Published for SISSA by Springer

Received: *April 5, 2018*
Accepted: *June 5, 2018*
Published: *June 20, 2018*

JHEP06(2018)099

Five-particle phase-space integrals in QCD

O. Gituliar, V. Magerya and A. Pikelner[1]

II. Institut für Theoretische Physik, Universität Hamburg,
Luruper Chaussee 149, D-22761 Hamburg, Germany

E-mail: oleksandr@gituliar.net, vitaly.magerya@tx97.net, andrey.pikelner@desy.de

Abstract: We present analytical expressions for the 31 five-particle phase-space master integrals in massless QCD as an ϵ-series with coefficients being multiple zeta values of weight up to 12. In addition, we provide computer code for the Monte-Carlo integration in higher dimensions, based on the RAMBO algorithm, that has been used to numerically cross-check the obtained results in 4, 6, and 8 dimensions.

Keywords: Perturbative QCD, Scattering Amplitudes, Renormalization Group

ArXiv ePrint: 1803.09084

[1] On leave of absence from Joint Institute for Nuclear Research, 141980 Dubna, Russia.

https://doi.org/10.1007/JHEP06(2018)099

Contents

1 Introduction

Nowadays, perturbative calculations play the key role in describing data from high-energy particle colliders, such as the LHC, as well as in improving the precision of numerical parameters in Standard Model and other models. It is clear now that higher-order calculations will play an even more crucial role in processing data from future high-luminosity colliders, like the FCC or the ILC, where theoretical errors will dominate over experimental statistical errors. These arguments motivate us to make one step forward beyond available fully-inclusive phase-space integrals for a four-particle decay [1] and calculate a set of yet unknown integrals that corresponds to a five-particle decay of a color-neutral off-shell particle in Quantum Chromodynamics. These results, among others, are necessary ingredients in calculating various exclusive quantities with the method of differential equations where they are needed to determine integration constants, as for example discussed in [2, 3] for the three-loop time-like splitting functions [4].

In this paper, we focus on the calculation of master integrals that can be used to express any other integral of the corresponding topology provided a set of integration-by-parts rules (IBP) [5] is known. Our approach is based on techniques for solving dimensional recurrence relations (DRR) [6] described in [7, 8]. In particular, we use `DREAM` package [7] to obtain numerical results for desired integrals with 2000-digit precision and restore analytical form in terms of multiple zeta values (MZV) [9–11] up to weight 12 using the PSLQ method [12] as implemented in Mathematica. We also present a Monte-Carlo code, based on the RAMBO algorithm [13], for numerical integration of the phase-space integrals in arbitrary (integer) number of dimensions that has been used to check consistency of the obtained results.

This paper is organized as following. In section 2 we introduce our notation and describe our calculational method in more details. In section 3 we provide complete results for four-particle integrals and discuss numerical cross-checks using Monte-Carlo integration. In section 4 we make final remarks. In appendix A we provide the complete list of master integrals.

Additionally we provide supplementary files attached to this paper containing the complete master integrals with MZV weight up to 12 as well as the Monte-Carlo integration routines with the corresponding results.

2 The method

We start by identifying a set of five-particle phase-space master integrals using two different approaches for consistency. As the first approach we exploit the equivalency of IBP rules for cut and ordinary propagators, and obtain the complete basis of phase-space master integrals by taking all five-particle cuts of the 28 master integrals for four-loop massless propagators found in [14], discarding those that do not correspond to the squared matrix elements of the $1 \to 5$ process, and reducing the remaining integrals with Laporta-style IBP reduction [15, 16] as implemented in `FIRE5` [17]. As an alternative approach, we construct the complete expression for the total cross-section of the $1 \to 5$ process in QCD using `QGRAF` [18] and `FORM` [19], and then reduce it with the help of `FIRE5`. Both methods give 31 master integrals listed in table 1, each having up to 6 unique propagators. Our notation for these integrals is

$$F_i = S_\Gamma \int \mathrm{dPS}_5 \, \frac{1}{D_1^{(i)} \, \dots \, D_n^{(i)}}, \tag{2.1}$$

where $D_j^{(i)}$ are propagators that take the form of invariant scalar products

$$s_{kl\dots q} = (p_k + p_l + \dots + p_q)^2, \tag{2.2}$$

dPS_5 is a five-particle phase-space element in D dimensions

$$\mathrm{dPS}_N = \left(\prod_{i=1}^{N} \mathrm{d}^D p_i \, \delta^+(p_i^2) \right) \delta^{(D)}\Big(q - p_1 - \dots - p_N\Big), \tag{2.3}$$

and S_Γ is a common normalization factor chosen for convenience[1] to be

$$S_\Gamma = (q^2)^{5-2D} \frac{(2\pi)^4}{\pi^{2D}} \Gamma\left(\frac{D}{2} - 1\right) \Gamma\left(3\frac{D}{2} - 3\right). \tag{2.4}$$

With this normalization and knowing the volume of the complete N-particle phase space[2]

$$\int \mathrm{dPS}_N = (q^2)^{\frac{D}{2}(N-1)-N} \frac{\pi^{\frac{D}{2}(N-1)}}{(2\pi)^{N-1}} \frac{\Gamma\left(\frac{D}{2}-1\right)^N}{\Gamma\Big(\left(\frac{D}{2}-1\right)(N-1)\Big)\Gamma\Big(\left(\frac{D}{2}-1\right)N\Big)} \tag{2.5}$$

we can already fix the value of F_1 as

$$F_1 = S_\Gamma \int \mathrm{dPS}_5 = \frac{\Gamma\left(\frac{D}{2}-1\right)^6 \Gamma\left(3\frac{D}{2}-3\right)}{\Gamma\left(4\frac{D}{2}-4\right)\Gamma\left(5\frac{D}{2}-5\right)}. \tag{2.6}$$

[1] This way we prevent additional constants (e.g, γ_E or $\ln \pi$) to appear in the final results, hence reducing its size as well as a size of the basis for PSLQ algorithm.

[2] The dependence on q^2 here is trivial, and can be restored by power counting. We will omit it from now on, setting q^2 to 1.

