gluing along $X_3 \times \mathbb{S}^1$. These moduli are, however, in one-to-one correspondence with the geometric deformations of the Calabi-Yau three-fold X_3 in the neck region, i.e. Kähler and complex structure deformations of X_3 . While compactification of M-Theory on $X_3 \times \mathbb{S}^1 \times \mathbb{S}^1$ results in a theory with $\mathcal{N} = 4$ supersymmetry in 3d, these multiplets are truncated to $\mathcal{N} = 1$ for M-Theory compactified on Z . Similar to the case of TCS G_2 manifolds [11], the eigenvalues un-

3d Multiplets	Origin
$\mathcal{N} = 2$ vector	N_+^2
$\mathcal{N} = 2$ chiral	$h^{3,1}(\tilde{Z}_+)$
$\mathcal{N} = 4$ vector	$b_2(X_3), b_3(X_3)$
$\mathcal{N} = 2$ vector	N_-^2
$\mathcal{N} = 2$ chiral	N_-^3

Table 1. Subsectors with enhanced supersymmetry and their topological origin. The $\mathcal{N} = 4$ vector multiple from the neck region is only massless in the infinite neck limit and away from this becomes an $\mathcal{N} = 1$ vector.

der the Laplace operator of the appropriate non-harmonic forms needed to lift these $\mathcal{N} = 1$ multiplets to $\mathcal{N} = 4$ multiplets go to zero in the limit in which the neck region is stretched.

4.5 GCS Spin(7)-manifolds as quotients of CY_4

Our construction has a natural connection to Spin(7)-manifolds constructed as quotients of Calabi-Yau four-folds by anti-holomorphic involutions, however we will see that there are key difference to the previous constructions in [18, 20]. Consider a compact Calabi-Yau four-fold X_4 , which is fibered by Calabi-Yau three-folds over a base $\mathbb{P}^1$ with homogeneous coordinates $[z_1 : z_2]$ and let $z = z_1/z_2$. Let us denote the fiber over a point p of the $\mathbb{P}^1$ base by $X_3(p)$. For appropriate fibrations, we may then tune the complex structure (defining equation) of X_4 such that

$$X_3(z) = X_3(1/\bar{z}). \quad (4.20)$$

For appropriate anti-holomorphic involutions σ acting on $X_3(p)$ for all p , we may then construct an anti-holomorphic involution

$$\Sigma = \sigma \circ \left\{ \begin{array}{l} z_1 \rightarrow \bar{z}_2 \\ z_2 \rightarrow \bar{z}_1 \end{array} \right\} \quad (4.21)$$

acting in X_4 . This will produce a singular Spin(7)-manifold $Z_s = X_4/\Sigma$, and we will assume that it can be resolved into a smooth Spin(7)-manifold Z .

In the $\mathbb{P}^1$ base the fixed locus of this involution will be given by the circle $\mathbb{S}_f^1 = \{|z|^2 = 1\}$ and the quotient effectively truncates the base $\mathbb{P}^1$ to a disc with boundary. We may make this disc very large and furthermore confine all of the singular fibers of the fibration of $X_3(p)$ near the origin. If the fibration of $X_3(p)$ over the fixed $\mathbb{S}_f^1$ is trivial, the fibers over $\mathbb{S}_f^1$ all become identical and we may simply denote them by X_3 . Cutting along a circle of fixed radius now produces one half near the boundary of the disc which may be described as

$$\mathbb{S}_f^1 \times (X_3 \times \mathbb{R})/\sigma. \quad (4.22)$$

In the limit in which the disc is very large, a resolution of Z_s to Z is equivalent to a resolution of $(X_3 \times \mathbb{R})/\sigma$ to a smooth acyl G_2 manifold Z_- . The other half of Z near the origin of the disc does not require resolution and becomes an acyl Calabi-Yau four-fold Z_+ which

asymptotes to X_3 . In the middle region, the two spaces Z_+ and $Z_- \times \mathbb{S}_f^1$ are glued along the product of an interval and $X_3 \times \mathbb{S}_f^1$. This is precisely the gluing construction we have given above. An illustration is given in figure 3.

Let us highlight two aspects of the topology of Z which are immediately recovered from this point of view. The resolution of Z_s to Z only introduces new cycles sitting over a circle, so that the topological Euler characteristic of Z is the same as that of Z_s . But the topological Euler characteristic of Z_s is simply given by half of the Euler characteristic of X_4 . In fact, we may think of X_4 as being glued from two copies of the acyl Calabi-Yau four-fold Z_+ , which implies that the Euler characteristic of Z is equal to that of Z_+ , which is the same result found from our gluing construction earlier. Furthermore, we have observed that we produce a barely $\text{Spin}(7)$ -manifold with fundamental group $\mathbb{Z}_2$, whenever Z_- is constructed from a free quotient of $X_3 \times \mathbb{R}$ by σ . This means that we may write Z as the free quotient X_4/Σ , so that we immediately recover $\pi_1(Z) = \mathbb{Z}_2$.

As we have explained, a GCS $\text{Spin}(7)$ -manifold can be described as a resolution of a quotient of a Calabi-Yau four-fold by an anti-holomorphic involution whenever the acyl G_2 manifold Z_- can be described as a resolution of $(X_3 \times \mathbb{R})/\sigma$ for an anti-holomorphic involution σ . In the absence of a different construction for acyl G_2 manifolds, our GCS construction is hence equivalent to forming quotients of a specific class of Calabi-Yau four-folds. In this context, it implies that a resolution of singularities of the quotient by Σ is already captured by resolving quotients $(X_3 \times \mathbb{R})/\sigma$. Furthermore, it gives a completely new point of view on such geometries; it shows how to distinguish subsectors of enhanced supersymmetry and, as we shall see, allows to construct dual heterotic backgrounds on TCS G_2 manifolds. Finally, the realization that many GCS $\text{Spin}(7)$ -manifolds are simply (resolutions of) quotients X_4/Σ in fact proves that our construction produces eight-dimensional manifolds with Ricci-flat metrics with holonomy group $\text{Spin}(7)$ (or $\text{SU}(4) \rtimes \mathbb{Z}_2$).

In [18] Joyce proposed another construction of $\text{Spin}(7)$ -manifolds base on non-free quotients of Calabi-Yau four-folds with orbifold singularities was given. Crucially, the resulting singular $\text{Spin}(7)$ orbifolds are taken to only have isolated singularities here, which can be resolved by gluing in appropriate ALE spaces. Even though similar in spirit, the anti-holomorphic involution appearing here are either free or have fixed loci of real dimension at least one, so these constructions are distinct. It seems likely, however, that there are examples for which the resolutions can be described by gluing in a acyl G_2 manifold times a circle, so that there will be many $\text{Spin}(7)$ -manifolds which can be found using both constructions.

4.6 A simple example

We now construct a new $\text{Spin}(7)$ -holonomy manifold using the GCS construction. There will be further examples in the next section, which are geared toward the application of M-theory/Heterotic duality.

Consider a smooth anti-canonical hypersurface in weighted projective space $\mathbb{P}_{11114}$. The resulting space is a Calabi-Yau three-fold $X_{1,149}$ with Hodge numbers $h^{1,1} = 1$ and $h^{2,1} = 149$. We now show how to construct an acyl G_2 -manifold Z_- which asymptotes to $X_{1,149} \times \mathbb{R}$ and an acyl Calabi-Yau four-fold Z_+ which asymptotes to $X_{1,149} \times \mathbb{S}^1 \times \mathbb{R}$,

and then glue Z_+ and $Z_- \times \mathbb{S}^1$ to a $\text{Spin}(7)$ -manifold Z as described above. Let us first describe the acyl G_2 -manifold Z_- . The Calabi-Yau three-fold $X_{1,149}$ admits a freely acting anti-holomorphic involution, which is obvious by choosing a Fermat type hypersurface

$$x_1^8 + x_2^8 + x_3^8 + x_4^8 + x_5^2 = 0, \quad (4.23)$$

where $[x_1 : x_2 : x_3 : x_4 : x_5]$ are the homogeneous coordinates of $\mathbb{P}_{11114}$. Letting $x_i \rightarrow \bar{x}_i$ then acts as a fixed-point free anti-holomorphic involution, as the above equation has no solutions purely over the reals. The odd/even Hodge numbers are

$$h_e^2 = 0, \quad h_o^2 = 1, \quad h_e^3 = h_o^3 = 149, \quad (4.24)$$

and $(X_{1,149} \times \mathbb{R})/\mathbb{Z}_2$ is an acyl G_2 -manifold. Note that $n_-^i = 0$ for all i as no resolution of singularities is required on Z_- .

