

Exact solution of the Izergin-Korepin model with general non-diagonal boundary terms

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ABSTRACT: The Izergin-Korepin model with general non-diagonal boundary terms, a typical integrable model beyond A-type and without U(1)-symmetry, is studied via the off-diagonal Bethe ansatz method. Based on some intrinsic properties of the R -matrix and the K -matrices, certain operator product identities of the transfer matrix are obtained at some special points of the spectral parameter. These identities and the asymptotic behaviors of the transfer matrix together allow us to construct the inhomogeneous $T - Q$ relation and the associated Bethe ansatz equations. In the diagonal boundary limit, the reduced results coincide exactly with those obtained via other methods.

KEYWORDS: Bethe Ansatz, Lattice Integrable Models

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1 Introduction

Exactly solvable models (or quantum integrable systems) [1] have provided important benchmarks for the non-perturbative analysis of quantum systems appearing in string and super-symmetric Yang-Mills (SYM) theories [2] (see also [3] and references therein), low-dimensional condensed matter physics [4, 5] and statistical physics [6–8]. However, it has been well known for many years that there exists a quite usual class of integrable models, which do not possess $U(1)$ -symmetry and thus make the conventional Bethe ansatz methods almost inapplicable. Some famous examples are the closed XYZ chain with odd number of sites [9], the anisotropic spin torus [10], the quantum spin chains with non-diagonal boundary fields [11–15] and their multi-component generalizations [16–21]. The broken $U(1)$ -symmetry in those models leads to the absence of an obvious reference state, which is crucial for the usage of the conventional Bethe ansatz (BA) methods [1, 9, 22–28]. There have been numerous efforts [6, 7, 11–15, 29–46] to approach the exact solutions¹ to this class of models over the last 20 years.

Very recently, based on the intrinsic properties of the R -matrix and the K -matrices for quantum integrable models, a systematic method for identifying the spectrum of integrable models without $U(1)$ -symmetry, i.e., the off-diagonal Bethe ansatz (ODBA) method was

¹It is also interesting that the eigenstates can be classified by the representation of the so-called q -Onsager algebra [47, 48] and that the half-infinite XXZ spin chain with a triangular boundary has been studied by q -vertex operator method [49].

proposed in [43–46] for the models associated with $su(2)$ algebra. Subsequently, the nested-version of ODBA for the models associated with $su(n)$ algebra was developed in [21]. Several long-standing models were then diagonalized exactly [21, 43–46, 50, 51] by the ODBA method.² Despite those progresses, an important issue, i.e., whether the ODBA method can be applied to other multi-component integrable models defined beyond the $su(n)$ algebra, is still open.

In this paper, we study the Izergin-Korepin (IK) model [57] with generic integrable boundaries. This model has played a fundamental role in quantum integrable models associated with algebras beyond $su(n)$ (or non A-type models). It was introduced as a quantum integrable model related to the Dodd-Bullough-Mikhailov or Jiber-Mikhailov-Shabat model [58, 59], one of two integrable relativistic models containing one scalar field (the other is sine-Gordon model). The R -matrix of the model corresponds to the simplest twisted affine algebra $A_2^{(2)}$. The IK model with open boundary condition is related to the loop models [60] and self-avoiding walks at a boundary [61]. The IK model with $U(1)$ -symmetry, i.e. with periodic boundary condition or with diagonal boundaries have been extensively studied [60, 62–72]. Even the most general integrable boundary condition (corresponding to the non-diagonal reflection matrix) has been known almost for 20 years [64, 69, 71], its exact solution is still missing. The purpose of this paper is to propose the spectrum of this model using the ODBA method.

The paper is organized as follows. Section 2 serves as an introduction to the model and our notations. In section 3, we derive certain operator product identities for the transfer matrix of the model with general non-diagonal boundary terms by using some intrinsic properties of the R -matrix and K -matrices. The asymptotic behaviors of the transfer matrix are also obtained. Section 4 is devoted to the construction of the inhomogeneous $T - Q$ relation and the corresponding Bethe ansatz equations. Section 5 is attributed to the reduction to case of diagonal boundaries. It is found that the reduced results coincide exactly with those obtained by other Bethe ansatz methods. In section 6, we summarize our results and give some discussions. Some detailed technical proof is given in appendix A.

2 Transfer matrix

Throughout, \mathbf{V} denotes a three-dimensional linear space and let $\{|i\rangle|i = 1, 2, 3\}$ be an orthonormal basis of it. We shall adopt the standard notations: for any matrix $A \in \text{End}(\mathbf{V})$, A_j is an embedding operator in the tensor space $\mathbf{V} \otimes \mathbf{V} \otimes \cdots$, which acts as A on the j -th space and as identity on the other factor spaces; for $B \in \text{End}(\mathbf{V} \otimes \mathbf{V})$, B_{ij} is an embedding operator of B in the tensor space, which acts as identity on the factor spaces except for the i -th and j -th ones.

²The completeness of the solutions for the (an-)isotropic spin- $\frac{1}{2}$ chain has been checked numerically within the Bethe ansatz method [52, 53], and holds by construction in the quantum separation variables method (SOV) [54, 55] or the q-Onsager algebra method [36]. An expression for the corresponding eigenvectors for the XXX open chain in the framework of algebraic Bethe ansatz was also proposed recently in [56].

The R -matrix $R(u) \in \text{End}(\mathbf{V} \otimes \mathbf{V})$ of the IK model is given by [57]

$$R_{12}(u) = \left(\begin{array}{c|c|c} c(u) & & \\ \hline & b(u) & e(u) \\ & d(u) & g(u) \\ \hline \bar{e}(u) & b(u) & f(u) \\ & \bar{g}(u) & g(u) \\ & a(u) & e(u) \\ \hline \bar{f}(u) & \bar{g}(u) & d(u) \\ & \bar{e}(u) & b(u) \\ & & c(u) \end{array} \right), \quad (2.1)$$

where the matrix elements are

$$\begin{aligned} a(u) &= \sinh(u - 3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh \eta, & b(u) &= \sinh(u - 3\eta) + \sinh 3\eta, \\ c(u) &= \sinh(u - 5\eta) + \sinh \eta, & d(u) &= \sinh(u - \eta) + \sinh \eta, \\ e(u) &= -2e^{-\frac{u}{2}} \sinh 2\eta \cosh\left(\frac{u}{2} - 3\eta\right), & \bar{e}(u) &= -2e^{\frac{u}{2}} \sinh 2\eta \cosh\left(\frac{u}{2} - 3\eta\right), \\ f(u) &= -2e^{-u+2\eta} \sinh \eta \sinh 2\eta - e^{-\eta} \sinh 4\eta, \\ \bar{f}(u) &= 2e^{u-2\eta} \sinh \eta \sinh 2\eta - e^{\eta} \sinh 4\eta, \\ g(u) &= 2e^{-\frac{u}{2}+2\eta} \sinh \frac{u}{2} \sinh 2\eta, & \bar{g}(u) &= -2e^{\frac{u}{2}-2\eta} \sinh \frac{u}{2} \sinh 2\eta. \end{aligned} \quad (2.2)$$