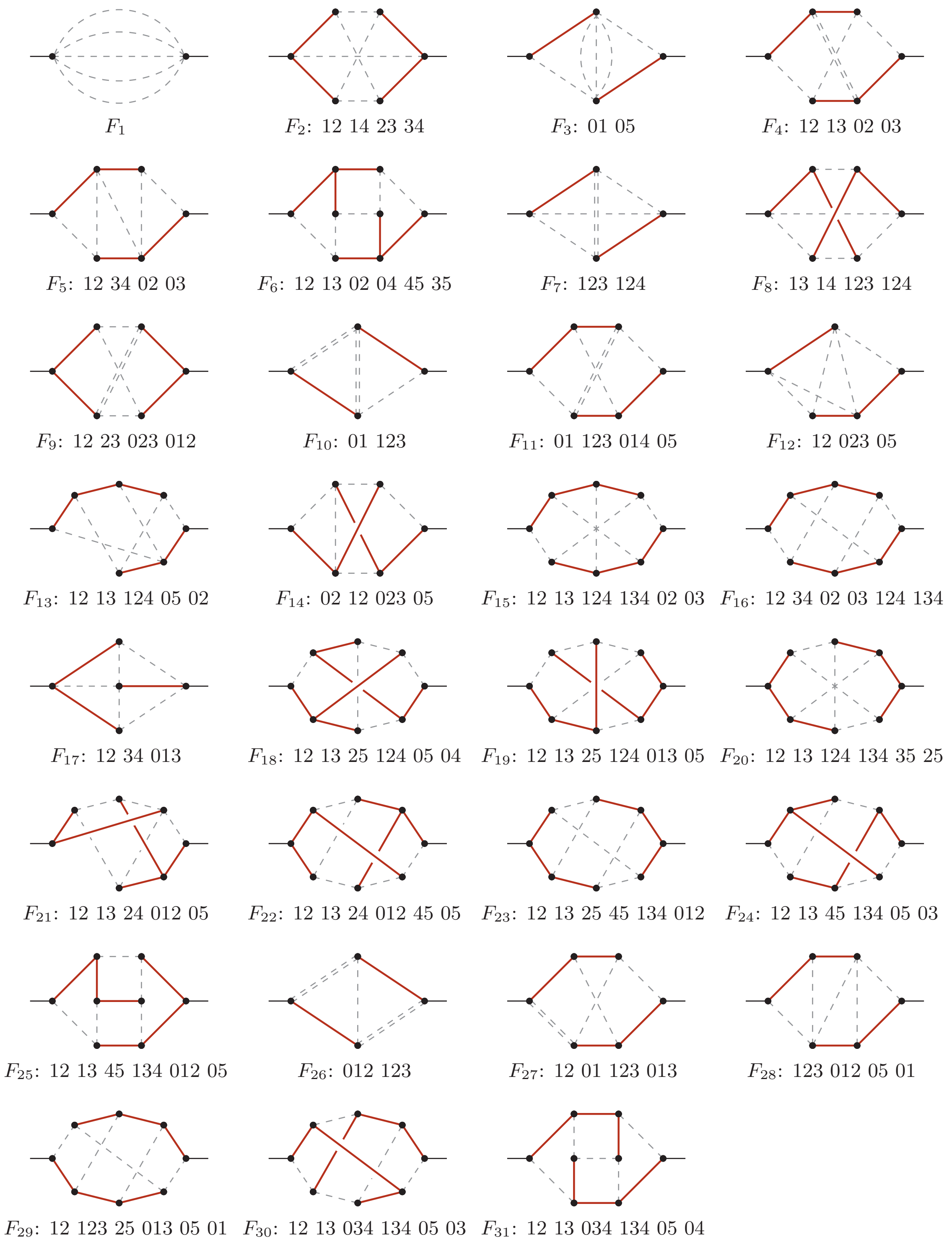

Table 1. Cut diagrams for five-particle phase-space master integrals in QCD. Dashed lines represent cut propagators and carry final-state momenta $p_1, \ldots, p_5$. Labels represent propagators, so that "123" corresponds to $p_1 + p_2 + p_3$ and "012" to $q - p_1 - p_2$ (where q is the initial-state momentum, i.e., $q = p_1 + \cdots + p_5$).

Next, with the help of LiteRed [20] and FIRE5 [17] we derive a set of lowering dimensional recurrence relations which express master integrals in $D+2$ dimensions in terms of master integrals in D dimensions:

$$F_i(D+2) = M_{ij}(D)\, F_j(D). \tag{2.7}$$

In our case M can be shuffled into triangular form, with each F_i only depending on itself and master integrals from lower sectors S_i:

$$F_i(D+2) = M_{ii}(D)\, F_i(D) + \sum_{k\in S_i} M_{ik}(D) F_k(D) \tag{2.8}$$

and the general solution being

$$F_i(D) = \omega_i(D)\, H_i(D) + R_i(D), \tag{2.9}$$

where H_i is a homogeneous solution of eq. (2.8), R_i is a partial solution that can be constructed numerically with DREAM [7] provided F_1 is known, and $\omega_i(D)$ is an arbitrary periodic function that needs to be determined from separate considerations.

We argue that all $\omega_{i>1}$ are zero. To see this, first let us look at the asymptotic behavior of F_i at large D. Rewriting eq. (2.1) as an integral over invariants s_{ij} gives

$$F_i = S_\Gamma \left(\prod_{k=1}^{N-1} \frac{\Omega_{D-k}}{2}\right) \int \left(\prod_{l<m} \frac{\mathrm{d}s_{lm}}{2}\right) (\Delta_N)^{\frac{D-N-1}{2}}\, \Theta(\Delta_N)\, \delta\left(1 - s_{1\ldots N}\right) \frac{1}{D_1^{(i)}\ \ldots\ D_n^{(i)}}, \tag{2.10}$$

where Δ_N is the Gram determinant defined as

$$\Delta_N = \frac{(-1)^{N+1}}{2^N} \begin{vmatrix} s_{11} & s_{12} & \cdots & s_{1N} \\ s_{12} & s_{22} & \cdots & s_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1N} & s_{2N} & \cdots & s_{NN} \end{vmatrix}, \tag{2.11}$$

and Ω_k is the surface area of a unit hypersphere in k-dimensional space

$$\Omega_k = 2\pi^{\frac{k}{2}}\, \Gamma\left(\frac{k}{2}\right)^{-1}. \tag{2.12}$$

If $\Delta_N(s_{ij})$ has a unique global maximum inside the integration region, we can apply Laplace's method to eq. (2.10) and find its asymptotic as

$$F_i(D\to\infty) = S_\Gamma \left(\prod_{k=1}^{N-1} \Omega_{D-k}\right) (\Delta_N^{\max})^{\frac{D}{2}} \left(\frac{2\pi}{D}\right)^{\frac{1}{2}\left(\frac{N(N-1)}{2}-1\right)} \left(\mathcal{C}_i + \mathcal{O}\left(D^{-1}\right)\right), \tag{2.13}$$

where $\mathcal{C}_i$ is a constant that depends on the location of the maximum and the denominators $D_j^{(i)}$, but not on D.

The global maximum of Δ_N is reached when all s_{ij} $(i \neq j)$ are identical and equal to $\frac{2}{N(N-1)}$. Geometrically this configuration corresponds to the vectors $\vec{p}_i$ pointing to the vertices of a regular N-hedron embedded into Euclidean space of $(N-1)$ dimensions. The maximum value is then

$$\Delta_N^{\max} = \frac{1}{N^N(N-1)^{N-1}} \tag{2.14}$$

and explicitly we get

$$F_i(D\to\infty) = \pi^{\frac{7}{2}} 2^{\frac{25}{2}} \frac{\left(4^4 5^5\right)^{-\frac{D}{2}} \Gamma\left(\frac{3D}{2}-3\right)}{D^{\frac{9}{2}} \Gamma\left(\frac{D-4}{2}\right)\Gamma\left(\frac{D-3}{2}\right)\Gamma\left(\frac{D-1}{2}\right)} \left(\mathcal{C}_i + \mathcal{O}\left(D^{-1}\right)\right). \tag{2.15}$$

It follows that all F_i have identical asymptotic behavior up to a constant $\mathcal{C}_i$. As a confirmation, it can be shown that eq. (2.15) is asymptotically the same expression as we had for F_1 in eq. (2.6).

Next, we can find the asymptotics of the homogeneous parts of eq. (2.9), $H_i(D)$, using e.g. the routine `FindAsymptotics` from `DREAM`. Comparing these to eq. (2.15), we determine that all $H_i(D)$ for $i>1$ are growing exponentially faster than $F_i(D)$, which can only happen if the corresponding periodic functions $\omega_i(D)$ are zero.