To find an acyl Calabi-Yau four-fold, we consider an algebraic four-fold $\tilde{Z}_+$ which is a fibration of $X_{1,149}$ over $\mathbb{P}^1$ and has $c_1(\tilde{Z}_+) = [X_{1,149}]$. Such a $\tilde{Z}_+$ can easily be constructed by as the zero locus of a generic section of $\mathcal{O}(8,1)$ over $\mathbb{P}_{11114} \times \mathbb{P}^1$. Such a four-fold has $h^{1,1}(\tilde{Z}_+) = 3$, which means that $n_+^2 = 1$. Furthermore, the resulting acyl Calabi-Yau four-fold $Z_+ = \tilde{Z}_+ \setminus X_{1,149}$ has $\chi(Z_+) = \chi(\tilde{Z}_+) - \chi(X_{1,149}) = 1680 = \chi(Z)$.

Note that we may choose an element of the algebraic family $\tilde{Z}_+$ such that the fiber over some chosen point in the base $\mathbb{P}^1$ is described by (4.23), so that Z_+ manifestly asymptotes to a Calabi-Yau three-fold which is isomorphic to the Calabi-Yau three-fold in the neck region of Z_- . For the resulting GCS $\text{Spin}(7)$ -manifold constructed from this choice of Z_+ and Z_- , it now follows that

$$b^2(Z) = 1 \quad b^3(Z) = 0 \quad b_-^4(Z) = 551 \quad b_+^4(Z) = 1125. \quad (4.25)$$

Using (2.4), this is most efficiently calculated by using $\chi(Z) = 1680$. One can also work out that $n_+^4 = 1374$, which reproduces $b^4(Z) = 1676$ using (4.14). Due to the free action of the anti-holomorphic involution on $X_{1,149}$, this example produces a barely $\text{Spin}(7)$ -manifold. By considering more general anti-holomorphic involutions and resolving the resulting singularities, similar examples with full holonomy $\text{Spin}(7)$ can be constructed from this procedure.

In agreement with our general discussion in section 4.5, Z may also be found as the *free* quotient of a Calabi-Yau four-fold X_4 realized as a hypersurface in $\mathbb{P}_{11114} \times \mathbb{P}^1$. The Hodge numbers of X_4 are

$$h^{1,1}(X_4) = 4 \quad h^{2,1}(X_4) = 0 \quad h^{3,1}(X_4) = 548 \quad h^{2,2}(X_4) = 2252 \quad (4.26)$$

and $\chi(X_4) = 3360$. The free involution Σ acts as

$$\begin{aligned} x_i &\rightarrow \bar{x}_i \\ z_1 &\rightarrow \bar{z}_2 \\ z_2 &\rightarrow \bar{z}_1, \end{aligned} \quad (4.27)$$

where $[z_1 : z_2]$ are homogeneous coordinates on the base $\mathbb{P}^1$ and x_i are the homogeneous coordinates on $\mathbb{P}_{11114}$. We may choose a smooth invariant hypersurface on which Σ acts without fixed points as

$$(z_1^2 + a_5 z_1 z_2 + z_2^2) x_5^2 + \sum_i^4 (z_1^2 + a_i z_1 z_2 + z_2^2) x_i^8 = 0, \quad (4.28)$$

for a set of real parameters a_i . A little thought reveals that a one-dimensional subspace of the forms in $H^{1,1}(X_4)$ is even under Σ , which together with $\chi(X_4) = 3360$ and $b^3(X_4) = 0$ implies (4.25).

5 M-theory on GCS Spin(7)/heterotic on TCS G_2

In this section we construct a new class of GCS Spin(7) manifolds Z following our general discussion in section 4, where the building blocks are an open acyl Calabi-Yau four-fold Z_+ and an open acyl G_2 -manifold Z_- (times a circle), which are both asymptote to a cylinder times the same Calabi-Yau three-fold X_3 . The additional input in this section is that we will consider these in the context of the duality to Heterotic strings theory on G_2 -manifolds, which we show to be TCS-manifolds, with building blocks $X_{\pm}$.

Our strategy in finding dual compactifications will be similar to the one we used in [12], i.e. using fiberwise duality between M-theory and heterotic string theory, we will identify dual pairs of building blocks, which can then be glued to find the dual Spin(7) and G_2 -manifolds. As this requires to carefully dissect geometries on both sides, the following discussion will unfortunately be rather technical. After motivating our construction for a dual pair of geometries, we will summarize our results in section 5.4 before checking the spectra. In our first example, the $E_8 \times E_8$ gauge symmetry is broken by a (quotient of a) Wilson line, and we can easily match the light degrees of freedom of M-theory to those of the dual heterotic string theory. In general, such a match is much more difficult, as it requires to determine the number of moduli associated with a vector bundle on a G_2 -manifold. We present a second closely related example in which we use the proposed duality to instead predict the number of bundle moduli on the G_2 manifold.

We begin with a brief summary of heterotic G_2 -systems in section 5.1 and a discussion of the duality in section 5.2, before then constructing examples of dual pairs.

5.1 Heterotic string theory on G_2 -manifolds

Before we begin with a detailed discussion of dual pairs, we should explain how to construct 3d $\mathcal{N} = 1$ theories from heterotic string theory on a G_2 -holonomy manifold together with a vector bundle. Here we will discuss briefly heterotic compactifications on G_2 -holonomy manifolds.

A compact orientable 7-dimensional manifold J which has a Ricci-flat metric g with holonomy group contained in G_2 supports a closed three-form Φ , the Hodge-* dual of which is closed as well, which can be expressed in local coordinates (in which g is the euclidean metric) as

$$\Phi = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}. \quad (5.1)$$

This three-form and its Hodge- $*$ -dual are preserved by the action of G_2 . Besides having a Ricci-flat metric, G_2 -manifolds support a single covariantly constant spinor.

In contrast to Spin(7)-manifolds, a necessary and sufficient condition for the holonomy group of a manifold J with a Ricci-flat metric g and $\text{hol}(g) \subseteq G_2$, to be equal to G_2 is that the fundamental group of J is finite rather than trivial. We will refer to manifolds which support a single covariantly constant spinor, but have holonomy group $\text{SU}(3) \rtimes Z_2$ as barely G_2 -manifolds. The only non-trivial Betti numbers of a G_2 -manifold are b^2 and b^3 as $b^1 = 0$, and the moduli space of Ricci-flat metrics for a G_2 -manifold has real dimension b_3 . For G_2 -manifolds, calibrated submanifolds can have dimension three (associative cycles), in which case they are calibrated by Φ , oder dimension four (co-associative cycles), in which case they are calibrated by $*\Phi$.

Let us now discuss compactification of heterotic $E_8 \times E_8$ strings on G_2 -manifolds. Earlier studies of this in the context of resolutions of Joyce orbifolds T^7/Γ have appeared in [22, 42, 43]. The most in depth analysis of the conditions that such a compactification needs to satisfy were derived in [44] and are given in terms of a *heterotic G_2 -system*. This is comprised of a G_2 -holonomy manifold and its tangent bundle (J, TJ) as well as an $E_8 \times E_8$ -vector bundle with connection (V, A) . The curvatures R and F of TJ and V satisfy

$$R \wedge *\Phi = 0, \quad F \wedge *\Phi = 0. \quad (5.2)$$

Furthermore there can be NS5-branes wrapped on associative three-cycles, subject to the anomaly condition

$$\frac{\alpha'}{4} (\text{Tr } F \wedge F - \text{Tr } R \wedge R) = dH, \quad (5.3)$$

where $dH = [\text{NS5}]$ the Poincaré dual of the homology class corresponding to associatives wrapped by NS5-branes.

5.2 GCS Spin(7)-manifolds and M-theory/heterotic duality

Up to this point, our construction of GCS Spin(7)-manifold was largely motivated for constructions of new geometries. An alternative motivation is to realize that these give rise to M-theory duals of heterotic strings on TCS G_2 -manifolds. A review of the TCS construction is given in appendix A. Let us hence consider heterotic strings compactified on a manifold of G_2 holonomy J , together with an appropriately chosen vector bundle, giving an effective 3d theory with $\mathcal{N} = 1$ supersymmetry. Fiberwise duality to M-theory is possible if J admits a calibrated fibration by three-tori T^3 , which are replaced by K3 surfaces in the dual compactification of M-theory. As 3d compactifications of M-theory with $\mathcal{N} = 1$ supersymmetry are found upon compactification on manifolds of Spin(7) holonomy, we expect the resulting geometry to be in this class.

