6D SCFTs and gravity

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ABSTRACT: We study how to couple a 6D superconformal field theory (SCFT) to gravity. In F-theory, the models in question are obtained working on the supersymmetric background $\mathbb{R}^{5,1} \times B$ where B is the base of a compact elliptically fibered Calabi-Yau threefold in which two-cycles have contracted to zero size. When the base has orbifold singularities, we find that the anomaly polynomial of the 6D SCFTs can be understood purely in terms of the intersection theory of fractional divisors: the anomaly coefficient vectors are identified with elements of the orbifold homology. This also explains why in certain cases, the SCFT can appear to contribute a "fraction of a hypermultiplet" to the anomaly polynomial. Quantization of the lattice of string charges also predicts the existence of additional light states beyond those captured by such fractional divisors. This amounts to a refinement to the lattice of divisors in the resolved geometry. We illustrate these general considerations with explicit examples, focusing on the case of F-theory on an elliptic Calabi-Yau threefold with base $\mathbb{P}^2/\mathbb{Z}_3$.

Contents

1 Introduction and summary

F-theory [1–3] provides the broadest known arena for constructing string vacua. Part of the utility of this formulation is that many stringy ingredients such as seven-branes are automatically packaged in terms of elliptic fibrations and Calabi-Yau geometry. For six-dimensional low energy effective field theories, this approach is particularly powerful, and has led to a characterization of virtually all known string vacua.[1]

Recently, there has also been renewed interest in using F-theory as a tool to systematically construct and study 6D superconformal field theories (SCFTs). Building on earlier work (see e.g. [5–16]), there is now a complete classification of 6D SCFTs without a Higgs branch [17], with steady progress on the classification of theories with a Higgs branch [18–20]. In this paper we consider the problem of coupling these systems to gravity.

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[1]For a recent review of the close correspondence between bottom-up and top-down constraints on 6D theories, see [4].

An important assumption in much of the literature on 6D supergravity theories is that the matter fields organize according to "conventional" supermultiplets. This includes the gravity multiplet, the tensor multiplet and vector multiplet, as well as (half) hypermultiplets. There are rather tight consistency conditions for the possible ways such ingredients can be combined. These requirements include 6D anomaly cancellation, as well the requirement that the lattice of BPS strings is properly quantized and unimodular[2] — see e.g. [21, 22].

The situation with 6D superconformal subsectors is much less understood. First of all, it is quite clear that a consistent theory of gravity cannot be obtained by coupling to arbitrary SCFTs,[3] and here we begin the task of determining which of these models might still be consistent upon coupling to gravity.

From the perspective of the 6D gravity theory, such an SCFT corresponds to the presence of a strongly coupled sector which exhibits approximate scale invariance (which is broken by Planck scale effects). We explain the way in which some of these models can indeed satisfy the anomaly cancellation and unimodularity constraints, giving rise to physically sound supergravity theories. To accomplish this, we shall use the F-theory description of 6D supergravity backgrounds. This consists of specifying a compact complex surface B, and the data of an elliptically fibered Calabi-Yau threefold $X \to B$. To generate the strongly coupled conformal subsectors of our 6D supergravity theory, we allow two-cycles in the base to degenerate to zero size. The strings obtained by wrapping D3-branes over such vanishing cycles then become tensionless, leading to the desired SCFTs.

It would be quite desirable to understand F-theory away from the limit in which the volume of all cycles in the base are large. Indeed, a potential weaknesses in the "large volume perspective" is the absence of a systematic α' expansion. The effects of short distance physics can often be recovered by taking various singular degeneration limits. In the physical theory, this corresponds to adding light degrees of freedom in the low energy effective field theory, namely D3-branes wrapping vanishing cycles. One might therefore ask whether one can provide an intrinsic formulation of an F-theory compactification away from the large volume limit. The benefits of having such a formulation would be considerable. For one, it would allow one to dispense with the assumption that there is a moduli space of vacua connecting different regimes of parameter space.[4] Another aim of this paper will be to take some preliminary steps in this direction. We focus on the case of 6D F-theory vacua with eight real supercharges. This is a particularly tractable example to study, because there are a number of universal strong consistency conditions. Additionally, there is typically a moduli space of vacua which will enable us to check our formalism by moving

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[2]Recall that a unimodular lattice is a lattice equipped with an integer valued quadratic form such that its determinant is ± 1.

[3]For example, it is well known that a sufficiently large number of M5-branes cannot be consistently recoupled to 6D gravity. Nevertheless, these are consistent field theories, and provide an M-theory realization of the A_N series of $(2,0)$ theories.

[4]For example, in 4D $\mathcal{N} = 1$ vacua, there can be obstructions to motion on the geometric moduli space of a compactification, trapping the theory at small volumes [23]. See also [24] for a discussion of how the open string metric may nevertheless remain at large volume, and the corresponding formulation in terms of non-commutative geometry.

back to the large radius limit. In more detail, we will be interested in giving a direct geometric formulation of 6D F-theory vacua where the (compact) base B of the elliptic fibration $X \to B$ contains orbifold singularities of the type considered in [17, 18].

Now, in the absence of any superconformal subsystem, the 6D effective theory supports a set of strings which can couple to various two-form potentials. Geometrically, the resulting lattice of charges Λ_{string} is identified with the second homology lattice of the F-theory base:[5]

$$\Lambda_{\text{string}} \equiv H_2(B_{\text{smth}}, \mathbb{Z}) . \tag{1.1}$$

Moreover, the intersection theory of $H_2(B_{\text{smth}}, \mathbb{Z})$ completely captures the anomaly polynomials of conventional matter. In the case where B has orbifold singularities corresponding to strongly coupled SCFTs, the second orbifold homology contains fractional divisors, that is divisors which have an intersection pairing valued over the rational numbers, rather than the integers. Quite surprisingly, the intersection theory of the lattice

$$\Lambda_{\text{frac}} \equiv H_2(B_{\text{orb}}, \mathbb{Z}) \tag{1.2}$$

still matches the anomaly coefficients of the corresponding SCFTs! Thus superconformal theories really behave as ordinary matter for this aspect of its F-theory realization. Naïvely, it is quite tempting, in view of this result, to expect that this correspondence persists, namely that the string charges fill out the corresponding homology lattice $H_2(B_{\text{orb}}, \mathbb{Z})$. As this contains fractional divisors, we might therefore expect the theory to contain fractional strings from D3-branes wrapped over such divisors. Indeed, the calculation of the anomaly polynomial for these theories leads to contributions which fill out fractions of those of hypermultiplets. This may appear puzzling: how can such fractional states be compatible with the condition of charge quantization? Much in the spirit of [25], the answer is that charge quantization *predicts* the existence of *additional* strings. Once we supplement the theory with these additional states, the full theory turns out to obey charge quantization (as it must). For this to work, there must exist a refinement Λ_{ref} of our lattice Λ_{frac}, and a surjective map $\Lambda_{\text{ref}} \to \Lambda_{\text{frac}}$: only the states spanning Λ_{ref} have to obey the standard Dirac quantization conditions. So where are these additional states in the low energy effective field theory? The answer is that they are the additional states associated with the SCFT itself. Indeed Dirac quantization and unimodularity are constraints for the tensorial Coulomb branch of the supergravity theory. As the SCFTs have their own Coulomb branches, these must be taken into account. In F-theory, the superconformal sectors come about from two-cycles in the base which have collapsed to zero size. The refined basis of string states comes from D3-branes wrapped over precisely these additional $\mathbb{P}^1$'s. The lattice Λ_{ref} is then identified with the second homology lattice of the resolved base: that being a smooth compact oriented 4-manifold, the lattice is unimodular by Poincaré duality [26].

To further support this picture, we study a particular example of F-theory compactification on a Calabi-Yau threefold with orbifold singularities in the base B. We take $B = \mathbb{P}^2/\mathbb{Z}_3$, and show that the effective theories can be consistently described by six-dimensional $\mathcal{N} = 1$ supergravity theories coupled to SCFTs. When the elliptically fibered

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[5]Throughout this work, we do not consider possible discrete torsional contributions to the lattice of charges.

Calabi-Yau manifold $X \to B$ is at a generic point in complex moduli space, the effective theory has three A_2 $(2,0)$ theories and 93 neutral hypermultiplets coupled to the $\mathcal{N} = 1$ supergravity theory. By tuning the complex structure moduli, the effective theory can develop gauge symmetries. We examine the various divisors a gauge symmetry can live on and describe the physics in each case. As expected, when the gauge brane hits the orbifold locus, there exists a strongly coupled SCFT living at the intersection that contributes to the gauge anomaly. We verify that the anomaly cancellation conditions hold in each of these cases.

The rest of this paper is organized as follows. First, in section 2, we spell out the constraints imposed by 6D anomaly cancellation and charge quantization both in 6D field theory terms, as well as in the geometry of an F-theory compactification. In section 3 we turn to the case where the base contains orbifold singularities, and therefore an SCFT sector. We show that anomaly cancellation can be understood in terms of the intersection theory of fractional divisors. Moreover, we explain how the conditions of charge quantization are obeyed in this case. In section 4, we study the case when $B = \mathbb{P}^2/\mathbb{Z}_3$ in detail. We conclude with comments and further directions of research in section 5. Some technical details are collected in the appendices.

2 F-theory on a smooth base

In this section we discuss some aspects of 6D supergravity theories for F-theory compactified on a *smooth* base B_{smth}. Most of the material we shall review is well-known, and can be found in the existing literature.

Recall that in F-theory, the type IIB axio-dilaton has a position dependent profile. To get an $\mathcal{N} = (1,0)$ theory in six dimensions, we work on the background $\mathbb{R}^{5,1} \times B_{\mathrm{smth}}$ where B_{smth} is the base of an elliptically fibered Calabi-Yau threefold, i.e. $X \to B_{\mathrm{smth}}$. In minimal Weierstrass form, the defining equation for X is:

$$y^2 = x^3 + fx + g \tag{2.1}$$

where f and g are respectively sections of $\mathcal{O}_{B_{\mathrm{smth}}}(-4K_{B_{\mathrm{smth}}})$ and $\mathcal{O}_{B_{\mathrm{smth}}}(-6K_{B_{\mathrm{smth}}})$. The elliptic fiber may contain singularities, and the discriminant locus $4f^3 + 27g^2 = 0$ tells us the locations of seven-branes wrapping curves in B_{smth}.

A hallmark of chiral 6D theories is the presence of self-dual and anti-self-dual three-form field strengths, and the corresponding BPS lattice of strings. These field strengths come about about from reduction of the 10D gravity multiplet, as well as reduction of the RR five-form flux to six-dimensional vacua. We get a single self-dual two-form potential $B_{\mu\nu}^+$ from the 6D gravity multiplet, and T anti-self-dual two-form potentials $B_{\mu\nu}^-$ from the 6D tensor multiplets. Altogether, the corresponding two-form potentials rotate as a vector of $\mathrm{SO}(1,T)$. We reach the tensorial Coulomb branch of the theory by giving vevs to the scalars in the tensor multiplets. This also generates a tension for the strings.

Anomaly cancellation via the Green-Schwarz-West-Sagnotti mechanism [27–30] dictates a delicate interplay between these tensor degrees of freedom, and the vector multiplets of the 6D theory. For example, the invariant field strengths of these tensor fields are

G	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
λ	1	2	1	2	6	12	60	6	2

Table 1. The Dynkin index for the fundamental representation of each group.

given by

$$H^M = dB^M + \frac{1}{2}a^M\omega_{3L} + \sum_i \frac{2b_i^M}{\lambda_i}\omega_{3Y}^i \,, \tag{2.2}$$

where the index M runs from 0 to T. The fields and parameters of the theory can in fact be written in an SO(1, T) invariant fashion, and the upper-case letters $M, N, \cdots$ are used to denote these indices. The index i labels the gauge group factors G_i of the theory, while ω_{3L} and ω_{3Y}^i are gravitational and gauge Chern-Simons three-forms, respectively. The numerical factor λ_i, tabulated in table 1, is the Dynkin index for the fundamental representation of the gauge group G_i — it normalizes the trace of the gauge group so that the minimum-charge instanton has unit charge with respect to the fundamental trace [21, 31].[6] We have assumed that the gauge group factors are all non-abelian, although incorporating abelian group factors is straightforward [33]. The various multiplets of the effective theory, which couple to the graviton and gauge fields, have gravitational, mixed and gauge anomalies. Given that the total anomaly polynomial, which is an eight-form I_A, factors in the form

$$I_A = \frac{1}{32}\Omega_{MN}X^M \wedge X^N \tag{2.3}$$

with the factors being the four-forms

$$X^M = \frac{1}{2}a^M \operatorname{tr} R^2 + \sum_i \frac{2b_i^M}{\lambda_i}\operatorname{tr} F_i^2 \,, \tag{2.4}$$

the anomaly can be cancelled by a local term

$$\mathcal{L}_{\mathrm{GS}} = -\frac{1}{32}\Omega_{MN}B^M \wedge X^N. \tag{2.5}$$

Here, R and F_i denote the Riemann curvature and the G_i-field strength, respectively. The trace "tr" without any index denotes the trace taken with respect to the fundamental representation. Given that the effective theory can be described by using conventional $(1, 0)$ supermultiplets in six-dimensions, the total anomaly polynomial I_A can be computed by adding up the anomaly polynomials of the individual multiplets, which we summarize in appendix A.

The symmetric matrix Ω_{MN} of (2.3) is a SO(1, T) metric, which can be understood as an integer-valued quadratic form on the string charge lattice [10, 22, 28, 34]. Geometrically, this is just the intersection pairing on $H_2(B_{\mathrm{smth}}, \mathbb{Z})$. It is convenient to use this SO(1, T) metric to raise and lower indices, i.e.,

$$\Omega^{MN} \equiv (\Omega^{-1})_{MN} \,. \tag{2.6}$$

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[6]We note that these differ from the group theoretical factors used in [32] by a factor of two.

The magnetic source $\widetilde{J}_M$ of the Mth tensor field is given by

$$\widetilde{J}^M = dH^M = \frac{1}{2}a^M \operatorname{tr} R^2 + \sum_i b_i^M \left(\frac{2}{\lambda_i} \operatorname{tr} F_i^2 \right). \tag{2.7}$$

Meanwhile, the self duality conditions of the theory can be written as

$$* \Omega_{MN} H^N = G_{MN} H^N \tag{2.8}$$

where the elements of the matrix G_{MN} are given by

$$G_{MN} = 2j_M j_N - \Omega_{MN}. \tag{2.9}$$

The star operator acts on differential forms by the Hodge dual operation. Here, the SO$(1, T)$ unit vector j^M, i.e.,

$$j^M j_M = \Omega_{MN} j^M j^N = 1, \tag{2.10}$$

parametrizes the vacuum expectation value of the T scalars in the T tensor multiplets of the six-dimensional theory. The electric source of the Mth tensor field is then given by [10, 28, 34]

$$J_M = d * G_{MN} H^N = \frac{1}{2}a_M \operatorname{tr} R^2 + \sum_i b_{i,M} \left(\frac{2}{\lambda_i} \operatorname{tr} F_i^2 \right). \tag{2.11}$$

We hence see that gauge instantons of the theory are electric/magnetic sources for the tensor fields of the theory. The anomaly coefficients encode the string charges of the BPS instantons — the tension of an instanton with minimum charge is given by the inverse gauge coupling

$$b_i \cdot j \equiv \Omega_{MN} b_i^M j^N. \tag{2.12}$$

The Dirac quantization condition for these strings impose that the vectors b_i must be elements of an integral lattice whose inner product matrix is given by Ω.

For F-theory models on a smooth compact base B_{smth}, we have $H_2(B_{\text{smth}}, \mathbb{Z}) = \Lambda_{\text{string}}$, the string charge lattice. The matrix Ω_{MN} is then the intersection pairing matrix of the homology cycles — this lattice is integral and unimodular, due to Poincaré duality. Then, a and b_i have the geometric interpretation as being the homology classes of the canonical divisor and the divisor the G_i seven-brane wraps [21, 29, 35]. The vector j can then be understood as the Kähler class of the base manifold.

2.1 Charge quantization and unimodularity

Let us say a few more words on the charge quantization conditions and their geometric avatars in an F-theory compactification. Geometrically, the lattice of string charges Λ_{string} is simply the homology lattice for the compact model B_{smth}, i.e. we have:

$$\Lambda_{\text{string}} = H_2(B_{\text{smth}}, \mathbb{Z}). \tag{2.13}$$

As we have remarked above $H_2(B_{\text{smth}}, \mathbb{Z})$ is automatically unimodular. This fact can also be understood purely in terms of the 6D supergravity theory [22] (see also [25, 36]). Along the tensorial Coulomb branch, Dirac quantization in flat $\mathbb{R}^{5,1}$ implies that the allowed string charges have to be integer valued [37].

Additional constraints can follow from studying a 6D effective theory on different backgrounds. In particular, the existence of a partition function imposes the condition that the lattice of string charges is in fact unimodular [22]. As explained in [22], to establish this condition, consider the 6D theory on a $\mathbb{R}^{1,1} \times \mathbb{CP}^2$ background to obtain a chiral 2D theory whose charge lattice is identified with the string charge lattice. Now, consider the partition function of such 2D theory on a torus $T^2 = S^1_a \times S^1_b$. The S transformation that exchanges S^1_a with S^1_b is always a symmetry of the theory, and therefore the partition function has to be invariant with respect to it. As explained in [22], the S-invariance of the 2D partition function is realized only if the charge lattice of the 2D model is self-dual, i.e. unimodular. This implies that the unimodularity of the string charge lattice in 6D is a necessary condition for the theory to have a well-defined partition function on $\mathbb{CP}^2 \times T^2$, and therefore a necessary condition to have a consistent supergravity theory. This is why the string charge lattice of the tensor fields in a consistent 6D supergravity theory must be unimodular.

Of course, some well-known 6D theories do not satisfy this condition of a unimodular intersection form. A notable class of examples are the $(2,0)$ theories of A_N type. Indeed, we must note that a theory of (anti-)self-dual fields that does not have a well-defined partition function on a manifold nevertheless can define a sensible quantum field theory [38–47]. Such a theory, referred to as a "relative quantum field theory" in [47], has a partition bundle (or a partition vector) over the geometric moduli space of the manifold as opposed to a partition function: additional topological data must be specified to fully characterize the behavior of the model in curved spacetimes. These theories, however, on their own cannot be coupled to gravity in a consistent way, as they can be thought of as having anomalies under large diffeomorphisms.

