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C_T for non-unitary CFTs in higher dimensions

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Abstract: The coefficient C_T of the conformal energy-momentum tensor two-point function is determined for the non-unitary scalar CFTs with four- and six-derivative kinetic terms. The results match those expected from large-N calculations for the CFTs arising from the $\mathrm{O}(N)$ non-linear sigma and Gross-Neveu models in specific even dimensions. C_T is also calculated for the CFT arising from $(n-1)$-form gauge fields with derivatives in $2n+2$ dimensions. Results for $(n-1)$-form theory extended to general dimensions as a non-gauge-invariant CFT are also obtained; the resulting C_T differs from that for the gauge-invariant theory. The construction of conformal primaries by subtracting descendants of lower-dimension primaries is also discussed. For free theories this also leads to an alternative construction of the energy-momentum tensor, which can be quite involved for higher-derivative theories.

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Contents

1 Introduction

It is a truth almost universally acknowledged that there are no non-trivial unitary conformal field theories in more than six dimensions. Indeed for superconformal theories this is a long established result [1], and it is often conjectured that the superconformal $(2,0)$ theory in six dimensions is the one theory to rule them all and in the light bind them.

However, there is now good evidence that there are interacting conformal theories, which contain an energy-momentum tensor, in more than four dimensions and, indeed, for non-unitary CFTs, for dimensions larger than six. The $\mathrm{O}(N)$ non-linear sigma model has a tractable $1/N$ expansion without restriction on the spatial dimension d [2–4], defining a conformal theory with calculable scaling dimensions, at least for sufficiently large N [5]. This suggests a non-trivial five-dimensional CFT which is also accessible by ε-expansion methods starting from $4+\varepsilon$ and $6-\varepsilon$ dimensions. Generalisations to higher dimensions were recently explored in [6, 7]. To leading order in $1/N$ the non-linear sigma model comprises an N-component scalar φ_i with dimension $\frac{1}{2}(d-2)$ and also a singlet σ with dimension 2, clearly violating the unitarity bound for scalars when $d>6$.

Apart from the scaling dimensions for conformal primary operators and the parameters determining the three-point functions and, hence, the operator product expansion, crucial data defining a CFT are given by the correlation functions involving the energy-momentum tensor. In any CFT C_T, the coefficient of the two-point function which is fixed up to an overall constant by conformal invariance, plays a crucial role. The scale of the energy-momentum tensor is determined by Ward identities and with our conventions

$$S_d{}^2 \big\langle T^{\mu\nu}(x)\, T^{\sigma\rho}(0)\big\rangle = C_T\, \frac{1}{(x^2)^d}\, \mathcal{I}^{\mu\nu,\sigma\rho}(x)\,, \tag{1.1}$$

for $S_d = 2\pi^{\frac{1}{2}d}/\Gamma(\frac{1}{2}d)$ and where $\mathcal{I}$ is the inversion tensor for symmetric traceless tensors, constructed in terms of the inversion tensor I for vectors

$$\mathcal{I}^{\mu\nu,\sigma\rho} = \frac{1}{2}(I^{\mu\sigma}I^{\nu\rho} + I^{\mu\rho}I^{\nu\sigma}) - \frac{1}{d}\,\eta^{\mu\nu}\eta^{\sigma\rho}\,, \quad I^{\mu\nu}(x) = \eta^{\mu\nu} - \frac{2}{x^2}\,x^\mu x^\nu\,. \tag{1.2}$$

C_T may be regarded as a measure of the number of degrees of freedom. It determines the contribution of the energy-momentum tensor in the conformal partial-wave expansion, and so is readily determined in bootstrap calculations. For the conventional free scalar and fermion theories C_T was determined for arbitrary d and for vector gauge theories for $d = 4$ some time ago in [8], and later C_T was calculated for $(n-1)$-form gauge in $d = 2n$ dimensions in [9]. For the $\mathrm{O}(N)$ sigma model results for C_T to first order in $1/N$ were obtained by Petkou [10, 11] by applying the operator product expansion to the four-point function for ϕ_i, and have recently been rederived by direct calculation and extended to the Gross-Neveu model in [12]. In the non-linear sigma model

$$C_T^{\mathrm{O}(N)} = C_{T,S}\Big(N + C_{T,1}^{\mathrm{O}(N)} + \mathrm{O}(N^{-1})\Big)\,, \qquad C_{T,S} = \frac{d}{d-1}\,, \tag{1.3}$$

where $C_{T,S}$ is the result for a free scalar in d dimensions. For general d, $C_{T,1}^{\mathrm{O}(N)}$ depends on the digamma function, $\psi(x) = \Gamma'(x)/\Gamma(x)$ but for $d = 4+2p$ only the contribution of the poles $\psi(x) \sim -1/(x+n)$, $n = 0, 1, \dots$ for some n are relevant. In consequence, the result for general d reduces to

$$C_{T,1}^{\mathrm{O}(N)}\Big|_{d=4+2p} = (-1)^{p-1}\,\frac{4\,(2p+1)!}{(p-1)!\,(p+3)!}\,, \tag{1.4}$$

which is an integer [6]. Thus, $C_{T,1}^{\mathrm{O}(N)}\big|_{d=4} = 0$, since then the theory reduces to N free scalars with σ non-dynamical, whereas $C_{T,1}^{\mathrm{O}(N)}\big|_{d=6} = 1$. The extra 1 was interpreted in [5] as the contribution of the dynamical free scalar σ and for $d = 6-\varepsilon$ the ε-expansion defines a CFT at a fixed point starting from the renormalisable $\mathrm{O}(N)$-invariant Lagrangian

$$\mathcal{L}_6 = -\frac{1}{2}\big(\partial^\mu\varphi_i\partial_\mu\varphi_i + \partial^\mu\sigma\partial_\mu\sigma + g\,\sigma\,\varphi_i\varphi_i\big) - \frac{1}{6}\lambda\,\sigma^3\,, \tag{1.5}$$

with $g, \lambda = \mathrm{O}(\varepsilon)$. In higher even dimensions there are corresponding renormalisable Lagrangians with higher-derivative kinetic terms for σ. For $d = 8-\varepsilon$ there is a perturbative fixed point starting from

$$\mathcal{L}_8 = -\frac{1}{2}\big(\partial^\mu\varphi_i\partial_\mu\varphi_i + \partial^2\sigma\partial^2\sigma + g\,\sigma\varphi_i\varphi_i + \lambda'\,\sigma^2\partial^2\sigma\big) - \frac{1}{24}\lambda\,\sigma^4\,. \tag{1.6}$$

The β functions for the couplings g, λ', λ have recently been calculated by Gracey [7]. In this case $C_{T,1}^{\mathrm{O}(N)}\big|_{d=8} = -4$, which we show arises from the higher-derivative σ contribution.

A similar narrative emerges for the Gross-Neveu model with N fermion fields ψ_i. There is also a self-consistent $1/N$ expansion as a conformal field theory for any dimension d. To leading order ψ_i has dimension $\frac{1}{2}(d-1)$, and there is a singlet scalar field σ with scale dimension 1, which is consequently below the unitary bound for $d > 4$. The leading $1/N$ correction to C_T has been recently determined in [12], and can be expressed in the form

$$C_T^{\mathrm{GN}} = \frac{1}{2}d\,\mathrm{tr}(\mathbb{1})\,N + C_{T,S}\big(C_{T,1}^{\mathrm{GN}} + \mathrm{O}(N^{-1})\big)\,, \tag{1.7}$$

where $\frac{1}{2}d\,\mathrm{tr}(\mathbb{1})$ is the contribution to C_T for a single fermion with $\mathrm{tr}(\mathbb{1})$ the sum over spinorial indices. For general d the result [12] for $C_{T,1}^{\mathrm{GN}}$ is similar to that for the sigma

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model; for even dimensions it reduces to

$$C_{T,1}^{\rm GN}\Big|_{d=2+2p} = (-1)^{p-1}\,\frac{(2p+1)!}{(p-1)!\,(p+2)!}\,. \tag{1.8}$$

In this case $C_{T,1}^{\rm GN}\big|_{d=4} = 1$, representing the contribution of the dynamical scalar σ whereas $C_{T,1}^{\rm GN}\big|_{d=6} = -5$. For $d = 4-\varepsilon$ equivalent results can be obtained as a perturbative ε expansion at the RG fixed point starting from the renormalisable Lagrangian

$$\mathcal{L}_{\rm GN,4} = -\bar\psi\,\partial\!\!\!/\psi - \frac{1}{2}\,\partial^\mu\sigma\partial_\mu\sigma - g\,\sigma\,\bar\psi\psi - \frac{1}{24}\lambda\,\sigma^4\,, \tag{1.9}$$

with N Dirac fields ψ.

In this paper we calculate the contributions to C_T corresponding to higher-derivative scalars, such as that for σ in (1.6), for general d. The energy-momentum tensor is determined from the corresponding local Weyl-invariant actions on curved space quadratic in a scalar field φ. The construction of such actions is equivalent to obtaining conformal differential operators starting from powers of the Laplacian. We then determine C_T for scalar theories with kinetic terms with $2p$ derivatives for $p = 2, 3$, and conjecture a result for general p.[1] The formula agrees with (1.4) and (1.8) for the relevant values of d.

In section 3 we consider $(n-1)$-form gauge theories with additional derivatives in $2n+2$ dimensions when they also define a CFT and obtain C_T in this case. Reflecting the lack of unitarity, $C_T < 0$. We also discuss in section 4 the $(n-1)$-form theory, without additional derivatives, extended to define a CFT away from $d = 2n$ dimensions when gauge invariance is lost.

The energy-momentum tensors in the higher-derivative theories are rather non-trivial. In section 5 we discuss an alternative construction which constructs a spin-two conformal primary by successively subtracting the descendants of lower-dimension primary operators. The final expressions thereby obtained are identical with those derived from curved-space Weyl-invariant actions; the subtractions are related to improvement terms which need to be added to the canonical energy-momentum tensor to obtain a tensor which is traceless as well as symmetric.

2 Higher-derivative scalar theories

The actions for higher-derivative free scalars considered here have the form

$$S_4[\varphi] = -\int {\rm d}^d x\,\frac{1}{2}\,\partial^2\varphi\,\partial^2\varphi\,, \qquad S_6[\varphi] = -\int {\rm d}^d x\,\frac{1}{2}\,\partial^\mu\partial^2\varphi\,\partial_\mu\partial^2\varphi\,. \tag{2.1}$$

Theories starting from such actions were considered in [13, 14]. A symmetric traceless energy-momentum tensor may be obtained by the usual Noether procedure or by extending (2.1) to a general curved space background so as to be invariant under Weyl rescalings of the metric. Assuming diffeomorphism invariance then reducing to flat space ensures that the resulting energy momentum tensor satisfies conformal Ward identities which ensure that it is a conformal primary.

[1] $C_{T,S}$ in (1.3) corresponds to $p = 1$.

