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On $4D$, $\mathcal{N}=1$ massless gauge superfields of arbitrary superhelicity[1]

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Abstract: We present an alternative method of exploring the component structure of an arbitrary super-helicity (integer $Y=s$, or half odd integer $Y=s+1/2$ for any integer s) irreducible representation of the Super-Poincaré group. We use it to derive the component action and the SUSY transformation laws. The effectiveness of this approach is based on the equations of motion and their properties, like the Bianchi identities. These equations are generated by the superspace action when it is expressed in terms of prepotentials. For that reason we reproduce the superspace action for arbitrary superhelicity, using unconstrained superfields. The appropriate, to use, superfields are dictated by the representation theory of the group and the requirement that there is a smooth limit between the massive and massless case.

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Contents

1 Introduction

Higher spin field theory has a very rich history driving the developments of modern theoretical physics and after many decades still remains a very active subject. It started with Dirac [1] trying to generalize his celebrated spin-$\frac{1}{2}$ equation. His comment in that paper "the underlying theory is of considerable interest" still resonates. After the classical work by Fierz and Pauli [2] there was an increasing number of papers formulating the theory of a massive arbitrary spin in four dimensions [3, 4] as well as developments for the massless arbitrary helicities using the 'principle' of gauge invariance [5, 6]. Since then there has been tremendous progress with generalizations of these results regarding irreducible representations of the little group in D-dimensions [7], derivations of the massive theories by means of dimensional reduction of the massless theories in $D+1$-dimensions [8], Stückelberg formulations [9], BRST [10], quantization and many other things.

The discussion of arbitrary spin gauge fields in the context of simple supersymmetry in four dimensions parallels this development of the general discussion. At the level of component fields this was initiated by Curtright [11], followed by the superfield discussion at the level of on-shell equations of motion [12], and finally followed by the off-shell discussions in the work of Kuzenko, et al. [13, 14]. These pioneering works on higher spin $4D$, $\mathcal{N}=1$ supermultiplets have also led to the creation of a growing literature [15, 16] on the subject.

A current generator of interest about higher spin theories has been created by string theory as its low-energy approximation leads to consideration of fields of unbounded spins since the spectrum of string and superstring theory includes an infinite tower of massive spin states. Therefore a limit must exist where (super)string theory is formulated as a field theory of interacting spins. That points to the interesting direction of extending all previous results to include supersymmetry. The tool to build $4D$, $\mathcal{N}=1$ manifestly SUSY invariant theories is superspace and the usage of superfields.

For the massless case such a construction exists [13, 14]. The theories presented in these works, were initially described in terms of constrained superfields. The purpose of the differential constraints is to achieve gauge invariance, however these constraints can easily be solved in terms of prepotentials. These prepotentials can play a role in the formulation of massive superspin theories and possibly in the description of interactions. In a subsequent work [17], the unconstrained prepotentials were introduced and used to show that the works of [18, 19] occur by applying a transformation to the original formulations.

In this current work we would like to show how representation theory of the Super-Poincaré group makes these prepotential variables, building blocks for massive and massless theories and then use them to reproduce the superspace actions that describe irreducible representations with arbitrary super-helicity. Then we explore the rich off-shell component structure of these theories and provide the corresponding supersymmetric transformation laws.

In the previous works, when discussion about the component field spectrum of the theories was given, it was based on θ-expansion of the superfields in the Wess-Zumino gauge. This implied that by using that ansatz for the components and the usual rules of projection, the component action and the SUSY-transformation laws can be derived.

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This process is straightforward but cumbersome. For this reason we exploit an alternative efficient way of defining components, using the superfield equations of motion. The action itself, with the help of the Bianchi identities, will guide us to efficient definitions of the components, the derivation of the component action and the SUSY-transformation laws. This approach builds naturally on [20] for the study of the component structure of super-helicity $Y = 1$ and discussions [21] on old-minimal supergravity.

However there is a key difference with both of these. The first one used the superfield strength as a guide for the definition of the components. This approach can not be generalized for the arbitrary super-helicity because of the mass dimensionality of the superfield strength is proportional to super-helicity and therefore can not appear in the action. In the second paper components were defined without finding the component action and SUSY-transformation laws. We will do both of these for the arbitrary super-helicity system.