It is nevertheless always possible to embed the string charges of a 6D SCFT in a unimodular lattice. A particularly important class of examples come from taking a singular limit of a 6D supergravity theory.[7] In such a situation, we can consider simultaneously contracting some subset of two-cycles. While the full lattice of string charges of the 6D supergravity theory is automatically unimodular, the particular subset associated with the 6D SCFT need not satisfy this property. Let us also point out that there are also perfectly healthy SCFTs with a unimodular lattice of string charges which are not expected to embed in any consistent 6D supergravity theory. An example of this type is the large N limit of the configuration of curves $1, 2 \ldots, 2$, i.e. the theory of N M5-branes probing an E_8 nine-brane.

3 The case of an orbifold base

Having reviewed the case of F-theory on a smooth base, we now turn to the study of F-theory on a base with orbifold singularities. Roughly speaking, the physical picture is that

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[7]We thank Y. Tachikawa and W. Taylor for explaining these points to us.

our base will now contain various "fractional divisors" which can be wrapped by seven-branes, as well as D3-branes. Geometrically, these fractional divisors will pass through the locus of the orbifold singularity. As such, it is important to understand whether we can still make sense of the resulting theory. The seven-branes will contribute gauge theory sectors, and the D3-branes will contribute BPS strings with tension. Owing to the fact that there is a singularity in the base, we can also expect there to be additional light states which contribute to the low energy effective theory. These states are the contribution from a 6D SCFT. The F-theory geometry provides a systematic way to couple these systems to gravity.

Remarkably, many aspects of the 6D effective theory can be understood purely in terms of the geometry of these fractional divisors. For example, we find that the anomaly polynomial for such theories can be understood purely in terms of the intersection theory of $H_2(B_{\text{orb}}, \mathbb{Z})$. On the other hand, we will also see that charge quantization *predicts* the existence of additional light states in the low energy effective field theory. These states are simply the contributions from the 'internal' degrees of freedom of the SCFTs.

3.1 Geometric preliminaries

Since it will form the core of our mathematical analysis, we first review some salient features of intersection theory on an orbifold base B_{orb}. The key point is that the intersection numbers for cycles in $H_2(B_{\text{orb}}, \mathbb{Z})$ will be rational numbers. To avoid cluttering the notation, we shall drop the "orb" from B_{orb} in what follows.

To begin, we shall always assume the existence of a smooth resolution $\hat{B} \to B$. Denote by e_M for $M = 0, \ldots, T$ the basis of divisors for $\hat{B}$. To reach the orbifold point, we shall blowdown some subset of these divisors. Denote this collection by:

$$D_m = D_m^M e_M, \qquad m = T_0 + 1, \cdots, T. \tag{3.1}$$

The orbifold point is reached by tuning the Kähler class j such that

$$j \cdot D_m = 0. \tag{3.2}$$

Viewed as a vector in $H_2(\hat{B}, \mathbb{R})$, j is thus restricted to lie in the orthogonal complement of the subspace V_S spanned by $\{D_m\}$ with respect to the inner product space $H_2(\hat{B}, \mathbb{R})$. We identify this orthogonal complement with the inner product space $H_2(B, \mathbb{R})$ of the surface B obtained by blowing down the divisors D_m. It is convenient to take an integral basis

$$u_\mu, \qquad \mu = 0, \cdots, T_0 \tag{3.3}$$

of the $\text{SO}(1, T_0)$ sublattice $\Lambda_0 = V_S^\perp \cap \Lambda$ of Λ. By definition,

$$u_\mu \cdot D_m = 0 \tag{3.4}$$

for any m and μ. We consistently use the labels m, n (resp. μ, ν) to label indices in the range $\{T_0 + 1, \cdots, T\}$ (resp. $\{0, \cdots, T_0\}$), respectively. Taking u_μ and D_m to be the new basis for $H_2(\hat{B}, \mathbb{R})$, the intersection matrix now factors into the form:

$$\Omega'_{M'N'} = \Omega^0_{\mu\nu} \oplus \Omega^S_{mn}, \tag{3.5}$$

with

$$\Omega_{\mu\nu}^0 \equiv u_\mu \cdot u_\nu \,, \qquad \Omega_{mn}^S \equiv D_m \cdot D_n \,. \tag{3.6}$$

As before, we raise and lower the μ, ν (resp. m, n) indices using the metric $\Omega_{\mu\nu}^0$ (resp. Ω_{mn}^S).

Though it is tempting to identify Λ_0 as the integral homology lattice of the orbifold B, this is not quite true. Letting V_0 to be the inner product space spanned by u_μ over the reals, the integral homology lattice $H_2(B, \mathbb{Z})$ of B is given by the orthogonal projection of the homology lattice $\Lambda = H_2(\hat{B}, \mathbb{Z})$ of its resolution to V_0. Due to the unimodularity of Λ, the homology lattice of B can be shown to be given by the dual of Λ_0:

$$\Lambda_0^* = \{\ell \in V_0 \,:\, \ell \cdot \ell' \in \mathbb{Z}, \text{ for all } \ell' \in \Lambda_0\} \,. \tag{3.7}$$

Since B is an orbifold, as opposed to being a smooth manifold, Λ_0^* is strictly larger than Λ_0. Hence, the lattice Λ_0 is not unimodular and Λ_0^* is not integral, but rational.

An equivalent, algebraic definition of $H_2(B, \mathbb{Z})$ can be given [48], since B can be treated as a rational surface ($H^{2,0}(B, \mathbb{C}) = 0$). $H_2(B, \mathbb{Z})$ is the group of divisors of B modulo algebraic equivalence — with suitable definitions of divisors and algebraic equivalence for orbifolds. A divisor on an orbifold can be $\mathbb{Q}$-Cartier, but not Cartier — that is, on its own, it may not have a good defining equation, while a multiple of it has one. $\mathbb{Q}$-Cartier divisors can be identified as the divisors that are Weil, but not Cartier. These divisors, which we shall often refer to as "fractional divisors", can have fractional intersection numbers with other divisors.

An operational definition of Weil and Cartier divisors can be given by the following: Weil divisors can be understood as divisors that have a well defined locus, while Cartier divisors are divisors whose defining equation, in each patch of the algebraic variety, lies in the ring of rational functions of that patch. The homology class of the Cartier divisors are elements of the integral lattice Λ_0, while the homology class of the fractional divisors are elements of Λ_0^* that do not lie on the integral lattice points. A more rigorous treatment of divisors of complex orbifolds can be found in section 4.4 of [48].

There is an intuitive way of describing the origin of the fractional divisors from the point of view of the homology lattice. The map of the homology class of a divisor on $\hat{B}$ to B, upon the birational map of blowing down the divisors D_m, is given by the projection from the homology lattice Λ to Λ_0^*. In particular, the canonical class of $\hat{B}$ maps to that of B in this way.[8] Since Λ_0 is not unimodular, there exist integral divisors D of $\hat{B}$ that have fractional coefficients when written as a linear combination of the basis $\{u_\mu\} \coprod \{D_m\}$. This is because the unimodularity of Λ_0 and the requirement that the projection of any lattice vector in Λ to V_0 lies in the integral lattice in Λ_0 are equivalent facts.[9] Such divisors become fractional upon projecting down to Λ_0^*, i.e., when $\hat{B}$ is blown down to B.

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[8]See, for example, proposition 4.4.15 and equation (4.4.2) of [48].

[9]Proof: the projection v_0 of a vector v in Λ to V_0 is given by

$$v_0 = (v \cdot u^\mu) u_\mu \,. \tag{3.8}$$

If Λ_0 is unimodular, v_0 obviously lies within Λ_0, as u^μ is an integral vector in Λ. Meanwhile, if $v \cdot u^\mu$ is integral for any v and μ, u^μ itself is an integral vector, which lies in Λ_0. Hence, Λ_0 must be unimodular.

3.2 Anomalies and fractional divisors

Having dispensed with the geometric preliminaries, we now turn to the study of 6D supergravity theories coupled to SCFTs. The big surprise is that the anomaly polynomial for these theories can be recast purely in terms of the intersection theoretic data on $H_2(B_{\text{orb}}, \mathbb{Z})$ alone. In other words, the SCFTs constitute a "black box" which effectively generalizes the case of contributions from more conventional matter fields such as 6D hypermultiplets.

Our starting point will be F-theory compactified on a smooth base $\hat{B}$ which degenerates to a base B that contains orbifold singularities. Denote by $\hat{X}$ the corresponding elliptic Calabi-Yau with base $\hat{B}$. We use the same notation as in the previous subsection, e.g. we let $\Lambda = H_2(\hat{B}, \mathbb{Z})$ denote the lattice of BPS strings for the smooth phase. We move to the case with orbifold singularities by collapsing a subset of the divisors of $\hat{B}$:

$$D_m = D_m^M e_M, \qquad m = T_0 + 1, \cdots, T. \tag{3.9}$$

We shall be interested in studying the physical theory defined by the divisors which remain at finite volume. Seven-branes wrapping such fractional divisors will support vector multiplets, and D3-branes wrapped over such fractional divisors correspond to strings. With the same notation of the previous section, in the new basis j can be written as

$$j = u_\mu j^\mu, \tag{3.10}$$

with

$$j^\mu = (u^\mu \cdot e_M) j^M. \tag{3.11}$$

The $(T_0 + 1)$ tensor fields $\mathcal{B}^\mu$ which are not part of the SCFT degrees of freedom are also aligned along the subspace of $H_2(\hat{B}, \mathbb{R})$ spanned by u_μ, i.e., the inner product space $H_2(B, \mathbb{R})$. $\mathcal{B}^\mu$ is related to B^M by

$$\mathcal{B}^\mu = (u^\mu \cdot e_M) B^M. \tag{3.12}$$

Meanwhile, the tensor fields $\mathcal{B}_S^m$, whose electrically charged strings become tensionless can be identified as

$$\mathcal{B}_S^m \equiv (D^m \cdot e_M) B^M. \tag{3.13}$$

These tensors are part of the SCFT. The corresponding gauge invariant field strengths of the tensors are denoted by $\mathcal{H}^\mu$ and $\mathcal{H}_S^m$. A consistency check that the tensors that are not part of the conformal subsector should be identified as (3.12) is to observe that under this identification, none of the tensionless strings of the SCFT carry electric charge under $\mathcal{B}^\mu$. Indeed the electric string current four-form is given by

$$J_\mu = d * (2j_\mu j_\nu - \Omega_{\mu\nu}^0) \mathcal{H}^\nu, \tag{3.14}$$

and our claim follow from the orthogonal splitting (3.5) of the lattice.[10]

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[10]We remind the reader that the indices μ, ν of equation (3.14) are the $SO(1, T_0)$ indices — the space-time indices in this equation are suppressed.

The (local) gauge group of the theory on $\hat{B}$ can be factored into two pieces,

$$\mathcal{G} = \mathcal{G}_0 \times \mathcal{G}_S = \prod_k G_{0,k} \times \prod_\kappa G_{S,\kappa} \,, \tag{3.15}$$

where the second factor denotes the gauge groups which become strongly coupled. The seven-branes responsible for the gauge symmetry wrap a linear combination of cycles that are being blown down. Hence, the gauge anomaly coefficients of the gauge groups $G_{S,\kappa}$ are linear combinations only of the shrinking cycles D_m,

$$b_{S,\kappa} = b_{S,\kappa}^m D_m \,, \tag{3.16}$$

where the coefficients $b_{S,\kappa}^m$ are all integral. Notice that from equation (3.11) it follows that

$$j \cdot b_{S,\kappa} = 0 \,, \tag{3.17}$$

so that the instantons (i.e. the strings) of the gauge group $\mathcal{G}_S$ become tensionless. Meanwhile, the anomaly coefficients of $G_{0,k}$ and the gravitational anomaly coefficient can in general have components in the D_m directions. We can decompose them in the following way:

$$
\begin{aligned}
a &= a^\mu u_\mu + a^m D_m \equiv a_0 + a_\mathcal{S} \,, \\
b_k &= b_k^\mu u_\mu + b_k^m D_m \equiv b_{0,k} + b_{\mathcal{S},k} \,.
\end{aligned}
\tag{3.18}
$$

a_0 and $b_{0,k}$ are projections of the coefficients a and b_k to $H_2(B,\mathbb{R})$.

The Green-Schwarz term of the effective theory on $\hat{X}$ now can be decomposed as

$$\mathcal{L}_{\text{GS}} = -\frac{1}{32}\Omega_{\mu\nu}\mathcal{B}^\mu X^\nu - \frac{1}{32}\Omega_{mn}\mathcal{B}_S^m X^n \equiv \mathcal{L}_0 + \mathcal{L}_S \,, \tag{3.19}$$

where

$$
\begin{aligned}
X^\mu &= d\mathcal{H}^\mu = \frac{1}{2}a_0^\mu \operatorname{tr} R^2 + \sum_k b_{0,k}^\mu \left(\frac{2}{\lambda_{0,k}} \operatorname{tr} F_{0,k}^2 \right) \\
X^m &= d\mathcal{H}_S^m = \frac{1}{2}a_\mathcal{S}^m \operatorname{tr} R^2 + \sum_k b_{\mathcal{S},k}^m \left(\frac{2}{\lambda_{0,k}} \operatorname{tr} F_{0,k}^2 \right) + \sum_\kappa b_{S,\kappa}^m \left(\frac{2}{\lambda_{S,k}} \operatorname{tr} F_{S,\kappa}^2 \right).
\end{aligned}
\tag{3.20}
$$

Recall that the effective theory on the tensor branch of the superconformal theory is a $(1,0)$ field theory with the tensor multiplets $\mathcal{B}_S^m$, the gauge group $\mathcal{G}_S$, hypermultiplets charged under $\mathcal{G}_S$ (some of which can carry charge under $\mathcal{G}_0$), and in certain cases, some neutral hypermultiplets [17–19, 32, 49, 50]. Let us denote the one-loop anomaly polynomial of the fields used to describe the SCFT, upon coupling its stress energy tensor and flavor currents to a background graviton and gauge fields, as $I_{S,1\ell}$. The total anomaly polynomial of the low energy supergravity theory on $\hat{X}$ then decomposes into

$$I_{\text{tot}} = I_{0,1\ell} + I_{S,1\ell} \,. \tag{3.21}$$

The one-loop anomalies $I_{0,1\ell}$ come from the supergravity multiplet, the T_0 tensor multiplets that are not part of the SCFT, gauge multiplets of $\mathcal{G}_0$ and hypermultiplets that are

either neutral or carry charge only under $\mathcal{G}_0$. Assuming that the low energy supergravity description of the F-theory compactification on $\hat{X}$ is consistent, we find that the anomaly cancellation condition

$$I_{0,1\ell} + I_{S,1\ell} = \frac{1}{32}\Omega_{\mu\nu}X^\mu X^\nu + \frac{1}{32}\Omega_{mn}X^m X^n \qquad (3.22)$$

is satisfied.

Now the piece $\mathcal{L}_S$ of (3.19) is precisely the Green-Schwarz term of the effective theory of the superconformal theory on the tensor branch, as it is the piece that only involves the tensors $\mathcal{B}_S^m$. Then the anomaly polynomial of the SCFT coupled to supergravity and gauge fields in gauge group $\mathcal{G}_0$ is given by [32, 50]

$$I_S = I_{S,1\ell} - \frac{1}{32}\Omega_{mn}X^m X^n. \qquad (3.23)$$

A simple consistency check of the fact that $\mathcal{L}_S$ is the correct Green-Schwarz term is that, due to (3.22), all the gauge and mixed anomalies involving the gauge symmetry group $\mathcal{G}_S$ are cancelled in (3.23) — the only remaining terms in I_S involve the metric curvature and gauge field strengths of $\mathcal{G}_0$.

The F-theory compactification on the singular base $X \to B$ leads to an effective theory that can be described by a supergravity theory with T_0 tensor multiplets and gauge symmetry $\mathcal{G}_0$ interacting with a strongly coupled superconformal system [18, 19]. Let us now show that the anomalies of this theory are cancelled by the GSSW mechanism with anomaly coefficient vectors a_0 and $b_{0,k}$. Note that in this effective theory, the Green-Schwarz term now becomes $\mathcal{L}_0$ of (3.19), which only involve the tensor fields $\mathcal{B}^\mu$. The total anomaly polynomial of the theory is given by

$$I_{0,\text{tot}} = I_{0,1\ell} + I_S. \qquad (3.24)$$

The first term is the contribution of the conventional fields of the supergravity theory, while the second term comes from the strongly coupled sector. Due to the computation (3.23), and the anomaly cancellation condition (3.22) on the compactification on $\hat{X}$, we find that

$$I_{0,\text{tot}} = \frac{1}{32}\Omega_{\mu\nu}X^\mu X^\nu, \qquad (3.25)$$

which is precisely cancelled by the Green-Schwarz term $\mathcal{L}_0$!

We also see that the gravitational and gauge anomaly coefficients in this equation are given by a_0 and $b_{0,k}$ defined in (3.18). Notice that this does not require us to pass onto the tensor branch of the 6D SCFT. Indeed, upon passing onto the tensor branch, we get additional gauge theory sectors, but none of the instantons of the gauge group $\mathcal{G}_S$ are charged under the tensor fields $\mathcal{B}^\mu$ of this theory.

The discussion above implies that the anomaly coefficients a_0 and $b_{0,k}$ of the effective theory of F-theory compactified on X still have the geometric interpretation as respectively specifying the homology class of the canonical class and the seven-brane loci. However, if a seven-brane carrying gauge group G is wrapping a fractional divisor β of B whose homology class is given by $b = [\beta]$, intersection numbers involving b, e.g.,

$$\Omega_{\mu\nu}b^\mu b^\nu, \qquad (3.26)$$

can be fractional. Since β comes from the projection of an integral divisor $\hat{\beta}$ in $\hat{B}$, $\hat{b} = [\hat{\beta}]$ must have components lying along the directions of the cycles blown down. In fact, a fractional divisor β intersects the orbifold loci where the resolution divisors contained in $\hat{\beta}$ are localized. Physically, this implies that the G-instantons are charged under tensor fields that are part of the strongly coupled subsector. The anomaly polynomial I_S of the superconformal theory has the proper fractional coefficients to offset the fractionality of the intersection number (3.26). However, this also implies that the string charge lattice of an F-theory background with a superconformal sector cannot be identified with the homology lattice of the singular base B, as this would violate the quantization of charges over the integers — it must be identified with the homology lattice of the base $\hat{B}$ obtained by resolving all the strongly coupled singularities, as we are going to argue in the following section.