For S_4 the extension to a Weyl invariant form on curved space is equivalent to constructing the Paneitz operator [15] (see also [16] and [17] for the $d=4$ version of the Paneitz operator) and for S_6 this involves the generalisation of the $d=6$ Branson operator [18] to general d. These operators provide extensions of $\nabla^2\nabla^2$ and $-\nabla^2\nabla^2\nabla^2$ to conformal differential operators. A convenient form for the Branson operator for general d was constructed in [19][2] by extending S_6 to a Weyl invariant form on an arbitrary curved background. A useful mathematical discussion for arbitrary powers of the Laplacian is contained in [20], such operators fail to exist in particular integer dimensions, for the Paneitz, Branson operators these are $d=2,4$.[3] Varying the metric about flat space gives (an alternative derivation based on a generalised Noether procedure is given in [25])

$$
\begin{aligned}
T^{\mu\nu}_{\varphi,4} = {}& 2\,\partial^\mu\partial^\nu\varphi\,\partial^2\varphi - \frac{1}{2}\,\eta^{\mu\nu}\,\partial^2\varphi\partial^2\varphi - \partial^\mu(\partial^\nu\varphi\,\partial^2\varphi) - \partial^\nu(\partial^\mu\varphi\,\partial^2\varphi) + \eta^{\mu\nu}\,\partial_\rho(\partial^\rho\varphi\,\partial^2\varphi) \\
& + 2\,\mathcal{D}^{\mu\nu\sigma\rho}(\partial_\sigma\varphi\,\partial_\rho\varphi) - \frac{1}{d-1}\,(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2)\Big(\partial^\rho\varphi\,\partial_\rho\varphi - \frac{1}{2}(d-4)\,\partial^2\varphi\,\varphi\Big)\,,
\end{aligned}
\tag{2.2}
$$

for

$$
\begin{aligned}
\mathcal{D}^{\mu\nu\sigma\rho} = {}& \frac{1}{d-2}\big(\eta^{\mu(\sigma}\partial^{\rho)}\partial^\nu + \eta^{\nu(\sigma}\partial^{\rho)}\partial^\mu - \eta^{\mu(\sigma}\eta^{\rho)\nu}\partial^2 - \eta^{\mu\nu}\partial^\sigma\partial^\rho\big) \\
& - \frac{1}{(d-2)(d-1)}\,\big(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2\big)\eta^{\sigma\rho}\,,
\end{aligned}
\tag{2.3}
$$

where $\partial_\mu\mathcal{D}^{\mu\nu\sigma\rho}=0$, $\eta_{\mu\nu}\mathcal{D}^{\mu\nu\sigma\rho} = -\partial^\sigma\partial^\rho$, and

$$
\begin{aligned}
T^{\mu\nu}_{\varphi,6} = {}& \partial^\mu\partial^2\varphi\,\partial^\nu\partial^2\varphi - 2\,\partial^\mu\partial^\nu\varphi\,\partial^2\partial^2\varphi - \frac{1}{2}\,\eta^{\mu\nu}\,\partial^\sigma\partial^2\varphi\,\partial_\sigma\partial^2\varphi \\
& + \partial^\mu(\partial^\nu\varphi\,\partial^2\partial^2\varphi) + \partial^\nu(\partial^\mu\varphi\,\partial^2\partial^2\varphi) - \eta^{\mu\nu}\,\partial_\rho(\partial^\rho\varphi\,\partial^2\partial^2\varphi) \\
& + 8\,\mathcal{D}^{\mu\nu\sigma\rho}(\partial_\sigma\partial_\rho\varphi\,\partial^2\varphi) - \frac{1}{d-1}\,(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2)O \\
& + \lambda\,\mathcal{D}_B^{\mu\nu\sigma\rho}\,(\partial_\sigma\varphi\partial_\rho\varphi)\,, \\
O = {}& \frac{1}{2}(d-6)\,\partial^2(\partial^2\varphi\,\varphi) + (10-d)\,\partial_\rho(\partial^\rho\varphi\,\partial^2\varphi) + \frac{3}{4}(d-2)\,\partial^2\varphi\,\partial^2\varphi\,,
\end{aligned}
\tag{2.4}
$$

where

$$
\mathcal{D}_B^{\mu\nu\sigma\rho} = \mathcal{D}^{\mu\nu\sigma\rho}\,\partial^2 - \frac{1}{d-1}\,\big(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2\big)\partial^\sigma\partial^\rho\,. \tag{2.5}
$$

It is useful to note that

$$
\mathcal{D}_B^{\mu\nu\sigma\rho}(\partial_\sigma v_\rho + \partial_\rho v_\sigma) = 0\,, \quad \mathcal{D}_B^{\mu\nu\sigma\rho}\eta_{\sigma\rho} = 0\,, \quad \partial_\mu\mathcal{D}_B^{\mu\nu\sigma\rho} = 0\,, \quad \eta_{\mu\nu}\mathcal{D}_B^{\mu\nu\sigma\rho} = 0\,. \tag{2.6}
$$

The terms in the expressions for $T^{\mu\nu}_{\varphi,4}$, $T^{\mu\nu}_{\varphi,6}$ involving the second- and higher-order derivative operators $\mathcal{D}^{\mu\nu\sigma\rho}$, $\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2$ and $\mathcal{D}_B^{\mu\nu\sigma\rho}$ arise from explicit curvature-dependent terms in the curved-space action and represent improvement terms to be added to the canonical

[2] See version 3 of [19].

[3] This is then an obstruction to relating Weyl and conformal invariance [21], but it also prevents the existence of a symmetric traceless energy-momentum tensor which is a conformal primary. Related discussions are given in [22] and for $d=2$ in [23, 24].

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energy-momentum tensor. In particular the contribution involving $\mathcal{D}_B^{\mu\nu\sigma\rho}$ comes from the reduction of a term in the curved-space result proportional to $\partial_\mu\varphi\partial_\nu\varphi\, B^{\mu\nu}$, with $B^{\mu\nu}$ the Bach tensor, and this gives

$$\lambda = -\frac{8}{d-4}\,. \tag{2.7}$$

The results for $T_{\varphi,4}^{\mu\nu}$ and $T_{\varphi,6}^{\mu\nu}$ in (2.2) and (2.4) obey the conservation and trace conditions,

$$\partial_\mu T_{\varphi,2p}^{\mu\nu} = (-1)^{p-1}(\partial^2)^p\varphi\,\partial^\nu\varphi\,, \quad \eta_{\mu\nu}T_{\varphi,2p}^{\mu\nu} = (-1)^{p-1}\Delta_{2p}\,(\partial^2)^p\varphi\,\varphi\,, \quad \Delta_{2p} = \frac{1}{2}(d-2p)\,, \tag{2.8}$$

which of course vanish on the relevant equations of motion $(\partial^2)^p\varphi = 0$.

Correlators and operator products in the free field theories are determined just by

$$\begin{aligned}\big\langle\varphi(x)\,\varphi(0)\big\rangle_4 &= \frac{1}{2(d-4)(d-2)S_d}\,\frac{1}{(x^2)^{\frac{1}{2}(d-4)}}\,,\\ \big\langle\varphi(x)\,\varphi(0)\big\rangle_6 &= \frac{1}{8(d-6)(d-4)(d-2)S_d}\,\frac{1}{(x^2)^{\frac{1}{2}(d-6)}}\,.\end{aligned} \tag{2.9}$$

These are respectively singular when $d\to 4,6$ but in (2.2), (2.4) the only terms not involving $\partial\varphi$ have overall factors $d-4$, $d-6$ in each case. From this term, for both $T_{\varphi,4}^{\mu\nu}$, $T_{\varphi,6}^{\mu\nu}$, we may verify for the leading term in the operator product

$$S_d\,T_{\varphi,2p}^{\mu\nu}(x)\,\varphi(0) \sim -\frac{d\,\Delta_{2p}}{d-1}\,\frac{1}{(x^2)^{\frac{1}{2}d}}\bigg(\frac{x^\mu x^\nu}{x^2} - \frac{1}{d}\,\eta^{\mu\nu}\bigg)\varphi(0)\,. \tag{2.10}$$

The coefficient is determined by Ward identities assuming $T_{\varphi,2p}^{\mu\nu}$ is canonically normalised.

In free field theories any local operator formed from φ with derivatives at the same point can be decomposed in terms of conformal primaries and descendants, or derivatives, of conformal primaries of lower dimension. Since $T^{\mu\nu}$ is a conformal primary the result in (1.1) is therefore unchanged for $T^{\sigma\rho}\to T^{\sigma\rho}+\partial_\tau X^{\sigma\rho\tau}$ for any local $X^{\sigma\rho\tau}$ expressible as a conformal primary or descendant. This ensures that, dropping also terms which vanish on the equations of motion,

$$\begin{aligned}&\big\langle T_{\varphi,4}^{\mu\nu}(x)\,T_{\varphi,4}^{\sigma\rho}(0)\big\rangle = 2\,\big\langle T_{\varphi,4}^{\mu\nu}(x)\,\partial^\sigma\partial^\rho\varphi\partial^2\varphi(0)\big\rangle\,,\\ &\big\langle T_{\varphi,6}^{\mu\nu}(x)\,T_{\varphi,6}^{\sigma\rho}(0)\big\rangle = -3\,\big\langle T_{\varphi,6}^{\mu\nu}(x)\,\partial^\sigma\partial^\rho\varphi\partial^2\partial^2\varphi(0)\big\rangle = 3\,\big\langle T_{\varphi,6}^{\mu\nu}(x)\,\partial^\sigma\partial^2\varphi\partial^\rho\partial^2\varphi(0)\big\rangle\,.\end{aligned} \tag{2.11}$$

We get, with λ as in (2.7),

$$C_{T,\varphi,4} = -\frac{2d(d+4)}{(d-2)(d-1)}\,, \qquad C_{T,\varphi,6} = \frac{3d(d+4)(d+6)}{(d-4)(d-2)(d-1)}\,. \tag{2.12}$$

There is an obvious generalisation of (2.1) to actions with more derivatives, S_{2p} formed from $(\partial^2)^r\varphi$ or $\partial_\mu(\partial^2)^r\varphi$ for $p=2r$ or $p=2r+1$. On the basis of the results in (2.12) we may guess

$$C_{T,\varphi,2p} = C_{T,S}\,\frac{p\,(\frac{1}{2}d+2)_{p-1}}{(-\frac{1}{2}d+1)_{p-1}}\,, \qquad p=1,2,\ldots\,, \tag{2.13}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol and $C_{T,S}$ the free scalar result given in (1.3). The result (2.13) agrees with (1.4), (1.8) for the particular cases of d.

