Published for SISSA by Springer

Received: April 20, 2016
Accepted: May 12, 2016
Published: June 1, 2016

JHEP06(2016)003

Stringy origin of 4d black hole microstates

M. Bianchi,[a] **J.F. Morales**[b,a] **and L. Pieri**[a]

[a] *Dipartimento di Fisica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, I-00133 Roma, Italy*
[b] *I.N.F.N. — Sezione di Roma 2, Via della Ricerca Scientifica, I-00133 Roma, Italy*

E-mail: bianchi@roma2.infn.it, morales@roma2.infn.it, lorenzo.pieri@roma2.infn.it

Abstract: We derive a precise dictionary between micro-state geometries and open string condensates for a large class of excitations of four dimensional BPS black holes realised in terms of D3-branes intersecting on a six-torus. The complete multipole expansion of the supergravity solutions at weak coupling is extracted from string amplitudes involving one massless closed string and multiple open strings insertions on disks with mixed boundary conditions.

Keywords: Black Holes in String Theory, D-branes

ArXiv ePrint: 1603.05169

doi:10.1007/JHEP06(2016)003

Contents

1 Introduction

The fuzzball proposal [1, 2] relates black-hole micro-states to smooth and horizon-less geometries with the same asymptotics (the same mass and charges) as the putative black hole but differing from it in its interior. The (semi)classical black-hole geometry only results from a coarse-graining approximation of the micro-state solutions. For black holes corresponding to D-brane bound-states, the micro-states can be realised and counted in terms of open-string excitations of the D-branes constituents [3, 4]. As a result, the fuzzball proposal suggests a one-to-one correspondence between open-string condensates and the geometries generated by them.

The best understood example is the two-charge BPS system in type IIB theory for which the whole family of micro-state geometries is known [5–7]. The solutions can be written in several different U-duality related frames. In [8], the asymptotic form of the general micro-state geometries associated to a system of D1- and D5-branes has been derived from disk amplitudes computing the emission rate of closed string states from open string condensates at the D1-D5 intersections. The two-charge system represents a somewhat degenerate example of a black hole in that its classical geometry has a singular horizon of zero size. Smooth horizons in the supergravity limit require systems with 3- or 4-charges. Unfortunately, despite significant progress (see [9–18] and references therein for recent results and reviews), a similar analysis for 3- and 4-charge black holes is still missing since a family of micro-state solutions large enough to account for the black hole entropy is not yet known.

Aim of this paper is to partially fill in this gap, providing a direct link between micro-state geometries and open string condensates for a large class of 4-charge black hole microstates. We realise the four-dimensional BPS black holes in the 'democratic' frame of four mutually intersecting stacks of D3-branes on T^6. We focus on a family of black hole solutions characterised by eight ('lovely' rather than 'hateful') harmonic functions in the type IIB theory. This family is dual to the extensively studied BPS solutions [19–24] describing D0-D2-D4-D6 brane bound-states in the type IIA theory or intersecting M2 and M5 branes in M-theory carrying non-trivial momentum and KK monopole charge. Amusingly, in the D3-brane frame, the eight harmonic functions are described entirely in terms of excitations of the metric and the four-form RR field. A subclass of this family of solutions involving two harmonic functions has been studied recently in [25] and shown to be regular and horizon-less.

The micro-state geometries will be derived from disk amplitudes computing the emission rate of massless closed string states from open string condensates binding the different stacks of D3 branes. The disk amplitude computes the leading correction to the supergravity solution in the limit of small string coupling g_s. In principle the supergravity solution can be recovered by plugging the result into the type IIB equations of motion and solving recursively order by order in g_s. Equivalently, one can expand the general micro-state solution to order g_s and match it against the result of the string amplitude. Proceeding in this way, we identify three class of micro-states depending on whether the open string condensate involves either one D3-brane stack or the intersection of two or four different stacks. We dub the three classes as L, K and M solutions referring to the names of the harmonic functions underlying the solutions. We remark that disk contributions add and at this linear order both the metric and the four-form potential can be written as a sum of harmonic functions. In particular, we find that four of the harmonic functions (L) specifying the solution are associated to the four different disks with a single boundary, three harmonic functions (K) are associated to the three inequivalent 2-charge intersections and the last one (M) is sourced by a 4-charge open string condensate. The three types of micro-state geometries fall off at infinity as Q_i/r, Q_iQ_j/r^2 and $Q_1Q_2Q_3Q_4/r^3$ respectively. Higher multipole modes are generated by insertions of untwisted (boundary-preserving) open strings on the boundary of the disk, much in the same way as for the

D1-D5 system [12]. Remarkably, as we will see, the micro-state angular momentum is fully accounted for by the type K component of the solution.

Solutions of type M are particularly interesting since the number of disks with four different boundaries grows with the product of the four charges $N = Q_1\, Q_2\, Q_3\, Q_4$, the same quartic invariant entering in the black-hole entropy formula $S = 2\pi\sqrt{N}$. It is tempting to speculate that after quantization the corresponding micro-states may account for the entropy of the putative black-hole much in the same way as the entropy of the 2-charge system with $N = Q_1 Q_2$ is reproduced by quantisation of the profile function describing D1-D5 intersections [5, 26, 27] or, equivalently, D3-D3' pair intersections, in the systems that we consider here.[1]

We remark that the class of 4-charge micro-state solutions generated by open string condensates that we find here is larger than the family of supergravity solutions explicitly known. This is at variant with the 2-charge systems, where the most general micro-state [7] is reproduced by a 2-charge intersection [8]. In the case of 4 charges we find new solutions looking pretty much like the K and M solutions, but tilted with respect to the background D3-brane geometry. It would be interesting to understand whether these could be accommodated in a larger class of micro-state geometries or should be excluded on the basis of supersymmetry or stability conditions (D- and F-flatness) for the open strings.

The plan of the paper is as follows. In section 1 we introduce the 'eight-ful' harmonic family of supergravity solutions in the 'democratic' frame of D3-branes. We identify three sub-classes of solutions that we dub L solutions, K solutions and M solutions. In section 2, we focus on solutions sourced by disks with a single type of boundary condition. We compute disk amplitudes involving the insertion of a closed string NS-NS or R-R vertex operator in the bulk and an arbitrary number of untwisted open string scalars. In section 3, we compute disk amplitudes with two twisted open string insertions that intertwine between two different boundary conditions. In section 4, we compute disk amplitudes with four different boundaries conditions. The computations are much more involved (they are equivalent to a 6-point open string amplitude) but one can numerically integrate the final correlators and get a finite result both for NS-NS and R-R closed string insertions in the bulk. Remarkably, NS-NS and R-R correlators are proportional to the same integrals, so their numeric values are irrelevant for the identification of the dual supergravity solution. We finally draw our conclusions and indicate directions for further research. In the appendices we collect our conventions for string correlators and present the derivation of the D3-brane solution starting from its T-dual description in Type IIA theory.

2 Supergravity solution

We consider four-dimensional BPS black hole solutions of type IIB supergravity consisting of D3-branes intersecting on T^6. The solutions are characterised by a non-trivial metric and a five-form flux with all the other fields of type IIB supergravity vanishing. In this

[1] Indeed, in our case, any pair of D3-branes intersect along a string in T^6 and can be mapped into D1-D5 or F1-P1 via U-dualities.

JHEP06(2016)003

sector, the type IIB equations of motion take the extremely simple and elegant form[2]

$$R_{MN} = \frac{1}{4\cdot 4!} F_{MP_1P_2P_3P_4}\, F_N{}^{P_1P_2P_3P_4}$$
$$F_5 = *_{10} F_5 \tag{2.1}$$

with

$$F_5 = dC_4 \tag{2.2}$$

A family of solutions can be found by acting with three T-dualities on the more familiar class [22, 24] describing systems of D0-D2-D4-D6 branes in type IIA theory (see appendix for details). The solutions can be written in terms of eight harmonic functions ($a = 1, \ldots 8$, $I = 1, 2, 3$)

$$H_a = \{V, L_I, K_I, M\} \tag{2.3}$$

on (flat) $\mathbb{R}^3$ associated to the four electric and four magnetic charges in the type IIA description of the system. These functions are conveniently combined into

$$\begin{aligned} P_I &= \frac{K_I}{V} \\ Z_I &= L_I + \frac{|\epsilon_{IJK}|}{2}\frac{K_J K_K}{V} \\ \mu &= \frac{M}{2} + \frac{L_I K_I}{2\,V} + \frac{|\epsilon_{IJK}|}{6}\frac{K_I K_J K_K}{V^2} \end{aligned} \tag{2.4}$$

Here ϵ_{IJK} characterise the triple intersections between two cycles on T^6.

In terms of these functions the type IIB supergravity solution can be written as

$$\begin{aligned} ds^2 &= -e^{2U}(dt+w)^2 + e^{-2U}\sum_{i=1}^{3} dx_i^2 + \sum_{I=1}^{3}\left[\frac{dy_I^2}{V e^{2U} Z_I} + V e^{2U} Z_I\, \tilde{e}_I^2\right] \\ C_4 &= \alpha_0 \wedge \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3 + \beta_0 \wedge dy_1 \wedge dy_2 \wedge dy_3 \\ &\quad + \frac{1}{2}\epsilon_{IJK}\, \left(\alpha_I \wedge dy_I \wedge \tilde{e}_J \wedge \tilde{e}_K + \beta_I \wedge \tilde{e}_I \wedge dy_J \wedge dy_K\right) \end{aligned} \tag{2.5}$$

with

$$\begin{aligned} e^{-4U} &= Z_1 Z_2 Z_3 V - \mu^2 V^2 \\ b_I &= P_I - \frac{\mu}{Z_I} \qquad\qquad \tilde{e}_I = d\tilde{y}_I - b_I\, dy_I \\ \alpha_0 &= A - \mu\, V^2\, e^{4U}\,(dt+w) \qquad\qquad \alpha_I = -\frac{(dt+w)}{Z_I} + b_I\, A + w_I \\ \beta_0 &= -v_0 + \frac{e^{-4U}}{V^2 Z_1 Z_2 Z_3}(dt+w) - b_I\, v_I + b_1\, b_2\, b_3\, A + \frac{|\epsilon_{IJK}|}{2}\, b_I\, b_J\, w_K \\ \beta_I &= -v_I + \frac{|\epsilon_{IJK}|}{2}\left(\frac{\mu}{Z_J Z_K}(dt+w) + b_J\, b_K\, A + 2 b_J\, w_K\right) \end{aligned} \tag{2.6}$$

[2] Pretty much as for gauge fields in $D = 4$, the stress tensor of a 4-form is (classically) trace-less in $D = 10$.

Brane	t	x_1	x_2	x_3	y_1	$\tilde{y}_1$	y_2	$\tilde{y}_2$	y_3	$\tilde{y}_3$
$D3_0$	$-$	.	.	.	$-$	.	$-$	.	$-$	.
$D3_1$	$-$	.	.	.	$-$	.	.	$-$	.	$-$
$D3_2$	$-$	.	.	.	.	$-$	$-$	.	.	$-$
$D3_3$	$-$	.	.	.	.	$-$	.	$-$	$-$	.