Thus, to find F_i we only need to find R_i, the inhomogeneous solutions to eq. (2.8). We compute them as a series in $\epsilon=(4-D)/2$ using `DREAM` with 2000-digit accuracy and then restore the analytical form of the series coefficients in terms of MZVs using PSLQ method [12]. In this way we obtain the analytical result for all master integrals up to MZVs of weight 12 using the corresponding bases from [9] and the `SummerTime` package [11] for their numerical evaluation. Corresponding expressions are presented in appendix A as well as in the supplementary files attached to this paper.

3 Crosschecks

3.1 Four-particle integrals

As the first consistency check of our method we reproduce results for four-particle phase-space integrals reported in [1]. We perform all the steps described in section 2. Generating the IBP rules with the help of `LiteRed` and then proceeding with `DREAM` we obtain the final result with 2000-digit accuracy and MZVs up to weight 12. The series reconstructed with PSLQ (using the original notation, and omitting S_Γ and q^2 factors) are:

$$\begin{aligned} R_6 = {} & -1+\zeta_2+\epsilon\left(-12+5\zeta_2+9\zeta_3\right)+\epsilon^2\left(-91+27\zeta_2+45\zeta_3+\frac{61}{5}\zeta_2^2\right) \\ & +\epsilon^3\left(-558+161\zeta_2+197\zeta_3+61\zeta_2^2-80\zeta_3\zeta_2+207\zeta_5\right) \\ & +\epsilon^4\left(-3025+939\zeta_2+897\zeta_3+\frac{1157}{5}\zeta_2^2-400\zeta_3\zeta_2+1035\zeta_5+\frac{288}{5}\zeta_2^3-153\zeta_3^2\right), \end{aligned} \tag{3.1}$$

$$R_{8,a} = \frac{5}{\epsilon^4}-\frac{40\zeta_2}{\epsilon^2}-\frac{126\zeta_3}{\epsilon}+14\zeta_2^2+\epsilon\left(1008\zeta_2\zeta_3-1086\zeta_5\right)+\epsilon^2\left(-\frac{272}{7}\zeta_2^3+1602\zeta_3^2\right), \tag{3.2}$$

$$R_{8,b} = \frac{3}{4\epsilon^4}-\frac{17\zeta_2}{2\epsilon^2}-\frac{44\zeta_3}{\epsilon}-\frac{183}{5}\zeta_2^2+\epsilon\left(376\zeta_2\zeta_3-790\zeta_5\right)+\epsilon^2\left(-\frac{19088}{105}\zeta_2^3+698\zeta_3^2\right). \tag{3.3}$$

JHEP06(2018)099

3.2 Numerical verification

As another cross-check we have calculated the leading terms of the ϵ-expansion of F_i numerically using the direct way: through Monte-Carlo integration of eq. (2.1) over the phase-space. While such a technique can not be easily applied to divergent integrals, we can sidestep such an issue by noting that our master integrals only suffer from IR divergences that disappear already at $D = 6$. In this way we can check several leading terms of the expansion at $D = 4 - 2\epsilon$ by calculating the corresponding integrals in $D \geq 6$ since both are connected by dimensional recurrence relations.

To calculate a finite integral of the form eq. (2.1) we choose a uniform mapping from a hypercube into momentum coordinates using an algorithm similar to RAMBO [13] but extended into arbitrary D. Then we calculate the integrand from scalar products of the momenta, and finally we integrate over the hypercube using the Vegas [21] implementation from Cuba [22].

Note that although the integrals we are calculating are finite, the integrands are not. Exposing an integration algorithm like Vegas to such infinities may lead to unpredictable behaviour, so as a precaution we choose to regulate these infinities by adding a small parameter α to the denominator of the integrand, and then to calculate the integral with progressively smaller values of α (from 2^{-30} to 2^{-100}), checking if convergence was reached afterwards.

The results of this method are summarized in table 2, and show good agreement between numerical and analytic results. Our integration program is written in C using the GNU Scientific Library [23] and Cuba. Its source code can be found at https://hg.tx97.net/rambo, and also in the supplementary files attached to this paper. With a requested accuracy of 0.1% the complete integration takes less than two days on a 12-core machine, with each integration taking between a minute and two hours.

4 Conclusions

In this paper we present analytical expressions for five-particle phase-space integrals expressed in terms of multiple zeta values up to weight 12. The results are calculated using dimensional recurrence relation method with a 2000-digit accuracy using the `DREAM` package. We also present computer code for the numerical integration of phase-space integrals in a higher-number of dimensions that has been used to cross-check the obtained results with an accuracy of 0.1%. The approach presented here shows excellent performance for calculating single-scale integrals without ultra-violet divergences and can be easily applied to other problems of this kind.

Acknowledgments

We are thankful to Sven Moch for numerous discussions and helpful suggestions concerning this work, and for proofreading this paper. We were pleased to use `Axodraw2` [24] to draw diagrams for this paper.

i	Numerical results			Analytic results		
	$D=4$	$D=6$	$D=8$	$D=4$	$D=6$	$D=8$
2	–	1708(2)00	4699(1)0	–	171085.62	47000.531
3	3.7823(4)	3.1704(2)	3.0221(1)	3.7823736	3.1704486	3.0221118
4	–	1504.7(8)	725.3(1)	–	1504.4507	725.26806
5	–	1007.4(5)	580.80(9)	–	1007.5235	580.76347
6	–	6191(5)00	14496(6)0	–	619633.25	144975.32
7	46.46(4)	18.533(2)	15.205(1)	46.435253	18.532303	15.205538
8	–	2031(2)0	5357(2)	–	20297.189	5355.3611
9	–	4313(3)	2406.7(4)	–	4312.8823	2406.7943
10	10.436(2)	7.1508(5)	6.5093(3)	10.435253	7.1507477	6.5092878
11	228.8(1)	62.67(1)	47.663(4)	229.11836	62.667046	47.663194
12	–	157.34(4)	102.26(1)	–	157.33521	102.26408
13	–	13729(8)	4000(1)	–	13732.166	4000.2779
14	–	268.46(8)	172.80(2)	–	268.45969	172.79805
15	–	6322(6)0	16048(5)	–	63316.356	16049.857
16	–	4414(3)0	12952(4)	–	44117.898	12951.443
17	–	1243.4(7)	709.6(1)	–	1243.1369	709.52840
18	–	3002(2)00	5899(3)0	–	300402.99	58965.517
19	–	4982(4)00	10637(4)0	–	498329.79	106357.81
20	–	2360(2)000	5777(2)00	–	2362594.9	577686.64
21	–	6312(5)0	20642(5)	–	63147.876	20642.071
22	–	8402(7)00	24407(8)0	–	840453.94	244075.75
23	–	1443(1)000	4556(1)00	–	1443198.3	455543.43
24	–	1391(1)00	3997(1)0	–	139263.92	39966.878
25	–	3347(3)00	8526(3)0	–	335128.10	85254.217
26	25.563(6)	15.376(1)	13.6042(7)	25.564747	15.376404	13.604247
27	–	697.3(3)	397.84(6)	–	697.18948	397.83514
28	143.9(1)	52.855(7)	42.917(3)	143.97886	52.853837	42.917424
29	–	4409(3)0	13702(3)	–	44117.898	13700.597
30	–	6327(6)0	16181(5)	–	63316.356	16178.566
31	–	8955(8)0	19055(8)	–	89611.062	19051.115

Table 2. Numerical results for the ratio F_i/F_1 with the corresponding uncertainties (standard deviations) indicated in the parenthesis. Missing entries correspond to divergent integrals.