Let us point out that β can be a Cartier divisor on B and its blow-up $\hat{\beta}$ still can have components of D_m in it. When β is a Cartier divisor that does not intersect any orbifold points, its homology class remains in the sublattice Λ_0 of Λ even after blowing up B into $\hat{B}$. On the other hand, when β intersects the orbifold point, $\hat{\beta}$ has components that are orthogonal to Λ_0 within Λ. Physically, this is because the seven-brane intersects the locus where an SCFT resides, and hence its instantons carry charge under the tensor fields of that SCFT.

3.3 Charge quantization and unimodularity

The results of the previous subsection are perhaps surprising. Without needing to specify *any* details of the microsopic theory generated by our SCFT, the coarse data of the homology lattice $\Lambda_0^* = H_2(B_{\mathrm{orb}}, \mathbb{Z})$ is sufficient to extract the details of the anomaly polynomial for the 6D theory. In this sense, one might loosely refer to Λ_0^* as the "anomaly lattice", since this suffices to fix these properties of the macroscopic theory.

However, Λ_0^* cannot be interpreted as the string charge lattice of the model, because the condition of charge quantization and unimodularity would be clearly violated. Recall that for a 6D field theory on the tensorial Coulomb branch, Dirac quantization imposes the condition that there is an integer valued pairing for the complete Hilbert space of states. Indeed, superconformal systems have their own tensor multiplets, and the charge quantization and unimodularity constraints are expected to hold only when *all* scalars belonging to tensor multiplets are given vevs. The fact that Λ_0^* cannot be interpreted as the string charge lattice of the model is not so surprising: such lattice has to include all strings coming from the SCFTs as well. In the geometry, this is the requirement that the intersection form on the resolved geometry $\hat{B}$ is valued in the integers. Said differently, the resolution $\hat{B}$ provides a refinement of the lattice of fractional divisors:

$$H_2(B, \mathbb{Z}) \subset H_2(\hat{B}, \mathbb{Z}) \tag{3.27}$$

or equivalently:

$$\Lambda_0^* \subset \Lambda_{\mathrm{string}} . \tag{3.28}$$

Now, upon projecting $\hat{B}$ to the homology lattice of B, we lose some data. That is, if we have two seven-branes wrapping different divisors in $\hat{B}$, the image under the projection map

may be identical. Nevertheless, the gauge instantons can have inequivalent charges with respect to the tensor multiplets that are part of the superconformal theory. The homology class of a divisor β of B is only *part* of the information that specifies β: a divisor is an algebro-geometric object rather than a topological one. In particular, by knowing β, we can also compute the homology class of the divisor $\hat{\beta}$ obtained from β upon the resolution of B to $\hat{B}$. When a seven-brane with gauge group G wraps the divisor β of B the anomaly coefficient of the gauge group G is identified with $[\beta] \in \Lambda_0^*$. However, the string charge of a unit G-instanton is given by $[\hat{\beta}] \in \Lambda$.

This "loss of data" is really a hallmark of having an SCFT coupled to gravity, and can occur even when there is no orbifold singularity present in the base. For example, the superconformal matter of [18, 19] comes about when we have a collision of two components of the discriminant locus, at which point the elliptic fiber becomes too singular to satisfy the Calabi-Yau condition. Introducing the requisite blowups at this collision point, we obtain additional curves in the base geometry. These additional curves produce a refinement of the original lattice of BPS charges. Observe, however, that by construction, the blowdown of these extra curves leads us back to a smooth base.

3.4 Example: the $\mathcal{T}_p(N, M)$ theories

To illustrate some of the above points, we will now turn to some explicit non-compact examples. We shall couple these examples to gravity in section 4. We introduce the theories $\mathcal{T}_p(N, M)$ which are defined by intersecting two collections of non-compact seven-branes with respective gauge groups $\mathrm{SU}(N)$ and $\mathrm{SU}(M)$ at a point in the base with an A_{p-1} singularity, which is a $\mathbb{Z}_p$ orbifold singularity of the form $\mathbb{C}^2/\mathbb{Z}_p$, where the groups acts on the holomorphic coordinates (u, v) of $\mathbb{C}^2$ as $(u, v) \mapsto (\omega u, \omega^{-1} v)$, where ω is a primitive p^{th} root of unity.

In the terminology of [18], the $\mathcal{T}_p(N, N)$ can be indentified with $T\big(\mathrm{SU}(N), p-1\big)$ theories, while the $\mathcal{T}_p(N, M)$ with $N \neq M$ are examples of $T\big(\mathrm{SU}(N), p-1\big)$ theories with decorations by T-brane data. Equivalently, these are engineered in Type IIA with a non-zero Romans mass — using Nahm pole boundary conditions for D8-D6-NS5 systems [51].

An important feature of these theories is that although they involve the collision of two non-compact seven-branes, the "matter" living at the intersection point is itself a strongly coupled superconformal theory. In other words, this is an example of a "superconformal matter" system in the sense of reference [18, 19]. As such, they are also an excellent test case for studying the structure of 6D anomaly cancellation and charge quantization.

Let us give more details on the geometric realization of these theories. We consider F-theory on the base $B_{\mathrm{orb}} = \mathbb{C}^2/\mathbb{Z}_p$. Since this base is already Calabi-Yau, we can actually consider a trivial fibration. Then, we get the A_{p-1} type $(2, 0)$ theory. When the elliptic fibration is non-trivial and contains non-abelian seven-branes we get a $(1, 0)$ SCFT with additional seven-branes in the geometry. These seven-branes can either wrap the compact cycles obtained by resolving the orbifold singularity, or can also correspond to non-compact divisors. In fact, 6D anomaly cancellation usually correlates these contributions. As we will see shortly, these theories turn out to have fractional anomaly coefficients, quantized in units of p^{-1}. This is compatible with the fact that the intersection numbers of Weil

divisors on a surface with an A_{p-1} singularity can have fractional intersection numbers in units of p^{-1}.

For expository purposes, we focus on the case where there is an $SU(N)$ seven-brane supported on a non-compact divisor $u = 0$, and an $SU(N + pk)$ seven-brane supported on another non-compact divisor $v = 0$. These seven-branes pass through the orbifold fixed point, and so to properly cancel all 6D anomalies, we can expect additional light degrees of freedom to be present. As a point of notation, let $\widetilde{B}$ denote the covering space for $B_{\rm orb} = \mathbb{C}^2/\mathbb{Z}_p$. Now let us consider the divisor D_U, defined by the equation $u = 0$. The locus of the divisor D_U is well defined — the locus $u = 0$ on the covering manifold $\widetilde{B}$ is Γ-invariant. The polynomial defining the divisor, u, however, is not Γ-invariant. In more mathematical terms, it does not lie in the ring of rational functions of B. The same goes for the divisor D_V, defined by the equation $v = 0$. These divisors are "fractional" — they are Weil, but not Cartier. The Cartier divisors of B are defined by elements of the ring of rational functions of B, which is generated by the Γ-invariant combinations

$$u^p, \quad v^p, \quad \text{and} \quad uv \tag{3.29}$$

of u and v.

Now let us consider an elliptic fibration over B with an A_N singularity along the divisor U. Writing the local Weierstrass model for the elliptic fibration, we see that when N is not divisible by p, an additional singularity must be present along the divisor D_V. That is, using the local coordinates of the cover $\widetilde{B}$, we see that in order for the Weierstrass model for the fibration to be Γ-invariant, it must be of the form

$$xy = u^N v^{N+pk}, \tag{3.30}$$

for some integer k such that $N + pk$ is non-negative. The SCFT lying at the orbifold point then has $SU(N) \times SU(N+pk)$ global symmetry, and we denote this theory as $\mathcal{T}_p(N, N+pk)$.

The SCFT can be taken to a generic point in its tensor branch by blowing up the A_{p-1} singularity, and arriving at the non-singular manifold $\hat{B}$. The resolution divisors

$$D_1, \cdots, D_{p-1} \tag{3.31}$$

are (-2) curves whose intersection matrix is given (up to an overall minus sign) by the Cartan matrix of $SU(p)$. Let D_1 be the divisor adjacent to the $SU(N)$ locus and D_{p-1} be adjacent to the $SU(N + pk)$ divisor in the resolved manifold. The effective theory of the SCFT on the tensor branch is a theory with $(p-1)$ tensor fields with

$$SU(N + k) \times SU(N + 2k) \times \cdots \times SU\big(N + (p-1)k\big) \tag{3.32}$$

gauge symmetry, where for each pair of adjacent gauge groups, there exists a bifundamental hypermultiplet. There are also N hypermultiplets of $SU(N+k)$ that can be thought of as a bifundamental between $SU(N+k)$ and the $SU(N)$ flavor group, and $N+pk$ hypermultiplets of $SU\big(N+(p-1)k\big)$ that can be thought in an analogous way. The geometry of the resolved singularity is shown in figure 1. The one-loop contribution of the effective fields to the total
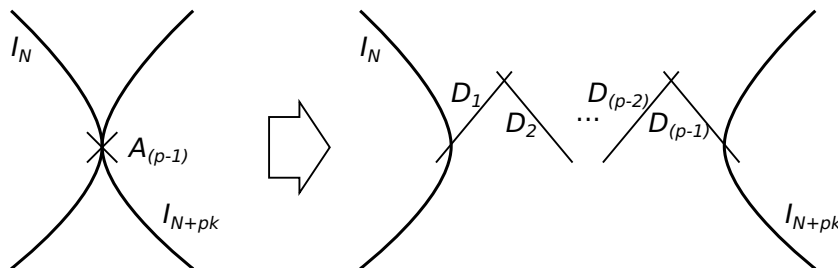
Figure 1. The tensor branch of the SCFT $\mathcal{T}_p(N, N+pk)$. On the left, $\mathcal{T}_p(N, N+pk)$ is localized at the A_{p-1} locus which two flavor branes, each of type I_N and I_{N+pk}, pass through. The A_{p-1} singularity in the base is resolved on the right, by introducing the resolution divisors D_m, hence moving on to the tensor branch of the theory. A singular fiber of type I_{N+mk} fibers over the divisor D_m. The effective gauge group of the tensor branch theory can thus be identified as $\prod_{m=1}^{p-1} \mathrm{SU}(N + mk)$.

anomaly polynomial is then given by

$$
\begin{aligned}
I_{S,1\ell} = {} & \frac{1}{5760}\left(30p - 30 + N^2 + Nkp + \frac{p(p-1)}{2}k^2\right)\left(\operatorname{tr} R^4 + \frac{5}{4}\operatorname{tr} R^2\right) \\
& - \frac{(p-1)}{128}(\operatorname{tr} R^2)^2 - \frac{1}{4}\sum_{m=1}^{p-1}(\operatorname{tr} F_m^2)^2 + \frac{1}{4}\sum_{m=1}^{p-2}\operatorname{tr} F_m^2 \operatorname{tr} F_{m+1}^2 \\
& + \frac{1}{4}\operatorname{tr} F_0^2 \operatorname{tr} F_1^2 + \frac{1}{4}\operatorname{tr} F_{p-1}^2 \operatorname{tr} F_p^2 \\
& - \frac{1}{96}(N + k)\operatorname{tr} R^2 \operatorname{tr} F_0^2 - \frac{1}{96}\big(N + (p-1)k\big)\operatorname{tr} R^2 \operatorname{tr} F_p^2 \\
& + \frac{1}{24}(N + k)\operatorname{tr} F_0^4 + \frac{1}{24}\big(N + (p-1)k\big)\operatorname{tr} F_p^4 \,,
\end{aligned}
\tag{3.33}
$$

where the first term comes from counting all the effective fields:

$$
29(p-1) - \sum_{m=1}^{p-1}\big((N + mk)^2 - 1\big) + \sum_{m=0}^{p-1}(N + mk)(N + mk + k)\,.
\tag{3.34}
$$

We have used F_m to denote the field strength of the $\mathrm{SU}(N + mk)$ gauge group. We have also included the anomalies of the flavor symmetries, whose background field strengths are denoted by F_0 and F_p.

Taking the basis of the homology lattice to be the divisors (3.31), the anomaly coefficients of the theory are given by

$$
D_m X^m = 2\hat{D}^1 \operatorname{tr} F_0^2 + 2\hat{D}^{p-1} \operatorname{tr} F_p^2 + \sum_{m=1}^{p-1} 2D_m \operatorname{tr} F_m^2\,.
\tag{3.35}
$$

We have also included the anomaly coefficients for the flavor symmetries. Note that there is no $\operatorname{tr} R^2$ term in (3.35). This is because the canonical class of the base manifold is trivial, as it is a Calabi-Yau resolution of an A_{p-1} singularity. Recall that the coefficient of the $\operatorname{tr} R^2$ term in the four-form $D_m X^m$ is given by the projection of the canonical class of the

base to the divisors D_m (3.20). The coefficient of $\text{tr}\,R^2$ obtained this way only depends on the local geometry of the resolution, and does not depend on the global embedding of the singularity. The divisor $\hat{D}^m$ denotes the dual divisor of D_m:

$$\hat{D}^m \cdot D_n = \delta_n^m\,. \tag{3.36}$$

$\hat{D}^1$ and $\hat{D}^{p-1}$ can be explicitly written as

$$\hat{D}^1 = \frac{1}{p}\sum_m (m-p)D_m\,, \qquad \hat{D}^{p-1} = -\frac{1}{p}\sum_m mD_m\,. \tag{3.37}$$

Not coincidentally, the choice for the flavor anomaly coefficients in (3.35) is the unique choice that cancels the one-loop anomalies involving matter jointly charged under the flavor and gauge symmetries — i.e., the third line of (3.33) — consistent with the picture of [32, 50]. Geometrically, $\hat{D}^1$ and $\hat{D}^p$ have the interpretation as the projection of the flavor brane locus on the resolved manifold $\hat{B}$ to the compact basis D_m. The total anomaly of the theory, then, is given by

$$
\begin{aligned}
I_S &= I_{S,1\ell} - \frac{1}{32}(D_m \cdot D_n)X^m X^n \\
&= \frac{1}{5760}\left(30p - 30 + N^2 + Nkp + \frac{p(p-1)}{2}k^2\right)\left(\text{tr}\,R^4 + \frac{5}{4}\,\text{tr}\,R^2\right) \\
&\quad - \frac{(p-1)}{128}(\text{tr}\,R^2)^2 - \frac{1}{96}(N+k)\,\text{tr}\,R^2\,\text{tr}\,F_0^2 - \frac{1}{96}\big(N+(p-1)k\big)\,\text{tr}\,R^2\,\text{tr}\,F_p^2 \\
&\quad + \left[\frac{1}{24}(N+k)\,\text{tr}\,F_0^4 + \frac{1}{8}\frac{(p-1)}{p}(\text{tr}\,F_0^2)^2\right] \\
&\quad + \left[\frac{1}{24}\big(N+(p-1)k\big)\,\text{tr}\,F_p^4 + \frac{1}{8}\frac{(p-1)}{p}(\text{tr}\,F_p^2)^2\right] + \frac{1}{4p}\,\text{tr}\,F_0^2\,\text{tr}\,F_p^2\,.
\end{aligned} \tag{3.38}
$$

Here, we have used the fact that

$$\hat{D}^1 \cdot \hat{D}^1 = \hat{D}^p \cdot \hat{D}^p = \frac{1}{p} - 1\,, \qquad \hat{D}^1 \cdot \hat{D}^p = -\frac{1}{p}\,. \tag{3.39}$$

All the terms involving the strongly coupled gauge fields are cancelled, as desired. Observe that the coefficients of the $(\text{tr}\,F^2)^2$ terms are quantized in units of $1/8p$. When the flavor symmetries are gauged, this coefficient contributes to the gauge anomaly of the gauge groups $\text{SU}(N)$ and $\text{SU}(N+pk)$. Recall that the $(\text{tr}\,F^2)^2$ terms of vector or hypermultiplets are quantized in units of $1/8$ [21]. Hence the contribution of the superconformal matter to the gauge anomaly can be interpreted as contributing in fractional units of p^{-1}. In the following section, we present examples where these SCFTs appear in global F-theory backgrounds.

4   F-theory on $X \to \mathbb{P}^2/\mathbb{Z}_3$

In this section, we study a particular example of an F-theory compactification on a manifold whose base has orbifold singularities. We consider an elliptically fibered Calabi-Yau

threefold $X \to B$, where $B = \mathbb{P}^2/\mathbb{Z}_3$. The orbifold group action is defined such that it acts on the projective coordinates (U, V, W) of the $\mathbb{P}^2$, which we often denote $\widetilde{B}$, by

$$(U, V, W) \to (\omega U, \omega^2 V, W) \tag{4.1}$$

for the cube-root of unity ω. This orbifold has three fixed-points at $U = V = 0$, $V = W = 0$ and $W = U = 0$, where the geometry is locally a

$$\mathbb{C}^2/\mathbb{Z}_3 \times T^2. \tag{4.2}$$

At each fixed point, the action involves both primitive third roots of unity, so this theory has three $(2, 0)$ A_2 theories sitting at these loci.[11]

This orbifold of $\mathbb{P}^2$ is useful to think about for a number of reasons. Its cohomology is simple to describe, and all the orbifold points are codimension-two. Furthermore, when the complex structure of the manifold is generic, the discriminant locus avoids all the orbifold singularities. Also, there are points in the complex structure moduli space where the gauge seven-brane does intersect these orbifold points. Hence, it is useful to investigate how certain superconformal sectors show up as we tune the complex structure moduli; for example, we can move a seven-brane on top of the orbifold fixed point. The methods for analyzing this particular model, however, are expected to generalize to other bases that have more complicated SCFTs generically.

The complex structure deformations of this model are represented by the allowed coefficients of the usual Weierstrass model of the elliptic fibration over $\mathbb{P}^2$ that are invariant under the orbifold action (4.1). Recall that the Weierstrass coefficients are given, in projective coordinates, by

$$f_{12} = \sum_{l+m+n=12} f_{l,m,n} U^l V^m W^n, \qquad g_{18} = \sum_{l+m+n=18} g_{l,m,n} U^l V^m W^n. \tag{4.3}$$

In order for f_{12} and g_{18} to be invariant under the $\mathbb{Z}_3$ action, only coefficients of terms with

$$l \equiv m \equiv n \mod 3 \tag{4.4}$$

are allowed to be nonzero. Thus, there are 95 nonzero coefficients in (4.3). To get the number of complex structure deformations of the orbifold, we must subtract the number of automorphisms of the fibration, which is given by a $(\mathbb{C}^*)^3$ action. Two of the automorphisms are those of the base that leave the fixed point invariant — they act on U/W and V/W. The other is an overall scaling of the base. The $(\mathbb{C}^*)^3$ action then can be understood as a rescaling of the three homogeneous coordinates. We hence arrive at 92 complex structure deformations, all of which can be identified with hypermultiplets. Then, the total number of hypermultiplets in the theory becomes 93, by adding the hypermultiplet controlling the size of the base manifold. Meanwhile, the homology $H_2(B, \mathbb{R})$ of the base manifold is generated by a single element, that can be lifted to three times of the hyperplane class $3H$ of the covering $\mathbb{P}^2$. Therefore, the effective theory of this compactification

---

[11]This is because the action could also be written in the two equivalent forms $(U, V, W) \to (U, \omega V, \omega^2 W)$ and $(U, V, W) \to (\omega^2 U, V, \omega W)$.

on a generic point in complex moduli space is given by a $(1,0)$ supergravity theory with no tensor multiplets, 93 hypermultiplets and three A_2 theories.

