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$N = 4$ super-Schwarzian theory on the coadjoint orbit and $\text{PSU}(1,1|2)$

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ABSTRACT: An $N = 4$ super-Schwarzian theory is formulated by the coadjoint orbit method. It is discovered that the action has symmetry under $\text{PSU}(1,1|2)$.

KEYWORDS: 2D Gravity, Differential and Algebraic Geometry, Extended Supersymmetry

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1 Introduction

The Schwarzian theory has been drawing a lot of attention giving a hope to be a theoretical gateway between the SYK model and the $D = 2$ effective gravity. The Liouville gravity, which is the simplest one for the $D = 2$ effective gravity, was extensively studied by various methods after the pioneering Polyakov's work. (See [1, 2] for an overview of the studies at the early stage.) Among them the coadjoint orbit method proposed by Alekseev and Shatashvili is the most geometrical [3]. In a recent paper [4] this coadjoint orbit method was revisited to study the Schwarzian theory. The Hamilton structure of the Schwarzian theory was clarified through the formulation by the coadjoint orbit method.

The first aim of this paper is to formulate an $N = 4$ super-Schwarzian theory by means of the coadjoint orbit method. The lower symmetric cases have been discussed in the past few years. (See [5, 6] for instance.) But a proper account on the differential geometry has been given only recently in [4], but for the non-supersymmetric case.

As for supersymmetrization of the coadjoint orbit method for the Liouville gravity it is opportune to give a brief summary of the studies. After the work [3] the coadjoint orbit method was generalized to formulate the (1,0) and (2,0) supersymmetric theories in [7] and [8, 9] respectively. The left-moving sector was extended so as to admit the (1,0) and (2,0) superconformal symmetry. In the right-moving sector the conformal symmetry remained non-supersymmetric, but the symmetry $SL(2)$ got promoted to $OSp(2|1)$ and $OSp(2|2)$ for the respective supersymmetric theories [10, 11]. A further extension of the

coadjoint orbit method to the $N = (4, 0)$ supersymmetric case has not been discussed, although the $N = 4$ superconformal algebra has been known since a long time ago [12].

Once formulated an $N = 4$ super-Schwarzian theory the second aim of this paper is to examine if it has a symmetry further generalizing $\text{OSp}(2|2)$. We show that the theory indeed has symmetry under $\text{PSU}(1,1|2)$.

The paper is organized as follows. In section 2 we give a short summary of the coadjoint orbit method. In section 3 we discuss the $N = 4$ superconformal diffeomorphism and give the $N = 4$ super-Schwarzian derivative, which is a key element for the paper. Following these arguments the coadjoint orbit method is worked out to construct the $N = 4$ super-Schwarzian theory in section 4. In section 5 we show that the theory admits symmetry under $\text{PSU}(1,1|2)$. Three appendices are devoted to help the calculations in the main body of the paper.

2 A short summary of the coadjoint orbit method

We shall start with a brief review about the general construction of the Kirillov-Kostant 2-form on the coadjoint orbit of a Lie-group G [3]. Let $\mathfrak{g}$ to be a Lie-algebra of G and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. An element $g \in G$ acts on an element $a \in \mathfrak{g}$ by $\text{Ad}(g)a = gag^{-1}$, while the group action on an element X of the dual space $\mathfrak{g}^*$ is defined by means of an invariant quadratic form

$$\langle \text{Ad}^*(g)X, a \rangle = \langle X, \text{Ad}(g^{-1})a \rangle. \quad (2.1)$$

Elements $a \in \mathfrak{g}$ and $X \in \mathfrak{g}^*$ are called adjoint and coadjoint vectors respectively. We define the Kirillov-Kostant 2-form as

$$\Omega_X = \frac{1}{2} \langle X, [Y, Y] \rangle, \quad (2.2)$$

where Y is a $\mathfrak{g}$ -valued 1-form related with X by

$$dX = \text{ad}^*(Y)X. \quad (2.3)$$

Here $\text{ad}^*(Y)$ is the infinitesimal coadjoint action on X determined according to (2.1), i.e.,

$$\langle \text{ad}^*(Y)X, a \rangle = -\langle X, \text{ad}(Y)a \rangle = -\langle X, [Y, a] \rangle. \quad (2.4)$$

(2.3) defines an orbit in the dual space $\mathfrak{g}^*$, called the coadjoint orbit O_X . Owing to the Jacobi identity we can show that $dY = \frac{1}{2}[Y, Y]$ and the 2-form Ω_X is closed. This 2-form is a central tool for the coadjoint orbit method.

3 The $N = 4$ superconformal diffeomorphism

The $N = 4$ superconformal group is a group of which elements are superdiffeomorphism in the $N = 4$ superspace. A general account of the N -extended superconformal group was given in [13, 14]. The case of $N = 4$ was studied in [15]. Here we elaborate their arguments. The $N = 4$ superspace is described by the supercoordinates

$$(x, \theta_1, \theta_2, \theta^1, \theta^2) \equiv (x, \theta).$$

Here x is a real coordinate. $\theta_a, a = 1, 2$, are fermionic ones, while $\theta^a, a = 1, 2$, their complex conjugates. The supercovariant derivatives are defined by

$$\begin{aligned} D_{\theta a} &= \frac{\partial}{\partial \theta^a} + \theta_a \frac{\partial}{\partial x} \equiv \partial_{\theta a} + \theta_a \partial_x, \\ D_{\theta}^a &= \frac{\partial}{\partial \theta_a} + \theta^a \frac{\partial}{\partial x} \equiv \partial_{\theta}^a + \theta^a \partial_x, \end{aligned} \quad (3.1)$$

so as to satisfy

$$\{D_{\theta a}, D_{\theta}^b\} = 2\delta_a^b \partial_x, \quad \{D_{\theta a}, D_{\theta b}\} = 0, \quad \{D_{\theta}^a, D_{\theta}^b\} = 0. \quad (3.2)$$

We consider $N = 4$ superdiffeomorphisms

$$x \longrightarrow f(x, \theta), \quad \theta_a \longrightarrow \varphi_a(x, \theta), \quad \theta^a \longrightarrow \varphi^a(x, \theta). \quad (3.3)$$

The supercovariant derivatives change as

$$D_{\theta a} = D_{\theta a} f \frac{\partial}{\partial f} + D_{\theta a} \varphi_b \frac{\partial}{\partial \varphi_b} + D_{\theta a} \varphi^b \frac{\partial}{\partial \varphi^b}, \quad (3.4)$$

$$D_{\theta}^a = D_{\theta}^a f \frac{\partial}{\partial f} + D_{\theta}^a \varphi_b \frac{\partial}{\partial \varphi_b} + D_{\theta}^a \varphi^b \frac{\partial}{\partial \varphi^b}, \quad (3.5)$$

by the chain rule. Impose the chirality conditions

$$D_{\theta a} \varphi_b = 0, \quad D_{\theta}^a \varphi^b = 0, \quad (3.6)$$

and the superconformal conditions

$$D_{\theta a} f = \varphi_b D_{\theta a} \varphi^b, \quad D_{\theta}^a f = \varphi^b D_{\theta}^a \varphi_b. \quad (3.7)$$

Then (3.4) and (3.5) become supercovariant derivatives as

$$\begin{aligned} D_{\theta a} &= D_{\theta a} \varphi^b D_{\varphi b} \equiv D_{\theta a} \varphi^b \left(\frac{\partial}{\partial \varphi^b} + \varphi_b \frac{\partial}{\partial f} \right), \\ D_{\theta}^a &= D_{\theta}^a \varphi_b D_{\varphi}^b \equiv D_{\theta}^a \varphi_b \left(\frac{\partial}{\partial \varphi_b} + \varphi^b \frac{\partial}{\partial f} \right). \end{aligned}$$

When the supercovariant derivatives satisfy these transformation properties, the transformations in (3.3) are called superconformal diffeomorphisms. Elements of the superconformal group consist of such diffeomorphisms.

