

## Frozen singularities in M and F theory

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**Yuji Tachikawa**

*Department of Physics, Faculty of Science,  
University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan  
Institute for the Physics and Mathematics of the Universe,  
University of Tokyo, Kashiwa, Chiba 277-8583, Japan*

*E-mail:* [yuji.tachikawa@ipmu.jp](mailto:yuji.tachikawa@ipmu.jp)

**ABSTRACT:** We revisit the duality between ALE singularities in M-theory and 7-branes on a circle in F-theory. We see that a frozen M-theory singularity maps to a circle compactification involving a rotation of the plane transverse to the 7-brane, showing an interesting correspondence between commuting triples in simply-laced groups and Kodaira's classification of singular elliptic fibrations. Our analysis strongly suggests that the  $O7^+$  plane is the only completely frozen F-theory singularity.

**KEYWORDS:** F-Theory, M-Theory, String Duality

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**1 Introduction and summary**

It is by now a textbook material that, in M-theory, a singularity locally of the form  $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$  where  $\Gamma_{\mathfrak{g}}$  is a finite subgroup of  $SU(2)$  corresponding to a simply-laced algebra  $\mathfrak{g}$  produces a 7d super Yang-Mills theory with gauge algebra  $\mathfrak{g}$  at the singular locus. It might be less-known that there are (partially) frozen variants of such a singularity, still preserving 16 supercharges, characterized by a non-zero value of

$$r := \int_{S^3/\Gamma_{\mathfrak{g}}} C = \frac{n}{d} \pmod{1} \tag{1.1}$$

around it [1–3]. Here,  $C$  is the M-theory 3-form and  $d$  is a label on the nodes of the Dynkin diagram of  $\mathfrak{g}$  and  $\gcd(n, d) = 1$ . On such an M-theory singularity, we have a gauge algebra  $\mathfrak{h}_r$ . We list possible values of  $r, \mathfrak{h}_r$  for all  $\mathfrak{g}$  in table 1.<sup>1</sup>

Let us now consider a situation where such a singularity is in a singular fiber of an elliptic fibration  $X_g$  over the complex plane<sup>2</sup>  $\mathbb{C}_z$ , such that there is an  $SL(2, \mathbb{Z})$  monodromy  $g$  around  $z = 0$  acting on the elliptic fiber. Possible conjugacy classes of monodromies were classified by Kodaira, see table 2. The central fiber can be appropriately blown up so that the whole geometry is smooth, but we mainly consider the case that there is a singularity there. When the monodromy is diagonalizable, the supergravity solution can be taken so that the base  $\mathbb{C}_z$  is conical, with a metric of the form  $dR^2 + R^2 d\theta^2$  with a nontrivial periodicity  $\theta \sim \theta + \theta_g$ . We call  $\theta_g$  the opening angle in this note.

Now let us first consider standard singularities which are not (partially) frozen. At each point on a base, we can reduce along an  $S^1$  of the fiber and take the T-dual of the other  $S^1$  to have a type IIB setup on  $S^1$ . The monodromy around  $z = 0$  now acts on the axiodilaton of the type IIB theory, therefore we are now in an F-theory configuration: we have a 7-brane at  $z = 0$ , compactified on  $S^1$ . As is well-known, when we shrink the elliptic fiber on the M-theory side, the  $S^1$  on the F-theory side opens up. We see that a 7-brane characterized by a monodromy  $g$ , obtained in a manner outlined above, has a gauge symmetry  $\mathfrak{g}$  on it [4, 5].

$\mathfrak{g}$	$\mathfrak{so}(2k+8)$	$\mathfrak{e}_6$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_7$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_8$
$r = \frac{n}{d}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$\frac{1}{6}, \frac{5}{6}$
$\mathfrak{h}_r$	$\mathfrak{sp}(2k)$	$\mathfrak{su}(3)$	*	$\mathfrak{so}(7)$	$\mathfrak{su}(2)$	*	$\mathfrak{f}_4$	$\mathfrak{g}_2$	$\mathfrak{su}(2)$	*	*

**Table 1.** Partially frozen half-BPS M-theory singularities of the form  $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$ .