3 Higher-order $(n-1)$-form gauge theories

In $d=2n$ dimensions free conformal theories can be formed from $(n-1)$-form gauge fields $A_{\mu_1\dots\mu_{n-1}}$ with n-form field strengths $F_{\mu_1\dots\mu_n}=n\,\partial_{[\mu_1}A_{\mu_2\dots\mu_n]}$ with the gauge invariant action $S_{n,0}[A]=-\frac{1}{2\,n!}\int \mathrm{d}^{2n+2}x\; F^{\mu_1\dots\mu_n}F_{\mu_1\dots\mu_n}$, generalising the conformal invariant Maxwell theory in four dimensions. Here we consider the corresponding theory with two additional derivatives given by the action in $d=2n+2$ dimensions

$$\begin{aligned} S_{n,2}[A] = &-\frac{1}{2n!}\int \mathrm{d}^{2n+2}x\;\partial^\lambda F^{\mu_1\dots\mu_n}\partial_\lambda F_{\mu_1\dots\mu_n} \\ = &-\frac{1}{2(n-1)!}\int \mathrm{d}^{2n+2}x\;\partial_\lambda F^{\lambda\mu_1\dots\mu_{n-1}}\partial^\rho F_{\rho\mu_1\dots\mu_{n-1}}\,. \end{aligned} \tag{3.1}$$

This may be extended to a general curved background metric $\gamma_{\mu\nu}$ so as to be invariant under Weyl rescalings in the form

$$\begin{aligned} S_{n,2}[A] = -\frac{1}{2\,n!}\int \mathrm{d}^{2n+2}x\sqrt{-\gamma}\,\Big(&\nabla^\lambda F^{\mu_1\dots\mu_n}\nabla_\lambda F_{\mu_1\dots\mu_n} \\ &+\big(2n\,P_{\lambda\rho}+(n+2)\,\gamma_{\lambda\rho}\,\hat R\big)F^{\lambda\mu_1\dots\mu_{n-1}}F^\rho{}_{\mu_1\dots\mu_{n-1}} \\ &+c_W\,n(n-1)\,W_{\mu\nu\lambda\rho}\,F^{\mu\nu\mu_1\dots\mu_{n-2}}F^{\lambda\rho}{}_{\mu_1\dots\mu_{n-2}}\Big)\,, \end{aligned} \tag{3.2}$$

where $P_{\lambda\rho}=\frac{1}{d-2}(R_{\lambda\rho}-\gamma_{\lambda\rho}\,\hat R)$ is the Schouten tensor, $\hat R=\frac{1}{2(d-1)}R$ a rescaled scalar curvature and $W_{\mu\nu\lambda\rho}$ the Weyl tensor. The Weyl tensor term is invariant under Weyl rescaling by itself and so, for $n>1$, has an arbitrary coefficient. The expression for the action may be written in various forms with the aid of the identity, for d arbitrary,

$$\begin{aligned} n\,\nabla_\lambda\nabla_\rho\big(&F^{\lambda\mu_1\dots\mu_{n-1}}F^\rho{}_{\mu_1\dots\mu_{n-1}}\big)-\nabla^2\big(F^{\mu_1\dots\mu_n}F_{\mu_1\dots\mu_n}\big) \\ &=-\nabla^\lambda F^{\mu_1\dots\mu_n}\nabla_\lambda F_{\mu_1\dots\mu_n}+n\,\nabla_\lambda F^{\lambda\mu_1\dots\mu_{n-1}}\nabla^\rho F_{\rho\mu_1\dots\mu_{n-1}} \\ &\quad -n\,\big((d-2n)\,P_{\lambda\rho}+\gamma_{\lambda\rho}\,\hat R\big)F^{\lambda\mu_1\dots\mu_{n-1}}F^\rho{}_{\mu_1\dots\mu_{n-1}} \\ &\quad +\frac{1}{2}n(n-1)\,W_{\mu\nu\lambda\rho}\,F^{\mu\nu\mu_1\dots\mu_{n-2}}F^{\lambda\rho}{}_{\mu_1\dots\mu_{n-2}}\,, \end{aligned} \tag{3.3}$$

depending on the Bianchi identity for F.[4]

Varying the metric in (3.3) determines the corresponding flat space energy momentum tensor for $(n-1)$-form gauge fields involving two derivatives

$$\begin{aligned} n!\,T^{\mu\nu}_{n,2} = n\,\partial^\lambda F^{\mu\mu_1\dots\mu_{n-1}}\,\partial_\lambda F^\nu{}_{\mu_1\dots\mu_{n-1}}+\partial^\mu F^{\mu_1\dots\mu_n}\,\partial^\nu F_{\mu_1\dots\mu_n} \\ -\frac{1}{2}\,\eta^{\mu\nu}\,\partial^\lambda F^{\mu_1\dots\mu_n}\partial_\lambda F_{\mu_1\dots\mu_n}+2n\,\partial_\lambda\big(F^{\lambda\mu_1\dots\mu_{n-1}}\overset{\leftrightarrow}{\partial}{}^{(\mu}F^{\nu)}{}_{\mu_1\dots\mu_{n-1}}\big) \end{aligned}$$

[4] In d-dimensions there is a conformal scalar

$$\begin{aligned} (4n+2-d)\big((d-2n)\,\nabla^\lambda F^{\mu_1\dots\mu_n}\nabla_\lambda F_{\mu_1\dots\mu_n}-n\,\nabla_\lambda F^{\lambda\mu_1\dots\mu_{n-1}}\nabla^\rho F_{\rho\mu_1\dots\mu_{n-1}}\big) \\ -(n+1)(d-2n)\,(\nabla^2-2n\,\hat R)\,\big(F^{\mu_1\dots\mu_n}F_{\mu_1\dots\mu_n}\big)\,, \end{aligned} \tag{3.4}$$

which generalises an expression obtained by Parker and Rosenberg [26]. For $d=4n+2$ this is just the conformal Laplacian acting on F^2.

$$
\begin{aligned}
&- \frac{1}{2} n\, \partial^2 \big(F^{\mu\mu_1\ldots\mu_{n-1}} F^\nu{}_{\mu_1\ldots\mu_{n-1}} \big) + n\, \mathcal{D}^{\mu\nu\sigma\rho} \big(F_\sigma{}^{\mu_1\ldots\mu_{n-1}} F_{\rho\mu_1\ldots\mu_{n-1}} \big) \\
&- \frac{n+2}{2(2n+1)} \left(\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial^2 \right) \big(F^{\mu_1\ldots\mu_n} F_{\mu_1\ldots\mu_n} \big) \\
&+ c_W\, n(n-1)\, \mathcal{E}_W{}^{\mu\sigma\nu\rho,\epsilon\eta\kappa\lambda}\, \partial_\sigma \partial_\rho \big(F_{\epsilon\eta}{}^{\mu_1\ldots\mu_{n-2}} F_{\kappa\lambda\mu_1\ldots\mu_{n-2}} \big) \,, \qquad (3.5)
\end{aligned}
$$

where $\overleftrightarrow{\partial} = \frac{1}{2}(\partial - \overleftarrow{\partial})$. In the last line $\mathcal{E}_W$ is the projector for traceless tensors satisfying the symmetries of the Weyl tensor and has the properties $\mathcal{E}_W{}^{\mu\sigma\nu\rho,\epsilon\eta\kappa\lambda} = \mathcal{E}_W{}^{[\mu\sigma][\nu\rho],[\epsilon\eta][\kappa\lambda]} = \mathcal{E}_W{}^{\epsilon\eta\kappa\lambda,\mu\sigma\nu\rho}$, $\eta_{\mu\nu}\, \mathcal{E}_W{}^{\mu\sigma\nu\rho,\epsilon\eta\kappa\lambda} = \mathcal{E}_W{}^{\mu[\sigma\nu\rho],\epsilon\eta\kappa\lambda} = 0$. The energy-momentum tensor in (3.5) satisfies, using the Bianchi identity,

$$
n!\, \partial_\mu T^{\mu\nu}_{n,2} = -n\, \partial^2 \partial_\mu F^{\mu\mu_1\ldots\mu_{n-1}}\, F^\nu{}_{\mu_1\ldots\mu_{n-1}} \,, \qquad n!\, \eta_{\mu\nu} T^{\mu\nu}_{n,2} = 0 \,, \qquad (3.6)
$$

and so $T^{\mu\nu}_{n,2}$ is conserved subject to the equation of motion $\partial^2 \partial_\mu F^{\mu\mu_1\ldots\mu_{n-1}} = 0$.

In a Feynman type gauge the action (3.1) reduces to

$$
S_{n,2}[A] = -\frac{1}{2(n-1)!} \int \mathrm{d}^{2n+2}x\; \partial^2 A^{\mu_1\ldots\mu_{n-1}}\, \partial^2 A_{\mu_1\ldots\mu_{n-1}} \,, \qquad (3.7)
$$

so that in this gauge

$$
\big\langle A_{\mu_1\ldots\mu_{n-1}}(x)\, A_{\nu_1\ldots\nu_{n-1}}(0) \big\rangle = \frac{(n-2)!}{8n\, S_{2n+2}} \frac{1}{(x^2)^{n-1}}\, \mathcal{E}^{(n-1)}{}_{\mu_1\ldots\mu_{n-1},\,\nu_1\ldots\nu_{n-1}} \,, \qquad (3.8)
$$

where

$$
\mathcal{E}^{(n)}{}_{\mu_1\ldots\mu_n,}{}^{\nu_1\ldots\nu_n} = \delta_{[\mu_1}{}^{\nu_1} \ldots \delta_{\mu_n]}{}^{\nu_n} \,, \qquad (3.9)
$$

is the projector on to rank n antisymmetric tensors. Then, with F defined in (3.1), the two-point function for F, which is gauge independent, is given by

$$
\big\langle F_{\mu_1\ldots\mu_n}(x)\, F_{\nu_1\ldots\nu_n}(0) \big\rangle = \frac{n!}{4\, S_{2n+2}} \frac{1}{(x^2)^n}\, \mathcal{E}^{(n)}{}_{\mu_1\ldots\mu_n,}{}^{\lambda_1\ldots\lambda_n} I_{\lambda_1\nu_1}(x) \ldots I_{\lambda_n\nu_n}(x) \,, \qquad (3.10)
$$

with $I_{\lambda\nu}(x)$ determined by (1.2). The result (3.10) has the expected form for $F_{\mu_1\ldots\mu_n}$ a conformal primary of dimension n.