Table 1. D3-brane configuration: Neumann (N) and Dirichlet (D) directions are represented by lines and dots respectively.

and

$$
\begin{aligned}
&*_3 dA = dV \qquad *_3 dw_I = -d(K_I) \qquad *_3 dv_0 = dM \qquad *_3 dv_I = dL_I \\
&*_3 dw = V d\mu - \mu dV - V Z_I dP_I
\end{aligned}
\tag{2.7}
$$

For $K_I = M = 0$, the supergravity solution (2.5) describes a system of four stacks of intersecting D3-branes aligned according to the table above. The four harmonic functions V, L_I describe the distribution of the D3-branes in the non compact spatial directions $x \in \mathbb{R}^3$.

Solutions with non-zero K_I and M will be associated to condensates involving twisted open strings. These excitations modify the geometry of the putative black hole in its interior while preserving the same asymptotic fall off at infinity. Consequently, K_I and M should decay faster than $|x|^{-1}$ at infinity. We thus focus on solutions involving harmonic functions with the following asymptotics

$$
\begin{aligned}
L_I &\approx 1 + \alpha_{D3}\,\frac{N_I}{|x|} + \dots \qquad & V &\approx 1 + \alpha_{D3}\,\frac{N_0}{|x|} + \dots \\
K_I &\approx \alpha_{D3}\, c_i^{K_I}\,\frac{x^i}{|x|^3} + \dots \qquad & M &\approx \alpha_{D3}\, c_i^{M}\,\frac{x^i}{|x|^3} + \dots
\end{aligned}
\tag{2.8}
$$

with c_i some constants, $\alpha_{D3} = 4\,\pi\, g_s\,(\alpha')^2/V_{T_3}$ the inverse D3-brane tension and V_{T_3} the volume of the 3-torus wrapped by the stack of D3-branes, that for simplicity we will take to be equal for all the stacks. We anticipate that the explicit string computations show that the coefficients c_i's entering in (2.8) satisfy the relation

$$
c_i^M + \sum_{I=1}^{3} c_i^{K_I} = 0 \tag{2.9}
$$

or equivalently the function μ defined in (2.4) starts at order $1/r^3$. Since at linear order in α_{D3}, the harmonic functions entering in the solution simply add, we can consider the contribution of each harmonic function separately. More precisely, we define three basic class of solutions: L solutions are associated to the harmonic functions L_I, V with $K_I = M = 0$. K solutions are generated by turning on a single K_I harmonic function for a given I and taking $M = -K_I$ for the given I. Finally M solutions are associated to the orthogonal combination $M + \sum_I K_I$ that is generated at order $1/r^3$. In other words the dipole modes in (2.8) vanish: $c_i^M = c_i^{K_I} = 0$.

JHEP06(2016)003

The family (2.5) can be seen as the "superposition" of these three basic classes of solutions. We remark that disks emissions simply add but massless closed string states interact with each other. This entails non-linearity, so the resulting solution will not be the naive "superposition" of its constituents as it is obvious from (2.5). In the rest of this section, we display the supergravity solutions for the three basic classes of micro-state geometries.

2.1 L solutions

A representative of the L class of micro-state solutions is given by taking

$$V = L(x) \qquad M = K_I = 0 \qquad L_I = 1 \tag{2.10}$$

in (2.5). This results into

$$\begin{aligned} &e^{-2U} = L^{1/2} \qquad \tilde{e}_I = d\tilde{y}_I \qquad \mu = 0 \\ &dA = *_3 dL \qquad \alpha_0 = A \qquad \beta_0 = L^{-1}\, dt \qquad w = \beta_I = 0 \qquad \alpha_I = -dt \end{aligned} \tag{2.11}$$

The Type IIB supergravity solution reduces to (discarding d-exact terms)

$$\begin{aligned} ds^2 &= L^{-1/2}\left(-dt^2 + \sum_{i=1}^{3} dy_i^2\right) + L^{1/2}\sum_{i=1}^{3}\left(dx_i^2 + d\tilde{y}_i^2\right) \\ C_4 &= L^{-1} \wedge dt \wedge dy_1 \wedge dy_2 \wedge dy_3 + A \wedge d\tilde{y}_1 \wedge d\tilde{y}_2 \wedge d\tilde{y}_3 \end{aligned} \tag{2.12}$$

This solution is generated by distributing D3-branes of type 0 in table 1 along $\mathbb{R}^3$. Similar solutions are found by turning on L_I associated to the other three types of D3-branes in the table.[3]

2.2 K solutions

Next we consider solutions with

$$K_3 = -M = K(x) \qquad \mu = 0 \qquad L_I = V = 1 \qquad K_1 = K_2 = 0 \tag{2.13}$$

For this choice one finds

$$\begin{aligned} &e^{-2U} = 1 \qquad *_3 dw = -dK \qquad \tilde{e}_1 = d\tilde{y}_1 \qquad \tilde{e}_2 = d\tilde{y}_2 \qquad \tilde{e}_3 = d\tilde{y}_3 - K\, dy_3 \\ &\alpha_0 = \beta_I = 0 \qquad \alpha_1 = \alpha_2 = -(dt + w) \qquad \alpha_3 = -dt \qquad \beta_0 = dt \end{aligned} \tag{2.14}$$

and the resulting supergravity solution reduces to (discarding d-exact terms)

$$\begin{aligned} ds^2 &= -(dt+w)^2 + \sum_{i=1}^{3}\left(dx_i^2 + dy_i^2 + d\tilde{y}_i^2\right) - 2\,K\, dy_3\, d\tilde{y}_3 + K^2\, dy_3^2 \\ C_4 &= -(dt+w) \wedge \tilde{e}_3 \wedge (dy_1 \wedge d\tilde{y}_2 + d\tilde{y}_1 \wedge dy_2) \end{aligned} \tag{2.15}$$

Similar solutions are found by turning on K_1 or K_2. We would like to stress that K solutions in general carry angular momentum, unlike L and M solutions.

[3] For a spherical symmetric harmonic function $L(r) = Q/r$ one finds $A = Q\, \cos\theta\, d\varphi$ in spherical coordinates where $dx_i^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$.

2.3 M solutions

Finally we consider solutions with

$$K_2 = M = M(x) \qquad \mu = M \qquad L_I = V = 1 \qquad K_1 = K_3 = 0 \tag{2.16}$$

Now one finds

$$\begin{aligned}
e^{-2U} &= \sqrt{1-M^2} \quad \tilde{e}_1 = d\tilde{y}_1 + M\,dy_1 \qquad \tilde{e}_2 = d\tilde{y}_2 \quad \tilde{e}_3 = d\tilde{y}_3 + M\,dy_3 \qquad w = 0 \\
\alpha_0 &= \frac{-M dt}{1-M^2} \quad \alpha_1 = \alpha_3 = -dt \qquad \alpha_2 = -dt + w_2 \qquad dw_2 = -*_3 dM \\
\beta_0 &= (1-M^2)\,dt + (1+M^2)\,w_2 \qquad \beta_1 = \beta_3 = M(dt - w_2) \qquad \beta_2 = M\,dt
\end{aligned} \tag{2.17}$$

The Type IIB supergravity solution reads (discarding d-exact terms)

$$ds^2 = -e^{2U} dt^2 + e^{-2U} \sum_{i=1}^{3} (dx_i^2 + dy_i^2) + e^{2U} \left[\sum_{i=1,3} (d\tilde{y}_i + M\,dy_i)^2 + d\tilde{y}_2^2 \right] \tag{2.18}$$

$$\begin{aligned}
C_4 = &-\frac{1}{1-M^2}\, dt \wedge \tilde{e}_3 \wedge (dy_1 \wedge d\tilde{y}_2 + M\, d\tilde{y}_1 \wedge d\tilde{y}_2) \\
&+ w_2 \wedge (dy_1 \wedge dy_2 \wedge dy_3 + d\tilde{y}_1 \wedge dy_2 \wedge d\tilde{y}_3)
\end{aligned} \tag{2.19}$$

2.4 Harmonic expansions

The harmonic functions H_a entering into the supergravity solutions can be conveniently expanded in multi-pole modes. The general expansion can be written as

$$H_a(x) = h_a + \sum_{n=0}^{\infty} c^a_{i_1\ldots i_n} P_{i_1\ldots i_n}(x) \tag{2.20}$$

with $P_{i_1\ldots i_n}(x)$ a totally symmetric and traceless rank n tensor providing a basis of harmonic functions on $\mathbb{R}^3$. The explicit form of $P_{i_1\ldots i_n}(x)$ can be found from the multi-pole expansion

$$\frac{1}{|x+a|} = \sum_{n=0}^{\infty} a_{i_1} \ldots a_{i_n} P_{i_1\ldots i_n}(x) \tag{2.21}$$

For example for $n = 0, 1, 2$ one finds

$$P(x) = \frac{1}{|x|} \qquad P_i(x) = -\frac{x_i}{|x|^3} \qquad P_{ij}(x) = \frac{3x_i x_j - \delta_{ij}|x|^2}{|x|^5} \tag{2.22}$$

and so on. We notice that even if the explicit form of $P_{i_1\ldots i_n}(x)$ can quickly become messy as n grows, the form of the Fourier transform is very simply. Indeed writing

$$P_{i_1\ldots i_n}(x) = \int \frac{d^3k}{(2\pi)^3} e^{ikx} \tilde{P}_{i_1\ldots i_n}(k) \tag{2.23}$$

and using

$$\frac{1}{|x+a|} = \sum_{n=0}^{\infty} a_{i_1} \ldots a_{i_n} P_{i_1\ldots i_n}(x) = 4\pi \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik(x+a)}}{k^2} \tag{2.24}$$

JHEP06(2016)003

one finds that the Fourier transform of the harmonic functions $P_{i_1 \ldots i_n}(x)$ are simply polynomials of the momenta divided by k^2

$$\tilde{P}_{i_1 \ldots i_n}(k) = \frac{4\pi\,(\mathrm{i})^n}{n!\,k^2} k_{i_1} \ldots k_{i_n} \tag{2.25}$$

Disk amplitudes will be conveniently written in this basis. We notice that even if the single harmonic components $P_{i_1 \ldots i_n}(x)$ are singular at the origin, in analogy with the 2-charge case [2, 28], one may expect that for an appropriate choice of the coefficients $c_{i_1 \ldots i_n}$ the infinite sum (2.20) produce a fuzzy and smooth geometry. This was explicitly shown in [25] for a class of solutions parametrized by one complex harmonic function $\mathcal{H} = \mathcal{H}_1 + \mathrm{i}\mathcal{H}_2$ obtained from our general solution by taking

$$\begin{aligned} \mathcal{H}_1 &= L_1 = L_2 = L_3 = V \\ \mathcal{H}_2 &= K_1 = K_2 = K_3 = -M \end{aligned} \tag{2.26}$$

3 String amplitudes and 1-charge microstates

At weak coupling, the gravitational background generated by a D-brane state can be extracted from a disk amplitude involving the insertion of a closed string states in the bulk and open string states specifying the D-brane micro-state on the boundary. The open string state, or condensate, is specified by a boundary operator $\mathcal{O}$ given by a trace along the boundary of the disk of a product of constant open string fields. The profile of a supergravity field Φ in the micro-state geometry can be reconstructed from the string amplitude $\mathcal{A}_{\Phi,\mathcal{O}}(k)$ after Fourier transform in the momentum k of the closed string state. Moreover, since we consider constant open string fields we can take all open string momenta to vanish. For simplicity, we take the only non vanishing components k_i of the closed string momentum to be along the 3 non compact space directions, i.e. $k_0 = k_{y_I} = k_{\tilde{y}_I} = 0$. In order to keep the mass-shell condition $k^2 = 0$, we analytically continue k_i to complex values, and require $k_i^2 \approx 0$. More precisely, the deviation $\delta\tilde{\Phi}(k)$ from flat space of a field $\tilde{\Phi}(k)$ is extracted from the string amplitude via the bulk-to-boundary formula

$$\delta\tilde{\Phi}(k) = \left(-\frac{\mathrm{i}}{k^2}\right) \frac{\delta \mathcal{A}_{\Phi,\mathcal{O}}(k)}{\delta\Phi} \tag{3.1}$$

with $(-\mathrm{i}/k^2)$ denoting the massless closed string propagator for the (transverse) physical modes.