In what follows, we focus on arbitrary super-helicity (integer and half odd integer) irreducible representation of the $4D$, $\mathcal{N} = 1$ Super-Poincaré group. The presentation is organized as follows: in section 2 we briefly review the representation theory of the little group of the $4D$, $\mathcal{N} = 1$ Super-Poincaré group, following [10]. This discussion will illuminate the proper superfields one should use in order to construct the desired representations. In section 3 we focus on the massless case and illustrate how the principle of gauge invariance emerges from the requirement to have a smooth transition between massive and massless theories. In section 4 we construct the integer superhelicity superspace action and explore the off-shell component structure of the theory. We present a self-contained method of defining the components, find the component action and give explicit expressions for the SUSY-transformation laws. In section 5 we repeat the procedure for the half odd integer superhelicity representations. In the last section 6 we present the map of free highest superhelicity irreducible representations and there is a short discussion about the clues it contains for $\mathcal{N} = 2$ theories. The main new results in this work involve the derivation of a complete component-level description that involves *no* explicit θ-expansion of superfields. The conventions used are the ones of [21].

2 Irreducible representations

As is well known the Super-Poincaré group has two Casimir operators that label the irreducible representations. The first one is the mass and the other one is a supersymmetric extension of the Poincaré Spin operator.

2.1 Massive case

For the massive case the second casimir operator takes the form

$$C_2 = \frac{W^2}{m^2} + \left(\frac{3}{4} + \lambda\right) P_{(o)} \tag{2.1}$$

where W^2 is the ordinary spin operator (the square of the Pauli-Lubanski vector), $P_{(o)}$ is the projection operator $P_{(o)} = -\frac{1}{m^2}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^2\mathrm{D}_{\gamma}$ and the parameter λ satisfies the equation

$$\lambda^2 + \lambda = \frac{W^2}{m^2}\ . \tag{2.2}$$

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In order to diagonalize C_2 we want to diagonalize both W^2, $P_{(o)}$. The superfield $\Phi_{\alpha(n)\dot{\alpha}(m)}$ that does this

$$\begin{aligned} W^2\Phi_{\alpha(n)\dot{\alpha}(m)} &= j(j+1)m^2\Phi_{\alpha(n)\dot{\alpha}(m)},\ j=\frac{n+m}{2}, \\ P_{(o)}\Phi_{\alpha(n)\dot{\alpha}(m)} &= \Phi_{\alpha(n)\dot{\alpha}(m)}, \end{aligned}$$

and describes the representation with the highest possible superspin

$$\begin{aligned} \lambda &= \frac{n+m}{2}, \\ C_2\Phi_{\alpha(n)\dot{\alpha}(m)} &= Y(Y+1)\Phi_{\alpha(n)\dot{\alpha}(m)},\ Y=\frac{n+m+1}{2}, \end{aligned}$$

has to satisfy the following:

$$\begin{aligned} \mathrm{D}^2\Phi_{\alpha(n)\dot{\alpha}(m)} &= 0 \\ \bar{\mathrm{D}}^2\Phi_{\alpha(n)\dot{\alpha}(m)} &= 0 \\ \mathrm{D}^{\gamma}\Phi_{\gamma\alpha(n-1)\dot{\alpha}(m)} &= 0 \\ \partial^{\gamma\dot{\gamma}}\Phi_{\gamma\alpha(n-1)\dot{\gamma}\dot{\alpha}(m-1)} &= 0 \\ \Box\Phi_{\alpha(n)\dot{\alpha}(m)} &= m^2\Phi_{\alpha(n)\dot{\alpha}(m)} \end{aligned} \tag{2.3}$$

where all dotted and undotted indices are fully symmetrized and the spin content of this supermultiplet is $j = Y+1/2,\ Y,\ Y,\ Y-1/2$.