For this theory the two-point function of the energy-momentum tensor is determined from

$$
\big\langle T^{\mu\nu}_{n,2}(x)\, T^{\sigma\rho}_{n,2}(0) \big\rangle = \big\langle T^{\mu\nu}_{n,2}(x)\, X^{\sigma\rho}_n(0) \big\rangle \,, \qquad (3.11)
$$

for, discarding total derivatives and terms which vanish on the equations of motion,

$$
n!\, X^{\sigma\rho}_n = n\, \partial^\lambda F^{\sigma\mu_1\ldots\mu_{n-1}}\, \partial_\lambda F^\rho{}_{\mu_1\ldots\mu_{n-1}} + \partial^\sigma F^{\mu_1\ldots\mu_n}\, \partial^\rho F_{\mu_1\ldots\mu_n} \,. \qquad (3.12)
$$

The c_W contribution in (3.5) can also be dropped since this term is a conformal primary descendant of a conformal primary and does not contribute to (3.11). Using the two-point function (3.10) the combinatorics for arbitrary n can be handled with the identities

$$
\begin{aligned}
&\mathcal{E}^{(n)}{}_{\mu_1\ldots\mu_n,\,\sigma_1\ldots\sigma_p\lambda_1\ldots\lambda_{n-p}}\; \mathcal{E}^{(n)\mu_1\ldots\mu_n,}{}_{\rho_1\ldots\rho_p}{}^{\lambda_1\ldots\lambda_{n-p}} = \mathcal{E}^{(n)}{}_{\sigma_1\ldots\sigma_p\lambda_1\ldots\lambda_{n-p},\,\rho_1\ldots\rho_p}{}^{\lambda_1\ldots\lambda_{n-p}} \\
&\quad = A^{(n)}_p\, \mathcal{E}^{(p)}{}_{\sigma_1\ldots\sigma_p,\,\rho_1\ldots\rho_p} \,, \quad p = 0,\ldots,n \,, \qquad A^{(n)}_p = \frac{p!}{n!} \frac{\Gamma(d-p+1)}{\Gamma(d-n+1)} \,, \qquad (3.13)
\end{aligned}
$$

and, if $n \geq 1$,

$$\mathcal{E}^{(n)}{}_{\mu\,\mu_1\ldots\mu_{n-1},\,\sigma_1\ldots\sigma_p\lambda_1\ldots\lambda_{n-p}}\ \mathcal{E}^{(n)}{}_{\nu}{}^{\mu_1\ldots\mu_{n-1},}{}_{\rho_1\ldots\rho_p}{}^{\lambda_1\ldots\lambda_{n-p}}$$
$$= B_p^{(n)}\,\delta_\mu{}^\lambda\,\delta_\nu{}^{\lambda'}\,\mathcal{E}^{(p)}{}_{\sigma_1\ldots\sigma_p,\,\lambda\lambda_1\ldots\lambda_{p-1}}\ \mathcal{E}^{(p)}{}_{\rho_1\ldots\rho_p,\,\lambda'}{}^{\lambda_1\ldots\lambda_{p-1}} + C_p^{(n)}\,\eta_{\mu\nu}\,\mathcal{E}^{(p)}{}_{\sigma_1\ldots\sigma_p,\,\rho_1\ldots\rho_p}\,, \quad (3.14)$$

where

$$B_p^{(n)} = \frac{p\,p!}{n\,n!}\,\frac{\Gamma(d-p)}{\Gamma(d-n)}\,, \qquad C_p^{(n)} = \frac{(n-p)\,p!}{n\,n!}\,\frac{\Gamma(d-p)}{\Gamma(d-n+1)}\,, \qquad p = 0,\ldots,n\,. \quad (3.15)$$

Consistency requires $B_p^{(n)} + d\,C_p^{(n)} = A_p^{(n)}$, $B_{p-1}^{(n)} = B_p^{(n)}\,B_{p-1}^{(p)}$, as well as $C_{p-1}^{(n)} = B_p^{(n)}\,C_{p-1}^{(p)} + B_p^{(n)}\,A_{p-1}^{(p)}$. Evaluating (3.11) then gives

$$C_{T,n,2}^{\text{gauge}} = -\frac{2n(n+1)(n+3)\,(2n)!}{(n+2)\,n!^2}\,. \quad (3.16)$$

Note that $C_{T,1,2}^{\text{gauge}} = C_{T,\varphi,4}$ for $d = 4$.

Large N methods, similar to those for the $\mathrm{O}(N)$ and Gross-Neveu models, have been extended to an Abelian gauge theory coupled to N fermions [27]. For $d = 4 + 2p$, $p = 0, 1, \ldots$, this becomes equivalent to a renormalisable theory with N fermions and a higher-derivative gauge theory with a Lagrangian $-\frac{1}{4}\,F^{\mu\nu}(-\partial^2)^p F_{\mu\nu}$. Using the large N results for C_T and subtracting the free fermion contribution [27], in the notation used above, predicted that for the free gauge theory $C_{T,2,2p}^{\text{gauge}} = (-1)^p\,2(p+2)(2p+4)!/((p+1)!(p+3)!)$. For $p = 0$ this is the standard result and the case $p = 1$ agrees with (3.16) when $n = 2$.

4 $(n-1)$-form theories away from integer dimensions

The usual gauge invariant action for $(n-1)$-form gauge fields is only conformally invariant in $d = 2n$ dimensions although it may be extended, as in the previous section, in higher even dimensions with additional derivatives. However abandoning gauge invariance the action may be extended to be conformal for general d. The corresponding Weyl-invariant action on a curved-space background was obtained by Erdmenger [28], following from the construction of a conformal second-order differential operator on k-forms obtained by Branson [18], and the corresponding flat-space action for vector fields or one-forms was given in [29].

The curved-space action obtained in [28] may be expressed, with a similar notation to (3.2), as

$$S_{n,0}[A] = -\frac{1}{2\,n!}\int \mathrm{d}^d x\sqrt{-\gamma}\,\Big(F^{\mu_1\ldots\mu_n}F_{\mu_1\ldots\mu_n} + \alpha\,\nabla_\lambda A^{\lambda\mu_1\ldots\mu_{n-2}}\,\nabla_\rho A^\rho{}_{\mu_1\ldots\mu_{n-2}}$$
$$+\frac{1}{2}n(d-2n)\big(\gamma_{\lambda\rho}\,\hat{R} - 2(n-1)\,P_{\lambda\rho}\big)A^{\lambda\mu_1\ldots\mu_{n-2}}A^\rho{}_{\mu_1\ldots\mu_{n-2}}\Big)\,, \quad (4.1)$$
$$\alpha = n(n-1)\,\frac{d-2n}{d-2n+4}\,.$$

On flat space this is tantamount to a particular choice of a covariant gauge fixing term [29]. The flat space energy-momentum tensor is then

$$
\begin{aligned}
n!\, T_{n,0}^{\mu\nu} = {} & n\, F^{\mu\mu_1\dots\mu_{n-1}} F^{\nu}{}_{\mu_1\dots\mu_{n-1}} - \frac{1}{2}\,\eta^{\mu\nu}\, F^{\mu_1\dots\mu_n} F_{\mu_1\dots\mu_n} \\
& + (n-2)\alpha\, \partial_\lambda A^{\lambda\mu\mu_1\dots\mu_{n-3}}\, \partial_\rho A^{\rho\nu}{}_{\mu_1\dots\mu_{n-3}} - 2\alpha\, A^{(\mu|\mu_1\dots\mu_{n-2}}\, \partial^{\nu)}\partial_\rho A^{\rho}{}_{\mu_1\dots\mu_{n-2}} \\
& - \frac{1}{2}\alpha\, \eta^{\mu\nu}\, \partial_\lambda A^{\lambda\mu_1\dots\mu_{n-2}}\, \partial_\rho A^{\rho}{}_{\mu_1\dots\mu_{n-2}} \\
& - 2(n-2)\alpha\, \partial_\lambda \big(A^{\lambda(\mu|\mu_1\dots\mu_{n-3}}\, \partial_\rho A^{\rho|\nu)}{}_{\mu_1\dots\mu_{n-3}} \big) \qquad (4.2) \\
& + \alpha\, \eta^{\mu\nu}\, \partial_\lambda \big(A^{\lambda\mu_1\dots\mu_{n-2}}\, \partial_\rho A^{\rho}{}_{\mu_1\dots\mu_{n-2}} \big) \\
& + \frac{1}{2} n(n-1)(d-2n)\, \mathcal{D}^{\mu\nu\sigma\rho} \big(A_\sigma{}^{\mu_1\dots\mu_{n-2}} A_{\rho\mu_1\dots\mu_{n-2}} \big) \\
& - \frac{n(d-2n)}{4(d-1)}\, (\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2) \big(A^{\mu_1\dots\mu_{n-1}} A_{\mu_1\dots\mu_{n-1}} \big)\,. \qquad (4.3)
\end{aligned}
$$

This satisfies

$$
\begin{aligned}
n!\, \partial_\mu T_{n,0}^{\mu\nu} = {} & \big(n\, \partial_\mu F^{\mu\mu_1\dots\mu_{n-1}} + \alpha\, \partial^{\mu_1}\partial_\rho A^{\rho\mu_2\dots\mu_{n-1}} \big) F^{\nu}{}_{\mu_1\dots\mu_{n-1}} \qquad (4.4) \\
& - \alpha\, \partial^2 \partial_\rho A^{\rho\mu_1\dots\mu_{n-2}} A^{\nu}{}_{\mu_1\dots\mu_{n-2}}\,, \\
n!\, \eta_{\mu\nu} T_{n,0}^{\mu\nu} = {} & -\frac{1}{4}(d-2n)\, \big(n\, \partial_\mu F^{\mu\mu_1\dots\mu_{n-1}} + \alpha\, \partial^{\mu_1}\partial_\rho A^{\rho\mu_2\dots\mu_{n-1}} \big) A_{\mu_1\dots\mu_{n-1}}\,, \qquad (4.5)
\end{aligned}
$$

and so the energy-momentum tensor is conserved and traceless on the equations of motion. Of course for $d = 2n$, $T_{n,0}^{\mu\nu}$ reduces to the usual gauge invariant form.

The two-point function for the $(n-1)$-form field determined by the action (4.1) on flat space was calculated in [28] by inverting the Fourier transform of the kinetic differential operator and then returning to x-space giving

$$
\begin{aligned}
& \big\langle A_{\mu_1\dots\mu_{n-1}}(x)\, A_{\nu_1\dots\nu_{n-1}}(0) \big\rangle \\
& \qquad = \frac{(n-1)!}{(d-2n)\, S_d}\, \frac{1}{(x^2)^{\frac{1}{2}(d-2)}}\, \mathcal{E}^{(n-1)}{}_{\mu_1\dots\mu_{n-1},}{}^{\lambda_1\dots\lambda_{n-1}} I_{\lambda_1\nu_1}(x) \dots I_{\lambda_n\nu_{n-1}}(x)\,, \qquad (4.6)
\end{aligned}
$$

which has the form required by conformal invariance for $A_{\mu_1\dots\mu_{n-1}}$ a conformal primary. From (4.6)

$$
\begin{aligned}
\big\langle F_{\mu_1\dots\mu_n}(x)\, A_{\nu_1\dots\nu_{n-1}}(0) \big\rangle = {} & \frac{n!}{(d-2n)\, S_d}\, \partial_\lambda \bigg(\frac{1}{(x^2)^{\frac{1}{2}(d-2)}}\, \delta_\kappa{}^\eta - 2(n-1)\, \frac{1}{(x^2)^{\frac{1}{2}d}}\, x_\kappa x^\eta \bigg) \\
& \times \mathcal{E}^{(n)}{}_{\mu_1\dots\mu_{n-1},}{}^{\lambda\kappa\lambda_1\dots\lambda_{n-2}}\, \mathcal{E}^{(n-1)}{}_{\eta\lambda_1\dots\lambda_{n-2},\nu_1\dots\nu_{n-1}} \\
= {} & -\frac{n!}{S_d}\, \frac{1}{(x^2)^{\frac{1}{2}d}}\, x^\lambda\, \mathcal{E}^{(n)}{}_{\mu_1\dots\mu_n,\lambda\nu_1\dots\nu_{n-1}}\,, \qquad (4.7)
\end{aligned}
$$

and

$$
\begin{aligned}
\big\langle F_{\mu_1\dots\mu_n}(x)\, F_{\nu_1\dots\nu_n}(0) \big\rangle = {} & \frac{n\, n!}{S_d}\, \partial_\rho \bigg(\frac{1}{(x^2)^{\frac{1}{2}d}}\, x^\lambda \bigg) \mathcal{E}^{(n)}{}_{\mu_1\dots\mu_{n-1},\lambda\lambda_1\dots\lambda_{n-1}}\, \mathcal{E}^{(n)\rho\lambda_1\dots\lambda_{n-1},}{}_{\nu_1\dots\nu_n} \\
= {} & \frac{n\, n!}{S_d}\, \frac{1}{(x^2)^{\frac{1}{2}d}} \bigg(\delta_\lambda{}^\rho - d\, \frac{1}{x^2}\, x_\lambda x^\rho \bigg)\, \mathcal{E}^{(n)}{}_{\mu_1\dots\mu_n,}{}^{\lambda\lambda_1\dots\lambda_{n-1}}\, \mathcal{E}^{(n)}{}_{\rho\lambda_1\dots\lambda_{n-1},\nu_1\dots\nu_n}\,. \qquad (4.8)
\end{aligned}
$$

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This has the conformally invariant form in terms of the inversion tensor only when $d = 2n$ and is identical in this case to the expression obtained for gauge choices for $A_{\mu_1 \dots \mu_{n-1}}$ other than that implicit in (4.6) [25].