We notice that the limit of zero momenta for open string states is well defined only in the case the disk diagram cannot factorize into two or more disks via the exchange of open string states. Indeed, in presence of factorization channels the string amplitude has poles in the open string momenta and the underlying world-sheet integral diverges. We will always carefullly choose the open string polarisations in such a way that no factorization channels be allowed. As a result, one finds that, up to $1/k^2$, $\delta\tilde{\Phi}(k)$ is a polynomial in k_i and therefore can be expressed as a linear combination of harmonic functions $\tilde{P}_{i_1 \ldots i_n}(k)$ in momentum space

$$\delta\widetilde{\Phi}(k) = \sum_n c^{\Phi}_{i_1 \ldots i_n} \tilde{P}_{i_1 \ldots i_n}(k) \tag{3.2}$$

In this section we illustrate the stringy description of the micro-state geometry for the very familiar case of parallel (wrapped) D3-branes. The vacua of this system are parametrised by the expectation values of the (three) scalar fields ϕ_i, along the non-compact directions. The general open string condensate involves an arbitrary number of insertions of the 'untwisted' field ϕ_i on the boundary of the disk. On the other hand, the associated micro-state geometry is described by a D3-brane solution characterised by a smooth multi-pole harmonic function determined by the open string condensate [29]. The string computation will allow us to fix the relative normalisation between the NS and RR vertices that will be used in the analysis of 2- and 4-charge intersections in the next sections.

3.1 L solutions at weak coupling

We start by considering the supergravity solution generated by a single stack of D3-branes, let us say of type 0. The gravity solution is given by (2.12). At linear order in α_{D3} one finds

$$\begin{aligned}\delta g_{MN} dx^M dx^N &= \frac{\delta L}{2}\left[dt^2 - \sum_{i=1}^{3}(dy_i^2 - dx_i^2 - d\tilde{y}_i^2)\right] + \dots \\ \delta C_4 &= -\delta L \wedge dt \wedge dy_1 \wedge dy_2 \wedge dy_3 + A \wedge d\tilde{y}_1 \wedge d\tilde{y}_2 \wedge d\tilde{y}_3 + \dots \end{aligned} \tag{3.3}$$

with $\delta L = L - 1$ and A both or order α_{D3}. In particular one can take[4]

$$L = 1 + \frac{\alpha_{D3} N_0}{|x|} + \dots \qquad\qquad *_3 dL = dA \tag{3.6}$$

corresponding to N_0 D3 branes on $\mathbb{R}^3$.

3.2 NS-NS amplitude

Let us start by considering a single D3-brane. The moduli space is parametrized in terms of vacuum expectation value of open string scalar fields ϕ_i describing the position of the brane in the transverse space. The associated supergravity solution can be extracted from disk amplitudes involving the insertion of one closed string field and a bunch of open scalar fields ϕ_i at zero momenta.

The open string background can be conveniently encoded in a generating function

$$\xi(\phi) = \sum_{n=0}^{\infty} \xi_{i_1 \dots i_n} \phi^{i_1} \dots \phi^{i_n} \tag{3.7}$$

[4]More generally for an axially symmetric harmonic function

$$L = d_0 + \frac{d_1}{r} + d_2 \frac{\cos\theta}{r^2} + d_3 \frac{3\cos^2\theta - 1}{r^3} \tag{3.4}$$

one finds

$$A = \left(d_1 \cos\theta - d_2 \frac{\sin^2\theta}{r} - d_3 \frac{3\cos\theta \sin^2\theta}{r^2}\right) d\varphi\,. \tag{3.5}$$

The relevant vertex operators are[5]

$$
\begin{aligned}
W_{\mathrm{NS-NS}}(z,\bar z) &= c_{\mathrm{NS}}\,(ER)_{MN}\, e^{-\varphi}\psi^M e^{ikX}(z)\, e^{-\varphi}\psi^N e^{ikRX}(\bar z)\\
V_{\xi(\phi)}(x_a) &= \sum_{n=0}^{\infty} \xi_{i_1\ldots i_n}\,\partial X^{i_1}(x_1)\prod_{a=2}^{n}\int_{-\infty}^{\infty}\frac{dx_a}{2\pi}\,\partial X^{i_a}(x_a)
\end{aligned}
\tag{3.8}
$$

with $E = h + b$ the polarization tensor containing the fluctuation h of the metric and b of the B-field and c_{NS} a normalisation constant. The momentum of the closed string state will be labeled by k. Henceforth we denote by $X(z)$ the holomorphic part of the closed string field $X(z,\bar z) = X(z) + RX(\bar z)$ with R the reflection matrix implementing the boundary conditions on the disk. More precisely, R is a diagonal matrix with plus one (+1) and minus one (-1) along the Neumann and Dirichelet directions, respectively. In particular for a disk of type 0, we have plus along t, y_I and minus along $x_i, \tilde y_I$. Consequently $kR = -k$. We choose length units whereby $\alpha' = 2$. Moreover we exploit $\mathrm{SL}(2,\mathbb{R})$ invariance of the disk to fix the positions of the closed string and the first of the open string insertions. Other open string vertices are integrated along the real line to take into account all possible orderings of the insertions.

The resulting string amplitude can be written as

$$
\mathcal{A}_{\mathrm{NS-NS},\xi(\phi)} = \langle c(z)c(\bar z)c(z_1)\rangle\,\left\langle W_{\mathrm{NS-NS}}(z,\bar z)V_{\xi(\phi)}\right\rangle \tag{3.9}
$$

The basic contributions to the correlators are

$$
\begin{aligned}
\langle c(z)c(\bar z)c(z_1)\rangle &= (z-\bar z)(\bar z - z_1)(z_1 - z)\\
\left\langle e^{\mathrm{i}kX}(z)e^{-\mathrm{i}kX}(\bar z)\,\partial X^{i_a}(x_a)\right\rangle &= \mathrm{i}k^{i_a}\left(\frac{1}{z-x_a}-\frac{1}{\bar z - x_a}\right) = \mathrm{i}k^{i_a}\,\frac{\bar z - z}{|z-x_a|^2}\\
\left\langle e^{-\varphi}\psi^M(z)e^{-\varphi}\psi^N(\bar z)\right\rangle &= \frac{\delta^{MN}}{(z-\bar z)^2}
\end{aligned}
\tag{3.10}
$$

Using

$$
\int_{-\infty}^{\infty} dx_a\frac{(\bar z - z)}{|z-x_a|^2} = -2\pi\mathrm{i} \tag{3.11}
$$

one finds

$$
\mathcal{A}_{\mathrm{NS-NS},\xi(\phi)} = \mathrm{i}\, c_{\mathrm{NS}}\;\mathrm{tr}(ER)\xi(k) \tag{3.12}
$$

The asymptotic deviation from the flat metric can be extracted from

$$
\delta\tilde g_{MN}(k) = \left(-\frac{\mathrm{i}}{k^2}\right)\sum_{n=0}^{\infty}\frac{\delta\mathcal{A}_{\mathrm{NS-NS},\phi^n}}{\delta h_{MN}} = c_{\mathrm{NS}}\frac{\xi(k)}{k^2}\,(\eta R)_{MN} \tag{3.13}
$$

After Fourier transform one finds

$$
\delta g_{MN} = \int\frac{d^3k}{(2\pi)^3}\delta\tilde g_{MN} = -\frac{1}{2}(\eta R)_{MN}\,\delta L(x) \tag{3.14}
$$

[5] Here the correlator is evaluated for generic $n \neq 0$ and the $n = 0$ term is obtained by extrapolation. A direct evaluation of this term is subtler since there are not enough insertions to completely fix the $\mathrm{SL}(2,\mathbb{R})$ world-sheet invariance.

with the identification

$$\delta L(x) = -2\, c_{\rm NS} \int \frac{d^3k}{(2\pi)^3} \frac{\xi(k)}{k^2} e^{ikx} \tag{3.15}$$

We conclude that the function $\xi(\phi)$ codifies the complete multipole expansion of $L(x)$ in terms of vacuum expectation values for untwisted scalar fields

$$\xi_{i_1\ldots i_n} = \langle {\rm tr}\, \phi_{i_1} \ldots \phi_{i_n} \rangle \tag{3.16}$$

In particular, the harmonic function for a single D3-brane at position a, is realized by taking

$$\xi(\phi) \sim e^{{\rm i}\, a\, \phi} \tag{3.17}$$

that reproduces the multipole expansion (2.21) of $L(x)$ according to (3.15). Similar results can be found by considering disks ending on D3-branes associated to the harmonic functions L_I.

We conclude by observing that D- and F-flatness conditions may restrict the choices for $\xi_{i_1\ldots i_n}$. Indeed for pointlike branes, the flatness conditions require $[\phi_i, \phi_j] = 0$, so the matrices ϕ_i can be simultaneously diagonalized and written in terms of N_0 positions in $\mathbb{R}^3$ [29].

3.3 R-R amplitude

Next we consider the R-R string amplitude

$$\mathcal{A}_{{\rm R-R},\xi(\phi)} = \langle c(z) c(\bar z) c(z_1) \rangle \left\langle W_{\rm R-R}(z,\bar z) V_{\xi(\phi)} \right\rangle \tag{3.18}$$

with

$$\begin{aligned} W_{\rm R-R}(z,\bar z) &= c_{\rm R}\, (\mathcal{F}\mathcal{R})_{\Lambda\Sigma}\, e^{-\frac{\varphi}{2}}\, S^\Lambda e^{ikX}(z)\, e^{-\frac{\varphi}{2}}\, S^\Sigma\, e^{ikRX}(\bar z) \\ V_{\xi(\phi)} &= \sum_{n=0}^{\infty} \xi_{i_1\ldots i_n} e^{-\varphi} \psi^{i_1}(z_1) \prod_{a=2}^{n} \int_{-\infty}^{\infty} \frac{dx_a}{2\pi}\, \partial X^{i_a}(x_a) \end{aligned} \tag{3.19}$$

Here

$$S^\Lambda = e^{\frac{\rm i}{2}(\pm\varphi_1 \pm\varphi_2 \ldots \pm\varphi_5)} \qquad \text{with even \# of } -'\, s \tag{3.20}$$

represents a ten-dimensional spin field of positive chirality with $\Lambda = 1, \ldots 16$ and

$$\mathcal{F} = \sum_n \frac{1}{n!} F_{M_1\ldots M_n} \Gamma^{M_1\ldots M_n} \qquad\qquad \mathcal{R} = \Gamma^{t y_1 y_2 y_3} \tag{3.21}$$

where F are the R-R field strengths and $\mathcal{R}$ is the reflection matrix in the spinorial representation.