A superfield with weight 0, denoted by $\Psi_0(x, \theta)$, transforms as

$$\Psi_0(x, \theta) \longrightarrow \Psi_0(f(x, \theta), \varphi(x, \theta)),$$

by the superconformal diffeomorphisms. Infinitesimally it reads

$$\delta \Psi_0(x, \theta) = [\delta x \partial_x + \delta \theta_c \partial_{\theta}^c + \delta \theta^c \partial_{\theta c}] \Psi_0(x, \theta), \quad (3.8)$$

which may be put in the form

$$\delta \Psi_0(x, \theta) = [v \partial_x + \delta \theta_c D_{\theta}^c + \delta \theta^c D_{\theta c}] \Psi_0(x, \theta), \quad (3.9)$$

by using the supercovariant derivatives (3.1) and an infinitesimal parameter $v = v(x, \theta)$ given by

$$v = \delta x + \theta_c \delta \theta^c + \theta^c \delta \theta_c.$$

When the superconformal conditions (3.7) are imposed, the infinitesimally small parameters δx , $\delta \theta$ and $\delta \theta$ are constrained as

$$\begin{aligned} D_{\theta_a} \delta x &= \delta \theta_a + \theta_c D_{\theta_a} \delta \theta^c = \delta \theta_a - D_{\theta_a} (\theta_c \delta \theta^c), \\ D_{\theta^a} \delta x &= \delta \theta^a + \theta^c D_{\theta^a} \delta \theta_c = \delta \theta^a - D_{\theta^a} (\theta^c \delta \theta_c), \end{aligned}$$

which become respectively

$$\delta \theta_a = \frac{1}{2} D_{\theta_a} v, \quad \delta \theta^a = \frac{1}{2} D_{\theta^a} v, \quad (3.10)$$

by using the chirality condition (3.6) in the infinitesimal form

$$D_{\theta_a} \delta \theta_b = D_{\theta_a} \delta \varphi_b \Big|_{(f, \varphi) = (x, \theta)} = 0. \quad (3.11)$$

Using this we write the transformation $\delta \Psi_0$, given by (3.9), in a supercovariant form as

$$\delta_v \Psi_0 = [v \partial_x + \frac{1}{2} D_{\theta_c} v D_{\theta^c} + \frac{1}{2} D_{\theta^c} v D_{\theta_c}] \Psi_0.$$

Hereinafter we do not write the arguments of superfields explicitly if they are simply (x, θ) as here. This transformation law can be generalized to define a superfield having arbitrary weight w and charge q as

$$\delta_v \Psi_w = \left[v \partial_x + \frac{1}{2} D_{\theta_c} v D_{\theta^c} + \frac{1}{2} D_{\theta^c} v D_{\theta_c} + w \partial_x v + q [D_{\theta_c}, D_{\theta^c}] v \right] \Psi_w. \quad (3.12)$$

However the charge part of this transformation drops out, since $[D_{\theta_c}, D_{\theta^c}] v = 0$ as discussed in appendix B.

A posteriori we recognize that the superconformal diffeomorphisms (3.3) may be given by superfields with weight 0, but the fermionic ones are constrained by the chirality condition (3.6). That is,

$$\delta_v f = [v \partial_x + \frac{1}{2} D_{\theta_c} v D_{\theta^c} + \frac{1}{2} D_{\theta^c} v D_{\theta_c}] f, \quad (3.13)$$

$$\delta_v \varphi_a = [v \partial_x + \frac{1}{2} D_{\theta_c} v D_{\theta^c}] \varphi_a, \quad (3.14)$$

$$\delta_v \varphi^a = [v \partial_x + \frac{1}{2} D_{\theta^c} v D_{\theta_c}] \varphi^a. \quad (3.15)$$

We now propose that the $N = 4$ super-Schwarzian derivative¹

$$\mathcal{S}(f, \varphi; x, \theta) = \log \det [D_{\theta_a} \varphi^b(x, \theta)] \quad (3.16)$$

¹The $N = 4$ super-Schwarzian derivative of this form appeared in [14].

with the above superdiffeomorphisms. When expanded in components by using the formulae in appendix A, the purely bosonic part takes the form

$$\mathcal{S}(f, \varphi; x, \theta) = \log \partial_x h + \frac{1}{2}(\theta_a \theta^a)^2 \left[-\frac{\partial_x^3 h}{\partial_x h} + 2 \left(\frac{\partial_x^2 h}{\partial_x h} \right)^2 \right] + O(\eta), \quad (3.17)$$

in which h is the lowest component of the superfield f and $O(\eta)$ indicates contributions of fermion fields. The top component does not coincide with the non-supersymmetric Schwarzian derivative. However using this $\mathcal{S}(f, \varphi; x, \theta)$ we will find an $N = 4$ super-Schwarzian action in the next section, of which purely bosonic part is the usual non-supersymmetric Schwarzian one. (See (4.14) and the argument thereafter.) Or without going through such an argument we may convince ourselves that $\mathcal{S}(f, \varphi; x, \theta)$ in this form is indeed the $N = 4$ super-Schwarzian derivative. Namely it obeys the anomalous superdiffeomorphism with weight 0

$$\mathcal{S}(F(f, \varphi), \Phi(f, \varphi); x, \theta) = \mathcal{S}(F, \Phi; f, \varphi) + \mathcal{S}(f, \varphi; x, \theta),$$

which can be easily checked by the chain rule. Infinitesimally it reads

$$\delta_v \mathcal{S}(f, \varphi; x, \theta) = [v \partial_x + \frac{1}{2} D_{\theta c} v D_{\theta}^c + \frac{1}{2} D_{\theta}^c v D_{\theta c}] \mathcal{S}(f, \varphi; x, \theta) + \partial_x v. \quad (3.18)$$

The last term is the conformal anomaly.

From (3.18) it follows that the quantity $\det[D_{\theta a} \varphi^b(x, \theta)]$ is a superfield transforming as Ψ_1 , given by (3.12). On the other hand we can easily show that the quantity

$$\Delta \equiv \partial_x f + \varphi_a \partial_x \varphi^a + \varphi^a \partial_x \varphi_a, \quad (3.19)$$

is also a superfield obeying the same transformation law as Ψ_1 . Hence showing

$$\Delta = \det[D_{\theta a} \varphi^b(x, \theta)], \quad (3.20)$$

would give an alternative check of (3.18). The relation (3.20) is indeed proved in appendix B as well as $\Delta = \det[D_{\theta}^a \varphi_b(x, \theta)]$.