We already discussed the 7d duality between M/K3 and Het/ T^3 in section 3.1. We now apply this fiberwise to construct dual pairs of TCS and GCS constructions, in cases when there is a K3-fibration on the GCS Spin(7)-manifold and dually on the heterotic side there is a T^3 -fibered TCS G_2 -manifold. In this section we make some initial observations about the general features of these compactifications.

If J is a TCS G_2 -manifold, mirror symmetry for Type II strings implies the existence of a calibrated T^3 -fibration, which realizes a mirror map in the spirit of Strominger-Yau-Zaslow (SYZ) [45] via three T-dualities [13]. Such a fibration exists if one of the two building blocks, say X_+ , used in the TCS-construction of J is fibered by K3 surfaces S_+ , which in turn admit an fibration by an elliptic curve $\mathbb{E}$

$$\mathbb{E} \hookrightarrow S_+ \rightarrow \mathbb{P}^1. \quad (5.4)$$

This implies that X_+ admits an elliptic fibration with fiber $\mathbb{E}$ as well, and the T^3 -fibers of J restricted to $\mathbb{E} \times \mathbb{S}_{e,+}^1$ on $X_+ \times \mathbb{S}_{e,+}^1$. The Donaldson matching (A.1) then maps $\mathbb{E}$ to a sLag fiber of the K3-surface S_- , and $\mathbb{S}_{e,+}^1$ to $\mathbb{S}_{b,-}^1$. Going away from the neck region of X_- , $\mathbb{E} \times \mathbb{S}_{b,-}^1$ becomes the SYZ fiber of X_- .

Next let us consider how this maps under Heterotic/M-theory duality:

- X_+ : we replace the T^3 -fibration given by $\mathbb{E} \times \mathbb{S}_{e,+}^1$ on $X_+ \times \mathbb{S}_{e,+}^1$, with a K3 surface S_+ . Then only two out of the three forms of the hyper-Kähler structure of S_+ have a non-trivial fibration over the base. In particular, we may choose to described the resulting K3 fibration algebraically and end up replacing the product of the acyl Calabi-Yau three-fold X_+ with $\mathbb{S}_{e,+}^1$ by an acyl Calabi-Yau four-fold Z_+ .
- X_- : here, the SYZ-fiber of X_- is fibered such that application of Heterotic/M-theory duality leads to a K3-fibration in which all three (1,1)-forms of the hyper-Kähler structure have a non-trivial variation over the base. As in the 4d $\mathcal{N} = 1$ duality between heterotic strings and M-theory, we replace a Calabi-Yau three-fold by a G_2 -manifold. In the present setup, this means replacing an acyl Calabi-Yau three-fold X_- with an acyl G_2 -manifold Z_- . In the cylinder region of J , where J is diffeomorphic to a product $S \times \mathbb{S}^1 \times \mathbb{S}^1 \times I$ for a K3-surface S with a calibrated T^2 -fibration,² we simply find $X_3 \times \mathbb{S}^1 \times I$ by application of Heterotic/M-theory duality.

We hence find the statement that M-theory duals of heterotic strings on TCS G_2 -manifolds are compactified on Spin(7)-manifolds, which allows a construction as proposed in our GCS-construction: an acyl Calabi-Yau four-fold Z_+ and an acyl G_2 -manifold Z_- times an $\mathbb{S}^1$ are glued along a cylinder times a Calabi-Yau three-fold.

In our discussion, we have so far ignored that the heterotic compactification comes equipped with a vector bundle on J , and we need to identify what this is mapped to on the dual M-theory geometry. For X_+ , where only $\mathbb{E}$ varies non-trivially, replacing $X_+ \times \mathbb{S}_{e,+}^1$ by Z_+ is simply a non-compact version of the 3d duality between heterotic strings and M-theory on Calabi-Yau four-folds. We can hence apply the usual logic of how bundles constructed from spectral covers are translated in F-theory [46]. Restricting J to $X_- \times \mathbb{S}_{e,-}^1$, holomorphic vector bundles on a (non-compact) Calabi-Yau three-fold get translated to the geometry of a (non-compact) G_2 -manifold, similar to discussed in [12]. Gluing Z_+ and $Z_- \times \mathbb{S}^1$ to a compact Spin(7)-manifold Z then implies that the geometry of Z determines a heterotic G_2 -system on the G_2 -manifold J . It is hence tempting to construct such vector bundles on twisted connected sum G_2 -manifolds by appropriately gluing

²Depending on a choice of complex structure, this is an elliptic or sLag fibration.

holomorphic vector bundles on the building blocks $X_{\pm}$, see [47] for a concrete realization of this idea. In section 5.5 we will use this logic to count the number of bundle moduli on a TCS G_2 -manifold and confirm a matching with the degrees of freedom of the dual M-theory compactification on a GCS Spin(7)-manifold.

5.3 Duality for the building blocks

Before heading to the full construction of M-theory/Spin(7) and Heterotic/ G_2 we first explore the dual pairs that are relevant for each building block. Recall:

- GCS Spin(7)-manifold Z is built out of an acyl CY₄ building block Z_- and an acyl G_2 -building block Z_+ .
- TCS G_2 -manifold J is build out of two acyl CY₃, $X_{\pm}$.

We will now construct dual pairs for CY₄ and G_2 and correspondingly on the heterotic side CY₃ in sections 5.3.2 and 5.3.3. These will then be used as building blocks to construct the GCS/TCS dual pairs. As a preparation we first need to construct pairs of G_2 and CY₃ and duals for M/het duality.

5.3.1 Geometric preparation: dual pairs of G_2 and CY₃

In this section we discuss a well-suited example for building the G_2 building block of the GCS-manifold, by consider M-theory on G_2 -manifold and the heterotic on CY₃ compactification. Similar to the strategy used in [16], we can find such a pair by forming an appropriate quotient of a dual pair of 4d $\mathcal{N} = 2$ compactifications, which are

$$\text{M}/X_{43,43} \times S^1 \longleftrightarrow \text{Het}/\text{K3} \times T_h^2. \quad (5.5)$$

On the heterotic side we choose a trivial vector bundle on the K3 surface and break $E_8 \times E_8$ to $U(1)^{16}$ by introducing Wilson lines on an the T_h^2 factor. Tadpole cancellation then necessitates the introduction of 24 NS5-branes wrapped on T_h^2 . Such a compactification of the heterotic string theory on $\text{K3} \times T_h^2$ is dual to M-theory compactified on $X_{43,43} \times S^1$, where $X_{43,43}$ is a CY₃-fold, chosen to be a fibration of K3-surfaces with polarizing lattice $N = U \oplus (-E_8) \times (-E_8)$ over $\mathbb{P}^1$. The resulting CY₃ is constructed as a hypersurface in a toric variety A associated to the pair of reflexive polytopes

$$\Delta^\circ = \begin{pmatrix} -1 & 0 & 2 & 2 & 2 \\ 0 & -1 & 3 & 3 & 3 \\ 0 & 0 & -12 & 0 & 12 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix} \quad \Delta = \begin{pmatrix} -2 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -6 & 6 & -6 \end{pmatrix} \quad (5.6)$$

and has $h^{1,1}(X_{43,43}) = h^{2,1}(X_{43,43}) = 43$. Besides the 4d $\mathcal{N} = 2$ supergravity multiplet (which also contains a graviphoton), both of these compactifications have 43 $\mathcal{N} = 2$ vector multiplets, 43 hypermultiplets and one universal hypermultiplet.

We can now quotient both theories in (5.5) by an involution to find dual compactifications of M-theory on a G_2 -manifold and heterotic strings on a Calabi-Yau three-fold with

a vector bundle. This truncates the light excitations to be described by a theory with 20 $\mathcal{N} = 1$ U(1) vectors and 67 $\mathcal{N} = 1$ chiral multiplets.

