Published for SISSA by Springer

Received: April 15, 2015
Accepted: May 16, 2015
Published: June 8, 2015

Hydrodynamics on the lowest Landau level

Michael Geracie and Dam Thanh Son

Kadanoff Center for Theoretical Physics, University of Chicago, Chicago, Illinois 60637, U.S.A.

E-mail: michael.geracie@uchicago.edu, dtson@uchicago.edu

Abstract: Using the recently developed approach to quantum Hall physics based on Newton-Cartan geometry, we consider the hydrodynamics of an interacting system on the lowest Landau level. We rephrase the non-relativistic fluid equations of motion in a manner that manifests the spacetime diffeomorphism invariance of the underlying theory. In the massless (or lowest Landau level) limit, the fluid obeys a force-free constraint which fixes the charge current. An entropy current analysis further constrains the energy response, determining four transverse response functions in terms of only two: an energy magnetization and a thermal Hall conductivity. Kubo formulas are presented for all transport coefficients and constraints from Weyl invariance derived. We also present a number of Středa-type formulas for the equilibrium response to external electric, magnetic and gravitational fields.

Keywords: Space-Time Symmetries, Thermal Field Theory

ArXiv ePrint: 1408.6843

doi:10.1007/JHEP06(2015)044

JHEP06(2015)044

Contents

1 Introduction

Since its discovery, the fractional quantum Hall (FQH) effect [1, 2] has been subjected to intense study and proven a fruitful playground for new concepts in both condensed matter and high energy physics. Beyond its quantized Hall conductance, FQH states exhibit a number of interesting features including anyonic excitations [3, 4], edge states [5] and are prime examples of topological phases of matter [6]. Beginning with the Laughlin wavefunction [2], it has been attacked with numerous approaches including Chern-Simons field theory [7, 8] and the composite-fermion approach [9].

This paper follows previous work [10, 11] with a focus on spacetime symmetries. This approach has allowed one to derive a number of new results, including a relationship between the Hall conductivity at finite wave numbers and the shift [10], sum rules involving the spectral densities of the stress tensor components [12], and a relationship between the

JHEP06(2015)044

density-curvature response and the chiral central charge [13]. These and related results have also been obtained independently using other methods [14–16].

The approach was developed further in ref. [17]. In the approach we will follow here, a FQH system is put on a curved spatial manifold whose metric may change with time. The microscopic theory is found to exhibit a coordinate invariance which can be interpreted as diffeomorphisms of a geometric structure called a Newton-Cartan spacetime. The FQH system thus is viewed as living in a 2+1 dimensional Newton-Cartan geometry. This structure was initially proposed by Cartan as a geometric description of Newtonian gravity [18–21] and is in fact the natural coordinate invariant setting for non-relativistic physics in general. A Newton-Cartan covariant formulation of non-relativistic superfluids was developed in ref. [22]. Our approach differs significantly from the latter in particular by the introduction of torsion and, in principle, can be applied generally to fluids of any type, although gapped quantum Hall states will be the focus of our attention here.

One of the most important features of the FQH problem is the presence of a large magnetic field separating the Landau levels, reducing the problem, in its most essential limit, to that of interacting particles confined to the lowest Landau level (LLL). The LLL limit can also be realized by taking the massless limit of a non-relativistic theory. From the point of view of the symmetries of a Newton-Cartan space, this massless limit is a regular limit. This important feature allows one to directly attack the LLL limit of the FQH problem. In this paper, we construct the hydrodynamic theory describing a finite-temperature FQH fluid in the LLL. (At finite temperature, the FQH plateaux will be smeared out, but for convenience we will continue to call any interacting systems of particles confined to the LLL a "FQH fluid.")

Hydrodynamics is an effective theory describing the long distance physics of a system that is in local thermodynamic equilibrium. In the standard hydrodynamic theory we then have as variables a locally defined temperature T and chemical potential μ, as well as a fluid velocity v^i that vary slowly in space and time. Their dynamics is given by conservation laws supplemented by constitutive relations expanded to some chosen order in derivatives. This construction of the hydrodynamic theory is simplest in relativistic physics, where covariance is manifest in the equations of motion

$$\nabla_\mu j^\mu = 0, \qquad \nabla_\nu T^{\mu\nu} = F^{\mu\nu} j_\nu \,, \tag{1.1}$$

and simple to implement in the constitutive relations, which are the most general expressions for j^μ and $T^{\mu\nu}$ in terms of the fluid degrees of freedom T, μ, and u^μ (the time-like four-velocity normalized so that $u_\mu u^\mu = 1$). One merely writes down all possible terms that have the correct index structure and may be obtained from T, μ, u^μ and $F_{\mu\nu}$ only through contraction and differentiation; although the second law of thermodynamics puts extra constraints on this expansion.

In non-relativistic physics, we have three conservation laws: those of particle number, momentum and energy (which are independent in this context)

$$\partial_0 j^0_{\rm nc} + \partial_i j^i_{\rm nc} = 0, \tag{1.2a}$$
$$\partial_0 \varepsilon^0_{\rm nc} + \partial_i \varepsilon^i_{\rm nc} = F_{i0} j^i, \tag{1.2b}$$
$$m\partial_0 j^i_{\rm nc} + \partial_j T^{ij}_{\rm nc} = F^{i0} j_{\rm nc0} + F^{ij} j_{\rm nc\, j}. \tag{1.2c}$$

We would like the most general constitutive relations for $j^i_{\rm nc}$, $\varepsilon^i_{\rm nc}$, and $T^{ij}_{\rm nc}$. The Galilean invariance of the equations is then imposed as an additional constraint.

Newton-Cartan geometry greatly simplifies the process of writing down the conservation laws and the constitutive relations. Currents that transform covariantly under diffeomorphism can be defined and covariant Ward identities derived ref. [17]. (In contrast, the energy current $\varepsilon^\mu_{\rm nc}$ and stress $T^{ij}_{\rm nc}$ are not spacetime covariant; the "nc"'s in eqs. (1.2) are to distinguish the standard currents from the covariant ones we will be using throughout.) That a Newton-Cartan geometry naturally includes a source for the energy current has been noted in ref. [23] and used to study energy transport in a recent paper [24].

Our paper is organized as follows. In section 2 we briefly recap the results of ref. [17]. The derivative expansion and entropy current analysis then proceed entirely along the lines of the relativistic case. Section 3 obtains the most general constitutive relations and derives results of the massless limit. We find the FQH system is constrained to be force-free, which has powerful implications on the dynamics. In particular, all first order charge transport is determined by thermodynamics.

Section 4 contains the entropy current analysis, completing the program outlined above. We find that what are in principle four independent parity odd response coefficients (on the basis of symmetries) are determined by only two: and energy magnetization M_E and the Righi-Leduc (or thermal Hall) coefficient c_{RL}. In all, on trivial Newton-Cartan backgrounds (i.e., in flat metric and zero field coupled to the energy density) we have

$$
\begin{aligned}
j^0_{\rm nc} &= n, \qquad \varepsilon^0_{\rm nc} = \epsilon, \\
j^i_{\rm nc} &= \epsilon^{ij}\left(\frac{n}{B}\left(E_j - \partial_j\mu\right) - \frac{s}{B}\partial_j T + \partial_j M\right), \\
\varepsilon^i_{\rm nc} &= \Sigma_T \partial^i T + \epsilon^{ij}\left(\frac{\epsilon + p}{B}\left(E_j - \partial_j\mu\right) - M\partial_j\mu - Tc_{RL}\partial_j T + \partial_j M_E\right), \\
T^{ij}_{\rm nc} &= \left(p - \zeta\Theta\right)\delta^{ij} - \eta\sigma^{ij} - \tilde{\eta}\tilde{\sigma}^{ij}.
\end{aligned}
\tag{1.3}
$$

Here T is the temperature, μ the chemical potential and E_i, B the external electric and magnetic fields. p, n, ϵ, s and M are identified with the internal pressure, number density, energy density, entropy density and magnetization density while ζ, η and $\tilde{\eta}$ are the usual bulk, shear and Hall viscosities and Σ_T the thermal conductivity. All are arbitrary functions of the thermodynamic variables T, μ and B except for constraints from the usual thermodynamic identities and several positivity conditions

$$
\zeta, \eta \geq 0, \qquad \Sigma_T \leq 0. \tag{1.4}
$$

The system is dissipationless if and only if all inequalities are saturated. Kubo formulas for all coefficients may be found in sections 5 (where they are presented in the Newton-Cartan formalism used throughout this paper) and 6.4 (where they are given in standard form).

A recent analysis of 2+1 dimensional gapped phases derives the most general set of transport coefficients for zero temperature nondissipative systems [25]. Equations (1.3) generalize this to to an arbitrary hydrodynamic theory with nonzero temperature and chemical potential (though they are assumed to be slowly varying and far below the gap) giving us the dissipative viscosities and Righi-Leduc coefficient.

JHEP06(2015)044

Finally, we present a set of generalized Středa formulas that characterize the equilibrium response to probing electric, magnetic and gravitational fields. A FQH fluid in thermodynamic equilibrium has nonzero electric and energy currents,

$$\begin{aligned} j^i_{\rm nc} &= \varepsilon^{ij}\left(\sigma^{\rm eq}_H E_j + \sigma^{B{\rm eq}}_H \partial_j B + \sigma^{G{\rm eq}}_H G_j\right), \\ \varepsilon^i_{\rm nc} &= \varepsilon^{ij}\left(\kappa^{\rm eq}_H E_j + \kappa^{B{\rm eq}}_H \partial_j B + \kappa^{G{\rm eq}}_H G_j\right), \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} \sigma^{\rm eq}_H &= \left(\frac{\partial n}{\partial B}\right)_{T,\mu}, & \sigma^{B{\rm eq}}_H &= \left(\frac{\partial M}{\partial B}\right)_{T,\mu}, \\ \sigma^{G{\rm eq}}_H &= T\left(\frac{\partial s}{\partial B}\right)_{T,\mu} + \mu\left(\frac{\partial n}{\partial B}\right)_{T,\mu} - M, \\ \kappa^{\rm eq}_H &= \left(\frac{\partial M_E}{\partial \mu}\right)_{T,B} - M, & \kappa^{B{\rm eq}}_H &= \left(\frac{\partial M_E}{\partial B}\right)_{T,\mu}, \\ \kappa^{G{\rm eq}}_H &= T\left(\frac{\partial M_E}{\partial T}\right)_{\mu,B} + \mu\left(\frac{\partial M_E}{\partial \mu}\right)_{T,B} - 2M_E, \end{aligned} \tag{1.6}$$

the first of which may be recognized as the usual Středa formula [26]. Here $G_i = \partial_i \Phi$ is the external force exerted by a gravitational potential $-\Phi$.

We give concluding remarks in section 7. The appendices contain additional constraints due to Weyl invariance and other materials of a technical character. In a companion paper [27] we present an alternative derivation of some of the results of this paper without the use of the Newton-Cartan formalism, compute the thermal Hall coefficient in the high-temperature regime and discuss the question of particle-hole symmetry of the hydrodynamic theory.