The R -matrix satisfies the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2), \quad (2.3)$$

and possesses the following properties,

$$\text{Initial condition :} \quad R_{12}(0) = (\sinh \eta - \sinh 5\eta)P_{12}, \quad (2.4)$$

$$\text{Unitarity relation :} \quad R_{12}(u)R_{21}(-u) = \rho_1(u) \times \text{id}, \quad (2.5)$$

$$\text{Crossing relation :} \quad R_{12}(u) = V_1 R_{12}^{t_2}(-u + 6\eta + i\pi) V_1^{-1}, \quad (2.6)$$

$$\text{PT-symmetry :} \quad R_{21}(u) = R_{12}^{t_1 t_2}(u), \quad (2.7)$$

$$\text{Periodicity :} \quad R_{12}(u + 2i\pi) = R_{12}(u). \quad (2.8)$$

Here $R_{21}(u) = P_{12}R_{12}(u)P_{12}$ with P_{12} being the usual permutation operator and t_i denotes transposition in the i -th space. The function $\rho_1(u)$ and the crossing matrix V are given by

$$\rho_1(u) = -4 \sinh\left(\frac{u}{2} - 2\eta\right) \sinh\left(\frac{u}{2} + 2\eta\right) \cosh\left(\frac{u}{2} - 3\eta\right) \cosh\left(\frac{u}{2} + 3\eta\right), \quad (2.9)$$

$$V = \begin{pmatrix} & -e^{-\eta} \\ 1 & \\ & \\ -e^{\eta} & \end{pmatrix}, \quad V^2 = 1. \quad (2.10)$$

The unitarity property (2.5) and crossing relation (2.6) of the R -matrix and expressions (2.9)–(2.10) of the function $\rho_1(u)$ and the V -matrix imply that the R -matrix satisfies

the crossing-unitarity relation

$$R_{12}^{t_1}(u) \mathcal{M}_1 R_{21}^{t_1}(-u + 12\eta) \mathcal{M}_1^{-1} = \rho_2(u) \times \text{id}, \quad (2.11)$$

$$\mathcal{M} = V^t V = \begin{pmatrix} e^{2\eta} & & \\ & 1 & \\ & & e^{-2\eta} \end{pmatrix}, \quad (2.12)$$

$$\rho_2(u) = -4 \cosh\left(\frac{u}{2} - 5\eta\right) \cosh\left(\frac{u}{2} - \eta\right) \sinh\frac{u}{2} \sinh\left(\frac{u}{2} - 6\eta\right). \quad (2.13)$$

It is easily to check that the R -matrix also satisfies the following relation

$$\mathcal{M}_1 \mathcal{M}_2 R_{12}(u) \mathcal{M}_1^{-1} \mathcal{M}_2^{-1} = R_{12}(u), \quad R_{12}(u) R_{21}(v) = R_{21}(v) R_{12}(u). \quad (2.14)$$

Let us introduce the “row-to-row” (or one-row) monodromy matrices $T(u)$ and $\hat{T}(u)$, which are 3×3 matrices with operator-valued elements acting on $\mathbf{V}^{\otimes N}$,

$$T_0(u) = R_{0N}(u - \theta_N) R_{0N-1}(u - \theta_{N-1}) \cdots R_{01}(u - \theta_1), \quad (2.15)$$

$$\hat{T}_0(u) = R_{10}(u + \theta_1) R_{20}(u + \theta_2) \cdots R_{N0}(u + \theta_N). \quad (2.16)$$

Here $\{\theta_j | j = 1, \dots, N\}$ are arbitrary free complex parameters which are usually called as inhomogeneous parameters. The transfer matrix can be constructed as follows [28, 73]. Let us introduce further a pair of K -matrices $K^-(u)$ and $K^+(u)$. The former satisfies the reflection equation (RE)

$$\begin{aligned} R_{12}(u_1 - u_2) K_1^-(u_1) R_{21}(u_1 + u_2) K_2^-(u_2) \\ = K_2^-(u_2) R_{12}(u_1 + u_2) K_1^-(u_1) R_{21}(u_1 - u_2), \end{aligned} \quad (2.17)$$

and the latter satisfies the dual RE

$$\begin{aligned} R_{12}(u_2 - u_1) K_1^+(u_1) \mathcal{M}_1^{-1} R_{21}(-u_1 - u_2 + 12\eta) \mathcal{M}_1 K_2^+(u_2) \\ = K_2^+(u_2) \mathcal{M}_2^{-1} R_{12}(-u_1 - u_2 + 12\eta) \mathcal{M}_2 K_1^+(u_1) R_{21}(u_2 - u_1). \end{aligned} \quad (2.18)$$

For open boundaries, one needs to consider the double-row monodromy matrix $\mathbb{T}(u)$

$$\mathbb{T}(u) = T(u) K^-(u) \hat{T}(u). \quad (2.19)$$

The double-row transfer matrix $t(u)$ is thus given by

$$t(u) = \text{tr}(K^+(u) \mathbb{T}(u)). \quad (2.20)$$

The QYBE (2.3) and (dual) REs (2.17) and (2.18) lead to the fact that the transfer matrices with different spectral parameters commute with each other [28]: $[t(u), t(v)] = 0$. Therefore $t(u)$ serves as the generating functional of the conserved quantities of the corresponding system.

In this paper we consider the generic non-diagonal K -matrix $K^-(u)$ found in [64] and was classified as type II³ in [69, 71]

$$K^-(u) = \begin{pmatrix} 1 + 2e^{-u-\epsilon} \sinh \eta & 0 & 2e^{-\epsilon+\sigma} \sinh u \\ 0 & 1 - 2e^{-\epsilon} \sinh(u - \eta) & 0 \\ 2e^{-\epsilon-\sigma} \sinh u & 0 & 1 + 2e^{u-\epsilon} \sinh \eta \end{pmatrix}, \quad (2.21)$$

³The generalization to the non-diagonal K -matrices of type I in [69, 71] is straightforward.

and non-diagonal $K^+(u)$ given by

$$K^+(u) = \mathcal{M}K^-(-u + 6\eta + i\pi) \Big|_{(\epsilon, \sigma) \rightarrow (\epsilon', \sigma')} . \quad (2.22)$$

Besides the crossing parameter η , the corresponding transfer matrix $t(u)$ given by (2.20) has four other free parameters $\{\epsilon, \sigma, \epsilon', \sigma'\}$ describing the boundary fields. The Hamiltonian of the Izergin-Korepin model with general non-diagonal boundary terms specified by the K -matrices given by (2.21) and (2.22) then is given in terms of the transfer matrix by