On the type IIA side we act with a free involution $\mathbb{Z}_2^o$ to produce a G_2 -manifold³ M as

$$M = (X_{43,43} \times \mathbb{S}_u^1) / \mathbb{Z}_2^o, \quad (5.7)$$

where $\mathbb{Z}_2^o$ sends the coordinate $u \rightarrow -u$ and acts on $X_{43,43}$ as an antiholomorphic involution. This anti-holomorphic involution is chosen such that it sends all of the homogeneous coordinates of the ambient toric space A for $X_{43,43}$ to their complex conjugate and furthermore swaps any homogeneous coordinate of A associated with the lattice point (a, b, c, d) on Δ° with the homogeneous coordinate associated with the lattice point $(a, b, -c, d)$ on Δ° . Some thought reveals that this is a symmetry of A and that one can choose the hypersurface equation of $X_{43,43}$ such that $\mathbb{Z}_2^o$ becomes a symmetry of $X_{43,43}$ (without enforcing singularities) and acts as $\mathbb{Z}_2^o : \Omega \rightarrow \bar{\Omega}$, as appropriate for an antiholomorphic involution. There are 47 divisors on $X_{43,43}$, which descend from A by intersecting $X_{43,43}$ with the vanishing loci of the homogeneous coordinates of A , out of which 43 are linearly independent. There are 21 pairs of homogeneous coordinates $\{z_{i\pm}; i = 1 \cdots 21\}$ on which $\mathbb{Z}_2^o$ acts as

$$\mathbb{Z}_2^o : \quad z_{i\pm} \rightarrow \pm \bar{z}_{i\mp} \quad (5.8)$$

and the remaining 5 homogeneous coordinates $\{w_j; j = 0 \cdots 5\}$ are acted on as

$$\mathbb{Z}_2^o : \quad w_j \rightarrow \bar{w}_j. \quad (5.9)$$

Such an involution acts freely, because a fixed point would require that $z_{i\pm} = \pm \bar{z}_{i\mp}$, up to the $\mathbb{C}^*$ action on the homogeneous coordinates of A . For our choice of $z_{i\pm}$ the $\mathbb{C}^*$ actions imply that fixed points require $z_{i+} = z_{i-} = 0$. For our choice, such intersections vanish for any pair $z_{i\pm}$, so that there are no fixed points. Out of the four linear relations among the divisors of A , only one involves the 42 divisors associated with $\{z_{i\pm}; i = 1 \cdots 21\}$ and the remaining three involve the $\{w_j; j = 0 \cdots 5\}$. Hence there are precisely 20 even combinations of such divisors and we find that⁴

$$h_+^{1,1}(X_{43,43}) = 20 \quad h_-^{1,1}(X_{43,43}) = 23. \quad (5.10)$$

As $\mathbb{Z}_2^o$ is an antiholomorphic involution, we also have that $b_\pm^3(X_{43,43}) = 44$, from which we find

$$\begin{aligned} b_2(M) &= h_+^{1,1}(X_3) &= 20 \\ b_3(M) &= h_-^{1,1}(X_3) + b_+^3(X_3) &= 67, \end{aligned} \quad (5.11)$$

which gives the number of $\mathcal{N} = 1$ U(1) vector and chiral multiplets for compactification of M-Theory on M .

³As the involution we consider acts freely, this produces only a ‘barely’ G_2 -manifold, the holonomy group of which is only the subgroup $SU(3) \rtimes \mathbb{Z}_2$ of G_2 .

⁴Another way to see this is as follows: the three-fold $X_{43,43}$ is fibered by elliptic K3 surfaces with two II^* over $\mathbb{P}^1$ and this involution acts as the antipodal map of the base of this elliptic fibration (another way to see why it acts freely) in particular identifying the two II^* fibers. It furthermore acts (non-freely) on the $\mathbb{P}^1$ base of the K3 fibration and identifies the 24 reducible K3 fibers on X_3 pairwise. This reproduces why there are $8 + 12 = 20$ even classes in $b^2(X_3)$.

On the dual heterotic side, the corresponding involution $\mathbb{Z}_2^h$ acts as a combination of $z \rightarrow -z$ (on the T_h^2 factor) together with the Enriques involution⁵ on the K3 surface to produce a Calabi-Yau three-fold

$$X_{11,11} = (\text{K3} \times T_h^2) / \mathbb{Z}_2^h \quad (5.12)$$

with Hodge numbers $h^{1,1}(X_{11,11}) = h^{2,1}(X_{11,11}) = 11$. Furthermore, it acts on the bundle data by twisting the two Wilson lines such as to break $E_8 \times E_8$ to $U(1)^8$. We can think about the surviving bundle data as an extension of a line bundle associated with a Wilson line breaking $E_8 \times E_8$ to $U(1)^8$, and there are eight complex degrees of freedom specifying this bundle.

The orbifold furthermore identifies the 24 NS5-branes on T_h^2 pairwise,⁶ so that the number of 4d $\mathcal{N} = 1$ $U(1)$ vector and chiral multiplets on the heterotic side is given by

$$\begin{aligned} n_v &= 8 + 12 &= 20 \\ n_c &= 1 + 2 \cdot 11 + 3 \cdot 12 + 8 &= 67 \end{aligned} \quad (5.13)$$

in perfect agreement with the dual M-theory on the G_2 -manifold M . With this preparation in hand we can now study the duality for the building blocks.

5.3.2 The acyl G_2 -manifold and its dual acyl CY₃

We can turn $X_{11,11}$ and M in the last subsection into a dual pair of an acyl Calabi-Yau three-fold and an acyl G_2 -manifold by cutting each of them in half. Let us start with M in (5.7). By cutting along the middle of the interval remaining of the circle $\mathbb{S}_u^1$, M is turned into an acyl (barely) G_2 -manifold Z_- which we may represent by

$$Z_- = (X_{43,43} \times \mathbb{R}_u) / \mathbb{Z}_2^o, \quad (5.14)$$

where $\mathbb{Z}_2^o$ acts as before on $X_{43,43}$ and as $u \rightarrow -u$ on $\mathbb{R}_u$. The asymptotic Calabi-Yau three-fold of Z_- is $X_{43,43}$ and there are $b_+^2(X_{43,43}) = 20$ classes in the image of the restriction map from $H^2(Z_-, \mathbb{Q})$ to $H^2(X_{43,43}, \mathbb{Q})$, as well as $b_+^3(X_{43,43}) = 44$ classes in the image of the restriction map from $H^3(Z_-, \mathbb{Q})$ to $H^3(X_{43,43}, \mathbb{Q})$. There are no elements in the kernels of these maps.

On the heterotic side, Calabi-Yau three-fold $X_{11,11}$ (5.12) carries a natural (almost trivial) K3 fibration

$$\text{K3} \hookrightarrow X_{11,11} \rightarrow \mathbb{P}_h^1, \quad (5.15)$$

where the base $\mathbb{P}_h^1$ is found as the quotient of T_h^2 . Over four points on this $\mathbb{P}_h^1$, the K3 fiber is truncated to an Enriques surface, but is constant otherwise. Note that this way of thinking reproduces the fact that the Euler characteristic of $X_{11,11}$ vanishes from

$$\chi(X_{11,11}) = 24 \cdot (2 - 4) + 4 \cdot 12 = 0, \quad (5.16)$$

⁵The Enriques involution is the unique fixed-point free non-symplectic involution on a K3 surface and is identified by the invariants $(r, a, \delta) = (10, 10, 0)$ in the classification of [48].

⁶Equivalently, one can compute that $ch_2(X_{11,11}) = 12[T_h^2]$.

where we have excised the four fibers which are Enriques surfaces from the base $\mathbb{P}_h^1$ in the first term and added them back in for the second. If we now cut $\mathbb{P}_h^1$ into two halves, each of which is $\mathbb{C}$ and contains two of the Enriques fibers over it, we produce an acyl Calabi-Yau three-fold X_- as the TCS G_2 building block

$$\begin{aligned} K(X_-) &= \emptyset \\ N(X_-) &= \mathrm{U}(2) \oplus (-E_8(2)), \end{aligned} \tag{5.17}$$

where $L(2)$ denotes a lattice L with its inner form rescaled by a factor of two. The expression for $N(X_-)$ is nothing but the even sublattice of $H^2(K3, \mathbb{Z})$ under the Enriques involution. Furthermore, X_- may be compactified to a compact building block $\tilde{X}_-$ in the sense of [5] by gluing in a single K3 fiber. It now follows from a computation as above that $\chi(\tilde{X}_-) = 24$, which together with $b^2(\tilde{X}_-) = 11$ (and $b^1(\tilde{X}_-) = 0$ implies that $b^3(\tilde{X}_-) = 0$). This is all of the data we will need in using X_- as a building block for a G_2 -manifold.

Note that we have indeed decomposed M and $X_{11,11}$ by cutting along the same S^1 . This becomes particularly clear, by noting that these dual compactifications descend from a compactification to five dimension, in which M-theory is put on $X_{43,43}$ and heterotic string on $K3 \times S^1$. The dual models we have constructed are then found by compactifying both side on a further S_u^1 and quotienting by $\mathbb{Z}_2$. It is the quotient of S_u^1 , which becomes one of the two circles on T_h^2 , which is cut in half to produce Z_- and X_- , respectively.

