

AdS₅ vacua from type IIB supergravity on $T^{1,1}$

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ABSTRACT: We study maximally supersymmetric Anti-de Sitter backgrounds in consistent $\mathcal{N} = 2$ truncations of type IIB supergravity compactified on the Sasaki-Einstein manifold $T^{1,1}$. In particular, we focus on truncations that contain fields coming from the nontrivial second and third cohomology forms on $T^{1,1}$. These give rise to $\mathcal{N} = 2$ supergravity coupled to two vector- and two hypermultiplets (Betti-vector truncation) or one vector- and three hypermultiplets (Betti-hyper truncation), respectively. We find that both truncations admit AdS₅ backgrounds with the gauge group always being broken but containing at least an $U(1)_R$ factor. Moreover, in both cases we show that the moduli space of AdS vacua is nontrivial and of maximal dimension. Finally, we explicitly compute the metrics on these moduli spaces.

KEYWORDS: Supergravity Models, AdS-CFT Correspondence

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1 Introduction

Supersymmetric solutions of type IIB supergravity with AdS factors preserving 8 real supercharges have been extensively studied in the past [1–6]. They are related to gauge theories on the boundary of AdS by the AdS/CFT correspondence.¹ On the other hand, references [10–16] studied AdS vacua in different dimensions and with varying amounts of supersymmetry purely from a supergravity perspective, without any relation to compactifications of higher-dimensional theories. It was found explicitly in every case that these AdS backgrounds have properties which agree with the associated dual gauge theories. For example, all studied backgrounds admit an R-symmetry in the background that precisely matches the gauge theory R-symmetry. Moreover, it was explicitly shown in each case that moduli spaces of AdS backgrounds satisfy the same conditions as the conformal manifolds of the dual gauge theories.

In this work we study five-dimensional $\mathcal{N} = 2$ AdS backgrounds obtained via consistent truncations from ten-dimensional compactifications of type IIB supergravity. In particular, we will focus on the well-known background of the form $AdS_5 \times T^{1,1}$, where $T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)}$ is the Sasaki-Einstein manifold underlying the conifold. Due to the coset structure of $T^{1,1}$, one can apply an argument of [17]: retaining only singlets of a transitively acting isometry subgroup in a Kaluza-Klein reduction is a necessary condition for the consistency of the truncation. Since $T^{1,1}$ is a coset space, the group $SU(2) \times SU(2)$ acts transitively and is a subgroup of the isometry. It was shown in [18, 19] that truncating to singlets of this group indeed is consistent and leads to a five-dimensional gauged $\mathcal{N} = 4$ supergravity theory coupled to three vector multiplets. In particular, to study $\mathcal{N} = 2$ AdS backgrounds we will be interested in certain $\mathcal{N} = 2$ subtruncations of this $\mathcal{N} = 4$ theory developed in [18, 20].

Since the $AdS_5 \times T^{1,1}$ background is dual to the Klebanov-Witten theory [21], this solution has been extensively studied in the past. For example, the moduli space of the

¹See [7, 8] for earlier work and [9] for a recent review.

full ten-dimensional theory is known to be complex five-dimensional [4, 5, 21–23] and the moduli have been identified. There is always one modulus corresponding to the Axion-Dilaton τ and another coming from the vacuum expectation value (VEV) of the complex B-field of type IIB supergravity integrated over the nontrivial two-cycle of $T^{1,1}$.² Moreover the remaining three complex moduli (one of which is the deformation identified in [24]) transform as a triplet under the $SU(2) \times SU(2)$ in the isometry group. However, it has been shown in [25] that it is inconsistent to keep non-singlet representations of $SU(2) \times SU(2)$ in a truncation and thus only the first two moduli are expected to show up in the truncated theories.

The aim of this paper is to find the conditions on AdS backgrounds and their moduli spaces in the consistent $\mathcal{N} = 2$ truncations of $T^{1,1}$ supergravity. To this end, we focus on the truncations known as the Betti-vector truncation³ and the Betti-hyper truncation in [20], since they contain multiplets associated with the topology of $T^{1,1} \cong S^2 \times S^3$. The Betti-vector truncation contains gravity coupled to two vector multiplets and two hypermultiplets, while the Betti-hyper truncation contains gravity coupled to one vector multiplet and three hypermultiplets. We then apply the methods developed in [15] to these truncations. We find that both truncations admit AdS vacua with an unbroken $U(1)_R$ symmetry in the background. Moreover, we explicitly compute the metric on the moduli spaces. In the case of the Betti-vector truncation, we find that the moduli space \mathcal{M}^{BV} is spanned by the Axion-Dilaton τ and $\mathcal{M}^{BV} = \mathcal{H}$ is the upper half-plane \mathcal{H} . For the Betti-hyper truncation we compute a complex two-dimensional moduli space \mathcal{M}^{BH} that is given by a torus bundle with base space parametrized again by τ . In particular, this reproduces the result of [21] that the moduli in question are the Axion-Dilaton and a complex scalar that parametrizes a torus. However, the metric on the moduli space is not a direct product but a nontrivial fibration known as the universal elliptic curve $\mathcal{E} = (\mathbb{C} \times \mathcal{H})/\mathbb{Z}^2$ [27].

The rest of this paper is organized as follows: in section 2 we briefly review gauged $\mathcal{N} = 2$ supergravity in five dimensions and the conditions on AdS_5 vacua in these theories. In section 3 we introduce the relevant $\mathcal{N} = 2$ truncations on $T^{1,1}$ and collect the relevant data. We then use this to compute the conditions on AdS vacua and moduli spaces in these truncations. Finally, we conclude and discuss our results in section 4.

2 AdS_5 vacua in $\mathcal{N} = 2$ gauged supergravity

Let us start by reviewing AdS_5 vacua of gauged $\mathcal{N} = 2$, $d = 5$ supergravity as discussed in [10, 15]. To this end we introduce gauged $\mathcal{N} = 2$ supergravity as discussed in [28–30]. In this paper we will consider theories with the following field content: the gravity multiplet

$$\{g_{\mu\nu}, \psi_\mu^A, A_\mu^0\}, \quad \mu, \nu = 0, \dots, 4, \quad A = 1, 2, \quad (2.1)$$

containing the spacetime metric $g_{\mu\nu}$, an $SU(2)_R$ doublet of symplectic Majorana fermions ψ_μ^A and a vector A_μ^0 called the graviphoton. Additionally, we allow for n_V vector multiplets

²This comes from the fact that $T^{1,1}$ is diffeomorphic to $S^2 \times S^3$ and thus has non-vanishing second Betti-number [18, 19].

³This truncations was recently studied in the context of holographic duals of c-extremization [26].

$\{A_\mu, \lambda^A, \phi\}$ that transform in the adjoint of the gauge group, containing a vector A_μ , a doublet of gauginos λ^A and a real scalar ϕ . Including the graviphoton, we label the vector fields by $I, J = 0, \dots, n_V$ and their associated scalars by $i, j = 1, \dots, n_V$.⁴ Note that there is no scalar associated to the graviphoton. Moreover, we include n_H hypermultiplets

$$\{q^u, \zeta^\alpha\}, \quad u = 1, 2, \dots, 4n_H, \quad \alpha = 1, 2, \dots, 2n_H, \quad (2.2)$$

containing $4n_H$ real scalars q^u and $2n_H$ hyperini ζ^α .