All the above constraints can be satisfied if

$$\Phi_{\alpha(n)\dot{\alpha}(m)} \sim \frac{1}{m^2}\mathrm{D}^{\gamma}W_{\alpha(n)\gamma\dot{\alpha}(m)}\,,\ W_{\alpha(n+1)\dot{\alpha}(m)} \sim \bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{n+1}}\Phi_{\alpha(n))\dot{\alpha}(m)} \tag{2.4}$$

with

$$\begin{aligned} \bar{\mathrm{D}}_{\dot{\beta}}W_{\gamma\alpha(m)\dot{\alpha}(n)} &= 0,\ \text{chiral} \\ \partial^{\beta\dot{\beta}}W_{\beta\alpha(m)\dot{\beta}\dot{\alpha}(n-1)} &= 0 \\ \Box W_{\alpha(m+1)\dot{\alpha}(n)} &= m^2W_{\alpha(m+1)\dot{\alpha}(n)} \end{aligned} \tag{2.5}$$

The superfield that describes the highest superspin Y system, has index structure such that $n+m=2Y-1$ where n, m are integers. This Diophintine equation has a finite number of different solutions for the various (n, m) pairs, but the corresponding superfields are all related because we can use the $\partial_{\beta\dot{\beta}}$ operator to convert one kind of index to another. So we can pick one of them to represent the entire class.

One last comment has to be made about the reality of the representation. Under a hermitian conjugation, a (n, m) representation realized by a superfield like $\Phi_{\alpha(n)\dot{\alpha}(m)}$ goes to a (m, n) representation, realized by $\bar{\Phi}_{\alpha(m)\dot{\alpha}(n)}$

$$(n,m)^* \to (m,n) \begin{cases} \text{if } m=n, & (n,n)^* \to (n,n) \text{ :reality} \\ \text{if } m\neq n, & (n,m)^* \to (m,n) \neq (n,m) \\ & \text{to make real representations} \\ & \text{we need to consider } (n,m)\oplus(m,n) \end{cases}$$

At the superfield level this mapping can be done by the dimensionless operator $\Delta_{\alpha\dot{\alpha}} \equiv -i\frac{\partial_{\alpha\dot{\alpha}}}{\Box^{1/2}}$ which if used in repetition will convert all the undotted indices to dotted ones and vice versa.

$$\bar{\Phi}_{\alpha(m)\dot{\alpha}(n)} = \Delta_{a_1}{}^{\dot{\gamma}_1} \dots \Delta_{a_m}{}^{\dot{\gamma}_m} \Delta^{\gamma_1}{}_{\dot{\alpha}_1} \dots \Delta^{\gamma_1}{}_{\dot{\alpha}_1} \Phi_{\gamma(n)\dot{\gamma}(m)}$$

For irreducible representations with $n = m$ (bosonic superfields) the reality condition becomes $\Phi_{\alpha(n)\dot{\alpha}(n)} = \bar{\Phi}_{\alpha(n)\dot{\alpha}(n)}$ and for fermionic superfields ($n = m + 1$) the reality condition is the Dirac equation $i\partial_{\alpha_n}{}^{\dot{\alpha}_n}\bar{\Phi}_{\alpha(n-1)\dot{\alpha}(n)} + m\Phi_{\alpha(n)\dot{\alpha}(n-1)} = 0$. The conclusion is that real bosonic superfields with $n = m = s$ ($H_{\alpha(s)\dot{\alpha}(s)}$, $H_{\alpha(s)\dot{\alpha}(s)} = \bar{H}_{\alpha(s)\dot{\alpha}(s)}$), have even total number of indices should be used to describe half odd integer superspin systems, $Y = s+1/2$. On the other hand fermionic superfields with $n = s+1 = m+1$ which satisfy the Dirac equation ($\Psi_{\alpha(s+1)\dot{\alpha}(s)}$, $i\partial_{\alpha_{s+1}}{}^{\dot{\alpha}_{s+1}}\bar{\Psi}_{\alpha(s)\dot{\alpha}(s+1)} + m\Psi_{\alpha(s+1)\dot{\alpha}(s)}$) should be used to describe integer superpsin systems, $Y = s + 1$.