5.3.3 The acyl CY_4 and its dual acyl CY_3

We now construct a dual pair of an acyl Calabi-Yau four-fold Z_+ (as a building block for the $\mathrm{Spin}(7)$ GCS) and the heterotic dual acyl Calabi-Yau three-fold X_+ (as a building block for the G_2 TCS). As we want Z_+ to asymptote to $X_{43,43}$ so that we can glue Z_+ with $Z_- \times S^1$, we construct

$$Z_+ = \tilde{Z}_+ \setminus X_{43,43}^0, \quad X_{43,43} \hookrightarrow \tilde{Z}_+ \rightarrow \mathbb{P}^1, \tag{5.18}$$

so that the algebraic three-fold $\tilde{Z}_+$ satisfies

$$c_1(\tilde{Z}_+) = [X_{43,43}^0]. \tag{5.19}$$

Note that we may also construct Z_+ by cutting a Calabi-Yau four-fold X_4 , which is a fibration of $X_{43,43}$ over $\mathbb{P}^1$ in two halves. This four-fold is a toric hypersurface defined by the pair

$$\begin{aligned} \Delta^\circ &= \begin{pmatrix} -1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & -1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & -12 & -12 & 12 & 12 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{pmatrix} \\ \Delta &= \begin{pmatrix} -2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -6 & 6 & 6 & -6 \\ 0 & 0 & 6 & -6 & -6 & 6 & 6 \end{pmatrix} \end{aligned} \tag{5.20}$$

of reflexive polytopes. Its Hodge numbers are

$$h^{1,1}(X_4) = 68, \quad h^{3,1}(X_4) = 292, \quad h^{2,1}(X_4) = 0, \quad h^{2,2}(X_4) = 1484, \quad (5.21)$$

so that $\chi(X_4) = 2208$. As a generalization of [9] to four-folds, $\tilde{Z}_+$ can also be constructed from one of the two isomorphic tops in Δ° corresponding to the fibration by $X_{43,43}$. Either way, it follows that the Euler characteristic of Z_+ is $\chi(Z_+) = 1104$, half of the Euler characteristic of X_4 .

The restriction maps $\beta^i : H^i(Z_+, \mathbb{Q}) \rightarrow H^i(X_{43,43}^0 \times \mathbb{S}^1, \mathbb{Q})$ are fairly easy because all of the divisors and none of the three-cycle on $X_{43,43}$ descend from Z_+ . Furthermore, there are 12 divisor classes on Z_+ associated with reducible fibers of the fibration by $X_{43,43}$. Hence we conclude

$$\begin{aligned} |\mathrm{im}(\beta^2)| &= 43, & |\mathrm{ker}(\beta^2)| &= 12, & |\mathrm{coker}(\beta^2)| &= 0, \\ |\mathrm{im}(\beta^3)| &= 0, & |\mathrm{ker}(\beta^3)| &= 0, & |\mathrm{coker}(\beta^3)| &= 131. \end{aligned} \quad (5.22)$$

This will be all the data from Z_+ we need in the following.

Let us now find the dual geometry for the heterotic string. As Z_+ and X_4 are fibered by elliptic K3 surfaces, this is fairly straightforward, as we simply need to apply the known rules of heterotic-F-theory duality. In particular, Z_+ and X_4 are fibered by elliptic K3 surfaces with Picard lattice $U \oplus (-E_8)^2$, so that the dual heterotic theory has no $E_8 \times E_8$ vector bundles turned on. Furthermore, Z_+ and X_4 are elliptic fibrations over (blow-ups of) $\mathbb{C} \times \mathbb{P}^1 \times \mathbb{P}^1$. These blowups capture the presence of the NS5-branes needed on the heterotic side to satisfy the Bianchi identity. Hence the dual heterotic geometry dual to is X_4 given by an elliptically fibered Calabi-Yau $X_{3,243}$ with base $\mathbb{P}^1 \times \mathbb{P}^1$ times a circle $\mathbb{S}^1$. The three-fold $X_{3,243}$ is constructed from a pair of reflexive polytopes with vertices

$$\Delta^\circ = \begin{pmatrix} -1 & 0 & 2 & 2 & 2 & 2 \\ 0 & -1 & 3 & 3 & 3 & 3 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad \Delta = \begin{pmatrix} -2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 6 & 6 & -6 & -6 & 0 & 6 \\ 0 & 0 & -6 & 0 & -6 & 0 & 6 \end{pmatrix}. \quad (5.23)$$

and it follows that $X_{3,243}$ has Hodge numbers $(h^{1,1}, h^{2,1}) = (3, 243)$. The dual acyl CY building block X_+ of the acyl Calabi-Yau four-fold Z_+ is constructed using by cutting $X_{3,243}$ along an S^1 in the base of its K3 fibration. Equivalently, we may glue in a K3 fiber to compactify X_+ to a building block $\tilde{X}_+$ which can be constructed using the methods of [9] from a pair of projecting tops

$$\diamond^\circ = \begin{pmatrix} -1 & 0 & 2 & 2 & 2 \\ 0 & -1 & 3 & 3 & 3 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \diamond = \begin{pmatrix} -2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 0 & 6 & 6 & -6 & -6 \\ 0 & 0 & -6 & 0 & -6 \end{pmatrix}. \quad (5.24)$$

It follows that

$$\begin{aligned} N(X_+) &= U \\ K(X_+) &= \emptyset \\ h^{2,1}(\tilde{X}_+) &= 112. \end{aligned} \quad (5.25)$$

Note that there are Wilson lines breaking $E_8 \times E_8$ to $U(1)^{16}$ on the product $\mathbb{S}^1$ on the heterotic side, which are dual to the volumes of the divisors associated with the $(-E_8)^{\oplus 2}$ in the Picard lattice of the K3 fibers of X_4 .

5.4 First example of dual pairs: new Spin(7)- and its dual G_2 -manifold

Above we have constructed a pair of an acyl Calabi-Yau four-fold Z_+ and an acyl G_2 -manifold Z_- which both asymptote to (a cylinder/an interval times) the same Calabi-Yau three-fold $X_{43,43}$, together with the two dual acyl Calabi-Yau three-folds X_+ and X_- together with the corresponding bundle data on the heterotic side. We will now glue Z_+ with $Z_- \times \mathbb{S}^1$ to a GCS Spin(7)-manifold Z and $X_{\pm} \times \mathbb{S}^1$ to a TCS G_2 -manifold J and verify that the light fields in the effective field theories indeed agree.

5.4.1 The M-theory on GCS Spin(7)-manifold

As detailed in section 4, the Spin(7)-manifold Z is constructed as

$$Z = Z_+ \cup (Z_- \times S^1), \quad (5.26)$$

where $Z_+ \cap (Z_- \times S^1) = X_3 \times S^1 =$ and $X_3 = X_{43,43}$. It immediately follows that $\chi(Z) = \chi(Z_+) = 1104$. The remaining Betti numbers are found from the Mayer-Vietoris sequence. First,

$$b^2(Z) = |\ker \gamma^2| + |\operatorname{coker} \gamma^1| = (20 + 12) + 0 = 32, \quad (5.27)$$

where we have used that 20 classes in $h^{1,1}(X_{43,43})$ are in the common image of the restriction maps $\gamma^2 : H^2(Z_+) \oplus H^2(Z_- \times S^1) \rightarrow H^2(X_{43,43} \times S^1)$ and that the kernel of $\beta_+^2 : H^2(Z_+) \rightarrow H^2(X_{43,43})$ is 12-dimensional. Furthermore,

$$b^3(Z) = |\ker \gamma^3| + |\operatorname{coker} \gamma^2| = 0, \quad (5.28)$$

as all of the classes in $h^{1,1}(X_{43,43})$ are in the image of β_+^2 , no class in $b^3(X_{43,43})$ is in the image of β_+^3 and the restriction maps $\beta_{\pm}^3$ have no kernels.

We can now use (2.9) to find

$$b_-^4(Z) = 327, \quad (5.29)$$

and it follows that there are $n_c = 360$ real degrees of freedom originating from the metric and three-form for compactifications of M-theory on Z . Out of these, 32 sit in 3d $\mathcal{N} = 1$ vector multiplets. Furthermore, a consistent compactification requires the introduction of $1104/24 = 46$ M2 branes. Each of these contributes a further 8 real moduli from moving on Z .

Alternatively, this Spin(7)-manifold can also be obtained as a free quotient of the Calabi-Yau four-fold X_4 specified as a toric hypersurface by (5.20).