The basic contributions to the correlators are given by the first two lines of (3.10) and

$$\left\langle e^{-\frac{\varphi}{2}}\, S^\Lambda(z) e^{-\frac{\varphi}{2}}\, S^\Sigma\, (\bar z) e^{-\varphi}\, \psi^M(z_1) \right\rangle = \frac{1}{\sqrt{2}} \frac{(\Gamma^M)^{\Lambda\Sigma}}{(z-\bar z)|z-z_1|^2} \tag{3.22}$$

Altogether one finds

$$\mathcal{A}_{{\rm R-R},\xi(\phi)} = {\rm i}\, \frac{c_{\rm R}}{\sqrt{2}}\, {\rm tr}_{16}(\mathcal{C}\mathcal{R}) \xi(k) = \frac{16\, {\rm i}\, c_{\rm R}}{\sqrt{2}}\, \xi(k) C_{t y_1 y_2 y_3} \tag{3.23}$$

where we wrote $\mathcal{F} = \mathrm{i}k_i\Gamma^i\mathcal{C}$. Here and below we will always restrict ourselves to the 'electric' components $C_{tM_1M_2M_3}$ since the remaining 'magnetic' components are determined by the self-duality of F_5. The R-R fields at the disk level is then determined from the corresponding variation of the string amplitudes

$$\delta\tilde{C}_{ty_1y_2y_3} = \left(-\frac{\mathrm{i}}{k^2}\right)\sum_{n=0}^{\infty}\frac{\delta\mathcal{A}_{\mathrm{R-R},\xi(\phi)}}{\delta C_{ty_1y_2y_3}} = 8\sqrt{2}\,c_{\mathrm{R}}\,\frac{\xi(k)}{k^2} \tag{3.24}$$

Choosing

$$c_{\mathrm{R}} = \frac{c_{\mathrm{NS}}}{4\sqrt{2}} \tag{3.25}$$

and using (3.15) one finds

$$\delta C_{ty_1y_2y_3} = -\delta L(x) \tag{3.26}$$

in agreement with (3.3) for an arbitrary harmonic function $L(x)$ specified by $\xi(k)$ via (3.15).

4 String amplitudes and 2-charge microstates

Next we consider K solutions. At leading order in α_{D3}, K solutions (2.15) reduce to

$$\delta g_{MN}dx^M dx^N = -2\,dt\,w - 2\,K\,dy_3 d\tilde{y}_3 + \ldots \tag{4.1}$$

$$\delta C_4 = (K\,dt\wedge dy_3 - w\wedge d\tilde{y}_3)\wedge(dy_1\wedge d\tilde{y}_2 + d\tilde{y}_1\wedge dy_2) \tag{4.2}$$

with

$$*_3 dw = -dK \tag{4.3}$$

and K a harmonic function starting at order $|x|^{-2}$. For example one can take K to be

$$K = \frac{v_i x_i}{|x|^3} + \ldots \tag{4.4}$$

while[6]

$$w = \epsilon_{ijk}\,v_i\frac{x_j\,dx_k}{|x|^3} + \ldots \tag{4.5}$$

4.1 NS-NS amplitude

In this section we compute the disk amplitude generating the metric of the K solutions. K solutions will be associated to string fermionic bilinears localised at the intersections of two D3-branes. The relevant string amplitude has been computed in [8] for the case of D1-D5-branes. Here we review this computation, adapting it to the case of D3-branes, and include the effect of untwisted open string insertions in order to account for higher multi-pole modes of the harmonic function K.

We consider a non-trivial open string condensate

$$\mathcal{O}^{AB} = \mathrm{tr}\,\bar{\mu}^{(A}\mu^{B)}\,\xi(\phi) \tag{4.6}$$

[6]For instance, for $v_1 = v_2 = 0$, $v_3 \neq 0$ one has $w = v_3(x_1\,dx_2 - x_2\,dx_1)/|x|^3$.

with $\bar{\mu}^A$ a fermionic excitation of the open string starting from $D3_0$ and ending on $D3_3$ and μ^B a fermion excitation of the open string with opposite orientation. Here and below we denote with trace the sum over Chan-Paton indices along the boundary of the disk. We organise states in representations of the $SO(1,5)$ Lorentz symmetry rotating the six-dimensional hyper-plane along which the two stacks of branes are NN or DD, i.e. the space-time and the $y_3, \tilde{y}_3$ directions. Upper (lower) indices $A = 1, \ldots 4$ and $M = 1, \ldots 6$ run over the right (left) spinor and vector representations of this group, respectively. The projection onto the symmetric part is required by the irreducibility of the string diagram since the anti-symmetric part of the fermionic bilinear produces a scalar field $\phi_{[AB]}$ and consequently a factorisation channel.

The relevant disk amplitude can be written as

$$\mathcal{A}^{\mathrm{NS-NS}}_{\mu^2,\xi(\phi)} = \int dz_4 \langle c(z_1)\, c(z_2)\, c(z_3)\rangle \left\langle V_{\bar{\mu}}(z_1)\, V_\mu(z_2)\, W(z_3,z_4)\, V_{\xi(\phi)}\right\rangle \tag{4.7}$$

with

$$\begin{aligned} V_{\bar{\mu}}(z_1) &= \bar{\mu}^A\, e^{-\varphi/2}\, S_A \sigma_2\sigma_3 \\ V_\mu(z_2) &= \mu^B\, e^{-\varphi/2}\, S_B \sigma_2\sigma_3 \\ V_{\xi(\phi)} &= \sum_{n=0}^{\infty} \xi_{i_1\ldots i_n} \prod_{a=1}^{n} \int_{-\infty}^{\infty} \frac{dx_a}{2\pi}\, \partial X^{i_a}(x_a) \\ W(z_3,z_4) &= c_{\mathrm{NS}}\,(ER)_{MN} e^{-\varphi}\psi^M\, e^{\mathrm{i}kX}(z_3)(\partial X^N + i\, k\psi\, \psi^N) e^{-\mathrm{i}kX}(z_4) \end{aligned} \tag{4.8}$$

Here σ_I denotes the $\mathbb{Z}_2$-twist field along the I^{th} T^2 inside T^6 with conformal dimension $1/8$ and

$$S_A = e^{\pm\frac{1}{2}(\mathrm{i}\varphi_3 \pm \mathrm{i}\varphi_4 \pm \mathrm{i}\varphi_5)} \qquad \text{even number } -'\mathrm{s} \tag{4.9}$$

the spin field on the six-dimensional plane along which D3-branes are NN or DD.

To evaluate the correlator one can consider a specific component and then use $SO(6)$ invariance to reconstruct its covariant form. For instance, if we take $A = B = \frac{1}{2}(+++)$, the open string condensate contributes a net charge $+2$ along the first three complex directions, so that only the $\psi{:}\psi\psi{:}$ term can contribute to the correlator. The relevant correlators are

$$\begin{aligned} dz_4\langle c(z_1)\, c(z_2)\, c(z_3)\rangle &= dz_4\, z_{12}\, z_{23}\, z_{31} = dw\,(z_{14}z_{23})^2 \\ \left\langle e^{ikX}(z_3) e^{-ikX}(z_4)\, V_{\xi(\phi)}\right\rangle &= \xi(k) \\ \left\langle e^{-\varphi/2}(z_1) e^{-\varphi/2}(z_2) e^{-\varphi}(z_3)\right\rangle &= z_{12}^{-1/4}(z_{13}z_{23})^{-1/2} \\ \langle \sigma_2\sigma_3(z_1)\sigma_2\sigma_3(z_2)\rangle &= z_{12}^{-1/2} \\ \left\langle S_A(z_1) S_B(z_2) \psi^M(z_3)\psi^N\psi^P(z_4)\right\rangle &= \frac{\Gamma^{MNP}_{AB}\, z_{12}^{3/4}}{2\sqrt{2}(z_{13}z_{23})^{1/2} z_{14}z_{24}} \end{aligned} \tag{4.10}$$

with $w = \frac{z_{13}z_{24}}{z_{14}z_{23}}$. We notice that the scalar fields factorize from the rest, since they are always oriented along the non compact space directions, while twist fields are along T^6.

The z_i dependence boils down to the worldsheet integral

$$\mathcal{I} = \int_\gamma \frac{dw}{w} = 2\pi i \tag{4.11}$$

where γ is a small contour around the origin. On the other hand the open string condensate can be written as

$$\left\langle \mathrm{tr}\, \bar{\mu}^{(A}\mu^{B)} \right\rangle = \frac{c_{\mathcal{O}}}{3!}\, v_{MNP}\, (\Gamma^{MNP})^{AB} \tag{4.12}$$

with v_{MNP} a self-dual tensor in six-dimensions and $c_{\mathcal{O}}$ a normalisation constant. Combining (4.12) with (4.10) and taking

$$c_{\mathcal{O}} = \frac{1}{\sqrt{2}\,\mathcal{I}\, c_{\mathrm{NS}}} \tag{4.13}$$

one finds

$$\mathcal{A}^{\mathrm{NS-NS}}_{\mu^2,\xi(\phi)} = \frac{1}{3!}\,(ER)_{MN} k_P\, v^{MNP}\, \xi(k) \tag{4.14}$$

As before, the factor $\xi(k)$ comes from untwisted insertions and is associated to higher multi-pole modes of the underlying harmonic function, so it is enough to match the leading term, i.e. we set $\xi(k) = 1$. Assuming a v_{IJK} of the form

$$v_{y_3\tilde{y}_3 3} = -v_{12t} = 4\pi\, v \tag{4.15}$$

and using (4.14) one finds

$$\delta\tilde{g}_{2t} = -4\pi v\, k_1 \qquad \delta\tilde{g}_{1t} = 4\pi v\, k_2 \qquad \delta\tilde{g}_{y_3\tilde{y}_3} = -4\pi v\, k_3 \tag{4.16}$$

After Fourier transform one finds

$$\delta g_{2t} = -v\,\frac{x_1}{|x|^3} \qquad \delta g_{1t} = v\,\frac{x_2}{|x|^3} \qquad \delta g_{y_3\tilde{y}_3} = -v\,\frac{x_3}{|x|^3} \tag{4.17}$$

in agreement with (4.1) for the choice (4.4) with $v_i = \delta_{i3}\, v$. We conclude that the harmonic function K_3 describes a particular component of the fermion bilinear condensates connecting two branes parallel along the $(y_3, \tilde{y}_3)$-plane. Similarly $K_{1,2}$ represents fermion bilinear condensates connecting branes parallel along the 1 and 2 planes (tori).

We conclude by noticing that other choices for the 2-charge condensate give rise to solutions that do not belong to the 'eight-ful' harmonic family (2.5). In particular, taking non-zero values of $v_{ty_3\tilde{y}_3}$, $v_{ty_3 i}$ or $v_{t\tilde{y}_3 i}$ give rise to solutions with non-trivial b_{ij}, $g_{y_3 t}$ or $b_{\tilde{y}_3 t}$ components, respectively. These components can be matched against a more general 2-charge solution obtained from [7] after two T-dualities. We refer the reader to [8] for a detailed match in the T-dual D1-D5 description.