JHEP06(2018)099

This work was supported in part by the German Research Foundation DFG through the Collaborative Research Centre No. SFB 676 *Particles, Strings and the Early Universe: the Structure of Matter and Space-Time.*

A Results

The main results of our work are listed below. For brevity, we truncate them up to MZVs of weight 6. Complete results with weight up to 12 are available in supplementary files attached to this paper.

$$
\begin{aligned}
F_1 =& \frac{1}{72}+\epsilon\frac{53}{288}+\epsilon^2\left(\frac{15961}{10368}-\frac{13}{72}\zeta_2\right)+\epsilon^3\left(\frac{436013}{41472}-\frac{689}{288}\zeta_2-\frac{13}{18}\zeta_3\right) \\
&+\epsilon^4\left(\frac{96102601}{1492992}-\frac{207493}{10368}\zeta_2-\frac{689}{72}\zeta_3+\frac{17}{240}\zeta_2^2\right)+\epsilon^5\left(\frac{2206279853}{5971968}-\frac{5668169}{41472}\zeta_2\right. \\
&\left.-\frac{207493}{2592}\zeta_3+\frac{901}{960}\zeta_2^2+\frac{169}{18}\zeta_3\zeta_2-\frac{65}{6}\zeta_5\right)+\epsilon^6\left(\frac{437728233961}{214990848}-\frac{1249333813}{1492992}\zeta_2\right. \\
&\left.-\frac{5668169}{10368}\zeta_3+\frac{271337}{34560}\zeta_2^2+\frac{8957}{72}\zeta_3\zeta_2-\frac{3445}{24}\zeta_5-\frac{4007}{5040}\zeta_2^3+\frac{169}{9}\zeta_3^2\right)
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
F_2 =& \frac{10}{3\epsilon^5}-\frac{55}{3\epsilon^4}+\frac{1}{\epsilon^3}\left(\frac{160}{3}-\frac{130}{3}\zeta_2\right)+\frac{1}{\epsilon^2}\left(-\frac{490}{3}+\frac{715}{3}\zeta_2-\frac{512}{3}\zeta_3\right) \\
&+\frac{1}{\epsilon}\left(\frac{1450}{3}-\frac{2080}{3}\zeta_2+\frac{2816}{3}\zeta_3+25\zeta_2^2\right)-\frac{4390}{3}+\frac{6370}{3}\zeta_2-\frac{8192}{3}\zeta_3-\frac{275}{2}\zeta_2^2 \\
&+\frac{6656}{3}\zeta_3\zeta_2-2504\zeta_5+\epsilon\left(\frac{13090}{3}-\frac{18850}{3}\zeta_2+\frac{25088}{3}\zeta_3+400\zeta_2^2-\frac{36608}{3}\zeta_3\zeta_2\right. \\
&\left.+13772\zeta_5-211\zeta_2^3+\frac{13136}{3}\zeta_3^2\right)
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
F_3 =& \frac{7}{8}-\frac{\zeta_2}{2}+\epsilon\left(\frac{219}{16}-\frac{7}{2}\zeta_2-6\zeta_3\right)+\epsilon^2\left(\frac{4231}{32}-\frac{227}{8}\zeta_2-42\zeta_3-\frac{111}{10}\zeta_2^2\right) \\
&+\epsilon^3\left(\frac{65347}{64}-\frac{3999}{16}\zeta_2-\frac{499}{2}\zeta_3-\frac{777}{10}\zeta_2^2+86\zeta_3\zeta_2-237\zeta_5\right)+\epsilon^4\left(\frac{887695}{128}\right. \\
&\left.-\frac{64219}{32}\zeta_2-\frac{6303}{4}\zeta_3-\frac{5967}{16}\zeta_2^2+602\zeta_3\zeta_2-1659\zeta_5-\frac{1827}{20}\zeta_2^3+204\zeta_3^2\right)
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
F_4 =& -\frac{\zeta_2}{\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{\zeta_2}{2}-13\zeta_3\right)+\frac{1}{\epsilon}\left(\frac{3}{2}\zeta_2+\frac{13}{2}\zeta_3-\frac{267}{10}\zeta_2^2\right)+4\zeta_2+\frac{39}{2}\zeta_3+\frac{267}{20}\zeta_2^2 \\
&+191\zeta_3\zeta_2-\frac{1129}{2}\zeta_5+\epsilon\left(10\zeta_2+52\zeta_3+\frac{801}{20}\zeta_2^2-\frac{191}{2}\zeta_3\zeta_2+\frac{1129}{4}\zeta_5-\frac{16547}{70}\zeta_2^3\right. \\
&\left.+487\zeta_3^2\right)
\end{aligned} \tag{A.4}
$$

JHEP06(2018)099

$$F_5 = \frac{7\zeta_2^2}{5\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{7}{2}\zeta_2^2 - 18\zeta_3\zeta_2 + 87\zeta_5\right) - \frac{7}{10}\zeta_2^2 + 45\zeta_3\zeta_2 - \frac{435}{2}\zeta_5 + \frac{647}{5}\zeta_2^3 - 108\zeta_3^2 \tag{A.5}$$

$$\begin{aligned} F_6 = & -\frac{35}{9\epsilon^5} - \frac{19}{6\epsilon^4} + \frac{1}{\epsilon^3}\left(-\frac{278}{3} - \frac{191}{3}\zeta_2\right) + \frac{1}{\epsilon^2}\left(\frac{1697}{3} + \frac{1687}{18}\zeta_2 - \frac{3236}{9}\zeta_3\right) \\ & + \frac{1}{\epsilon}\left(-\frac{7793}{3} + \frac{10450}{9}\zeta_2 + \frac{2386}{3}\zeta_3 - \frac{4927}{18}\zeta_2^2\right) + \frac{31259}{3} - \frac{67705}{9}\zeta_2 + 4296\zeta_3 \\ & + \frac{209681}{180}\zeta_2^2 + \frac{14708}{3}\zeta_3\zeta_2 - \frac{28148}{3}\zeta_5 + \epsilon\left(-\frac{117869}{3} + \frac{317353}{9}\zeta_2 - 31444\zeta_3\right. \\ & \left. - \frac{66809}{45}\zeta_2^2 - \frac{101858}{9}\zeta_3\zeta_2 + \frac{84602}{3}\zeta_5 - \frac{357871}{126}\zeta_2^3 + \frac{96472}{9}\zeta_3^2\right) \end{aligned} \tag{A.6}$$

$$\begin{aligned} F_7 = & -1 + \zeta_2 + \epsilon\left(-17 + 10\zeta_2 + 9\zeta_3\right) + \epsilon^2\left(-\frac{351}{2} + \frac{163}{2}\zeta_2 + 90\zeta_3 + \frac{36}{5}\zeta_2^2\right) \\ & + \epsilon^3\left(-\frac{5709}{4} + \frac{2495}{4}\zeta_2 + \frac{1337}{2}\zeta_3 + 72\zeta_2^2 - 151\zeta_3\zeta_2 + 207\zeta_5\right) + \epsilon^4\left(-\frac{80649}{8}\right. \\ & \left. + \frac{35823}{8}\zeta_2 + \frac{18035}{4}\zeta_3 + \frac{4881}{10}\zeta_2^2 - 1510\zeta_3\zeta_2 + 2070\zeta_5 - \frac{387}{10}\zeta_2^3 - 387\zeta_3^2\right) \end{aligned} \tag{A.7}$$