5.4.2 The dual heterotic model on TCS G_2 -manifold

The dual heterotic side has various components: geometry, vector bundle and NS5-branes. We will determine all the dual data, and compare with the spectrum obtained in the M-theory compactification — and find agreement.

Geometry. M-theory on Z is dual to heterotic on a G_2 -manifold J which can be obtained by fiber-wise duality as

$$J = (X_+ \times \mathbb{S}_{e,+}^1) \cup (X_- \times \mathbb{S}_{e,-}^1), \quad (5.30)$$

where $X_{\pm}$ are the acyl Calabi-Yau three-folds introduced in sections 5.3.3 and 5.3.2 above. Recall that the lattices $N(X_{\pm})$ are

$$\begin{aligned} N(X_+) &= U \\ N(X_-) &= U(2) \oplus E_8(2), \end{aligned} \quad (5.31)$$

and there is a matching such that $N(X_+) \cap N(X_-) = \emptyset$. Together with $K(X_{\pm})$ and $h^{2,1}(X_{\pm})$ we hence immediately find [6]

$$\begin{aligned} b_2(J) &= K(X_+) + K(X_-) = 0 \\ b_2(J) + b_3(J) &= 23 + 2(K(X_+) + K(X_-)) + 2(h^{2,1}(X_+) + h^{2,1}(X_-)) = 247, \end{aligned} \quad (5.32)$$

giving $b_3(J) = 247$. We hence find 247 real degrees of freedom from the metric of a heterotic compactification on J and no moduli associated with the B -field.

Bundle data. Recall from section 5.3.1 that the heterotic compactification on $X_{11,11} = X_- \cup X_+$ dual to M-theory on $M = Z_- \cup Z_+$ has eight complex bundle degrees of freedom, associated with a bundle which is an extension of appropriate Wilson lines on the double cover of $X_{11,11}$. The origin of the corresponding degrees of freedom in the dual M-theory suggests to associate 4 complex bundle degrees of freedom with each of the S^1 s in the $K3 \times T_h^2$ double cover of $X_{11,11}$. For X_- , one of these two S^1 s has disappeared, so that we are left with 8 real bundle degrees of freedom there. In particular, these give the non-trivial holonomies in $E_8 \times E_8$ along $\mathbb{S}_{b,-}^1$ in the base $\mathbb{C}$ of the K3 fibration on X_- . In the TCS G_2 -manifold, these get mapped to Wilson lines on $\mathbb{S}_{e,-}^1$. Note that this is precisely what is expected for the duality between M-theory on the Calabi-Yau four-fold X_4 and heterotic strings on $X_{3,243} \times \mathbb{S}_e^1$: the non-zero volumes of the divisors on Z_+ associated with the $(-E_8)^{\oplus 2}$ in the Picard lattice of the K3 fibers of Z_+ are mapped to Wilson lines on $\mathbb{S}_{e,+}^1$ on the heterotic side. We hence conclude that J carries a vector bundle V inherited from a bundle on X_- which breaks $E_8 \times E_8 \rightarrow U(1)^8$ and has eight real moduli. The data of this bundle is captured by the volumes of the curves in $(-E_8^{\oplus 2})$ of the K3 fibers of $Z_{\pm}$.

NS5-branes. Furthermore, our compactifications of the heterotic string require the addition of NS5-branes for consistency as the gauge bundle V does not have a second Chern character. We will infer the NS5-branes which we need to include by exploiting piecewise duality. In particular, we can think about a representative of the first Pontrjagin class of J as being glued from representatives of $c_2(\tilde{X}_{\pm})$.

The second Chern class of $\tilde{X}_+$ is

$$\begin{aligned} c_2(\tilde{X}_+) &= 46H\hat{H}_+ + 46H_+\sigma_+ + 23\hat{H}_+\sigma_+ + 11\sigma_+^2 \\ &= 46H_+\hat{H}_+ + 24H_+\sigma_+ + 12\hat{H}_+\sigma_+, \end{aligned} \quad (5.33)$$

where $\hat{H}_+$ is the divisor class corresponding to fixing a point on the base of the K3 fibration on $\tilde{X}_+$ and H_+ is the class corresponding to fixing a point on the other $\mathbb{P}^1$ in the $\mathbb{P}^1 \times \hat{P}^1$

base of the elliptic fibration on $\tilde{X}_+$. The divisor class σ_+ is represented by the section of this elliptic fibration and these classes satisfy the relation $\sigma_+(\sigma_+ + 2H_+ + \hat{H}_+) = 0$. Note that $c_1(\tilde{X}_+) = [\hat{H}_+]$.

The 46 NS5-branes on the curve $H\hat{H}$ are wrapped on the elliptic fiber of X_+ times the auxiliary $\mathbb{S}_{+,e}^1$ multiplying X_+ in the construction of J . As such, they are wrapped precisely on the conjectured fiber of the T^3 fibration of J used for the duality and we can associated them with the 46 M2 branes present on the M-theory side. Let us confirm this by counting the number of moduli. Each of these NS5-branes has 4 real degrees of freedom by displacing the elliptic fiber on X_+ . Furthermore, there is one real degree of freedom from the scalar ϕ in the tensor multiplet (the position along the 11th direction in heterotic M-theory) and three real degrees of freedom from the self-dual two-form B on the $b^1(T^3) = b^2(T^3) = 3$ cycles of the T^3 they are wrapped on. This makes 8 degrees of freedom matching the counting for the M2 branes.

The 12 NS5-branes on $\hat{H}_+\sigma_+$ are points on the $\mathbb{P}^1$ base of the K3 fibration of $\tilde{X}_+$ so that they are wrapped on a three-manifold with the topology $S^2 \times S^1$ in J . Within X_+ , each such NS5-brane has 2 real moduli associated with displacement,⁷ one modulus associated with the worldvolume scalar ϕ , and one modulus from the self-dual two-form B . At least in the Kovalev limit, we are hence led to associate 4 real moduli with each such NS5-brane.

Finally, there are the 24 NS5-branes in the class $H_+\sigma_+$ of $\tilde{X}_+ \times \mathbb{S}_{e,+}^1$. On $X_+ = \tilde{X}_+ \setminus S_+^0$, they are wrapped on the whole of the open base $\mathbb{C} \times \mathbb{S}_{e,+}^1$ of the K3 fibration on $X_+ \times \mathbb{S}_{e,+}^1$. Similarly, there are 12 NS5-branes wrapped on a double cover the open base $\mathbb{C} \times \mathbb{S}_{e,-}^1$ of the K3 fibration on $X_- \times \mathbb{S}_{e,-}^1$ as we have seen in section 5.3.1. Note that this sector of NS5-branes becomes 24 copies of the T^2 factor times the interval in $(X_+ \times \mathbb{S}_{e,+}^1) \cap (X_- \times \mathbb{S}_{e,-}^1) = I \times T^2 \times K3$. On the G_2 -manifold, these NS5-branes can hence be joined to form 12 irreducible NS5-branes. This has several effects. First of all, the relative positions of the 24 branes on $X_+ \times \mathbb{S}_{+,e}^1$, which are points on the K3 fiber S_+ of $X_+ \times$, are pairwise fixed to be symmetric under the Enriques involution acting on S_- . Second, each such pair of NS5-branes only has two real moduli of deformation. Because they are wrapped on the $\mathbb{C}$ -base of the K3 fibration on X_+ and the elliptic fibration of X_+ is non-trivial there, they cannot be displaced in the direction of the elliptic curve of X_+ . Third, the two branches of each pair of NS5-branes are swapped when encircling two special points in the base of the K3 fibration on X_- , so that we should think of each such pair of NS5-branes as a single NS5-brane wrapped on a three-manifold L which is a double cover of S^3 branched along two unlinked S^1 s. We conclude that there are 12 NS5-branes wrapped on three-manifolds L inside J with two real deformations each. It is not hard to see that $b^1(L) = b^2(L) = 1$, so that each of these NS5-branes is associated with 4 real degrees of freedom: 2 real moduli of displacement together with 2 real moduli from ϕ and B adding up to 4 real moduli each.

Counting of degrees of freedom. We are now ready to count the number of degrees of freedom on the heterotic side. To do so, we can neglect the 46 NS5-branes wrapped on the T^3 fiber of J , as these are mapped to M2 branes in M-theory and we have already matched

⁷The associated holomorphic curves which lift to associatives are fixed to lie on the section of the elliptic fibration of X_+ .

their degrees of freedom. Summarizing the different contributions discussed above, the remaining light fields for heterotic strings on J are hence the 247 moduli of the Ricci-flat metric on J , the 8 real degrees of freedom in the bundle V , 8 $U(1)$ vectors which remain massless in $E_8 \times E_8$, 24 NS5-branes with 4 real moduli each, and the dilaton, giving a total of 360 real degrees of freedom. Out of these, there are 32 3d $\mathcal{N} = 1$ vector multiplets, originating from the 8 surviving $U(1)$ s together with the 24 NS5-branes wrapped on three-cycles with $b^1 = 1$. We have hence verified that the number of light fields between heterotic string theory on J and M-theory on Z precisely agrees.