2 Ward identities

We begin with a brief recap of recent work on the Ward identities of non-relativistic systems. For details we refer the reader to ref. [17]. In this paper we derive covariant Ward identities using the Newton-Cartan structure of non-relativistic theories [18–21]. In considering response to a perturbing gravitational scalar potential we will need a torsionful version of this geometry (this has also been considered in ref. [28]). This involves a degenerate metric $g^{\mu\nu}$ that measures spatial distances. It's degeneracy direction is spanned by a one-form n_μ satisfying $n \wedge dn = 0$ that provides an absolute notion of space through it's integral submanifolds. It's convenient to also define an auxiliary "velocity" field v^μ satisfying $n_\mu v^\mu = 1$ that allows one to invert the metric to a transverse projector

$$g_{\mu\lambda} g^{\lambda\nu} = P_\mu{}^\nu \qquad \text{where} \qquad P_\mu{}^\nu = \delta_\mu{}^\nu - n_\mu v^\nu. \tag{2.1}$$

The connection ∇_μ is then uniquely specified by

$$\nabla_\mu n_\nu = 0, \qquad \nabla_\lambda g^{\mu\nu} = 0, \qquad g_{\lambda[\mu}\nabla_{\nu]} v^\lambda = 0, \tag{2.2}$$

and has torsion $T^{\lambda}{}_{\mu\nu} = v^{\lambda}(dn)_{\mu\nu}$. The velocity field is unphysical and may be chosen in whatever manner is convenient for a particular problem.

In ref. [17] we demonstrate that for systems constrained to the LLL, the Ward identities following from gauge and diffeomorphism invariance in a nonrelativistic theory take the covariant form

$$(\nabla_\mu - G_\mu)\, j^\mu = 0, \tag{2.3}$$

$$(\nabla_\mu - G_\mu)\, \varepsilon^\mu = -F_{\mu\nu} v^\mu j^\nu + G_{\mu\nu} v^\mu \varepsilon^\nu - \frac{1}{2}\tau_{\mu\nu} T^{\mu\nu}, \tag{2.4}$$

$$(\nabla_\nu - G_\nu)\, T^{\mu\nu} = F^\mu{}_\nu j^\nu - G^\mu{}_\nu \varepsilon^\nu. \tag{2.5}$$

Ward identities for Newton-Cartan diffeomorphisms have also been considered in refs. [29] and [23]. The above is a covariant generalization of these equations to arbitrary backgrounds, subjected to a LLL projection in the form of a massless limit. These identities also assume a spinful fluid of spin $s = 1$.

Here j^μ and ε^μ are the particle and energy currents and $T^{\mu\nu}$ a transverse symmetric stress

$$T^{\mu\nu} n_\nu = 0. \tag{2.6}$$

The stress is conserved except for the action of external forces. The first of these is exerted by the familiar electromagnetic field strength $F_{\mu\nu} = (dA)_{\mu\nu}$, but there is also a torsional field strength $G_{\mu\nu} = (dn)_{\mu\nu}$ that couples to the energy current. Before the LLL projection the equation for stress conservation contains terms involving the momentum current. These however drop out upon taking the massless limit $m \to 0$ and stress conservation becomes the force balance (2.5).

The first equation expresses conservation of charge current while the second is the work-energy equation. Here

$$\tau_{\mu\nu} = \pounds_v g_{\mu\nu} \tag{2.7}$$

is the shear tensor. Although the Ward identities appear to depend on a choice of v^μ, one can demonstrate that the implicit and explicit dependence cancel and they are in fact invariant under v^μ redefinitions. Finally note in all cases the divergence operator takes the form $\nabla_\mu - G_\mu$ where $G_\mu = T^\nu{}_{\nu\mu}$ which is the correct form of the divergence on a torsionful manifold.

In writing these formulas, we have chosen g-factor $g = 2$ and spin $s = 1$ as we are always free to do. The former is necessary for a regular massless limit, the later is a matter of convenience. A given system may not satisfy these conditions, but in ref. [17] we present a precise dictionary that allows one to translate our results to the general case.

2.1 Coordinate expressions

To aid in the interpretation of eqs. (2.3)–(2.5) and comparison to the usual treatment of non-relativistic fluid dynamics, we collect here a number of coordinate dependent expressions for the above structure. Because we demand $n \wedge dn = 0$, a Newton-Cartan geometry

JHEP06(2015)044

admits a convenient set of coordinates called global time coordinates (GTC) in which

$$n_\mu = \left(e^{-\Phi},\, 0\right), \tag{2.8}$$

for some scalar potential Φ. It is instructive to have a few coordinate expressions for the structure outlined above in GTC. In these coordinates we may generally parameterize the metric and velocity vector as

$$g^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix}, \qquad v^\mu = e^\Phi \begin{pmatrix} 1 \\ v^i \end{pmatrix}. \tag{2.9}$$

It's then a matter of calculation to show that

$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} v^2 & -v_j \\ -v_i & g_{ij} \end{pmatrix}, & G^\mu &= \begin{pmatrix} 0 \\ \partial^i \Phi \end{pmatrix}, \\ \tau^{\mu\nu} &= \begin{pmatrix} 0 & 0 \\ 0 & e^\Phi \left(\nabla^i v^j + \nabla^j v^i - \dot{g}^{ij}\right) \end{pmatrix}, & \nabla_\mu v^\mu &= e^\Phi \left(\nabla_i\, v^i + \frac{1}{2} g^{ij} \dot{g}_{ij}\right), \end{aligned} \tag{2.10}$$

∇_i being the standard spatial connection.

There is a unique volume element $\varepsilon_{\mu\nu\lambda}$ that is compatible with the connection,

$$\nabla_\rho\, \varepsilon_{\mu\nu\lambda} = 0, \tag{2.11}$$

where we specialize to $2+1$ dimensions from this point forward. If we define

$$\varepsilon_{\mu\nu} = \varepsilon_{\mu\nu\lambda} v^\lambda, \tag{2.12}$$

then $\varepsilon^{\mu\nu}$ plays the role of the spatial volume element. Again in GTC we have

$$\varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon^{ij} \end{pmatrix}, \qquad \varepsilon_{012} = \sqrt{g} e^{-\Phi}, \tag{2.13}$$

where ε^{ij} is the antisymmetric tensor with $\varepsilon^{12} = \frac{1}{\sqrt{g}}$.

In GTC the Ward identities then read

$$\begin{aligned} \frac{1}{\sqrt{g}} e^\Phi \partial_0 (\sqrt{g} e^{-\Phi} j^0_{\text{nc}}) + e^\Phi \nabla_i (e^{-\Phi} j^i_{\text{nc}}) &= 0, \\ \frac{1}{\sqrt{g}} \partial_0 \left(\sqrt{g} \varepsilon^0_{\text{nc}}\right) + \nabla_i \varepsilon^i_{\text{nc}} &= E_i j^i_{\text{nc}} + G_i \varepsilon^i_{\text{nc}} - \frac{1}{2} T^{ij}_{\text{nc}} \dot{g}_{ij}, \\ \nabla_j T_{\text{nc}i}{}^j &= j^0_{\text{nc}} E_i + \varepsilon_{ij} j^j_{\text{nc}} B + \left(\varepsilon^0_{\text{nc}} \delta_i{}^j + T_i{}^j\right) G_j. \end{aligned} \tag{2.14}$$

We see that $G_i = \partial_i \Phi$ plays the role of an external gravitational field that couples to the energy density so we may think of $-\Phi$ as the non-relativistic gravitational potential.

3 Constitutive relations

In $2+1$ dimensions there are four independent one-point Ward identities: current conservation, the work-energy equation and Newton's second law. In the low energy, long wavelength limit, we expect that the system admits a fluid description, that is, the remaining degrees of freedom are also four-fold: two thermodynamic variables, which we take to be the temperature T and chemical potential μ, and the fluid velocity. The Ward identities then suffice to determine the evolution of the system and serve as equations of motion.

However, in the massless limit we lose two of these degrees of freedom. The momentum current drops out of the final Ward identity

$$(\nabla_\nu - G_\nu)T^{\mu\nu} = F^\mu{}_\nu j^\nu - G^\mu{}_\nu \varepsilon^\nu, \tag{3.1}$$

which now contains no time derivatives. What is typically a dynamical equation for the momentum flow reduces to a force-free constraint: since the fluid is massless, it is obliged to flow in such a manner that the applied forces cancel. We will use this in what follows to solve for the charge flow. What remains is two equations of motion

$$(\nabla_\mu - G_\mu)j^\mu = 0, \tag{3.2}$$

$$(\nabla_\mu - G_\mu)\varepsilon^\mu = -F_{\mu\nu}v^\mu j^\nu + G_{\mu\nu}v^\mu j^\nu - \frac{1}{2}\tau^{\mu\nu}T_{\mu\nu}, \tag{3.3}$$

that will determine T and μ for all time given initial conditions.

Of course for these to say anything we need to specify constitutive relations, that is $T^{\mu\nu}$, ε^μ and j^μ in terms of the fluid degrees of freedom T, μ, and the external fields. In the long wavelength, low energy limit when the fluid description is assumed to hold, we can assume that only low powers in the derivatives of these variables are important. In this section, we present the most general constitutive relations consistent with non- relativistic diffeomorphism covariance to first order in a derivative expansion.

Our derivative counting scheme for the background fields is as follows. The FQH problem assumes a large, nonvanishing magnetic field, which we will take to vary slowly in space and time. The fluid is also assumed to be moving in a nearly flat geometry and to have only slightly departed from thermodynamic equilibrium; that is, $F_{\mu\nu}$, $g^{\mu\nu}$, T and μ are all $\mathcal{O}(0)$. By it's definition $(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)f = -T^\lambda{}_{\mu\nu}\nabla_\lambda f$, the torsion must already be at $\mathcal{O}(1)$.

To organize the independent data appearing at each order we first note a few convenient facts. To begin, any vector w^μ may be uniquely decomposed into a part parallel to v^μ and perpendicular to n_μ

$$w^\mu = av^\mu + b^\mu, \qquad \text{where} \qquad n_\mu b^\mu = 0. \tag{3.4}$$

A similar decomposition may be carried out for tensors of all types. In particular for $(2,0)$ tensors we have

$$t^{\mu\nu} = av^\mu v^\nu + v^\mu b^\mu + c^\mu v^\nu + d^{\mu\nu}, \tag{3.5}$$

for some spatial vectors b^μ and c^μ and a spatial tensor $d^{\mu\nu}$. As a result, we need only consider scalars, transverse vectors, and transverse tensors in our classification. Since transverse 2-tensors may be further decomposed into a trace, a symmetric traceless part and an antisymmetric part (which we will not need), we are left in the end with scalars, transverse vectors and transverse symmetric tensors. We further subdivide this classification by evenness or oddness under parity.

3.1 Zeroth order

Let's begin by analyzing the force-free constraint (3.1). To our order we have

$$F^\mu{}_\nu j^\nu = 0. \tag{3.6}$$

The charge current must be proportional to the unique zero eigenvector of $F_{\mu\nu}$. We make a "choice of frame" so that v^μ tracks this equilibrium charge current

$$v^\mu = \frac{1}{2B}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda} = e^\Phi \begin{pmatrix} 1 \\ \frac{\varepsilon^{ij}E_j}{B} \end{pmatrix} \qquad \Longrightarrow \qquad j^\mu = nv^\mu, \tag{3.7}$$

where n will be some function of the zeroth order data

	Data	
Scalar	T	μ
Pseudoscalar	B	

Here $B = \frac{1}{2}\varepsilon^{\mu\nu}F_{\mu\nu}$ is of course the magnetic field. Note that in this frame we have

$$F_{\mu\nu} = B\varepsilon_{\mu\nu}\,, \qquad \nabla_\mu(Bv^\mu) = 0. \tag{3.8}$$

In the reference frame comoving with the charge, we expect that the equilibrium state is invariant under spatial rotations. This implies that the energy current must coincide with the charge current and the stress must be pure trace. Hence

$$\varepsilon^\mu = \epsilon v^\mu, \qquad T^{\mu\nu} = pg^{\mu\nu}, \tag{3.9}$$

where ϵ and p are again functions of T, μ and B.