As in other cases, evaluating the energy-momentum tensor two-point function can be simplified to

$$
\begin{aligned}
\big\langle T^{\mu\nu}_{n,0}(x)\, T^{\sigma\rho}_{n,0}(0)\big\rangle &= \big\langle T^{\mu\nu}_{n,0}(x)\, Y^{\sigma\rho}_{n}(0)\big\rangle\,, \\
n!\, Y_n^{\sigma\rho} &= n\, F^{\sigma\mu_1\dots\mu_{n-1}} F^{\rho}{}_{\mu_1\dots\mu_{n-1}} + (n-2)\alpha\, \partial_\lambda A^{\lambda\sigma\mu_1\dots\mu_{n-3}}\, \partial_\kappa A^{\kappa\rho}{}_{\mu_1\dots\mu_{n-3}} \\
&\quad - 2\alpha\, A^{(\sigma|\mu_1\dots\mu_{n-2}}\, \partial^{\rho)} \partial_\kappa A^{\kappa}{}_{\mu_1\dots\mu_{n-2}}\,.
\end{aligned}
\tag{4.9}
$$

This then determines

$$
C_{T,n} = \frac{d}{d-1}\, \frac{(d-n+2)_{n-1}}{(n-1)!}\,, \qquad n = 1, 2, \dots\,. \tag{4.10}
$$

As expected $C_{T,1} = C_{T,S}$. The corresponding result for $(n-1)$-form gauge fields in $d = 2n$ dimensions, whose energy-momentum tensor is obtained just from the FF terms in (4.5), is $C^{\rm gauge}_{T,n,0} = 2\, n^2 (2n-2)!/(n-1)!^2$, [9, 25]. This is not equal to $C_{T,n}$ in (4.10) when $d = 2n$, although (4.3) apparently reduces to the required form for this d.[5] The difference arises since the $\langle AA\rangle$ two-point function in (4.6) is also singular when $d = 2n$. The representation of the conformal group generated from a conformal primary $A_{\mu_1\dots\mu_{n-1}}$ is reducible when $d = 2n$ and an irreducible representation for the associated gauge theory is obtained by quotienting by the invariant subspace corresponding to gauge transformations. Since C_T is related to the number of degrees of freedom it is expected to differ between the gauge theory and that corresponding to the $(n-1)$-form $A_{\mu_1\dots\mu_{n-1}}$.

To demonstrate this further we may consider the Fourier transform of the two-point function in (4.6) letting $\frac{1}{2}(d-2) \to \Delta$, $k = n-1$ and also setting the overall coefficient to 1,

$$
\pi^{\frac{1}{2}d}\, \frac{\Gamma(\Delta - \frac{1}{2}d + 1)}{\Gamma(\Delta+1)}\, F(p^2)\, A_{\mu_1\dots\mu_k,\,\nu_1\dots\nu_k}(p)\,, \qquad F(p^2) = -\frac{\pi}{\sin \pi(\Delta - \frac{1}{2}d)} \left(\frac{1}{4}\, p^2\right)^{\Delta - \frac{1}{2}d}\,,
$$

$$
\begin{aligned}
&A_{\mu_1\dots\mu_k,\,\nu_1\dots\nu_k}(p) = \\
&\quad \bigg((\Delta - k)\, \delta_\lambda{}^\rho - 2k\Big(\Delta - \frac{1}{2}d\Big)\, \frac{1}{p^2}\, p_\lambda\, p^\rho \bigg)\, \mathcal{E}^{(k)}{}_{\mu_1\dots\mu_k,}{}^{\lambda\lambda_1\dots\lambda_{k-1}}\, \mathcal{E}^{(k)}{}_{\rho\lambda_1\dots\lambda_{k-1},\,\nu_1\dots\nu_k}\,.
\end{aligned}
\tag{4.11}
$$

Noting that $\big(F(p^2 - i\epsilon) - F(p^2 + i\epsilon)\big)/2\pi i = \theta(-p^2)\, \big(-\frac{1}{4}\, p^2\big)^{\Delta - \frac{1}{2}d}$ unitarity requires that the matrix $A_{\mu_1\dots\mu_k,\,\nu_1\dots\nu_k}(p)$ should be positive definite for $p^2 < 0$. The eigenvalues are $\Delta - k$, $d - \Delta - k$, but for the the second case the eigenvectors $p_{[\mu_1}\, \epsilon_{\mu_2\dots\mu_k]}$ have negative norm for $p^2 < 0$ so that we must have, for a unitary CFT of k-forms,

$$
\Delta > k\,, \qquad \Delta > d - k\,. \tag{4.12}
$$

[5] If the energy-momentum tensor (4.3) is restricted to the gauge-invariant FF terms and we use (4.8) then the resulting two-point function is not of the required conformal form (1.1). If C_T is identified through the coefficient of the $xxxx$ terms for $n = 2$ then $C_{T,2} = \frac{1}{2} d^2 (d-2)$ as obtained in [27]. This prescription in general gives $C_{T,n} = \frac{1}{2}\, d^2 (d-2) \cdots (d-n)/(n-1)!$.

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When $\Delta = k$ or $\Delta = d - k$ there are zero modes related to the reducibility of the representation. For the case of interest above $\Delta = \frac{1}{2}(d-2)$ and $\Delta = k$ corresponds to $d = 2n$. For d an integer it should be noted that the conditions (4.12) are invariant under duality $A_{\mu_1 \dots \mu_k} \to ({}^*A)_{\mu_1 \dots \mu_{d-k}}$.

5 Conformal primary operators

The energy-momentum tensor is a conformal primary operator. The detailed expressions in (2.2) and (2.4) are necessary to ensure this and we show here how they can be recovered by requiring $T_{\varphi,n,\mu\nu}$ to be a conformal primary, and that this determines the parameter λ in accord with (2.7), although this term is conserved and traceless by itself.

For a local tensor operator $X_{\alpha_1 \dots \alpha_n}$ formed from multinomials in φ and derivatives at $x = 0$ we define

$$\big[K_b, \partial_\mu\big] X_{\alpha_1 \dots \alpha_n} = b_\mu \, D \, X_{\alpha_1 \dots \alpha_n} + \textstyle\sum_i \big(b_{\alpha_i} \, X_{\alpha_1 \dots \mu \dots \alpha_n} - \eta_{\mu \alpha_i} \, b^\lambda X_{\alpha_1 \dots \lambda \dots \alpha_n}\big) \, , \tag{5.1}$$

with

$$D \, \partial_\mu = \partial_\mu (D+1) \, , \qquad D \, X_{\alpha_1 \dots \alpha_n} = \Delta_X \, X_{\alpha_1 \dots \alpha_n} \, , \tag{5.2}$$

where Δ_X is determined just by counting the number of derivatives and fields φ in $X_{\alpha_1 \dots \alpha_n}$. For any conformal primary X_A, $A = \{\alpha_1 \dots \alpha_n\}$, then $K_b \, X_A = 0$ (this is of course the usual condition $K_\mu X_A(0) = 0$). Otherwise X_A is not a conformal primary and generates a reducible representation of the conformal group. Acting with K_b removes derivatives so that for some finite N, ${K_b}^{N+1} X_A = 0$ and hence we can write ${K_b}^N X_A = \sum_{r,I} f_{A,rI}(b) \, O_I$, where $\{O_I\}$ is a basis of conformal primaries with $\Delta_{O_I} = \Delta_X - N$ and $f_{A,rI}(b) = \mathrm{O}(b^N)$. For $Y_A = \sum_{r,I} \mathcal{D}_{A,rI}(\partial) \, O_I$ then ${K_b}^N (X_A - Y_A) = 0$ gives $\sum_{s,J} M_{rI,sJ} \, \mathcal{D}_{A,sJ}(b) = f_{A,rI}(b)$. Extending $\{f_{A,rI}(b)\}$ to include all possible rotationally covariant forms, with $f_{A,rI} = 0$ for some r if necessary, M is a square matrix and we may solve for $\mathcal{D}_{A,rI}(b)$ unless $\{O_I\}$ have conformal primary descendants with N derivatives so that $\det M = 0$. For generic Δ_X this does not arise. Iterating this construction then gives the conformal primary $X_A - \sum_Y Y_A$ which is the lowest weight state for an irreducible representation. If $\det M = 0$, for particular Δ_X, the representation space obtained from X_A is reducible but not decomposable.

The result (5.1) can be extended successively to multiple derivatives. For a scalar conformal primary φ with scale dimension δ, so that $D \, \varphi = \delta \, \varphi$, and an arbitrary vector a, we have

$$K_b \, (a \cdot \partial)^n \varphi = n(\delta + n - 1) \, a \cdot b \, (a \cdot \partial)^{n-1} \varphi - \frac{1}{2} n(n-1) \, a^2 \, b \cdot \partial \, (a \cdot \partial)^{n-2} \varphi \, , \tag{5.3}$$

from which, by acting with $\partial_a \cdot \partial_a$,

$$\begin{aligned} K_b \, (a \cdot \partial)^n \partial^2 \varphi = {} & n(\delta + n + 1) \, a \cdot b \, (a \cdot \partial)^{n-1} \partial^2 \varphi + (2\delta + 2 - d) \, b \cdot \partial \, (a \cdot \partial)^n \varphi \\ & - \frac{1}{2} n(n-1) \, a^2 \, b \cdot \partial \, (a \cdot \partial)^{n-2} \partial^2 \varphi \, . \end{aligned} \tag{5.4}$$

From (5.3) then

$$\Phi_n(a) = a^{\mu_1} \dots a^{\mu_n} \, \Phi_{\mu_1 \dots \mu_n} = \sum_{r=0}^{n} \binom{n}{r} \frac{(-1)^r}{(\delta)_r \, (\delta)_{n-r}} \, (a \cdot \partial)^r \varphi \, (a \cdot \partial)^{n-r} \varphi \tag{5.5}$$

satisfies $K_b\,\Phi_n(a) = \mathrm{O}(a^2)$ and so taking $a^2 = 0$, which projects out the traces, this demonstrates that $\Phi_n(a)$ defines a symmetric traceless conformal primary with $\Delta_{\Phi_n} = 2\delta+n$ and twist 2δ [30, 31]. For higher twist results are more complicated. In the following we will work out $\mathrm{O}(a^2)$ terms in a few examples and obtain some results for higher twist. Note that (5.5) of course gives $\Phi_n(a) = 0$ for n odd.