Let us confirm that the anomalies are cancelled in this effective theory. The computation of [52, 53] implies that for Q coincident M5-branes (see also [54]),

$$I_Q = \frac{Q}{48}\left(-p_2 + \frac{1}{4}p_1^2\right). \tag{4.5}$$

The anomaly of the A_2 theory can be obtained from $I_{Q=3}$ by subtracting the anomaly of a free $(2,0)$ tensor multiplet — it is given by

$$I_{A_2} = I_{Q=3} - I_{Q=1} = \frac{1}{24}\left(-p_2 + \frac{1}{4}p_1^2\right) = \frac{1}{96}\left(\operatorname{tr} R^4 - \frac{1}{4}(\operatorname{tr} R^2)^2\right) \tag{4.6}$$

which is precisely the anomaly polynomial of two $(2,0)$ tensor multiplets. Then, the total anomaly polynomial of the theory is given by

$$I_{\text{tot}} = \frac{1}{5760}(93 - 273)\left(\operatorname{tr} R^4 + \frac{5}{4}(\operatorname{tr} R^2)^2\right) + \frac{9}{128}(\operatorname{tr} R^2)^2 + 3I_{A_2} = \frac{3}{128}(\operatorname{tr} R^2)^2. \tag{4.7}$$

This anomaly polynomial exactly coincides with

$$\frac{1}{32}\left(\frac{K}{2}\cdot\frac{K}{2}\right)(\operatorname{tr} R^2)^2 = \frac{1}{128}\left(\frac{\widetilde{K}\cdot\widetilde{K}}{3}\right)(\operatorname{tr} R^2)^2, \tag{4.8}$$

where we use tilded variables to denote divisors in the manifold before orbifolding (which in this case is $\mathbb{P}^2$), while untilded variables are used to denote divisors in the orbifolds. In this equation, K is the canonical class of the base orbifold while $\widetilde{K}$ is the canonical class of $\mathbb{P}^2$. Recall that $\widetilde{K} = -3H$, so that

$$\widetilde{K}\cdot\widetilde{K} = 9. \tag{4.9}$$

Now given that the divisor C_i of B can be lifted to a divisor $\widetilde{C}_i$ in its covering space $\widetilde{B}$, the relation

$$C_i \cdot C_j = \frac{1}{3}\widetilde{C}_i \cdot \widetilde{C}_j \tag{4.10}$$

holds. Thus we see that the anomalies of the theory are cancelled by the GSSW anomaly cancellation mechanism, with the gravitational anomaly coefficient given by the canonical class of the base, just as we have explained in the previous section. Note that the formula (4.10) suggests that certain divisors in the base have fractional intersection numbers. We explore the case when there is enhanced gauge symmetry over such loci shortly.

Upon moving to special points in the complex structure moduli space of X, the theory can acquire enhanced gauge symmetry. For the sake of concreteness, we only concern ourselves with $\mathrm{SU}(N)$ gauge symmetry — generalizations of our results to other gauge groups is expected to be straightforward. The effective supergravity theory with $\mathrm{SU}(N)$ gauge symmetry differs depending on the nature of the seven-brane locus σ that carries the gauge symmetry. In classifying the behavior of σ, it is useful to examine the behavior of the divisor $\widetilde{\sigma}$ on $\widetilde{B}$ obtained by lifting σ to the cover of B. The irreducible divisor σ can then be one of the following:

1. The divisor σ does not intersect any orbifold points.

   (a) $\widetilde{\sigma}$ is also irreducible on $\widetilde{B}$.

   (b) $\widetilde{\sigma}$ is reducible on $\widetilde{B}$.

2. The divisor σ intersects an orbifold point.

   (a) σ is a Cartier divisor.

   (b) σ is Weil, but not Cartier.

When σ does not intersect any of the orbifold points, the supergravity theory develops an SU(N) gauge symmetry whose charged matter only consist of hypermultiplets. As we show shortly, when $\widetilde{\sigma}$ is irreducible on $\widetilde{B}$, its genus is at least one. Consequently, the genus of σ is also at least one, and the theory has an adjoint hypermultiplet. An interesting phenomenon happens when $\widetilde{\sigma}$ is not irreducible on $\widetilde{B}$. In this case, $\widetilde{\sigma}$ factors into three divisors, which project down to the single irreducible divisor σ on B. In this case, σ develops double points. Then, one of the global adjoint hypermultiplets becomes localized at this point along with a neutral hypermultiplet.

While a divisor σ that does not intersect any orbifold points is always Cartier, in the event that σ intersects an orbifold point, its defining equation might not be a well-defined element of the ring of rational functions on the manifold. In the case that σ is a Cartier divisor, the point in complex structure moduli space where σ hits the orbifold point can be approached from case 1-(a) by tuning the complex structure modulus that controls the location of σ. As σ hits the orbifold point, the A_2 theory is enhanced to the SCFT $\mathcal{T}_3(N, N)$ with SU(N) × SU(N) global symmetry, whose diagonal group is gauged by the SU(N) gauge group. In fact, the A_2 SCFT, an adjoint and a neutral hypermultiplet are traded for this new SCFT. It is interesting to understand the string charge of the SU(N) instantons of the theory. These instantons are charged under the tensor degrees of freedom in the SCFT, and hence the string charge lies in the homology lattice of $\hat{B}$. When σ is not Cartier, something more drastic happens. The gauge divisor now has a fractional self-intersection number — these fractional anomaly coefficients cancel the anomalies of the SCFTs sitting at the orbifold loci, which come in fractional units.

In the following, we examine each case in more detail. Before doing so, however, we first describe the geometry and topology of the manifold B and its resolution $\hat{B}$ in more detail in subsection 4.1. In subsection 4.2, we investigate the case when the gauge divisor does not cross the orbifold locus. In subsection 4.3, we discuss the case when it does.

4.1 The geometry and topology of $\mathbb{P}^2/\mathbb{Z}_3$

Let us review the geometry of the orbifold B. The integral sublattice of the homology lattice of the orbifold is spanned by a single element h, which lifts to three times the homology class of the hyperplane in $\mathbb{P}^2$:

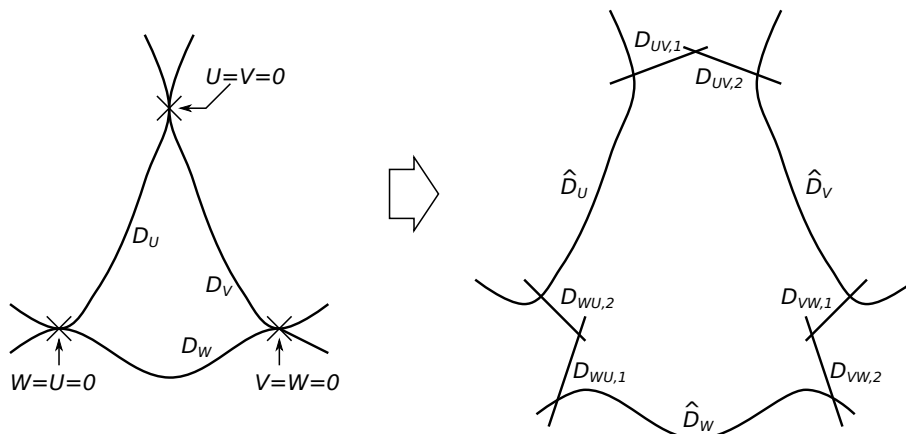$$\widetilde{h} = 3H. \tag{4.11}$$

Figure 2. $B = \mathbb{P}^2/\mathbb{Z}_3$ and its resolution $\hat{B}$. $\hat{B}$ is a dP$_6$. There are six resolution divisors $D_{xy,a}$ that resolve the three A_2 singularities of B. The divisors D_x of B are mapped to divisors $\hat{D}_x$. Each pair of adjacent divisors in the diagram have intersection number 1.

The self-intersection number of h is given by

$$h \cdot h = \frac{1}{3}(3H \cdot 3H) = 3 \,. \tag{4.12}$$

Thus, the integral sublattice of the homology lattice of B is not unimodular. B has fractional divisors whose homology class come in fractions of h. In fact, the basis vector for the full homology lattice is given by $h/3$.

We focus our attention on the Weil divisors D_U, D_V and D_W that come from projections of the divisors U, V and W of $\widetilde{B}$. The homology class of these divisors are given by

$$[D_U] = [D_V] = [D_W] = \frac{1}{3}h \,. \tag{4.13}$$

We use the square brackets to denote the homology class of a divisor. It is simple to see that

$$D_x \cdot D_y = \frac{1}{3}(H \cdot H) = \frac{1}{3} \tag{4.14}$$

for any pair of $x, y \in \{U, V, W\}$, which is consistent with (4.13).

Upon resolving the geometry by blowing up the three A_2 singularities of B, we arrive at $\hat{B}$, which is a del Pezzo surface of degree three, or equivalently, a dP$_6$ manifold. This blow up can be interpreted as going on the tensor branch of the three SCFTs localized at the three orbifold points. There are different resolutions of the orbifold singularities that are related to each other by flops. These flops correspond to going to different chambers of the tensor branch of the SCFT. To be unambiguous, we choose a particular resolution in the succeeding discussions, but it is straightforward to incorporate flops into the picture. Upon resolving the singularities, the divisors D_x map into divisors $\hat{D}_x$ of the manifold $\hat{B}$. Let us denote the two resolution divisors that come from resolving the singularity at $x = y = 0$ by $D_{xy,a}$ with $a = 1, 2$. The tensor branch parameters, in this particular chamber, can be identified with the sizes of the six cycles $D_{xy,a}$. This resolution is depicted in figure 2.

The seven homology classes

$$[\hat{D}_U], \; [D_{UV,1}], \; [D_{UV,2}], \cdots, \; [D_{WU,2}] \tag{4.15}$$

form a basis for $H_2(\hat{B}, \mathbb{R})$, though it does not quite span the full integral homology lattice of $\hat{B}$. $\hat{D}_x$ are rational curves with self-intersection (-1), while the resolution divisors are rational curves with self-intersection (-2). Hence the intersection matrix of the divisors (4.15) can be read off of the diagram on the right-hand-side of figure 2. A more relevant basis for $H_2(\hat{B}, \mathbb{R})$ for our discussion, of course, is one that is spanned by $D_{xy,a}$ and an integral homology class orthogonal to $D_{xy,a}$, namely

$$h = 3[\hat{D}_U] + 2[D_{WU,2}] + [D_{WU,1}] + 2[D_{UV,1}] + [D_{UV,2}]. \tag{4.16}$$

This class has self-intersection 3, while it is orthogonal to all the resolution divisors, and hence can be identified with the basis of the integral sublattice of the homology lattice of B. Since all the divisors $D_{xy,a}$ are (-2) rational curves, the canonical class $\hat{K}$ of $\hat{B}$ is orthogonal to these — in fact, its homology class is given by

$$[\hat{K}] = -h. \tag{4.17}$$

The integral homology lattice $H_2(\hat{B}, \mathbb{Z})$ is spanned by seven elements, e_i with $i = 0, \cdots, 6$. A dP$_6$ surface can be thought of as a smooth $\mathbb{P}^2$ blown up at six generic points. The element e_0 is the hyperplane class of the original $\mathbb{P}^2$, while the six elements e_i are the exceptional cycles coming from the blow-ups. The intersection matrix between these elements is given by

$$(e_i \cdot e_j) = \text{diag}(1, -1, -1, -1, -1, -1, -1). \tag{4.18}$$

The elements h and $[D_{xy,a}]$ can be related to e_i by

$$\begin{aligned} h &= 3e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6, \\ [D_{UV,1}] &= -e_1 + e_2, \quad [D_{UV,2}] = -e_2 + e_3, \quad [D_{VW,1}] = e_4 - e_5, \quad [D_{VW,2}] = e_5 - e_6, \\ [D_{WU,1}] &= -e_0 + e_1 + e_2 + e_3, \qquad\qquad\qquad [D_{WU,2}] = -e_0 + e_4 + e_5 + e_6. \end{aligned} \tag{4.19}$$

Now while the homology classes of D_U, D_V and D_W were the same in B, we observe that the homology classes of $\hat{D}_U$, $\hat{D}_V$ and $\hat{D}_W$ differ. In fact,

$$[\hat{D}_U] = \frac{1}{3}h - \frac{1}{3}[D_{WU,1}] - \frac{2}{3}[D_{WU,2}] - \frac{2}{3}[D_{UV,1}] - \frac{1}{3}[D_{UV,2}] \tag{4.20}$$

$$[\hat{D}_V] = \frac{1}{3}h - \frac{1}{3}[D_{UV,1}] - \frac{2}{3}[D_{UV,2}] - \frac{2}{3}[D_{VW,1}] - \frac{1}{3}[D_{VW,2}] \tag{4.21}$$

$$[\hat{D}_W] = \frac{1}{3}h - \frac{1}{3}[D_{VW,1}] - \frac{2}{3}[D_{VW,2}] - \frac{2}{3}[D_{WU,1}] - \frac{1}{3}[D_{WU,2}]. \tag{4.22}$$

It is a simple exercise to check that the self-intersection numbers of these homology classes are indeed given by (-1). The fractional coefficients reflect the fact that h and $D_{xy,a}$ do not span the full integral homology lattice of $\hat{B}$. When these divisors are written as a linear

combinations of the basis elements e_i, in fact, the coefficients are integral. It is evident from this formula that the projection of $\hat{D}_x$ down to the sublattice of $H_2(\hat{B}, \mathbb{Z})$ spanned by h all become $h/3$. The homology class of Cartier divisors D of B that do not intersect orbifold loci in the class $[D] = nh$ remain the same through the blow up. Meanwhile, as we see in section 4.3, Cartier divisors that intersect an orbifold locus may contain the resolution divisors as componenets upon blowing up the orbifold points.

Let us explore the physical implications of the facts presented. Given a generic elliptic fibration $X \to B$, there exists BPS strings of the six-dimensional F-theory compactification on X obtained by wrapping D3-branes on divisors of B. If the D3-brane is wrapping a divisor D that intersects an orbifold locus, it is not enough to know the homology class of D in B to determine its full string charge. The full string charge is given by the homology class of $\hat{D}$ in $H_2(\hat{B}, \mathbb{Z})$. While the homology class of the three divisors $D_{U,V,W}$ in B are equivalent, we see explicitly from equation (4.22) that the homology classes of $\hat{D}_{U,V,W}$ differ.

Meanwhile, as we have shown in the previous section, given that there is a gauge brane wrapping a divisor of B, the anomaly coefficient of the gauge group still can be identified with the homology class of that divisor within B. For example, the anomaly coefficient of a gauge group obtained by wrapping a brane on D_U is given by $h/3$. The string charge of a unit instanton of that gauge group, however, is given by the homology class of $\hat{D}_U$ in (4.22). As noted before, when a gauge brane is wrapping a divisor intersecting an orbifold point, the anomaly coefficient cannot be identified with the string charge of the unit instanton of that gauge group.

4.2 Enhanced gauge symmetry without charged superconformal matter

Let us now consider loci of the complex moduli space of X where there is an SU(N) gauge symmetry along the locus σ. The lift of $\widetilde{\sigma}$ of σ to $\widetilde{B} = \mathbb{P}^2$ must be of the form:

$$\widetilde{\sigma} = p(u^3, v^3, uv) \,, \tag{4.23}$$

where p is a polynomial in three variables, and u and v are local coordinates of the $\mathbb{P}^2$ in the chart $W = 1$. The class of $\widetilde{\sigma}$ hence is always given by a multiple of $3H$, consistent with the fact that then

$$\sigma \cdot \sigma = \widetilde{\sigma} \cdot \widetilde{\sigma}/3 \,, \qquad \sigma \cdot K = \widetilde{\sigma} \cdot \widetilde{K}/3 \tag{4.24}$$

is integral. The genus g of σ is then given by

$$g = \frac{1}{2}(\sigma \cdot \sigma + K \cdot \sigma + 2) = \frac{1}{6}(\widetilde{\sigma} \cdot \widetilde{\sigma} - 3H \cdot \widetilde{\sigma} + 6) \,. \tag{4.25}$$

The gauge theory on σ can be thought of as a "quotient" of the gauge theory on $\widetilde{\sigma}$ living on $\widetilde{B}$, in the following sense. The gauge theory on $\widetilde{\sigma}$ has

$$\widetilde{g} = \frac{1}{2}(\widetilde{\sigma} \cdot \widetilde{\sigma} - 3H \cdot \widetilde{\sigma} + 2) = 3g - 2 \tag{4.26}$$

global adjoint hypermultiplets [55, 56]. The other matter come from loci where $\widetilde{\sigma}$ intersects the rest of the discriminant locus, or where $\widetilde{\sigma}$ itself develops a singularity. Now the

Weierstrass equation of the theory on $\widetilde{B}$ must be restricted to be invariant under the $\mathbb{Z}_3$ action. Then, since the $\mathbb{Z}_3$ action acts freely on the locus $\widetilde{\sigma}$, as it does not cross through any orbifold points, such loci come in triplets. Hence the rest of the matter, other than the $\widetilde{g}$ adjoint matter, come in triplets. Upon quotienting by $\mathbb{Z}_3$, the gauge theory on σ has g adjoint hypermultiplets. The rest of the charged matter spectrum can be obtained by quotienting the charged matter of the theory on $\widetilde{B}$ that come from codimension-two singularities by three, as each triplet of singular loci on $\widetilde{B}$ reduces to a single codimension-two locus on B.