However, p, ϵ and n are not entirely arbitrary being constrained by thermodynamics. It can be shown from statistical considerations (see appendix A) that ϵ is the energy density and n number density of the fluid. The hydrodynamic pressure p is sometimes called "internal pressure" and is related to the grand potential density (sometimes called the "thermodynamic pressure") $p_{\rm thm} = p_{\rm thm}(T, \mu, B)$ by a Legendre transformation

$$p = p_{\rm thm} - B\partial_B p_{\rm thm}. \tag{3.10}$$

To simplify some of our formulae we prefer to work with the internal pressure $p = p(T, \mu, M)$ which is naturally a function of T, μ and the magnetization density $M = \partial_B p_{\rm thm}$. We will thus exchange B for M as the independent variable in what follows. The functions p, ϵ and n satisfy the thermodynamic identities

$$\epsilon + p = Ts + \mu n - MB, \qquad d\epsilon = Tds + \mu dn - MdB. \tag{3.11}$$

Only one of these functions (say p) is independent. It is called the equation of state.

JHEP06(2015)044

3.2 First order

We now seek the most general corrections to $T^{\mu\nu}$, ε^μ and j^μ to first order. Denote these as

$$j^\mu = nv^\mu + \nu^\mu, \qquad \varepsilon^\mu = \epsilon v^\mu + \xi^\mu, \qquad T^{\mu\nu} = pg^{\mu\nu} + \pi^{\mu\nu}. \tag{3.12}$$

The complete set of first order data is

	Independent Data			
Scalar	Θ	$(v^\mu \nabla_\mu T)$	$(v^\mu \nabla_\mu \mu)$	
Pseudoscalar	–			
Vector	$\nabla^\mu T$	$\nabla^\mu \mu$	$\varepsilon^{\mu\nu}\nabla_\nu M$	G^μ
Pseudovector	$\varepsilon^{\mu\nu}\nabla_\nu T$	$\varepsilon^{\mu\nu}\nabla_\nu \mu$	$\nabla^\mu M$	$\varepsilon^{\mu\nu} G_\nu$
Traceless Symmetric Tensor	$\sigma^{\mu\nu}$			
Traceless Symmetric Pseudotensor	$\tilde{\sigma}^{\mu\nu}$			

Here $\Theta = \nabla_\mu v^\mu$ is the expansion and $\sigma^{\mu\nu} = \tau^{\mu\nu} - \Theta g^{\mu\nu}$ the traceless shear. The "tilde" operation is defined for symmetric two tensors as $\tilde{A}^{\mu\nu} = \frac{1}{2}\big(A^{\mu\lambda}\varepsilon_\lambda{}^\nu + A^{\nu\lambda}\varepsilon_\lambda{}^\mu\big)$. We do not include the material derivative of all three thermodynamic variables since one may always be eliminated by the constraint

$$\nabla_\mu (B v^\mu) = 0 \qquad \Longrightarrow \qquad v^\mu \nabla_\mu B = -B\Theta. \tag{3.13}$$

Not all of this data is independent on-shell and we may choose to eliminate some in favor of the others by solving the equations of motion. In our case, there are two scalar equations: the continuity equation and the work-energy equation. We use these to eliminate the material derivatives of T and μ, as indicated by parentheses.

Before we continue, a few comments on fluid frames are due. Since we will be considering small departures from thermal equilibrium there is an inherent ambiguity at first order in derivatives in how we define T and μ. This is a problem extensively discussed in the literature on nonequilibrium fluids [30, 31]. We differ from the usual case only in that we do not have any independent definition of a fluid velocity that would require additional fixing. Hence we have a two parameter ambiguity which we choose to fix by going to the Landau frame

$$n_\mu \nu^\mu = n_\mu \xi^\mu = 0. \tag{3.14}$$

Note that we have $\pi^{\mu\nu} n_\nu = 0$ for free since the stress is a transverse tensor.

The most general first order constitutive relations are then

$$\begin{aligned} \nu^\mu &= \chi_T \nabla^\mu T + \chi_\mu \nabla^\mu \mu + \tilde{\chi}_M \nabla^\mu M + \chi_G G^\mu \\ &\quad + \tilde{\chi}_T \varepsilon^{\mu\nu}\nabla_\nu T + \tilde{\chi}_\mu \varepsilon^{\mu\nu}\nabla_\nu \mu + \chi_M \varepsilon^{\mu\nu}\nabla_\nu M + \tilde{\chi}_G \varepsilon^{\mu\nu} G_\nu, \end{aligned} \tag{3.15a}$$

$$\begin{aligned} \xi^\mu &= \Sigma_T \nabla^\mu T + \Sigma_\mu \nabla^\mu \mu + \tilde{\Sigma}_M \nabla^\mu M + \Sigma_G G^\mu \\ &\quad + \tilde{\Sigma}_T \varepsilon^{\mu\nu}\nabla_\nu T + \tilde{\Sigma}_\mu \varepsilon^{\mu\nu}\nabla_\nu \mu + \Sigma_M \varepsilon^{\mu\nu}\nabla_\nu M + \tilde{\Sigma}_G \varepsilon^{\mu\nu} G_\nu, \end{aligned} \tag{3.15b}$$

$$\pi^{\mu\nu} = -\zeta \Theta g^{\mu\nu} - \eta \sigma^{\mu\nu} - \tilde{\eta}\tilde{\sigma}^{\mu\nu}, \tag{3.15c}$$

where a tilde denotes oddness under parity. We derive Kubo formulas for these coefficients in section 5.

3.3 Force-free flows

As mentioned previously, we may use the force balance constraint to completely solve for the charge current

$$\begin{gathered}\nabla^\mu p = B\varepsilon^{\mu\nu}\nu_\nu + (\epsilon+p)G^\mu \qquad \Longrightarrow \\ \tilde{\chi}_T = -\frac{s}{B}\,, \qquad \tilde{\chi}_\mu = -\frac{n}{B}\,, \qquad \chi_M = 1, \qquad \tilde{\chi}_G = \frac{\epsilon+p}{B}\,. \end{gathered} \tag{3.16}$$

All charge transport coefficients are thus determined by the equation of state. Also note that all longitudinal responses are zero. This is because the Lorentz force must cancel forces from pressure gradients and the magnetic field always produces a force perpendicular to the current; hence the current must be perpendicular to pressure gradients.

4 Entropy current analysis

The constitutive relations (3.15) subject to the restrictions (3.16) are the most general possible that are consistent with the equations of motion and constraint. However, it is still possible to generate flows that violate the second law of thermodynamics. For example, it is well known that a negative shear viscosity allows one to remove entropy from an isolated system and so we should have $\eta \geq 0$ [32]. To derive all such restrictions, we perform an entropy current analysis along the lines of ref. [30]. Lacking a spacetime picture of non-relativistic physics, previous analyses were restricted to the Lorentzian case and in particular did not include an independent energy current. Our results reproduce theirs for those coefficients that we have in common as well as derive new results for energy transport.

The canonical entropy current is

$$s^\mu_{\text{can}} = sv^\mu - \frac{\mu}{T}\nu^\mu + \frac{1}{T}\xi^\mu, \tag{4.1}$$

but out of equilibrium we should in principle once again expand in first order data

$$s^\mu = s^\mu_{\text{can}} + \zeta^\mu, \tag{4.2}$$

where

$$\begin{aligned}\zeta^\mu = {}& \zeta_\Theta \Theta v^\mu + \zeta_T \nabla^\mu T + \zeta_\mu \nabla^\mu \mu + \tilde{\zeta}_M \nabla^\mu M + \zeta_G G^\mu \\ & + \tilde{\zeta}_T \varepsilon^{\mu\nu}\nabla_\nu T + \tilde{\zeta}_\mu \varepsilon^{\mu\nu}\nabla_\nu \mu + \zeta_M \varepsilon^{\mu\nu}\nabla_\nu M + \tilde{\zeta}_G \varepsilon^{\mu\nu} G_\nu .\end{aligned} \tag{4.3}$$

Now we impose the second law. For non-negative entropy production between all spatial slices, we must have

$$(\nabla_\mu - G_\mu)s^\mu \geq 0. \tag{4.4}$$

Using the equations of motion in the form

$$v^\mu \nabla_\mu n + n\Theta = -\nabla_\mu \nu^\mu + G_\mu \nu^\mu, \tag{4.5a}$$

$$v^\mu \nabla_\mu \epsilon + (\epsilon+p)\Theta = -\nabla_\mu \xi^\mu - \frac{1}{2}\Theta\pi - \frac{1}{2}\sigma^{\mu\nu}\pi_{\mu\nu} + 2G_\mu \xi^\mu, \tag{4.5b}$$

JHEP06(2015)044

one may check that the divergence of the canonical entropy current is a quadratic form in first order data

$$(\nabla_\mu - G_\mu)s^\mu_{\text{can}} = -\nu^\mu \nabla_\mu\Big(\frac{\mu}{T}\Big) - \frac{1}{2T}\Theta g^{\mu\nu}\pi_{\mu\nu} - \frac{1}{2T}\sigma^{\mu\nu}\pi_{\mu\nu} - \frac{1}{T^2}\xi^\mu(\nabla_\mu T - TG_\mu), \quad (4.6)$$

and so the only genuine second order data in (4.4) is

$$\nabla_\mu \zeta^\mu\Big|_{2-\partial} = \zeta_\Theta v^\mu \nabla_\mu \Theta + \zeta_T \nabla^2 T + \zeta_\mu \nabla^2 \mu + \tilde{\zeta}_M \nabla^2 M + \zeta_G \nabla_\mu G^\mu, \quad (4.7)$$

where we have used the Newton-Cartan identities $\varepsilon^{\mu\nu}G_{\mu\nu} = 0$ and $\varepsilon^{\mu\nu}\nabla_\mu G_\nu = 0$. Since each term may be independently varied to have either sign, all coefficients appearing in this equation must be zero.

The remaining first order data is then

$$\begin{aligned}(\nabla_\mu - G_\mu)s^\mu = &\frac{1}{T}\zeta\Theta^2 + \frac{1}{2T}\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{1}{T}\Sigma_G G_\mu G^\mu \\ &+ \frac{1}{T}\Big(\Sigma_T - \frac{1}{T}\Sigma_G\Big)G^\mu\nabla_\mu T + \frac{1}{T}\Sigma_\mu G^\mu \nabla_\mu \mu + \frac{1}{T}\tilde{\Sigma}_M G^\mu \nabla_\mu M \\ &- \Big(\partial_T\tilde{\zeta}_G + \tilde{\zeta}_T - \frac{1}{T}\tilde{\Sigma}_T - \frac{1}{T^2}\tilde{\Sigma}_G + \frac{\mu}{T^2}\tilde{\chi}_G\Big)\varepsilon^{\mu\nu}G_\mu\nabla_\nu T \\ &- \Big(\partial_\mu\tilde{\zeta}_G + \tilde{\zeta}_\mu - \frac{1}{T}\tilde{\Sigma}_\mu - \frac{1}{T}\tilde{\chi}_G\Big)\varepsilon^{\mu\nu}G_\mu\nabla_\nu\mu - \Big(\partial_M\tilde{\zeta}_G + \zeta_M - \frac{1}{T}\Sigma_M\Big)\varepsilon^{\mu\nu}G_\mu\nabla_\nu M \\ &- \frac{1}{T^2}\Sigma_T\nabla_\mu T\nabla^\mu T - \frac{1}{T^2}\Sigma_\mu\nabla_\mu T\nabla^\mu\mu - \frac{1}{T^2}\tilde{\Sigma}_M\nabla_\mu T\nabla^\mu M \\ &+ \Big(\partial_T\tilde{\zeta}_\mu - \partial_\mu\tilde{\zeta}_T + \frac{1}{T}\tilde{\chi}_T + \frac{\mu}{T^2}\tilde{\chi}_\mu - \frac{1}{T^2}\tilde{\Sigma}_\mu\Big)\varepsilon^{\mu\nu}\nabla_\mu T\nabla_\nu\mu \\ &+ \Big(\partial_T\zeta_M - \partial_M\tilde{\zeta}_T + \frac{\mu}{T^2}\chi_M - \frac{1}{T^2}\Sigma_M\Big)\varepsilon^{\mu\nu}\nabla_\mu T\nabla_\nu M \\ &+ \Big(\partial_\mu\zeta_M - \partial_M\tilde{\zeta}_\mu - \frac{1}{T}\chi_M\Big)\varepsilon^{\mu\nu}\nabla_\mu\mu\nabla_\nu M \\ \geq\, & 0. \end{aligned} \quad (4.8)$$

Note that by ∂_μ we mean the partial derivative with respect to the chemical potential, not a spatial derivative. For clarity we will always use ∇_μ for the spatial derivative when there is the possibility of confusion.