4 $N = 4$ super-Schwarzian theory

Now we are in a position to discuss the coadjoint orbit method to apply for the $N = 4$ superconformal algebra. The superconformal algebra $\mathfrak{g}$ and the dual space $\mathfrak{g}^*$ are centrally extended. Their elements are given by

$$(u, k) \in \mathfrak{g}, \quad (b, c) \in \mathfrak{g}^*.$$

Here k and c are central elements. u and b are bosonic superfields, obeying the superconformal transformations of Ψ_{-1} and Ψ_0 given by (3.12) respectively. The transformation of the latter may be centrally extended. The volume element of the $N = 4$ superspace, $dx d^4 \theta$, has weight 1, so that the invariant quadratic form is defined by

$$\langle (b, c), (u, k) \rangle = \int dx d^4 \theta \, bu + ck. \quad (4.1)$$

The centrally extended superconformal algebra $\mathfrak{g}$ is given by the infinitesimal adjoint action $\text{ad}(v, l)$ on $(u, k) \in \mathfrak{g}$

$$\begin{aligned} \text{ad}(v, l)(u, k) &= \left(v\partial_x u - u\partial_x v + \frac{1}{2}D_{\theta c}vD_{\theta}^c u + \frac{1}{2}D_{\theta}^c vD_{\theta c}u, \int dx d^4\theta v\partial_x u \right) \\ &\equiv [(u, k), (v, l)]. \end{aligned} \quad (4.2)$$

Then using the relation (2.4) yields the corresponding coadjoint action $\text{ad}^*(v, l)$ on $(b, c) \in \mathfrak{g}^*$ is given by

$$\text{ad}^*(v, l)(b, c) = \left([v\partial_x + \frac{1}{2}D_{\theta c}vD_{\theta}^c + \frac{1}{2}D_{\theta}^c vD_{\theta c}]b + c\partial_x v, 0 \right). \quad (4.3)$$

We think of a coadjoint orbit $O_{(b, c)}$, whose initial point is $(b, c) \in \mathfrak{g}^*$. The finite form of (4.3) is generated on the coadjoint orbit by the superdiffeomorphism (3.3) as²

$$\text{Ad}^*(f, \varphi)(b, c) \equiv (b(f, \varphi) + c\mathcal{S}(f, \varphi; x, \theta), c). \quad (4.4)$$

Here $\mathcal{S}(f, \varphi; x, \theta)$ is the super-Schwarzian derivative given by (3.16).

Now the Kirillov-Kostant 2-form (2.2) can be given by

$$\Omega_{(b, c)} = \frac{1}{2} \langle \text{Ad}^*(f, \varphi)(b, c), [(y, 0), (y, 0)] \rangle, \quad (4.5)$$

on the coadjoint orbit $O_{(b, c)}$. Here the commutator was given by (4.2). $(y, 0)$ is a centrally extended $\mathfrak{g}$ -valued 1-form in a space parameterizing the coadjoint orbit. Namely we think of the superfields f, φ_c, φ^c in a fictitious space beyond the $D = 1, N = 4$ superspace as $f(x, \theta, t_1, t_2, \dots)$ etc. The 1-form y is a function of them. We should have written it as $y(f, \varphi)$ according to our convention. But we would not like to do it for simplicity hereinafter too. y is determined so that the exterior derivative of the quantity (4.4), which is an element of $\mathfrak{g}^*$, is induced by the infinitesimal coadjoint action (4.3) on it along the orbit $O_{(b, c)}$ as

$$d\text{Ad}^*(f, \varphi)(b, c) = \text{ad}^*(y, 0) \left(b(f, \varphi) + c\mathcal{S}(f, \varphi; x, \theta), c \right). \quad (4.6)$$

Keep in mind that the exterior derivative acts only on the coordinates $t_1, t_2, \dots$. It is the most important step in our arguments to find an explicit form of y by solving this equation. It turns out that the solution is given by

$$y = \frac{1}{\Delta} (df + \varphi_c d\varphi^c + \varphi^c d\varphi_c), \quad (4.7)$$

with Δ defined by (3.19). Once found y as a solution to (4.6), the centrally extended commutator in (4.5) becomes

$$[(y, 0), (y, 0)] = \left(2y\partial_x y + D_{\theta c}yD_{\theta}^c y, \int dx d^4\theta y\partial_x y \right), \quad (4.8)$$

from (4.2).

²We are sticking to the convention employed below (3.11), that is, superfields always depend on (x, θ, t) , if the arguments are not written explicitly. So this convention is applied to the superfields $b, f, \varphi_c, \varphi^c$ herein.

We shall verify that y given by (4.7) indeed solves the equation (4.6). Using (4.3) we may rewrite (4.6) by a pair of the equations

$$db(f, \varphi) = [y\partial_x + \frac{1}{2}D_{\theta c}yD_{\theta}^c + \frac{1}{2}D_{\theta}^cyD_{\theta c}]b(f, \varphi), \quad (4.9)$$

$$d\mathcal{S}(f, \varphi; x, \theta) = [y\partial_x + \frac{1}{2}D_{\theta c}yD_{\theta}^c + \frac{1}{2}D_{\theta}^cyD_{\theta c}]\mathcal{S}(f, \varphi; x, \theta) + \partial_x y. \quad (4.10)$$

Compare these with the respective superdiffeomorphisms

$$\delta_v b(f, \varphi) = [v\partial_x + \frac{1}{2}D_{\theta c}vD_{\theta}^c + \frac{1}{2}D_{\theta}^cvD_{\theta c}]b(f, \varphi) \quad (4.11)$$

and $\delta_v \mathcal{S}(f, \varphi; x, \theta)$ given by (3.18). The former one can be verified by the infinitesimal variation (3.8), in which $\Psi_0(x, \theta) = b(f(x, \theta), \varphi(x, \theta))$. The equations (4.9) and (4.10) require that the exterior derivatives of $b(f, \varphi)$ and $\mathcal{S}(f, \varphi; x, \theta)$ coincide with their superconformal diffeomorphisms, if the infinitesimal parameter v is replaced by y . In a mathematical language we can put it as

$$i_v db(f, \varphi) = \delta_v b(f, \varphi), \quad i_v d\mathcal{S}(f, \varphi; x, \theta) = \delta_v \mathcal{S}(f, \varphi; x, \theta).$$

Here i_v is the anti-derivative of the differential form, implying the operation

$$i_v df = \delta_v f, \quad i_v d\varphi_a = \delta_v \varphi_a, \quad i_v d\varphi^a = \delta_v \varphi^a, \quad (4.12)$$

of which the r.h.s.s have been given by (3.13)~(3.15). Or the equation (4.6) boils down to the following simple equation

$$i_v y = v. \quad (4.13)$$

The 1-form y given by (4.7) indeed satisfies this equation by the operation (4.12). Thus we have proved that it is a right solution for (4.6). The above arguments might have become too abstract. In appendix B we show that the equations in (4.12) are obtained from (4.7) by elementary calculations.

The Kirillov-Kostant 2-form (4.5) is invariant under the $N = 4$ superdiffeomorphism by the definition of the quadratic form (4.1). Therefore we have $(di_v + i_v d)\Omega_{(b,c)} = 0$. $\Omega_{(b,c)}$ is closed so that there exists a quantity such as $i_v \Omega_{(b,c)} = dH$. We shall show that it takes the form

$$H = \int dx d^4\theta \, v (b(f, \varphi) + c\mathcal{S}(f, \varphi; x, \theta)), \quad (4.14)$$

with the $N = 4$ super-Schwarzian derivative (3.16). To check the claim let us put the Kirillov-Kostant 2-form (4.5) in an explicit form as

$$2\Omega_{(b,c)} = \int dx d^4\theta \left[(b(f, \varphi) + c\mathcal{S}(f, \varphi; x, \theta)) (2y\partial_x y + D_{\theta c}yD_{\theta}^cy) + cy\partial_x y \right],$$

by (4.1) with (4.4) and (4.8). Take the anti-derivative and use (4.13). By integration by part we get

$$i_v (2\Omega_{(b,c)}) = \int dx d^4\theta \, 2vd (b(f, \varphi) + c\mathcal{S}(f, \varphi; x, \theta)),$$

owing to by (4.9) and (4.10). Thus (4.14) has been shown. It is worth knowing about non-supersymmetric approximation of $\mathcal{S}(f, \varphi; x, \theta)$. By using (3.17) and the expanding formula of v , given in appendix A, we find the top component of the integrand as

$$v\mathcal{S}(f, \varphi; x, \theta) = \dots + \frac{1}{2}(\theta \cdot \theta)^2 \left\{ -(\partial^2 \alpha) \log \partial_x h + \alpha \left[-\frac{\partial_x^3 h}{\partial_x h} + 2 \left(\frac{\partial_x^2 h}{\partial_x h} \right)^2 \right] + O(\eta) \right\}$$

Here α is the lowest component of v and $dv = 0$. Upon integrating the first term by part the top component of the integrand becomes the ordinary Schwarzian derivative multiplied by -2 . So there is nothing wrong to have claimed that $\mathcal{S}(f, \varphi; x, \theta)$ given by (3.16) is the $N = 4$ super-Schwarzian derivative of which purely bosonic part is given by (3.17). Finally putting $v = 1$ in (4.14) leads us to the desired super-Schwarzian action.