$$\begin{aligned} F_8 = & \frac{1}{2\,\epsilon^5} - \frac{11}{4\,\epsilon^4} + \frac{1}{\epsilon^3}\left(8 - \frac{49}{6}\zeta_2\right) + \frac{1}{\epsilon^2}\left(-\frac{49}{2} + \frac{539}{12}\zeta_2 - \frac{127}{3}\zeta_3\right) + \frac{1}{\epsilon}\left(\frac{145}{2} - \frac{392}{3}\zeta_2\right. \\ & \left. + \frac{1397}{6}\zeta_3 - \frac{823}{60}\zeta_2^2\right) - \frac{439}{2} + \frac{2401}{6}\zeta_2 - \frac{2032}{3}\zeta_3 + \frac{9053}{120}\zeta_2^2 + \frac{1829}{3}\zeta_3\zeta_2 \\ & - \frac{2381}{3}\zeta_5 + \epsilon\left(\frac{1309}{2} - \frac{7105}{6}\zeta_2 + \frac{6223}{3}\zeta_3 - \frac{3292}{15}\zeta_2^2 - \frac{20119}{6}\zeta_3\zeta_2 + \frac{26191}{6}\zeta_5\right. \\ & \left. + \frac{48983}{1260}\zeta_2^3 + 1397\zeta_3^2\right) \end{aligned} \tag{A.8}$$

$$\begin{aligned} F_9 = & -\frac{\zeta_2}{\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{\zeta_2}{2} - 7\zeta_3\right) + \frac{1}{\epsilon}\left(\frac{3}{2}\zeta_2 + \frac{7}{2}\zeta_3 + \frac{17}{10}\zeta_2^2\right) + 4\zeta_2 + \frac{21}{2}\zeta_3 - \frac{17}{20}\zeta_2^2 + 131\zeta_3\zeta_2 \\ & - \frac{127}{2}\zeta_5 + \epsilon\left(10\zeta_2 + 28\zeta_3 - \frac{51}{20}\zeta_2^2 - \frac{131}{2}\zeta_3\zeta_2 + \frac{127}{4}\zeta_5 + \frac{7677}{70}\zeta_2^3 + 328\zeta_3^2\right) \end{aligned} \tag{A.9}$$

$$\begin{aligned} F_{10} = & \zeta_2 - \frac{3}{2} + \epsilon\left(-\frac{47}{2} + \frac{15}{2}\zeta_2 + 11\zeta_3\right) + \epsilon^2\left(-\frac{903}{4} + 58\zeta_2 + \frac{165}{2}\zeta_3 + \frac{171}{10}\zeta_2^2\right) \\ & + \epsilon^3\left(-\frac{13795}{8} + \frac{951}{2}\zeta_2 + \frac{1003}{2}\zeta_3 + \frac{513}{4}\zeta_2^2 - 159\zeta_3\zeta_2 + \frac{739}{2}\zeta_5\right) + \epsilon^4\left(-\frac{184655}{16}\right. \\ & \left. + \frac{14539}{4}\zeta_2 + 3092\zeta_3 + \frac{6507}{10}\zeta_2^2 - \frac{2385}{2}\zeta_3\zeta_2 + \frac{11085}{4}\zeta_5 + \frac{1293}{14}\zeta_2^3 - 374\zeta_3^2\right) \end{aligned} \tag{A.10}$$

JHEP06(2018)099

$$F_{11}=10\zeta_2\zeta_3-16\zeta_5+\epsilon\left(-5\zeta_3\zeta_2+8\zeta_5-\frac{48}{35}\zeta_2^3+43\zeta_3^2\right) \tag{A.11}$$

$$F_{12}=-\frac{2\zeta_3}{\epsilon}-13\zeta_3-\frac{37}{5}\zeta_2^2+\epsilon\left(-76\zeta_3-\frac{481}{10}\zeta_2^2+30\zeta_3\zeta_2-115\zeta_5\right)+\epsilon^2\left(-422\zeta_3\right.$$
$$\left.-\frac{1406}{5}\zeta_2^2+195\zeta_3\zeta_2-\frac{1495}{2}\zeta_5-\frac{158}{7}\zeta_2^3+112\zeta_3^2\right) \tag{A.12}$$

$$F_{13}=\frac{1}{6\epsilon^5}-\frac{11}{12\epsilon^4}+\frac{1}{\epsilon^3}\left(\frac{8}{3}+\frac{\zeta_2}{2}\right)+\frac{1}{\epsilon^2}\left(-\frac{49}{6}-\frac{11}{4}\zeta_2+\frac{86}{3}\zeta_3\right)+\frac{1}{\epsilon}\left(\frac{145}{6}+8\zeta_2-\frac{473}{3}\zeta_3\right.$$
$$\left.+\frac{5251}{60}\zeta_2^2\right)-\frac{439}{6}-\frac{49}{2}\zeta_2+\frac{1376}{3}\zeta_3-\frac{57761}{120}\zeta_2^2-466\zeta_3\zeta_2+1794\zeta_5+\epsilon\left(\frac{1309}{6}\right.$$
$$\left.+\frac{145}{2}\zeta_2-\frac{4214}{3}\zeta_3+\frac{21004}{15}\zeta_2^2+2563\zeta_3\zeta_2+-9867\zeta_5+\frac{1349771}{1260}\zeta_2^3-1372\zeta_3^2\right) \tag{A.13}$$

$$F_{14}=-\frac{8\zeta_2^2}{5\epsilon}-\frac{28}{5}\zeta_2^2-48\zeta_5+\epsilon\left(-\frac{164}{5}\zeta_2^2-168\zeta_5-\frac{6248}{105}\zeta_2^3+6\zeta_3^2\right) \tag{A.14}$$

$$F_{15}=\frac{1}{6\epsilon^6}+\frac{41}{36\,\epsilon^5}+\frac{1}{\epsilon^4}\left(-\frac{311}{36}-\frac{73}{18}\zeta_2\right)+\left(\frac{445}{18}-\frac{563}{36}\zeta_2-\frac{281}{9}\zeta_3\right)\frac{1}{\epsilon^3}+\frac{1}{\epsilon^2}\left(-\frac{689}{9}\right.$$
$$\left.+\frac{5273}{36}\zeta_2-\frac{907}{18}\zeta_3-\frac{7103}{180}\zeta_2^2\right)+\frac{1}{\epsilon}\left(\frac{2024}{9}-\frac{7759}{18}\zeta_2+\frac{13933}{18}\zeta_3+\frac{10553}{120}\zeta_2^2\right.$$
$$\left.+\frac{1489}{3}\zeta_3\zeta_2-\frac{3257}{3}\zeta_5\right)-\frac{6158}{9}+\frac{12437}{9}\zeta_2-\frac{22193}{9}\zeta_3+\frac{28621}{120}\zeta_2^2+\frac{14065}{18}\zeta_3\zeta_2$$
$$+\frac{3631}{6}\zeta_5-\frac{134489}{420}\zeta_2^3+1189\zeta_3^2 \tag{A.15}$$