The $\nabla_\mu T\nabla^\mu T$, $G^\mu\nabla_\mu T$ and $G_\mu G^\mu$ terms need not be separately constrained. We obtain a less stringent condition by setting $\Sigma_T = -\frac{1}{T}\Sigma_G$, in which case they arrange into a perfect square

$$-\frac{1}{T^2}\Sigma_T(\nabla_\mu T - TG_\mu)(\nabla^\mu T - TG^\mu). \quad (4.9)$$

We note in passing that in thermal equilibrium there can be no entropy production. This implies

$$\nabla^\mu T = TG^\mu, \quad (4.10)$$

or $\partial_i T = T\partial_i\Phi$ in coordinates. The physics of this clear: $-\Phi$ is the source that couples to the energy density and plays the role of a Newtonian gravitational potential. Heat will tend to flow from regions of higher $-\Phi$ to lower $-\Phi$. Equilibrium is reached once the temperature profile is such that (4.10) is satisfied. This result is also follows from the treatment of equilibrium statistical mechanics in appendix A. In general relativity this is known as the Tolman- Ehrenfest effect which states that the redshifted temperature $T||\xi||$ is constant in thermal equilibrium for ξ a timelike killing field [33]. In the non-relativistic case we have $Tn_\mu\xi^\mu = \text{const}$.

From (4.8) we immediately obtain the expected signs of the parity even viscosities and thermal conductivity

$$\zeta \geq 0, \qquad \eta \geq 0, \qquad \Sigma_T \leq 0. \tag{4.11}$$

The remaining terms place new restrictions on the energy and entropy coefficients

$$\Sigma_\mu = \tilde{\Sigma}_M = 0, \qquad \begin{pmatrix} \partial_\mu\zeta_M - \partial_M\tilde{\zeta}_\mu \\ \partial_M\tilde{\zeta}_T - \partial_T\zeta_M \\ \partial_T\tilde{\zeta}_\mu - \partial_\mu\tilde{\zeta}_T \end{pmatrix} = \begin{pmatrix} \frac{1}{T}\chi_M \\ -\frac{1}{T^2}\Sigma_M + \frac{\mu}{T^2}\chi_M \\ \frac{1}{T^2}\tilde{\Sigma}_\mu - \frac{1}{T}\tilde{\chi}_T - \frac{\mu}{T^2}\tilde{\chi}_\mu \end{pmatrix},$$
$$\begin{pmatrix} \partial_T\tilde{\zeta}_G \\ \partial_\mu\tilde{\zeta}_G \\ \partial_M\tilde{\zeta}_G \end{pmatrix} = \begin{pmatrix} -\tilde{\zeta}_T + \frac{1}{T}\tilde{\Sigma}_T + \frac{1}{T^2}\tilde{\Sigma}_G - \frac{\mu}{T^2}\tilde{\chi}_G \\ -\tilde{\zeta}_\mu + \frac{1}{T}\tilde{\Sigma}_\mu + \frac{1}{T}\tilde{\chi}_G \\ -\zeta_M + \frac{1}{T}\Sigma_M \end{pmatrix}. \tag{4.12}$$

We seek the most general solution to these constraints. Begin by eliminating the entropy coefficients by taking the curl of the third equation and plugging in the second

$$\begin{pmatrix} \partial_\mu\left(\frac{1}{T}\Sigma_M\right) - \partial_M\left(\frac{1}{T}\tilde{\Sigma}_\mu\right) \\ \partial_M\left(\frac{1}{T}\tilde{\Sigma}_T\right) - \partial_T\left(\frac{1}{T}\Sigma_M\right) \\ \partial_T\left(\frac{1}{T}\tilde{\Sigma}_\mu\right) - \partial_\mu\left(\frac{1}{T}\tilde{\Sigma}_T\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{T}\left(\chi_M + \partial_M\tilde{\chi}_G\right) \\ -\frac{1}{T^2}\left(\Sigma_M + \partial_M\tilde{\Sigma}_G\right) + \frac{\mu}{T^2}\left(\chi_M + \partial_M\tilde{\chi}_G\right) \\ \frac{1}{T^2}\left(\tilde{\Sigma}_\mu + \partial_\mu\tilde{\Sigma}_G\right) - \frac{1}{T}\left(\tilde{\chi}_T + \partial_T\tilde{\chi}_G\right) - \frac{\mu}{T^2}\left(\tilde{\chi}_\mu + \partial_\mu\tilde{\chi}_G\right) \end{pmatrix}. \tag{4.13}$$

Since the left hand side is the curl of a vector, the right hand side is divergenceless and it appears as if we might obtain another constraint. However one may check that this is automatically satisfied by virtue of the constraints (3.16) and the thermodynamic identities (3.11).

We may simplify the partial differential equation (4.13) by a substitution that isolates the energy response's dependence on the equation of state and $\tilde{\Sigma}_G$

$$\tilde{\Sigma}_T = -\frac{1}{T}\tilde{\Sigma}_G + \frac{\mu}{T}\frac{Ts+\mu n}{B} + T^2\tilde{g}_T, \qquad \tilde{\Sigma}_\mu = -\frac{Ts+\mu n}{B} + T^2\tilde{g}_\mu, \qquad \Sigma_M = T^2 g_M,$$
$$\implies \qquad \begin{pmatrix} \partial_\mu g_M - \partial_M\tilde{g}_\mu \\ \partial_M\tilde{g}_T - \partial_T g_M \\ \partial_T\tilde{g}_\mu - \partial_\mu\tilde{g}_T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.14}$$

We see that since $(\tilde{g}_T, \tilde{g}_\mu, g_M)$ is curl free, it must be the gradient of some function

$$\tilde{g}_T = \partial_T\tilde{g}, \qquad \tilde{g}_\mu = \partial_\mu\tilde{g}, \qquad g_M = \partial_M\tilde{g}. \tag{4.15}$$

Summary

This completes the entropy current analysis. For convenience, we collect our results in this section. FQH fluids may be generally viewed as massless fluids in a Newton-Cartan geometry. For the special values $g = 2$, $s = 1$ of the parity breaking parameters we have the following constitutive relations:

The charge-current response is purely transverse

$$j^\mu = nv^\mu + \tilde{\chi}_T \varepsilon^{\mu\nu}\nabla_\nu T + \tilde{\chi}_\mu \varepsilon^{\mu\nu}\nabla_\nu \mu + \chi_M \varepsilon^{\mu\nu}\nabla_\nu M + \tilde{\chi}_G \varepsilon^{\mu\nu} G_\nu, \tag{4.16}$$

where all coefficients are determined in terms of thermodynamics

$$\tilde{\chi}_T = -\frac{s}{B}, \qquad \tilde{\chi}_\mu = -\frac{n}{B}, \qquad \chi_M = 1, \qquad \tilde{\chi}_G = \frac{\epsilon + p}{B}. \tag{4.17}$$

Since $v^\mu = e^\Phi(1, \frac{\varepsilon^{ij}E_j}{B})$, we have a pure Hall conductivity $\sigma_H = e^\Phi \frac{n}{B}$.

The energy-current takes the form

$$\varepsilon^\mu = \epsilon v^\mu + \Sigma_T\big(\nabla^\mu T - TG^\mu\big) + \tilde{\Sigma}_T \varepsilon^{\mu\nu}\nabla_\nu T + \tilde{\Sigma}_\mu \varepsilon^{\mu\nu}\nabla_\nu \mu + \Sigma_M \varepsilon^{\mu\nu}\nabla_\nu M + \tilde{\Sigma}_G \varepsilon^{\mu\nu} G_\nu. \tag{4.18}$$

There is one longitudinal response, the thermal conductivity

$$\Sigma_T \leq 0. \tag{4.19}$$

The remaining four coefficients are all transverse and depend only on the equation of state and two arbitrary functions $\tilde{\Sigma}_G$ and $\tilde{g}$ of T, μ and M

$$\tilde{\Sigma}_T = -\frac{1}{T}\tilde{\Sigma}_G + T^2\partial_T \tilde{g} + \frac{\mu}{T}\frac{Ts + \mu n}{B}, \tag{4.20a}$$

$$\tilde{\Sigma}_\mu = T^2 \partial_\mu \tilde{g} - \frac{Ts + \mu n}{B}, \qquad \Sigma_M = T^2 \partial_M \tilde{g}. \tag{4.20b}$$

Using (4.12) we find the entropy current is determined by $\tilde{g}$ and $\tilde{\zeta}_G$

$$s^\mu = s^\mu_{\text{can}} + \tilde{\zeta}_T \varepsilon^{\mu\nu}\nabla_\nu T + \tilde{\zeta}_\mu \varepsilon^{\mu\nu}\nabla_\nu \mu + \zeta_M \varepsilon^{\mu\nu}\nabla_\nu M + \tilde{\zeta}_G \varepsilon^{\mu\nu} G_\nu, \tag{4.21}$$

where

$$\tilde{\zeta}_T = T\partial_T \tilde{g} - \partial_T \tilde{\zeta}_G + \frac{\mu M}{T^2}, \tag{4.22a}$$

$$\tilde{\zeta}_\mu = T\partial_\mu \tilde{g} - \partial_\mu \tilde{\zeta}_G - \frac{M}{T}, \tag{4.22b}$$

$$\zeta_M = T\partial_M \tilde{g} - \partial_M \tilde{\zeta}_G. \tag{4.22c}$$

Finally, the stress is determined by the internal pressure and three viscosities

$$T^{\mu\nu} = pg^{\mu\nu} - \zeta\Theta g^{\mu\nu} - \eta\sigma^{\mu\nu} - \tilde{\eta}\tilde{\sigma}^{\mu\nu}. \tag{4.23}$$

The bulk and shear viscosities must be non-negative

$$\zeta \geq 0, \qquad \eta \geq 0, \tag{4.24}$$

whereas the Hall viscosity $\tilde{\eta}$ is unconstrained. In a Weyl invariant theory, the bulk viscosity must vanish. The complete set of restrictions imposed by Weyl on the coefficients considered above are given in appendix B.