2.2 Massless case

For the masssless case, the supersymmetric analogue to the Pauli-Lubanski vector $W_{\gamma\dot{\gamma}}$ takes the form

$$Z_{\gamma\dot{\gamma}} = W_{\gamma\dot{\gamma}} + \frac{1}{4}[\mathrm{D}_\gamma, \bar{\mathrm{D}}_{\dot{\gamma}}] \tag{2.6}$$

and our goal is to make it proportional to momentum. The superfield $F_{\alpha(n)\dot{\alpha}(m)}$ which does that and describes the highest super-helicity representation

$$\begin{aligned} W_{\gamma\dot{\gamma}}F_{\alpha(n)\dot{\alpha}(m)} &= hP_{\gamma\dot{\gamma}}F_{\alpha(n)\dot{\alpha}(m)}\,, \qquad\qquad h = \frac{n-m}{2} \\ Z_{\gamma\dot{\gamma}}F_{\alpha(n)\dot{\alpha}(m)} &= \left(Y + \frac{1}{4}\right)P_{\gamma\dot{\gamma}}F_{\alpha(n)\dot{\alpha}(m)}\,,\ Y = \frac{n-m}{2} \end{aligned} \tag{2.7}$$

must satisfy the following:

$$\begin{aligned} \bar{\mathrm{D}}_{\dot{\gamma}}F_{\alpha(n)\dot{\alpha}(m)} &= 0,\ \text{chiral} \\ \mathrm{D}^\beta F_{\beta\alpha(n-1)\dot{\alpha}(m)} &= 0 \\ \partial_\gamma{}^{\dot{\beta}}F_{\alpha(n)\dot{\beta}\dot{\alpha}(m-1)} &= 0 \end{aligned} \tag{2.8}$$

where all dotted and undotted indices are fully symmetrized and the helicity content is $h = Y + 1/2,\ Y$

The superfield that describes a system with super-helicity Y, must have index structure such that $n - m = 2Y$. Unlike the massive case this Diophintine equation has infinite many different solutions with an increasing number of indices. Nevertheless all of them can be generated by acting with $\partial_{\beta\dot{\beta}}$ on the superfield with the fewest indices $F_{\alpha(2Y)}$ and symmetrize. Therefore we can choose a chiral superfield $F_{\alpha(2s)}$ for the description of integer superhelicity systems, $Y = s$ and a chiral superfield $F_{\alpha(2s+1)}$ for the description of half odd integer superhelicity systems, $Y = s + 1/2$.

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3 Massless theories

Now that we know the basic building blocks for the various representations and the constraints they have to satisfy, the next logical step is to attempt to construct superspace actions that will dynamically generate all the above. This is easier said than done. For the massive arbitrary superspin case the construction of a superspace action is still an open question, but some small progress has been done [25–28]. We will focus on the description of massless irreducible representations. Massless theories have their own special features that we will attempt to present in a unifying and pedagogical way.

3.1 Non supersymmetric sector, $\mathcal{N} = 0$

First of all it is a fact about physics that there is a discontinuous difference between the massive spin states and the massless helicity states. Also if we want to describe massless helicity theories in a lagrangian way, respecting locality and Lorentz invariance we are forced to introduce redundancies (gauge symmetry).

In principle we could have two separate classes of theories, one for the massive and one for massless states, that do not communicate. But if we have a theory for a massive spin, there must be a mass parameter and therefore we should be able to ask and answer the question, what happens as we gradually reduce the mass and eventually take the massless limit. The answer is that the limit exists and it is the corresponding irreducible massless theory. This connection between the lagrangian formulation of massive irreducible spin theories and massless irreducible helicity theories can be used as the cornerstone for the construction of $4D$ non-supersymmetric ($\mathcal{N} = 0$) free spin theories and makes contact with various other ideas such as the gauge invariant description of massive spin states and Stüeckelberg formulations.

3.2 Supersymmetric sector, $\mathcal{N} = 1$

Because the representations of the $4D$, $\mathcal{N} = 1$ Super-Poincaré group include all the above structure, it seems reasonable to assume that this smooth transition between massive and massless theories holds in $\mathcal{N} = 1$ superspace as well. Having that in mind we get the following.