The scalar fields can be interpreted as coordinate charts from spacetime M_5 to a target space \mathcal{T} , $\phi^i \otimes q^u : M_5 \rightarrow \mathcal{T}$. Locally $\mathcal{T} \cong \mathcal{T}_V \times \mathcal{T}_H$ where the first factor is a very special real manifold of dimensions n_V . This is defined as a hypersurface in an $(n_V + 1)$ -dimensional real manifold \mathcal{H} by

$$P(h^I(\phi)) = C_{IJK} h^I h^J h^K = 1, \quad (2.3)$$

where h^I are the coordinates on \mathcal{H} and $P(h^I(\phi))$ is a cubic homogeneous polynomial with C_{IJK} constant and completely symmetric. The index I can be lowered via

$$h_I = C_{IJK} h^J h^K. \quad (2.4)$$

The second factor of \mathcal{T} is a quaternionic Kähler manifold of real dimension $4n_H$ (see [31] for a more detailed introduction). This is equipped with a metric G_{uv} and a triplet of almost complex structures \vec{J} such that $J^1 J^2 = J^3$. In particular, G_{uv} is hermitian with respect to all of these almost complex structures and one can define the associated two-forms $\vec{\omega}_{uv} := G_{uv} \vec{J}_v^w$. In what follows we need to know more about the action of the gauge group G on the target space of the hypermultiplet scalars. This is given by Killing vectors k_I^u on \mathcal{T}_H that respect the quaternionic structure, i.e. that are triholomorphic. This implies the existence of a triplet of moment maps $\vec{\mu}_I$ which satisfy

$$\frac{1}{2} \vec{\omega}_{uv} k_I^v = -\nabla_u \vec{\mu}_I, \quad (2.5)$$

where ∇_u is a connection that combines the Levi-Civita connection for G_{uv} and an $SU(2)$ -connection. Moreover the covariant derivatives of the scalars q^u are given by

$$\mathcal{D}_\mu q^u = \partial_\mu q^u + k_I^u A_\mu^I. \quad (2.6)$$

Equipped with the above data one can then define the scalar potential $V(\phi^i, q^u)$ and the supersymmetry variations of the fermions. In a maximally supersymmetric AdS background the latter have to vanish, i.e.

$$\langle \delta \psi_\mu^A \rangle = \langle \delta \lambda^A \rangle = \langle \delta \zeta^\alpha \rangle = 0. \quad (2.7)$$

Then the scalar potential is nonzero in the background, $\langle V \rangle = \Lambda$ where $\Lambda \in \mathbb{R}$ is the cosmological constant. It has been shown in [15] that the conditions (2.7) can be expressed in terms of the moment maps and Killing vectors on the scalar manifold,

$$\langle h^I k_I \rangle = 0, \quad \langle \vec{\mu}_I \rangle = \lambda \langle h_I \rangle \vec{v}, \quad (2.8)$$

⁴In [15] we also considered tensor multiplets which arise by dualising vector fields that transform in a different representation of the gauge group. Since these do not appear in the truncations that we will analyze later, we do not include them here.

where $\vec{v} \in S^2$ is a constant unit vector and $\lambda = \frac{1}{2}\sqrt{\Lambda}$. In particular, one can perform an $SU(2)_R$ -rotation to set $\vec{v} = \vec{e}_3$. The first equation in (2.8) then implies that a $U(1)_R$ symmetry is always unbroken in the vacuum, i.e. the gauge group in the AdS background is of the form $H \times U(1)_R$. Moreover, the second equation shows that this R -symmetry is gauged by the graviphoton $\langle h_I \rangle A_\mu^I$ [15].

We are now interested in the moduli space \mathcal{M} of these vacua. To this end we define the space \mathcal{D} of scalar deformations $\phi^i \rightarrow \langle \phi^i \rangle + \delta\phi^i$, $q^u \rightarrow \langle q^u \rangle + \delta q^u$ that leave (2.8) invariant. Moreover we define the space of Goldstone bosons, given by $\mathcal{G} = \text{span}_{\mathbb{R}}\{\langle k_I \rangle\}$.⁵ If the gauge group is spontaneously broken, the corresponding Goldstone bosons are always among the invariant deformations of (2.8), $\mathcal{G} \subset \mathcal{D}$. However, they should not be counted as physical moduli. Thus we define the moduli space of the AdS background to be the quotient $\mathcal{M} = \mathcal{D}/\mathcal{G}$. This space is in particular a submanifold of the scalar target space \mathcal{T} . It was shown in [15] that $\mathcal{M} \subset \mathcal{T}_H$, i.e. all vector multiplet scalars are fixed while scalars from the hypermultiplets can be moduli. Moreover \mathcal{M} carries a Kähler structure that naturally descends from the quaternionic Kähler structure of the ambient manifold \mathcal{T}_H . Finally, if we denote by $n_G = \text{rk}\langle k_I^u \rangle$ the number of Goldstone bosons, then the real dimension of the moduli space is

$$\dim \mathcal{M} \leq 2n_H - 2n_G . \tag{2.9}$$

3 AdS₅ vacua from $T^{1,1}$ compactifications

In this section we review consistent truncations of type IIB supergravity compactified on the Sasaki-Einstein manifold $T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)}$. Moreover, we study AdS backgrounds which preserve $\mathcal{N} = 2$ supersymmetry in these examples and explicitly compute their moduli spaces. Before we turn to the special case of truncations that preserve $\mathcal{N} = 2$ supersymmetry in five dimensions (8 real supercharges), let us first discuss the general case. Compactifications of type IIB supergravity, their supersymmetry and their consistent truncations have been extensively studied, see for example [32–34, 36]. Starting with type IIB supergravity, one has 32 real supercharges in ten dimensions. The amount of supersymmetry that is preserved in a consistent truncation to five dimensions depends on the number of linearly independent spinors on the compact manifold. In the case at hand we are interested in compactifications on five-dimensional Sasaki-Einstein manifolds SE_5 . Such a Sasaki-Einstein manifold admits two linearly independent Killing spinors which are charge conjugate [37], i.e. there exists a spinor η on SE_5 such that

$$\nabla_m^{\text{SE}} \eta = \alpha \gamma_m \eta, \quad m = 1, \dots, 5, \tag{3.1}$$

for some constant $\alpha \in \mathbb{R}$. Then the charge conjugate η^c satisfies the same equation with constant $-\alpha$. This fact was used in [33, 34] to prove that a consistent truncation on a Sasaki-Einstein manifold can be described by an $\mathcal{N} = 4$ supergravity in five dimensions, i.e. the number of supercharges reduces to 16. One can expand the ten-dimensional gravitino

⁵This is due to the fact that the kinetic terms for the scalar fields can induce a mass term for the gauge bosons of the form $M_{IJ} \propto \langle K_{uv} k_I^u k_J^v \rangle$ for some invertible matrix K_{uv} . See [15] for an explicit computation.

in terms of the spinors on the Sasaki-Einstein manifold as $\Psi^A = \psi_1^A \otimes \eta \otimes \theta + \psi_2^A \otimes \eta^c \otimes \theta^c$, giving rise to four symplectic Majorana gravitini. Here θ is a two-component spinor that appears due to the decomposition of the ten-dimensional Clifford algebra into five-dimensional ones [1].