$$\begin{aligned} H &= \frac{\partial \ln t(u)}{\partial u} \Big|_{u=0, \{\theta_j=0\}} \\ &= \sum_{j=1}^{N-1} \frac{2P_{jj+1}R'_{jj+1}(0)}{\sinh \eta - \sinh 5\eta} + \frac{\text{tr}K^{+'}(0)}{\text{tr}K^+(0)} + \frac{2\text{tr}_0[K_0^+(0)P_{N0}R'_{N0}(0)]}{(\sinh \eta - \sinh 5\eta)\text{tr}K^+(0)} + \frac{K_1^{-'}(0)}{1 + 2e^{-\epsilon} \sinh \eta} \\ &= \frac{2}{\sinh \eta - \sinh 5\eta} \sum_{j=1}^{N-1} \left\{ \cosh 5\eta (E_j^{11}E_{j+1}^{11} + E_j^{33}E_{j+1}^{33}) \right. \\ &\quad + \sinh 2\eta(\sinh 3\eta - \cosh 3\eta) (E_j^{11}E_{j+1}^{22} + E_j^{22}E_{j+1}^{33}) \\ &\quad + \sinh 2\eta(\sinh 3\eta + \cosh 3\eta) (E_j^{22}E_{j+1}^{11} + E_j^{33}E_{j+1}^{22}) \\ &\quad + 2\sinh \eta \sinh 2\eta (e^{-2\eta}E_j^{11}E_{j+1}^{33} + e^{2\eta}E_j^{33}E_{j+1}^{11}) + \cosh \eta (E_j^{13}E_{j+1}^{31} + E_j^{31}E_{j+1}^{13}) \\ &\quad + \cosh 3\eta (E_j^{12}E_{j+1}^{21} + E_j^{21}E_{j+1}^{12} + E_j^{22}E_{j+1}^{22} + E_j^{23}E_{j+1}^{32} + E_j^{32}E_{j+1}^{23}) \\ &\quad \left. - e^{-2\eta} \sinh 2\eta (E_j^{12}E_{j+1}^{32} + E_j^{21}E_{j+1}^{23}) + e^{2\eta} \sinh 2\eta (E_j^{23}E_{j+1}^{21} + E_j^{32}E_{j+1}^{12}) \right\} \\ &\quad - \frac{2e^{-\epsilon}}{1 + 2e^{-\epsilon} \sinh \eta} [\sinh \eta (E_1^{11} - E_1^{33}) + \cosh \eta E_1^{22} - e^\sigma E_1^{13} - e^{-\sigma} E_1^{31}] \\ &\quad + 2 \left[(\sinh \eta - \sinh 5\eta) \left(2 \cosh 2\eta - 4e^{-\epsilon'} \sinh \eta \cosh 4\eta + 1 + 2e^{-\epsilon'} \sinh 5\eta \right) \right]^{-1} \\ &\quad \times \left\{ \left[(e^{2\eta} - 2e^{-4\eta-\epsilon'} \sinh \eta) \cosh 5\eta + (1 + 2e^{-\epsilon'} \sinh 5\eta) \right. \right. \\ &\quad \times \sinh 2\eta(\sinh 3\eta - \cosh 3\eta) + 2 \left(e^{-4\eta} - 2e^{2\eta-\epsilon'} \sinh \eta \right) \sinh \eta \sinh 2\eta \Big] E_N^{11} \\ &\quad + \left[(e^{2\eta} - 2e^{-4\eta-\epsilon'} \sinh \eta) \sinh 2\eta(\sinh 3\eta + \cosh 3\eta) + (1 + 2e^{-\epsilon'} \sinh 5\eta) \right. \\ &\quad \times \cosh 3\eta + 2 \left(e^{-2\eta} - 2e^{4\eta-\epsilon'} \sinh \eta \right) \sinh 2\eta(\sinh 3\eta - \cosh 3\eta) \Big] E_N^{22} \\ &\quad + \left[2 \left(e^{4\eta} - 2e^{-2\eta-\epsilon'} \sinh \eta \right) \sinh \eta \sinh 2\eta + (1 + 2e^{-\epsilon'} \sinh 5\eta) \right. \\ &\quad \times \sinh 2\eta(\sinh 3\eta + \cosh 3\eta) + \left(e^{-2\eta} - 2e^{4\eta-\epsilon'} \sinh \eta \right) \cosh 5\eta \Big] E_N^{33} \\ &\quad \left. - 2e^{-\epsilon'} \sinh 6\eta \cosh \eta \left[e^{2\eta+\sigma'} E_N^{13} + e^{-2\eta-\sigma'} E_N^{31} \right] \right\} \\ &\quad + \frac{2e^{-\epsilon'} (2 \sinh \eta \sinh 4\eta - \cosh 5\eta)}{2 \cosh 2\eta - 4e^{-\epsilon'} \sinh \eta \cosh 4\eta + 1 + 2e^{-\epsilon'} \sinh 5\eta} , \end{aligned} \quad (2.23)$$

where $E_j^{\mu\nu}$ is the Weyl matrix or the Hubbard operator

$$E^{\mu\nu} = |\mu\rangle\langle\nu| ,$$

and

$$R'_{ij}(u) = \frac{\partial}{\partial u} R'_{ij}(u), \quad K^{\pm'}_j(u) = \frac{\partial}{\partial u} K^{\pm}_j(u). \quad (2.24)$$

3 Operator identities of the transfer matrix

Following [21] we apply the fusion technique to study the the present model. In this case, we need to use the fusion techniques both for R -matrices [74–78] and for K -matrices [66, 79]. Similar as that in [21], the fusion procedure will lead to the desired operator identities to determine the spectrum of the transfer matrix $t(u)$ given by (2.20). For this purpose, let us introduce the following vectors in the tensor space $\mathbf{V} \otimes \mathbf{V}$

$$|\Phi_0\rangle = \frac{1}{\sqrt{2 \cosh 2\eta + 1}}(e^{-\eta}|1, 3\rangle - |2, 2\rangle + e^{\eta}|3, 1\rangle), \quad (3.1)$$

$$|\Phi_1\rangle = \frac{1}{\sqrt{2 \cosh 2\eta}}(e^{-\eta}|1, 2\rangle - e^{\eta}|2, 1\rangle), \quad (3.2)$$

$$|\Phi_2\rangle = \frac{1}{\sqrt{2 \cosh 2\eta}}(|1, 3\rangle - 2 \sinh \eta |2, 2\rangle - |3, 1\rangle), \quad (3.3)$$

$$|\Phi_3\rangle = \frac{1}{\sqrt{2 \cosh 2\eta}}(e^{-\eta}|2, 3\rangle - e^{\eta}|3, 2\rangle), \quad (3.4)$$

and the associated projectors⁴

$$P_{12}^{(1)} = |\Phi_0\rangle\langle\Phi_0|, \quad P_{12}^{(3)} = \sum_{i=1}^3 |\Phi_i\rangle\langle\Phi_i|. \quad (3.5)$$

Direct calculation shows that the R -matrix given by (2.1) at some degenerate points is proportional to the projectors,

$$R_{12}(6\eta + i\pi) = P_{12}^{(1)} \times S_{12}^{(1)}, \quad R_{12}(4\eta) = P_{12}^{(3)} \times S_{12}^{(3)}, \quad (3.6)$$

where $S_{12}^{(i)}$ are some non-degenerate matrices $\in \text{End}(\mathbf{V} \otimes \mathbf{V})$. After some calculations, we find

$$P_{12}^{(1)} R_{23}(u) R_{13}(u + 6\eta + i\pi) P_{12}^{(1)} = \rho_1(u) P_{12}^{(1)} \otimes \text{id} = \text{Det}_q(R(u)) P_{12}^{(1)} \otimes \text{id}, \quad (3.7)$$

$$P_{12}^{(1)} R_{31}(u) R_{32}(u + 6\eta + i\pi) P_{12}^{(1)} = \rho_1(u) P_{12}^{(1)} \otimes \text{id}, \quad (3.8)$$

$$P_{12}^{(3)} R_{23}(u) R_{13}(u + 4\eta) P_{12}^{(3)} = \rho_3(u) R_{13}(u + 2\eta + i\pi), \quad (3.9)$$

$$P_{12}^{(3)} R_{31}(u) R_{32}(u + 4\eta) P_{12}^{(3)} = \rho_3(u) R_{31}(u + 2\eta + i\pi), \quad (3.10)$$

where the function $\rho_3(u)$ is

$$\rho_3(u) = -2 \sinh\left(\frac{u}{2} + 2\eta\right) \cosh\left(\frac{u}{2} - 3\eta\right). \quad (3.11)$$