4.2 R-R amplitude

Next we consider the disk amplitude with the insertion of a R-R vertex operator

$$\mathcal{A}^{\mathrm{R-R}}_{\mu^2,\xi(\phi)} = \int dz_4 \langle c(z_1)\, c(z_2)\, c(z_3)\rangle \left\langle V_{\bar{\mu}}(z_1) V_\mu(z_2) W_{\mathrm{R-R}}(z_3, z_4) V_{\xi(\phi)} \right\rangle \tag{4.18}$$

with

$$
\begin{aligned}
W_{\mathrm{R-R}}(z_3, z_4) &= c_{\mathrm{R}}\, (\mathcal{FR})_{CD} (e^{-\frac{\varphi}{2}} C^C C^{\dot\alpha} e^{\mathrm{i}kX})(z_3)(e^{-\frac{\varphi}{2}} C^D C_{\dot\alpha} e^{-\mathrm{i}kX})(z_4) \\
C^A &= e^{\pm\frac{1}{2}(\mathrm{i}\varphi_3 \pm \mathrm{i}\varphi_4 \pm \mathrm{i}\varphi_5)} \qquad \text{odd number } -' \text{s} && (4.19)\\
C_{\dot\alpha} &= e^{\pm\frac{\mathrm{i}}{2}(\varphi_4 - \varphi_5)} && (4.20)
\end{aligned}
$$

The ghost and untwisted bosonic correlators are given again by the first two lines in (4.10) while the fermionic and twist-field correlators are

$$
\begin{aligned}
\langle S_{(A}(z_1) S_{B)}(z_2) C^C(z_3) C^D(z_4)\rangle &= \delta^C_{(A}\delta^D_{B)} \left(\frac{z_{12} z_{34}}{z_{13} z_{14} z_{23} z_{24}}\right)^{3/4} \\
\langle \sigma_2\sigma_3(z_1)\sigma_2\sigma_3(z_2)\rangle &= z_{12}^{-1/2} \\
\langle \prod_{i=1}^{4} e^{-\varphi/2}(z_i)\rangle &= \prod_{i<j}^{4} z_{ij}^{-1/4} \\
\langle C^{\dot\alpha}(z_3) C_{\dot\alpha}(z_4)\rangle &= 2\, z_{34}^{-1/2} && (4.21)
\end{aligned}
$$

Again one finds that the amplitude is proportional to $\mathcal{I}$ given in (4.11) and can be written in the form

$$
\mathcal{A}^{\mathrm{R-R}}_{\mu^2,\xi(\phi)} = \frac{1}{3!\,4}\, v_{MNP}\, \mathrm{tr}_4 \left(\mathcal{FR}\Gamma^{MNP}\right) \xi(k) \tag{4.22}
$$

where we used (4.12), (4.13) and (3.25). Taking $\xi(k) = 1$, specializing to the condensate in (4.15) and focussing on the $C_{tM_1M_2M_3}$ components one finds

$$
\mathcal{A}^{\mathrm{R-R}}_{\mu^2,\xi(\phi)} = 4\pi v\, (F_{t3y_1\tilde{y}_2y_3} + F_{t3\tilde{y}_1y_2y_3}) \tag{4.23}
$$

Varying with respect to the four-form potential one finds

$$
\delta\tilde{C}_{ty_1\tilde{y}_2y_3} = \delta\tilde{C}_{t\tilde{y}_1y_2y_3} = k_3\, 4\pi v \tag{4.24}
$$

reproducing the Fourier transform of (4.1) for the choice (4.4) with $v_i = \delta_{i3}\, v$.

5 String amplitudes and 4-charge microstates

Finally we consider M solutions. At leading order in α_{D3}, M solutions (2.18) reduce to

$$
\begin{aligned}
\delta g_{MN} dx^M dx^N &= 2M \left(dy_1\, d\tilde{y}_1 + dy_3\, d\tilde{y}_3\right) + \ldots \\
\delta C_4 &= -M\, dt \wedge (dy_1 \wedge d\tilde{y}_2 \wedge dy_3 + d\tilde{y}_1 \wedge d\tilde{y}_2 \wedge d\tilde{y}_3) \\
&\quad + w_2 \wedge (dy_1 \wedge dy_2 \wedge dy_3 + d\tilde{y}_1 \wedge dy_2 \wedge d\tilde{y}_3) + \ldots && (5.1)
\end{aligned}
$$

with M a harmonic function starting at order $|x|^{-3}$. For example one can take M to be of the form

$$
M = v_{ij} \frac{3\, x_i\, x_j - \delta_{ij}|x|^2}{|x|^5} + \ldots \tag{5.2}
$$

5.1 NS-NS amplitude

We consider now string amplitudes on a disk with boundary on all four types of D3-branes. In particular we consider the insertions of four fermions μ_a starting on a D3-brane of type (a) and ending on a D3-branes of type $(a+1)$ with $a = 0,1,2,3$ (mod 4), in a cyclic order. We notice that unlike the case of two boundaries now the condensate is complex. Indeed, even if each intersection preserves $\mathcal{N} = 2$ SUSY (1/4 BPS), so that each fermion μ_a comes together with its charge conjugate $\bar{\mu}_a$, the overall configuration preserves only $\mathcal{N} = 1$ SUSY (1/8 BPS) and fermions connecting all four type of D3 brane form two pairs of opposite chirality. The charge conjugate condensate can be defined by replacing each μ_a with its charge conjugate field $\bar{\mu}_a$ and running along the boundary of the disk with the same cyclic order but in reversed sense. The real and imaginary parts of the string amplitude can be selected by turning on the real and imaginary parts of the condensate respectively.

We consider the following NS-NS amplitude

$$\begin{aligned}\mathcal{A}^{\rm NS-NS}_{\mu^4,\xi(\phi)} = \langle c(z_1)\, c(z_2)\, c(z_4)\rangle \int dz_3 dz_5 dz_6 \\ \left\langle V_{\mu_1}(z_1)V_{\mu_2}(z_2)V_{\mu_3}(z_3)V_{\mu_4}(z_4)W_{NSNS}(z_5,z_6)V_{\xi(\phi)}\right\rangle\end{aligned} \tag{5.3}$$

with

$$\begin{aligned}V_{\mu_1}(z_1) &= \mu_1^{\alpha}\, e^{-\varphi/2}\, S_\alpha S_1\, \sigma_2\sigma_3(z_1) \\ V_{\mu_2}(z_2) &= \mu_2^{\beta}\, e^{-\varphi/2}\, S_\beta S_3 \sigma_1\sigma_2(z_2) \\ V_{\mu_3}(z_3) &= \mu_3^{\dot\alpha}\, e^{-\varphi/2}\, C_{\dot\alpha}\bar{S}_1\sigma_2\sigma_3(z_3) \\ V_{\mu_4}(z_4) &= \mu_4^{\dot\beta}\, e^{-\varphi/2}\, C_{\dot\beta}\bar{S}_3\sigma_1\sigma_2(z_4) \\ V_{\xi(\phi)} &= \sum_{n=0}^{\infty} \xi_{i_1\ldots i_n}\prod_{a=1}^{n}\int_{-\infty}^{\infty}\frac{dx_a}{2\pi}\,\partial X^{i_a}(x_a) \\ W_{\rm NSNS}(z_5,z_6) &= c_{\rm NS}(ER)_{MN}(\partial X^M - {\rm i}k\cdot\psi\psi^M)e^{{\rm i}kX}(z_5)(\partial X^N + {\rm i}k\cdot\psi\psi^N)e^{-{\rm i}kX}(z_6)\end{aligned} \tag{5.4}$$

We use the notation

$$S_I = e^{\frac{{\rm i}\varphi_I}{2}} \qquad \bar{S}_I = e^{-\frac{{\rm i}\varphi_I}{2}} \qquad S_\alpha = e^{\pm\frac{\rm i}{2}(\varphi_4+\varphi_5)} \qquad C_{\dot\alpha} = e^{\pm\frac{\rm i}{2}(\varphi_4-\varphi_5)} \tag{5.5}$$

for internal and spacetime spin fields. The condensate is now the tensor

$$\mathcal{O}^{\alpha\beta\dot\alpha\dot\beta} = {\rm tr}\,\mu_1^{(\alpha}\,\mu_2^{\beta)}\,\bar{\mu}_3^{(\dot\alpha}\,\bar{\mu}_4^{\dot\beta)}\,\xi(\phi) \tag{5.6}$$

where the sum over all the Chan-Paton indices $\mu^{1\ i_2}_{i_1}\mu^{2\ i_3}_{i_2}\bar{\mu}^{3\ i_4}_{i_3}\bar{\mu}^{4\ i_1}_{i_4}$ is understood. As before we discard components proportional to $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot\alpha\dot\beta}$ to ensure the irreducibility of the string diagram. As a result $\mathcal{O}^{\alpha\beta\dot\alpha\dot\beta}$ transforms in the $(\mathbf{3},\mathbf{3})$ of the ${\rm SU}(2)_L\times{\rm SU}(2)_R$ Lorentz group and can be better viewed as a symmetric and traceless tensor $v^{\mu\nu}$. We notice that even though the presence of the four different D3-branes breaks the Lorentz symmetry down to SO(4), the string amplitude is invariant under ${\rm SO}(4)\times{\rm U}(1)^3$ since the ${\rm U}(1)^3$ rotations of the three internal tori do not change the relative angles between branes. We notice also that space-time chirality determines the internal $\mathrm{U}(1)^3$ charges of the twisted fermions. In particular μ_1^α and $\bar{\mu}_3^{\dot{\alpha}}$ have opposite charges with respect to the first U(1) while μ_2^α and $\bar{\mu}_4^{\dot{\alpha}}$ have opposite charges with respect to the third internal U(1). Therefore the open string condensate $v_{\mu\nu}$ carries no $\mathrm{U}(1)^3$ charges.

To evaluate the correlator we specialize to $\alpha = \beta = \frac{1}{2}(++)$ and $\dot{\alpha} = \dot{\beta} = \frac{1}{2}(-+)$. Only the four fermion piece of the closed string vertex can compensate for the net $+2$ charge of the open string condensate and two of these fermions have to be taken along the fifth plane. In addition since the net internal $\mathrm{U}(1)^3$ charge of the condensate is zero, the only choices for the $\psi^M \psi^N$ fermions in the closed string vertex are $(M,N) = (I, \bar{I})$ with $I = 1,2,3,4$