5 PSU(1,1|2) symmetry

In this section we show symmetry of the action (4.14) under PSU(1,1|2). The action depends on the initial point b of the coadjoint orbit. We discuss the issue dividing the dependence into two cases. For each case we are involved in different realization of the PSU(1,1|2) symmetry.

i) $b = 0$. We expect it to be realized on a supermanifold whose local coordinates are the superdiffeomorphism f, φ_a, φ^a discussed in section 3 and their complex conjugate $\bar{f}, \bar{\varphi}_a, \bar{\varphi}^a$. Such a supermanifold is given by the coset space $\text{PSU}(1,1|2)/\{\text{SU}(2) \otimes \text{U}(1)\}$ for which the generators of PSU(1,1|2) are decomposed as

$$\{T^A\} = \underbrace{\{L, F_a, F^a, \bar{L}, \bar{F}_a, \bar{F}^a\}}_{\substack{\text{PSU}(1,1|2) \\ \text{SU}(2) \otimes \text{U}(1)}} \underbrace{\{L^0, R_b^a\}}_{\text{SU}(2) \otimes \text{U}(1)}. \quad (5.1)$$

The coset generators L, F^a, F_a correspond to the coordinates f, φ_a, φ^a . The fermionic coordinates φ_a and φ^a are doublets of the subgroup SU(2). It is well-known that PSU(1,1|2) can be embedded in the larger supergroup D(2,1; γ). We may write the fermionic generators and the corresponding coordinates by using the notation of D(2,1; γ) as

$$F^{a\dot{\alpha}} = \begin{pmatrix} F^{1\dot{1}} & F^{1\dot{2}} \\ F^{2\dot{1}} & F^{2\dot{2}} \end{pmatrix}, \quad \varphi_{a\dot{\alpha}} = \begin{pmatrix} \varphi_{1\dot{1}} & \varphi_{1\dot{2}} \\ \varphi_{2\dot{1}} & \varphi_{2\dot{2}} \end{pmatrix}.$$

with the identifications

$$\begin{aligned} F^a &= \frac{1}{\sqrt{2}} \begin{pmatrix} F^{1\dot{1}} \\ F^{2\dot{1}} \end{pmatrix}, & F_a &= \frac{1}{\sqrt{2}} \begin{pmatrix} F^{2\dot{2}} \\ -F^{1\dot{2}} \end{pmatrix}, \\ \varphi_a &= \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_{1\dot{1}} \\ \varphi_{2\dot{1}} \end{pmatrix}, & \varphi^a &= \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_{2\dot{2}} \\ -\varphi_{1\dot{2}} \end{pmatrix}. \end{aligned} \quad (5.2)$$

See [16] for the more precise relation between the generators of PSU(1,1|2) and D(2,1; γ). Knowing the Lie algebra of D(2,1; γ) given in a rather simple form, we can write down that

of $\text{PSU}(1,1|2)$ as

$$\begin{aligned}
 [R_b^a, R_d^c] &= -\delta_b^c R_d^a + \delta_d^a R_b^c, & [R_b^a, L^0] &= 0, \\
 [\bar{L}, L] &= 2L^0, & [\bar{L}, L^0] &= \bar{L}, \\
 [L, L^0] &= -L, & [\bar{F}^{a\dot{\alpha}}, L^0] &= \frac{1}{2}\bar{F}^{a\dot{\alpha}}, \\
 [F^{a\dot{\alpha}}, L^0] &= -\frac{1}{2}F^{a\dot{\alpha}}, & [\bar{F}^{a\dot{\alpha}}, L] &= F^{a\dot{\alpha}}, \\
 [F^{a\dot{\alpha}}, L] &= 0, & [\bar{F}^{a\dot{\alpha}}, \bar{L}] &= 0, \\
 [F^{a\dot{\alpha}}, \bar{L}] &= -\bar{F}^{a\dot{\alpha}}, & [\bar{F}^{a\dot{\alpha}}, R_c^b] &= \delta_c^a \bar{F}^{b\dot{\alpha}} - \frac{1}{2}\delta_c^b \bar{F}^{a\dot{\alpha}}, \\
 [F^{a\dot{\alpha}}, R_c^b] &= \delta_c^a F^{b\dot{\alpha}} - \frac{1}{2}\delta_c^b F^{a\dot{\alpha}}, & \{\bar{F}^{a\dot{\alpha}}, \bar{F}^{b\dot{\beta}}\} &= -\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{L}, \\
 \{F^{a\dot{\alpha}}, F^{b\dot{\beta}}\} &= -\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}L, & \{F^{a\dot{\alpha}}, \bar{F}^{b\dot{\beta}}\} &= -\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}L^0 + \varepsilon^{ac}\varepsilon^{\dot{\alpha}\dot{\beta}}R_c^b.
 \end{aligned} \tag{5.3}$$

By means of these commutation relations we can calculate the Killing vectors on the coset space $\text{PSU}(1,1|2)/\{\text{SU}(2)\otimes\text{U}(1)\}$ following the general method developed in [17]. They were worked out in [18]³

$$\begin{aligned}
 \delta_\epsilon f &\equiv -i\epsilon_A R^A \\
 &= \epsilon_L + f\epsilon_{L^0} + \frac{1}{2}\varphi_{a\dot{\alpha}}\epsilon_{Fb\dot{\beta}}\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}} + \frac{1}{2}\left(2f^2\epsilon_{\bar{L}} + f\varphi_{a\dot{\alpha}}\epsilon_{\bar{F}b\dot{\beta}}\varepsilon^{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}\right) \\
 &\quad - \frac{1}{12}\varphi_{a\dot{\alpha}}\varphi_{b\dot{\beta}}\varphi_{c\dot{\gamma}}\epsilon_{\bar{F}d\dot{\delta}}\varepsilon^{cb}\varepsilon^{\dot{\gamma}\dot{\delta}}\varepsilon^{ad}\varepsilon^{\dot{\alpha}\dot{\beta}} - \frac{1}{24}\varphi_{a\dot{\alpha}}\varphi_{b\dot{\beta}}\varphi_{c\dot{\gamma}}\varphi_{d\dot{\delta}}\epsilon_{\bar{L}}\varepsilon^{ac}\varepsilon^{\dot{\gamma}\dot{\delta}}\varepsilon^{db}\varepsilon^{\dot{\alpha}\dot{\beta}}, \tag{5.4}
 \end{aligned}$$

$$\begin{aligned}
 \delta_\epsilon \varphi_{a\dot{\alpha}} &\equiv -i\epsilon_A R_{a\dot{\alpha}}^A \\
 &= \epsilon_{Fa\dot{\alpha}} + f\epsilon_{\bar{F}a\dot{\alpha}} + \frac{1}{2}\varphi_{a\dot{\alpha}}\epsilon_{L^0} - \varphi_{b\dot{\alpha}}\epsilon_{R^b{}_a} \\
 &\quad + \frac{1}{2}\left(2f\varphi_{a\dot{\alpha}}\epsilon_{\bar{L}} + \varphi_{b\dot{\alpha}}\varphi_{c\dot{\gamma}}\epsilon_{\bar{F}a\dot{\beta}}\varepsilon^{bc}\varepsilon^{\dot{\beta}\dot{\gamma}}\right) + \frac{1}{6}\varphi_{b\dot{\alpha}}\varphi_{c\dot{\gamma}}\varphi_{a\dot{\beta}}\epsilon_{\bar{L}}\varepsilon^{bc}\varepsilon^{\dot{\beta}\dot{\gamma}}. \tag{5.5}
 \end{aligned}$$