Noting that the rank of the projector $P_{12}^{(1)}$ is one, the quantum determinant [80] $\text{Det}_q(R(u))$ is defined by

$$\text{Det}_q(R(u)) = \text{tr}_{12} \left\{ P_{12}^{(1)} R_{23}(u) R_{13}(u + 6\eta + i\pi) P_{12}^{(1)} \right\}, \quad (3.12)$$

⁴In contrast to most of rational models, here $P_{12}^{(i)} \neq P_{21}^{(i)}$.

which plays the role of the generating function of the centers of the associated quantum algebras [81]. In our particular case the corresponding quantum determinant is proportional to the identity operator,

$$\text{Det}_q(R(u)) = \rho_1(u) \text{id}. \quad (3.13)$$

The above relations (3.7)–(3.13) imply that the associated one-row monodromy matrices satisfy the following relations

$$P_{12}^{(1)} T_2(u) T_1(u + 6\eta + i\pi) P_{12}^{(1)} = \prod_{l=1}^N \rho_1(u - \theta_l) P_{12}^{(1)} \otimes \text{id}, \quad (3.14)$$

$$P_{12}^{(1)} \hat{T}_1(u) \hat{T}_2(u + 6\eta + i\pi) P_{12}^{(1)} = \prod_{l=1}^N \rho_1(u + \theta_l) P_{12}^{(1)} \otimes \text{id}, \quad (3.15)$$

$$P_{12}^{(3)} T_2(u) T_1(u + 4\eta) P_{12}^{(1)} = \prod_{l=1}^N \rho_3(u - \theta_l) T_1(u + 2\eta + i\pi), \quad (3.16)$$

$$P_{12}^{(3)} \hat{T}_1(u) \hat{T}_2(u + 4\eta) P_{12}^{(1)} = \prod_{l=1}^N \rho_3(u + \theta_l) \hat{T}_1(u + 2\eta + i\pi), \quad (3.17)$$

where $\{\theta_l | l = 1, \dots, N\}$ are the inhomogeneous parameters defined in (2.15)–(2.16).

Now let us construct the corresponding fusion procedures for the K -matrices following [66, 79]. The properties (3.6) of the R -matrix at the degenerate points and the RE (2.17) and its dual RE (2.18) allow us to construct the fused K -matrices from the origin ones (2.21) and (2.22). After some tedious calculations, we find that

$$P_{21}^{(1)} K_1^-(u) R_{21}(2u + 6\eta + i\pi) K_2^-(u + 6\eta + i\pi) P_{12}^{(1)} = \text{Det}_q(K^-(u)) P_{21}^{(1)}, \quad (3.18)$$

$$P_{12}^{(1)} K_2^+(u + 6\eta + i\pi) \mathcal{M}_1 R_{12}(-2u + 6\eta + i\pi) \mathcal{M}_1^{-1} K_1^+(u) P_{21}^{(1)} = \text{Det}_q(K^+(u)) P_{12}^{(1)}, \quad (3.19)$$

$$P_{21}^{(3)} K_1^-(u) R_{21}(2u + 4\eta) K_2^-(u + 4\eta) P_{12}^{(3)} = f_-(u) K_1^-(u + 2\eta + i\pi), \quad (3.20)$$

$$P_{12}^{(3)} K_2^+(u + 4\eta) \mathcal{M}_1 R_{12}(-2u + 8\eta) \mathcal{M}_1^{-1} K_1^+(u) P_{21}^{(3)} = f_+(u) K_1^+(u + 2\eta + i\pi), \quad (3.21)$$

where the functions $f_{\pm}(u)$ and $\text{Det}_q(K^{\pm}(u))$ are

$$f_-(u) = -2(1 - 2e^{-\epsilon} \sinh(u - \eta)) \cosh(u - \eta) \sinh(u + 4\eta), \quad (3.22)$$

$$f_+(u) = 2(1 - 2e^{-\epsilon'} \sinh(u - \eta)) \cosh(u - \eta) \sinh(u - 6\eta), \quad (3.23)$$

$$\begin{aligned} \text{Det}_q(K^-(u)) &= \text{tr}_{12} \left\{ P_{21}^{(1)} K_1^-(u) R_{21}(2u + 6\eta + i\pi) K_2^-(u + 6\eta + i\pi) P_{12}^{(1)} \right\} \\ &= -2(1 - 2e^{-\epsilon} \sinh(u - \eta)) (1 + e^{-\epsilon} \sinh(u + \eta)) \\ &\quad \times \sinh(u + 6\eta) \cosh(u + \eta), \end{aligned} \quad (3.24)$$

$$\begin{aligned} \text{Det}_q(K^+(u)) &= \text{tr}_{12} \left\{ P_{12}^{(1)} K_2^+(u + 6\eta + i\pi) \mathcal{M}_1 R_{12}(-2u + 6\eta + i\pi) \mathcal{M}_1^{-1} K_1^+(u) P_{21}^{(1)} \right\} \\ &= 2(1 - 2e^{-\epsilon'} \sinh(u - \eta)) (1 + e^{-\epsilon'} \sinh(u + \eta)) \\ &\quad \times \sinh(u - 6\eta) \cosh(u - \eta). \end{aligned} \quad (3.25)$$

Following the method in [21] and using the relations (3.14)–(3.21), after a long calculation, we find the transfer matrix given by (2.20) satisfies the following operator identities

$$t(\pm\theta_j)t(\pm\theta_j + 6\eta + i\pi) = \frac{\delta_1(u) \times \text{id}}{\rho_1(2u)} \Big|_{u=\pm\theta_j}, \quad j = 1, \dots, N, \quad (3.26)$$

$$t(\pm\theta_j)t(\pm\theta_j + 4\eta) = \frac{\delta_2(u) \times t(u + 2\eta + i\pi)}{\rho_2(-2u + 8\eta)} \Big|_{u=\pm\theta_j}, \quad j = 1, \dots, N. \quad (3.27)$$

Here the functions $\delta_1(u)$ and $\delta_2(u)$ are

$$\delta_1(u) = \text{Det}_q(K^-(u))\text{Det}_q(K^+(u)) \prod_{l=1}^N \rho_1(u - \theta_l)\rho_1(u + \theta_l), \quad (3.28)$$

$$\delta_2(u) = f_-(u)f_+(u) \prod_{l=1}^N \rho_3(u - \theta_l)\rho_3(u + \theta_l). \quad (3.29)$$

Following the method in [82] and using the the explicit expressions (2.21) and (2.22) of the K -matrices, we find that the transfer matrix (2.20) possesses the crossing symmetry

$$t(u) = t(-u + 6\eta + i\pi). \quad (3.30)$$

The proof of the above relation is given in appendix A. We remark that the very operator identities (3.26)–(3.27) at the points $-\theta_j$ is guaranteed by their equalities at the points θ_j and the crossing symmetry (3.30) of the transfer matrix. Therefore, in the following we will use the operator identities (3.26)–(3.27) at the points θ_j and the crossing symmetry (3.30) to determine the eigenvalues of the transfer matrix.