For example, taking $(M,N) = (1,\bar{1})$ the relevant correlators are

$$
\begin{aligned}
\langle c(z_1)\, c(z_2)\, c(z_4)\rangle &= z_{12}\, z_{24}\, z_{41} \\
\left\langle e^{ikX}(z_3) e^{-ikX}(z_4)\, V_{\xi(\phi)}\right\rangle &= \xi(k) \\
\langle \sigma_3(z_1)\sigma_1(z_2)\sigma_3(z_3)\sigma_1(z_4)\rangle &= (z_{13} z_{24})^{-1/4} \\
\left\langle \prod_{j=1}^{4} e^{-\varphi/2}(z_j) \right\rangle &= \prod_{i<j} z_{ij}^{-1/4} \\
\left\langle S_1(z_1)\bar{S}_1(z_3)\psi^1(z_5)\psi^{\bar{1}}(z_6)\right\rangle \langle S_3(z_2)\bar{S}_3(z_4)\rangle &= \frac{1}{2} z_{13}^{-\frac{1}{4}} z_{56}^{-1} \left(\frac{z_{15} z_{36}}{z_{16} z_{35}}\right)^{1/2} (z_{24})^{-1/4} \\
\langle \sigma_2(z_1)\sigma_2(z_2)\sigma_2(z_3)\sigma_2(z_4)\rangle &= f\left(\frac{z_{14} z_{23}}{z_{13} z_{24}}\right) \left(\frac{z_{13} z_{24}}{z_{12} z_{23} z_{34} z_{41}}\right)^{1/4} \\
\left\langle S_{(\alpha}(z_1) S_{\beta)}(z_2) C_{(\dot{\alpha}}(z_3) C_{\dot{\beta})}(z_4) \psi^\mu(z_5)\psi^\nu(z_6)\right\rangle &= \frac{(z_{12} z_{34})^{1/2} z_{56}}{2\prod_{i=1}^4 (z_{i5} z_{i6})^{1/2}} \sigma^{(\mu}_{\alpha\dot{\alpha}} \sigma^{\nu)}_{\beta\dot{\beta}}
\end{aligned}
\tag{5.7}
$$

with $f(x)$ the four twist correlator [30–32]

$$
f(x) = \frac{\Lambda(x)}{(F(x)F(1-x))^{1/2}} \tag{5.8}
$$

where $F(x) = {}_2F_1(1/2, 1/2; 1; x)$ is a hypergeometric function[7] and

$$
\Lambda(x) = \sum_{n_1, n_2} e^{-\frac{2\pi}{\alpha'}\left[\frac{F(1-x)}{F(x)} n_1^2 R_1^2 + \frac{F(x)}{F(1-x)} n_2^2 R_2^2\right]} \tag{5.9}
$$

accounts for the classical contribution associated to world-sheet instantons.[8] Assembling all pieces together and taking

$$
z_1 = -\infty \qquad z_2 = 0 \qquad z_3 = x \qquad z_4 = 1 \qquad z_5 = z \qquad z_6 = \bar{z} \tag{5.10}
$$

one finds that the string amplitude is proportional to the integral

$$
I_1 = \int_0^1 dx\, f(x)\, \mathcal{I}_1(x) \tag{5.11}
$$

[7] Equivalently, $F(x) = \vartheta_2(it(x))/\vartheta_3(it(x))$ with $t(x) = F(1-x)/F(x)$.

[8] In the supergravity limit $\sqrt{\alpha'} \ll R_{1,2}$, worldsheet instantons are exponentially suppressed and one can simply take $\Lambda(x) \to 1$.

with

$$\mathcal{I}_1(x) = \int_{\mathbb{C}^+} \frac{d^2 z}{|z(1-\bar{z})|(x-z)} \tag{5.12}$$

Similarly for other choices of $(M,N) = (I,\bar{I})$, $I = 2,3,4$, one finds integrals of the form (5.11) with $\mathcal{I}_1$ replaced by

$$\begin{aligned}
\mathcal{I}_2(x) &= \int_{\mathbb{C}^+} \frac{d^2 z}{|z(1-\bar{z})(x-z)|} \\
\mathcal{I}_3(x) &= \int_{\mathbb{C}^+} \frac{d^2 z}{\bar{z}(1-z)|x-z|} \\
\mathcal{I}_4(x) &= \int_{\mathbb{C}^+} \frac{d^2 z}{\bar{z}(1-z)(x-z)}
\end{aligned} \tag{5.13}$$

The integrals (5.11), (5.12), (5.13) can be computed numerically for arbitrary values of the radii and one always finds a finite result, with $\mathcal{I}_2, \mathcal{I}_4$ real and $\mathcal{I}_1, \mathcal{I}_3$ purely imaginary.[9] Moreover the following relation holds

$$\mathcal{I}_1 = \mathcal{I}_3 \tag{5.14}$$

Noticing that real and imaginary parts of the amplitude contribute to symmetric $(ER)_{(MN)}$ and antisymmetric parts $(ER)_{[MN]}$ we conclude that a purely imaginary v^{ij} generates the string amplitude

$$\mathcal{A}^{\mathrm{NS-NS}}_{\mu^4,\xi(\phi)} = \left[(ER)_{[1\bar{1}]} + (ER)_{[3\bar{3}]}\right] k_i k_j\, v^{ij}\, \xi(k) \tag{5.15}$$

with

$$\left\langle \mathrm{tr}\, \mu_1^{(\alpha}\, \mu_2^{\beta)}\, \bar{\mu}_3^{(\dot{\alpha}}\, \bar{\mu}_4^{\dot{\beta})} \right\rangle = \frac{2\pi v^{ij}}{c_{\mathrm{NS}}\, \mathcal{I}_1}\, \sigma_i^{\alpha\dot{\alpha}} \bar{\sigma}_j^{\beta\dot{\beta}} \tag{5.16}$$

We notice that $(ER)_{[1\bar{1}]}$ and $(ER)_{[3\bar{3}]}$ always involve a Neumann and a Dirichlet direction, so only the metric contributes to the antisymmetric part of the matrix. Varying with respect to the symmetric part h_{ij} of the polarization tensor $E = h + b$ one finds

$$\delta\tilde{g}_{1\bar{1}} = \delta\tilde{g}_{3\bar{3}} = -2\pi \mathrm{i}\, v^{ij}\, \frac{k_i k_j}{k^2}\, \xi(k) \tag{5.17}$$

that reproduces (5.1) after Fourier transform.

We conclude this section by noticing that there other choices for the 4-charge condensate besides (5.6) lead to solutions that go beyond the eight harmonic function family (2.5). On the one hand one can consider different orderings of the branes along the disk. These condensates lead again to solutions of type M but with internal coordinates that are permuted among one another. Notice that some of these solutions can look very different from each other since L_I and V do not enter in a symmetric fashion. On the other hand, turning on the real part of the condensate (5.6) will produce solutions with a non-trivial B-field given in terms of the real worldsheet integrals $\mathcal{I}_2$ and $\mathcal{I}_4$. Finally turning one can consider

[9]Similar integrals appear in [33].

condensates of type[10]

$$\widetilde{\mathcal{O}}^{\alpha\dot\alpha\beta\dot\beta} = \mathrm{tr}\, \mu_1^{(\alpha}\, \bar\mu_2^{(\dot\alpha}\, \mu_3^{\beta)}\, \bar\mu_4^{\dot\beta)}$$
$$\hat{\mathcal{O}}^{(\alpha\beta\gamma)\dot\beta} = \mathrm{tr}\, \mu_1^{(\alpha}\, \mu_2^{\beta}\, \mu_3^{\gamma)}\, \bar\mu_4^{\dot\beta} \tag{5.18}$$

These condensates generate a new class of solutions with off-diagonal terms mixing either two different T^2's inside T^6 or space-time and internal components. We leave it as an open question to establish whether or not an extended family of supergravity solution accounting for these brane excitations can be explicitly written.

5.2 R-R amplitude

Finally we consider the amplitude with the insertion of a R-R vertex operator

$$\mathcal{A}^{\mathrm{R-R}}_{\mu^4,\xi(\phi)} = \langle c(z_1)\, c(z_2)\, c(z_4)\rangle \int dz_3 dz_5 dz_6 \tag{5.19}$$
$$\left\langle V_{\mu_1}(z_1) V_{\mu_2}(z_2) V_{\mu_3}(z_3) V_{\mu_4}(z_4) W_{\mathrm{R-R}}(z_5,z_6) V_{\xi(\phi)}\right\rangle$$

with

$$W^{(-1/2,+1/2)}_{\mathrm{R-R}}(z_5,z_6) = c_{\mathrm{R}}\, (\mathcal{F}\mathcal{R}\Gamma_M)_\Lambda{}^\Sigma\, e^{-\frac{\varphi}{2}} S^\Lambda\, e^{ikX}(z_5) e^{\frac{\varphi}{2}} (\partial X^M + ik\psi\psi^M) C_\Sigma\, e^{-ikX}(z_6)$$

the R-R vertex operator in the asymmetric (-1/2,+1/2) super-ghost picture.[11] Notice that $\Psi^M_\Sigma(k) = {:}k\psi\psi^M C_\Sigma{:}$ is a world-sheet primary field of dimension 13/8, belonging to the **128** of $SO(1,9)$, with C_Σ the spin field of negative chirality. Again taking $\alpha = \beta = \frac{1}{2}(++)$, $\dot\alpha = \dot\beta = \frac{1}{2}(-+)$, it is easy to see that the net charge -2 of the condensate can be only compensated by the $k\psi\psi^M$ term in the closed string vertex with S^Λ and C_Σ both carrying charge $-\frac{1}{2}$ along the fifth direction. This leads to eight choices for S^Λ. On the other hand, once S^Λ is chosen, the charges of $\Psi^M_\Sigma(k) = {:}k\psi\psi^M C_\Sigma{:}$ are completely determined by charge conservation. For example, for the choice

$$S^\Lambda = e^{\frac{i}{2}(\varphi_1+\varphi_2-\varphi_3+\varphi_4-\varphi_5)}$$
$$k\psi\, \psi^M\, C_\Sigma = k_{\bar 5}\psi^{\bar 5}\psi^{\bar 1} e^{\frac{i}{2}(\varphi_1-\varphi_2+\varphi_3-\varphi_4-\varphi_5)} \tag{5.20}$$

one finds again the first four lines in (5.7), while the last four lines are replaced by

$$\langle \prod_{i=1}^{5} e^{-\frac{\varphi}{2}}(z_i) e^{\frac{\varphi}{2}}(z_6)\rangle = \prod_{j=1}^{5} z_{j6}^{\frac{1}{4}} \prod_{i<j}^{5} z_{ij}^{-\frac{1}{4}} \tag{5.21}$$

$$\left\langle S_1(z_1) S_3(z_2) \bar S_1(z_3) \bar S_3(z_4) S_1 S_2 \bar S_3(z_5) \bar S_1 \bar S_2 S_3(z_6)\right\rangle = \left(\frac{z_{15} z_{45} z_{26} z_{36}}{z_{13} z_{24} z_{25} z_{35} z_{16} z_{46} z_{56}^3}\right)^{\frac{1}{4}}$$

$$\left\langle S_{(\alpha}(z_1) S_{\beta)}(z_2) C_{(\dot\alpha}(z_3) C_{\dot\beta)}(z_4) C_{\dot\gamma}(z_5) \psi^\mu S_\gamma(z_6)\right\rangle = \frac{1}{\sqrt{2}} \left(\frac{z_{12} z_{34} z_{56}}{z_{35} z_{45} z_{36} z_{46} z_{16}^2 z_{26}^2}\right)^{\frac{1}{2}} \epsilon_{\gamma(\alpha} \sigma^\mu_{\beta)(\dot\beta} \epsilon_{\dot\alpha)\dot\gamma}$$

[10] The open string condensate $\check{\mathcal{O}}^{(\alpha\beta\gamma\delta)}$ does not couple to massless closed string states.

[11] Similarly to the 'canonical' (-1/2,-1/2) picture, thanks to the extra Γ^M, $W^{(-1/2,+1/2)}_{\mathrm{R-R}}$ can be expressed in terms of field-strengths F_n solely, unlike in the (-3/2,-1/2) picture that involves the potentials C_{n-1} [34].

JHEP06(2016)003

	$\mathcal{I}_1$	$\mathcal{I}_2$	$\mathcal{I}_3$	$\mathcal{I}_4$
Λ	$\frac{1}{2}(++-+-)$	$\frac{1}{2}(+++--)$	$\frac{1}{2}(-+++-)$	$\frac{1}{2}(+-++-)$
	$\bar{\mathcal{I}}_1$	$\bar{\mathcal{I}}_2$	$\bar{\mathcal{I}}_3$	$\bar{\mathcal{I}}_4$
Λ	$\frac{1}{2}(--+--)$	$\frac{1}{2}(---+-)$	$\frac{1}{2}(+----)$	$\frac{1}{2}(-+---)$

Table 2. Contributions to the string correlator of the various spinor components.