	$g$	$\theta_g/\frac{\pi}{6}$	$\mathfrak{g}$	ord( $a$ )	ord( $b$ )	ord( $\Delta$ )	sample equation
$I_k$	$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	$12 - k$	$\mathfrak{su}(k)$	0	0	$k$	$y^2 = x^3 - 3x + 2 + z^k$
$II$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	10	*	1	1	2	$y^2 = x^3 + z$
$III$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	9	$\mathfrak{su}(2)$	1	2	3	$y^2 = x^3 + zx$
$IV$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	8	$\mathfrak{su}(3)$	2	2	4	$y^2 = x^3 + z^2$
$I_k^*$	$\begin{pmatrix} -1 & -k \\ 0 & -1 \end{pmatrix}$	$6 - k$	$\mathfrak{so}(2k+8)$	2	3	$k+6$	$y^2 = x^3 - 3z^2x + 2z^3(1+z^k)$
$IV^*$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	4	$\mathfrak{e}_6$	3	4	8	$y^2 = x^3 + z^4$
$III^*$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	3	$\mathfrak{e}_7$	3	5	9	$y^2 = x^3 + z^3x$
$II^*$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	2	$\mathfrak{e}_8$	4	5	10	$y^2 = x^3 + z^5$

**Table 2.** Possible monodromies  $g$  of elliptic fibrations over the base  $\mathbb{C}_z$ . Here,  $\mathfrak{g}$  is the type of the singularity and  $\theta_g$  is the opening angle. The equation of the elliptic fiber is given by  $y^2 = x^3 + a(z)x + b(z)$  where  $z$  is the coordinate of the base, and  $\Delta = 4a^3 + 27b^2$ .

It is known, however, that the monodromy alone does not completely characterize a 7-brane in F-theory. For example, the singular fiber of type  $I_4^*$  can be realized in perturbative type IIB string theory by putting 8 D7-branes<sup>3</sup> on top of an  $O7^-$  plane,<sup>4</sup> and also by just an  $O7^+$  plane. The former has  $\mathfrak{so}(16)$  gauge algebra on it but the latter does not have any, and therefore they are clearly distinct. Correspondingly, the former  $I_4^*$  singularity can be deformed, but the latter  $I_4^*$  singularity is somehow completely frozen, probably due to an effect of a discrete flux [6].

This begs a natural question: *are there other (partially) frozen variants of F-theory 7-branes?* Clearly  $I_{4+k}^*$  singularities have two versions: one given by an  $O7^-$  plane plus

<sup>1</sup>Our convention is  $\mathfrak{sp}(2) = \mathfrak{su}(2)$ .

<sup>2</sup>In the following the subscript on  $\mathbb{C}$  denotes the symbol for its coordinate.

<sup>3</sup>Without counting mirror images.

<sup>4</sup>Our convention is that  $O^-$  planes give orthogonal symmetries and  $O^+$  planes give symplectic symmetries.

$8 + k$  D7-branes, another given by an  $O7^+$  plane plus  $k$  D7-branes. The main objective of this note is to argue that *there are no other (partially) frozen half-BPS 7-branes*.

We approach this question by first studying an M-theory configuration on an elliptic fibration  $X_g$  over  $\mathbb{C}_w$  with a singularity of type  $\mathfrak{g}$  at the central fiber, now with a nonzero value of  $r = \frac{n}{d}$  defined in (1.1). In section 2, we will see that, a fiber-wise duality to the type IIB description, we have an F-theory configuration on  $(\mathbb{C}_z \times S^1)/\mathbb{Z}_d$  where  $w = z^d$  with monodromy  $g^d$  around  $z = 0$ , such that the quotient is given by  $z \mapsto e^{2\pi i n/d} z$  together with a  $\frac{1}{d}$  shift along  $S^1$ .

We then run the arguments in reverse in section 3. Namely, we take a putative (partially) frozen half-BPS 7-brane in F-theory, and compactify it on  $S^1$ . We will take a fiber-wise duality to go to an M-theory frame. This should result in an elliptic fibration with singularities at the origin of the base  $\mathbb{C}$ . Using the list of (partially) frozen half-BPS singularities in M-theory, we conclude that the  $O7^+$  planes with D7-branes on top are the only (partially) frozen half-BPS 7-branes in F-theory. We conclude with a brief discussion in section 4.

## 2 Frozen singularities in M-theory and their F-theory duals

Let us start by commenting further on the structure of the (partially) frozen singularities of the form  $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$  in M-theory [1, 2]. As already recalled, they are characterized by a non-zero value of  $r = \int_{S^3/\Gamma_{\mathfrak{g}}} C \pmod{1}$  around the singularity. The allowed choices of  $r$  correspond to the Chern-Simons invariants of flat  $G$  bundles on  $T^3$ , where  $G$  is the connected and simply-connected group for  $\mathfrak{g}$ . A flat  $G$  bundle on  $T^3$  corresponds to a commuting triple of elements of  $G$  up to conjugacy, and therefore this information  $r$  is often called a *triple*.