Now let us derive some properties of the transfer matrix. Firstly the expressions of the K -matrices $K^\pm(u)$ given by (2.21) and (2.22) imply that $K^\pm(u)$ have the following periodicity

$$K^\pm(u + 2i\pi) = K^\pm(u).$$

This leads to the periodicity of the transfer matrix $t(u)$

$$t(u + 2i\pi) = t(u). \quad (3.31)$$

Moreover the explicit expressions of the R -matrix and the K -matrices allow us to have the following asymptotic behaviors of the transfer matrix

$$\lim_{u \rightarrow \pm\infty} t(u) = \left(\frac{1}{2}\right)^{2N} e^{-\epsilon - \epsilon'} e^{\pm 2(N+1)(u-3\eta)} (1 + 2 \cosh(\sigma' - \sigma + 2\eta)) \times \text{id} + \dots \quad (3.32)$$

The explicit expressions of the K -matrices given by (2.21) and (2.22) also imply that $K^\pm(u)$ satisfy the following properties

$$K^-(0) = (1 + 2e^{-\epsilon} \sinh \eta), \quad K^+(6\eta + i\pi) = (1 + 2e^{-\epsilon'} \sinh \eta) \mathcal{M}, \quad (3.33)$$

$$K^-(i\pi) = (1 - 2e^{-\epsilon} \sinh \eta), \quad K^+(6\eta) = (1 - 2e^{-\epsilon'} \sinh \eta) \mathcal{M}. \quad (3.34)$$

The unitarity (2.5) and the crossing unitarity (2.11) of the R -matrix imply that the one-row monodromy matrices $T(u)$ given by (2.15) and $\hat{T}(u)$ given by (2.16) satisfy the following relations

$$T_0(u)\hat{T}_0(-u) = \prod_{l=1}^N \rho_1(u - \theta_l) \times \text{id}, \quad (3.35)$$

$$T_0^{t_0}(u)\mathcal{M}_0\hat{T}_0^{t_0}(-u + 12\eta)\mathcal{M}_0^{-1} = \prod_{l=1}^N \rho_2(u - \theta_l) \times \text{id}. \quad (3.36)$$

The above relations allow us to work out the transfer matrix at special points $0, i\pi, 6\eta, 6\eta + i\pi$ as follows

$$t(0) = t(6\eta + i\pi) = (1 + 2e^{-\epsilon} \sinh \eta) \text{tr} \{K^+(0)\} \prod_{l=1}^N \rho_1(-\theta_l) \times \text{id}, \quad (3.37)$$

$$t(i\pi) = t(6\eta) = (1 - 2e^{-\epsilon} \sinh \eta) \text{tr} \{K^+(i\pi)\} \prod_{l=1}^N \rho_1(i\pi - \theta_l) \times \text{id}. \quad (3.38)$$

4 Functional relations and the T-Q relation

The property $[t(u), t(v)] = 0$ implies that the eigenstates of $t(u)$ are independent of u . Suppose $|\Psi\rangle$ is an eigenstate of $t(u)$ with an eigenvalue $\Lambda(u)$, namely,

$$t(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle.$$

The very operator identities (3.26)–(3.27) of the transfer matrix at the points θ_j imply the corresponding eigenvalue $\Lambda(u)$ satisfies the similar relations

$$\Lambda(\theta_j)\Lambda(\theta_j + 6\eta + i\pi) = \frac{\delta_1(u)}{\rho_1(2u)} \Big|_{u=\theta_j}, \quad j = 1, \dots, N, \quad (4.1)$$

$$\Lambda(\theta_j)\Lambda(\theta_j + 4\eta) = \frac{\delta_2(u) \times \Lambda(u + 2\eta + i\pi)}{\rho_2(-2u + 8\eta)} \Big|_{u=\theta_j}, \quad j = 1, \dots, N, \quad (4.2)$$

where the functions $\delta_i(u)$ are given by (3.28) and (3.29). The crossing symmetry (3.30) of the transfer matrix leads to the crossing symmetry of its eigenvalues

$$\Lambda(u) = \Lambda(-u + 6\eta + i\pi). \quad (4.3)$$

The properties of the transfer matrix $t(u)$ given by (3.31), (3.32) and (3.37)–(3.38) imply that the corresponding eigenvalue $\Lambda(u)$ satisfies the following relations:

$$\Lambda(u + 2i\pi) = \Lambda(u), \quad (4.4)$$

$$\Lambda(0) = (1 + 2e^{-\epsilon} \sinh \eta) \text{tr} \{K^+(0)\} \prod_{l=1}^N \rho_1(-\theta_l), \quad (4.5)$$

$$\Lambda(i\pi) = (1 - 2e^{-\epsilon} \sinh \eta) \text{tr} \{K^+(i\pi)\} \prod_{l=1}^N \rho_1(i\pi - \theta_l), \quad (4.6)$$

$$\lim_{u \rightarrow \pm\infty} \Lambda(u) = \left(\frac{1}{2}\right)^{2N} e^{-\epsilon - \epsilon'} e^{\pm 2(N+1)(u-3\eta)} (1 + 2 \cosh(\sigma' - \sigma + 2\eta)) + \dots \quad (4.7)$$

The periodicity (4.4) and the asymptotic behavior (4.7) of the eigenvalue $\Lambda(u)$, the analyticity of the R-matrix and K-matrices and the u -free eigenstate lead to the fact that the eigenvalue $\Lambda(u)$ further possesses the property

$$\Lambda(u), \text{ as an entire function of } u, \text{ is a trigonometric polynomial of degree } 2N + 2. \quad (4.8)$$

Namely, $\Lambda(u)$ is a Laurent polynomial of e^u with $4(N + 1) + 1$ unknown coefficients. The crossing relation (4.3) reduces the number of the independent unknown coefficients to $2(N + 1) + 1$. Thus one needs $2(N + 1) + 1$ conditions to determine these coefficients. Therefore the relations (4.1)–(4.8) completely characterize the spectrum of the transfer matrix of the model.

4.1 T-Q ansatz

Based on the properties (4.1)–(4.8) of the eigenvalues $\Lambda(u)$, following the ODBA method developed in [21, 43–46], let us propose the following conjecture for the eigenvalues,