$$F_{16}=\frac{1}{6\epsilon^6}+\frac{7}{12\epsilon^5}+\frac{1}{\epsilon^4}\left(-\frac{185}{36}-\frac{65}{18}\zeta_2\right)+\frac{1}{\epsilon^3}\left(\frac{209}{18}-\frac{157}{12}\zeta_2-\frac{289}{9}\zeta_3\right)+\frac{1}{\epsilon^2}\left(-\frac{76}{3}\right.$$
$$\left.+\frac{3563}{36}\zeta_2-\frac{215}{2}\zeta_3-\frac{10927}{180}\zeta_2^2\right)+\frac{1}{\epsilon}\left(\frac{239}{9}-\frac{2123}{18}\zeta_2+\frac{14431}{18}\zeta_3-\frac{7041}{40}\zeta_2^2\right.$$
$$\left.+463\zeta_3\zeta_2-1411\zeta_5\right)+\frac{857}{9}-\frac{688}{3}\zeta_2-\frac{5623}{9}\zeta_3+\frac{170591}{120}\zeta_2^2+\frac{8963}{6}\zeta_3\zeta_2$$
$$-\frac{8349}{2}\zeta_5-\frac{134141}{180}\zeta_2^3+\frac{4169}{3}\zeta_3^2 \tag{A.16}$$

$$F_{17}=\frac{4\zeta_3}{\epsilon^2}+\frac{1}{\epsilon}\left(10\zeta_3+\frac{84}{5}\zeta_2^2\right)+48\zeta_3+42\zeta_2^2-52\zeta_3\zeta_2+280\zeta_5+\epsilon\left(236\zeta_3+\frac{1008}{5}\zeta_2^2\right.$$
$$\left.-130\zeta_3\zeta_2+700\zeta_5+\frac{588}{5}\zeta_2^3-160\zeta_3^2\right) \tag{A.17}$$

$$F_{18}=-\frac{5}{12\epsilon^6}+\frac{1}{8\epsilon^5}+\frac{1}{\epsilon^4}\left(\frac{205}{12}+\frac{323}{36}\zeta_2\right)+\frac{1}{\epsilon^3}\left(-\frac{1937}{18}-\frac{71}{24}\zeta_2+\frac{223}{3}\zeta_3\right)+\frac{1}{\epsilon^2}\left(\frac{4019}{9}\right. \tag{A.18}$$

$$-\frac{3137}{12}\zeta_2-\frac{443}{18}\zeta_3+\frac{46931}{360}\zeta_2^2\Big)+\frac{1}{\epsilon}\Big(-\frac{3047}{2}+\frac{26681}{18}\zeta_2-1465\zeta_3-\frac{5473}{144}\zeta_2^2$$
$$-\frac{9113}{9}\zeta_3\zeta_2+\frac{27953}{9}\zeta_5\Big)+\frac{39460}{9}-\frac{54149}{9}\zeta_2+\frac{60682}{9}\zeta_3-\frac{33061}{24}\zeta_2^2$$
$$+\frac{6415}{18}\zeta_3\zeta_2-\frac{15967}{18}\zeta_5+\frac{12313867}{7560}\zeta_2^3-2302\zeta_3^2$$

$$F_{19}=-\frac{1}{4\epsilon^6}+\frac{19}{8\epsilon^5}+\frac{1}{\epsilon^4}\Big(\frac{64}{9}+\frac{173}{36}\zeta_2\Big)+\frac{1}{\epsilon^3}\Big(-\frac{1895}{18}-\frac{2767}{72}\zeta_2+\frac{371}{9}\zeta_3\Big)+\frac{1}{\epsilon^2}\Big(\frac{9157}{18} \tag{A.19}$$
$$-\frac{1451}{18}\zeta_2-\frac{3439}{18}\zeta_3+\frac{32357}{360}\zeta_2^2\Big)+\frac{1}{\epsilon}\Big(-\frac{5798}{3}+\frac{24635}{18}\zeta_2-\frac{7103}{18}\zeta_3-\frac{17023}{720}\zeta_2^2$$
$$-\frac{4127}{9}\zeta_3\zeta_2+\frac{18073}{9}\zeta_5\Big)+\frac{111845}{18}-\frac{120679}{18}\zeta_2+\frac{49330}{9}\zeta_3-\frac{3664}{5}\zeta_2^2+\frac{50743}{18}\zeta_3\zeta_2$$
$$-\frac{53357}{18}\zeta_5+\frac{10750309}{7560}\zeta_2^3-\frac{7078}{9}\zeta_3^2$$

$$F_{20}=\frac{20}{9\epsilon^6}+\frac{85}{6\epsilon^5}+\frac{1}{\epsilon^4}\Big(-\frac{775}{12}-\frac{325}{9}\zeta_2\Big)+\frac{1}{\epsilon^3}\Big(-\frac{985}{18}-\frac{1705}{9}\zeta_2-\frac{1595}{9}\zeta_3\Big) \tag{A.20}$$
$$+\frac{1}{\epsilon^2}\Big(\frac{6445}{9}-\frac{88}{9}\zeta_2^2+\frac{3815}{4}\zeta_2-\frac{4775}{6}\zeta_3\Big)+\frac{1}{\epsilon}\Big(-3635+\frac{8935}{18}\zeta_2+\frac{26455}{6}\zeta_3$$
$$-\frac{743}{36}\zeta_2^2+\frac{23485}{9}\zeta_3\zeta_2-\frac{7535}{3}\zeta_5\Big)+\frac{121310}{9}-\frac{80005}{9}\zeta_2+\frac{7415}{9}\zeta_3+\frac{9299}{24}\zeta_2^2$$
$$+\frac{189155}{18}\zeta_3\zeta_2-\frac{26065}{2}\zeta_5+\frac{134581}{126}\zeta_2^3+\frac{54985}{9}\zeta_3^2$$

$$F_{21}=\frac{1}{2\epsilon^5}-\frac{11}{4\epsilon^4}+\frac{1}{\epsilon^3}\Big(8+\frac{3}{2}\zeta_2\Big)+\frac{1}{\epsilon^2}\Big(-\frac{49}{2}-\frac{33}{4}\zeta_2+\frac{158}{3}\zeta_3\Big)+\frac{1}{\epsilon}\Big(\frac{145}{2}+24\zeta_2 \tag{A.21}$$
$$-\frac{869}{3}\zeta_3+\frac{1205}{12}\zeta_2^2\Big)-\frac{439}{2}-\frac{147}{2}\zeta_2+\frac{2528}{3}\zeta_3-\frac{13255}{24}\zeta_2^2-\frac{2326}{3}\zeta_3\zeta_2+\frac{5614}{3}\zeta_5$$
$$+\epsilon\Big(\frac{1309}{2}+\frac{435}{2}\zeta_2-\frac{7742}{3}\zeta_3+\frac{4820}{3}\zeta_2^2+\frac{12793}{3}\zeta_3\zeta_2-\frac{30877}{3}\zeta_5+\frac{565729}{1260}\zeta_2^3$$
$$-\frac{5108}{3}\zeta_3^2\Big)$$

$$F_{22}=\frac{23}{9\epsilon^6}+\frac{55}{9\epsilon^5}+\frac{1}{\epsilon^4}\Big(-\frac{2521}{36}-\frac{127}{3}\zeta_2\Big)+\frac{1}{\epsilon^3}\Big(\frac{1777}{9}-\frac{190}{3}\zeta_2-\frac{692}{3}\zeta_3\Big) \tag{A.22}$$
$$+\frac{1}{\epsilon^2}\Big(-\frac{11117}{18}+\frac{11419}{12}\zeta_2-\frac{400}{3}\zeta_3-\frac{11507}{90}\zeta_2^2\Big)+\frac{1}{\epsilon}\Big(\frac{32441}{18}-\frac{7993}{3}\zeta_2$$
$$+\frac{12061}{3}\zeta_3+\frac{6109}{18}\zeta_2^2+\frac{28472}{9}\zeta_3\zeta_2-\frac{15326}{3}\zeta_5\Big)-\frac{99143}{18}+\frac{50243}{6}\zeta_2-\frac{33388}{3}\zeta_3$$