5.5 Second example of dual pairs

Let us now make consider a variation of the previous example and work out the topology of the $Spin(7)$ associated with putting an $E_8 \times E_8$ vector bundle on J . To describe such a situation, we intend to replace Z_- and Z_+ by an acyl G_2 -manifolds and acyl CY_4 originating from a K3 fibration with Picard lattice U instead of $U \oplus (-E_8)^{\oplus 2}$.

5.5.1 The acyl Calabi-Yau four-fold Z_+

Following the usual rules of heterotic-M-theory (F-theory) duality Z_+ is now found as one half of a compact Calabi-Yau four-fold X_4 described by a generic Weierstrass elliptic fibration over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $Z_+ \cap (Z_- \times \mathbb{S}^1) = X_{3,243} \times \mathbb{S}^1 \times I$. Here, $X_{3,243}$ has already appeared in section 5.3.3 and X_4 is found from the pair of reflexive polytope:

$$\begin{aligned} \Delta^\circ &= \begin{pmatrix} -1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & -1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix} \\ \Delta &= \begin{pmatrix} -2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -6 & -6 & -6 & -6 & 6 & 6 & 6 & 6 \\ 0 & 0 & -6 & -6 & 6 & 6 & -6 & -6 & 6 & 6 \\ 0 & 0 & -6 & 6 & -6 & 6 & -6 & 6 & -6 & 6 \end{pmatrix}, \end{aligned} \quad (5.34)$$

which implies

$$h^{1,1}(X_4) = 4, \quad h^{3,1}(X_4) = 2916, \quad h^{2,1}(X_4) = 0, \quad h^{2,2}(X_4) = 11724, \quad (5.35)$$

and $\chi(X_4) = 17568$. We can write $Z_+ \cup Z_+ = X_4$ with $Z_+ \cap Z_+ = X_{3,243} \times \mathbb{S}^1 \times I$ and the data of Z_+ relevant for its use in the construction of Z are

$$\begin{aligned} |\mathrm{im} \beta_+^2| &= 3, & |\mathrm{ker} \beta_+^2| &= 0 \\ |\mathrm{im} \beta_+^3| &= 0, & |\mathrm{ker} \beta_+^3| &= 0 \end{aligned} \quad (5.36)$$

and $\chi(Z_+) = 8784$.

5.5.2 The acyl G_2 -manifold Z_-

To construct Z_- , we first consider $M = Z_- \cup Z_- = X_{3,243} \times \mathbb{S}_u^1/\mathbb{Z}_2$. The three-fold $X_{3,243}$ was constructed as a toric hypersurface associated with the pair of reflexive polytopes shown in (5.23). Its homogeneous coordinates have the toric weights

$$\begin{array}{|c|c|c|c|c|c|c|} \hline y & x & w & z_1 & z_2 & \hat{z}_1 & \hat{z}_2 \\ \hline 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 6 & 4 & 0 & 1 & 1 & 0 & 0 \\ 6 & 4 & 0 & 0 & 0 & 1 & 1 \\ \hline \end{array} . \quad (5.37)$$

Similar to our procedure in section 5.3.1 we can introduce an antiholomorphic involution on $X_{3,243}$ by letting⁸

$$\begin{aligned} (y, x, w, \hat{z}_1, \hat{z}_2) &\rightarrow (\bar{y}, \bar{x}, \bar{w}, \hat{\bar{z}}_1, \hat{\bar{z}}_2) \\ (z_1, z_2) &\rightarrow (-\bar{z}_2, \bar{z}_1) . \end{aligned} \quad (5.38)$$

This involution acts as the antipodal map on the $\mathbb{P}^1$ with homogeneous coordinates $[z_1, z_2]$ and hence acts freely on $X_{3,243}$ such that

$$\begin{aligned} h_+^{1,1}(X_{3,243}) &= 0 & h_-^{1,1}(X_{3,243}) &= 3 \\ h_+^{2,1}(X_{3,243}) &= 244 & h_-^{2,1}(X_{3,243}) &= 244 \end{aligned} \quad (5.39)$$

The resulting barely G_2 -manifold M hence has Betti numbers

$$b^2(M) = 0, \quad b^3(M) = 247. \quad (5.40)$$

We can now form an acyl G_2 -manifold Z_- by cutting $\mathbb{S}_u^1$ in the middle, i.e. $M = Z_- \cup Z_-$ and $Z_- \cup Z_- = \mathbb{S}^1 \times X_{3,243}$. The restriction maps are

$$\begin{aligned} |\mathrm{im} \beta_-^2| &= 0, & |\mathrm{ker} \beta_-^2| &= 0 \\ |\mathrm{im} \beta_-^3| &= 244, & |\mathrm{ker} \beta_-^3| &= 0. \end{aligned} \quad (5.41)$$

5.5.3 M-theory spectrum on the GCS Spin(7)-manifold

From the acyl Calabi-Yau four-fold Z_+ and the acyl G_2 -manifold $Z_- \times \mathbb{S}$ we can now form a Spin(7)-manifold $Z = Z_+ \cup Z_- \times \mathbb{S}$. It follows from the restriction maps that

$$b^2(Z) = b^3(Z) = 0. \quad (5.42)$$

Furthermore $\chi(Z) = 8784$. Hence

$$b_-^4(Z) = 2919 \quad (5.43)$$

and compactifications of M-theory on Z have 2920 real degrees of freedom from the metric which all sit in real multiplets, as well $8784/24 = 366$ space-time filling M2-branes.

Again, this Spin(7)-manifold can also be obtained as a free quotient of the Calabi-Yau four-fold X_4 specified as a toric hypersurface by (5.34).

⁸In fact, $X_{43,43}$ is connected to $X_{3,243}$ by several singular transitions and the actions (5.8) and (5.9) imply the one given here.

5.5.4 The dual heterotic model

The dual heterotic compactification lives on the same G_2 -manifold J as discussed in section 5.4.2. Here, the metric contributes 247 real degrees of freedom. Now, however, we choose to switch on a vector bundle V completely breaking $E_8 \times E_8$, such that V is flat on the T^3 fibers of J for the duality to hold fiberwise. It now follows from the proposed duality that such a bundle must have

$$m_V = 2920 - 247 - 1 = 2672 \quad (5.44)$$

real moduli. Furthermore, the heterotic Bianchi identity in this context must force to include 366 NS5-branes wrapped on the T^3 fiber of J .

We can reproduce the number of bundle moduli by studying the moduli of holomorphic vector bundles on $X_{\pm}$ together with a gluing condition. For X_- , such bundles are inherited from $E_8 \times E_8$ vector bundles on K3 symmetric under the Enriques involution. From the point of view of the K3 fibration on X_- , the bundle data are constant over the base $\mathbb{C}$ of X_- . A holomorphic vector bundle on K3 has $4 \cdot 112$ real moduli (they all sit in hypermultiplets for heterotic string theory on K3) and the restriction to be invariant imposes a restriction fixing half, i.e. 224 real moduli of such a bundle.

For X_+ , the number of bundle moduli can be inferred as follows. Consider heterotic strings on $X_{3,243} = X_+ \cup X_-$. By a straightforward application of duality to F-theory (which is compactified on the manifold X_4 discussed in section 5.5.1) immediately gives that a generic vector $E_8 \times E_8$ model W , which is flat on the elliptic fiber of $X_{3,243}$, has $m_W = 5344$ real moduli [46, 49]. An $E_8 \times E_8$ vector bundle on $X_+ \cap X_- = I \times \mathbb{S}^1 \times K3$ has $4 \cdot 112$ real moduli, which gives the number of conditions when matching two generic $E_8 \times E_8$ bundles $W_{\pm}$ on $X_{\pm}$. We hence expect

$$m_{W_+} + m_{W_-} - 448 = m_V. \quad (5.45)$$

As furthermore $m_{W_+} = m_{W_-}$, we find that $m_{W_+} = 2896$.