JHEP06(2015)044

5 Kubo formulas

Fractional quantum Hall transport is determined by p, ζ, η, $\tilde{\eta}$, Σ_T, $\tilde{\Sigma}_G$ and $\tilde{g}$, some of which are subject to positivity constraints, but are otherwise arbitrary functions of T, μ and M. In this section we provide Kubo formula's for these functions. For concreteness, perturb around a flat background

$$n_\mu = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{ij} \end{pmatrix}, \qquad F_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_{ij}B \end{pmatrix}, \qquad v^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{5.1}$$

with wavevector k_μ. The wavevector may be decomposed into temporal and transverse parts $k_\mu = \omega n_\mu + q_\mu$. We will be considering response in both the "rapid" ($q^\mu \to 0$) and "slow" ($\omega \to 0$) cases. The two-point functions of interest are

$$G^{\mu,\nu}_{\varepsilon\varepsilon}(x) = \frac{\delta\langle\varepsilon^\mu(x)\rangle}{\delta n_\nu(0)}, \qquad G^{\mu,\nu}_{\varepsilon j}(x) = \frac{\delta\langle\varepsilon^\mu(x)\rangle}{\delta A_\nu(0)}, \qquad G^{\mu\nu,\lambda\rho}(x) = \frac{\delta\langle T^{\mu\nu}(x)\rangle}{\delta h_{\lambda\rho}(0)}. \tag{5.2}$$

Explicitly, these are

$$G^{\mu,\nu}_{\varepsilon\varepsilon}(x) = \left\langle \frac{\delta\varepsilon^\mu(x)}{\delta n_\nu(0)} \right\rangle - i\theta(x^0)\left\langle[\varepsilon^\mu(x), \varepsilon^\nu(0)]\right\rangle, \tag{5.3a}$$

$$G^{\mu,\nu}_{\varepsilon j}(x) = \left\langle \frac{\delta\varepsilon^\mu(x)}{\delta A_\nu(0)} \right\rangle + i\theta(x^0)\left\langle[\varepsilon^\mu(x), j^\nu(0)]\right\rangle, \tag{5.3b}$$

$$G^{\mu\nu,\lambda\rho}(x) = \left\langle \frac{\delta T^{\mu\nu}(x)}{\delta g_{\lambda\rho}(0)} \right\rangle + \frac{i}{2}\theta(x^0)\left\langle[T^{\mu\nu}(x), T^{\lambda\rho}(0)]\right\rangle. \tag{5.3c}$$

The contact terms in these equations do not contribute to the imaginary parts of the respective Green's functions in momentum space, which will appear later in Kubo's formulas.

In this section we prefer to take all coefficients as functions of T, μ and B rather than T, μ and M as it is less awkward to deal with electromagnetic perturbations. It's a straightforward matter to translate the T, μ, M dependence of equations (4.16) and (4.18) to T, μ, B by plugging in $M(T,\mu,B) = \partial_B p_{\text{thm}}(T,\mu,B)$ and use of the chain rule.

5.1 Viscosities

The viscosities have already been discussed at length in the literature [14, 30], but we rederive their Kubo formulas in our language for completeness. Our treatment is particularly close to that of ref. [30]. Consider a rapid metric perturbation $\delta h_{\mu\nu}$. Using the definition (2.7) of the shear, we find

$$\begin{aligned}\delta T^{\mu\nu} = {}& (\partial_T p\delta T + \partial_\mu p\delta\mu + \partial_B p\delta B - \zeta\delta\Theta)g^{\mu\nu} \\ & - p\delta h^{\mu\nu} - i\omega\eta\Pi^{\mu\nu}{}_{\lambda\rho}\delta h^{\lambda\rho} - i\omega\tilde{\eta}\tilde{\Pi}^{\mu\nu}{}_{\lambda\rho}\delta h^{\lambda\rho}.\end{aligned} \tag{5.4}$$

δT and $\delta\mu$ may of course be solved for using the linearized equations of motion but we will not need to do so here. $\Pi^{\mu\nu}{}_{\lambda\rho}$ and $\tilde{\Pi}^{\mu\nu}{}_{\lambda\rho}$ are the even and odd symmetric traceless projectors

$$\Pi^{\mu\nu\lambda\rho} = g^{\mu(\lambda}g^{\rho)\nu} - \frac{1}{2}g^{\mu\nu}g^{\lambda\rho}, \qquad \tilde{\Pi}^{\mu\nu\lambda\rho} = \frac{1}{2}\big(g^{\mu(\lambda}\varepsilon^{\rho)\nu} + g^{\nu(\lambda}\varepsilon^{\rho)\mu}\big). \tag{5.5}$$

JHEP06(2015)044

They are traceless in the first and second pairs of indices, have cross traces

$$\Pi^{\mu\nu}{}_{\mu\nu} = 2, \qquad \tilde{\Pi}^{\mu\nu}{}_{\mu\nu} = 0 \tag{5.6}$$

and satisfy the algebra

$$\Pi^{\mu\nu}{}_{\alpha\beta}\Pi^{\alpha\beta}{}_{\lambda\rho} = \Pi^{\mu\nu}{}_{\lambda\rho}, \qquad \Pi^{\mu\nu}{}_{\alpha\beta}\tilde{\Pi}^{\alpha\beta}{}_{\lambda\rho} = \tilde{\Pi}^{\mu\nu}{}_{\lambda\rho}, \qquad \tilde{\Pi}^{\mu\nu}{}_{\alpha\beta}\tilde{\Pi}^{\alpha\beta}{}_{\lambda\rho} = -\Pi^{\mu\nu}{}_{\lambda\rho}. \tag{5.7}$$

Using these identities and the symmetry properties $\Pi^{\mu\nu\lambda\rho} = \Pi^{\lambda\rho\mu\nu}$ and $\tilde{\Pi}^{\mu\nu\lambda\rho} = -\tilde{\Pi}^{\lambda\rho\mu\nu}$, it is then straightforward to verify that

$$\eta = -\lim_{\omega\to 0} \mathrm{Im}\,\frac{\Pi_{\mu\nu\lambda\rho}G^{\mu\nu,\lambda\rho}(\omega)}{2\omega}\,, \tag{5.8a}$$

$$\tilde{\eta} = -\lim_{\omega\to 0} \frac{\tilde{\Pi}_{\mu\nu\lambda\rho}G^{\mu\nu,\lambda\rho}(\omega)}{2i\omega}\,. \tag{5.8b}$$

Here and further, whenever we write $\lim_{\omega\to 0}$, we assume that spatial momentum is put to zero ($q = 0$) before the limit is taken. Vice versa, when we write $\lim_{q\to 0}$ we implicitly assume that the frequency has been put to zero ($\omega = 0$) befor the limit is taken.

To get at the bulk viscosity, use $\delta\Theta = \frac{1}{2}i\omega g^{\mu\nu}\delta h_{\mu\nu}$ and take the trace and imaginary part of (5.4)

$$\zeta = -\lim_{\omega\to 0} \mathrm{Im}\,\frac{G_{\mu}{}^{\mu,}{}_{\nu}{}^{\nu}(\omega)}{2\omega}\,. \tag{5.9}$$

5.2 Thermal conductivities

Before deriving the remaining Kubo formulas, we would like to make some comments on the relation $\Sigma_T = -\frac{1}{T}\Sigma_G$ obtained from the entropy current analysis and rederive it from an alternative point of view that highlights the underlying physics. The Einstein relation identifies the conductivity and dissipation $\sigma = -\chi_\mu$ of any charged fluid where

$$j_i = \sigma E_i + \chi_\mu \partial_i \mu + \cdots\,. \tag{5.10}$$

These seemingly unrelated coefficients are connected by the following physical consideration. Apply a static but spatially varying electric potential δA_0. Charges will flow, but give the system time to relax and the current will again vanish. The chemical potential will adjust to match the profile of the electric potential $\delta\mu = \delta A_0$. Consistency then demands that $\sigma = -\chi_\mu$.

$\Sigma_T = -\frac{1}{T}\Sigma_G$ follows along similar lines. In the presence of the gravitational potential $-\Phi$, energy will flow from regions of large potential to small potential until equilibrium is reached. From appendix A we have for a static background

$$\beta = \int_c n, \qquad \mu = T\int_c A, \tag{5.11}$$

where c is the time circle passing through the fluid element under consideration. Now add a time independent perturbation $\delta n_\mu = -\delta\Phi n_\mu$ and the temperature and chemical potential adjust by

$$\delta T = T\delta\Phi, \qquad \delta\mu = \mu\delta\Phi. \tag{5.12}$$

Altogether we have

$$\delta\varepsilon^\mu = (\epsilon + T\partial_T\epsilon + \mu\partial_\mu\epsilon)\delta\Phi v^\mu + iq^\mu(T\Sigma_T + \Sigma_G)\delta\Phi + i(T\tilde{\Sigma}_T + \mu\tilde{\Sigma}_\mu)\varepsilon^{\mu\nu}q_\nu\delta\Phi. \tag{5.13}$$

We now impose consistency with the linearized equation of motion

$$\nabla_\mu\delta\varepsilon^\mu = -q^2(T\Sigma_T + \Sigma_G)\delta\Phi = 0, \tag{5.14}$$

which gives the gravitational Einstein relation $\Sigma_T = -\frac{1}{T}\Sigma_G$.

Luttinger first used the gravitational Einstein relation to obtain a Kubo formula for Σ_T [34]. We perform a derivation in our language for completeness. Take a rapid transverse perturbation δn_μ such that $v^\mu\delta n_\mu = 0$,

$$\delta\varepsilon^\mu = (\partial_T\epsilon\delta T + \partial_\mu\epsilon\delta\mu)v^\mu + i\omega\Sigma_G\delta n^\mu + i\omega\tilde{\Sigma}_G\varepsilon^{\mu\nu}\delta n_\nu, \tag{5.15}$$

where we have used $\varepsilon^{\mu\nu}k_\nu = 0$. Upon application of projectors we have

$$\begin{aligned}&P^\mu{}_\lambda P^\nu{}_\rho G^{\lambda,\rho}_{\varepsilon\varepsilon}(\omega) = i\omega(\Sigma_G g^{\mu\nu} + \tilde{\Sigma}_G\varepsilon^{\mu\nu})\\ \Longrightarrow\qquad &\Sigma_T = -\lim_{\omega\to 0}\operatorname{Im}\frac{1}{T}\frac{g_{\mu\nu}G^{\mu,\nu}_{\varepsilon\varepsilon}(\omega)}{2\omega}, \qquad \tilde{\Sigma}_G = \lim_{\omega\to 0}\frac{\varepsilon_{\mu\nu}G^{\mu,\nu}_{\varepsilon\varepsilon}(\omega)}{2i\omega}.\end{aligned} \tag{5.16}$$

Finally, we derive a Kubo formula for the function $\tilde{g}$. Under a slow perturbation $\delta n_\mu = -\delta\Phi n_\mu$ we have

$$\delta T = T\delta\Phi, \qquad \delta\mu = \mu\delta\Phi, \qquad \delta G^\mu = iq^\mu\delta\Phi, \qquad \delta v^\mu = \delta\Phi v^\mu. \tag{5.17}$$

The energy current varies as

$$\delta\varepsilon^\mu = (\epsilon + T\partial_T\epsilon + \mu\partial_\mu\epsilon)\delta\Phi v^\mu + i(T\tilde{\Sigma}_T + \mu\tilde{\Sigma}_\mu + \tilde{\Sigma}_G)\varepsilon^{\mu\nu}q_\nu\delta\Phi. \tag{5.18}$$

Plugging in the explicit form of $\tilde{\Sigma}_T$ and $\tilde{\Sigma}_\mu$, we obtain

$$T\partial_T\tilde{g} + \mu\partial_\mu\tilde{g} = \lim_{q^2\to 0}\frac{\varepsilon_{\mu\nu}q^\mu G^{\nu,\lambda}_{\varepsilon\varepsilon}(q)n_\lambda}{iT^2q^2}. \tag{5.19}$$

This only determines $\tilde{g}$ up to a function $f(\frac{\mu}{T})$. We can fix this ambiguity by response to a slow electric potential perturbation $\delta A_\mu = \delta a_0 n_\mu$. We then have

$$\delta T = 0, \qquad \delta\mu = \delta a_0, \qquad \delta v^\mu = \frac{i}{B}\varepsilon^{\mu\nu}q_\nu\delta a_0, \tag{5.20}$$

$$\delta\varepsilon^\mu = (\partial_\mu\epsilon\delta\mu)\,v^\mu + i\left(\tilde{\Sigma}_\mu + \frac{\epsilon}{B}\right)\varepsilon^{\mu\nu}q_\nu\delta a_0 \tag{5.21}$$

giving

$$T^2\partial_\mu\tilde{g} - \frac{p}{B} = \lim_{q\to 0}\frac{i\varepsilon_{\mu\nu}q^\mu G^{\nu,\lambda}_{\varepsilon j}(q)n_\lambda}{q^2}. \tag{5.22}$$

Recall here that derivatives are taken at constant B rather than at constant M. The Kubo formulas for $\tilde{\Sigma}_G$ and $\tilde{g}$ completely determine the parity odd energy transport.