On such an M-theory singularity with non-zero  $r$ , we have a 7d super Yang-Mills theory with gauge algebra  $\mathfrak{h}_r$  which is given by the Langlands dual of the subalgebra of  $\mathfrak{g}$  commuting with this flat bundle, or equivalently the commuting triple.<sup>5</sup> If  $\mathfrak{h}_r$  is empty the singularity is completely frozen. If  $\mathfrak{h}_r$  is nonempty, the singularity is only partially frozen and can be deformed to a completely frozen one, whose type is a minimal one compatible with a given value of  $r$ .

For example, take  $\mathfrak{g} = \mathfrak{e}_8$  and  $r = \frac{1}{3}$ . The minimal algebra compatible with this value of  $r$  is  $\mathfrak{e}_6$ , whose nontrivial commuting triple is in fact contained in  $\mathfrak{f}_4$ . The commutant of this  $\mathfrak{f}_4$  is  $\mathfrak{g}_2$ , whose Langlands dual gives  $\mathfrak{h}_r = \mathfrak{g}_2$ . As another example, take  $\mathfrak{g} = \mathfrak{so}(2k + 8)$  and  $r = \frac{1}{2}$ . The minimal algebra compatible with this  $r$  is  $\mathfrak{so}(8)$ , whose nontrivial commuting triple is in fact contained in  $\mathfrak{so}(7)$ . Its commutant within  $\mathfrak{g}$  is  $\mathfrak{so}(2k + 1)$ , whose Langlands dual is  $\mathfrak{h}_r = \mathfrak{sp}(2k)$ .

Take now an elliptic fibration  $X_g$  over  $\mathbb{C}_w$  with a singularity of type  $\mathfrak{g}$  in the singular fiber at  $w = 0$ . We remind the reader that  $g$  stands for the  $SL(2, \mathbb{Z})$  monodromy at  $w = 0$ . We furthermore put a nonzero value of  $r$  to (partially) freeze the singularity. We would like to construct an F-theoretic dual description of this setup.

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<sup>5</sup>These facts can be understood via the fractionation of M5-branes and its relation to instantons on  $T^3 \times \mathbb{R}$ , see e.g. [7] and section 3.1 of [8]. Readable accounts on triples can be found in the last section of [9] and the last subsection of [2].

It is easiest to start with the case of the  $I_k^*$  fiber with a  $D_{k+4}$  singularity, (partially) frozen with  $r = \frac{1}{2}$ . When we reduce it to type IIA, this becomes an  $O6^+$  plane plus  $k$  D6-branes [6, 10]. Since the monodromy preserves an  $S^1$  of the elliptic fiber up to a multiplication by  $-1$ , we can reduce the whole setup globally along this  $S^1$  to a genuine type IIA configuration on  $(\mathbb{C}_z \times S_{\text{IIA}}^1)/\mathbb{Z}_2$  with an  $O6^+$  plane plus  $k$  D6-branes on one of the fixed points. On the other fixed point, we should have an  $O6^-$  plane, which is known to lift to a smooth configuration in M-theory. Note that we have  $w = z^2$ , since the two points  $\pm z$  on  $\mathbb{C}_z$  are identified by the orientifolding action.

Now, take the T-dual to obtain a type IIB configuration. We have a so-called shift-orientifold on  $(\mathbb{C}_z \times S_{\text{IIB}}^1)/\mathbb{Z}_2$ , where the orientifold action on  $\mathbb{C}_z$  is accompanied by a  $\frac{1}{2}$  shift of  $S_{\text{IIB}}^1$  with  $2k$  D7-branes on the locus  $z = 0$ , see e.g. [11]. Note that there is  $\mathfrak{su}(2k)$  gauge algebra locally on the D7-branes, which is broken to  $\mathfrak{sp}(2k)$  by the compactification on  $S^1$  involving the orientifolding action, that acts as an outer automorphism of  $\mathfrak{su}(2k)$ .