3.2.1 Integer case

The superspace action for the massive integer superspin ($Y = s$) representation must be constructed in terms of a fermionic superfield $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ which can also be defined in terms of a chiral superfield $W_{\alpha(s+1)\dot{\alpha}(s-1)} \sim \bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{s+1}}\Psi_{\alpha(s))\dot{\alpha}(s-1)}$. On the other hand the theory of massless integer super-helicity must be described in terms of a chiral superfield $F_{\alpha(2s)}$ and also it must be the massless limit of the massive theory. These theories are described by different objects, how can the one be the massless limit of the other? For that to happen we have to able to construct an object like $F_{\alpha(2s)}$ out of $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ / $W_{\alpha(s+1)\dot{\alpha}(s-1)}$. Given the chirality property of F and W and their index structure, we could guess a mapping that could do the trick.

$$F_{\alpha(2s)} \sim \partial_{(\alpha_{2s}}{}^{\dot{\alpha}_{s-1}} \dots \partial_{\alpha_{s+2}}{}^{\dot{\alpha}_1}\bar{\mathrm{D}}^2\mathrm{D}_{\alpha_{s+1}}\Psi_{\alpha(s))\dot{\alpha}(s-1)} \tag{3.1}$$

But there is a problem with this map. The problem is that $F_{\alpha(2s)}$ which describes the system and carries the physical degrees of freedom seems to be defined in terms of another object $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$. Also F as defined above seems to have the on-shell degrees of freedom of Ψ which is more than needed. If this is going to work we have to find a way to 1) remove the physical (observable) status of Ψ and 2) remove its extra degrees of freedom.

There is a mechanism that can do both at the same time. That is to introduce a redundancy. We identify $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ with $\Psi_{\alpha(s)\dot{\alpha}(s-1)} + R_{\alpha(s)\dot{\alpha}(s-1)}$ and instead of talking about $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ we talk about equivalence classes of $\Psi_{\alpha(s)\dot{\alpha}(s-1)} \sim \Psi_{\alpha(s)\dot{\alpha}(s-1)} + R_{\alpha(s)\dot{\alpha}(s-1)}$. This redundancy, whatever it is has to respect the physical - propagating degrees of freedom of F and leave them unchanged. Hence

$$\partial_{(\alpha_{2s}}{}^{\dot{\alpha}_{s-1}} \dots \partial_{\alpha_{s+2}}{}^{\dot{\alpha}_1} \bar{\mathrm{D}}^2 \mathrm{D}_{\alpha_{s+1}} R_{\alpha(s))\dot{\alpha}(s-1)} = 0 \tag{3.2}$$

The most general solution to that is

$$R_{\alpha(s)\dot{\alpha}(s-1)} = \frac{1}{s!}\mathrm{D}_{(\alpha_s} K_{\alpha(s-1))\dot{\alpha}(s-1)} + \frac{1}{(s-1)!}\bar{\mathrm{D}}_{(\dot{\alpha}_{s-1}} \Lambda_{\alpha(s)\dot{\alpha}(s-2))} \tag{3.3}$$

where $K_{\alpha(s-1)\dot{\alpha}(s-1)}$, $\Lambda_{\alpha(s)\dot{\alpha}(s-2)}$ are completely unconstrained superfields. It is obvious that this redundancy will be the starting point for the gauge invariance story.

3.2.2 Half odd integer case

Similar discussion can be done for the half odd integer scenario. The massive theory is constructed by a real bosonic superfield $H_{\alpha(s)\dot{\alpha}(s)}$ which can be defined as well by a chiral superfield $W_{\alpha(s+1)\dot{\alpha}(s)} \sim \bar{\mathrm{D}}^2 \mathrm{D}_{(\alpha_{s+1}} H_{\alpha(s))\dot{\alpha}(s)}$. The massless theory is based on a chiral superfield $F_{\alpha(2s+1)}$ and it is the massless limit of the massive theory. For that to happen we must have