However, we are interested in solutions that contain an AdS₅ factor and preserve all supercharges. As discussed in the five-dimensional case above, this imposes $\langle \delta_\epsilon \Psi^A \rangle = 0$ for ϵ being the supersymmetry parameter. It was shown in [1, 35] that for supersymmetric backgrounds of the form AdS₅ × SE₅, ϵ has to be a complex spinor of the form

$$\epsilon = \psi \otimes \eta \otimes \theta, \tag{3.2}$$

where ψ is a Killing spinor on AdS₅. Since the smallest spinor representations of Spin(1, 4) are eight-dimensional, the background only preserves 8 real supercharges, i.e. $\mathcal{N} = 2$ supersymmetry in five dimensions. Due to this, one can expect AdS₅ solutions of the truncated theory to preserve at most $\mathcal{N} = 2$ supersymmetry. This was indeed shown to be true in [18].

The consistent truncations of [33, 34] were generalized to include nontrivial second cohomology forms on $T^{1,1}$ in [18, 19]. There it was shown that the truncated theory can again be interpreted as $\mathcal{N} = 4$, $d = 5$ supergravity, coupled to 3 vector multiplets. In particular, these truncations contain an additional $\mathcal{N} = 4$ vector multiplet compared to the truncations discussed in [33, 34], called the $\mathcal{N} = 4$ Betti-vector multiplet because it originates from the fact that $T^{1,1}$ has nontrivial second and third cohomology classes. The reduction then proceeds as follows: type IIB supergravity contains the spacetime metric, the Axion-Dilaton, a doublet of three-forms and the Ramond-Ramond five form. These fields are then expanded in terms of the invariant tensors on $T^{1,1}$ [18, 19, 35]. Let us describe the five-dimensional field content gained from this expansion: the reduction of the type IIB metric yields the five-dimensional spacetime metric $g_{\mu\nu}$, the graviphoton A_μ^0 , three real scalars $\{u_1, u_2, u_3\}$ and a complex scalar v . Moreover, the reduction of the two three-forms gives two doublets of vectors $\{B_\mu^a, C_\mu^a\}$, two doublets of real scalars $\{e^a, c^a\}$ and one doublet of complex scalars b^a for $a = 1, 2$. Finally, the reduction of the RR five-form contributes a complex vector L_μ , two real vectors $\{A_\mu^1, A_\mu^2\}$ and a real scalar k . Combined with the reduction of the Axion-Dilaton τ , the five-dimensional bosonic field content is thus given by the metric $g_{\mu\nu}$, the graviphoton A_μ^0 , eight vectors $\{A_\mu^1, A_\mu^2, B_\mu^a, C_\mu^a, L_\mu, \bar{L}_\mu\}$, $a = 1, 2$, and 16 real scalars

$$\{u_1, u_2, u_3, c^a, e^a, k, \tau, \bar{\tau}, v, \bar{v}, b^a, \bar{b}^a\}. \tag{3.3}$$

The scalar manifold is the coset space

$$\mathcal{M}_{\text{scalar}} = \text{SO}(1, 1) \times \frac{\text{SO}(5, 3)}{\text{SO}(5) \times \text{SO}(3)}. \tag{3.4}$$

Before we proceed, let us discuss the transformation of the five-dimensional fields under the global SL(2, ℝ)-symmetry of type IIB supergravity. Since the three-forms \hat{F}_3^a in type IIB transform linearly under SL(2, ℝ) [38], the same holds for all the five-dimensional fields appearing in their expansion. In particular, for $a, b, c, d \in \mathbb{R}$ and $A \in \text{SL}(2, \mathbb{R})$ such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \tag{3.5}$$

an $SL(2, \mathbb{R})$ -doublet V^a coming from the reduction of the type IIB three-forms transforms

$$\begin{pmatrix} V^1 \\ V^2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}. \quad (3.6)$$

Moreover, the Axion-Dilaton τ transforms under the global $SL(2, \mathbb{R})$ -symmetry non-linearly via [38]

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}. \quad (3.7)$$

All remaining fields are invariant under $SL(2, \mathbb{R})$.

Let us now discuss the supersymmetry breaking from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ of this compactification. The $\mathcal{N} = 4$ gravity multiplet decomposes into an $\mathcal{N} = 2$ gravity multiplet, an $\mathcal{N} = 2$ gravitino multiplet and an $\mathcal{N} = 2$ vector multiplet. Moreover, each $\mathcal{N} = 4$ vector multiplet decomposes into an $\mathcal{N} = 2$ vector multiplet and a hypermultiplet, giving in total four vector multiplets and three hypermultiplets. The truncation to $\mathcal{N} = 2$ supergravity is then done by removing the massive $\mathcal{N} = 2$ gravitino multiplet. However, by examining the resulting equations of motion one finds that this alone would not lead to a consistent truncation. In fact, one can show [18] that for the truncation to be consistent, one additionally has to either truncate the $\mathcal{N} = 2$ Betti-vector multiplet or the Betti-hypermultiplet. These theories were extensively studied in [20]. There the data we used in section 2 to determine the conditions on AdS vacua in a given supergravity were explicitly computed for these $\mathcal{N} = 2$ truncations.

In principle, it would also be consistent to completely truncate the $\mathcal{N} = 4$ Betti multiplet from the spectrum. Since the resulting theories are precisely the ones analyzed in [33, 34], we will not discuss these truncations here. Moreover, reference [20] also computes moment maps and Killing vectors for the so-called NS truncation, which is done by removing the whole RR sector of type IIB supergravity before reducing to five dimensions. However, this means that the NS truncations does not contain the Axion-Dilaton and thus is not expected to have supersymmetric AdS vacua [5]. We checked this explicitly by inserting the data for the Killing vectors and moment maps from [20] into equation (2.8). One finds that the resulting equations can not be satisfied and thus no supersymmetric AdS backgrounds exist in this truncation.

In the rest of this section we review the data for the Betti-vector and Betti-hyper truncation [20] and compute the conditions on AdS backgrounds and their moduli spaces in these five-dimensional theories. Before we proceed, let us collect some differences in notation between this work and [20]. In general, we replace the indices $i, j = 1, 2$ with $a, b = 1, 2$. Moreover, we replaced

$$\begin{aligned} P_I &\rightarrow \mu_I, & b_0^i &\rightarrow b^a, & v_i &\rightarrow f_a, & A_1 &\rightarrow A_0 \\ k_{11} &\rightarrow A_1, & k_{12} &\rightarrow A_2, & e_0^i &\rightarrow e^a. \end{aligned} \quad (3.8)$$

3.1 The Betti-vector truncation

We first discuss the Betti-vector truncation, i.e. the case where the Betti-hypermultiplet is truncated out of the spectrum. This leads to an $\mathcal{N} = 2$ theory that contains gravity

coupled to two vector multiplets and two hypermultiplets [18]. In this truncation, 10 scalars $\{u_1, u_2, u_3, k, \tau, \bar{\tau}, b^a, \bar{b}^a\}$ from (3.3) are kept. The vector in the gravity multiplet is A_μ^0 while the other vectors are given by $A_\mu^{1,2}$ with associated one-forms A_I , $I = 0, 1, 2$. The scalars $\{u_2, u_3\}$ of the vector multiplets span the manifold

$$\mathcal{T}_V^{\text{BV}} = \text{SO}(1, 1) \times \text{SO}(1, 1) . \tag{3.9}$$