$$\begin{aligned} \Lambda(u) = & \prod_{l=1}^N c(u - \theta_l) c(u + \theta_l) (1 - 2e^{-\epsilon} \sinh(u - \eta)) \left(1 - 2e^{-\epsilon'} \sinh(u - \eta) \right) \\ & \times \frac{\sinh(u - 6\eta) \cosh(u - \eta)}{\sinh(u - 2\eta) \cosh(u - 3\eta)} \frac{Q_1(u + 4\eta)}{Q_2(u)} \\ & + \prod_{l=1}^N d(u - \theta_l) d(u + \theta_l) (1 - 2e^{-\epsilon} \sinh(u - 5\eta)) \left(1 - 2e^{-\epsilon'} \sinh(u - 5\eta) \right) \\ & \times \frac{\sinh u \cosh(u - 5\eta)}{\sinh(u - 4\eta) \cosh(u - 3\eta)} \frac{Q_2(u - 6\eta + i\pi)}{Q_1(u - 2\eta + i\pi)} \\ & + \prod_{l=1}^N b(u - \theta_l) b(u + \theta_l) (1 + 2e^{-\epsilon} \sinh(u - 3\eta)) \left(1 + 2e^{-\epsilon'} \sinh(u - 3\eta) \right) \\ & \times \frac{\sinh u \sinh(u - 6\eta)}{\sinh(u - 2\eta) \sinh(u - 4\eta)} \frac{Q_1(u + 2\eta + i\pi) Q_2(u - 4\eta)}{Q_2(u - 2\eta + i\pi) Q_1(u)} \\ & + 4^{1-N} c \frac{\sinh u \sinh(u - 6\eta)}{\cosh(u - 3\eta)} \prod_{l=1}^N c(u - \theta_l) c(u + \theta_l) d(u - \theta_l) d(u + \theta_l) \\ & \times \left[\frac{Q_1(u + 2\eta + i\pi) \sinh^{l_1}(u - 3\eta) \sinh^{l_1}(u - \eta) \cosh^{l_2}(u - 2\eta)}{Q_1(u) Q_2(u)} \right. \\ & \left. - (-1)^{l_2} \frac{Q_2(u - 4\eta) \sinh^{l_1}(u - 3\eta) \sinh^{l_1}(u - 5\eta) \cosh^{l_2}(u - 4\eta)}{Q_1(u - 2\eta + i\pi) Q_2(u - 2\eta + i\pi)} \right], \quad (4.9) \end{aligned}$$

where the functions $c(u)$, $b(u)$ and $d(u)$ are the elements of the R -matrix given by (2.2) and the functions $Q_i(u)$ and the constant c are given by

$$Q_1(u) = \prod_{k=1}^{\bar{N}} \sinh \left[\frac{u - \lambda_k}{2} - \eta \right], \quad \bar{N} = 4(N + l_1) + 2(l_2 - 1), \quad (4.10)$$

$$Q_2(u) = \prod_{k=1}^{\bar{N}} \sinh \left[\frac{u + \lambda_k}{2} - \eta \right], \quad (4.11)$$

$$c = (-1)^{(l_2-1)} e^{-\epsilon-\epsilon'} \left\{ \frac{\cosh(\sigma' - \sigma + 2\eta) - \cosh\left(\delta_{l_1, l_2} \eta - \sum_{j=1}^{\bar{N}} \lambda_j\right)}{\cosh\left(\frac{\delta_{l_1, l_2} \eta}{2} - \frac{1}{2} \sum_{j=1}^{\bar{N}} \lambda_j\right)} \right\}, \quad (4.12)$$

$$\delta_{l_1, l_2} = 4N + 8l_1 + 4l_2 - 2. \quad (4.13)$$

Here l_1 and l_2 are arbitrary non-negative integers.⁵ It is easy to check that the functions $Q_i(u)$ possess the following properties

$$Q_i(u + 2i\pi) = Q_i(u), \quad \text{for } i = 1, 2, \quad \text{and} \quad Q_2(u) = Q_1(-u + 4\eta). \quad (4.14)$$

With the help of the above relations, we have checked that the $T - Q$ ansatz (4.9) indeed satisfies the relations (4.3)–(4.6). Moreover the special choice of the constant c given by (4.12) implies that the ansatz also satisfies the asymptotic behavior (4.7). The explicit expressions of the functions $b(u)$, $c(u)$ and $d(u)$ given by (2.1)–(2.2) imply that

$$b(0) = d(0) = c(6\eta + i\pi) = b(6\eta + i\pi) = c(4\eta) = d(2\eta + i\pi) = 0. \quad (4.15)$$

These properties give rise to the fact that the $T - Q$ ansatz (4.9) satisfies the very functional relations (4.1)–(4.2). This means that our ansatz (4.9) actually satisfies the relations (4.1)–(4.7) except the analytic properties (4.8).

From the explicit expression (4.9) of $\Lambda(u)$, one may find that the $T - Q$ ansatz (4.9) might have some apparent simple poles at the following points:

$$2\eta, 4\eta, 3\eta + i\frac{\pi}{2}, \quad \text{mod}(i\pi), \quad (4.16)$$

and

$$\lambda_j + 2\eta, -\lambda_j + 2\eta, \lambda_j + 4\eta + i\pi, -\lambda_j + 4\eta + i\pi, \quad \text{mod}(2i\pi), \quad j = 1, \dots, \bar{N}. \quad (4.17)$$

Direct calculation shows that the residues of the $T - Q$ ansatz (4.9) at the points given by (4.16) vanishes, or the ansatz has no singularity at these points. Moreover we have checked that the $T - Q$ ansatz (4.9) also has no singularity at the points given by (4.17)

⁵The results of anisotropic spin- $\frac{1}{2}$ chains [43–46] strongly suggest that fixed l_1 and l_2 might give a complete set of eigenvalues of the transfer matrix. In such a sense, different l_1 and l_2 might only give different parameterizations of the eigenvalues but not different states.

provided that the \bar{N} parameters $\{\lambda_j|j = 1, \dots, \bar{N}\}$ satisfy the following BAEs

$$\begin{aligned} & \frac{(1 + 2e^{-\epsilon} \sinh(\lambda_j - \eta)) \left(1 + 2e^{-\epsilon'} \sinh(\lambda_j - \eta)\right) \cosh(\lambda_j - \eta) Q_2(\lambda_j - 2\eta) Q_2(\lambda_j + 2\eta)}{4 \sinh \lambda_j \sinh(\lambda_j - 2\eta) \sinh^{l_1}(\lambda_j + \eta) \sinh^{l_1}(\lambda_j - \eta) \cosh^{l_2} \lambda_j} \\ &= -c Q_2(\lambda_j + i\pi) \prod_{l=1}^N \sinh \left[\frac{\lambda_j - \theta_l}{2} - \eta \right] \sinh \left[\frac{\lambda_j + \theta_l}{2} - \eta \right] \cosh \left[\frac{\lambda_j - \theta_l}{2} \right] \cosh \left[\frac{\lambda_j + \theta_l}{2} \right], \\ & j = 1, \dots, \bar{N}, \end{aligned} \tag{4.18}$$

where the function $Q_2(u)$ is given by (4.11). The non-singular property of the $T - Q$ ansatz (4.9) at the points (4.17) can be verified by directly calculating the residues of the ansatz at these points.

Finally we conclude that the $T - Q$ ansatz (4.9) indeed satisfies (4.1)–(4.8) as it is required, if the the \bar{N} parameters $\{\lambda_j|j = 1, \dots, \bar{N}\}$ satisfy the associated BAEs (4.18). Thus the $\Lambda(u)$ given by (4.9), when the \bar{N} parameters $\{\lambda_j|j = 1, \dots, \bar{N}\}$ satisfy the associated BAEs (4.18), becomes the eigenvalue of the transfer matrix $t(u)$ given by (2.20) for the Izergin-Korepin model with the general non-diagonal boundary terms specified by the non-diagonal K -matrices $K^\pm(u)$ given by (2.21) and (2.22).