$$F_{\alpha(2s+1)} \sim \partial_{(\alpha_{2s+1}}{}^{\dot{\alpha}_s} \dots \partial_{\alpha_{s+2}}{}^{\dot{\alpha}_1} \bar{\mathrm{D}}^2 \mathrm{D}_{\alpha_{s+1}} H_{\alpha(s))\dot{\alpha}(s)} \tag{3.4}$$

and to solve the problem of the physical degrees of freedom as mentioned above we must identify $H_{\alpha(s)\dot{\alpha}(s)}$ with $H_{\alpha(s)\dot{\alpha}(s+)} + R_{\alpha(s)\dot{\alpha}(s)}$ where $R_{\alpha(s)\dot{\alpha}(s)}$ is constrained

$$\partial_{(\alpha_{2s+1}}{}^{\dot{\alpha}_s} \dots \partial_{\alpha_{s+2}}{}^{\dot{\alpha}_1} \bar{\mathrm{D}}^2 \mathrm{D}_{\alpha_{s+1}} R_{\alpha(s))\dot{\alpha}(s)} = 0 \tag{3.5}$$

The most general solution[1] to this is

$$R_{\alpha(s)\dot{\alpha}(s)} = \frac{1}{s!}\mathrm{D}_{(\alpha_s} \bar{L}_{\alpha(s-1))\dot{\alpha}(s)} - \frac{1}{s!}\bar{\mathrm{D}}_{(\dot{\alpha}_s} L_{\alpha(s)\dot{\alpha}(s-1))} \,. \tag{3.6}$$

4 Ineteger superhelicity theory

Using the equivalency class characterized by $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ and redundancy $R_{\alpha(s)\dot{\alpha}(s-1)}$ we will show by construction that there is a unique superspace action that will describe the irreducible representation of integer super-helicity, $Y = s$.

[1] R must be real since H is real.

4.1 The superspace action

Superfield Ψ must have mass dimensions $1/2$,[2] and the action must involve two covariant derivatives.[3] The most general action that can be written is:

$$\begin{aligned} S = \int d^8z\; & a_1 \Psi^{\alpha(s)\dot\alpha(s-1)} \mathrm{D}^2 \Psi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\ & + a_2 \Psi^{\alpha(s)\dot\alpha(s-1)} \bar{\mathrm{D}}^2 \Psi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\ & + a_3 \Psi^{\alpha(s)\dot\alpha(s-1)} \bar{\mathrm{D}}^{\dot\alpha_s} \mathrm{D}_{\alpha_s} \bar\Psi_{\alpha(s-1)\dot\alpha(s)} \\ & + a_4 \Psi^{\alpha(s)\dot\alpha(s-1)} \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}^{\dot\alpha_s} \bar\Psi_{\alpha(s-1)\dot\alpha(s)} \end{aligned}$$

The goal is to find an action that respects the redundancy. That is the starting point for gauge invariance $\delta_G S = 0$. The strategy to obtain this is to pick the free parameters in a special way. If this is not possible then we introduce auxiliary superfields, compensators and/or impose constraints on the parameters of the redundancy (gauge parameters). It is reasonable to expect any compensators introduced, if necessary, will not introduce degrees of freedom with spin higher or equal than the one we wish to describe. Thus, they must have less indices than Ψ.

For this case we obtain the following expression for the modification of the action due to the redundancy,

$$\begin{aligned} \delta_G S = \int d^8z\, & \Big\{ -2a_1 \mathrm{D}_{\alpha_s} \Psi^{\alpha(s)\dot\alpha(s-1)} \\ & + a_4 \bar{\mathrm{D}}_{\dot\alpha_s} \bar\Psi^{\alpha(s-1)\dot\alpha(s)} \Big\} \mathrm{D}^\beta \bar{\mathrm{D}}_{\dot\alpha_{s-1}} \Lambda_{\beta\alpha(s-1)\dot\alpha(s-2)} \\ & + \Big\{ -a_3 \left[\frac{s-1}{s}\right] \bar{\mathrm{D}}_{\dot\alpha_s} \mathrm{D}_{\alpha_{s-1}} \bar\Psi^{\alpha(s-1)\dot\alpha(s)} \\ & + \left[-a_3 + \frac{s+1}{s} a_4 \right] \mathrm{D}_{\alpha_{s-1}} \bar{\mathrm{D}}_{\dot\alpha_s} \bar\Psi^{\alpha(s-1)\dot\alpha(s)} \Big\} \mathrm{D}^\beta K_{\beta\alpha(s-2)\dot\alpha(s-1)} \\ & + \Big\{ 2a_2 \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}^2 \Psi^{\alpha(s)\dot\alpha(s-1)} - a_3 \bar{\mathrm{D}}_{\dot\alpha_s} \mathrm{D}^2 \bar\Psi^{\alpha(s-1)\dot\alpha(s)} \Big\} K_{\alpha(s-1)\dot\alpha(s-1)} \\ & + c.c. \end{aligned} \tag{4.1}$$