Here R^A and $R_{a\dot{\alpha}}^A$ are the Killing vectors satisfying the Lie algebra of $\text{PSU}(1,1|2)$. ϵ_A are infinitesimal parameters of the transformation corresponding to the generators of $\text{PSU}(1,1|2)$, given by (5.1).

It is not guaranteed at all that the $\text{PSU}(1,1|2)$ transformations generated by these Killing vectors (5.4) and (5.5) respect the chirality conditions (3.6) as well as the superconformal conditions (3.7). So we claim that

$$D_{\theta a}\delta_\epsilon \varphi_b = 0, \quad D_\theta^a \delta_\epsilon \varphi^b = 0, \tag{5.6}$$

and

$$D_{\theta a}\delta_\epsilon f = \delta_\epsilon \varphi_b D_{\theta a}\varphi^b + \varphi_b D_{\theta a}\delta_\epsilon \varphi^b, \quad D_\theta^a \delta_\epsilon f = \delta_\epsilon \varphi^b D_\theta^a \varphi_b + \varphi^b D_\theta^a \delta_\epsilon \varphi_b. \tag{5.7}$$

³Precisely speaking, it was the Killing vectors of the coset space $\text{PSU}(2|2)/\{\text{SU}(2)\otimes\text{U}(1)\}$ that were calculated in [18]. There use was made of the Lie algebra of $\text{PSU}(2|2)$, which is given by (5.3) with $\bar{L}$ replaced by $-\bar{L}$. The Killing vectors given below in this paper can be obtained from those given by (2.43) and (2.44) in [18] with the replacement $\epsilon_{\bar{L}} \rightarrow -\epsilon_{\bar{L}}$.

This claim will be verified in appendix C. Therefore it makes perfect sense to study the transformation property of the $N = 4$ super-Schwarzian action by the Killing vectors (5.4) and (5.5). Remarkably we find the quantity Δ , given by (3.19), to obey a fairly simple transformation as

$$\delta_\epsilon \Delta = \left(\epsilon_{L^0} + 2f\epsilon_{\bar{L}} + \varphi_{c\dot{\gamma}}\epsilon_{\bar{F}d\dot{\delta}}\epsilon^{cd}\epsilon^{\dot{\gamma}\dot{\delta}} \right) \Delta. \quad (5.8)$$

This follows by a straightforward calculation with the use of Δ written in the notation of $D(2, 1; \gamma)$ as

$$\Delta = \partial_x f + \frac{1}{2} \varphi_{c\dot{\gamma}} \partial_x \varphi_{d\dot{\delta}} \epsilon^{cd} \epsilon^{\dot{\gamma}\dot{\delta}}.$$

As the result the action (4.14) with $b(f, \varphi) = 0$ transforms as

$$\delta_\epsilon H|_{v=1, b=0} = c \int dx d^4\theta \delta_\epsilon \log \Delta = c \int dx d^4\theta \left(\epsilon_{L^0} + 2f\epsilon_{\bar{L}} + \varphi_{c\dot{\gamma}}\epsilon_{\bar{F}d\dot{\delta}}\epsilon^{cd}\epsilon^{\dot{\gamma}\dot{\delta}} \right), \quad (5.9)$$

in which $\mathcal{S}(f, \varphi; x, \theta) = \log \Delta$ owing to (3.16) and (3.20). We find that the top component of the integrand is of the form $\partial_x(\dots)$, when the superfields f and $\varphi_{c\dot{\gamma}}$ are expanded in components as in appendix A and use is made of the second equation in (A.1). Therefore the Schwarzian action $H|_{v=1, b=0}$ is invariant under the $\text{PSU}(1, 1|2)$ transformations generated by the Killing vectors (5.4) and (5.5).

ii) $b \neq 0$. The infinitesimal parameter v of the $N = 4$ superdiffeomorphism is expanded in components in appendix A. The modes of the components

$$\alpha = e^{\pm i n x} \alpha_{\pm n}, \quad \alpha_0, \quad (5.10)$$

$$\beta_a = e^{\pm \frac{1}{2} i n x} \beta_{a \pm \frac{1}{2} n}, \quad (5.11)$$

$$\beta^a = e^{\pm \frac{1}{2} i n x} \beta^a_{\pm \frac{1}{2} n}, \quad (5.12)$$

$$t^i = t_0^i, \quad (5.13)$$

span the $N = 4$ superconformal algebra [12]. The $\text{PSU}(1, 1|2)$ symmetry is realized also by the modes of the diffeomorphisms with n odd. They sequentially correspond to the generators

$$\begin{aligned} &L, \quad \bar{L}, \quad L^0 \\ &F_a, \quad \bar{F}_a, \\ &F^a, \quad \bar{F}^a, \\ &R^i, \end{aligned}$$

in (5.3).⁴ It is wise to write the Schwarzian action (4.14) as

$$\begin{aligned} H_{v=1} &= \int dx d^4\theta (b(f, \varphi) + c\mathcal{S}(f, \varphi; x, \theta)) \\ &\equiv \int dx d^4\theta \text{Ad}^*(f, \varphi) b(x, \theta). \end{aligned} \quad (5.14)$$

⁴Note that $R^i = (\sigma^i)^a_b R^b_a$.

In the second line we have abused the definition (4.4) since the c -dependence of the initial point of the coadjoint orbit $O_{(b,c)}$ is implicit. But as for the arguments of the initial point b we have made it explicit as $b(x, \theta)$ against the convention employed below (3.10). Now the question is if there exists a certain configuration of $b(x, \theta)$ with which the Schwarzian action is invariant by the superdiffeomorphism given by (5.10)~(5.13) with n odd. It may be examined at the initial point of the coadjoint orbit $O_{b,c}$, i.e.,

$$\begin{aligned} [\delta_v \text{Ad}^*(f, \varphi) b(x, \theta)] \Big|_{(f, \varphi)=(x, \theta)} &= [\delta_v (b(f, \varphi) + c\mathcal{S}(f, \varphi; x, \theta))] \Big|_{(f, \varphi)=(x, \theta)} \\ &= v[\partial_x + \frac{1}{2}D_{\theta a}vD_{\theta}^a + \frac{1}{2}D_{\theta}^a vD_{\theta a}]b(x, \theta) + c\partial_x v. \end{aligned} \quad (5.15)$$

Here use was made of (3.18) and (4.11).