4.2 Reduction to the conventional $T - Q$ ansatz

It follows from (4.12) that the parameter c does depend on not only the boundary parameters but also the parameters $\{\lambda_j\}$ (such a dependence also appeared in the anisotropic spin- $\frac{1}{2}$ chains with arbitrary boundary fields [43–46]). The vanishing condition of c , i.e. $c = 0$, will lead to the constraint (see (4.23) below) among the boundary parameters, in this case one might find a proper “local vacuum” to apply the conventional Bethe ansatz [15, 83]. The Bethe ansatz equations (4.18) imply that for this case the parameters $\{\lambda_j\}$ have to form two types of pairs:

$$(\lambda_j, -\lambda_j), \quad (\lambda_j, -\lambda_j + 4\eta). \tag{4.19}$$

Suppose the number of the first type pairs is M (M being a non-negative integer such that $0 \leq M \leq \frac{\bar{N}}{2}$), the resulting $T - Q$ relation (4.9) becomes the conventional one [1]

$$\begin{aligned} \Lambda(u) &= \prod_{l=1}^N c(u - \theta_l) c(u + \theta_l) (1 - 2e^{-\epsilon} \sinh(u - \eta)) \left(1 - 2e^{-\epsilon'} \sinh(u - \eta)\right) \\ &\quad \times \frac{\sinh(u - 6\eta) \cosh(u - \eta)}{\sinh(u - 2\eta) \cosh(u - 3\eta)} \frac{Q(u + 4\eta)}{Q(u)} \\ &+ \prod_{l=1}^N d(u - \theta_l) d(u + \theta_l) (1 - 2e^{-\epsilon} \sinh(u - 5\eta)) \left(1 - 2e^{-\epsilon'} \sinh(u - 5\eta)\right) \\ &\quad \times \frac{\sinh u \cosh(u - 5\eta)}{\sinh(u - 4\eta) \cosh(u - 3\eta)} \frac{Q(u - 6\eta - i\pi)}{Q(u - 2\eta + i\pi)} \\ &+ \prod_{l=1}^N b(u - \theta_l) b(u + \theta_l) (1 + 2e^{-\epsilon} \sinh(u - 3\eta)) \left(1 + 2e^{-\epsilon'} \sinh(u - 3\eta)\right) \\ &\quad \times \frac{\sinh u \sinh(u - 6\eta)}{\sinh(u - 2\eta) \sinh(u - 4\eta)} \frac{Q(u + 2\eta + i\pi) Q(u - 4\eta)}{Q(u - 2\eta + i\pi) Q(u)}, \end{aligned} \tag{4.20}$$

where the resulting function $Q(u)$ is

$$Q(u) = \prod_{j=1}^M \sinh \left[\frac{u - \lambda_j}{2} - \eta \right] \sinh \left[\frac{u + \lambda_j}{2} - \eta \right]. \quad (4.21)$$

The resulting BAEs become

$$\begin{aligned} & \prod_{l=1}^N \frac{\sinh \left[\frac{\lambda_j - \theta_l}{2} - \eta \right] \sinh \left[\frac{\lambda_j + \theta_l}{2} - \eta \right] (1 - 2e^{-\epsilon} \sinh(\lambda_j + \eta)) (1 - 2e^{-\epsilon'} \sinh(\lambda_j + \eta))}{\sinh \left[\frac{\lambda_j - \theta_l}{2} + \eta \right] \sinh \left[\frac{\lambda_j + \theta_l}{2} + \eta \right] (1 + 2e^{-\epsilon} \sinh(\lambda_j - \eta)) (1 + 2e^{-\epsilon'} \sinh(\lambda_j - \eta))} \\ &= - \frac{\sinh(\lambda_j + 2\eta) \cosh(\lambda_j - \eta) Q(-\lambda_j - 2\eta) Q(-\lambda_j + 4\eta + i\pi)}{\sinh(\lambda_j - 2\eta) \cosh(\lambda_j + \eta) Q(\lambda_j - 2\eta) Q(\lambda_j + 4\eta + i\pi)}, \quad j = 1, \dots, M. \end{aligned} \quad (4.22)$$

On the other hand, the form of the pairs (4.19) implies that

$$\sum_{j=1}^{\bar{N}} \lambda_j = \sum_{j=1}^{\frac{\bar{N}-2M}{2}} (\lambda_j - \lambda_j + 4\eta) = (2\bar{N} - 4M) \eta.$$

The relation (4.12) and the definition (4.10) of \bar{N} give rise to the following constraint between the boundary parameters⁶

$$\cosh(\sigma' - \sigma + 2\eta) = \cosh(4M\eta - 4N\eta + 2\eta), \quad 0 \leq M. \quad (4.23)$$

4.2.1 For a generic η

For a generic η , there exist three solutions to the constraint (4.23). If the boundary parameters σ and σ' satisfy the constraint

$$\sigma' - \sigma = -4k\eta \pmod{2i\pi}, \quad k \leq -N, \quad \text{and } k \in \mathbb{Z}, \quad (4.24)$$

the corresponding M can take only one allowed value,

$$M = N - k. \quad (4.25)$$

If the boundary parameters σ and σ' satisfy the following constraint

$$\sigma' - \sigma = -4k\eta \pmod{2i\pi}, \quad 1 - N \leq k \leq N, \quad \text{and } k \in \mathbb{Z}, \quad (4.26)$$

then M can take the two allowed values denoted by M^\pm respectively,

$$M^- = N - k, \quad M^+ = N + k - 1. \quad (4.27)$$

If the boundary parameters σ and σ' satisfy

$$\sigma' - \sigma = -4k\eta \pmod{2i\pi}, \quad N + 1 \leq k, \quad \text{and } k \in \mathbb{Z}, \quad (4.28)$$

⁶One can always choose the non-negative integers l_1 and l_2 such that M becomes any non-negative integer since that $0 \leq M \leq 2(N + l_1) + l_2 - 1 = \bar{N}/2$.

M can take only one allowed value,

$$M = N + k - 1. \quad (4.29)$$

For the case that the boundary parameters satisfy the relation (4.24) (or (4.28)), the eigenvalue of the transfer matrix is characterized by a fixed M in (4.25) (or (4.29)). This $\Lambda(u)$ itself might give the complete set of the eigenvalues of the transfer matrix. However, if the boundary parameters obey the relation (4.26), there exist two $\Lambda^\pm(u)$ corresponding to the two different allowed M in (4.27). Similar as that of the anisotropic spin- $\frac{1}{2}$ chain with arbitrary boundary fields [15, 37, 84, 85], these two $\Lambda^\pm(u)$ together might constitute the complete set of the eigenvalues of the transfer matrix.

4.2.2 For some degenerate η

Similar as that in the closed XYZ chain and the anisotropic spin- $\frac{1}{2}$ chain with arbitrary boundary fields [43–46], if the isotropic (or crossing) parameter η takes the following discrete value

$$\eta = \frac{\sigma - \sigma'}{4N - 4M} + \frac{2i\pi m}{4N - 4M}, \quad m, M \in Z, \quad \text{and } 0 \leq M, \quad (4.30)$$

the constraint (4.23) is automatically satisfied for arbitrary boundary parameters $\epsilon, \epsilon', \sigma$ and σ' . Thus in this case the eigenvalue of the transfer matrix is given by (4.20) and the corresponding BAEs are given by (4.22). It should be emphasized that these degenerate points (4.30) become dense in the thermodynamic limit ($N, m, M \rightarrow \infty$). This enables one to obtain the thermodynamic properties (up to the order of $O(N^{-2})$) of the model for generic values of η via the conventional thermodynamic Bethe ansatz methods [86, 87]. This method has been proven to be very successful in the derivation of the surface energy of the XXZ spin chain with arbitrary boundary fields [88].