The relevant special geometric data is given by [20]

$$h^0 = e^{4u_3} \quad h^1 = e^{2u_2 - 2u_3}, \quad h^2 = e^{-2u_2 - 2u_3}, \tag{3.10}$$

with $C_{012} = \frac{1}{6}$ and all others zero.⁶ Lowering the index I according to (2.4) one finds

$$h_0 = \frac{1}{3}e^{-4u_3}, \quad h_1 = \frac{1}{3}e^{-2u_2 + 2u_3}, \quad h_2 = \frac{1}{3}e^{2u_2 + 2u_3}. \tag{3.11}$$

The hypermultiplet scalars $\{u_1, k, \tau, \bar{\tau}, b^a, \bar{b}^a\}$ for $a = 1, 2$ can be shown to span the quaternionic Kähler manifold [18]

$$\mathcal{T}_H^{\text{BV}} = \frac{\text{SO}(4, 2)}{\text{SO}(4) \times \text{SO}(2)}. \tag{3.12}$$

Note that the scalars u_1 and k are real while all others are complex. In particular, $\tau = a + ie^{-\phi}$ is the reduction of the Axion-Dilaton of type IIB supergravity. The metric on $\mathcal{T}_H^{\text{BV}}$ can be read off from the kinetic terms of the hypermultiplets in the Lagrangian [20],

$$\begin{aligned} \mathcal{L}_{Hyper}^{\text{BV}} = & -4e^{-4u_1 + \phi} M_{ab} D b_0^a \wedge * D \bar{b}_0^b - 8du_1 \wedge * du_1 \\ & - \frac{1}{2}e^{-8u_1} K \wedge * K - \frac{1}{2}d\phi \wedge * d\phi - \frac{1}{2}e^{2\phi} da \wedge * da, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} D b^a &= db^a - 3ib^a A_0, \\ K &= Dk + 2\epsilon_{ab}[b^a D\bar{b}^b + \bar{b}^a D b^b], \\ Dk &= dk - Q A_0 - 2A_1 - 2A_2, \end{aligned} \tag{3.14}$$

for $Q \in \mathbb{R}^+$ and

$$M_{ab} = e^\phi \begin{pmatrix} a^2 + e^{-2\phi} & -a \\ -a & 1 \end{pmatrix} = \frac{1}{\text{Im } \tau} \begin{pmatrix} |\tau|^2 & -\text{Re } \tau \\ -\text{Re } \tau & 1 \end{pmatrix}. \tag{3.15}$$

For $A \in \text{SL}(2, \mathbb{R})$, the matrix M_{ab} transforms under the symmetry discussed in (3.6) as [38]

$$M_{ab} \mapsto (A^{-1})_a^c M_{cd} (A^{-1})_b^d. \tag{3.16}$$

The gauge group is realized on the hypermultiplet scalar manifold via the Killing vectors $k_I = k_I^u \partial_u$ [20]

$$\begin{aligned} k_0 &= -3ib^a \partial_{b^a} + 3i\bar{b}^a \partial_{\bar{b}^a} - Q \partial_k, \\ k_1 &= 2\partial_k, \\ k_2 &= 2\partial_k. \end{aligned} \tag{3.17}$$

⁶Note that we use conventions for the special real geometry which are different from [20], see [30] for details.

The associated moment maps [20] are⁷

$$\begin{aligned}\vec{\mu}_0 &= 6e^{-2u_1} f_a b^a \vec{e}_1 - 6e^{-2u_1} \bar{f}_a \bar{b}^a \vec{e}_2 + \left(\frac{1}{2} e^{-4u_1} e^Z - 3 \right) \vec{e}_3, \\ \vec{\mu}_1 &= -e^{-4u_1} \vec{e}_3, \\ \vec{\mu}_2 &= -e^{-4u_1} \vec{e}_3,\end{aligned}\tag{3.18}$$

where $e^Z = Q - 6i\epsilon_{ab}(b^a \bar{b}^b - \bar{b}^a b^b)$ and f_a is defined via $f_a b^a = \frac{1}{\sqrt{\text{Im}\tau}}(b^2 - \tau b^1)$. From this we can immediately compute the Lie algebra \mathfrak{g} spanned by the k_I . Since $k_1 = k_2$, we only have to compute the respective Lie brackets with k_0 . We find

$$[k_0, k_1] = [k_0, k_2] = 0,\tag{3.19}$$

since the vectors $\{\partial_a, \bar{\partial}_a, \partial_k\}$ are linearly independent and Q is a constant. Thus by observing that the Killing vectors (3.17) have compact and non-compact parts, the gauge group is $G = \text{U}(1) \times \text{U}(1) \times \mathbb{R}$. Note that the scalars are only charged under an $\text{U}(1) \times \mathbb{R}$ subgroup of G with associated gauge fields A_0 and $QA_0 - 2A_1 - 2A_2$.

To find the AdS vacua of the five-dimensional theory coming from the Betti-vector truncation we have to solve the equations (2.8). For the first equation we use (3.11), (3.17) and find

$$\langle h^I k_I \rangle = -(3i\langle b^a \rangle \partial_{b^a} - 3i\langle \bar{b}^a \rangle \partial_{\bar{b}^a}) e^{4\langle u_3 \rangle} + 2e^{2\langle u_3 \rangle} \left(e^{2\langle u_2 \rangle} + e^{-2\langle u_2 \rangle} - \frac{Q}{2} e^{6\langle u_3 \rangle} \right) \partial_k = 0.\tag{3.20}$$

Due to linear independence of the vectors $\{\partial_{b^a}, \partial_{\bar{b}^a}, \partial_k\}$, this implies $\langle b^a \rangle = \langle \bar{b}^a \rangle = 0$ and

$$e^{2\langle u_2 \rangle} + e^{-2\langle u_2 \rangle} = \frac{Q}{2} e^{6\langle u_3 \rangle}.\tag{3.21}$$

In particular, the background values of the vector multiplet scalars are not independent of each other. Inserting these results into the moment maps (3.18), we find that only the third component $\langle \mu_I \rangle := \langle \mu_I^3 \rangle$ is nonzero in the background. These components are given by

$$\langle \mu_0 \rangle = \frac{Q}{2} e^{-4\langle u_1 \rangle} - 3, \quad \langle \mu_1 \rangle = \langle \mu_2 \rangle = -e^{-4\langle u_1 \rangle},\tag{3.22}$$

where we used the fact that $\langle e^Z \rangle = Q$ in the AdS background.

We are thus left with solving the second equation in (2.8). Since $\langle \mu_1 \rangle = \langle \mu_2 \rangle$, (2.8) implies that $\langle h_1 \rangle = \langle h_2 \rangle$ and thus using (3.33) we find

$$e^{-2\langle u_2 \rangle + 2\langle u_3 \rangle} = e^{2\langle u_2 \rangle + 2\langle u_3 \rangle}.\tag{3.23}$$

This fixes $\langle u_2 \rangle = 0$. Also, by using (3.21) we find $\langle u_3 \rangle = \frac{1}{6} \log \frac{4}{Q}$. Note that this fixes $\langle u_3 \rangle$, since Q is a constant. Now consider the zero-component $\langle \mu_0 \rangle$. Using again (3.11) and (3.18) and inserting this into the second equation of (2.8), we find

$$\frac{Q}{2} e^{-\langle u_1 \rangle} - 3 = \frac{\lambda}{3} e^{-4\langle u_3 \rangle} = \frac{Q^{2/3} \lambda}{6 \sqrt[3]{2}},\tag{3.24}$$

⁷We converted the moment maps according to $(\mu_I)_B^A = i\vec{\mu}_I(\vec{\sigma})_B^A$ where $(\vec{\sigma})_B^A$ are the Pauli matrices, see [30] for more details.

fixing the background value of the scalar $\langle u_1 \rangle = -\log\left(\frac{\lambda}{3\sqrt[3]{2}Q^{1/3}} + \frac{6}{Q}\right)$. To summarize, the conditions for AdS₅ vacua from the Betti-vector truncation fix all the scalars $\{u_2, u_3\}$ from the vector multiplets and moreover the scalars $\{u_1, b^a, \bar{b}^a\}$ from the hypermultiplets.