Combining the various correlators and taking

$$z_1 = -\infty \qquad z_2 = 0 \qquad z_3 = x \qquad z_4 = 1 \qquad z_5 = z \qquad z_6 = \bar{z} \tag{5.22}$$

one finds that the string amplitude is again given by the integral (5.11) involving $\mathcal{I}_1$. A similar analysis can be performed for other choices of spin-field components (labelled by Λ, Σ) leading to similar answers in terms of the four characteristic integrals $\mathcal{I}_1, \dots \mathcal{I}_4$ on $\mathbb{C}$. The results are summarised in table 2. Since $\mathcal{I}_2$, $\mathcal{I}_4$ are real and $\mathcal{I}_1, \mathcal{I}_3$ purely imaginary, a purely imaginary condensate selects the $\mathcal{I}_1$, $\mathcal{I}_3$ and $\bar{\mathcal{I}}_1$, $\bar{\mathcal{I}}_3$ components. The resulting string amplitude can then be written as

$$\mathcal{A}^{RR}_{\mu^4,\xi(\phi)} = \frac{1}{4}\mathrm{tr}_{16}(\mathcal{F}\,\mathcal{R}\,\mathcal{P}\,\Gamma^i) 2\pi v_{ij} k_j\, \xi(k) \tag{5.23}$$

with

$$\mathcal{P} = \frac{1}{2}(1 - \Gamma^{y_1\tilde{y}_1 y_3\tilde{y}_3})\Gamma^{y_2\tilde{y}_2} \tag{5.24}$$

a projector on $\mathcal{I}_1$, $\mathcal{I}_3$ and $\bar{\mathcal{I}}_1$, $\bar{\mathcal{I}}_3$ components with $+$i and $-$i eigenvalues respectively. These are precisely the eigenvalues of the matrix $\mathcal{P}$ justifying our claim. Using $\mathcal{R} = \Gamma^{ty_1y_2y_3}$ one finds

$$\mathcal{A}^{RR}_{\mu^4,\xi(\phi)} = 2\pi \mathrm{i}\, k_i v_{ij} k_j\, \xi(k)\, (C_{ty_1\tilde{y}_2 y_3} + C_{t\tilde{y}_1\tilde{y}_2\tilde{y}_3}) \tag{5.25}$$

Taking derivatives with respect to C_4 one finds agreement with (5.1) after Fourier transform.

6 Conclusions and outlook

We have provided a direct link between open string condensates and a large class of micro-state geometries associated to four-dimensional BPS black holes consisting of D3-branes intersecting on T^6. For specific choices of the condensates, we match the order g_s of the micro-state solutions (2.5) against disk amplitudes involving a NS-NS or R-R closed string state and zero, two or four twisted open string fermions besides an arbitrary number of untwisted scalars. Each mixed disk is associated to a harmonic function that for the class of supergravity solutions under consideration here can be written as a linear combination of eight harmonic functions $H_a = \{L_I, V, K_I, M\}$. The eight functions describe the distribution of single, 2- and 4-intersections in the $\mathbb{R}^3$ non compact space directions.

The M class of micro-states is particularly interesting since the number of disks with four different boundaries grows with the product of the four charges $N = Q_1\, Q_2\, Q_3\, Q_4$. In a first approximation, the function M can then be viewed as an N-center harmonic function with moduli space spanned by the N positions on $\mathbb{R}^3$ up to S_N-permutations. One may then expect that he number of micro-states in this class grow as $e^{2\pi\sqrt{N/6}}$, the number of conjugacy classes of S_N or equivalently the partitions of N, at large N. This suggests that the 4-charge micro-states identified in this work may account for a large number of states contributing to the (putative) black hole entropy. A quantitative study of this issue remains one of the most challenging and exciting questions left open by our work.

Although we have worked mostly with D3-branes that are either parallel (along some directions) or perpendicular (along other directions), our analysis can easily be adapted to branes intersecting at arbitrary angles and preserving the same amount of supersymmetry (1/8 of the original 32 super-charges) [35, 36] or branes intersecting in orbifolds [37, 38]. The open strings living at the intersections will be described in terms of twist-fields [39–41] that generalise the $\mathbf{Z}_2$ twist fields that appeared in the present analysis. We have explicitly checked that one-, two- and four- open string insertions source for classical supergravity profiles similar to those obtained in this paper. We defer a detailed analysis to future work. For the time being we would like to stress that off-diagonal components of the internal metric can mimic the effect of tilting the branes thus changing their intersection angles. Furthermore, it is amusing to observe that the 5 harmonic functions that appear in the static solutions found in [35–38] can be identified with V, L_I and M. This is consistent with the fact that setting $K_I = 0$ produces micro-state geometries with no angular momentum.

Our solution generating technique have useful physical applications, both for understanding the physics of multi-center extremal solutions and for constructing new smooth micro-states geometries for under-rotating extremal black holes in four and higher dimensions. Indeed, in addition to reproducing all known supergravity solutions in the 'eightful' harmonic family for certain choices of the condensates, we have shown that other choices are possible that can generate new 4-charge micro-state geometries beyond the ones available in the literature.

Acknowledgments

We are particular grateful to S. Giusto, S. Mathur and R Russo for illuminating discussions and comments on the manuscript. We would also like to thank A. Addazi, M. Bertolini, R. Emparan, F. Fucito, A. Guerrieri, G. Horowitz, O. Lunin, L. Martucci, I. Pesando, Ya. Stanev, M. Trigiante, C.-K. Wen for interesting discussions.

A Conventions

The basic bosonic and ghost correlators are ($\alpha' = 2$)

$$\begin{aligned}
\langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w})\rangle &= -\eta^{\mu\nu} \log|z-w|^2 \\
\langle \partial X^M(x) e^{ik_L X_L}(z) e^{ik_R X_R}(\bar{z})\rangle &= \frac{\mathrm{i}\, k_L^M}{z-x} + \frac{\mathrm{i}\, k_R^M}{\bar{z}-x} \\
\langle e^{q_1\varphi}(z) e^{q_2\varphi}(w)\rangle &= (z-w)^{-q_1 q_2} \\
\langle c(z) c(w)\rangle &= (z-w)
\end{aligned} \tag{A.1}$$

Fermions are bosonized according to

$$\Psi^I = e^{i\varphi_I} = \frac{1}{\sqrt{2}}\left(\psi_{2I-1} + i\psi_{2I}\right) \quad \Rightarrow \quad \Psi_M\, S^\Lambda \sim \frac{1}{\sqrt{2z}}\left(\Gamma_M\right)^{\Lambda\Sigma} C_\Sigma \tag{A.2}$$

and fermionic correlators can be computed with the help of

$$\begin{aligned} \langle e^{iq_1\varphi_I}(z) e^{iq_2\varphi_J}(w)\rangle &= \delta_{IJ}(z-w)^{q_1 q_2} \\ \langle \psi^M(z)\psi^N(w)\rangle &= \frac{\eta^{\mu\nu}}{(z-w)} \end{aligned} \tag{A.3}$$

B The supergravity solution

B.1 D0-D2-D4-D6 solution

A family of under-rotating multi-center black hole solutions corresponding to a general system of D0-D2-D4-D6 branes in type IIA theory wrapped on T^6 was constructed in [22, 24]. The solutions are parametrised by eight harmonic functions ($a = 1, \ldots 8$, $I = 1, 2, 3$)

$$H_a = \{V, K_I, L_I, M\} \tag{B.1}$$

It is convenient to combine these functions into

$$\begin{aligned} P_I &= \frac{K_I}{V} \\ Z_I &= L_I + \frac{C_{IJK}}{2}\frac{K_J K_K}{V} \\ \mu &= \frac{M}{2} + \frac{L_I K_I}{2\,V} + \frac{C_{IJK}}{6}\,\frac{K_I K_J K_K}{V^2} \end{aligned} \tag{B.2}$$

In term of these functions the metric in the string frame and the various p-form potentials are written as [24]

$$\begin{aligned} ds^2_{10} &= -e^{2U}(dt+w)^2 + e^{-2U}\sum_{i=1}^{3} dx_i^2 + \sum_{I=1}^{3}\frac{e^{-2U}}{VZ_I}(dy_I^2 + d\tilde{y}_I^2) \\ e^{-2\phi} &= e^{6U}V^3 Z_1 Z_2 Z_3 \qquad\qquad B_2 = \sum_{I=1}^{3} b_I\, dy_I \wedge d\tilde{y}_I \\ C_1 &= \alpha_0 \qquad\qquad C_7 = \beta_0 \prod_{I=1}^{3} dy_I \wedge d\tilde{y}_I \\ C_3 &= \sum_{I=1}^{3}\alpha_I \wedge dy_I \wedge d\tilde{y}_I \qquad\qquad C_5 = \frac{|\epsilon_{IJK}|}{2}\,\beta_I \wedge dy_J \wedge d\tilde{y}_J \wedge dy_K \wedge d\tilde{y}_K \end{aligned} \tag{B.3}$$

JHEP06(2016)003

with

$$
\begin{aligned}
e^{-4U} &= Z_1 Z_2 Z_3 V - \mu^2 V^2 & b_I &= \left(P_I - \frac{\mu}{Z_I}\right) \\
\alpha_0 &= A - \mu\, V^2\, e^{4U}\,(dt+w) & \alpha_I &= \left[-\frac{(dt+w)}{Z_I} + b_I\, A + w_I\right] \\
\beta_0 &= -v_0 + \frac{e^{-4U}}{V^2 Z_1 Z_2 Z_3}(dt+w) - b_I\, v_I + b_1\, b_2\, b_3\, A + \frac{|\epsilon_{IJK}|}{2}\, b_I\, b_J\, w_K \\
\beta_I &= -v_I + \frac{|\epsilon_{IJK}|}{2}\left(\frac{\mu}{Z_J Z_K}(dt+w) + b_J\, b_K\, A + 2b_J\, w_K\right)
\end{aligned}
\tag{B.4}
$$

and

$$
\begin{aligned}
&*_3 dA = dV \qquad *_3 dw_I = -d(K_I) \qquad *_3 dv_0 = dM \qquad *_3 dv_I = dL_I \\
&*_3 dw = V d\mu - \mu dV - V Z_I dP_I
\end{aligned}
\tag{B.5}
$$

Here A, w, w_I, v_0, v_I, α_0 and α_I are one-forms in the (flat) 3-dimensional x-space. The R-R field strengths are defined by

$$
F_n = dC_{n-1} - H_3 \wedge C_{n-3}, \tag{B.6}
$$

and satisfy the duality relations

$$
*_{10} F_2 = F_8 \qquad *_{10} F_4 = -F_6 \tag{B.7}
$$

We stress that the inclusion of C_5 and C_7 is required in order to get a self-dual five-form field $F_5 = dC_4$ after T-dualities. Only after the inclusion of these terms, the Buscher's rules map solutions of type IIA to solutions of IIB and viceversa.