Obviously we can not make all this terms vanish just by picking values for the a's without setting them all to zero and also we can't introduce compensators with proper mass dimensionality and index structure. The way out is to give some structure to the gauge parameter K. So let us choose

$$\begin{aligned} a_1 &= a_4 = 0 \\ \mathrm{D}^\beta K_{\beta\alpha(s-2)\dot\alpha(s-1)} &= 0 \to K_{\alpha(s-1)\dot\alpha(s-1)} = \mathrm{D}^{\alpha_s} L_{\alpha(s)\dot\alpha(s-1)} \\ 2a_2 &= -a_3 \end{aligned} \tag{4.2}$$

So we find

$$\begin{aligned} \delta_G S = -a_3 \int d^8z \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}^2 \Psi^{\alpha(s)\dot\alpha(s-1)} \left(\mathrm{D}^\beta L_{\beta\alpha(s-1)\dot\alpha(s-1)} + \bar{\mathrm{D}}^{\dot\beta} \bar L_{\alpha(s-1)\dot\beta\dot\alpha(s-1)} \right) \\ + c.c. \end{aligned} \tag{4.3}$$

[2]Its highest spin component is a propagating fermion.
[3]The action must be quadratic in Ψ and dimensionless.

This suggests we introduce a real bosonic compensator $V_{\alpha(s-1)\dot\alpha(s-1)}$ which transforms like $\delta_G V_{\alpha(s-1)\dot\alpha(s-1)} = \mathrm{D}^{\alpha_s}L_{\alpha(s)\dot\alpha(s-1)} + \bar{\mathrm{D}}^{\dot\alpha_s}\bar{L}_{\alpha(s-1)\alpha(s)}$ and couples with the real piece of $\mathrm{D}^{\alpha_s}\bar{\mathrm{D}}^2\Psi_{\alpha(s)\dot\alpha(s-1)}$.

In order to achieve invariance, we add to the action two new pieces, a coupling term of V with Ψ and a kinetic energy term for V. The full action takes the form

$$\begin{aligned} S = \int d^8z \, & -\frac{1}{2}a_3\Psi^{\alpha(s)\dot\alpha(s-1)}\bar{\mathrm{D}}^2\Psi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\ & + a_3\Psi^{\alpha(s)\dot\alpha(s-1)}\bar{\mathrm{D}}^{\dot\alpha_s}\mathrm{D}_{\alpha_s}\bar\Psi_{\alpha(s-1)\dot\alpha(s)} \\ & - a_3V^{\alpha(s-1)\dot\alpha(s-1)}\mathrm{D}^{\alpha_s}\bar{\mathrm{D}}^2\Psi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\ & + b_1V^{\alpha(s-1)\dot\alpha(s-1)}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^2\mathrm{D}_{\gamma}V_{\alpha(s-1)\dot\alpha(s-1)} \\ & + b_2V^{\alpha(s-1)\dot\alpha(s-1)}\left\{\mathrm{D}^2,\bar{\mathrm{D}}^2\right\}V_{\alpha(s-1)\dot\alpha(s-1)} \\ & + b_3V^{\alpha(s-1)\dot\alpha(s-1)}\mathrm{D}_{\alpha_{s-1}}\bar{\mathrm{D}}^2\mathrm{D}^{\gamma}V_{\gamma\alpha(s-2)\dot\alpha(s-1)} + c.c. \\ & + b_4V^{\alpha(s-1)\dot\alpha(s-1)}\mathrm{D}_{\alpha_{s-1}}\bar{\mathrm{D}}_{\dot\alpha_{s-1}}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^{\dot\gamma}V_{\gamma\alpha(s-2)\dot\gamma\dot\alpha(s-2)} + c.c. \end{aligned} \tag{4.4}$$