Let us now turn to the moduli space \mathcal{M}^{BV} described in section 2. From the above analysis we know that the hypermultiplet scalars $\{k, \tau, \bar{\tau}\}$ are not constrained by the conditions (2.8) for the AdS background. However, the moduli space was proven to be a Kähler manifold and thus has to be even-dimensional. This can be understood by considering the background values of the Killing vectors (3.17),

$$\langle k_0 \rangle = -Q\partial_k, \quad \langle k_1 \rangle = \langle k_2 \rangle = 2\partial_k. \quad (3.25)$$

From this we see that the space of Goldstone bosons is one-dimensional and spanned by ∂_k . The respective scalar k then gets eaten by the vector field $QA_0 - 2A_1 - 2A_2$, which becomes massive as a result of the symmetry breaking. In particular, we explained in section 2 that the $U(1)_R$ symmetry always remains unbroken in the vacuum and is gauged by the graviphoton

$$\langle h_I \rangle A_\mu^I = \frac{Q^{2/3}}{6\sqrt[3]{2}} A_\mu^0 + \frac{2^{2/3}}{3\sqrt[3]{Q}} (A_\mu^1 + A_\mu^2). \quad (3.26)$$

Thus we find that the gauge group in the supersymmetric AdS background is broken according to $U(1) \times U(1) \times \mathbb{R} \rightarrow U(1) \times U(1)_R$.

We have shown that the moduli space of the AdS vacuum is two-dimensional and spanned by the Axion-Dilaton $\{\tau, \bar{\tau}\}$. This agrees with the bound (2.9) on the dimension of the moduli space for $n_H = 2, n_G = 1$,

$$\dim \mathcal{M}^{\text{BV}} \leq 2 \cdot 2 - 2 \cdot 1 = 2, \quad (3.27)$$

i.e. the moduli space is of maximal dimension. To compute the metric g^{BV} on the moduli space, note that the coordinate one-forms of the fixed scalars vanish on \mathcal{M}^{BV} , i.e.

$$db^a|_{\mathcal{M}^{\text{BV}}} = d\bar{b}^a|_{\mathcal{M}^{\text{BV}}} = du_1|_{\mathcal{M}^{\text{BV}}} = dk|_{\mathcal{M}^{\text{BV}}} = 0. \quad (3.28)$$

Using this fact we can read off the metric from the Lagrangian (3.13),

$$g^{\text{BV}} = d\phi^2 + e^{2\phi} da^2 = \frac{1}{\text{Im} \tau^2} d\tau d\bar{\tau}. \quad (3.29)$$

This is precisely the metric on the upper half plane \mathcal{H} which is well-known to be a Kähler manifold. Thus the moduli space of the AdS₅ vacua in the Betti-vector truncation is

$$\mathcal{M}^{\text{BV}} = \mathcal{H}. \quad (3.30)$$

3.2 The Betti-hyper truncation

Let us now turn to the Betti-hyper truncation from which one obtains a five-dimensional $\mathcal{N} = 2$ theory that contains gravity coupled to one vector multiplet and three hypermultiplets [18]. Here, 13 scalars $\{u_1, u_3, k, e^a, \tau, \bar{\tau}, v, \bar{v}, b^a, \bar{b}^a\}$ are kept from (3.3). While

the graviphoton is still A_μ^0 , the vector in the vector multiplet is given as the combination $\frac{1}{2}(A_\mu^1 + A_\mu^2)$. The single scalar u_3 in the vector multiplet parametrizes the manifold

$$\mathcal{T}_V^{\text{BH}} = \text{SO}(1, 1) . \quad (3.31)$$

Its special real geometry is given by [20]

$$h^0 = e^{4u_3} , \quad h^1 = e^{-2u_3} , \quad (3.32)$$

with $C_{011} = \frac{1}{3}$ and all others zero. Moreover we obtain

$$h_0 = \frac{1}{3}e^{-4u_3} , \quad h_1 = \frac{2}{3}e^{2u_3} , \quad (3.33)$$

by lowering the index according to (2.4).

In the hypermultiplets, the scalars $\{u_1, k, e^a, b^a, \bar{b}^a, \tau, \bar{\tau}, v, \bar{v}\}$ span the quaternionic Kähler manifold [18]

$$\mathcal{T}_H^{\text{BH}} = \frac{\text{SO}(4, 3)}{\text{SO}(4) \times \text{SO}(3)} . \quad (3.34)$$

Here b^a and v are complex while all others are real.⁸ The hypermultiplet Lagrangian is given by [20]

$$\begin{aligned} \mathcal{L}_{\text{Hyper}}^{\text{BH}} = & -e^{-4u_1} M_{ab} \left[\frac{1}{2} D e^a \wedge * D e^b + \frac{1}{2} G^a \wedge * G^b + 2(F^a \wedge * \bar{F}^b + \bar{F}^a \wedge * F^b) \right] \\ & - 8 d u_1 \wedge * d u_1 - d|v| \wedge * d|v| + |v|^2 D \theta \wedge * D \theta \\ & - \frac{1}{2} e^{-8u_1} K \wedge * K - \frac{1}{2} d\phi \wedge * d\phi - \frac{1}{2} e^{2\phi} da \wedge * da , \end{aligned} \quad (3.35)$$

where

$$\begin{aligned} G^a &= (1 + |v|^2) D e^a - 4 \text{Im}(v D b^a) , \\ F^a &= D b^a - \frac{i}{2} \bar{v} D e^a , \\ D e^a &= d e^a - j^a A_0 , \\ D \theta &= d \theta + 3 A_0 , \end{aligned} \quad (3.36)$$

j^a are constant charges and $\theta = \theta(v, \bar{v})$ is a function of v and \bar{v} whose explicit form is not important for our purposes. Moreover $D b^a$ and M_{ab} are defined in (3.14) and (3.15), respectively. From this Lagrangian one can read off the metric for the hypermultiplet scalars.