B.2 D3^4-solution

The D0-D2-D4-D6 system can be mapped to a system of solely D3-branes applying three T-dualities along $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_3$. T-duality mixes the supergravity fields according to the generalised Buscher rules [5]:

$$
\begin{aligned}
g'_{zz} &= \frac{1}{g_{zz}}, \qquad e^{2\phi'} = \frac{e^{2\phi}}{g_{zz}}, \qquad g'_{zm} = \frac{B_{zm}}{g_{zz}}, \qquad B'_{zm} = \frac{g_{zm}}{g_{zz}} \\
g'_{mn} &= g_{mn} - \frac{g_{mz}\, g_{nz} - B_{mz}\, B_{nz}}{g_{zz}}, \qquad B'_{mn} = B_{mn} - \frac{B_{mz}\, g_{nz} - g_{mz}\, B_{nz}}{g_{zz}} \\
C'^{(n)}_{m_1\ldots m_n} &= C^{(n+1)}_{m_1\ldots m_n z} - n\, C^{(n-1)}_{[m_1\ldots m_{n-1}}\, B_{m_n]z} - n(n-1)\frac{C^{(n-1)}_{[m_1\ldots m_{n-2}|z}\, B_{|m_{n-1}|z}\, g_{|m_n]z}}{g_{zz}} \\
C'^{(n)}_{m_1\ldots m_{n-1}z} &= C^{(n-1)}_{m_1\ldots m_{n-1}} - (n-1)\,\frac{C^{(n-1)}_{[m_1\ldots m_{n-2}|z}\, g_{z|m_{n-1}]}}{g_{zz}}
\end{aligned}
\tag{B.8}
$$

where by z we denote the direction along with T-duality is performed and the metric is understood in the string frame. Applying this rules one finds that after T-dualities along $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_3$ the dilaton and the B-field becomes trivial while the metric maps to

$$
ds^2 = -e^{2U}(dt+w)^2 + e^{-2U}\sum_{i=1}^{3} dx_i^2 + \sum_{I=1}^{3}\left[\frac{dy_I^2}{V e^{2U} Z_I} + V e^{2U} Z_I\, \tilde{e}_I^2\right] \tag{B.9}
$$

with

$$\tilde{e}_I = d\tilde{y}_I - \left(P_I - \frac{\mu}{Z_I} \right) dy_I \tag{B.10}$$

Now let us consider the RR fields. It is easy to see that the two terms involving the metric g in the T-duality transformations of RR fields in (B.8) do not contribute to the metric (B.9). More precisely, the term proportional to $C^{(n-1)}_{[m_1 \dots m_{n-2}|z} B_{|m_{n-1}|z}$ cancels because C and B always share two indices while the term proportional to $C^{(n-1)}_{[m_1 \dots m_{n-2}|z} g_{z|m_{n-1}]}$ cancels because the original metric is always diagonal along the two-torus along one of whose two coordinates the following T-duality transformation is performed. As a result, the T-duality transformations can be written in the simple form

$$C'^{(n)} = C^{(n-1)}_{\perp} \wedge \tilde{e}_I + C^{(n+1)}_{||} \tag{B.11}$$

where we decompose the R-R forms

$$C^{(n)} = C^{(n-1)}_{||} \wedge d\tilde{y}_I + C^{n}_{\perp} \tag{B.12}$$

into a components containing or not the direction $d\tilde{y}_I$ along which the T-duality is performed. After T-dualities along $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_3$, one finds that all RR fields are mapped to the four-form C_4 that can be written in the form

$$\begin{aligned} C_4 &= \alpha_0 \wedge \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3 + \frac{1}{2} \epsilon_{IJK}\, \alpha_I \wedge dy_I \wedge \tilde{e}_J \wedge \tilde{e}_K \\ &\quad + \beta_0 \wedge dy_1 \wedge dy_2 \wedge dy_3 + \frac{1}{2} \epsilon_{IJK}\, \beta_I \wedge \tilde{e}_I \wedge dy_J \wedge dy_K \end{aligned} \tag{B.13}$$

One can check that the five form flux

$$F_5 = d\, C_4 \tag{B.14}$$

obtained from (B.13) is self-dual as required.

References

[1] S.D. Mathur, *The information paradox: A pedagogical introduction*, *Class. Quant. Grav.* **26** (2009) 224001 [arXiv:0909.1038] [INSPIRE].

[2] S.D. Mathur, *The fuzzball proposal for black holes: An elementary review*, *Fortsch. Phys.* **53** (2005) 793 [hep-th/0502050] [INSPIRE].

[3] A. Sen, *Extremal black holes and elementary string states*, *Mod. Phys. Lett.* **A 10** (1995) 2081 [hep-th/9504147] [INSPIRE].

[4] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029] [INSPIRE].

JHEP06(2016)003

[5] O. Lunin and S.D. Mathur, *Metric of the multiply wound rotating string*, Nucl. Phys. **B 610** (2001) 49 [hep-th/0105136] [INSPIRE].

[6] O. Lunin, S.D. Mathur and A. Saxena, *What is the gravity dual of a chiral primary?*, Nucl. Phys. **B 655** (2003) 185 [hep-th/0211292] [INSPIRE].

[7] I. Kanitscheider, K. Skenderis and M. Taylor, *Fuzzballs with internal excitations*, JHEP **06** (2007) 056 [arXiv:0704.0690] [INSPIRE].

[8] S. Giusto, J.F. Morales and R. Russo, *D1D5 microstate geometries from string amplitudes*, JHEP **03** (2010) 130 [arXiv:0912.2270] [INSPIRE].

[9] I. Bena and N.P. Warner, *Black holes, black rings and their microstates*, Lect. Notes Phys. **755** (2008) 1 [hep-th/0701216] [INSPIRE].

[10] K. Skenderis and M. Taylor, *The fuzzball proposal for black holes*, Phys. Rept. **467** (2008) 117 [arXiv:0804.0552] [INSPIRE].

[11] S.D. Mathur, *Fuzzballs and the information paradox: A summary and conjectures*, arXiv:0810.4525 [INSPIRE].

[12] S. Giusto, R. Russo and D. Turton, *New D1-D5-P geometries from string amplitudes*, JHEP **11** (2011) 062 [arXiv:1108.6331] [INSPIRE].

[13] S. Giusto, O. Lunin, S.D. Mathur and D. Turton, *D1-D5-P microstates at the cap*, JHEP **02** (2013) 050 [arXiv:1211.0306] [INSPIRE].

[14] I. Bena, J. de Boer, M. Shigemori and N.P. Warner, *Double, Double Supertube Bubble*, JHEP **10** (2011) 116 [arXiv:1107.2650] [INSPIRE].

[15] S. Giusto and R. Russo, *Perturbative superstrata*, Nucl. Phys. **B 869** (2013) 164 [arXiv:1211.1957] [INSPIRE].

[16] S. Giusto and R. Russo, *Superdescendants of the D1D5 CFT and their dual 3-charge geometries*, JHEP **03** (2014) 007 [arXiv:1311.5536] [INSPIRE].

[17] I. Bena, M. Shigemori and N.P. Warner, *Black-Hole Entropy from Supergravity Superstrata States*, JHEP **10** (2014) 140 [arXiv:1406.4506] [INSPIRE].

[18] I. Bena, S. Giusto, R. Russo, M. Shigemori and N.P. Warner, *Habemus Superstratum! A constructive proof of the existence of superstrata*, JHEP **05** (2015) 110 [arXiv:1503.01463] [INSPIRE].

[19] F. Denef, *Supergravity flows and D-brane stability*, JHEP **08** (2000) 050 [hep-th/0005049] [INSPIRE].

[20] F. Denef, *Quantum quivers and Hall/hole halos*, JHEP **10** (2002) 023 [hep-th/0206072] [INSPIRE].

[21] B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, JHEP **11** (2011) 127 [hep-th/0304094] [INSPIRE].

[22] V. Balasubramanian, E.G. Gimon and T.S. Levi, *Four Dimensional Black Hole Microstates: From D-branes to Spacetime Foam*, JHEP **01** (2008) 056 [hep-th/0606118] [INSPIRE].

[23] J. de Boer, F. Denef, S. El-Showk, I. Messamah and D. Van den Bleeken, *Black hole bound states in $AdS_3 \times S^2$*, JHEP **11** (2008) 050 [arXiv:0802.2257] [INSPIRE].

[24] G. Dall'Agata, S. Giusto and C. Ruef, *U-duality and non-BPS solutions*, JHEP **02** (2011) 074 [arXiv:1012.4803] [INSPIRE].

JHEP06(2016)003

[25] O. Lunin, *Bubbling geometries for $AdS_2 \times S^2$*, *JHEP* **10** (2015) 167 [arXiv:1507.06670] [INSPIRE].

[26] V.S. Rychkov, *D1-D5 black hole microstate counting from supergravity*, *JHEP* **01** (2006) 063 [hep-th/0512053] [INSPIRE].

[27] C. Krishnan and A. Raju, *A Note on D1-D5 Entropy and Geometric Quantization*, *JHEP* **06** (2015) 054 [arXiv:1504.04330] [INSPIRE].

[28] O. Lunin, J.M. Maldacena and L. Maoz, *Gravity solutions for the D1-D5 system with angular momentum*, hep-th/0212210 [INSPIRE].

[29] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B 556** (1999) 89 [hep-th/9905104] [INSPIRE].

[30] M. Cvetič and I. Papadimitriou, *Conformal field theory couplings for intersecting D-branes on orientifolds*, *Phys. Rev.* **D 68** (2003) 046001 [*Erratum ibid.* **D 70** (2004) 029903] [hep-th/0303083] [INSPIRE].

[31] M. Bianchi and J.F. Morales, *Unoriented D-brane Instantons vs Heterotic worldsheet Instantons*, *JHEP* **02** (2008) 073 [arXiv:0712.1895] [INSPIRE].

[32] P. Anastasopoulos, M. Bianchi and R. Richter, *Light stringy states*, *JHEP* **03** (2012) 068 [arXiv:1110.5424] [INSPIRE].

[33] S. Stieberger, *Open & Closed vs. Pure Open String Disk Amplitudes*, arXiv:0907.2211 [INSPIRE].

[34] M. Bianchi, G. Pradisi and A. Sagnotti, *Toroidal compactification and symmetry breaking in open string theories*, *Nucl. Phys.* **B 376** (1992) 365 [INSPIRE].

[35] M. Bertolini, M. Trigiante and P. Fré, *$N = 8$ BPS black holes preserving $1/8$ supersymmetry*, *Class. Quant. Grav.* **16** (1999) 1519 [hep-th/9811251] [INSPIRE].

[36] M. Bertolini, P. Fré and M. Trigiante, *The generating solution of regular $N = 8$ BPS black holes*, *Class. Quant. Grav.* **16** (1999) 2987 [hep-th/9905143] [INSPIRE].

[37] M. Bertolini and M. Trigiante, *Regular BPS black holes: Macroscopic and microscopic description of the generating solution*, *Nucl. Phys.* **B 582** (2000) 393 [hep-th/0002191] [INSPIRE].

[38] M. Bertolini and M. Trigiante, *Microscopic entropy of the most general four-dimensional BPS black hole*, *JHEP* **10** (2000) 002 [hep-th/0008201] [INSPIRE].

[39] I. Pesando, *Correlators of arbitrary untwisted operators and excited twist operators for N branes at angles*, *Nucl. Phys.* **B 886** (2014) 243 [arXiv:1401.6797] [INSPIRE].

[40] I. Pesando, *Green functions and twist correlators for N branes at angles*, *Nucl. Phys.* **B 866** (2013) 87 [arXiv:1206.1431] [INSPIRE].

[41] I. Pesando, *The generating function of amplitudes with N twisted and L untwisted states*, *Int. J. Mod. Phys.* **A 30** (2015) 1550121 [arXiv:1107.5525] [INSPIRE].

JHEP06(2016)003