We may proceed the argument quite analogously to the non-supersymmetric case, but in a much simpler way. The Schwarzian action is found as

$$H|_{v=1} = \int dx ((\partial_x h)^2 b(h) + c\mathcal{S}(h; x)) \equiv \int dx \text{Ad}^*(h)b(x). \quad (5.16)$$

Having conformal weight 2 the field $b(x)$ gets scaled with a factor $(\partial_x h)^2$ by the coadjoint action. Assuming $b(x) \neq 0$ we require that

$$\delta_{\alpha} \text{Ad}^*(h)b(x) = [\alpha \partial_x + 2\partial_x \alpha]b(x) + c\partial_x^3 \alpha = 0 \quad (5.17)$$

under the diffeomorphism with an infinitesimal parameter α . It is important to observe that this is a third-order equation for α . If $b(x)$ is constant, then it is solved by any constant α . It implies that the action is invariant under $U(1)$ symmetry generated by L^0 . If $b(x)$ is fixed to be $\frac{1}{2}cn^2$, then (5.17) admits three independent solutions of the form (5.10). The symmetry of the action is enhanced to $SL(2)$. This result is well-known in [3, 8, 19, 20] as well as [4]. For the case of $b(x) = 0$ refer to a comment in the end of the paper.

Let us turn to the $N = 4$ Schwarzian action (5.14). The superfield $b(x, \theta)$ is expanded in components as the superfield f was done in appendix A, i.e.,

$$\begin{aligned} b(x, \theta) &= a + \theta \cdot \gamma + \gamma \cdot \theta + \theta \cdot \theta i + (\theta \sigma^i \theta) s^i \\ &\quad + \frac{1}{2}\epsilon_{ab}\theta^a\theta^b j + \frac{1}{2}\epsilon^{ab}\theta_a\theta_b k + (\theta \cdot \theta)(\theta \cdot \sigma) + (\theta \cdot \theta)(\sigma \cdot \theta) + (\theta \cdot \theta)^2 d. \end{aligned} \quad (5.18)$$

Here the arguments of the component fields have been omitted according to our convention. Put this expansion as well as that of v , also given in appendix A, into the second line of (5.15). Calculating its top component we have⁵

$$\begin{aligned} [\delta_v H_{v=1}] \Big|_{(f, \varphi)=(x, \theta)} &= \int dx \left\{ 2\partial_x d\alpha + 4d\partial_x \alpha + \partial_x a \partial_x^2 \alpha - c\partial_x^3 \alpha \right. \\ &\quad + \frac{1}{2} \left(-(\partial_x \sigma \cdot \beta) - (\partial_x \gamma \cdot \partial_x \beta) - 3(\sigma \cdot \partial_x \beta) + (\gamma \cdot \partial_x^2 \beta) \right. \\ &\quad \left. \left. + (\beta \cdot \partial_x \sigma) - (\partial_x \beta \cdot \partial_x \gamma) + 3(\partial_x \beta \cdot \sigma) + (\partial_x^2 \beta \cdot \gamma) \right) - 4s^i \partial_x t^i \right\}. \end{aligned}$$

⁵Our convention is that $\int d^4\theta (\theta \cdot \theta)^2 = 2$.

We find that it is vanishing by the diffeomorphism (5.10)~(5.13) when the initial point b has a configuration such as

$$\begin{aligned} a &= 0, & d &= -\frac{1}{4}cn^2 & s^i &= s_0^i, \\ \gamma_a &= e^{\pm \frac{\sqrt{3}}{2}nx} \gamma_{a0}, & \gamma^a &= e^{\pm \frac{\sqrt{3}}{2}nx} \gamma^a_0, \end{aligned} \quad (5.19)$$

with

$$\sigma_a = -\frac{1}{3}\partial_x \gamma_a, \quad \sigma^a = \frac{1}{3}\partial_x \gamma^a.$$

Thus the Schwarzian action is invariant under $\text{PS}(1,1|2)$. But it is worth remarking that we do not encounter boundary terms at all in examining the symmetry of the integrand. It is also worth recognize that the solution contains the non-supersymmetric one in the previous paragraph by setting $2d = -b$.

The reader may ask about symmetry for the density of the Schwarzian action (5.14). Then the variation (5.15) is required to vanish at lower orders of θ as well. The resulting differential equations are too stringent to be satisfied by the above solution. For instance at the lowest order of θ it reads

$$\partial_x a \alpha + c \partial_x \alpha + \frac{1}{2}(\beta \cdot \gamma) + \frac{1}{2}(\gamma \cdot \beta) = 0.$$

More stringent equations come out at higher orders. Nonetheless it is not hard to see that all the equations are satisfied by the subset of the modes

$$\alpha = \alpha_0, \quad \beta_a = \beta^a = 0, \quad t^i = t_0^i,$$

in (5.10)~(5.13), when b has a configuration such as

$$d = d_0, \quad \text{others} = 0.$$

Therefore the subgroup $\text{SU}(2) \otimes \text{U}(1)$ is also a symmetry of the density of the Schwarzian action (5.14).

We content ourselves with these solutions, although our analysis of the differential equations is not exhaustive at all. In summary, the partition function of the $N = 4$ super-Schwarzian theory is given by

$$Z = \int_{\mathcal{M}} \mathcal{D}f \mathcal{D}\varphi_a \mathcal{D}\varphi^a \exp(H|_{v=1}), \quad \mathcal{M} = \text{superdiff}/\text{PSU}(1,1|2),$$

when the action is symmetric under $\text{PSU}(1,1|2)$.

6 Conclusions

In this paper we have formulated an $N = 4$ super-Schwarzian action by means of the coadjoint orbit method. The action is dependent on the initial point b of the orbit. For the case of $b = 0$ it has been shown to have symmetry under $\text{PSU}(1,1|2)$ realized by the Killing vectors for the coset space $\text{PSU}(1,1|2)/\{\text{SU}(2) \otimes \text{U}(1)\}$. When $b \neq 0$ we have also

shown that it becomes invariant by a set of modes of the superdiffeomorphism realizing PSU(1,1|2). For that we have found a configuration of b such as given by (5.19).

We comment on the case of $b = 0$ for the non-supersymmetric Schwarzian action (5.16), which we have not discussed in section 5. The non-supersymmetric Schwarzian derivative $\mathcal{S}(h; x)$ is invariant under SL(2) realized by the Killing vectors for the coset space SL(2)/U(1). However (5.9) implies that the Schwarzian derivative $\mathcal{S}(f, \varphi; x, \theta)$ for the $N = 4$ case is invariant only modulo boundary terms $\partial_x(\dots)$ by the same transformation. This discrepancy is not a problem because the purely bosonic part of $\mathcal{S}(f, \varphi; x, \theta)$ is given by (3.17) and the top component giving the action reads

$$-\mathcal{S}(h; x) + \frac{1}{2} \left(\frac{\partial_x^2 h}{\partial_x h} \right)^2.$$

The additional term is invariant modulo the boundary term $2\epsilon_L \partial_x^2 h$ under SL(2) realized by the Killing vectors. It is consistent with (5.9).

It is desirable to study quantum dynamics of the $N = 4$ super-Schwarzian action. Our study on this is in progress. It is also desirable to extend the $D = 2$ Liouville gravity to the $N = 4$ supersymmetric one. It will be reported in [21].