JHEP06(2015)044

6 Physical interpretation

To compare with physical results, we first need deal with two issues. First, the covariant currents ε^μ and $T^{\mu\nu}$ have implicit dependence on v^μ that must be removed. This can be done by instead considering the noncovariant currents defined by

$$\delta S = \int d^3x \sqrt{g} e^{-\Phi} \left(\frac{1}{2} T^{ij}_{\rm nc} \delta g_{ij} + \varepsilon^0_{\rm nc} \delta \Phi + \varepsilon^i_{\rm nc} \delta C_i + j^\mu_{\rm nc} \delta A_\mu \right). \tag{6.1}$$

In ref. [17] we demonstrate that these noncovariant currents are simply

$$j^\mu_{\rm nc} = j^\mu, \qquad \varepsilon^0_{\rm nc} = e^{-\Phi} \varepsilon^0, \qquad \varepsilon^i_{\rm nc} = e^{-\Phi} \varepsilon^i + T^{ij} v_j, \qquad T^{ij}_{\rm nc} = T^{ij} \tag{6.2}$$

(these relations are greatly simplified by our use of the massless limit and selection of $s = 1$). The only change from the above is in the energy current, which we defer discussion of until later. Written out explicitly, we have

$$j^0_{\rm nc} = e^\Phi n, \tag{6.3}$$

$$j^i_{\rm nc} = e^\Phi \varepsilon^{ij} \left(\frac{n}{B} \Big(E_j - \partial_j (e^{-\Phi} \mu) \Big) - \frac{s}{B} \partial_j (e^{-\Phi} T) + \partial_j (e^{-\Phi} M) \right), \tag{6.4}$$

$$T^{ij}_{\rm nc} = (p - \zeta \Theta)\, g^{ij} - \eta \sigma^{ij} - \tilde{\eta} \tilde{\sigma}^{ij}. \tag{6.5}$$

The second issue is that to perform the LLL projection we have taken the g-factor to be $g = 2$ (and the spin to be $s = 1$ though this is not essential). To compare to standard expressions used in literature we need to transform back to the values commonly assumed, $g = s = 0$. The result turns out to be rather trivial, in the end giving us back (6.3) with shifted transport coefficients, but it is worth demonstrating how this comes about. In the process we find simple formulas that demonstrate how to recover the physical transport coefficients from those calculated in the massless limit. The general procedure for how to do this is explained in ref. [17] and we merely outline the results here.

Note that to simplify the resulting formulas we assume that E_i is $\mathcal{O}(1)$ in derivatives. In the above, the electric field was potentially large; however, since it's variations are assumed to be small, a frame where E_i is small everywhere may always be obtained. In such a frame a large number of terms are higher order and neglected. Indeed we have already used this in (6.2) to neglect terms that involve the mass which we are otherwise restoring.

The $g = s = 0$ currents are then

$$j^0_{\rm nc} = e^\Phi n, \tag{6.6a}$$

$$j^i_{\rm nc} = e^\Phi \varepsilon^{ij} \left(\frac{n}{B} \left(E_j - \partial_j \left(e^{-\Phi} \left(\mu + \frac{B}{2m} \right) \right) \right) - \frac{s}{B} \partial_j (e^{-\Phi} T) + \partial_j \left(e^{-\Phi} \left(M - \frac{n}{2m} \right) \right) \right), \tag{6.6b}$$

$$T^{ij}_{\rm nc} = \left(p + \frac{Bn}{2m} - \zeta \Theta \right) g^{ij} - \eta \sigma^{ij} - \left(\tilde{\eta} + \frac{1}{2} n \right) \tilde{\sigma}^{ij}. \tag{6.6c}$$

This has a simple interpretation and with a little physical insight we could have guessed the form given here. Recall from ref. [17] that redefining g involves a shift to the electric potential

$$A_0^{g=2} = A_0 - \frac{1}{2m} e^{-\Phi} B \tag{6.7}$$

This also shifts the ground state energy of the system and so the chemical potential changes

$$\mu^{g=2} = \mu - \frac{B}{2m}. \tag{6.8}$$

$\mu^{g=2}$ is the chemical potential that appears in (6.6b).

Similarly, setting $g = 2$ alters the intrinsic magnetic moment of the fluid: each particle carries an excess magnetic dipole moment of $\frac{1}{2m}$. The $g = 2$ and physical magnetizations are then related by

$$M^{g=2} = M + \frac{n}{2m}, \tag{6.9}$$

accounting for the final term in (6.6b) and the shift to the internal pressure in (6.6c) since $p = p_{\text{thm}} - MB$. Finally, setting $s = 1$ overestimates the intrinsic angular momentum per particle by 1, giving the observed shift in the Hall viscosity. In the end, the constitutive relations simply revert to the form (6.3) where we are using the $g = s = 0$ values of μ, M, and $\tilde{\eta}$.

The non-covariant currents then satisfy equations of motion [17]

$$\frac{1}{\sqrt{g}} e^{\Phi} \partial_0 (\sqrt{g} e^{-\Phi} j^0_{\text{nc}}) + e^{\Phi} \nabla_i (e^{-\Phi} j^i_{\text{nc}}) = 0, \tag{6.10a}$$

$$\frac{1}{\sqrt{g}} \partial_0 \left(\sqrt{g} \varepsilon^0_{\text{nc}}\right) + e^{\Phi} \nabla_i (e^{-\Phi} \varepsilon^i_{\text{nc}}) = E_i j^i_{\text{nc}} - \frac{1}{2} T^{ij}_{\text{nc}} \dot{g}_{ij}, \tag{6.10b}$$

$$\frac{e^{\Phi}}{\sqrt{g}} \partial_0 \left(\sqrt{g}\, m j^i_{\text{nc}}\right) + e^{\Phi} \nabla_j (e^{-\Phi} T_{\text{nc}i}{}^{j}) = j^0_{\text{nc}} E_i + \varepsilon_{ij} j^j_{\text{nc}} B + \varepsilon^0_{\text{nc}} \nabla_i \Phi. \tag{6.10c}$$

6.1 The charge current

Now consider the current response to the electric field $\mathbf{E}$, the gravitational field $\mathbf{G} = \boldsymbol{\nabla}\Phi$ and gradients of T, μ and B

$$\mathbf{j}_{\text{nc}} = \left(\sigma_H \mathbf{E} + \sigma_H^T \boldsymbol{\nabla} T + \sigma_H^\mu \boldsymbol{\nabla} \mu + \sigma_H^B \boldsymbol{\nabla} B + \sigma_H^G \boldsymbol{G}\right) \times \hat{\mathbf{z}}. \tag{6.11}$$

We find a Hall conductance

$$\sigma_H = e^{\Phi} \frac{n}{B}. \tag{6.12}$$

This equation is can be obtained trivially by going to the coordinate system moving with the velocity $(\mathbf{E} \times \hat{\mathbf{z}})/B$, in which the electric field vanishes.

The Hall diffusivity is

$$\sigma_H^\mu = \left(\frac{\partial M}{\partial \mu}\right)_{T,B} - \frac{n}{B} \tag{6.13}$$

which using Maxwell's relations can be written as

$$\sigma_H^\mu = \left(\frac{\partial n}{\partial B}\right)_{T,\mu} - \frac{n}{B}. \tag{6.14}$$

From this equation it is easy to argue the existence of Hall plateaus when the chemical potential lies in a gap. In the $T \to 0$ limit, small variations in the chemical potential cannot induce electron transport and so $\sigma_H^\mu = 0$. Equation (6.14) then immediately implies

$$n = \nu B \qquad \Longrightarrow \qquad \sigma_H = \nu, \tag{6.15}$$

where we have taken $\Phi = 0$ and ν is some constant (which we of course know to be the filling fraction). (We are working in units where $e = \hbar = 1$.)

Similarly we also find

$$\sigma_H^T = \left(\frac{\partial s}{\partial B}\right)_{T,\mu} - \frac{s}{B}\,, \qquad \sigma_H^B = \left(\frac{\partial M}{\partial B}\right)_{T,\mu}\,, \qquad \sigma_H^G = \frac{\epsilon + p}{B}\,, \tag{6.16}$$

so in particular σ_H^B is simply the magnetic susceptibility.

6.2 The energy current

We now turn to energy transport. A redefinition

$$\tilde{\Sigma}_G \equiv \frac{\mu}{B}\,(Ts + \mu n) + T^2 c_{RL} - 2M_E, \qquad \tilde{g} \equiv \frac{M_E}{T^2}\,, \tag{6.17}$$

will aid in the physical interpretation of the formulas that follow. Including the $T^{ij}v_j$ shift to the covariant energy current we have

$$\begin{aligned}
\varepsilon^0_{\rm nc} &= \epsilon \\
\varepsilon^i_{\rm nc} &= \varepsilon^{ij}\left(\frac{\epsilon + p}{B}\,(E_j - \partial_j(e^{-\Phi}\mu)) - M\partial_j(e^{-\Phi}\mu) - T c_{RL}\partial_j(e^{-\Phi}T) + e^{\Phi}\partial_j(e^{-2\Phi}M_E)\right) \\
&\quad + \Sigma_T \partial^i\left(e^{-\Phi}T\right).
\end{aligned} \tag{6.18}$$

The $g = 0$ values of the energy density ϵ, energy magnetization M_E, and Righi-Leduc coefficient c_{RL} that are used in this formula are related to the $g = 2$ values by

$$\epsilon = \epsilon^{g=2} + \frac{Bn}{2m}, \qquad M_E = M_E^{g=2} + \frac{M^{g=2}B}{2m}, \qquad c_{RL} = c_{RL}^{g=2} + \frac{s/T}{2m}. \tag{6.19}$$

Defining thermal conductivities

$$\boldsymbol{\varepsilon}_{\rm nc} = \kappa \boldsymbol{\nabla} T + \kappa^{\Phi}\boldsymbol{G} + \left(\kappa_H \mathbf{E} + \kappa_H^T \boldsymbol{\nabla} T + \kappa_H^\mu \boldsymbol{\nabla}\mu + \kappa_H^B \boldsymbol{\nabla} B + \kappa_H^G \boldsymbol{G}\right) \times \hat{\mathbf{z}} \tag{6.20}$$

we have

$$\begin{aligned}
&\kappa = \Sigma_T \qquad \kappa^{\Phi} = -T\Sigma_T && \kappa_H = \frac{\epsilon + p}{B} \\
&\kappa_H^T = e^{-\Phi}\,(\partial_T M_E - T c_{RL}) && \kappa_H^\mu = e^{-\Phi}\left(\partial_\mu M_E - \frac{Ts + \mu n}{B}\right) \\
&\kappa_H^B = e^{-\Phi}\partial_B M_E && \kappa_H^G = e^{-\Phi}\left(T^2 c_{RL} - 2M_E + \frac{\mu}{B}(Ts + \mu n)\right).
\end{aligned} \tag{6.21}$$

6.3 Středa formulas

One notable feature about the formulas (6.4) and (6.18) is the charge and energy currents that persist in thermal equilibrium. We now turn to these, deriving a set of Středa-like formulas for two dimensional fluids. First note from the definitions of the temperature and chemical potential in appendix A that in thermal equilibrium we have $\partial_i \mu = e^{\Phi} E_i + \mu G_i$ and $\partial_i T = T G_i$ where $G_i = \partial_i \Phi$ is the gravitational field exerted by the potential $-\Phi$. The equilibrium currents are then

$$j^i_{\mathrm{nc}} = \varepsilon^{ij} e^{\Phi} \partial_j (e^{-\Phi} M) \qquad \varepsilon^i_{\mathrm{nc}} = \varepsilon^{ij} \left(-M \partial_j (e^{-\Phi} \mu) + e^{\Phi} \partial_j (e^{-2\Phi} M_E) \right) . \tag{6.22}$$