In the remainder of this section we apply the procedure outlined above for constructing conformal primaries. Initially we construct a conformal primary starting from $\partial_\mu\varphi\,\partial_\nu\varphi$. Using (5.1) we get

$$K_b^2(\partial_\mu\varphi\,\partial_\nu\varphi) = 2\delta^2\, b_\mu b_\nu\,\varphi^2\,, \tag{5.6}$$

and also

$$K_b^2(\partial_\mu\partial_\nu\varphi^2) = 2\delta\big(2(2\delta+1)\,b_\mu b_\nu - \eta_{\mu\nu}\,b^2\big)\varphi^2\,. \tag{5.7}$$

Hence, a symmetric tensor conformal primary with dimension $\Delta_{O_2} = 2\delta+2$ and twist 2δ is given by

$$\begin{aligned} O_{2,\mu\nu} &= \partial_\mu\varphi\,\partial_\nu\varphi - \frac{\delta}{2(2\delta+1)}\bigg(\partial_\mu\partial_\nu + \frac{1}{4\delta+2-d}\,\eta_{\mu\nu}\,\partial^2\bigg)\varphi^2 \\ &= -\,\partial_\mu\partial_\nu\varphi\,\varphi + \frac{1}{2(2\delta+1)}\bigg((\delta+1)\,\partial_\mu\partial_\nu - \frac{\delta}{4\delta+2-d}\,\eta_{\mu\nu}\,\partial^2\bigg)\varphi^2\,, \end{aligned} \tag{5.8}$$

so that $K_b\,O_{2,\mu\nu} = 0$. A scalar conformal primary is then

$$\eta^{\mu\nu}O_{2,\mu\nu} = -\partial^2\varphi\,\varphi + \frac{2\delta+2-d}{2(4\delta+2-d)}\,\partial^2\varphi^2\,. \tag{5.9}$$

For $\delta = \frac{1}{2}(d-2)$ we have

$$T_{\varphi,2,\mu\nu} = O_{2,\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,O_{2,\sigma\sigma}\,. \tag{5.10}$$

With more derivatives the construction becomes more lengthy as the descendants of more conformal primaries have to be subtracted. To construct conformal primary symmetric tensors with four derivatives we start from

$$K_b^4\big((a\cdot\partial)^4\varphi\,\varphi\big) = 6\delta(\delta+1)\big(4(\delta+2)(\delta+3)\,(a\cdot b)^4 - 12(\delta+2)\,(a\cdot b)^2a^2b^2 + 3(a^2b^2)^2\big)\varphi^2\,. \tag{5.11}$$

This can be cancelled by terms involving four derivatives acting on φ^2 so that

$$\begin{aligned} K_b^2\big((a\cdot\partial)^4\varphi\,\varphi - \mathcal{D}_{4,\varphi^2}\,\varphi^2\big) = {}& -6(\delta+2)\big(2(\delta+3)(a\cdot b)^2 - a^2b^2\big)O_{2,aa} \\ & + 24(\delta+2)\,a\cdot b\,a^2\,O_{2,ab} + 6(a^2)^2\,O_{2,bb}\,, \end{aligned} \tag{5.12}$$

where

$$\begin{aligned} \mathcal{D}_{4,\varphi^2} = \frac{1}{4(2\delta+1)(2\delta+3)}\bigg(&(\delta+2)(\delta+3)\,(a\cdot\partial)^4 - \frac{6\delta(\delta+2)}{4\delta+2-d}\,a^2(a\cdot\partial)^2\partial^2 \\ & + \frac{3\delta(\delta+1)}{(4\delta+2-d)(4\delta+4-d)}\,(a^2)^2(\partial^2)^2\bigg)\,, \end{aligned} \tag{5.13}$$

and $O_{2,aa} = a^\mu a^\nu O_{2,\mu\nu}$ with $O_{2,\mu\nu}$ the conformal primary given by (5.8). By adding extra contributions with two derivatives acting on $O_{2,\mu\nu}$ the remaining terms in (5.12) may be cancelled so as to obtain a four index conformal primary

$$a^\mu a^\nu a^\sigma a^\rho\,O_{4,\mu\nu\sigma\rho} = (a\cdot\partial)^4\varphi\,\varphi - \mathcal{D}_{4,\varphi^2}\,\varphi^2 - \mathcal{D}_4{}^{\mu\nu}O_{2,\mu\nu}\,, \tag{5.14}$$

where

$$
\begin{aligned}
\mathcal{D}_4{}^{\mu\nu} = & -\frac{3}{2\delta+5}\bigg((\delta+3)(a\cdot\partial)^2-\frac{\delta+2}{4\delta+6-d}\,a^2\,\partial^2\bigg)\,a^\mu a^\nu \\
& -\frac{3}{(2\delta+5)(4\delta+6-d)}\bigg(2\,a^2a\cdot\partial\,a^{(\mu}\partial^{\nu)}+\frac{1}{2\delta+3-d}\,(a^2)^2\,\partial^\mu\partial^\nu\bigg) \\
& +\frac{3}{(2\delta+3)(2\delta+5)(4\delta+6-d)}\bigg(a^2(a\cdot\partial)^2+\frac{1}{2\delta+3-d}\,(a^2)^2\partial^2\bigg)\,\eta^{\mu\nu}\,.
\end{aligned}
\tag{5.15}
$$

It is straightforward to check that $K_b\,O_{4,\mu\nu\rho\sigma}=0$, as guaranteed by the fact that there are no primaries with three derivatives and two φ's. By acting with $\partial_a\cdot\partial_a$ we obtain a spin-two primary with twist $2\delta+2$,

$$
\begin{aligned}
&\eta^{\sigma\rho}a^\mu a^\nu\,O_{4,\mu\nu\sigma\rho} \\
&\quad=(a\cdot\partial)^2\partial^2\varphi\,\varphi \\
&\qquad-\frac{2\delta+2-d}{4(2\delta+1)(4\delta+2-d)}\bigg((\delta+2)\,(a\cdot\partial)^2\partial^2-\frac{\delta}{4\delta+4-d}\,a^2\,(\partial^2)^2\bigg)\varphi^2 \\
&\qquad+\frac{1}{2(4\delta+6-d)}\bigg((2\delta+2-d)\,\partial^2\,a^\mu a^\nu \\
&\qquad\qquad+2(4\delta+8-d)\,a\cdot\partial\,a^\mu\partial^\nu+\frac{2}{2\delta+3-d}\,a^2\,\partial^\mu\partial^\nu\bigg)O_{2,\mu\nu} \\
&\qquad+\frac{1}{2(2\delta+3)(4\delta+6-d)}\bigg(\big(2(\delta+1)(2\delta+5)-d(\delta+2)\big)\,(a\cdot\partial)^2 \\
&\qquad\qquad-\frac{2\delta^2+5\delta+5-d(\delta+1)}{2\delta+3-d}\,a^2\partial^2\bigg)\,\eta^{\mu\nu}O_{2,\mu\nu}\,,
\end{aligned}
\tag{5.16}
$$

as well as a scalar primary,

$$
\begin{aligned}
\eta^{\mu\nu}\eta^{\sigma\rho}O_{4,\mu\nu\sigma\rho}=(\partial^2)^2\varphi\,\varphi&-\frac{(\delta+1)(2\delta+2-d)(2\delta+4-d)}{2(2\delta+1)(4\delta+2-d)(4\delta+4-d)}\,(\partial^2)^2\varphi^2 \\
&+\frac{2\delta+4-d}{2\delta+3-d}\bigg(\partial^\mu\partial^\nu+\frac{2\delta+2-d}{4\delta+6-d}\,\partial^2\,\eta^{\mu\nu}\bigg)O_{2,\mu\nu}\,.
\end{aligned}
\tag{5.17}
$$

For $\delta=\frac{1}{2}(d-4)$,

$$
T_{\varphi,4,\mu\nu}=2\,O_{4,\mu\nu\sigma\sigma}-\frac{1}{2}\,\eta_{\mu\nu}\,O_{4,\sigma\sigma\rho\rho}\,.
\tag{5.18}
$$

In this case $4\delta+6-d=d-2$ and the construction of $T_{4,\mu\nu}$ fails when $d=2$ since then $O_{2,\mu\nu}$ has a spin two conformal primary descendant with two derivatives.

For six derivatives (5.11) is extended to

$$
\begin{aligned}
K_b^6\big((a\cdot\partial)^6\varphi\,\varphi\big)=90\delta(\delta+1)(\delta+2)\big(&8(\delta+3)(\delta+4)(\delta+5)\,(a\cdot b)^6 \\
&-60(\delta+3)(\delta+4)\,(a\cdot b)^4a^2b^2 \\
&+90(\delta+3)(a\cdot b)^2(a^2b^2)^2-15(a^2b^2)^3\big)\varphi^2\,.
\end{aligned}
\tag{5.19}
$$

Then,

$$\begin{aligned}
&K_b{}^4\big((a\cdot\partial)^6\varphi\,\varphi-\mathcal{D}_{6,\varphi^2}\,\varphi^2\big)\\
&\quad=\,-90(\delta+2)(\delta+3)\Big(4(a\cdot b)^2(\delta+4)\big((\delta+5)(a\cdot b)^2-3\,a^2b^2\big)O_{2,aa}\\
&\qquad\qquad+3(a^2b^2)^2O_{2,aa}-8\,a\cdot b\,a^2\big(2(a\cdot b)^2-3\,a^2b^2\big)O_{2,ab}\\
&\qquad\qquad+6(a^2)^2\big(2(\delta+3)(a\cdot b)^2-a^2b^2\big)O_{2,bb}\Big)\,,
\end{aligned}\tag{5.20}$$

with

$$\begin{aligned}
\mathcal{D}_{6,\varphi^2}=\frac{1}{8(2\delta+5)}\bigg(&(\delta+3)(\delta+4)\bigg(\frac{\delta+5}{(2\delta+1)(2\delta+3)}\,(a\cdot\partial)^2\\
&-\frac{15\delta}{\big(4\delta(\delta+2)+3\big)(4\delta+2-d)}\,a^2\partial^2\bigg)(a\cdot\partial)^4\\
&+\frac{15\delta(\delta+1)}{(4\delta+2-d)(4\delta+4-d)}\bigg(\frac{3(\delta+3)}{4\delta(\delta+2)+3}\,(a\cdot\partial)^2\\
&-\frac{\delta+2}{(2\delta+1)(2\delta+3)(4\delta+6-d)}\,a^2\partial^2\bigg)(a^2\partial^2)^2\bigg).
\end{aligned}\tag{5.21}$$