A Superfields in components

In the body of the paper the $N = 4$ super-Schwarzian derivative $\mathcal{S}$ was needed to be expanded in components. We give here only the expansion for the basic ones. The superfields f, φ_c, φ^c which describe the $N = 4$ superdiffeomorphism are expanded as

$$\begin{aligned} f(x, \theta) &= h(x) + \theta \cdot \psi(x) + \psi(x) \cdot \theta + \theta \cdot \theta l(x) + (\theta \sigma^i \theta) t^i(x) \\ &\quad + \frac{1}{2} \epsilon_{ab} \theta^a \theta^b m(x) + \frac{1}{2} \epsilon^{ab} \theta_a \theta_b n(x) + (\theta \cdot \theta)(\theta \cdot \omega(x)) + (\theta \cdot \theta)(\omega(x) \cdot \theta) + (\theta \cdot \theta)^2 g(x), \\ \varphi_c(x, \theta) &= \rho(x + \theta \cdot \theta) \left[\theta_c + \eta_c(x + \theta \cdot \theta) + \frac{1}{2} \epsilon^{ab} \theta_a \theta_b \alpha_c(x)(x + \theta \cdot \theta) \right], \\ \varphi^c(x, \theta) &= \xi(x - \theta \cdot \theta) \left[\theta^c + \eta^c(x - \theta \cdot \theta) + \frac{1}{2} \epsilon_{ab} \theta^a \theta^b \alpha^c(x - \theta \cdot \theta) \right], \end{aligned}$$

with $\theta \cdot \psi \equiv \theta_a \psi^a$, $\psi \cdot \theta \equiv \psi_a \theta^a$ etc. Note that the component fields of φ_c and φ^c got the argument x shifted so that the chirality conditions (3.6) are satisfied. By imposing the superconformal conditions (3.7) they become

$$\begin{aligned} f(x, \theta) &= h + \rho \xi [\theta \cdot \eta - \eta \cdot \theta] \\ &\quad + \theta \cdot \theta \partial_x (\rho \eta \cdot \xi \eta) + 2 \frac{\xi}{\rho} (\rho \eta \cdot \theta) (\partial_x (\rho \eta) \cdot \theta) + 2 \frac{\rho}{\xi} (\theta \cdot \xi \eta) (\theta \cdot \partial_x (\xi \eta)) \\ &\quad + \theta \cdot \theta [(\theta \cdot \partial_x (\rho \xi \eta)) + (\partial_x (\xi \rho \eta) \cdot \theta)] \\ &\quad + \frac{1}{2} (\theta \cdot \theta)^2 [- (\rho \eta \cdot \partial_x^2 (\xi \eta)) + (\partial_x^2 (\rho \eta) \cdot \xi \eta) + \xi \partial_x \rho + \rho \partial_x \xi], \\ \varphi_c(x, \theta) &= \rho \eta_c + \rho \theta_c + \theta \cdot \theta \partial_x (\rho \eta_c) + \frac{1}{2} \epsilon^{ab} \theta_a \theta_b \rho \alpha_c + \theta \cdot \theta \theta_c \partial_x \rho + \frac{1}{2} (\theta \cdot \theta)^2 \partial_x^2 (\rho \eta_c), \\ \varphi^c(x, \theta) &= \xi \eta^c + \xi \theta^c - \theta \cdot \theta \partial_x (\xi \eta^c) + \frac{1}{2} \epsilon_{ab} \theta^a \theta^b \xi \alpha^c - \theta \cdot \theta \theta^c \partial_x \xi + \frac{1}{2} (\theta \cdot \theta)^2 \partial_x^2 (\xi \eta^c), \end{aligned}$$

with the remaining constraints

$$\begin{aligned}\xi\partial_x\rho &= \rho\partial_x\xi, & \partial_x h + (\rho\eta \cdot \partial_x(\xi\eta)) - (\partial_x(\rho\eta) \cdot \xi\eta) &= \rho\xi, \\ \xi\alpha_a &= 2\epsilon_{ab}\partial_x(\xi\eta^b), & \rho\alpha^a &= 2\epsilon^{ab}\partial_x(\rho\eta_b).\end{aligned}\tag{A.1}$$

Now the component fields have the argument x , which has been omitted for simplicity. It is important to note that all of their top components are of the form $\partial_x(\cdots)$. Use the second equation of (A.1) in order to see this for the one of f .

In the end of section 4 we also need to expand the infinitesimal parameter $v(x, \theta)$ of the $N = 4$ superdiffeomorphism. By (3.10) and (3.11) it satisfies

$$D_{\theta a}D_{\theta b}v = 0, \quad D_{\theta}^a D_{\theta}^b v = 0,$$

so that

$$v(x, \theta) = \alpha + \theta \cdot \beta - \beta \cdot \theta + (\theta\sigma^i\theta)t^i - (\theta \cdot \theta)(\theta \cdot \partial_x\beta) - (\theta \cdot \theta)(\partial_x\beta \cdot \theta) - \frac{1}{2}(\theta \cdot \theta)^2\partial_x^2\alpha,$$

in which $\alpha, \beta_a, \beta^a, t^i$ are independent parameters of the superdiffeomorphisms.

B Proofs of some formulae in sections 3 and 4

We prove the various formulae required for the arguments in sections 3 and 4. We begin by the following formulae

$$(D_{\theta a}\varphi^c)(D_{\theta}^b\varphi_c) = \delta_a^b(\partial_x f + \varphi_c\partial_x\varphi^c + \varphi^c\partial_x\varphi_c) \equiv \delta_a^b\Delta, \tag{B.1}$$

$$(D_{\theta}^c\varphi_a)(D_{\theta c}\varphi^b) = \delta_a^b(\partial_x f + \varphi_c\partial_x\varphi^c + \varphi^c\partial_x\varphi_c) \equiv \delta_a^b\Delta, \tag{B.2}$$

$$(D_{\theta a}\varphi^b)(D_{\theta}^a\varphi_b) = 2(\partial_x f + \varphi_c\partial_x\varphi^c + \varphi^c\partial_x\varphi_c) \equiv 2\Delta, \tag{B.3}$$

$$2D_{\theta a}\varphi^b\partial_x\varphi_b = D_{\theta a}\Delta, \quad 2D_{\theta}^a\varphi_b\partial_x\varphi^b = D_{\theta}^a\Delta, \tag{B.4}$$

$$\det[D_{\theta a}\varphi^b]\det[D_{\theta}^a\varphi_b] = \Delta^2. \tag{B.5}$$

They were studied in [15]. (B.2) and (B.3) follow from (B.1). (B.1) can be shown by taking the supercovariant derivative of (3.7) and using the algebra (3.2) and the chirality condition (3.6) as

$$\begin{aligned}D_{\theta}^b D_{\theta a} f &= (D_{\theta}^b\varphi_c)(D_{\theta a}\varphi^c) - 2\delta_a^b\varphi_c\partial_x\varphi^c, \\ D_{\theta a} D_{\theta}^b f &= (D_{\theta a}\varphi^c)(D_{\theta}^b\varphi_c) - 2\delta_a^b\varphi^c\partial_x\varphi_c.\end{aligned}$$

(B.4) can be shown by similarly taking the supercovariant derivative of (B.3). Then calculate the terms $(D_{\theta c}D_{\theta a}\varphi^b)(D_{\theta}^a\varphi_b)$ or $(D_{\theta a}\varphi^b)(D_{\theta}^c D_{\theta}^a\varphi_b)$ in the resulting equation as

$$\begin{aligned}(D_{\theta c}D_{\theta a}\varphi^b)(D_{\theta}^a\varphi_b) &= -D_{\theta c}\Delta + 4(D_{\theta c}\varphi^b)\partial_x\varphi_b, \\ (D_{\theta a}\varphi^b)(D_{\theta}^c D_{\theta}^a\varphi_b) &= -D_{\theta}^c\Delta + 4(D_{\theta}^c\varphi^b)\partial_x\varphi_b,\end{aligned}$$

by the successive use of (3.2), (B.1) and (3.6). We then get (B.4). (B.5) is now obvious from (B.1) and (B.2). It can be factorized to become

$$\Delta = \det[D_{\theta a}\varphi^b] = \det[D_{\theta}^a\varphi_b]. \tag{B.6}$$

We have checked this identity in components by using the expansion formulae in appendix A.