Expressing these in terms of the externally applied fields E_i, B and G_i we have

$$\begin{aligned} j^i_{\mathrm{nc}} &= \varepsilon^{ij} \left(e^{\Phi} \partial_\mu M E_j + \partial_B M \partial_j B + (T \partial_T M + \mu \partial_\mu M - M) G_j \right) \\ \varepsilon^i_{\mathrm{nc}} &= \varepsilon^{ij} \left((\partial_\mu M_E - M) E_j + e^{-\Phi} \partial_B M_E \partial_j B + e^{-\Phi} (T \partial_T M_E + \partial_\mu M_E - 2 M_E) G_j \right) . \end{aligned} \tag{6.23}$$

Defining equilibrium responses by

$$j^i_{\mathrm{nc}} = \varepsilon^{ij} \left(\sigma^{\mathrm{eq}}_H E_j + \sigma^{B\mathrm{eq}}_H \partial_j B + \sigma^{G\mathrm{eq}}_H G_j \right), \tag{6.24}$$

and using some Maxwell relations, we have

$$\begin{aligned} \sigma^{\mathrm{eq}}_H &= \left(\frac{\partial n}{\partial B} \right)_{T,\mu}, \qquad \sigma^{B\mathrm{eq}}_H = \left(\frac{\partial M}{\partial B} \right)_{T,\mu}, \\ \sigma^{G\mathrm{eq}}_H &= T \left(\frac{\partial s}{\partial B} \right)_{T,\mu} + \mu \left(\frac{\partial n}{\partial B} \right)_{T,\mu} - M , \end{aligned} \tag{6.25}$$

where we have set $\Phi = 0$ in these formulas. The first is the well-known Středa formula [26]. The following two are Středa-like formulas for currents induced by inhomogeneities in B and external gravitational forces.

Similarly working with the energy current

$$\varepsilon^i_{\mathrm{nc}} = \varepsilon^{ij} \left(\kappa^{\mathrm{eq}}_H E_j + \kappa^{B\mathrm{eq}}_H \partial_j B + \kappa^{G\mathrm{eq}}_H G_j \right), \tag{6.26}$$

we find that all equilibrium currents are determined by the magnetization M and energy magnetization M_E.

$$\begin{aligned} \kappa^{\mathrm{eq}}_H &= \left(\frac{\partial M_E}{\partial \mu} \right)_{T,B} - M, \qquad \kappa^{B\mathrm{eq}}_H = \left(\frac{\partial M_E}{\partial B} \right)_{T,\mu}, \\ \kappa^{G\mathrm{eq}}_H &= T \left(\frac{\partial M_E}{\partial T} \right)_{\mu,B} + \mu \left(\frac{\partial M_E}{\partial \mu} \right)_{T,B} - 2 M_E . \end{aligned} \tag{6.27}$$

A similar collection of Středa formulas was recently presented in ref. [24]; however they do not agree with ours.

Finally, we note that these Středa formulas in no way depend on the LLL projection that has been implicit throughout this paper. Indeed, they follow only from knowledge of the equilibrium persistent currents (6.23), which may be derived on entirely general grounds. Assuming a static background and dynamic equilibrium, current conservation implies that $j^i_{\rm nc}$ is the curl of a function which we identify as the magnetization density

$$\nabla_i j^i = 0 \qquad \Longrightarrow \qquad j^i_{\rm nc} = \varepsilon^{ij}\partial_j M, \tag{6.28}$$

where we have taken $\Phi = 0$. We can then retrieve the energy current by demanding a static energy density $\partial_0 \varepsilon^0_{\rm nc} = 0$

$$\nabla_i \varepsilon^i_{\rm nc} = E_i j^i_{\rm nc} \qquad \Longrightarrow \qquad \nabla_i \varepsilon^i_{\rm nc} = -\nabla_i (M\varepsilon^{ij}E_j), \tag{6.29}$$

where we have used that $\varepsilon^{ij}\nabla_i E_j = -\dot{B} = 0$. We then similarly obtain

$$\varepsilon^i_{\rm nc} = \varepsilon^{ij}\big(-ME_j + \partial_j M_E\big) \tag{6.30}$$

for some function M_E, reproducing (6.23). This equilibrium current is also found in ref. [35]. Were we to carry this out for nonzero Φ, we would again obtain the correct result, but the normalization of M and M_E with factors of $e^{-\Phi}$ has to be determined from other considerations.

6.4 Noncovariant Kubo formulas

For the reader's convenience, we restate here the Kubo formulas found above in terms of the energy magnetization M_E and thermal Hall coefficient c_{RL} without the use of the Newton-Cartan formalism. They are expressed in terms of two-point correlators of the non- covariant currents

$$\begin{aligned}
\mathcal{G}^{\mu,\nu}_{\varepsilon\varepsilon}(x) &= \left\langle \frac{\delta \varepsilon^\mu_{\rm nc}(x)}{\delta n_\nu(0)} \right\rangle - i\theta(x^0)\left\langle [\varepsilon^\mu_{\rm nc}(x), \varepsilon^\nu_{\rm nc}(0)]\right\rangle, \\
\mathcal{G}^{\mu,\nu}_{\varepsilon j}(x) &= \left\langle \frac{\delta \varepsilon^\mu_{\rm nc}(x)}{\delta A_\nu(0)} \right\rangle + i\theta(x^0)\left\langle [\varepsilon^\mu_{\rm nc}(x), j^\nu_{\rm nc}(0)]\right\rangle, \\
\mathcal{G}^{ij,kl}(x) &= \left\langle \frac{\delta T^{ij}_{\rm nc}(x)}{\delta g_{kl}(0)} \right\rangle + \frac{i}{2}\theta(x^0)\left\langle [T^{ij}_{\rm nc}(x), T^{kl}_{\rm nc}(0)]\right\rangle.
\end{aligned} \tag{6.31}$$

These are

$$\begin{aligned}
&\eta = -\lim_{\omega\to 0} \mathrm{Im}\frac{\Pi_{ijkl}\mathcal{G}^{ij,kl}(\omega)}{2\omega}, \qquad \tilde{\eta} = -\lim_{\omega\to 0}\frac{\tilde{\Pi}_{ijkl}\mathcal{G}^{ij,kl}(\omega)}{2i\omega}, \qquad \zeta = -\lim_{\omega\to 0}\mathrm{Im}\frac{\mathcal{G}_i{}^{i,}{}_j{}^j(\omega)}{2\omega}, \\
&\partial_\mu M_E - M = \lim_{q\to 0}\frac{i\varepsilon_{ij}q^i\mathcal{G}^{j,0}_{\varepsilon j}(q)}{q^2}, \quad T\partial_T M_E + \mu\partial_\mu M_E - 2M_E = -\lim_{q\to 0}\frac{i\epsilon_{ij}q^i\mathcal{G}^{j,0}_{\varepsilon\varepsilon}(q)}{q^2}, \\
&\Sigma_T = \frac{i}{T}\lim_{\omega\to 0}\frac{\delta_{ij}\mathcal{G}^{ij}_{\varepsilon\varepsilon}(\omega)}{2\omega}, \qquad T^2 c_{RL} + \frac{\mu}{B}(Ts+\mu n) - 2M_E = \lim_{\omega\to 0}\frac{\epsilon_{ij}\mathcal{G}^{ij}_{\varepsilon\varepsilon}(\omega)}{2i\omega},
\end{aligned} \tag{6.32}$$

the correlators being evaluated on the trivial background $g_{ij} = \delta_{ij}$, $\Phi = 0$, $E_i = 0$ and $\partial_i B = 0$.

JHEP06(2015)044

7 Conclusion

The proper coordinate invariant description of non-relativistic physics is that of a Newton-Cartan geometry, which naturally includes a source n_μ for the energy current in addition to those present for the stress and charge current. As discussed in recent work, with some care, diffeomorphism covariant currents may then be defined and 1-point Ward identities follow naturally as in the nonrelativistic case.

In a fluid dynamical description, the Ward identities become equations of motion once constitutive relations have been supplied. We have given the most general constitutive relations consistent with diffeomorphism covariance and derived their Kubo formulas. We argue that a fractional quantum Hall fluid is distinguished as being a *force-free* fluid in 2+1 dimensions. The force-free condition immediately gives powerful constraints on fractional quantum Hall transport, determining all charge transport in terms of thermodynamics.

A straightforward entropy current analysis was then performed. The expected restrictions on the signs of parity even viscosities and thermal conductivity are obtained, in addition to new constraints on the transverse energy response. These four coefficients are not independent but are instead determined by two free functions of T, μ and M: the thermal Hall conductivity and energy magnetization. The derived constitutive relations imply a set of formulas for the equilibrium response that generalize the well-known Středa formula. These new formulas characterize the system's response to Newtonian gravitational fields and inhomogeneous magnetic backgrounds.

It is our hope that the approach outlined here to non-relativistic fluids finds further use. In this approach spacetime coordinate invariance is automatic, just as in the standard treatment of relativistic fluids and computations are streamlined. Here we brought our formalism to bear on FQH fluids, but it is sufficiently general to treat arbitrary fluids in any dimension.

Acknowledgments

We would like to thank Alexander Abanov, Andrei Gromov, Kristan Jensen and and Paul Wiegmann for discussions. This work is supported, in part, by the US DOE grant No. DE-FG02-13ER41958, and the ARO-MURI 63834-PH-MUR grant, and a Simon Investigator grant from the Simons Foundation.

A Magneto-thermodynamics

We make a few comments on thermodynamics in curved backgrounds and nonzero magnetic fields to motivate the identifications of ϵ, n and p in (3.7) and (3.9) as the thermodynamic energy density, particle density and internal pressure, particularly the magnetization contribution whose presence is not obvious. These issues are discussed at length in ref. [36] in the case of free field theory, but we take a more general view, assuming only local thermodynamic equilibrium and a local free energy. This is the zeroth order part of an analysis along the lines of that found in ref. [37], from which we differ only in so far as our treatment is non- relativistic and includes a background magnetic field that gives rise to the magnetization currents in question.

Consider the thermal partition function Z associated to some microscopic quantum field theory and it's corresponding effective action

$$W = -\ln Z. \tag{A.1}$$

We assume nothing about the detailed dynamics other than the existence of a gap so that W is a local function of n_μ, $g^{\mu\nu}$ and A_μ. Specialize to a time independent but curved background geometry and gauge field whose spatial variations are small. We may then assume after some time local thermodynamic equilibrium is reached and each fluid element is characterized entirely by a temperature and a chemical potential

$$\beta = \int_c n, \qquad \frac{\mu}{T} = \int_c A, \tag{A.2}$$

where c is the time circle passing through that element. Note that T and μ may depend on space.

To zeroth order in derivatives, we then have

$$W = \int d^3x\, \sqrt{g}\, e^{-\Phi} p_{\text{thm}}(T, \mu, B), \tag{A.3}$$

B being the only other covariant scalar that may be constructed at zeroth order. The detailed form of p_{thm} will depend on the microscopic physics but will not be needed here. Had we assumed spatial homogeneity, this would merely be the elementary relation $\Omega = p_{\text{thm}} V$ that connects thermodynamics with statistical physics ($\Omega = TW$ is the grand potential). Thus p_{thm} is the grand potential density which, in the absence of the magnetic field, would coincide with the pressure that appears in the stress. We define local energy, entropy, particle and magnetization densities by

$$dp_{\text{thm}} = sdT + nd\mu + MdB \qquad \text{and} \qquad \epsilon + p_{\text{thm}} = Ts + \mu n, \tag{A.4}$$

which are merely the fundamental thermodynamic relations (3.11).