To be more concrete, let us consider the class of supergravity theories with $\mathrm{SU}(N)$ gauge symmetry on B that can be obtained by quotienting a theory on $\mathbb{P}^2$ with the following matter content in the non-abelian sector [57]:

$$\mathrm{SU}(N): \quad (72 - 9N) \times \square + 9 \times \boxminus + 1 \times \mathrm{Adj}, \qquad [\widetilde{\sigma}] = 3H, \tag{4.27}$$

for $N \leq 8$. As indicated, the cohomology class of $\widetilde{\sigma}$ in this case, is given by three-times the hyperplane class. The divisor $\widetilde{\sigma}$ can be written as

$$\widetilde{\sigma}: \quad au^3 + bv^3 + c + duv = 0. \tag{4.28}$$

We use $\widetilde{\sigma}$ to both denote the divisor itself as well at its defining equation. The genera $\widetilde{g}$ and g are given by

$$\widetilde{g} = g = 1. \tag{4.29}$$

Hence, the theory on $\mathbb{P}^2$ has a single adjoint global multiplet, while the rest of the matter come from codimension-two singularities that can be organized into triplets by acting on the $\mathbb{P}^2$ with the $\mathbb{Z}_3$ action. The non-abelian gauge group of the theory on B then, is given by $\mathrm{SU}(N)$ under which the representations of the charged matter content are given by

$$\mathrm{SU}(N): \quad (24 - 3N) \times \square + 3 \times \boxminus + 1 \times \mathrm{Adj}, \qquad [\sigma] = h. \tag{4.30}$$

Recall that h is the homology class on B that lifts to the homology class of $3H$ on $\widetilde{B}$, i.e.,

$$\widetilde{h} = 3H. \tag{4.31}$$

Note that

$$h \cdot h = 3, \qquad h \cdot K = -3. \tag{4.32}$$

The theory has three A_2 SCFTs coupled to the gravity theory as well. The gauge and mixed anomaly cancellation conditions

$$\frac{1}{16}(K \cdot h)\, \mathrm{tr}\, R^2\, \mathrm{tr}\, F^2 = -\frac{1}{96}\, \mathrm{tr}\, R^2 \left[(24 - 3N)\, \mathrm{tr}\, F^2 + 3\, \mathrm{tr}_{\square} F^2 + \mathrm{tr}_{\mathrm{Adj}} F^2 - \mathrm{tr}_{\mathrm{Adj}} F^2 \right]$$
$$\frac{1}{8}(h \cdot h)(\mathrm{tr}\, F^2)^2 = \frac{1}{24}\left[(24 - 3N)\, \mathrm{tr}\, F^4 + 3\, \mathrm{tr}_{\square} F^4 + \mathrm{tr}_{\mathrm{Adj}} F^4 - \mathrm{tr}_{\mathrm{Adj}} F^4 \right] \tag{4.33}$$

involving the $\mathrm{SU}(N)$ gauge group is then satisfied. The relations

$$\mathrm{tr}_{\boxminus} F^2 = (N - 2)\, \mathrm{tr}\, F^2, \qquad \mathrm{tr}_{\boxminus} F^4 = (N - 8)\, \mathrm{tr}\, F^4 + 3(\mathrm{tr}\, F^2)^2 \tag{4.34}$$

are needed to show the equality (4.33). We also note that

$$\mathrm{tr}_{\mathrm{Adj}}\, F^2 = 2N\, \mathrm{tr}\, F^2, \qquad \mathrm{tr}_{\mathrm{Adj}}\, F^4 = 2N\, \mathrm{tr}\, F^4 + 6(\mathrm{tr}\, F^2)^2 \tag{4.35}$$

for future reference. The anomaly coefficient of the $\mathrm{SU}(N)$ gauge group can hence be identified as h, which is the class of the $\mathrm{SU}(N)$ divisor σ.

It is possible to verify the gauge and mixed anomaly equations for any theory on B whose gauge brane σ is smooth, irreducible and does not cross the orbifold locus. This is because the lifted theory to $\widetilde{B}$, which is a theory with gauge group G, $\widetilde{g}$ global adjoint hypermultiplets and $3n_{\mathbf{R}}$ hypermultiplets in the representation $\mathbf{R}$, is also a consistent theory that must satisfy the gauge and mixed anomaly equations

$$\frac{1}{16}(\widetilde{K}\cdot\widetilde{\sigma})\,\mathrm{tr}\, R^2\, \mathrm{tr}\, F^2 = -\frac{1}{96}\,\mathrm{tr}\, R^2 \left[\sum_{\mathbf{R}} 3n_{\mathbf{R}}\, \mathrm{tr}_{\mathbf{R}}\, F^2 + (\widetilde{g}-1)\, \mathrm{tr}_{\mathrm{Adj}}\, F^2 \right]$$
$$\frac{1}{8}(\widetilde{\sigma}\cdot\widetilde{\sigma})(\mathrm{tr}\, F^2)^2 = \frac{1}{24}\left[\sum_{\mathbf{R}} 3n_{\mathbf{R}}\, \mathrm{tr}_{\mathbf{R}}\, F^4 + (\widetilde{g}-1)\, \mathrm{tr}_{\mathrm{Adj}}\, F^4 \right]. \tag{4.36}$$

The gauge and mixed anomaly cancellation equations for the theory on B can be obtained from these equations by dividing both sides of the equations by three. This follows from the fact that the charged matter content of B consists of

$$g = \frac{1}{3}(\widetilde{g}-1) + 1 \tag{4.37}$$

global adjoint hypermultiplets and $n_{\mathbf{R}}$ hypermultiplets that come from local singularities, and that the intersection numbers between the anomaly coefficients σ and K of the theory are related to those of $\widetilde{\sigma}$ and $\widetilde{K}$ by (4.24). This proof generalizes straightforwardly to any supergravity theory on B, with any gauge group (with multiplet semi-simple factors) whose gauge divisors are smooth, irreducible in $\widetilde{B}$ and avoid all the orbifold points.

The number of neutral hypermultiplets ν_H of the theories (4.30) can be computed using the gravitational anomaly constraint:

$$H - V + 29T + \Delta_S = 273\,, \tag{4.38}$$

where $H/V/T$ are the number of hyper/vector/tensor multiplets of the theory, and Δ_S is the contribution of the strongly coupled sector to the gravitational anomaly. In the event that none of the gauge branes cross the orbifold loci, we have computed

$$\Delta_S = 3 \times 60 = 180\,, \tag{4.39}$$

since each of the three $(2,0)$ A_2 theories contribute to the gravitational anomaly as much as two hyper and two tensor multiplets do.

Now the F-theory models (4.30) can have abelian factors, given that the abelian gauge group cannot be Higgsed away without breaking the non-abelian gauge group. As we explain in more detail in appendix B, it turns out some members of the family of F-theory models (4.30) *automatically* have an additional U(1) factor, all of whose charged particles

N	2	3	4	5	6	7	8
w	56	38	26	17	12	9	8
ν_H	57	39	27	18	13	10	9
r_{MW}	0	0	0	0	1	1	0

Table 2. The number of free complex parameters of Weierstrass models w vs. the number of neutral hypermultiplets ν_H for F-theory models on $\mathbb{P}^2/\mathbb{Z}_3$ whose non-abelian gauge group is given by $SU(N)$. The total gauge group of the theory is given by $SU(N) \times U(1)^{r_{MW}}$. For the $SU(6)$ and $SU(7)$ cases, the abelian gauge group is non-trivial.

have non-trivial non-abelian representations (see table 2). Hence, for this class of theories, the number of vector and hypermultiplets are given by

$$V = r_{MW} + N^2 - 1, \qquad H = \begin{cases} \nu_H + 36, & \text{when } N = 2 \\ \nu_H - \frac{1}{2}N^2 + \frac{45}{2}N - 1, & \text{when } N \geq 3 \end{cases}, \qquad (4.40)$$

where r_{MW} is the number of abelian factors, or equivalently, the Mordell-Weil rank of the elliptic fibration [3]. The counting of charged matter is slightly different for the theory with an $SU(2)$ gauge group, as the antisymmetric representation is neutral in that case. The gravitational anomaly constraint shows that the number of neutral hypermultiplets is given by

$$\nu_H = \begin{cases} 57, & \text{when } N = 2 \\ 93 + \frac{3}{2}N(N - 15) + r_{MW}, & \text{when } N \geq 3. \end{cases} \qquad (4.41)$$

It is quite interesting to compare the number of neutral hypermultiplets of the $SU(N)$ theories with the number of free complex coefficients of the Weierstrass model. As before, the number of complex coefficients can be enumerated by counting the number of complex coefficients of Weierstrass models over the covering space $\mathbb{P}^2$ that are $\mathbb{Z}_3$-invariant. The general form of Weierstrass models with enhanced $SU(N)$ symmetry has been systematically studied in [57].

We conclude this section by enumerating the number of free complex parameters for the $SU(2)$ theory and comparing with the number of neutral hypermultiplets (4.41), following [57]. The details of the general counting for $SU(N)$ with $N \leq 8$ is presented in appendix B, while the results are collected in table 2. By the usual arguments [2, 35, 57], the complex degrees of freedom of the Weierstrass model can represent at most $(\nu_H - 1)$ neutral hypermultiplets. The additional hypermultiplet comes from the overall scaling of the base. We see that this bound is saturated for all of the models we examine. According to [57], Weierstrass coefficients of a manifold that has an $SU(2)$ singularity along the locus $\widetilde{\sigma}$ must be of the form:

$$\begin{aligned} f &= -\frac{1}{48}\phi^2 + f_1\widetilde{\sigma} + f_2\widetilde{\sigma}^2 \\ g &= \frac{1}{864}\phi^3 - \frac{1}{12}\phi f_1\widetilde{\sigma} + g_2\widetilde{\sigma}^2. \end{aligned} \qquad (4.42)$$

$\mathbb{Z}_3$-invarance imposes that all individual factors appearing in this expression should also be $\mathbb{Z}_3$-invariant, since $\widetilde{\sigma}$ is chosen to be a $\mathbb{Z}_3$-invariant locus by assumption.

Now we can fix the coefficients of ϕ and f_1 such that it is of the form

$$\phi = \varphi_0 u^2 v^2 + \varphi_3(v^3)uv + \varphi_6(v^3)$$
$$f_1 = f_{1,3}(v^3)u^2 v^2 + f_{1,6}(v^3)uv + f_{1,9}(v^3)\,. \tag{4.43}$$

This is done by replacing any multiple of u^3 appearing in ϕ or f_1 by using the relation (4.28). The polynomials φ_{3n}, $f_{1,3n}$ are polynomials of v^3 of degree $3n$ in v. For example, $\varphi_6(v^3)$ is of the form

$$\varphi_6 = p_6 v^6 + p_3 v^3 + p_0\,, \tag{4.44}$$

and so on. Meanwhile, f_2 and g_2 are generic $\mathbb{Z}_3$ invariant polynomials with maximum degree 6 and 12 in the local coordinate variables, respectively;

$$f_2(u^3, v^3, uv)\,, \qquad g_2(u^3, v^3, uv)\,. \tag{4.45}$$

Summing all the number of complex coefficients present in the model, including the coefficients of $\widetilde{\sigma}$ (4.28), we find 60 complex coefficients in total. Now there is a rescaling symmetry that leaves f and g invariant, given by

$$\widetilde{\sigma} \to t\widetilde{\sigma}\,, \qquad f_1 \to t^{-1}f_1\,, \qquad f_2 \to t^{-2}f_2\,, \qquad g_2 \to t^{-2}g_2\,. \tag{4.46}$$

This symmetry, along with the $(\mathbb{C}^*)^3$ automorphism group of the elliptic fibration, cuts down the number of free complex coordinates to 56, which agrees with $(\nu_H - 1)$ of the theory.

The discriminant locus of the Weierstrass model with the coefficients (4.42) is of the form

$$\frac{1}{16}\widetilde{\sigma}^2 \left\{ \phi^2 \left(\frac{1}{12}f_2\phi^2 + g_2\phi - f_1^2 \right) + \mathcal{O}(\widetilde{\sigma}) \right\}\,. \tag{4.47}$$

In $\widetilde{B} = \mathbb{P}^2$, the A_1 singularity on $\widetilde{\sigma}$ is enhanced to A_2 at the $18 \times 3 = 54$ points where $\widetilde{\sigma}$ and

$$\frac{1}{12}f_2\phi^2 + g_2\phi - f_1^2 = 0 \tag{4.48}$$

intersect. These points come in triplets, which are exchanged amongst themselves upon acting with the $\mathbb{Z}_3$ action. Hence, in B, there are 18 points where the I_2 fiber along σ enhances to an I_3 fiber. These points are where the 18 fundamental matter of the SU(2) group are localized on. There is no additional matter lying at the loci where $\widetilde{\sigma}$ and ϕ meet, as the I_2 fiber along $\widetilde{\sigma}$ becomes a type III fiber at these loci — there is no increase of rank in this case.

When $\widetilde{\sigma}$ becomes factorizable, i.e., when the coefficients of its defining equation (4.28) satsify

$$d^3 + 27abc = 0\,, \tag{4.49}$$

an interesting situation occurs. The factors of $\widetilde{\sigma}$,

$$\widetilde{\sigma} = (a^{1/3}u + b^{1/3}v + c)(\omega a^{1/3}u + \omega^2 b^{1/3}v + c)(\omega^2 a^{1/3}u + \omega b^{1/3}v + c)\,, \tag{4.50}$$

are not invariant divisors on B. Therefore the divisor σ is still irreducible on B. Meanwhile, each pair of the three factors of $\widetilde{\sigma}$ meet with each other at a point. The three intersection
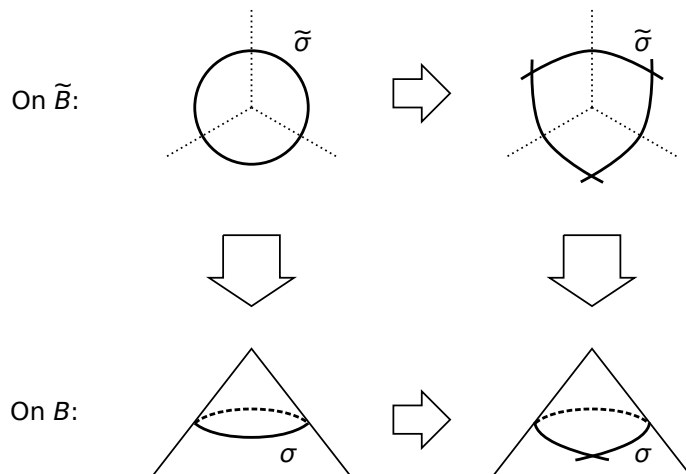
Figure 3. A schematic picture of $\widetilde{\sigma}$ as it becomes reducible in $\widetilde{B}$. The upper diagrams depict the locus of the divisor $\widetilde{\sigma}$ (bold curves) on $\widetilde{B}$, while the lower diagrams depict its projection, σ (also bold curves) on B. The orbifold B is depicted as a cone, while the dotted lines on $\widetilde{B}$ are used to show the fundamental domain of $\widetilde{B}$ under the orbifold action. When $\widetilde{\sigma}$ is irreducible (left), its projection is a smooth divisor on B. Meanwhile, when $\widetilde{\sigma}$ become reducible (right), it factors into three copies of divisors related by the $\mathbb{Z}_3$ action. Upon projection to B, σ develops a double-point.

points of these divisors on $\widetilde{B}$ project to a single double point of σ in B. The situation is sketched in figure 3. In principle, when a gauge brane is wrapping a curve with a double point, the matter localized at the double point locus can either be given by a pair of hypermultiplets in the adjoint and the trivial representations, or in the symmetric and anti-symmetric representations [57]. In this case, however, the double point locus can be reached by a continuous deformation of the parameters in σ. We can therefore conclude that the matter localized at the double point is an adjoint and a neutral hypermultiplet. The neutral hypermultiplet can be identified with the combination (4.49) of coefficients of $\widetilde{\sigma}$. The effective theory on B hence remains the same at this locus, despite the development of the singularity on σ.

4.3 Enhanced gauge symmetry with charged superconformal matter

We now examine the case when the SU(N) gauge brane passes through an orbifold locus. Let us first consider the case that the gauge brane is wrapping a Cartier divisor σ. This situation arises by starting at a point in the complex structure of the moduli space of X where the gauge brane locus $\sigma(c)$ is a Cartier divisor that does not intersect the orbifold point. Here we have used c to denote the parameter of $\sigma(c)$ that needs to be tuned to reach the orbifold point. Then, we can make the gauge brane cross the orbifold singularity by tuning the coefficient c to a particular value c_0. For example, for the class of SU(N) theories studied in the previous subsection, we can tune the coefficient c of the equation (4.28) to zero so that

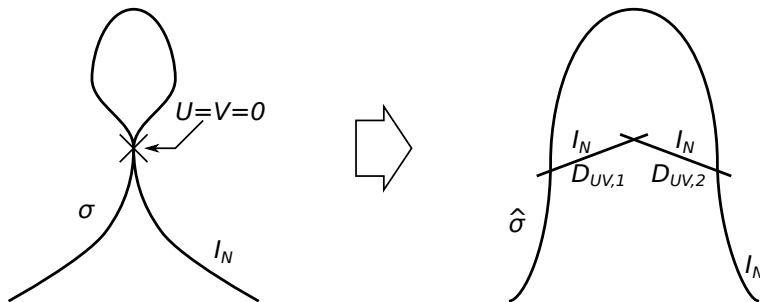$$\widetilde{\sigma}(c = 0): \quad au^3 + bv^3 + duv = 0 \tag{4.51}$$

Figure 4. A diagram depicting the resolution of the A_2 singularity at $U = V = 0$ when the gauge brane σ carrying an I_N singularity passes through. The two resolution divisors $D_{UV,1}$ and $D_{UV,2}$ each have a I_N singularity along them.

passes through the orbifold singularity at $u = v = 0$. In fact, given that we want to make the divisor hit this orbifold point, there is a single coefficient c that we need to tune to zero to do so — the constant term in $\widetilde{\sigma}(c)$, when $\widetilde{\sigma}$ is written as a polynomial in u and v.

Let us assume that the supergravity theory with non-zero c had g global adjoint hypermultiplets and $n_{\mathbf{R}}$ local hypermultiplets for each representation $\mathbf{R}$ of the gauge group $SU(N)$. Upon tuning c to zero, a global adjoint hypermultiplet, the A_2 theory sitting at the orbifold point, along with the neutral hypermultiplet degree of freedom parametrized by c, enhances into a $\mathcal{T}_3(N, N)$ SCFT whose diagonal $SU(N)$ global symmetry group is gauged. The rest of the matter remains the same, as the local codimension-two singularities merely shift their positions as c is taken to zero.