The orientifolding action  $z \mapsto -z$  when we go along  $S_{\text{IIB}}^1$  can also be understood as follows: we had  $\int_{S^3/\Gamma_g} C = \frac{1}{2}$  around the singularity in the M-theory description. This 3-cycle can be deformed into a 3-cycle  $T$  given by a large  $S_{\text{big}}^1 \subset \mathbb{C}_z$  times the elliptic fiber. In the type IIA reduction, this means that there is  $\int_{S_{\text{big}}^1 \times S_{\text{IIA}}^1} B = \frac{1}{2}$ , which turns into a  $\frac{1}{2}$  rotation in the type IIB setup.

Let us summarize what we have found in an F-theoretic language. We started from the  $I_k^*$  fiber with monodromy  $g = \begin{pmatrix} -1 & -k \\ 0 & -1 \end{pmatrix}$  on  $\mathbb{C}_w$ , with  $r = \frac{1}{2}$  in M-theory. The result is an F-theory configuration on  $\mathbb{C}_{z=w^{1/2}}$  with the  $I_{2k}$  fiber whose monodromy is  $g^2 = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$ , further compactified on  $S^1$  such that  $z \mapsto -z$  when we go around  $S^1$ . Now, from table 2, we find that the local singularity has the form  $st = z^{2k}$  where  $s, t$  are suitable combinations of  $x$  and  $y$ . The action  $z \mapsto -z$  can be lifted to  $(s, t, z) \mapsto (s, -t, -z)$ , which is known to correspond to a  $\mathbb{Z}_2$  outer automorphism of the  $\mathfrak{su}(2k)$  gauge algebra of the 7-brane of type  $I_{2k}$ .

Stated in this manner, it is straightforward to generalize this observation to any other elliptic fibration  $X_g$  with a compatible choice of  $r = \frac{n}{d}$ . Namely, the F-theory dual is given by a 7-brane on  $(\mathbb{C}_z \times S_{\text{IIB}}^1)/\mathbb{Z}_d$  whose monodromy around  $z = 0$  is  $g^d$ , where the  $\mathbb{Z}_d$  action is given by  $z \mapsto e^{2\pi i n/d} z$  together with a  $\frac{1}{d}$  shift along  $S_{\text{IIB}}^1$ . The results are tabulated in table 3.

We find that the opening angles  $\theta_g$  of  $\mathbb{C}_w$  and  $\theta_{g^d}$  of  $\mathbb{C}_z$  satisfy the relation  $\theta_{g^d} = d\theta_g$ , as it should be for the metric to be consistent. We also see that the action  $z \mapsto e^{2\pi i n/d} z$  corresponds to the outer automorphism of a required order, as can be checked using the explicit equation of the elliptic fibrations given in table 2. This reduction of the gauge symmetry due to the outer automorphism is known in F-theory configurations on complex surfaces. A small novelty here is that the base is real three-dimensional.

Before proceeding, we pause to mention that the rotation of the phase of  $z$  by  $2\pi \frac{n}{d}$  on the plane  $\mathbb{C}_z$  transverse to the 7-brane on the F-theory side can be derived by a further compactification on  $S^1$ . Let us start from the M-theory side. We compactify the whole setup on  $S^1$ . This is a type IIA configuration on an elliptic fibration  $X_g$  with a non-zero  $\int_T C_{(3)} = \frac{n}{d}$ , where, as before,  $T$  is a big circle  $S_{\text{big}}^1 \subset \mathbb{C}_w$  times the elliptic fiber, and  $C_{(3)}$