It's now a simple matter to calculate the equilibrium j^0, ε^0 and T^{ij}. To clarify the Φ dependence, parameterize the time circle by some interval $x^0 \in (0, \frac{1}{T_0})$. We then have

$$T = e^{\Phi} T_0, \qquad \mu = e^{\Phi} A_0. \tag{A.5}$$

Varying A_0, Φ and g^{ij} we find

$$j^0 = e^{\Phi} n, \qquad \varepsilon^0 = e^{\Phi}\epsilon, \qquad T^{ij} = (p_{\text{thm}} - MB) g^{ij} \tag{A.6}$$

The magnetization contribution to the internal pressure arises due to the magnetic flux density's metric dependence $B = \frac{1}{\sqrt{g}}(\partial_1 A_2 - \partial_2 A_1)$.

JHEP06(2015)044

B Weyl invariance

In special cases the theory may exhibit Weyl invariance. This happens, for example, when the interaction is a purely contact interaction [17]. In this case the functional form of the transport coefficients considered above will be constrained. We derive these constraints in this appendix.

A Weyl invariant theory is unchanged under the transformation

$$g'_{ij} = e^{-2\alpha} g_{ij}, \qquad \Phi' = \Phi + 2\alpha. \tag{B.1}$$

Since we have $S[g_{ij}, \Phi] = S[g'_{ij}, \Phi']$, varying the action we find

$$T'^{ij}_{\text{nc}} = e^{6\alpha} T^{\mu\nu}_{\text{nc}}, \qquad \varepsilon'^{\mu}_{\text{nc}} = e^{4\alpha} \varepsilon^{\mu}_{\text{nc}}, \tag{B.2}$$

for a Weyl invariant theory. From the equilibrium definitions of the thermodynamic variables we also find

$$T' = e^{2\alpha} T, \qquad \mu' = e^{2\alpha} \mu, \qquad B' = e^{2\alpha} B. \tag{B.3}$$

Let's first turn to the stress tensor

$$T^{ij}_{\text{nc}} = (p - \zeta\Theta) g^{ij} - \eta\sigma^{ij} - \tilde{\eta}\tilde{\sigma}^{ij}. \tag{B.4}$$

One may show from their definitions that the expansion and shear tensors transform as

$$\Theta' = e^{2\alpha} \left(\Theta - 2v^{\mu} \nabla_{\mu} \alpha\right), \qquad \sigma'^{ij} = e^{4\alpha} \sigma^{ij}, \qquad \tilde{\sigma}'^{ij} = e^{4\alpha} \tilde{\sigma}^{ij}. \tag{B.5}$$

To satisfy the scaling rule (B.2) the bulk viscosity must vanish: $\zeta = 0$ [38]. Furthermore, the equation of state and viscosities must be homogeneous functions of the thermodynamic variables

$$\begin{aligned} p_{\text{thm}}(\lambda T, \lambda\mu, \lambda B) &= \lambda^2 p_{\text{thm}}(T, \mu, B), \\ \eta(\lambda T, \lambda\mu, \lambda B) &= \lambda \eta(T, \mu, B), \\ \tilde{\eta}(\lambda T, \lambda\mu, \lambda B) &= \lambda \tilde{\eta}(T, \mu, B). \end{aligned} \tag{B.6}$$

Similar restrictions arise for the energy current without complication. The thermal conductivity, thermal Hall conductivity, and energy magnetization are also homogeneous functions,

$$\begin{aligned} \Sigma_T(\lambda T, \lambda\mu, \lambda B) &= \lambda \Sigma_T(T, \mu, B), \\ c_{RL}(\lambda T, \lambda\mu, \lambda B) &= c_{RL}(T, \mu, B), \\ M_E(\lambda T, \lambda\mu, \lambda B) &= \lambda^2 M_E(T, \mu, B). \end{aligned} \tag{B.7}$$

JHEP06(2015)044

References

[1] D.C. Tsui, H.L. Stormer and A.C. Gossard, *Two-dimensional magnetotransport in the extreme quantum limit*, *Phys. Rev. Lett.* **48** (1982) 1559 [INSPIRE].

[2] R.B. Laughlin, *Anomalous quantum Hall effect: An incompressible quantum fluid with fractionallycharged excitations*, *Phys. Rev. Lett.* **50** (1983) 1395 [INSPIRE].

[3] B.I. Halperin, *Statistics of quasiparticles and the hierarchy of fractional quantized Hall states*, *Phys. Rev. Lett.* **52** (1984) 1583 [*Erratum ibid.* **52** (1984) 2390] [INSPIRE].

[4] D. Arovas, J.R. Schrieffer and F. Wilczek, *Fractional Statistics and the Quantum Hall Effect*, *Phys. Rev. Lett.* **53** (1984) 722 [INSPIRE].

[5] X.-G. Wen, *Theory of the edge states in fractional quantum Hall effects*, *Int. J. Mod. Phys.* **B 6** (1992) 1711 [INSPIRE].

[6] X.-G. Wen, *Quantum Field Theory of Many Body Systems*, Oxford University Press, Oxford, U.K. (2004).

[7] S.C. Zhang, T.H. Hansson and S. Kivelson, *An effective field theory model for the fractional quantum hall effect*, *Phys. Rev. Lett.* **62** (1988) 82 [INSPIRE].

[8] B.I. Halperin, P.A. Lee and N. Read, *Theory of the half filled Landau level*, *Phys. Rev.* **B 47** (1993) 7312 [INSPIRE].

[9] J.K. Jain, *Composite fermion approach for the fractional quantum Hall effect*, *Phys. Rev. Lett.* **63** (1989) 199 [INSPIRE].

[10] C. Hoyos and D.T. Son, *Hall Viscosity and Electromagnetic Response*, *Phys. Rev. Lett.* **108** (2012) 066805 [arXiv:1109.2651] [INSPIRE].

[11] D.T. Son, *Newton-Cartan Geometry and the Quantum Hall Effect*, arXiv:1306.0638 [INSPIRE].

[12] S. Golkar, D.X. Nguyen and D.T. Son, *Spectral Sum Rules and Magneto-Roton as Emergent Graviton in Fractional Quantum Hall Effect*, arXiv:1309.2638 [INSPIRE].

[13] A. Gromov and A.G. Abanov, *Density-curvature response and gravitational anomaly*, *Phys. Rev. Lett.* **113** (2014) 266802 [arXiv:1403.5809] [INSPIRE].

[14] B. Bradlyn, M. Goldstein and N. Read, *Kubo formulas for viscosity: Hall viscosity, Ward identities and the relation with conductivity*, *Phys. Rev.* **B 86** (2012) 245309 [arXiv:1207.7021] [INSPIRE].

[15] T. Can, M. Laskin and P. Wiegmann, *Fractional Quantum Hall Effect in a Curved Space: Gravitational Anomaly and Electromagnetic Response*, *Phys. Rev. Lett.* **113** (2014) 046803 [arXiv:1402.1531] [INSPIRE].

[16] G.Y. Cho, Y. You and E. Fradkin, *Geometry of Fractional Quantum Hall Fluids*, *Phys. Rev.* **B 90** (2014) 115139 [arXiv:1406.2700] [INSPIRE].

[17] M. Geracie, D.T. Son, C. Wu and S.-F. Wu, *Spacetime Symmetries of the Quantum Hall Effect*, *Phys. Rev.* **D 91** (2015) 045030 [arXiv:1407.1252] [INSPIRE].

[18] E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie)*, *Ann. Scient. Ec. Norm. Sup.* **40** (1923) 325.

JHEP06(2015)044

[19] E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie)(Suite)*, *Ann. Scient. Ec. Norm. Sup.* **41** (1924) 1.

[20] H.P. Kuenzle, *Galilei and lorentz structures on space-time - comparison of the corresponding geometry and physics*, *Ann. Inst. Henri Poincaré* **17** (1972) 337.

[21] J. Ehlers, *Über den Newtonschen Grenzwert der Einsteinschen Gravitationstheorie*, in *Grundlagenprobleme der modernen Physik*, Bibliographisches Institute AG, Zürich, (1981) p. 65.

[22] B. Carter and I.M. Khalatnikov, *Canonically covariant formulation of Landau's Newtonian superfluid dynamics*, *Rev. Math. Phys.* **6** (1994) 277 [INSPIRE].

[23] S. Janiszewski and A. Karch, *Non-relativistic holography from Hořava gravity*, *JHEP* **02** (2013) 123 [arXiv:1211.0005] [INSPIRE].

[24] A. Gromov and A.G. Abanov, *Thermal Hall Effect and Geometry with Torsion*, *Phys. Rev. Lett.* **114** (2015) 016802 [arXiv:1407.2908] [INSPIRE].

[25] B. Bradlyn and N. Read, *Low-energy effective theory in the bulk for transport in a topological phase*, *Phys. Rev.* **B 91** (2015) 125303 [arXiv:1407.2911] [INSPIRE].

[26] P. Středa, *Theory of quantised Hall conductivity in two dimensions*, *J. Phys.* **C 15** (1982) L717.

[27] M. Geracie and D.T. Son, *Thermal Transport in a Noncommutative Hydrodynamics*, *J. Exp. Theor. Phys.* **120** (2015) 444 [arXiv:1407.4460] [INSPIRE].

[28] M.H. Christensen, J. Hartong, N.A. Obers and B. Rollier, *Boundary Stress-Energy Tensor and Newton-Cartan Geometry in Lifshitz Holography*, *JHEP* **01** (2014) 057 [arXiv:1311.6471] [INSPIRE].

[29] O. Andreev, M. Haack and S. Hofmann, *On Nonrelativistic Diffeomorphism Invariance*, *Phys. Rev.* **D 89** (2014) 064012 [arXiv:1309.7231] [INSPIRE].

[30] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz and A. Yarom, *Parity-Violating Hydrodynamics in 2+1 Dimensions*, *JHEP* **05** (2012) 102 [arXiv:1112.4498] [INSPIRE].

[31] J. Bhattacharya, S. Bhattacharyya, S. Minwalla and A. Yarom, *A Theory of first order dissipative superfluid dynamics*, *JHEP* **05** (2014) 147 [arXiv:1105.3733] [INSPIRE].

[32] L.D. Landau and E.M. Lifshitz *Course of Theoretical Physics, Vol.* **6***: Fluid Mechanics*, Pergamon Press, Oxford U.K. (1987).

[33] R. Tolman and P. Ehrenfest, *Temperature Equilibrium in a Static Gravitational Field*, *Phys. Rev.* **36** (1930) 1791 [INSPIRE].

[34] J.M. Luttinger, *Theory of Thermal Transport Coefficients*, *Phys. Rev.* **135** (1964) A1505.

[35] N.R. Cooper, B.I. Halperin and I.M. Ruzin, *Thermoelectric response of an interacting two-dimensional electron gas in a quantizing magnetic field*, *Phys. Rev.* **B 55** (1997) 2344.

[36] S.A. Hartnoll, P.K. Kovtun, M. Muller and S. Sachdev, *Theory of the Nernst effect near quantum phase transitions in condensed matter and in dyonic black holes*, *Phys. Rev.* **B 76** (2007) 144502 [arXiv:0706.3215] [INSPIRE].

[37] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz and A. Yarom, *Towards hydrodynamics without an entropy current*, *Phys. Rev. Lett.* **109** (2012) 101601 [arXiv:1203.3556] [INSPIRE].

[38] D.T. Son, *Vanishing bulk viscosities and conformal invariance of unitary Fermi gas*, *Phys. Rev. Lett.* **98** (2007) 020604 [cond-mat/0511721] [INSPIRE].

JHEP06(2015)044