$$-\frac{16091}{360}\zeta_2^2+\frac{13690}{9}\zeta_3\zeta_2+\frac{5165}{3}\zeta_5-\frac{539759}{630}\zeta_2^3+\frac{61504}{9}\zeta_3^2$$

$$F_{23}=\frac{115}{18\epsilon^6}-\frac{115}{36\epsilon^5}+\frac{1}{\epsilon^4}\left(-\frac{2645}{36}-\frac{1345}{18}\zeta_2\right)+\frac{1}{\epsilon^3}\left(\frac{3565}{18}+\frac{1345}{36}\zeta_2-\frac{2255}{9}\zeta_3\right) \quad (A.23)$$
$$+\frac{1}{\epsilon^2}\left(-\frac{5750}{9}+\frac{30935}{36}\zeta_2+\frac{2255}{18}\zeta_3+\frac{1235}{12}\zeta_2^2\right)+\frac{1}{\epsilon}\left(\frac{16445}{9}-\frac{41695}{18}\zeta_2\right.$$
$$\left.+\frac{51865}{18}\zeta_3-\frac{1235}{24}\zeta_2^2+\frac{26705}{9}\zeta_3\zeta_2-\frac{9910}{3}\zeta_5\right)-\frac{50945}{9}+\frac{67250}{9}\zeta_2-\frac{69905}{9}\zeta_3$$
$$-\frac{28405}{24}\zeta_2^2-\frac{26705}{18}\zeta_3\zeta_2+\frac{4955}{3}\zeta_5-\frac{280151}{252}\zeta_2^3+\frac{45385}{9}\zeta_3^2$$

$$F_{24}=\frac{1}{3\epsilon^6}+\frac{17}{9\epsilon^5}+\frac{1}{\epsilon^4}\left(-\frac{179}{12}-\frac{67}{9}\zeta_2\right)+\frac{1}{\epsilon^3}\left(\frac{370}{9}-\frac{89}{3}\zeta_2-\frac{568}{9}\zeta_3\right)+\frac{1}{\epsilon^2}\left(-\frac{2197}{18}\right. \quad (A.24)$$
$$\left.+\frac{9253}{36}\zeta_2-\frac{466}{3}\zeta_3-\frac{9659}{90}\zeta_2^2\right)+\frac{1}{\epsilon}\left(\frac{669}{2}-\frac{5840}{9}\zeta_2+\frac{14671}{9}\zeta_3-\frac{6923}{90}\zeta_2^2\right.$$
$$\left.+\frac{2836}{3}\zeta_3\zeta_2-2610\zeta_5\right)-\frac{16739}{18}+\frac{3441}{2}\zeta_2-\frac{33956}{9}\zeta_3+\frac{628781}{360}\zeta_2^2+\frac{18284}{9}\zeta_3\zeta_2$$
$$-3109\zeta_5-\frac{746701}{630}\zeta_2^3+\frac{7664}{3}\zeta_3^2$$

$$F_{25}=\frac{14}{9\epsilon^5}-\frac{1}{\epsilon^4}+\frac{1}{\epsilon^3}\left(-\frac{358}{9}-\frac{203}{9}\zeta_2\right)+\frac{1}{\epsilon^2}\left(\frac{2182}{9}+\frac{187}{6}\zeta_2-\frac{1159}{9}\zeta_3\right) \quad (A.25)$$
$$+\frac{1}{\epsilon}\left(-\frac{10072}{9}+\frac{3946}{9}\zeta_2+\frac{5333}{18}\zeta_3-\frac{6451}{45}\zeta_2^2\right)+\frac{40714}{9}-\frac{25759}{9}\zeta_2+1442\zeta_3$$
$$+\frac{48637}{90}\zeta_2^2+\frac{13613}{9}\zeta_3\zeta_2-3923\zeta_5+\epsilon\left(-\frac{154768}{9}+\frac{122299}{9}\zeta_2-10793\zeta_3\right.$$
$$\left.-\frac{19313}{45}\zeta_2^2-\frac{21757}{6}\zeta_3\zeta_2+\frac{22801}{2}\zeta_5-\frac{3443939}{1890}\zeta_2^3+\frac{25937}{9}\zeta_3^2\right)$$

$$F_{26}=2-\zeta_2+\epsilon\left(32-9\zeta_2-10\zeta_3\right)+\epsilon^2\left(313-\frac{165}{2}\zeta_2-90\zeta_3-\frac{111}{10}\zeta_2^2\right)+\epsilon^3\left(\frac{4855}{2}\right. \quad (A.26)$$
$$\left.-\frac{2905}{4}\zeta_2-669\zeta_3-\frac{999}{10}\zeta_2^2+164\zeta_3\zeta_2-268\zeta_5\right)+\epsilon^4\left(\frac{65815}{4}-\frac{45421}{8}\zeta_2\right.$$
$$\left.-\frac{9533}{2}\zeta_3-\frac{12339}{20}\zeta_2^2+1476\zeta_3\zeta_2-2412\zeta_5+\frac{1383}{70}\zeta_2^3+430\zeta_3^2\right)$$

$$F_{27}=\frac{2\zeta_3}{\epsilon^2}+\frac{1}{\epsilon}\left(-\zeta_3+\frac{52}{5}\zeta_2^2\right)-3\zeta_3-\frac{26}{5}\zeta_2^2-8\zeta_3\zeta_2+178\zeta_5+\epsilon\left(-8\zeta_3-\frac{78}{5}\zeta_2^2\right. \quad (A.27)$$
$$\left.+4\zeta_3\zeta_2-89\zeta_5+\frac{5342}{35}\zeta_2^3+10\zeta_3^2\right)$$