The gauge group acts on $\mathcal{T}_H^{\text{BH}}$ via the Killing vectors [20]

$$\begin{aligned} k_0 &= -(Q + \epsilon_{ab} j^a e^b) \partial_k - 3 i b^a \partial_{b^a} + 3 i \bar{b}^a \partial_{\bar{b}^a} + \frac{3}{2} (1 + \rho^2) \partial_\rho + \frac{3}{2} (1 + \bar{\rho}^2) \partial_{\bar{\rho}} - j^a \partial_{e^a} , \\ k_1 &= 4 \partial_k , \end{aligned} \quad (3.37)$$

⁸We replace $(e_1, e_2) \mapsto (-e_2, e_1)$ compared to [20].

where we introduced a complex scalar $\rho \in \mathcal{H}$ in the upper half plane that is related to v via

$$v = -\frac{(i-\rho)(i-\bar{\rho})}{1+|\rho|^2}. \quad (3.38)$$

The associated moment maps are then given by [20]

$$\begin{aligned} \vec{\mu}_0 &= \left(\frac{1}{2} e^{\tilde{Z}} e^{-4u_1} - \frac{3}{2\text{Im}\rho} (1+|\rho|^2) \right) \vec{e}_3 \\ &\quad - \frac{1}{2\rho} e^{-2u_1} f_a (-3i(\bar{\rho}-i)^2 \bar{b}^a - 3i(\bar{\rho}+i)^2 b^a + i(1-i\bar{\rho})(1+i\bar{\rho})j^a) \vec{e}_1 \\ &\quad + \frac{1}{2\rho} e^{-2u_1} \bar{f}_a (3i(\rho-i)^2 b^a + 3i(\rho+i)^2 \bar{b}^a + i(1-i\rho)(1+i\rho)j^a) \vec{e}_2, \\ \vec{\mu}_1 &= -2e^{-4u_1} \vec{e}_3, \end{aligned} \quad (3.39)$$

where $e^{\tilde{Z}} = e^Z + \epsilon_{ab}(j^a e^b - e^a j^b) = Q - 6i\epsilon_{ab}(b^a \bar{b}^b - \bar{b}^a b^b) + \epsilon_{ab}(j^a e^b - j^b e^a)$. Since $[k_0, k_1] = 0$, the gauge group in this case is $G = \text{U}(1) \times \mathbb{R}$.

Let us now solve (2.8) in this compactification. To do this, we insert (3.37) and (3.32) into the first equation of (2.8),

$$\begin{aligned} 0 = \langle h^I k_I \rangle &= (4e^{-6\langle u_3 \rangle} - Q - \epsilon_{ab} j^a \langle e^b \rangle) e^{4\langle u_3 \rangle} \partial_k - (3i\langle b^a \rangle \partial_{b^a} - 3i\langle \bar{b}^a \rangle \partial_{\bar{b}^a}) e^{4\langle u_3 \rangle} \\ &\quad + \frac{3}{2}(1 + \langle \rho^2 \rangle) e^{4\langle u_3 \rangle} \partial_\rho + \frac{3}{2}(1 + \langle \bar{\rho}^2 \rangle) e^{4\langle u_3 \rangle} \partial_{\bar{\rho}}. \end{aligned} \quad (3.40)$$

Using the linear independence of the basis vectors $\{\partial_{b^a}, \partial_{\bar{b}^a}, \partial_k, \partial_\rho, \partial_{\bar{\rho}}\}$ we immediately find $\langle b^a \rangle = \langle \bar{b}^a \rangle = j^a = 0$.⁹ Moreover, we learn that

$$\langle \rho^2 \rangle = \langle \bar{\rho}^2 \rangle = -1, \quad Q = 4e^{-6\langle u_3 \rangle}, \quad (3.41)$$

where we used $j^a = 0$. In particular, we find $\langle e^{\tilde{Z}} \rangle = Q$ and $\langle u_3 \rangle = -\frac{1}{6} \log \frac{Q}{4}$. The first equation is solved by $\langle \rho \rangle = i$ which implies $\langle v \rangle = \langle \bar{v} \rangle = 0$ and

$$\langle |\rho|^2 \rangle = 1. \quad (3.42)$$

Thus the vector multiplet scalar u_3 and the hypermultiplet scalars $\{b^a, \bar{b}^a, v, \bar{v}\}$ are fixed by the first equation in (2.8). Using the above results, we find that the moment maps (3.39) are only nontrivial in the \vec{e}_3 -direction and read

$$\langle \mu_0 \rangle = \frac{Q}{2} e^{-4\langle u_1 \rangle} - 3, \quad \langle \mu_1 \rangle = -2e^{-4\langle u_1 \rangle}. \quad (3.43)$$

Inserting these expressions into the second equation of (2.8) and using (3.33), we find

$$\left(\frac{Q}{2} e^{-4\langle u_1 \rangle} - 3 \right) = 3\lambda e^{4\langle u_3 \rangle}, \quad e^{-4\langle u_1 \rangle} = \frac{2}{3} \lambda e^{-2\langle u_3 \rangle}. \quad (3.44)$$

These equations fix the scalar $\langle u_1 \rangle = -\frac{1}{4} \log\left(\frac{2^{5/3}\lambda}{3\sqrt{Q}}\right)$. In conclusion, we have shown that the AdS₅ conditions for the Betti-hyper truncation fix the vector multiplet scalar u_3 and the hypermultiplet scalars $\{b^a, \bar{b}^a, v, \bar{v}, u_1\}$.

⁹The vanishing of the topological charges j^a shows that the backgrounds we are discussing are indeed related to the Klebanov-Witten theory [21] and not the Klebanov-Strassler solutions [39].

To compute the moduli space, we observe that the scalars $\{k, e^a, \tau, \bar{\tau}\}$ are not restricted by the AdS conditions (2.8) and thus their associated deformations leave the vacuum invariant. However, as before we find that the Killing vectors have nontrivial background values,

$$\langle k_0 \rangle = -Q\partial_k, \quad \langle k_1 \rangle = 4\partial_k, \quad (3.45)$$

and thus the space of Goldstone bosons is again one-dimensional. The gauge group is broken, $U(1) \times \mathbb{R} \rightarrow U(1)_R$, and the vector field $QA_0 - 2(A_1 + A_2)$ becomes massive by “eating” the scalar k . Note that, as discussed in section 2, the $U(1)_R$ symmetry of the background is still present after the spontaneous symmetry breaking. It is gauged by the graviphoton

$$\langle h_0 \rangle A_\mu^0 + \frac{1}{2} \langle h_1 \rangle (A_\mu^1 + A_\mu^2) = \frac{Q^{2/3}}{6\sqrt[3]{2}} A_\mu^0 + \frac{2^{2/3}}{3\sqrt[3]{Q}} (A_\mu^1 + A_\mu^2). \quad (3.46)$$

The moduli space is thus four-dimensional and spanned by the hypermultiplet scalars $\{e^a, \tau, \bar{\tau}\}$. This is in agreement with the bound (2.9) for $n_H = 3, n_G = 1$,

$$\dim \mathcal{M}^{\text{BH}} \leq 2 \cdot 3 - 2 \cdot 1 = 4. \quad (3.47)$$

Note that $\dim \mathcal{M}^{\text{BH}} = 4$ is again the maximal dimension possible for the given number of hypermultiplets and Goldstone bosons.