Further, we have

$$\begin{aligned}
K_b^2\big((a\cdot\partial)^6\varphi\,\varphi-\mathcal{D}_6{}^{\mu\nu}\,O_{2,\mu\nu}-\mathcal{D}_{6,\varphi^2}\,\varphi^2\big)=\;&15(\delta+4)\big(2(\delta+5)(a\cdot b)^2-a^2b^2\big)O_{4,aaaa}\\
&-120(\delta+4)\,a\cdot b\,a^2O_{4,aaab}+90(a^2)^2\,\mathcal{O}_{4,aabb}\,,
\end{aligned}\tag{5.22}$$

with

$$\begin{aligned}
\mathcal{D}_6{}^{\mu\nu}=&-\frac{15}{4(2\delta+5)(2\delta+7)}\bigg((\delta+4)(\delta+5)(a\cdot\partial)^4-\frac{6(\delta+2)(\delta+4)}{4\delta+6-d}\,a^2(a\cdot\partial)^2\partial^2\\
&\qquad+\frac{3(\delta+2)(\delta+3)}{(4\delta+6-d)(4\delta+8-d)}(a^2)^2(\partial^2)^2\bigg)a^\mu a^\nu\\
&-\frac{45}{(2\delta+5)(2\delta+7)(4\delta+6-d)}\bigg((\delta+4)(a\cdot\partial)^2-\frac{\delta+2}{4\delta+8-d}\,a^2\partial^2\bigg)a^2a\cdot\partial\,a^{(\mu}\partial^{\nu)}\\
&-\frac{45}{2(2\delta+5)(2\delta+7)(2\delta+3-d)(4\delta+6-d)(4\delta+8-d)}\\
&\qquad\times\bigg(\big(4\delta^2+26\delta+36-d(\delta+5)\big)(a\cdot\partial)^2-(\delta+2)\,a^2\partial^2\bigg)(a^2)^2\partial^\mu\partial^\nu\\
&+\frac{45}{2(2\delta+3)(2\delta+5)(2\delta+7)(4\delta+6-d)}\bigg((\delta+4)(a\cdot\partial)^4\\
&\qquad+\frac{2\delta^2+19\delta+30-3d}{(2\delta+3-d)(4\delta+8-d)}\,a^2(a\cdot\partial)^2\partial^2\\
&\qquad-\frac{\delta+2}{(2\delta+3-d)(4\delta+8-d)}\,(a^2)^2(\partial^2)^2\bigg)a^2\eta^{\mu\nu}\,.
\end{aligned}\tag{5.23}$$

A two-derivative operator on $O_{4,\mu\nu\sigma\rho}$ can now be constructed to obtain a conformal primary with six indices, namely

$$a^\mu a^\nu a^\sigma a^\rho a^\tau a^\omega\,O_{6,\mu\nu\sigma\rho\tau\omega}=-(a\cdot\partial)^6\varphi\,\varphi+\mathcal{D}_{6,\varphi^2}\,\varphi^2+\mathcal{D}_6{}^{\mu\nu}O_{2,\mu\nu}+\mathcal{D}_6{}^{\mu\nu\sigma\rho}\,O_{4,\mu\nu\sigma\rho}\,,\tag{5.24}$$

where

$$\mathcal{D}_6{}^{\mu\nu\sigma\rho} = \frac{15}{2(2\delta+9)}\bigg((\delta+5)(a\cdot\partial)^2 - \frac{\delta+4}{4\delta+10-d}a^2\partial^2\bigg)a^\mu a^\nu a^\sigma a^\rho$$
$$+ \frac{30}{(2\delta+9)(4\delta+10-d)}\, a\cdot\partial\, a^{(\mu}a^\nu a^\sigma\partial^{\rho)}$$
$$+ \frac{45}{(2\delta+9)(2\delta+3-d)(4\delta+10-d)}\,(a^2)^2 a^{(\mu}a^\nu\partial^\sigma\partial^{\rho)}$$
$$- \frac{45}{(2\delta+7)(2\delta+9)(4\delta+10-d)}\bigg((a\cdot\partial)^2 + \frac{1}{2\delta+3-d}\,a^2\partial^2\bigg)a^{(\mu}a^\nu\eta^{\sigma\rho)}$$
$$- \frac{180}{(2\delta+7)(2\delta+9)(2\delta+3-d)(4\delta+10-d)}\,(a^2)^2 a\cdot\partial\, a^{(\mu}\partial^\nu\eta^{\sigma\rho)}$$
$$- \frac{90}{(2\delta+7)(2\delta+9)(2\delta+3-d)(2\delta+5-d)(4\delta+10-d)}\,(a^2)^3\partial^{(\mu}\partial^\nu\eta^{\sigma\rho)}$$
$$+ \frac{90}{(2\delta+5)(2\delta+7)(2\delta+9)(2\delta+3-d)(4\delta+10-d)}$$
$$\times\bigg((a\cdot\partial)^2 + \frac{1}{2\delta+5-d}a^2\partial^2\bigg)(a^2)^2\eta^{\mu(\nu}\eta^{\sigma)\rho}\,. \tag{5.25}$$

We have $K_b\, O_{6,\mu\nu\sigma\rho\tau\omega} = 0$ since there are no conformal primaries with five derivatives and two φ's.

A four-index as well as a two-index and a scalar conformal primary can be obtained from O_6 by acting with $\partial_a\cdot\partial_a$. To avoid even more lengthy expressions we only list here the scalar primary,

$$\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\omega}O_{6,\mu\nu\sigma\rho\tau\omega} = -(\partial^2)^3\varphi\,\varphi$$
$$+ \frac{(\delta+2)(2\delta+2-d)(2\delta+4-d)(2\delta+6-d)}{4(2\delta+1)(4\delta+2-d)(4\delta+4-d)(4\delta+6-d)}\,(\partial^2)^3\varphi^2$$
$$- \frac{3(2\delta+4-d)(2\delta+6-d)}{2(2\delta+3-d)(4\delta+6-d)}\bigg(\partial^\mu\partial^\nu + \frac{2\delta^2+5\delta+1-(\delta+2)d}{(2\delta+3)(4\delta+8-d)}\,\partial^2\,\eta^{\mu\nu}\bigg)\partial^2 O_{2,\mu\nu}$$
$$+ \frac{3(2\delta+6-d)}{2(2\delta+5-d)}\bigg(2\,\partial^\mu\partial^\nu + \frac{2\delta+3-d}{4\delta+10-d}\,\partial^2\,\eta^{\mu\nu}\bigg)O_{4,\mu\nu\sigma\sigma}\,. \tag{5.26}$$

For $\delta = \frac{1}{2}(d-6)$ we can express

$$T_{\varphi,6,\mu\nu} = 3\,O_{6,\mu\nu\sigma\sigma\rho\rho} - \frac{1}{2}\,\eta_{\mu\nu}\,O_{6,\tau\tau\sigma\sigma\rho\rho}\,, \qquad \text{if} \quad \lambda = -\frac{8}{d-4}\,. \tag{5.27}$$

Thus, we see that the requirement that $T_{\varphi,6,\mu\nu}$ be a conformal primary determines λ, independently and consistently with the result (2.7) obtained from the curved-space action contribution of the Bach tensor.

The poles in (5.13), (5.15), (5.21), (5.23), (5.25) at $4\delta = 4-2k$, for $k = 1,2,\ldots$, arise for these δ since there are corresponding differential operators generating conformal primary descendants given by, for $O_{\mu_1\ldots\mu_\ell}$ a symmetric traceless tensor,

$$\mathcal{D}_{n,\ell}^{\mu_1\ldots\mu_\ell} O_{\mu_1\ldots\mu_\ell} = \sum_{r=0,-n}^{\ell} (-1)^r \frac{(\frac{1}{2}d+\ell+n-1)_r}{2^r\, r!\, (\ell-r)!\, (n+r)!} \, (a\cdot\partial)^{\ell-r} (\partial^2)^{n+r} a^{\mu_1} \ldots a^{\mu_r} \partial^{\mu_{r+1}} \partial^{\mu_\ell} O_{\mu_1\ldots\mu_\ell} \,,$$
$$a^2 = 0, \quad n+\ell \ge 1, \quad \Delta_O = \frac{1}{2}d - n - \ell \,, \tag{5.28}$$

is a conformal primary symmetric traceless tensor of rank ℓ and $\Delta = \frac{1}{2}d + n$. Thus $(\partial^2)^n O$ is a conformal primary scalar for $\Delta_O = \frac{1}{2}d - n$. The poles (5.15), (5.23), (5.25) at $2\delta = d-n$ for $n = 3, 5$ correspond to conformal primary descendants

$$\partial^{\mu_k} \ldots \partial^{\mu_\ell} O_{\mu_1\ldots\mu_\ell} \,, \qquad k = 1, \ldots, \ell \,, \qquad \Delta_O = d+k-2 \,, \tag{5.29}$$

which are symmetric traceless tensors of rank $\ell - k$. There are also conformal primary traceless tensor descendants of the form

$$(a\cdot\partial)^k a^{\mu_1} \ldots a^{\mu_\ell} O_{\mu_1\ldots\mu_\ell} \,, \quad a^2 = 0 \,, \quad k = 1, 2, \ldots \,, \quad \Delta_O = 1 - \ell - k \,, \tag{5.30}$$

of rank $k + \ell$ and correspond to the poles at $2\delta = -n$ for $n = 1, 3, \ldots$.

6 Conclusion

In this paper we have computed C_T for various free field theories outside the usual range. Although such theories involving higher derivatives in general correspond to non-unitary quantum field theories, they appear to be relevant in understanding some CFTs for large N numbers of component fields where the $1/N$ expansion remains valid for arbitrary dimension d. Of course there are additional parameters, such as those associated with the energy-momentum tensor three-point function, one of which is related to C_T by Ward identities. For the usual free CFTs these were also calculated in [8]. The theories discussed here might also be extended to determine the energy-momentum tensor three-point function, but the complexity of the expressions for $T_{\mu\nu}$ makes this a rather formidable task, as are the corresponding large N calculations.

In general in even dimensions C_T in a CFT is related to a particular term quadratic in the Weyl tensor in the energy-momentum tensor trace on a curved space background. In four and six dimensions, where the trace anomaly coefficients are c and c_3, the relations are [19]

$$C_T = \frac{4}{3} \times 5!\, c \,, \qquad C_T = \frac{3}{5} \times 7!\, c_3 \,, \tag{6.1}$$

with a normalisation chosen so that for conventional free field theories c, c_3 are given by $5!\, c = n_S + 3\, n_W + 12\, n_A$, $7!\, c_3 = 2\, n_S + 40\, n_W + 180\, n_B$ with n_S scalars, n_W Weyl fermions and n_A, n_B the number of vector, 2-form gauge fields in four, six dimensions. There is of course complete agreement between the results for C_T and the curved space results based on using the heat kernel for second-order conformal differential operators. The results obtained here allow some contributions to the heat kernel for higher-order operators on curved backgrounds to be obtained. These have been discussed for Ricci flat backgrounds in [32, 33].