$$
\begin{aligned}
F_{28} &= -\frac{\zeta_2^3}{5} + 2\zeta_3^2 && \text{(A.28)} \\
F_{29} &= \frac{5}{9\epsilon^6} - \frac{25}{18\epsilon^5} + \frac{1}{\epsilon^4}\left(-\frac{5}{18} - \frac{55}{9}\zeta_2\right) + \frac{1}{\epsilon^3}\left(-\frac{5}{9} + \frac{371}{18}\zeta_2 - \frac{10}{9}\zeta_3\right) + \frac{1}{\epsilon^2}\left(-\frac{10}{9}\right. && \text{(A.29)} \\
&\quad \left. - \frac{425}{18}\zeta_2 + \frac{949}{9}\zeta_3 + \frac{9167}{90}\zeta_2^2\right) + \frac{1}{\epsilon}\left(-\frac{20}{9} + \frac{595}{9}\zeta_2 - \frac{4597}{9}\zeta_3 + \frac{11021}{180}\zeta_2^2 + \frac{578}{9}\zeta_3\zeta_2\right. \\
&\quad \left. + \frac{4357}{3}\zeta_5\right) - \frac{40}{9} - \frac{946}{9}\zeta_2 + \frac{10234}{9}\zeta_3 - \frac{292403}{180}\zeta_2^2 - \frac{10673}{9}\zeta_3\zeta_2 + \frac{4225}{2}\zeta_5 \\
&\quad + \frac{797803}{630}\zeta_2^3 + \frac{430}{9}\zeta_3^2 \\
F_{30} &= \frac{1}{6\epsilon^6} + \frac{13}{12\epsilon^5} + \frac{1}{\epsilon^4}\left(-\frac{25}{3} - \frac{25}{6}\zeta_2\right) + \frac{1}{\epsilon^3}\left(\frac{143}{6} - \frac{181}{12}\zeta_2 - \frac{98}{3}\zeta_3\right) + \frac{1}{\epsilon^2}\left(-\frac{443}{6}\right. && \text{(A.30)} \\
&\quad \left. + \frac{431}{3}\zeta_2 - \frac{169}{3}\zeta_3 - \frac{163}{4}\zeta_2^2\right) + \frac{1}{\epsilon}\left(\frac{1301}{6} - \frac{833}{2}\zeta_2 + 802\zeta_3 + \frac{5887}{120}\zeta_2^2 + \frac{1606}{3}\zeta_3\zeta_2\right. \\
&\quad \left. - \frac{3344}{3}\zeta_5\right) - \frac{3959}{6} + \frac{7979}{6}\zeta_2 - \frac{7286}{3}\zeta_3 + \frac{5891}{15}\zeta_2^2 + 797\zeta_3\zeta_2 + \frac{8}{3}\zeta_5 - \frac{348377}{1260}\zeta_2^3 \\
&\quad + \frac{4240}{3}\zeta_3^2 \\
F_{31} &= \frac{7}{9\epsilon^5} - \frac{17}{18\epsilon^4} + \frac{1}{\epsilon^3}\left(-\frac{143}{9} - \frac{125}{9}\zeta_2\right) + \frac{1}{\epsilon^2}\left(\frac{902}{9} + \frac{133}{6}\zeta_2 - \frac{236}{3}\zeta_3\right) + \frac{1}{\epsilon}\left(-\frac{4190}{9}\right. && \text{(A.31)} \\
&\quad \left. + \frac{716}{3}\zeta_2 + \frac{1418}{9}\zeta_3 - \frac{265}{6}\zeta_2^2\right) + \frac{16892}{9} - \frac{4709}{3}\zeta_2 + \frac{9718}{9}\zeta_3 + \frac{3373}{20}\zeta_2^2 + 1228\zeta_3\zeta_2 \\
&\quad - \frac{17612}{9}\zeta_5 + \epsilon\left(-\frac{63902}{9} + \frac{22181}{3}\zeta_2 - \frac{68062}{9}\zeta_3 - \frac{377}{5}\zeta_2^2 - \frac{23666}{9}\zeta_3\zeta_2 + \frac{48610}{9}\zeta_5\right. \\
&\quad \left. - \frac{688249}{1890}\zeta_2^3 + \frac{27128}{9}\zeta_3^2\right)
\end{aligned}
$$

References

[1] A. Gehrmann-De Ridder, T. Gehrmann and G. Heinrich, *Four particle phase space integrals in massless QCD*, *Nucl. Phys.* **B 682** (2004) 265 [hep-ph/0311276] [INSPIRE].

[2] O. Gituliar, *Master integrals for splitting functions from differential equations in QCD*, *JHEP* **02** (2016) 017 [arXiv:1512.02045] [INSPIRE].

[3] O. Gituliar and S. Moch, *Towards three-loop QCD corrections to the time-like splitting functions*, *Acta Phys. Polon.* **B 46** (2015) 1279 [arXiv:1505.02901] [INSPIRE].

[4] A.A. Almasy, S. Moch and A. Vogt, *On the Next-to-Next-to-Leading Order Evolution of Flavour-Singlet Fragmentation Functions*, Nucl. Phys. **B 854** (2012) 133 [arXiv:1107.2263] [INSPIRE].

[5] K.G. Chetyrkin and F.V. Tkachov, *Integration by Parts: The Algorithm to Calculate β-functions in 4 Loops*, Nucl. Phys. **B 192** (1981) 159 [INSPIRE].

[6] O.V. Tarasov, *Connection between Feynman integrals having different values of the space-time dimension*, Phys. Rev. **D 54** (1996) 6479 [hep-th/9606018] [INSPIRE].

[7] R.N. Lee and K.T. Mingulov, *DREAM, a program for arbitrary-precision computation of dimensional recurrence relations solutions and its applications*, arXiv:1712.05173 [INSPIRE].

[8] R.N. Lee and K.T. Mingulov, *Meromorphic solutions of recurrence relations and DRA method for multicomponent master integrals*, JHEP **04** (2018) 061 [arXiv:1712.05166] [INSPIRE].

[9] A.S. Schwarz, *Gauge theories on noncommutative spaces*, hep-th/0011261 [INSPIRE].

[10] J. Blumlein, D.J. Broadhurst and J.A.M. Vermaseren, *The Multiple Zeta Value Data Mine*, Comput. Phys. Commun. **181** (2010) 582 [arXiv:0907.2557] [INSPIRE].

[11] R.N. Lee and K.T. Mingulov, *Introducing SummerTime: a package for high-precision computation of sums appearing in DRA method*, Comput. Phys. Commun. **203** (2016) 255 [arXiv:1507.04256] [INSPIRE].

[12] H. Ferguson, D. Bailey and S. Arno, *Analysis of PSLQ, an integer relation finding algorithm*, Math. Comp. **68** (1999) 351.

[13] R. Kleiss, W.J. Stirling and S.D. Ellis, *A New Monte Carlo Treatment of Multiparticle Phase Space at High-energies*, Comput. Phys. Commun. **40** (1986) 359 [INSPIRE].

[14] P.A. Baikov and K.G. Chetyrkin, *Four Loop Massless Propagators: An Algebraic Evaluation of All Master Integrals*, Nucl. Phys. **B 837** (2010) 186 [arXiv:1004.1153] [INSPIRE].

[15] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, Int. J. Mod. Phys. **A 15** (2000) 5087 [hep-ph/0102033] [INSPIRE].

[16] S. Laporta, *High-precision calculation of the 4-loop contribution to the electron g-2 in QED*, Phys. Lett. **B 772** (2017) 232 [arXiv:1704.06996] [INSPIRE].

[17] A.V. Smirnov, *FIRE5: a C++ implementation of Feynman Integral REduction*, Comput. Phys. Commun. **189** (2015) 182 [arXiv:1408.2372] [INSPIRE].

[18] P. Nogueira, *Automatic Feynman graph generation*, J. Comput. Phys. **105** (1993) 279.

[19] B. Ruijl, T. Ueda and J. Vermaseren, *FORM version 4.2*, arXiv:1707.06453 [INSPIRE].

[20] R.N. Lee, *LiteRed 1.4: a powerful tool for reduction of multiloop integrals*, J. Phys. Conf. Ser. **523** (2014) 012059 [arXiv:1310.1145] [INSPIRE].

[21] G.P. Lepage, *A New Algorithm for Adaptive Multidimensional Integration*, J. Comput. Phys. **27** (1978) 192 [INSPIRE].

[22] T. Hahn, *CUBA: A Library for multidimensional numerical integration*, Comput. Phys. Commun. **168** (2005) 78 [hep-ph/0404043] [INSPIRE].

[23] B. Gough. *GNU Scientific Library Reference Manual*, 3rd edition, Network Theory Ltd. (2009).

[24] J.C. Collins and J.A.M. Vermaseren, *Axodraw Version 2*, arXiv:1606.01177 [INSPIRE].

JHEP06(2018)099