Now let us check the anomaly cancellation conditions are satisfied for this theory. Assuming that the anomaly of the theory with $c \neq 0$ is cancelled, it follows that the anomaly of the theory with $\mathcal{T}_3(N, N)$ is also cancelled with the same anomaly coefficients, due to the relation

$$
\begin{aligned}
I_S = {} & \frac{1}{5760}(60 + N^2)\operatorname{tr} R^4 - \frac{2}{128}(\operatorname{tr} R^2)^2 \\
& - \frac{1}{96}(2N)\operatorname{tr} R^2 \operatorname{tr} F^2 + \frac{1}{24}\big(2N \operatorname{tr} F^4 + 6(\operatorname{tr} F^2)^2\big) = I_{A_2} + I_{H,\mathrm{Adj}} + I_{H,\mathrm{neutral}}\,,
\end{aligned}
\tag{4.52}
$$

where I_S is the anomaly polynomial of the SCFT $\mathcal{T}_3(N, N)$, computed in section 3.2. The anomaly polynomials for the traded fields add up precisely to the anomaly polynomial of the SCFT! The anomaly polynomial I_S can be obtained from equation (3.38) by setting

$$
F_0 = F_p = F\,,
\tag{4.53}
$$

for the gauge fields strength F of the $SU(N)$ group.

Let us now move to a generic point in the tensor branch of the SCFT $\mathcal{T}_3(N, N)$. By doing so, we resolve the A_2 singularity at $U = V = 0$, as shown in figure 4. As noted previously, when a Cartier divisor of B does not intersect an orbifold locus, its homology class remains in the sublattice of $\hat{B}$ spanned by h even after the blow up. In the case we are considering, however, σ intersects the orbifold locus — the divisor $\hat{\sigma}$ obtained by blowing up σ contains $D_{UV,a}$ as its components. This is signified by the fact that $\hat{\sigma}$ has non-zero

intersection numbers with the resolution divisors, as can be seen in figure 4. The homology class of $\hat\sigma$ is given by

$$[\hat\sigma] = h - [D_{UV,1}] - [D_{UV,2}]. \tag{4.54}$$

Note that

$$\hat\sigma \cdot \hat\sigma = 1, \qquad \hat\sigma \cdot \hat K = -3, \tag{4.55}$$

The charged matter with respect to the SU(N) gauge symmetry living on $\hat\sigma$ is given by the $(24 - 3N)$ fundamentals and the three antisymmetrics along with two bifundamentals each living at the intersection point between $\hat\sigma$ and $D_{UV,1}$, $D_{UV,2}$. The gauge and mixed anomaly cancellation conditions for this gauge group component is consistent with the intersection numbers (4.55).

By inspection of the geometry, we see that while the anomaly coefficient of the SU(N) gauge group on B should be identified with $h \in H_2(B, \mathbb{Z})$, the string charge of the instantons of the gauge group cannot lie within $H_2(B, \mathbb{Z})$. The instantons are charged under tensor multiplets of the SCFT $\mathcal{T}_3(N, N)$, as we see that the divisor $\hat\sigma$ intersects the resolution divisors $D_{UV,a}$. This should be contrasted to the case when a Cartier divisor of B does not intersect an orbifold locus, in which the divisor does not contain any resolution divisors as a component. The string charge of the unit SU(N) instanton should be identified with the homology cycle of $\hat\sigma$ given by the element (4.55) of the homology lattice of $\hat B$.

Let us now discuss what happens when the gauge brane wraps a Weil divisor σ that is not Cartier. In this case, the intersection numbers of the brane become fractional. For the remainder of the section, we explore the case where the Calabi-Yau manifold has I-type singularities along the simplest fractional divisors — those that can be lifted to $U = 0$, $V = 0$ and $W = 0$. As before, let us denote these divisors on B, D_U, D_V and D_W, respectively. Recall that these divisors have fractional intersection numbers:

$$D_x \cdot D_y = \frac{1}{3}, \qquad D_x \cdot K = -1 \tag{4.56}$$

for any pair of $x, y \in \{U, V, W\}$. The homology class of D_x are all given by $h/3$. Upon lifting to the covering manifold $\mathbb{P}^2$, we obtain models with I-type singularities along divisors in the hyperplane class H. Such SU(N) models have been studied in [57, 58]. A particular model that is simple to engineer is one with the following matter:

$$\text{SU}(N): \quad (24 - N) \times \square + 3 \times \boxminus. \tag{4.57}$$

Note that there is no adjoint hypermultiplet, as the seven-brane wraps a $\mathbb{P}^1$. For example, such a theory can be engineered on $\mathbb{P}^2$ with an I_N singularity fibered over $U = 0$. As explained in the previous section, a brane configuration on B cannot be lifted to a configuration on $\widetilde B$ with only an I_N singularity at $U = 0$. In fact, given that there is an I_N fiber along $U = 0$, there must be corresponding singular fibers at the other loci $V = 0$ and $W = 0$. Owing to the $\mathbb{Z}_3$ invariance of the elliptic fibration, the total seven-brane charges must be the same mod 3, so we can introduce integers k and k', and assign an I_{N+3k} fiber along $V = 0$ and an $I_{N+3k'}$ fiber along $W = 0$. The discriminant locus Δ of this model on $\mathbb{P}^2$ becomes

$$\Delta = U^N V^{N+3k} W^{N+3k'} \mathcal{F}, \tag{4.58}$$

in projective coordinates, where $\mathcal{F}$ can be written as

$$\mathcal{F} = \phi_{0,U}^4 \Phi_U + \mathcal{O}(U) = \phi_{0,V}^4 \Phi_V + \mathcal{O}(V) = \phi_{0,V}^4 \Phi_W + \mathcal{O}(W). \qquad (4.59)$$

Here, $\phi_{0,x}$ are sections of $3H$, while Φ_x are sections of $3(8 - N - k - k')H$. F-theory compactified on this Calabi-Yau fibration over $\mathbb{P}^2$ would yield a supergravity theory with gauge group

$$G_U \times G_V \times G_W \equiv \mathrm{SU}(N) \times \mathrm{SU}(N + 3k) \times \mathrm{SU}(N + 3k') \qquad (4.60)$$

and the following matter:

$$
\begin{aligned}
(24 - 3N - 3k - 3k') &\times \left\{ (\square, \cdot, \cdot) + (\cdot, \square, \cdot) + (\cdot, \cdot, \square) \right\} \\
&+ 3 \times \left\{ (\boxminus, \cdot, \cdot) + (\cdot, \boxminus, \cdot) + (\cdot, \cdot, \boxminus) \right\} + (\square, \square, \cdot) + (\cdot, \square, \square) + (\square, \cdot, \square).
\end{aligned} \qquad (4.61)
$$

The fundamental matter come from the intersections of gauge branes D_x and Φ_x, while the antisymmetrics come from the intersections between D_x and $\phi_{0,x}$. The bifundamental matter lie at the intersections between gauge branes. As expected, the values of k and k' are bounded above and below. For example, we have the upper bound:

$$3N + 3k + 3k' \leq 24 \qquad (4.62)$$

and the lower bounds:

$$N \geq 0, \qquad N + 3k \geq 0, \qquad N + 3k' \geq 0. \qquad (4.63)$$

Upon orbifolding this theory, we arrive at a theory on B with an I_N, I_{N+3k} and $I_{N+3k'}$ singularity along D_U, D_V and D_W, respectively. The divisors ϕ_0 and F become sections of h and $(8 - N - k - k')h$. Each fractional divisor meets the projection of ϕ_0 at a point where an antifundmental hypermultiplet lies, and the projection of F at $(8 - N - k - k')$ points, where fundamental multiplets are localized. The charged hypermultiplet spectrum is thus given by

$$(8 - N - k - k') \times \left\{ (\square, \cdot, \cdot) + (\cdot, \square, \cdot) + (\cdot, \cdot, \square) \right\} + \left\{ (\boxminus, \cdot, \cdot) + (\cdot, \boxminus, \cdot) + (\cdot, \cdot, \boxminus) \right\}. \qquad (4.64)$$

Meanwhile, at the orbifold points $U = V = 0$, $V = W = 0$ and $W = U = 0$, lies the strongly coupled SCFTs $\mathcal{T}_3(N, N + 3k)$, $\mathcal{T}_3(N + 3k, N + 3k')$, and $\mathcal{T}_3(N, N + 3k')$, whose global symmetries have now been weakly gauged. A schematic diagram of the configuration of the divisors is given in figure 5.

Let us now check the gauge and mixed anomaly equations for the gauge groups. We neglect the gravitational anomaly to avoid further cluttering of equations, but comment on it later on. The contribution to the gauge and mixed anomalies from the vector and hypermultiplets can be computed to be

$$
\begin{aligned}
I_{1\ell}^{g,m} = &-\frac{1}{16} \operatorname{tr} R^2 (\operatorname{tr} F_U^2 + \operatorname{tr} F_V^2 + \operatorname{tr} F_W^2) - \frac{1}{8} \left((\operatorname{tr} F_U^2)^2 + (\operatorname{tr} F_V^2)^2 + (\operatorname{tr} F_W^2)^2 \right) \\
&+ \frac{1}{96} \operatorname{tr} R^2 \left\{ (2N + k + k') \operatorname{tr} F_U^2 + (2N + 4k + k') \operatorname{tr} F_V^2 + (2N + k + 4k') \operatorname{tr} F_W^2 \right\} \\
&- \frac{1}{24} \left\{ (2N + k + k') \operatorname{tr} F_U^4 + (2N + 4k + k') \operatorname{tr} F_V^4 + (2N + k + 4k') \operatorname{tr} F_W^4 \right\}.
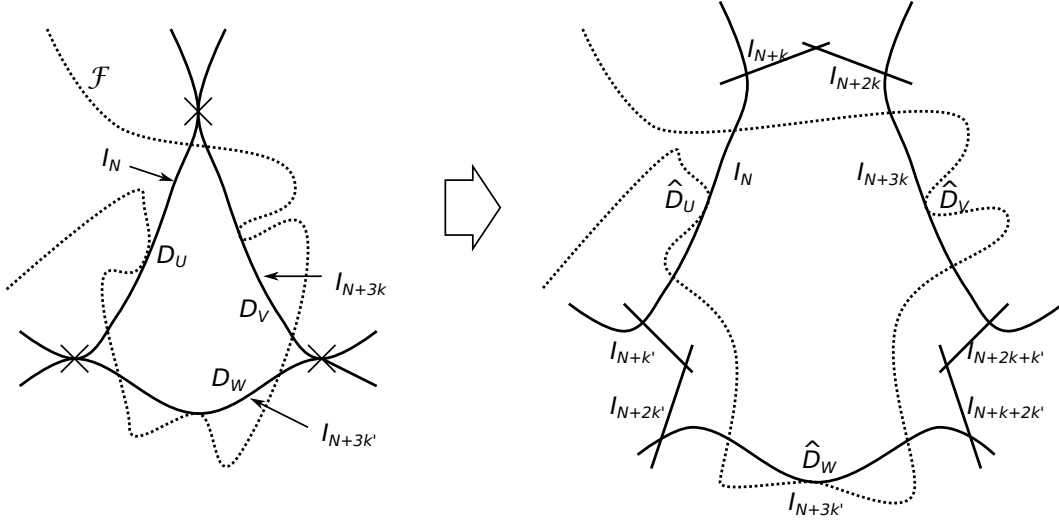\end{aligned} \qquad (4.65)
$$

Figure 5. A schematic diagram of the configuration of divisors on B (left) and $\hat{B}$ (right). The singularity type of the elliptic fiber over each divisor is indicated. On the left, D_x are the fractional divisors, while the dotted line represents $\mathcal{F}$, the residual divisor of the discriminant of the elliptic fibration. The points where the fundamentals matter of G_x lie are represented by points where $\mathcal{F}$ meets D_x transversally, and the points where antisymmetrics lie are represented by the points where $\mathcal{F}$ meets D_x tangentially. Each pair of divisors D_x and D_y meet at a single orbifold point, where superconformal matter jointly charged under $G_x \times G_y$ lie. On the right, $\hat{D}_x$ are integral divisors on $\hat{B}$ obtained by resolving D_x. The theory now has only ordinary matter. In particular, there exist bifundamental matter at the intersection loci of adjacent divisors.

The field strength F_x is that of the gauge symmetry that lies above the divisor D_x. Using the anomaly polynomial for the strongly coupled SCFTs computed in (3.38), we find that the total gauge and mixed anomaly contribution is given by

$$
\begin{aligned}
I_S^{g,m} = &\frac{1}{6}\big\{(\operatorname{tr} F_U^2)^2 + (\operatorname{tr} F_V^2)^2 + (\operatorname{tr} F_W^2)^2\big\} + \frac{1}{12}(\operatorname{tr} F_U^2 \operatorname{tr} F_V^2 + \operatorname{tr} F_V^2 \operatorname{tr} F_W^2 + \operatorname{tr} F_W^2 \operatorname{tr} F_U^2) \\
&- \frac{1}{96}\operatorname{tr} R^2\big\{(2N + k + k')\operatorname{tr} F_U^2 + (2N + 4k + k')\operatorname{tr} F_V^2 + (2N + k + 4k')\operatorname{tr} F_W^2\big\} \\
&+ \frac{1}{24}\big\{(2N + k + k')\operatorname{tr} F_U^4 + (2N + 4k + k')\operatorname{tr} F_V^4 + (2N + k + 4k')\operatorname{tr} F_W^4\big\}.
\end{aligned}
$$

(4.66)

The second and third lines for the expressions of $I_{1\ell}$ and I_S cancel upon summing the two contributions of the total anomaly:

$$
I_{1\ell}^{g,m} + I_S^{g,m} = -\frac{1}{16}\operatorname{tr} R^2(\operatorname{tr} F_U^2 + \operatorname{tr} F_V^2 + \operatorname{tr} F_W^2) + \frac{1}{8} \times \frac{1}{3}(\operatorname{tr} F_U^2 + \operatorname{tr} F_V^2 + \operatorname{tr} F_W^2)^2. \quad (4.67)
$$

We have isolated the factor of $1/3$ to emphasize the fractional quantization of the gauge anomaly term. Recall that in theories with only conventional multiplets contributing to the anomaly, the gauge anomaly term is quantized in units of $1/8$. The gauge and mixed anomaly terms are cancelled by the GSSW mechanism with the gauge anomaly coefficients of the $SU(N)$, $SU(N + 3k)$ and $SU(N + 3k')$ groups taken to be D_U, D_V and D_W,

respectively:

$$I_{GS} = \frac{1}{32} \left\{ \frac{K}{2} \operatorname{tr} R^2 + 2D_U \operatorname{tr} F_U^2 + 2D_V \operatorname{tr} F_V^2 + 2D_W \operatorname{tr} F_W^2 \right\}^2. \tag{4.68}$$

Using the intersection relations (4.56), we find that the $\operatorname{tr} R^2 \operatorname{tr} F^2$ and $(\operatorname{tr} F^2)^2$ terms of I_{GS} agrees precisely with equation (4.67). The gravitational anomaly cancellation condition is satisfied with

$$\nu_H = 6 + \frac{3}{2}(8 - N - k - k')(7 - N - k - k') \tag{4.69}$$

neutral hypermultiplets.

Upon moving to a generic point in the tensor branch of all the SCFTs in the chamber studied in section 4.1, we can resolve the manifold B to $\hat{B}$. Now the effective theory is a supergravity theory with six tensor multiplets, and a gauge group that consists of nine special-unitary components. The divisor configuration on $\hat{B}$ is depicted on the right panel of figure 5. The string charge of the instanton of gauge groups G_x lies in the homology lattice of this manifold — it is given by $\hat{D}_x$, which have been written out explicitly in terms of the basis $\{h\} \coprod \{D_{xy,a}\}$ in (4.22). Notice that while we have shown that the anomaly coefficients of G_x all can be identified with $[D_x] = h/3$, the string charge of the G_x-instantons all differ.

Let us denote the gauge group with support above the resolution divisor $D_{xy,a}$ as $G_{xy,a}$. Then the matter content of the theory on $\hat{B}$ is simple to specify. The hypermultiplet matter that are only charged under

$$G_U \times G_V \times G_W = \mathrm{SU}(N) \times \mathrm{SU}(N + 3k) \times \mathrm{SU}(N + 3k') \tag{4.70}$$

is equivalent to that of the theory on B. The difference is that the superconformal theories are gone, and that there is a bifundamental hypermultiplet for each pair of adjacent divisors. Using this matter content, we find that the anomaly cancellation conditions are satisfied with the gauge anomaly coefficients given by the homology class of the loci of the gauge branes. In particular, it is simple to verify that the divisors $\hat{D}_x$ have self-intersection (-1) while $D_{xy,a}$ have self-intersection (-2), using the anomaly cancellation conditions. This is a simple consistency check that the field theory computation agrees with the geometry described in section 4.1.

5 Conclusions and future directions

In this paper we have studied the question of how to couple a 6D superconformal field theory to gravity. To accomplish this, we have studied 6D F-theory vacua compactified on an elliptic Calabi-Yau threefold in which two-cycles of the base collapse to zero size. In particular, we have shown that when the base has orbifold singularities, the data of the anomaly polynomial is correctly reproduced by the intersection theory of the orbifold base. We have also seen how charge quantization predicts the existence of additional light states — namely those of the SCFT — and how this can be interpreted as a refinement of the lattice spanned by the fractional divisors of the orbifold theory. We have also presented a

compact model which illustrates all of these elements. In the remainder of this section we discuss some potential directions for future investigation.

Our primary focus has been on recoupling a particular class of 6D SCFTs to gravity. Now, one result from [17] is a classification of all possible orbifold singularities which a non-compact F-theory model could possess. Locally, these are all of the form $\mathbb{C}^2/\Gamma$ for Γ a discrete subgroup of U(2). In particular, the exact list of possible Γ's has been determined. It would be very interesting to determine the subset of such SCFTs which can be recoupled to gravity.

One of the motivations for the present work has been to see how much of F-theory can be phrased purely in terms of the intersection theory of fractional divisors. We anticipate that this feature will be particularly important in the context of 4D vacua, where there may be obstructions to motion on the geometric moduli space. Along these lines, it would be quite interesting to develop a similar analysis for orbifolds of the form $\mathbb{C}^3/\Gamma$ for Γ a discrete subgroup of U(3).

Finally, there is a conceptual point connected with the coupling of a 6D SCFT to gravity. On the one hand, this is straightforward to realize using the geometry of an F-theory compactification. On the other hand, the absence of a Lagrangian description for these theories renders a purely field theoretic analysis quite subtle. It would be interesting to study in more detail how a 6D effective field theorist would infer (perhaps along the lines of [59, 60]) that conformal symmetry has been broken by having a finite Planck scale.