The moduli space is thus spanned by the reduction of the Axion-Dilaton τ and a doublet of real scalars e^a coming from the Betti-hypermultiplet. To compute the metric, we use the fact that the coordinate one-forms of the fixed scalars vanish on \mathcal{M}^{BH} . Inserting this into the Lagrangian (3.35) for the hypermultiplets, we find

$$g^{\text{BH}} = \gamma M_{ab} de^a de^b + \frac{1}{\text{Im } \tau^2} d\tau d\bar{\tau}, \quad (3.48)$$

where $\gamma = 2e^{-4\langle u_1 \rangle} = \frac{2}{3}(2Q)^{1/3}\lambda$ and M_{ab} is defined in (3.15). We immediately recognize the second term in (3.48) as the metric of the Axion-Dilaton on the upper half plane (3.29). Let us first discuss the isometries of the metric (3.48). Clearly, (3.48) is invariant under shifts in the scalars e^a , i.e. $e^a \mapsto e^a + w^a$ for some constants w^a . Moreover, the metric has an $SL(2, \mathbb{R})$ -isometry induced by the global $SL(2, \mathbb{R})$ -symmetry of type IIB supergravity [38]. For the term $M_{ab} de^a de^b$, this follows from the fact that the transformations (3.6) and (3.16) exactly cancel each other. Moreover, the metric $\frac{1}{\text{Im } \tau^2} d\tau d\bar{\tau}$ on the upper half plane is known to have an $SL(2, \mathbb{R})$ -isometry given by (3.7). Thus the metric (3.48) has an $\mathbb{R}^2 \times SL(2, \mathbb{R})$ isometry group.

We already discussed that the moduli space of AdS₅ vacua should be Kähler and in particular complex. To this end, let us define a complex structure for the scalars $\{e^a, a, \phi\}$ and construct the Kähler potential associated to the metric g^{BH} , i.e. a real function K such that $(g^{\text{BH}})_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ for $i, j = 1, 2$ complex indices on \mathcal{M}^{BH} . For the scalars $\{a, \phi\}$, the complex structure is naturally given by the Axion-Dilaton $\tau = a + ie^{-\phi}$. To define a complex structure on the scalars e^a , recall that the three-forms \hat{F}_3^a in type IIB supergravity can be combined into a complex three-form [38],

$$\hat{G}_3 := \hat{F}_3^2 - \tau \hat{F}_3^1. \quad (3.49)$$

Translating this to the scalars e^a , we may define a complex scalar z by

$$z := e^2 - \tau e^1, \quad \bar{z} := e^2 - \bar{\tau} e^1. \quad (3.50)$$

In particular, this implies $e^1 = -\frac{\text{Im } z}{\text{Im } \tau}$. The associated coordinate one-forms then introduce a twist between z and τ ,

$$dz = de^2 - \tau de^1 - e^1 d\tau, \quad d\bar{z} = de^2 - \bar{\tau} de^1 - e^1 d\bar{\tau}. \quad (3.51)$$

Using these we can rewrite the metric of the e^a in terms of z and \bar{z} ,

$$M_{ab} de^a de^b = \frac{\text{Im } z^2}{\text{Im } \tau^3} d\tau d\bar{\tau} + \frac{1}{\text{Im } \tau} dz d\bar{z} - \frac{\text{Im } z}{\text{Im } \tau^2} (d\tau d\bar{z} + d\bar{\tau} dz). \quad (3.52)$$

Thus the full complex metric reads

$$g^{\text{BH}} = \left(\frac{1}{\text{Im } \tau^2} + \gamma \frac{\text{Im } z^2}{\text{Im } \tau^3} \right) d\tau d\bar{\tau} - \gamma \frac{\text{Im } z}{\text{Im } \tau^2} (d\tau d\bar{z} + d\bar{\tau} dz) + \frac{\gamma}{\text{Im } \tau} dz d\bar{z}. \quad (3.53)$$

This metric is derived from the Kähler potential

$$K = -4 \log(\tau - \bar{\tau}) - i\gamma \frac{(z - \bar{z})^2}{\tau - \bar{\tau}}, \quad (3.54)$$

and the associated Kähler form $\omega = \frac{i}{2} \partial \bar{\partial} K$ is closed. Thus the moduli space is a Kähler manifold with Kähler structure defined by K .

Let us now show that (3.53) extends to a globally well-defined metric and identify the manifold \mathcal{M}^{BH} . Examining the metric (3.48), we already observed that the second part is the metric $\frac{1}{\text{Im } \tau^2} d\tau d\bar{\tau}$ on the upper half plane \mathcal{H} . The first term in (3.48) is the metric of a torus \mathbb{C}/Λ_τ with complex structure parameter τ . Here $\Lambda_\tau = \mathbb{Z} \oplus \tau \mathbb{Z}$ is a lattice spanned by $(1, \tau)$. However, this description only holds locally. Globally, the moduli space is not a direct product of a complex torus with the upper half plane, since the complex structure (3.50) on the torus varies with τ . Thus the global metric is the metric on the total space of a complex torus bundle over the upper half plane, i.e.

$$\mathbb{C}/\Lambda_\tau \hookrightarrow \mathcal{M}^{\text{BH}} \longrightarrow \mathcal{H}, \quad (3.55)$$

Note that this agrees with the results stated in [21]; the moduli are the Axion-Dilaton and a complex scalar parametrizing a torus. However, it turns out that the metric is in general not a product metric but the metric of a non-trivial fibration. To identify the total space of the fibration (3.55), consider the universal elliptic curve \mathcal{E} over the upper half plane.¹⁰ This is defined as the quotient

$$\mathcal{E} = (\mathbb{C} \times \mathcal{H}) / \mathbb{Z}^2, \quad (3.56)$$

where $(m, n) \in \mathbb{Z}^2$ acts as

$$(z, \tau) \mapsto (z + m + n\tau, \tau). \quad (3.57)$$

¹⁰For an introduction to elliptic curves, their moduli spaces and the universal elliptic curve, see [27].

Since this action is free and proper, the quotient \mathcal{E} is a two-dimensional complex manifold [27]. In particular, the fibers of the projection $\mathcal{E} \rightarrow \mathcal{H}$ are precisely the complex tori \mathbb{C}/Λ_τ . To see that g^{BH} gives a well-defined metric on \mathcal{E} , we have to show that it is compatible with the quotient by the action (3.57). Since τ is fixed by (3.57), only the second term in the Kähler potential (3.54) transforms non-trivially. In particular, we find for the transformation of the Kähler potential,

$$K \mapsto K' = K + 2i\gamma n(z - \bar{z}) + i\gamma n^2(\tau - \bar{\tau}), \tag{3.58}$$

which is just a Kähler transformation $K \mapsto K' = K + f(\tau, z) + \bar{f}(\bar{\tau}, \bar{z})$ for a holomorphic function $f(\tau, z) = 2i\gamma n z + i\gamma n^2 \tau$. Thus both potentials give rise to the same Kähler metric and g^{BH} is a well-defined global metric on \mathcal{E} .¹¹ In conclusion, the moduli space of AdS vacua in the Betti-hyper truncation is given by the total space of the universal elliptic curve,

$$\mathcal{M}^{\text{BH}} = \mathcal{E} = (\mathbb{C} \times \mathcal{H})/\mathbb{Z}^2. \tag{3.59}$$

This manifold is in particular a homogeneous space, since it has a transitive group action given by the isometries of the metric (3.48), $\mathbb{R}^2 \times \text{SL}(2, \mathbb{R}) \cong \mathbb{C} \times \text{SL}(2, \mathbb{R})$. Because the upper half plane \mathcal{H} can be written as the quotient $\text{SL}(2, \mathbb{R})/\text{SO}(2)$, we find

$$\mathcal{M}^{\text{BH}} = [\mathbb{C} \times \text{SL}(2, \mathbb{R})/\text{SO}(2)]/\mathbb{Z}^2. \tag{3.60}$$

Thus for both truncations with homogeneous scalar target spaces, also the AdS moduli space is a homogeneous space.