$g$		$\theta_g/\frac{\pi}{6}$	alg.	$r = \frac{n}{d}$	$g^d$		$\theta_{g^d}/\frac{\pi}{6}$	alg.	outer	fixed
$\begin{pmatrix} -1 & -k \\ 0 & -1 \end{pmatrix}$	$I_k^*$	$6 - k$	$\mathfrak{so}(2k + 8)$	$\frac{1}{2}$	$\begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$	$I_{2k}$	$12 - 2k$	$\mathfrak{su}(2k)$	$\mathbb{Z}_2$	$\mathfrak{sp}(2k)$
$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$IV^*$	4	$\mathfrak{e}_6$	$\frac{1}{2}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$IV$	8	$\mathfrak{su}(3)$	$\mathbb{Z}_2$	$\mathfrak{su}(2)$
$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$IV^*$	4	$\mathfrak{e}_6$	$\frac{1}{3}, \frac{2}{3}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$I_0$	12	*	*	*
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$III^*$	3	$\mathfrak{e}_7$	$\frac{1}{2}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$I_0^*$	6	$\mathfrak{so}(8)$	$\mathbb{Z}_2$	$\mathfrak{so}(7)$
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$III^*$	3	$\mathfrak{e}_7$	$\frac{1}{3}, \frac{2}{3}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$III$	9	$\mathfrak{su}(2)$	*	$\mathfrak{su}(2)$
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$III^*$	3	$\mathfrak{e}_7$	$\frac{1}{4}, \frac{3}{4}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$I_0$	12	*	*	*
$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$II^*$	2	$\mathfrak{e}_8$	$\frac{1}{2}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$IV^*$	4	$\mathfrak{e}_6$	$\mathbb{Z}_2$	$\mathfrak{f}_4$
$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$II^*$	2	$\mathfrak{e}_8$	$\frac{1}{3}, \frac{2}{3}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$I_0^*$	6	$\mathfrak{so}(8)$	$\mathbb{Z}_3$	$\mathfrak{g}_2$
$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$II^*$	2	$\mathfrak{e}_8$	$\frac{1}{4}, \frac{3}{4}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$IV$	8	$\mathfrak{su}(3)$	$\mathbb{Z}_2$	$\mathfrak{su}(2)$
$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$II^*$	2	$\mathfrak{e}_8$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$II$	10	*	*	*
$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$II^*$	2	$\mathfrak{e}_8$	$\frac{1}{6}, \frac{5}{6}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$I_0$	12	*	*	*

**Table 3.** M-theory on an elliptic fibration with monodromy  $g$  and a discrete flux  $r = \frac{n}{d}$ , and its F-theory realization characterized by the monodromy  $g^d$  and an outer-automorphism.

is the RR 3-form. We then take a double T-dual along the elliptic fiber. This is again a type IIA configuration on an elliptic fibration with a non-zero  $\int_{S_{\text{big}}^1} C_{(1)} = \frac{n}{d}$ , where  $C_{(1)}$  is now the RR 1-form.<sup>6</sup> This lifts to a new M-theory configuration on

$$\left( (\text{an elliptic fibration } X_{g^d} \text{ over } \mathbb{C}_z) \times S^1 \right) / \mathbb{Z}_d$$

such that the generator of  $\mathbb{Z}_d$  is the rotation of the  $z$  plane by  $2\pi\frac{n}{d}$  together with the  $\frac{1}{d}$  shift of the M-theory circle.<sup>7</sup> Clearly this is the  $S^1$  compactification of the F-theory configuration described above.

<sup>6</sup>Such backgrounds were first considered in [12].

<sup>7</sup>On the one hand, close to the singularity in the singular fiber, this is essentially the configuration studied in [13]. On the other hand, if we replace the elliptic fibration  $X_g$  by a compact K3, such a background was first considered in [14].

### 3 Frozen F-theory 7-branes and their M-theory duals

Let us now study (partially) frozen F-theory 7-branes, by running the argument of the previous section in reverse. Again, it is easiest to start with the case which has a perturbative type IIB realization.

Let us consider an  $O7^+$  plane. It has a monodromy of type  $I_4^*$ . We compactify the whole system on  $S^1$ , and take the T-duality. We now have a type IIA system on  $(\mathbb{C}_w \times S^1)/\mathbb{Z}_2$ , with  $O6^+$  planes on both fixed points. Therefore, its M-theory lift has two frozen singularities of type  $D_4$ . Note that these two singularities are both on the same singular fiber of the elliptic fibration. Since the sum of two Dynkin diagrams of type  $D_4$  is contained in an affine Dynkin diagram of type  $D_8$ , we see that the singular fiber has the type  $I_4^*$ , as it should be.

Note that each of the  $D_4$  singularities has  $\int_{T_a} C = \frac{1}{2} \pmod 1$  around it, where  $T_{a=1,2}$  is the quotient of  $S^3$  around each of the singularities. The three-cycle  $T$  given by  $S_{\text{big}}^1 \subset \mathbb{C}_z$  times the elliptic fiber is their sum  $T_1 + T_2$ , and therefore has  $\int_T C = 0 \pmod 1$ . This is compatible with the fact that the  $S^1$  compactification on the F-theory side does not involve any rotation.