Acknowledgments

A Anomaly polynomials of supergravity multiplets

In this section, we review anomaly polynomials of 6D $(1, 0)$ supergravity multiplets. We use normalization conventions of [61], but with an overall minus sign. This is to make our conventions consistent with the anomaly calculations in the literature [32, 33, 49, 50, 52, 53, 62]. The list of "conventional multiplets" of six-dimensional supergravity whose anomaly polynomials we consider is given in table 3.

The anomaly polynomial of the gravity multiplet can be obtained by summing contributions from the self-dual tensor and the gravitino. It is given by

$$I_G = -\frac{273}{5760} \operatorname{tr} R^4 + \frac{17}{1536} (\operatorname{tr} R^2)^2 = -\frac{273}{5760} \left(\operatorname{tr} R^4 + \frac{5}{4} (\operatorname{tr} R^2)^2 \right) + \frac{9}{128} (\operatorname{tr} R^2)^2. \quad \text{(A.1)}$$

Multiplet	Field content
Gravity	$(g_{\mu\nu}, \psi_\mu^+, B_{\mu\nu}^+)$
Tensor	$(\phi, \chi^-, B_{\mu\nu}^-)$
Vector	(A_μ, λ^+)
Hyper	$(4\varphi, \psi^-)$

Table 3. Conventional multiplets of 6D $(1,0)$ supergravity theories. The superscripts on the fermions denote the chirality, while those on the antisymmetric tensors indicate self-duality/anti-self-duality.

The anomaly polynomial of the tensor multiplet can be obtained from contributions from its anti-self-dual tensor and fermion:

$$I_T = \frac{29}{5760} \operatorname{tr} R^4 - \frac{7}{4608} (\operatorname{tr} R^2)^2 = \frac{29}{5760} \left(\operatorname{tr} R^4 + \frac{5}{4} (\operatorname{tr} R^2)^2 \right) - \frac{1}{128} (\operatorname{tr} R^2)^2. \qquad (A.2)$$

Meanwhile, the contribution to the anomaly polynomials of the vector and hypermultiplets comes solely from its fermions. For a vector multiplet, the anomaly polynomial is given by

$$I_{V,G} = -\frac{1}{5760} \left(\operatorname{tr} R^4 + \frac{5}{4} (\operatorname{tr} R^2)^2 \right) (\operatorname{tr}_{\mathrm{Adj}} \mathbf{1}) + \frac{1}{96} \operatorname{tr} R^2 \operatorname{tr}_{\mathrm{Adj}} F^2 - \frac{1}{24} \operatorname{tr}_{\mathrm{Adj}} F^4, \qquad (A.3)$$

while the anomaly polynomial for a hypermultiplet charged in the representation $\mathbf{R}$ of the gauge group is given by

$$I_{H,\mathbf{R}} = \frac{1}{5760} \left(\operatorname{tr} R^4 + \frac{5}{4} (\operatorname{tr} R^2)^2 \right) (\operatorname{tr}_{\mathbf{R}} \mathbf{1}) - \frac{1}{96} \operatorname{tr} R^2 \operatorname{tr}_{\mathbf{R}} F^2 + \frac{1}{24} \operatorname{tr}_{\mathbf{R}} F^4. \qquad (A.4)$$

We omit the subscript R when taking the trace in the fundamental representation.

B SU(N) models on $B = \mathbb{P}^2/\mathbb{Z}_3$ with no charged strongly coupled sector

In this appendix, we count the degrees of freedom in the Weierstrass models that engineer the SU(N) theories of section 4.2. To be more precise, we write down a generic Weierstrass model of an elliptically fibered manifold over $B = \mathbb{P}^2/\mathbb{Z}_3$ whose low energy theory is a supergravity theory coupled to three $(2,0)$ A_2 theories that has an SU(N) gauge group with the following matter content:

$$\mathrm{SU}(N): \quad (24 - 3N) \times \square + 3 \times \boxminus + 1 \times \mathrm{Adj}, \qquad [\sigma] = h. \qquad (B.1)$$

The gauge and mixed anomaly cancellation involving the SU(N) gauge group for these models have been verified in equation (4.33). The Weierstrass model is given by

$$y^2 = x^3 + fx + g \qquad (B.2)$$

where f and g are sections of $12H$ and $18H$ of $\mathbb{P}^2$ that are invariant under the $\mathbb{Z}_3$ action, respectively. A section of nH is represented by a degree-n homogeneous polynomial of the

projective coordinates U, V and W. The SU(N) locus is represented by the divisor σ that lifts to the divisor

$$\widetilde{\sigma} = aU^3 + bV^3 + c + dUVW \,, \tag{B.3}$$

of the cover $\widetilde{B} = \mathbb{P}^2$ of B. We can follow the analysis of [57] that we have already used in section 4.2 to count the number of Weierstrass coefficients for SU(2) models. We continue applying this analysis to SU(N) models for $3 \leq N \leq 8$ in section B.1.

An interesting phenomenon can be observed in this particular class of models in F-theory. Given that there is a SU(6) or SU(7) non-abelian gauge symmetry, the F-theory model automatically turns out to have abelian gauge symmetry as well. In fact, there is an additional U(1) factor for both models. This can be confirmed by identifying a rational section of the elliptic fibration, and also by successively Higgsing the SU(8) model to arrive at the SU(6) and SU(7) model. We discuss issues related to the abelian gauge symmetry of the theory in section B.2.

B.1 Complex degrees of freedom in Weierstrass models

The number of complex degrees of freedom of the Weierstrass models can be systematically computed by expanding the Weierstrass coefficients f and g with respect to the SU(N) locus $\widetilde{\sigma}$ [57]:

$$f = \sum_i f_i \widetilde{\sigma}^i, \qquad g = \sum_i g_i \widetilde{\sigma}^i. \tag{B.4}$$

Then, the coefficients Δ_i of the discriminant locus Δ of the model in $\widetilde{\sigma}$,

$$\Delta = 4f^3 + 27g^2 = \sum_i \Delta_i \widetilde{\sigma}^i, \tag{B.5}$$

can be written in terms of f_i and g_i. Imposing that Δ has vanishing coefficients up to order $\widetilde{\sigma}^{N-1}$, we obtain constraints on the coefficients f_i and g_i. The degrees of freedom of the Weierstrass model is obtained by counting by the number of degrees of freedom in the solutions of these constraints.

SU(3): 38 complex degrees of freedom. For an SU(3) theory, the Weierstrass coefficients f and g must be of the form

$$\begin{aligned}
f &= -\frac{1}{48}\phi_0^4 + \frac{1}{2}\phi_0\psi_1\widetilde{\sigma} + f_2\widetilde{\sigma}^2 + f_3\widetilde{\sigma}^3 \\
g &= \frac{1}{864}\phi_0^6 - \frac{1}{24}\phi_0^3\psi_1\widetilde{\sigma} + \left(\frac{1}{4}\psi_1^2 - \frac{1}{12}\phi_0^2 f_2\right)\widetilde{\sigma}^2 + g_3\widetilde{\sigma}^3
\end{aligned} \tag{B.6}$$

ϕ_0, ψ_1, f_2, f_3 and g_3 are sections of $3H$, $6H$, $6H$, $3H$ and $9H$, respectively. While f_3 and g_3 are generic sections, the other degree-n homogeneous polynomials of the projective coordinates can always be reduced to the form,

$$W^n\big(p_0(v^3) + p_1(v^3)uv + p_2(v^3)u^2v^2\big) \tag{B.7}$$

where $u = U/W$, $v = V/W$, and $p_{0,1,2}$ are polynomials. We denote any polynomial of the form (B.7) to be in a "reduced form". This is due to the fact that these sections are $\mathbb{Z}_3$

invariant, and that any factor of u^3 can be replaced using the divisor $\widetilde{\sigma}$. For example, ϕ_0 can be written as

$$\phi_0 = W^3\big(\varphi_{0,0}(v^3) + \varphi_1 uv\big) \tag{B.8}$$

which has three complex degrees of freedom. Meanwhile, a generic $\mathbb{Z}_3$ invariant section of $3nH$ has

$$\binom{n+2}{2} + \binom{n+1}{2} + \binom{n}{2} \tag{B.9}$$

complex degrees of freedom. Summing up all the complex degrees of freedom available, including those in $\widetilde{\sigma}$, we see that there are 42 free coefficients in (B.6).

We must now subtract the number of symmetries and automorphisms available to arrive at the number of complex degrees of freedom of the Weierstrass model. On top of the $(\mathbb{C}^*)^3$ automorphism, there exists the $\mathbb{C}^*$ symmetry

$$\widetilde{\sigma} \to t\widetilde{\sigma}, \qquad \psi_1 \to t^{-1}\psi_1, \qquad f_n \to t^{-n}f_n, \qquad g_n \to t^{-n}g_n. \tag{B.10}$$

of the Weierstrass model. The final number of complex degrees of freedom is given by $42 - 4 = 38$.

SU(4): 26 complex degrees of freedom. The Weierstrass coefficients f and g must be of the form

$$f = -\frac{1}{48}\phi_0^4 - \frac{1}{6}\phi_0^2\phi_1\widetilde{\sigma} + f_2\widetilde{\sigma}^2 + f_3\widetilde{\sigma}^3 + f_4\widetilde{\sigma}^4$$

$$g = \frac{1}{864}\phi_0^6 + \frac{1}{72}\phi_0^4\phi_1\widetilde{\sigma} + \left(\frac{1}{36}\phi_0^2\phi_1^2 - \frac{1}{12}\phi_0^2 f_2\right)\widetilde{\sigma}^2 + \left(-\frac{1}{12}\phi_0^2 f_3 - \frac{1}{3}\phi_1 f_2 - \frac{1}{27}\phi_1^3\right)\widetilde{\sigma}^3 + g_4\widetilde{\sigma}^4. \tag{B.11}$$

ϕ_0, ϕ_1, f_2 and f_3 are sections of $3H$, $3H$, $6H$ and $3H$ in reduced form, while f_4 is a constant, and g_4 is a generic section of $6H$. The total number of complex coefficients in (B.11) is 30. Meanwhile, in addition to the $(\mathbb{C}^*)^3$ automorphism, there exists the $\mathbb{C}^*$ symmetry

$$\widetilde{\sigma} \to t\widetilde{\sigma}, \qquad \phi_1 \to t^{-1}\phi_1, \qquad f_n \to t^{-n}f_n, \qquad g_n \to t^{-n}g_n. \tag{B.12}$$

of the Weierstrass model. The number of complex degrees of freedom is given by $30 - 4 = 26$.

SU(5): 17 complex degrees of freedom. The Weierstrass coefficients f and g must be of the form

$$f = -\frac{1}{48}\phi_0^4 - \frac{1}{6}\phi_0^2\phi_1\widetilde{\sigma} + \left(\frac{1}{2}\phi_0\psi_2 - \frac{1}{3}\phi_1^2\right)\widetilde{\sigma}^2 + f_3\widetilde{\sigma}^3 + f_4\widetilde{\sigma}^4$$

$$g = \frac{1}{864}\phi_0^6 + \frac{1}{72}\phi_0^4\phi_1\widetilde{\sigma} + \left(\frac{1}{18}\phi_0^2\phi_1^2 - \frac{1}{24}\phi_0^3\psi_2\right)\widetilde{\sigma}^2$$

$$+ \left(-\frac{1}{12}\phi_0^2 f_3 - \frac{1}{6}\phi_0\phi_1\psi_2 + \frac{2}{27}\phi_1^3\right)\widetilde{\sigma}^3 + \left(\frac{1}{4}\psi_2^2 - \frac{1}{12}\phi_0^2 f_4 - \frac{1}{3}\phi_1 f_3\right)\widetilde{\sigma}^4 + g_5\widetilde{\sigma}^5. \tag{B.13}$$

ϕ_0, ϕ_1, ψ_2 and f_3 are sections of $3H$ of reduced form, while f_4 is a constant and g_5 is a generic section of $3H$. The total number of complex coefficients in (B.13) is 21. As before, in addition to the $(\mathbb{C}^*)^3$ automorphism, there is a $\mathbb{C}^*$ symmetry

$$\widetilde{\sigma} \to t\widetilde{\sigma}, \qquad \phi_1 \to t^{-1}\phi_1, \qquad \psi_2 \to t^{-2}\psi_2, \qquad f_n \to t^{-n}f_n, \qquad g_n \to t^{-n}g_n. \tag{B.14}$$

of the Weierstrass model. The number of complex degrees of freedom is given by $21 - 4 = 17$.

SU(6): 12 complex degrees of freedom. The Weierstrass coefficients f and g must be of the form

$$f = -\frac{\beta^4}{48}\alpha^4 - \frac{\beta^3}{6}\alpha^2\nu\widetilde{\sigma} + \left(-\frac{\beta\phi_2}{6}\alpha^2 - \frac{\beta^2}{3}\nu^2\right)\widetilde{\sigma}^2 + \left(-(3\beta)\lambda - \frac{\phi_2}{3}\nu\right)\widetilde{\sigma}^3 + f_4\widetilde{\sigma}^4$$

$$g = \frac{\beta^6}{864}\alpha^6 + \frac{\beta^5}{72}\alpha^4\nu\widetilde{\sigma} + \left(\frac{\beta^3\phi_2}{72}\alpha^4 + \frac{\beta^4}{18}\alpha^2\nu^2\right)\widetilde{\sigma}^2$$

$$+ \left(\frac{\beta^3}{4}\alpha^2\lambda + \frac{\beta^2\phi_2}{12}\alpha^2\nu + \frac{2\beta^3}{27}\nu^3\right)\widetilde{\sigma}^3$$

$$+ \left(\left(\frac{1}{36}\phi_2^2 - \frac{1}{12}\beta^2 f_4\right)\alpha^2 + (\beta^2)\lambda\nu + \frac{\beta\phi_2}{9}\nu^2\right)\widetilde{\sigma}^4 + \left((\phi_2)\lambda - \frac{\beta f_4}{3}\nu\right)\widetilde{\sigma}^5 + g_6\widetilde{\sigma}^6.$$

(B.15)

α, ν, λ are sections of $3H$ of reduced form, while β, ϕ_2, f_4 and g_6 are constants. The total number of complex coefficients in (B.15) is 17. On top of the $(\mathbb{C}^*)^3$ automorphism, there is a $(\mathbb{C}^*)^2$ symmetry

$$\widetilde{\sigma} \to t\widetilde{\sigma}\,, \quad \nu \to t^{-1}\nu\,, \quad \phi_2 \to t^{-2}\phi_2\,, \quad \lambda \to t^{-3}\lambda\,, \quad f_4 \to t^{-4}f_4\,, \quad g_6 \to t^{-6}g_6$$

$$\alpha \to s\alpha\,, \quad \beta \to s^{-1}\beta\,, \quad \nu \to s\nu\,, \quad \lambda \to s\lambda\,, \quad \phi_2 \to s^{-1}\phi_2$$

(B.16)

of the Weierstrass model. The number of complex degrees of freedom is given by $17-5 = 12$.

SU(7): 9 complex degrees of freedom. The Weierstrass coefficients f and g must be of the form (B.15) with

$$\lambda = \frac{1}{9\beta}\nu\phi_2 - \frac{1}{6}\psi_3\alpha\,, \qquad g_6 = -\frac{1}{27\beta^3}\phi_2^3 + \frac{1}{4}\psi_3^2 - \frac{1}{3\beta}\phi_2 f_4\,.$$

(B.17)

Now the moving pieces are α and ν, which are sections of $3H$ in reduced form, and the constants β, ϕ_2, ψ_3 and f_4, and of course, the brane locus $\widetilde{\sigma}$. The total number of complex coefficients is given by (B.15) is 14. As usual, there is a the $(\mathbb{C}^*)^3$ automorphism, and the $(\mathbb{C}^*)^2$ symmetry

$$\widetilde{\sigma} \to t\widetilde{\sigma}\,, \quad \nu \to t^{-1}\nu\,, \quad \phi_2 \to t^{-2}\phi_2\,, \quad \psi_3 \to t^{-3}\psi_3\,, \quad f_4 \to t^{-4}f_4\,,$$

$$\alpha \to s\alpha\,, \quad \beta \to s^{-1}\beta\,, \quad \nu \to s\nu\,, \quad \lambda \to s\lambda\,, \quad \phi_2 \to s^{-1}\phi_2$$

(B.18)

of the Weierstrass model. The number of complex degrees of freedom is given by $14-5 = 9$.

SU(8): 8 complex degrees of freedom. The SU(8) theory can be obtained from the SU(7) theory by taking

$$\psi_3 = 0\,.$$

(B.19)

Therefore the complex degrees of freedom of this theory is one less than that of the SU(7) theory, which is $9 - 1 = 8$.

The SU(8) model is maximal — it is the model with maximum rank in the family of models (B.1). The Weierstrass coefficient for such models is given by [57]

$$f = -\frac{1}{3}\Phi^2 + F_4\widetilde{\sigma}^4$$

$$g = \frac{2}{27}\Phi^3 - \frac{1}{3}F_4\Phi\widetilde{\sigma}^4$$

$$\Phi = \frac{1}{4}\phi_0^2 + \phi_1\widetilde{\sigma} + \phi_2\widetilde{\sigma}^2\,.$$

(B.20)

It is simple to see verify that this is indeed the case with the identification

$$\Phi = \frac{\alpha^2 \beta^2}{4} + \beta \nu \widetilde{\sigma} + \frac{\phi_2}{\beta} \widetilde{\sigma}^2, \qquad F_4 = f_4 + \frac{\phi_2^2}{3\beta^2}. \tag{B.21}$$

B.2 Abelian factors and the Mordell-Weil group

The Weierstrass models of the theories with SU(6) (B.15) and SU(7) (B.17) gauge symmetry both have a non-trivial Mordell-Weil group of rank one. This implies that the theory has a U(1) gauge group in addition to the non-abelian SU(N) group. This can come as a surprise from the way we have arrived at the models (B.15) and (B.17). We have successively tuned the Weierstrass coefficients to get higher-rank non-abelian gauge symmetry, and have not aimed at producing an elliptic fibration with a rational section.

From the top-down point of view, however, the existence of the U(1) factor is inevitable. The family of models (B.1) in F-theory are obtained from Higgsing the adjoint hypermultiplet of the SU(8) theory. When we Higgs the SU(8) theory using the adjoint field and preserve an SU(6) or SU(7) gauge symmetry, there always exists a U(1) subgroup whose charged matter all are in non-trivial representations of the non-abelian gauge group. This U(1) subgroup thus cannot be Higgsed away without disrupting the non-abelian gauge symmetry, and remains unbroken.