Before we conclude this section, let us briefly note the following: consider the Kähler potential L of $\text{SU}(2, 1)/\text{U}(2)$, given by

$$L = -\log(\tau - \bar{\tau} + i\epsilon(z - \bar{z})^2), \tag{3.61}$$

where $\epsilon \in \mathbb{R}$ is a constant. To make contact with the potential (3.54), we want to split off a term of the form $\log(\tau - \bar{\tau})$ from L . To this end, we separate a factor $\tau - \bar{\tau}$ inside the logarithm,

$$L = -\log(\tau - \bar{\tau} + i\epsilon(z - \bar{z})^2) = -\log(\tau - \bar{\tau}) - \log\left(1 + i\epsilon\frac{(z - \bar{z})^2}{\tau - \bar{\tau}}\right). \tag{3.62}$$

For small ϵ the second term can be expanded,

$$-\log\left(1 + i\epsilon\frac{(z - \bar{z})^2}{\tau - \bar{\tau}}\right) \simeq -i\epsilon\frac{(z - \bar{z})^2}{\tau - \bar{\tau}} + \mathcal{O}(\epsilon^2). \tag{3.63}$$

Thus we can write

$$L = -\log(\tau - \bar{\tau} + i\epsilon(z - \bar{z})^2) \simeq \frac{1}{4}K + \mathcal{O}(\epsilon^2), \tag{3.64}$$

for suitable ϵ and find that the Kähler potential (3.54) of the AdS moduli space appears as a first order term in the ϵ -expansion. To interpret this result we first note the following:

¹¹A different way to see the invariance of the metric under the \mathbb{Z}^2 -action is to realize (3.57) via the \mathbb{R}^2 -isometry of the metric (3.48) by $e^1 \mapsto e^1 - n$, $e^2 \mapsto e^2 + m$.

we can reinstall the five-dimensional gravitational constant κ into the metric (3.53) by $\gamma \mapsto \kappa^2 \gamma$. Thus the ϵ -expansion performed above actually corresponds to an expansion in the gravitational constant κ for $\epsilon = 4\kappa^2 \gamma$ and fixed γ . Since the limit $\kappa \rightarrow 0$ corresponds to the large N limit of the dual field theory,¹² we can interpret the metric $g_{\mathcal{M}}^{\text{BH}}$ on the moduli space as the first order contribution in a large N expansion of the metric on $\text{SU}(2, 1)/\text{U}(2)$.

4 Conclusions

Let us summarize the results obtained. We computed the conditions on AdS vacua in five-dimensional $\mathcal{N} = 2$ supergravity for both the $\mathcal{N} = 2$ Betti-vector truncation and the Betti-hyper truncation coming from ten-dimensional type IIB supergravity compactified on $T^{1,1}$. Since these truncations are consistent, our results lift to solutions of the full ten-dimensional supergravity. We find that both truncations admit AdS_5 backgrounds which retain an $\text{U}(1)_R$ symmetry in the vacuum. This is in agreement with the $\text{U}(1)_R$ symmetry coming from the dual field theories predicted by the AdS/CFT correspondence [7, 8, 21].

The moduli space of the full ten-dimensional solution $\text{AdS}_5 \times T^{1,1}$ is known to be complex five-dimensional [4, 5, 21–23]. However, only two of those moduli transform as singlets under the $\text{SU}(2) \times \text{SU}(2)$ factor in the isometry group and are thus accessible via consistent truncations [25]; those are the Axion-Dilaton τ and the complex modulus z related to the topology of $T^{1,1}$, i.e. coming from the fact that $b_2(T^{1,1}) = 1$. In [4, 5] the moduli spaces for type IIB solutions of the form $\text{AdS}_5 \times \text{SE}_5$ were computed from generalized geometry. In particular, it was shown that the Axion-Dilaton is always a modulus, independent from the topology of the Sasaki-Einstein manifold used for the compactification. Moreover, it was shown in [15] that the moduli of five-dimensional AdS backgrounds must always be recruited out of the hypermultiplets. Our present results agree with these predictions; the moduli space of the Betti-vector truncation is spanned only by the Axion-Dilaton residing in one of the hypermultiplets and the metric is the expected one on the upper half-plane. This can be understood as follows: the fact that $T^{1,1}$ has nontrivial second cohomology leads to the presence of an additional two-form in the reduction ansatz [18, 19]. This additional two-form gives rise to an additional $\mathcal{N} = 4$ vector multiplet which splits into an $\mathcal{N} = 2$ vector multiplet and an $\mathcal{N} = 2$ hypermultiplet. However, only one of the two can be retained in a consistent $\mathcal{N} = 2$ truncation [18]. Thus in the Betti-vector truncation the only hypermultiplet related to the topology of $T^{1,1} \cong S^2 \times S^3$ was removed and one would not expect to find the modulus z in this truncation in the first place.

In the case of the Betti-hyper truncation the situation is different. Here the topology of $T^{1,1}$ contributes to the five-dimensional hypermultiplets of the truncation and gives rise to an additional complex modulus z . This modulus parametrizes a complex torus with complex structure parameter given by the Axion-Dilaton τ . Thus the complex two-dimensional moduli space \mathcal{M}^{BH} of the Betti-hyper truncation contains all the moduli of $\text{AdS}_5 \times T^{1,1}$ that are expected to arise in a consistent truncation. However, even though the Axion-Dilaton is completely unrelated to the geometry of the compact Sasaki-Einstein manifold, it turns out that the metric on the moduli space \mathcal{M}^{BH} is not a product metric

¹²The AdS/CFT correspondence relates $\kappa \propto 1/N$, see [9] for a review.

with the Axion-Dilaton split off from the moduli corresponding to the geometry. Indeed, we find that \mathcal{M}^{BH} is a torus bundle with base space given by the upper half-plane \mathcal{H} . We showed that this bundle is the universal elliptic curve $\mathcal{E} \rightarrow \mathcal{H}$ with total space given by $\mathcal{E} = (\mathbb{C} \times \mathcal{H})/\mathbb{Z}^2$. Thus since the Betti-hyper truncation is consistent and lifts to the full ten-dimensional supergravity, the AdS/CFT correspondence relates this result to the metric on a submanifold of the conformal manifold of the Klebanov-Witten theory in four dimensions.¹³ It would be interesting to check this result on the field theory side.

Since $T^{1,1}$ is a member of the infinite family of Sasaki-Einstein manifolds called $Y^{p,q}$ [40], a natural question would be to extend our computations to these manifolds. On first glance, this might seem easily possible; $Y^{p,q} \cong S^2 \times S^3$ have the same topology as $T^{1,1}$ and a reduced isometry algebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$. Moreover the moduli of the ten-dimensional solutions $\text{AdS}_5 \times Y^{p,q}$ also contain the Axion-Dilaton and the modulus z from the VEV of the B-field integrated over the nontrivial two-cycle. Additionally, there only exists one other complex modulus transforming in a triplet under the $\text{SU}(2)$ in the isometry group, making the full moduli space complex three-dimensional. However, the $Y^{p,q}$ manifolds are in general not coset spaces but only admit a cohomogeneity-one action of the isometry group. Since the techniques used in [18, 19] rely heavily on the transitivity of the $\text{SU}(2) \times \text{SU}(2)$ action on the $T^{1,1}$ coset, a similar reduction for the $Y^{p,q}$ manifolds might not be consistent.

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