A direct calculation shows that the quantity $\det[D_{\theta a}\varphi^b]$ transforms by the superconformal transformations (3.13)~(3.15) as

$$\begin{aligned}\delta_v \log \det[D_{\theta a}\varphi^b] &= [v\partial_x + \frac{1}{2}D_{\theta c}\xi D_{\theta}^c + \frac{1}{2}D_{\theta}^c v D_{\theta c}] \log \det[D_{\theta a}\varphi^b] \\ &\quad + \frac{1}{2}D_{\theta c}D_{\theta}^c v.\end{aligned}$$

Similarly we can show that the quantity Δ , defined by (B.1), transforms as a superfield Ψ_1 given by (3.12). Both quantities should transform in the same way. Therefore the relation (B.6) implies that $[D_{\theta c}, D_{\theta}^c]v = 0$.

By using above formulae we can prove (4.12). Suppose that y is given by (4.7) and take the supercovariant derivative of it. We then get

$$\Delta D_{\theta a}y = -yD_{\theta a}\Delta + D_{\theta a}(df + \varphi_c d\varphi^c + \varphi^c d\varphi_c). \quad (\text{B.7})$$

Calculate the second term in the r.h.s. as

$$D_{\theta a}(df + \varphi_c d\varphi^c + \varphi^c d\varphi_c) = 2D_{\theta a}\varphi^c d\varphi_c,$$

by (3.6) and (3.7). Put this into (B.7) and contract both sides with $D_{\theta}^a\varphi_b$. Using (B.1) and (B.4) we then find

$$d\varphi_a = [y\partial_x + \frac{1}{2}D_{\theta c}y D_{\theta}^c]\varphi_a.$$

For $d\varphi^a$ the analogous formula can be shown. Substitute $d\varphi_a$ and $d\varphi^a$ in (4.7) by these formulae. We solve the resulting equation for df using the superconformal conditions (3.7). The solution is

$$df = [y\partial_x + \frac{1}{2}D_{\theta c}y D_{\theta}^c + \frac{1}{2}D_{\theta}^c y D_{\theta c}]f.$$

Thus all of the equations in (4.12) have been proved.

C Proof of (5.6) and (5.7) in section 5

We show the formulae (5.6) and (5.7) following from the chirality and superconformal conditions respectively. To this end it is convenient to write the Killing vectors in the

doublet notation

$$\begin{aligned}\delta_\epsilon f &\equiv -i\epsilon_A R^A \\ &= \epsilon_L + f\epsilon_{L^0} + (\varphi_c\epsilon_{\bar{F}}^c + \varphi^c\epsilon_{F_c}) + (f^2\epsilon_{\bar{L}} + f(\varphi_c\epsilon_{\bar{F}}^c + \varphi^c\epsilon_{\bar{F}_c})) \\ &\quad + (\varphi_b\varphi^b)(\varphi_c\epsilon_{\bar{F}}^c - \varphi^c\epsilon_{\bar{F}_c}) + (\varphi_c\varphi^c)^2\epsilon_{\bar{L}},\end{aligned}\tag{C.1}$$

$$\begin{aligned}\delta_\epsilon\varphi_a &\equiv -i\epsilon_A R^A_a \\ &= \epsilon_{Fa} + f\epsilon_{\bar{F}a} + \frac{1}{2}\varphi_a\epsilon_{L^0} - \varphi_c\epsilon_{R^c_a} \\ &\quad + (f\varphi_a\epsilon_{\bar{L}} + (\varphi_c\varphi^c\epsilon_{\bar{F}a} + 2\varphi_c\epsilon_{\bar{F}}^c\varphi_a)) + \varphi_c\varphi^c\varphi_a\epsilon_{\bar{L}},\end{aligned}\tag{C.2}$$

$$\begin{aligned}\delta_\epsilon\varphi^a &\equiv -i\epsilon_A R^{Aa} \\ &= \epsilon_F^a + f\epsilon_{\bar{F}}^a + \frac{1}{2}\varphi^a\epsilon_{L^0} + \varphi^c\epsilon_{R^a_c} \\ &\quad + (f\varphi^a\epsilon_{\bar{L}} - (\varphi_c\varphi^c\epsilon_{\bar{F}}^a - 2\varphi^c\epsilon_{\bar{F}_c}\varphi^a)) - \varphi_c\varphi^c\varphi^a\epsilon_{\bar{L}}.\end{aligned}\tag{C.3}$$

Here we have used the same doublet notation also for $\epsilon_{Fa\dot{\alpha}}$, $\epsilon_{\bar{F}a\dot{\alpha}}$ and $R^A_{a\dot{\alpha}}$ as given for $\varphi_{a\dot{\alpha}}$ by (5.2). Then it is immediate to see that (5.6) holds owing to (3.6) and (3.7). (5.7) can be also checked by a few of calculations. We do it explicitly for the first equation of (5.7) as an example. From (C.3) it follows that

$$\begin{aligned}D_{\theta a}\delta_\epsilon\varphi^b &= 2D_{\theta a}f\epsilon_{\bar{F}}^b + \frac{1}{2}D_{\theta a}\varphi^b\epsilon_{L^0} + D_{\theta a}\varphi^c\epsilon_{R^b_c} \\ &\quad + \left(2D_{\theta a}f\varphi^b + fD_{\theta a}\varphi^b - \varphi_c\varphi^cD_{\theta a}\varphi^b\right)\epsilon_{\bar{L}} + 2D_{\theta a}(\varphi^c\epsilon_{\bar{F}_c}\varphi^b),\end{aligned}\tag{C.4}$$

by using (3.6) and (3.7). For the case of $\epsilon = \epsilon_{\bar{F}}$ we have

$$\varphi_b D_{\theta a} \delta_\epsilon \varphi^b \Big|_{\epsilon=\epsilon_{\bar{F}}} = -2D_{\theta a}f \cdot \varphi_b \epsilon_{\bar{F}}^b - 2D_{\theta a}\varphi^c \epsilon_{\bar{F}_c} \cdot \varphi_b \varphi^b + 2\varphi^c \epsilon_{\bar{F}_c} \cdot \varphi_b D_{\theta a} \varphi^b.$$

By the same calculation we have also

$$\begin{aligned}\delta_\epsilon\varphi_b D_{\theta a}\varphi^b \Big|_{\epsilon=\epsilon_{\bar{F}}} &= ((f + \varphi_c\varphi^c)\epsilon_{\bar{F}b} + 2\varphi_c\epsilon_{\bar{F}}^c \cdot \varphi_b) D_{\theta a}\varphi^b, \\ D_{\theta a}\delta_\epsilon f \Big|_{\epsilon=\epsilon_{\bar{F}}} &= 2\varphi_b D_{\theta a}\varphi^b \cdot \varphi^c \epsilon_{\bar{F}_c} + fD_{\theta a}\varphi^c \epsilon_{\bar{F}_c} - \varphi_b \varphi^b \cdot D_{\theta a}\varphi^c \epsilon_{\bar{F}_c},\end{aligned}\tag{C.5}$$

from (C.2) and (C.1) respectively. It is now clear that the first relation of (5.7) is satisfied for the case of $\epsilon = \epsilon_{\bar{F}}$. It can be checked similarly for other cases than $\epsilon = \epsilon_{\bar{F}}$.

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