On the G_2 -manifold J , we can hence construct a suitable vector bundle V by letting $V|_{X_+ \times \mathbb{S}^1_{e,+}} = W_+$. Such a bundle restricts to a vector bundle on S_+ which then has to be appropriately restricted to match $V|_{X_- \times \mathbb{S}^1_{e,-}}$. As we have seen V_{S_-} is not a generic vector bundle on a K3 surface, but there are 224 conditions arising at it must be symmetric under the Enriques involution. Once this condition is met, V_{S_-} uniquely defines $V|_{X_- \times \mathbb{S}^1_{e,-}}$. As we have seen, the Enriques involution gives 224 real restrictions, so that we conclude that $m_V = 2896 - 224 = 2672$, which matches the expectation from the counting of degrees of freedom on the Spin(7)-manifold Z . Ignoring the 366 NS5-branes wrapped on the T^3 fiber of J , all of the moduli of this model sit in real multiplets, matching the result that $b^2(Z) = 0$ on the M-Theory side.

6 Discussion and outlook

We proposed a new construction of eight-manifolds with Spin(7)-holonomy, based on a generalized connected sum (GCS), where two building blocks — a Calabi-Yau four-fold and

a G_2 -holonomy manifold times S^1 — are glued together along an asymptotic region that is a Calabi-Yau three-fold times a cylinder. This construction is in part inspired by the recent twisted connected sum (TCS) realization of G_2 -holonomy manifolds, which has resulted in a multitude of new examples of compact G_2 -manifolds, and thereby a resurgence of interest in the string/M-theory context. Likewise the GCS-construction that we propose, provides an avenue to construct large classes of compact $\text{Spin}(7)$ -manifolds systematically. In particular the construction of acyl Calabi-Yau three-folds using semi-Fano three-folds [5] or tops [9] which is useful for TCS-constructions, has an obvious generalization to acyl Calabi-Yau four-folds, which is useful for expanding the set of examples of GCS-constructions.

We gave an alternative description of the GCS $\text{Spin}(7)$ -manifolds in terms of a quotient by an anti-holomorphic involution of a Calabi-Yau three-fold fibered Calabi-Yau four-fold in section 4.5, which is similar in spirit to the constructions of Joyce, however the key difference is that instead of gluing in ALE-spaces at point-like orbifold singularities, we glue in G_2 -manifolds with suitable asymptotics.

There is a multitude of future directions to consider:

1. M/F-duality for $\text{Spin}(7)$: M-theory on $\text{Spin}(7)$ results 3d $\mathcal{N} = 1$ theories, and we have seen that there are subsectors of the effective field theory, which in the limit of infinite asymptotic region enjoy enhanced supersymmetry, as discussed in section 4.4. It would clearly be very interesting to apply M/F-duality to the GCS $\text{Spin}(7)$ -manifolds and determine how the supersymmetry breaking in the 4d F-theory vacuum is realized [1]. Needless to say it is not difficult to construct GCS-examples that have an elliptic fibration and we will return to this shortly elsewhere. Again key for this will be also to understand the fluxes in the GCS $\text{Spin}(7)$ -manifolds.
2. M/Heterotic-duality and heterotic G_2 -systems: as was exemplified in section 5, some GCS-constructions have a K3-fibration, so that M-theory compactification on these has a dual description in terms of Heterotic on G_2 -manifolds with a T^3 -fibration with vector bundle. We have seen that the GCS-decomposition of the $\text{Spin}(7)$ -manifold maps in the dual G_2 -compactification to a TCS-decomposition. Heterotic on G_2 -manifolds has been studied only very sparsely, and this approach may very well provide further insight into the construction and moduli of vector bundles for so-called heterotic G_2 -systems. Furthermore, this duality also gives evidence for the existence of associative T^3 -fibrations on G_2 -manifolds as conjecture from G_2 mirror symmetry in [13], and these may indeed be less rare as was proposed in [23] despite the obstructions of associatives.
3. Non-abelian gauge groups: furthermore we expect, again through the duality to heterotic, that the GCS- $\text{Spin}(7)$ manifolds can give rise to non-abelian gauge symmetries. Locally, these will have a description in the form of an ADE singularities over Cayley four-cycle (for a discussion of such non-compact examples see [29]). It would indeed be interesting to develop this further and understand e.g. a Higgs bundle description of the effective theories and its relation to the compact $\text{Spin}(7)$ manifolds — much like what has been done in recent years for M/F-theory on CY_4 .

4. Mirror symmetry: mirror symmetry for TCS- G_2 manifolds as studied in [10, 13] by applying the mirror map to the Calabi-Yau three-fold building blocks. It would be interesting to see whether there is a similar way to study mirror symmetry for $\text{Spin}(7)$ manifolds (as proposed in [50]) by applying the mirror map to the building blocks.

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A TCS-construction of G_2 -manifolds

The TCS G_2 -manifolds [4–6] have the topology of a K3-fibration over an S^3 and are obtained by gluing two building blocks $Z_{\pm}$ that are algebraic three-folds, which are K3-fibered over S^2 . We will denote the generic K3-fibers of each building block by $S_{\pm}$. Excising a fiber, the two building blocks asymptote to $\text{K3} \times S^1 \times S^1 \times I$, where I is an interval. The building blocks are not Calabi-Yau but have $c_1(Z_{\pm}) = [S_{\pm}]$, which means that removing a fiber gives rise to acyl Calabi-Yau three-folds $X_{\pm}$. These are shown in figure 1. Here we summarize the salient points and introduce some notation that will be used in the main text. We refer the reader for a more detailed exposition to [4–6, 12].

The TCS-construction glues these two building blocks times an S^1 together, with a hyper-Kähler rotation (HKR), or Donaldson matching, ϕ acting on the Kähler forms ω and holomorphic $(2,0)$ -forms Ω as

$$\phi : \begin{cases} \omega_{\pm} & \longleftrightarrow & \text{Re } \Omega_{\mp}^{(2,0)} \\ \text{Im } \Omega_{+}^{(2,0)} & \longleftrightarrow & -\text{Im } \Omega^{(2,0)} \end{cases} \quad (\text{A.1})$$

and an exchange of the circles in the base $\mathbb{S}_{e,\pm}^1 = \mathbb{S}_{b,\mp}^1$ as shown in figure 1. In particular ϕ induces an isometry on the cohomologies of the fibers: $H^2(S_+, \mathbb{Z}) \cong H^2(S_-, \mathbb{Z})$.

What will be most important for this paper is how to determine the cohomologies of the G_2 -manifold in terms of data of the TCS gluing. For each building block we define the restriction map on the second cohomology as

$$H^2(Z_{\pm}, \mathbb{Z}) \xrightarrow{\rho_{\pm}} H^2(S_{\pm}, \mathbb{Z}) \cong \Lambda^{3,19} = U^3 \oplus (-E_8)^2, \quad (\text{A.2})$$

where we used the cohomology of the K3-surfaces, where U is the hyperbolic lattice of rank 2. The cohomology of the G_2 -manifold, can then be written in terms of the following lattices

$$\begin{aligned} N_{\pm} &= \text{im}(\rho_{\pm}) \\ T_{\pm} &= N_{\pm}^{\perp} \subset H^2(S_{\pm}, \mathbb{Z}) \\ K(Z_{\pm}) &= \ker(\rho_{\pm})/[S_{\pm}]. \end{aligned} \quad (\text{A.3})$$

Applying the Mayer-Vietoris sequence to this problem gives the cohomology of the G_2 manifold J as

$$\begin{aligned}
H^1(J, \mathbb{Z}) &= 0 \\
H^2(J, \mathbb{Z}) &= (N_+ \cap N_-) \oplus K(Z_+) \oplus K(Z_-) \\
H^3(J, \mathbb{Z}) &= \mathbb{Z}[S] \oplus \Lambda^{3,19}/(N_+ + N_-) \oplus (N_- \cap T_+) \oplus (N_+ \cap T_-) \\
&\quad \oplus H^3(Z_+) \oplus H^3(Z_-) \oplus K(Z_+) \oplus K(Z_-) \\
H^4(J, \mathbb{Z}) &= H^4(S) \oplus (T_+ \cap T_-) \oplus \Lambda^{3,19}/(N_- + T_+) \oplus \Lambda^{3,19}/(N_+ + T_-) \\
&\quad \oplus H^3(Z_+) \oplus H^3(Z_-) \oplus K(Z_+)^* \oplus K(Z_-)^* \\
H^5(J, \mathbb{Z}) &= \Lambda^{3,19}/(T_+ + T_-) \oplus K(Z_+)^* \oplus K(Z_-)^*.
\end{aligned} \tag{A.4}$$

We will determine a similar relation for the Spin(7) GCS-construction.

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