With this warm-up, let us consider a general (partially) frozen half-BPS 7-brane at  $z = 0$  of  $\mathbb{C}_z$ , with monodromy  $g$  around  $z = 0$ . We assume it preserves 16 supercharges. Compactify it on  $S^1$  and take the M-theory dual. This operation should be possible away from  $z = 0$  fiber-wise. We then have an M-theory configuration of an elliptic fibration away from  $z = 0$ , with the same monodromy  $g$ . Given that it preserves 16 supercharges, it is strongly likely that the M-theory geometry is given by an elliptic fibration with singularities no worse than orbifolds of  $\mathbb{C}^2$  by finite subgroups of  $SU(2)$ . Let us say there are  $m$  singularities of type  $\mathfrak{g}_1, \dots, \mathfrak{g}_m$  at the central fiber. At least two out of these  $m$  singularities should be (partially) frozen; otherwise we can change the Kähler parameter to have just zero or one frozen singularity, and we know those cases do not correspond to (partially) frozen 7-branes.

Therefore, at least two of  $\mathfrak{g}_1, \dots, \mathfrak{g}_m$  are of type  $D$  or  $E$ , thus with three prongs. We also know that the sum of Dynkin diagrams of type  $\mathfrak{g}_i$  is contained in an affine Dynkin diagram whose type is determined by the monodromy  $g$ . Now, by direct inspection, we can easily see that the only affine Dynkin diagrams that can contain more than one finite Dynkin diagrams with three prongs are of type  $D_{8+k}$  with  $k \geq 0$ . The two finite Dynkin diagrams are necessarily of the type  $D_{4+k_1}$  and  $D_{4+k_2}$ , with  $k_{1,2} \geq 0$ . By our assumption both are (partially) frozen.

This is a type IIA configuration on  $(\mathbb{C}_w \times S^1)/\mathbb{Z}_2$ , with an  $O6^+$  with  $k_1$  D6-branes on one fixed point, and an  $O6^+$  with  $k_2$  D6-branes on another. Taking the T-dual, we have a type IIB configuration on  $(\mathbb{C}_w/\mathbb{Z}_2) \times S^1 = \mathbb{C}_z \times S^1$ . We have an  $O7^+$  with  $k_1 + k_2$  D7-branes at  $z = 0$ , and the whole system is further compactified on  $S^1$  with a Wilson line around it, so that  $\mathfrak{sp}(2k_1 + 2k_2)$  is broken to  $\mathfrak{sp}(2k_1) \oplus \mathfrak{sp}(2k_2)$ . We conclude that *a half-BPS (partially) frozen 7-brane is necessarily an  $O7^+$  plane, possibly with an integral number of D7-branes on top*. Note that our analysis does *not* allow a stuck  $\frac{1}{2}$  D7-brane on top of an  $O7^+$ , thus precluding the existence of half-BPS  $\widetilde{O7}^+$ . This is consistent with the analysis in [1].

## 4 Discussions

In this short note, we argued that there is a duality between

- M-theory configurations on an elliptic fibration on  $\mathbb{C}_w$  with monodromy  $g$  around  $w = 0$  with a (partially) frozen singularity with  $\int_{S^3/\Gamma_g} C = \frac{n}{d} \pmod{1}$ , and
- F-theory configurations of a 7-brane on  $\mathbb{C}_{z=w^{1/d}}$  with monodromy  $g^d$  around  $z = 0$ , further compactified on  $S^1$  so that  $z$  is rotated as  $z \mapsto e^{2\pi i n/d} z$  when we go around  $S^1$ .

We then argued that, using the same logic, a (partially) frozen half-BPS 7-brane is necessarily a combination of an  $O7^+$  plane plus an integral number of D7-branes.

Note that in our argument, we assumed that table 1 exhausted the list of (partially) frozen half-BPS codimension-4 singularities of M-theory. Therefore, we can state our conclusion in a slightly different way: if there is a (partially) frozen half-BPS 7-brane other than the  $O7^+$  plane plus D7-branes, there should also be a new, hitherto-unknown (partially) frozen half-BPS codimension-4 singularity in M-theory. The author considers this extremely unlikely.

F-theory has been used in various different constructions in the string theory literature. Very often, it is implicitly assumed that the holomorphically varying axiodilaton corresponds to an elliptic fibration with a section and that 7-branes are not (partially) frozen, and it was not clear how serious the unintended consequences were. In the last two years, genus-one fibrations without a section have been actively investigated, starting with [15], but there are very few works on the  $O7^+$  plane in the recent years, a notable exception being [16]. It may be the time to start investigating F-theory setups with (partially) frozen singularities seriously. The author hopes that this short note is useful as a first step in that direction, by showing that there is no other frozen 7-brane than the  $O7^+$ -plane.

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