Let us first verify the existence of a rational section for the SU(6) and SU(7) models. Recall that the Weierstrass model of an elliptic fibration over $\mathbb{P}^2$ with a rational section can be written in the form [63]

$$\begin{aligned} y^2 = x^3 &+ (2f_{3+n}f_{9-n} - 3f_6^2 - b_n^2 f_{12-2n})x \\ &+ (2f_6^3 - 2f_{3+n}f_6 f_{9-n} + f_{3+n}^2 f_{12-2n} - 2b_n^2 f_6 f_{12-2n} + b_n^2 f_{9-n}^2), \end{aligned} \tag{B.22}$$

where b_n is a section of nH while f_k are sections of kH.[12] A rational section of this elliptic fibration is then given by

$$x = \frac{f_{3+n}^2}{b_n^2} - 2f_6, \qquad y = -\frac{f_{3+n}^3}{b_n^3} + 3\frac{f_6 f_{3+n}}{b_n} - b_n f_{9-n}. \tag{B.23}$$

For the SU(6) and SU(7) models, $n = 0$ in equation (B.22) and f_k and b_0 are given by

$$\begin{aligned} b_0 &= \sqrt{G_6}, & f_3 &= -\frac{3}{2}\Lambda, & f_6 &= \frac{1}{3}\Phi, \\ f_9 &= \widetilde{\sigma}^3 - \frac{3}{G_6}\Phi\Lambda + \frac{27}{4G_6^2}\Lambda^3, & f_{12} &= -\frac{3}{G_6}\Lambda\left(2\widetilde{\sigma}^3 - \frac{3}{G_6}\Phi\Lambda + \frac{27}{4G_6^2}\Lambda^3\right). \end{aligned} \tag{B.24}$$

Here, Λ, Φ and G_6 are related to the sections in equation (B.15) by

$$\begin{aligned} \Phi &= \frac{\alpha^2\beta^2}{4} + \beta\nu\widetilde{\sigma} + \frac{\phi_2}{\beta}\widetilde{\sigma}^2, \\ \Lambda &= \frac{1}{9}\nu\phi_2 - \beta\lambda + \left(\frac{\phi_2^2}{3\beta^2} + f_4\right)\widetilde{\sigma}, & G_6 &= \frac{1}{4}\left(g_6 + \frac{f_4\phi_2}{3\beta} + \frac{\phi_2^3}{27\beta^3}\right). \end{aligned} \tag{B.25}$$

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[12]Elliptic fibrations with non-zero Mordell-Weil rank have recently been used to construct phenomenologically interesting F-theory models. A small sample of such work is collected in the bibliography [64–68].

The SU(7) model is obtained from the SU(6) model by tuning

$$G_6 = \frac{1}{4}\psi_3^2, \qquad \Lambda = \frac{1}{2}\psi_3\beta\alpha + \left(f_4 + \frac{\phi_2^2}{3\beta^2}\right)\widetilde{\sigma}. \tag{B.26}$$

Note that in the SU(7) model

$$b_0 = \frac{1}{2}\psi_3. \tag{B.27}$$

Hence, by taking $\psi_3 \to 0$, we can enhance the abelian factor present in the theory [63]. But by taking this limit, we precisely arrive at the SU(8) model (B.20). This is consistent with the fact that the SU(7) model is obtained from the SU(8) model by Higgsing the adjoint hypermultiplet. The vacuum expectation value of the Higgs field can be identified with the parameter ψ_3.

Counting the number of complex degrees of freedom of the Weierstrass model and comparing with the expected number of neutral hypermultiplets (table 2), it is clear that the U(1) factors only exist for the SU(6) and SU(7) models. This has a simple explanation in terms of Higgsing from the SU(8) model (B.20). As noted before, the SU(7) (B.17) and SU(6) (B.15) models are obtained from the SU(8) model by giving a vacuum expectation value to the adjoint hypermultiplet of the theory. The Higgsing that preserves the SU(7) symmetry breaks the gauge group down to SU(7) × U(1) rather than SU(7):

$$\text{SU}(7) \times \text{U}(1): \quad 3 \times \left(\square, -6\right) + 3 \times \left(\boxminus, 2\right) + 1 \times (\text{Adj}, 0). \tag{B.28}$$

The U(1) factor is represented by the Cartan matrix

$$\text{diag}(1, 1, 1, 1, 1, 1, 1, -7) \tag{B.29}$$

in the fundamental representation of SU(8). Now there is no way we can break the U(1) gauge symmetry further without breaking the SU(7) symmetry, as all the matter charged under U(1) is also charged under SU(7). The F-theory model is hence forced to have an additional abelian gauge group factor.

Higgsing the theory further down using the adjoint hypermultiplet to preserve an SU(6) symmetry, the gauge group is broken down to SU(6) × U(1)^2 where the charged matter is given by

$$\begin{aligned}\text{SU}(6) \times \text{U}(1)_1 \times \text{U}(1)_2: \quad &3 \times (\cdot, 0, -6) + 3 \times \left(\square, -1, -2\right) + 3 \times \left(\square, 1, -2\right) \\ &+ 3 \times \left(\boxminus, 0, 2\right) + 1 \times (\text{Adj}, 0, 0).\end{aligned} \tag{B.30}$$

Here the U(1) factors are represented by the following two Cartan matrices of SU(8):

$$\text{diag}(0, 0, 0, 0, 0, 0, 1, -1), \qquad \text{diag}(1, 1, 1, 1, 1, 1, -3, -3). \tag{B.31}$$

It can be seen that in model (B.30) there exist hypermultiplets neutral under SU(6) but charged under U(1)$_2$ that can be used to Higgs away this abelian gauge symmetry. The gauge group U(1)$_1$, however, cannot be broken without breaking SU(6). The phenomenon that the SU(6) model (B.15) automatically has a rational section is a reflection of this fact.

Upon further Higgsing, we find that there are always enough hypermultiplets neutral under the non-abelian gauge group that can be used to Higgs away all the abelian gauge group factors. Hence a generic F-theory model (B.1) with $N \le 5$ does not have a non-trivial abelian gauge group.

References

[1] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [`hep-th/9602022`] [ɪɴSPIRE].

[2] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 1*, *Nucl. Phys.* **B 473** (1996) 74 [`hep-th/9602114`] [ɪɴSPIRE].

[3] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 2*, *Nucl. Phys.* **B 476** (1996) 437 [`hep-th/9603161`] [ɪɴSPIRE].

[4] W. Taylor, *TASI lectures on supergravity and string vacua in various dimensions*, [`arXiv:1104.2051`] [ɪɴSPIRE].

[5] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [`hep-th/9503124`] [ɪɴSPIRE].

[6] E. Witten, *Some comments on string dynamics*, [`hep-th/9507121`] [ɪɴSPIRE].

[7] E. Witten, *Small instantons in string theory*, *Nucl. Phys.* **B 460** (1996) 541 [`hep-th/9511030`] [ɪɴSPIRE].

[8] A. Strominger, *Open p-branes*, *Phys. Lett.* **B 383** (1996) 44 [`hep-th/9512059`] [ɪɴSPIRE].

[9] O.J. Ganor and A. Hanany, *Small E_8 instantons and tensionless noncritical strings*, *Nucl. Phys.* **B 474** (1996) 122 [`hep-th/9602120`] [ɪɴSPIRE].

[10] N. Seiberg and E. Witten, *Comments on string dynamics in six-dimensions*, *Nucl. Phys.* **B 471** (1996) 121 [`hep-th/9603003`] [ɪɴSPIRE].

[11] M. Bershadsky and A. Johansen, *Colliding singularities in F-theory and phase transitions*, *Nucl. Phys.* **B 489** (1997) 122 [`hep-th/9610111`] [ɪɴSPIRE].

[12] J.D. Blum and K.A. Intriligator, *Consistency conditions for branes at orbifold singularities*, *Nucl. Phys.* **B 506** (1997) 223 [`hep-th/9705030`] [ɪɴSPIRE].

[13] P.S. Aspinwall and D.R. Morrison, *Point-like instantons on K3 orbifolds*, *Nucl. Phys.* **B 503** (1997) 533 [`hep-th/9705104`] [ɪɴSPIRE].

[14] K.A. Intriligator, *New string theories in six-dimensions via branes at orbifold singularities*, *Adv. Theor. Math. Phys.* **1** (1998) 271 [`hep-th/9708117`] [ɪɴSPIRE].

[15] I. Brunner and A. Karch, *Branes at orbifolds versus Hanany Witten in six-dimensions*, *JHEP* **03** (1998) 003 [`hep-th/9712143`] [ɪɴSPIRE].

[16] A. Hanany and A. Zaffaroni, *Branes and six-dimensional supersymmetric theories*, *Nucl. Phys.* **B 529** (1998) 180 [`hep-th/9712145`] [ɪɴSPIRE].

[17] J.J. Heckman, D.R. Morrison and C. Vafa, *On the classification of 6D SCFTs and generalized ADE orbifolds*, *JHEP* **05** (2014) 028 [`arXiv:1312.5746`] [ɪɴSPIRE].

[18] M. Del Zotto, J.J. Heckman, A. Tomasiello and C. Vafa, *6d conformal matter*, *JHEP* **02** (2015) 054 [`arXiv:1407.6359`] [ɪɴSPIRE].

[19] J.J. Heckman, *More on the matter of 6D SCFTs*, *Phys. Lett.* **B 747** (2015) 73 [`arXiv:1408.0006`] [ɪɴSPIRE].

[20] J.J. Heckman, D.R. Morrison, T. Rudelius and C. Vafa, to appear.

[21] V. Kumar, D.R. Morrison and W. Taylor, *Global aspects of the space of 6D $\mathcal{N} = 1$ supergravities*, *JHEP* **11** (2010) 118 [arXiv:1008.1062] [INSPIRE].

[22] N. Seiberg and W. Taylor, *Charge lattices and consistency of 6D supergravity*, *JHEP* **06** (2011) 001 [arXiv:1103.0019] [INSPIRE].

[23] C. Cordova, *Decoupling gravity in F-theory*, *Adv. Theor. Math. Phys.* **15** (2011) 689 [arXiv:0910.2955] [INSPIRE].

[24] J.J. Heckman and H. Verlinde, *Evidence for F(uzz) theory*, *JHEP* **01** (2011) 044 [arXiv:1005.3033] [INSPIRE].

[25] T. Banks and N. Seiberg, *Symmetries and strings in field theory and gravity*, *Phys. Rev.* **D 83** (2011) 084019 [arXiv:1011.5120] [INSPIRE].

[26] J. Milnor, *On simply connected 4-manifolds*, in Proceedings of the *Symposio Internacional de Topologia Algebrica*, UNAM and UNESCO, Mexico City, Mexico (1958), pp. 122–128.

[27] M.B. Green, J.H. Schwarz and P.C. West, *Anomaly free chiral theories in six-dimensions*, *Nucl. Phys.* **B 254** (1985) 327 [INSPIRE].

[28] A. Sagnotti, *A note on the Green-Schwarz mechanism in open string theories*, *Phys. Lett.* **B 294** (1992) 196 [hep-th/9210127] [INSPIRE].

[29] V. Sadov, *Generalized Green-Schwarz mechanism in F-theory*, *Phys. Lett.* **B 388** (1996) 45 [hep-th/9606008] [INSPIRE].

[30] A. Grassi and D.R. Morrison, *Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds*, *Commun. Num. Theor. Phys.* **6** (2012) 51 [arXiv:1109.0042] [INSPIRE].

[31] C.W. Bernard, N.H. Christ, A.H. Guth and E.J. Weinberg, *Instanton parameters for arbitrary gauge groups*, *Phys. Rev.* **D 16** (1977) 2967 [INSPIRE].

[32] K. Ohmori, H. Shimizu, Y. Tachikawa and K. Yonekura, *Anomaly polynomial of general 6d SCFTs*, *Prog. Theor. Exp. Phys.* **2014** (2014) 103B07 [arXiv:1408.5572] [INSPIRE].

[33] J. Erler, *Anomaly cancellation in six-dimensions*, *J. Math. Phys.* **35** (1994) 1819 [hep-th/9304104] [INSPIRE].

[34] S. Ferrara, R. Minasian and A. Sagnotti, *Low-energy analysis of M and F theories on Calabi-Yau threefolds*, *Nucl. Phys.* **B 474** (1996) 323 [hep-th/9604097] [INSPIRE].

[35] V. Kumar, D.R. Morrison and W. Taylor, *Mapping 6D $\mathcal{N} = 1$ supergravities to F-theory*, *JHEP* **02** (2010) 099 [arXiv:0911.3393] [INSPIRE].

[36] S. Hellerman and E. Sharpe, *Sums over topological sectors and quantization of Fayet-Iliopoulos parameters*, *Adv. Theor. Math. Phys.* **15** (2011) 1141 [arXiv:1012.5999] [INSPIRE].

[37] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, *P-brane dyons and electric magnetic duality*, *Nucl. Phys.* **B 520** (1998) 179 [hep-th/9712189] [INSPIRE].

[38] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [hep-th/9610234] [INSPIRE].

[39] O. Aharony and E. Witten, *Anti-de Sitter space and the center of the gauge group*, *JHEP* **11** (1998) 018 [hep-th/9807205] [INSPIRE].

[40] E. Witten, *AdS/CFT correspondence and topological field theory*, *JHEP* **12** (1998) 012 [hep-th/9812012] [INSPIRE].

[41] G.W. Moore, *Anomalies, Gauss laws and Page charges in M-theory*, *C. R. Phys.* **6** (2005) 251 [hep-th/0409158] [INSPIRE].

[42] D. Belov and G.W. Moore, *Holographic action for the self-dual field*, hep-th/0605038 [INSPIRE].

[43] D.S. Freed, G.W. Moore and G. Segal, *Heisenberg groups and noncommutative fluxes*, *Annals Phys.* **322** (2007) 236 [hep-th/0605200] [INSPIRE].

[44] E. Witten, *Conformal field theory in four and six dimensions*, arXiv:0712.0157 [INSPIRE].

[45] E. Witten, *Geometric Langlands from six dimensions*, arXiv:0905.2720 [INSPIRE].

[46] M. Henningson, *The partition bundle of type A_{N-1} (2,0) theory*, *JHEP* **04** (2011) 090 [arXiv:1012.4299] [INSPIRE].

[47] D.S. Freed and C. Teleman, *Relative quantum field theory*, *Commun. Math. Phys.* **326** (2014) 459 [arXiv:1212.1692] [INSPIRE].

[48] C.P. Boyer and K. Galicki, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, U.K. (2008).

[49] K. Ohmori, H. Shimizu and Y. Tachikawa, *Anomaly polynomial of E-string theories*, *JHEP* **08** (2014) 002 [arXiv:1404.3887] [INSPIRE].

[50] K. Intriligator, *6d, $\mathcal{N} = (1,0)$ Coulomb branch anomaly matching*, *JHEP* **10** (2014) 162 [arXiv:1408.6745] [INSPIRE].

[51] D. Gaiotto and A. Tomasiello, *Holography for (1,0) theories in six dimensions*, *JHEP* **12** (2014) 003 [arXiv:1404.0711] [INSPIRE].

[52] D. Freed, J.A. Harvey, R. Minasian and G.W. Moore, *Gravitational anomaly cancellation for M-theory five-branes*, *Adv. Theor. Math. Phys.* **2** (1998) 601 [hep-th/9803205] [INSPIRE].

[53] J.A. Harvey, R. Minasian and G.W. Moore, *Non-Abelian tensor multiplet anomalies*, *JHEP* **09** (1998) 004 [hep-th/9808060] [INSPIRE].

[54] M.J. Duff, J.T. Liu and R. Minasian, *Eleven-dimensional origin of string-string duality: a one loop test*, *Nucl. Phys.* **B 452** (1995) 261 [hep-th/9506126] [INSPIRE].

[55] E. Witten, *Phase transitions in M-theory and F-theory*, *Nucl. Phys.* **B 471** (1996) 195 [hep-th/9603150] [INSPIRE].

[56] S.H. Katz, D.R. Morrison and M.R. Plesser, *Enhanced gauge symmetry in type II string theory*, *Nucl. Phys.* **B 477** (1996) 105 [hep-th/9601108] [INSPIRE].

[57] D.R. Morrison and W. Taylor, *Matter and singularities*, *JHEP* **01** (2012) 022 [arXiv:1106.3563] [INSPIRE].

[58] V. Kumar, D.S. Park and W. Taylor, *6D supergravity without tensor multiplets*, *JHEP* **04** (2011) 080 [arXiv:1011.0726] [INSPIRE].

[59] J.J. Heckman, *Statistical inference and string theory*, arXiv:1305.3621 [INSPIRE].

[60] V. Balasubramanian, J.J. Heckman and A. Maloney, *Relative entropy and proximity of quantum field theories*, *JHEP* **05** (2015) 104 [arXiv:1410.6809] [INSPIRE].

[61] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory. Vol. 2: Loop amplitudes, anomalies and phenomenology*, Cambridge University Press, Cambridge, U.K. (1987) [INSPIRE].

[62] K.A. Intriligator, *Anomaly matching and a Hopf-Wess-Zumino term in 6d, $\mathcal{N} = (2,0)$ field theories*, *Nucl. Phys.* **B 581** (2000) 257 [hep-th/0001205] [INSPIRE].

[63] D.R. Morrison and D.S. Park, *F-theory and the Mordell-Weil group of elliptically-fibered Calabi-Yau threefolds*, *JHEP* **10** (2012) 128 [arXiv:1208.2695] [INSPIRE].

[64] C. Mayrhofer, E. Palti and T. Weigand, U(1) *symmetries in F-theory GUTs with multiple sections*, *JHEP* **03** (2013) 098 [arXiv:1211.6742] [INSPIRE].

[65] V. Braun, T.W. Grimm and J. Keitel, *New global F-theory GUTs with* U(1) *symmetries*, *JHEP* **09** (2013) 154 [arXiv:1302.1854] [INSPIRE].

[66] M. Cvetič, D. Klevers and H. Piragua, *F-theory compactifications with multiple* U(1)*-factors: constructing elliptic fibrations with rational sections*, *JHEP* **06** (2013) 067 [arXiv:1303.6970] [INSPIRE].

[67] I. Antoniadis and G.K. Leontaris, *F-GUTs with Mordell-Weil* U(1)*'s*, *Phys. Lett.* **B 735** (2014) 226 [arXiv:1404.6720] [INSPIRE].

[68] M. Esole, M.J. Kang and S.-T. Yau, *A new model for elliptic fibrations with a rank one Mordell-Weil group: I. Singular fibers and semi-stable degenerations*, arXiv:1410.0003 [INSPIRE].