5 Reduction to the case with diagonal K -matrices

Now let us consider the diagonal K -matrices (i.e., taking the limits $\epsilon, \epsilon' \rightarrow +\infty$ of (2.21) and (2.22)). The resulting K -matrices read

$$K^-(u) = \text{id}, \quad K^+(u) = \mathcal{M}, \quad (5.1)$$

where the matrix \mathcal{M} is given by (2.12). The $T - Q$ ansatz (4.20) is reduced to the one [66] obtained by analytic Bethe ansatz method

$$\begin{aligned} \Lambda(u) = & \prod_{l=1}^N c(u - \theta_l) c(u + \theta_l) \frac{\sinh(u - 6\eta) \cosh(u - \eta)}{\sinh(u - 2\eta) \cosh(u - 3\eta)} \frac{Q(u + 4\eta)}{Q(u)} \\ & + \prod_{l=1}^N d(u - \theta_l) d(u + \theta_l) \frac{\sinh u \cosh(u - 5\eta)}{\sinh(u - 4\eta) \cosh(u - 3\eta)} \frac{Q(u - 6\eta + i\pi)}{Q(u - 2\eta + i\pi)} \\ & + \prod_{l=1}^N b(u - \theta_l) b(u + \theta_l) \frac{\sinh u \sinh(u - 6\eta)}{\sinh(u - 2\eta) \sinh(u - 4\eta)} \\ & \times \frac{Q(u - 4\eta) Q(u + 2\eta + i\pi)}{Q(u - 2\eta + i\pi) Q(u)}, \end{aligned} \quad (5.2)$$

where the function $Q(u)$ is still given by (4.21). In particular, in this case the vanishing condition of c is automatically satisfied thanks to the relation (4.12). In this case the $U(1)$ -symmetry is recovered and M can take any of the following values,⁷

$$M = 0, 1, \dots, 2N. \tag{5.3}$$

For each of the above allowed values, the resulting BAEs become

$$\begin{aligned} \prod_{l=1}^N \frac{\sinh\left[\frac{\lambda_j - \theta_l}{2} - \eta\right] \sinh\left[\frac{\lambda_j + \theta_l}{2} - \eta\right] \sinh(\lambda_j - 2\eta) \cosh(\lambda_j + \eta)}{\sinh\left[\frac{\lambda_j - \theta_l}{2} + \eta\right] \sinh\left[\frac{\lambda_j + \theta_l}{2} + \eta\right] \sinh(\lambda_j + 2\eta) \cosh(\lambda_j - \eta)} \\ = -\frac{Q(-\lambda_j - 2\eta)Q(-\lambda_j + 4\eta + i\pi)}{Q(\lambda_j - 2\eta)Q(\lambda_j + 4\eta + i\pi)}, \quad j = 1, \dots, M. \end{aligned} \tag{5.4}$$

6 Conclusions

The Izergin-Korepin model with general non-diagonal boundary terms specified by the most general non-diagonal K matrices given by (2.21) and (2.22) has been studied by the off-diagonal Bethe ansatz method. Based on some intrinsic properties of the R -matrix and K -matrices, we derive the very functional relations (3.26) and (3.27) of the transfer matrix. These relations, together with other properties, allow us to construct an off-diagonal (or inhomogeneous) $T - Q$ relation (4.9) of the eigenvalue of the transfer matrix and the associated BAEs (4.18). When the boundary parameters satisfy one constraint (4.23), the resulting $T - Q$ relation is reduced to the conventional one (4.20), which might allow one to use the method developed in [88] to study the thermodynamic properties (up to the order of $O(N^{-2})$) of the model for generic values of η via the conventional thermodynamic Bethe ansatz methods [86, 87]. Taking the limit $\epsilon, \epsilon' \rightarrow +\infty$, the corresponding K -matrices become diagonal ones and the resulting $T - Q$ relation is then reduced to that in [66].

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⁷The allowed values (5.3) of M can be deduced by studying the asymptotic behavior of the eigenvalues of transfer matrix [66].

A Proof of the crossing symmetry

Direct calculation shows that the K -matrices given by (2.21)–(2.22) satisfy the following relations

$$K^-(u)K^-(-u) = \Delta_-(u) \times \text{id}, \quad (\text{A.1})$$

$$V^t K^+(-u + 6\eta + i\pi) V V^t K^+(u + 6\eta + i\pi) V = \Delta_+(u) \times \text{id}, \quad (\text{A.2})$$

where the matrix V is given by (2.10) and the functions $\Delta_{\pm}(u)$ are

$$\Delta_-(u) = (1 - 2e^{-\epsilon} \sinh(u - \eta)) (1 + 2e^{-\epsilon} \sinh(u + \eta)), \quad (\text{A.3})$$

$$\Delta_+(u) = (1 - 2e^{-\epsilon'} \sinh(u - \eta)) (1 + 2e^{-\epsilon'} \sinh(u + \eta)). \quad (\text{A.4})$$

Similar as those in [82], let us introduce the following matrices $\bar{K}^{\pm}(u)$

$$\bar{K}_1^-(u) = \text{tr}_2 \left\{ P_{12} R_{21}(-2u) V_2 K_2^{-t_2}(u + 6\eta + i\pi) V_2^{t_2} \right\}, \quad (\text{A.5})$$

$$\bar{K}_1^+(u + 6\eta + i\pi) = \text{tr}_2 \left\{ P_{12} R_{12}(2u) K_2^{+t_2}(u) \right\}. \quad (\text{A.6})$$

Substituting the expressions (2.21)–(2.22) of the K -matrices into the above relations, we have that the K -matrices satisfy the following crossing relation

$$\bar{K}^-(u) = \frac{\text{Det}_q(K^-(u))}{\Delta_-(u)} K^-(u), \quad (\text{A.7})$$

$$\bar{K}^+(u + 6\eta + i\pi) = \frac{\text{Det}_q(K^+(u))}{\Delta_+(u)} V^t K^+(-u + 6\eta + i\pi) V, \quad (\text{A.8})$$

where the functions $\text{Det}(K^-(u))$ and $\text{Det}(K^+(u))$ are given respectively by (3.24) and (3.25). Combining the crossing relation (2.6) and PT-symmetry (2.7) of the R -matrix, one may derive the following relations

$$R_{21}(u) = V_1^{t_1} R_{12}^{t_1}(-u + 6\eta + i\pi) V_1^{t_1}, \quad (\text{A.9})$$

$$R_{21}(u) = V_2 R_{12}^{t_2}(-u + 6\eta + i\pi) V_2. \quad (\text{A.10})$$

Then the above relations, (2.6) and the expression (2.10) give rise to the following relation between the one-row monodromy matrices $T(u)$ and $\hat{T}(u)$ give by (2.15) and (2.16)

$$T_0^{t_0}(-u + 6\eta + i\pi) = V_0^{t_0} \hat{T}_0(u) V_0^{t_0}, \quad (\text{A.11})$$

$$\hat{T}_0^{t_0}(-u + 6\eta + i\pi) = V_0 T_0(u) V_0. \quad (\text{A.12})$$

Following the method in [82] and using the QYBE (2.3), we can prove that the transfer matrix specified by the R -matrix given by (2.1) and the K -matrices given by (2.21) and (2.22) satisfies the crossing symmetry (3.30).

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