Heat kernel techniques allow a perturbation expansion for arbitrary curved backgrounds so as to determine the leading corrections to c, c_3. In six dimensions for a cubic interaction $\frac{1}{6}\,\lambda_{ijk}\phi_i\phi_j\phi_k$ then results in [19, 34] give to lowest order

$$7!\, c_{3,1} = -\frac{7}{36}\,\hat{\lambda}_{ijk}\hat{\lambda}_{ijk}\,, \qquad \hat{\lambda}_{ijk} = \lambda_{ijk}/(4\pi)^{\frac{3}{2}}\,. \tag{6.2}$$

For the theory defined by (1.5), where $\phi_i \to (\sigma, \varphi_i)$, $\hat{\lambda}_{ijk}\hat{\lambda}_{ijk} = 3N\,\hat{g}^2 + \hat{\lambda}^2$ and to lowest order $\beta_{\hat{g}} = -\frac{1}{2}\varepsilon\,\hat{g} + \frac{1}{12}(N-8)\,\hat{g}^3 - \hat{g}^2\hat{\lambda} + \frac{1}{12}\,\hat{g}\,\hat{\lambda}^2$, $\beta_{\hat{\lambda}} = -\varepsilon\,\hat{\lambda} - N\,\hat{g}^3 + \frac{1}{4}N\,\hat{g}^2\hat{\lambda} - \frac{3}{4}\,\hat{\lambda}^3$, so that at the fixed point, to leading order in $\varepsilon, 1/N$, $\hat{g}^2 = 6\varepsilon/N$, $\hat{\lambda}^2 = 6^3\varepsilon/N$ and hence in (1.3), (1.5) gives for the CFT at $d = 6-\varepsilon$ for large N

$$C_{T,1} = 1 - \frac{7}{4}\,\varepsilon\,, \tag{6.3}$$

agreeing with the perturbative flat space calculation in [12] and also the expansion of the large N result. The corresponding results for four-dimensional renormalisable theories were obtained some time ago. For n_A gauge fields, with a simple gauge group and coupling g, Dirac fermions and Yukawa, scalar interactions $\bar{\psi}Y_i\psi\,\phi_i$, $\frac{1}{24}\,\lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l$ the results obtained in [35–38] by expanding about flat space and, using heat kernel methods for a curved background, [39, 40] give[6]

$$c_1 = -\frac{2}{9}\Big(C - \frac{7}{8}\,R_\psi - \frac{1}{4}R_\phi\Big) n_A\,\hat{g}^2 - \frac{1}{24}\,\mathrm{tr}(\hat{Y}_i\hat{Y}_i) - \frac{1}{32\times 27}\,\hat{\lambda}_{ijkl}\hat{\lambda}_{ijkl}\,, \tag{6.4}$$

for $\hat{g} = g/4\pi$, $\hat{Y}_i = Y_i/4\pi$, $\hat{\lambda}_{ijkl} = \lambda_{ijkl}/16\pi^2$ and we take the spinorial trace $\mathrm{tr}(\mathbb{1}) = 4$. The conventions in (6.4) for C, R_ψ, R_ϕ are such that the lowest order gauge β-function becomes $\beta_{\hat{g}} = -\frac{1}{2}\varepsilon\,\hat{g} + \frac{1}{3}(11C - 4R_\psi - \frac{1}{2}R_\phi)\hat{g}^3$. For the $\mathrm{O}(N)$ scalar theory, with interaction $\frac{1}{8}\,\lambda(\varphi^2)^2$ and $\beta_{\hat{\lambda}} = -\varepsilon\,\hat{\lambda} + (N+8)\,\hat{\lambda}^2$, this was shown in [11] to give a $\mathrm{O}(\varepsilon^2)$ contribution to c at the RG fixed point in $4-\varepsilon$ dimensions, $5!\,c_1 = -\frac{5}{12}N(N+2)/(N+8)^2\,\varepsilon^2$, consistent with large N results. For the Gross-Neveu model starting from (1.9) the one-loop β-functions are $\beta_{\hat{g}} = -\frac{1}{2}\varepsilon\,\hat{g} + (2N+3)\hat{g}^3$, $\beta_{\hat{\lambda}} = -\varepsilon\,\hat{\lambda} + 3\,\hat{\lambda}^2 + 8N\,\hat{\lambda}\,\hat{g}^2 - 48N\,\hat{g}^4$. At the fixed point to leading order for large N from the Yukawa terms in (6.4) $5!\,c_1 = -5N/(4N+6)\,\varepsilon$, which agrees with explicit calculations and the expansion of the large N Gross-Neveu result for C_T in [12].

For scalar and fermion theories large N methods allow non-trivial CFTs to be formally defined for general dimensions d which interpolate between physical theories for d an integer. The situation is less clear for gauge theories since maintaining conformal invariance and gauge invariance is more difficult, as was demonstrated in section 4. For a gauge-invariant quantum field theory the energy-momentum tensor in general contains contributions arising from the gauge fixing and ghost terms in the action. However these are BRS exact and do not contribute to correlation functions for gauge-invariant operators

[6] In terms of some previous literature $c = -16\pi^2\beta_a$. In [39] in the final result a misprint in (4.16) is corrected by taking $\frac{1}{9} \to \frac{2}{9}$.

so that the calculations of C_T in [8, 9] did not take account of them. In section 4 the gauge fixing terms made a contribution to the energy-momentum tensor (4.3) whose effect did not disappear in correlation functions when they notionally decoupled. It is an open question whether the ghost contributions to $T^{\mu\nu}$ could be extended to general d while maintaining conformal invariance. Their contributions to C_T should account for the difference between C_T in (4.10) and the corresponding gauge theory result when $d = 2n$.

Note added: for higher-derivative scalar theories calculations of C_T have also been carried out by Guerrieri et al. [41], who further considered the theory with eight derivatives. Prompted by their discussion there is a quick derivation of $C_{T,\varphi,2p}$, agreeing with (2.13) for any p, based on known results for conformal partial-wave expansions of four-point functions as a sum over conformal primaries. Using Mellin transform methods Fitzpatrick and Kaplan [42] obtained a conformal partial-wave expansion for the four-point function of generalised free fields in any dimension d. Their results give the expansion

$$u^\delta = \sum_{\ell,\tau=0}^{\infty} c_{\tau,\ell}\, G_{2\delta+2\tau+\ell,\ell}(u,v)\,, \tag{6.5}$$

where u, v are the standard conformal invariants and $G_{\Delta,\ell}(u,v)$ are the conformal partial waves for a conformal primary operator with scaling dimension Δ and spin ℓ and its descendants. The partial waves are normalised here so that $G_{\Delta,\ell}(u,v) \sim u^{\frac{1}{2}(\Delta-\ell)}(-1+v)^\ell$ as $u \to 0$, $v \to 1$. From [42],

$$c_{\tau,\ell} = \frac{\big((\delta - \frac{1}{2}d + 1)_\tau\, (\delta)_{\ell+\tau}\big)^2}{\ell!\,\tau!\,(\ell + \frac{1}{2}d)_\tau\,(2\delta + \tau - d + 1)_\tau\,(2\delta + \ell + \tau - \frac{1}{2}d)_\tau\,(2\delta + 2\tau + \ell - 1)_\ell}\,. \tag{6.6}$$

The four-point function for free fields $\langle \varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\,\varphi(x_4)\rangle$ defines a function of the conformal invariants $F(u,v) = N_\varphi{}^2\big(1 + u^{\Delta_\varphi} + (u/v)^{\Delta_\varphi}\big)$, where N_φ is the coefficient for the two-point function, which may be expanded in terms of conformal partial waves using (6.5), for $u \to u/v$ $c_{\tau,\ell} \to (-1)^\ell c_{\tau,\ell}$. Coresponding to the operator product (2.9),

$$\varphi(x)\,\varphi(0) \sim -\frac{N_\varphi}{C_{T,\varphi}}\,\frac{d\,\Delta_\varphi}{d-1}\,\frac{1}{(x^2)^{\Delta_\varphi - \frac{1}{2}d + 1}}\,x_\mu x_\nu\,T_\varphi^{\mu\nu}(0)\,. \tag{6.7}$$

Since $x_{12\,\mu} x_{34\,\nu}\, I^{\mu\nu}(x_{24})/(x_{12}{}^2 x_{34}{}^2)^{\frac{1}{2}} \sim -(1-v)/(2\sqrt{u})$ as $x_{12}, x_{34} \to 0$, we must have

$$c_{p-1,2}\Big|_{\delta=\frac{1}{2}(d-2p)} = \bigg(\frac{d\,\Delta_{2p}}{d-1}\bigg)^2\,\frac{1}{8\,C_{T,\varphi,2p}} = (-1)^{p-1}\,\frac{d(d-2p)^2}{32\,p(d-1)}\,\frac{(\frac{1}{2}d - p + 1)_{p-1}}{(\frac{1}{2}d + 2)_{p-1}}\,. \tag{6.8}$$

This gives a result for $C_{T,\varphi,2p}$ identical to that in (2.13). The restriction on δ is of course necessary to ensure that the expansion contains a conformal primary contribution which may be identified with the energy-momentum tensor, for this case in (6.5) $\tau \le p-1$ and the operators which contribute have $\Delta - \ell = d + 2(\tau - p)$ and are expressible as $\varphi(\overleftrightarrow{\partial})^\ell\,(\partial^2)^\tau\varphi$.

These results can easily be extended to consider the differing conserved currents present in these free theories. If we allow $\varphi \to \varphi_i$, there is a conserved current $J^\mu_{\varphi,ij} = -J^\mu_{\varphi,ji}$ with

$$\big\langle J^\mu_{\varphi,ij}(x)\,J^\nu_{\varphi,kl}(0)\big\rangle = C_{J,\varphi}\,\frac{1}{(x^2)^{d-1}}\,I^{\mu\nu}(x)\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,. \tag{6.9}$$

For its contribution in the operator product expansion we have

$$\varphi_i(x)\,\varphi_j(0) \sim -\frac{N_\varphi}{C_{J,\varphi}}\,\frac{1}{(x^2)^{\Delta_\varphi-\frac{1}{2}d+1}}\,x_\mu\,J^\mu_{\varphi,ij}(0)\,. \tag{6.10}$$

The four-point function in this case has the form $F_{ijkl}(u,v) = N_\varphi{}^2\big(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}\,u^{\Delta_\varphi} + \delta_{il}\delta_{jk}\,(u/v)^{\Delta_\varphi}\big)$, and using (6.5) we find

$$\frac{1}{C_{J,\varphi,2p}} = 2\,c_{p-1,1}\Big|_{\delta=\frac{1}{2}(d-2p)} = (-1)^{p-1}\,\frac{1}{p}\,\frac{(\frac{1}{2}d-p)_p}{(\frac{1}{2}d+1)_{p-1}}\,. \tag{6.11}$$

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