and it has to be invariant under

$$\delta_G\Psi_{\alpha(s)\dot\alpha(s-1)} = -\mathrm{D}^2L_{\alpha(s)\dot\alpha(s-1)} + \left[\frac{1}{(s-1)!}\right]\bar{\mathrm{D}}_{(\dot\alpha_{s-1}}\Lambda_{\alpha(s)\dot\alpha(s-2))} \tag{4.5a}$$

$$\delta_G V_{\alpha(s-1)\dot\alpha(s-1)} = \mathrm{D}^{\alpha_s}L_{\alpha(s)\dot\alpha(s-1)} + \bar{\mathrm{D}}^{\dot\alpha_s}\bar{L}_{\alpha(s-1)\dot\alpha(s)} \tag{4.5b}$$

The equations of motion of the superfields are the variation of the action with respect to the corresponding superfield

$$T_{\alpha(s)\dot\alpha(s-1)} = \frac{\delta S}{\delta\Psi^{\alpha(s)\dot\alpha(s-1)}} \tag{4.5c}$$

$$G_{\alpha(s-1)\dot\alpha(s-1)} = \frac{\delta S}{\delta V^{\alpha(s-1)\dot\alpha(s-1)}} \tag{4.5d}$$

and the invariance of the action gives the following Bianchi Identities

$$\mathrm{D}^2T_{\alpha(s)\dot\alpha(s-1)} + \frac{1}{s!}\mathrm{D}_{(\alpha_s}G_{\alpha(s-1))\dot\alpha(s-1)} = 0 \tag{4.5e}$$

$$\bar{\mathrm{D}}^{\dot\alpha_{s-1}}T_{a(s)\dot\alpha(s-1)} = 0 \tag{4.5f}$$

The satisfaction of the Bianchi identities fix all the coefficients

$$b_1 = \frac{1}{2}a_3 \qquad b_3 = 0$$
$$b_2 = 0 \qquad b_4 = 0$$

and the action takes the form[4]

$$\begin{aligned} S = \int d^8z \, \Big\{ & -\frac{1}{2}c\Psi^{\alpha(s)\dot\alpha(s-1)}\bar{\mathrm{D}}^2\Psi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\ & + c\Psi^{\alpha(s)\dot\alpha(s-1)}\bar{\mathrm{D}}^{\dot\alpha_s}\mathrm{D}_{\alpha_s}\bar\Psi_{\alpha(s-1)\dot\alpha(s)} \\ & - cV^{\alpha(s-1)\dot\alpha(s-1)}\mathrm{D}^{\alpha_s}\bar{\mathrm{D}}^2\Psi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\ & + \frac{1}{2}cV^{\alpha(s-1)\dot\alpha(s-1)}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^2\mathrm{D}_{\gamma}V_{\alpha(s-1)\dot\alpha(s-1)}\Big\} \end{aligned} \tag{4.6}$$

[4] Here c is an overall unconstrained parameter which can be absorbed into the definition of Ψ. We leave it as it is for now and fix it later in the component discussion.

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The equations of motion are

$$\begin{aligned} T_{\alpha(s)\dot\alpha(s-1)} &= -c\bar{\mathrm{D}}^2\Psi_{\alpha(s)\dot\alpha(s-1)} + \frac{c}{s!}\bar{\mathrm{D}}^{\dot\alpha_s}\mathrm{D}_{(\alpha_s}\bar\Psi_{\alpha(s-1))\dot\alpha(s)} \\ &\quad + \frac{c}{s!}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_s}V_{\alpha(s-1))\dot\alpha(s-1)} \qquad (4.7a) \\ G_{\alpha(s-1)\dot\alpha(s-1)} &= -c\left(\mathrm{D}^{\alpha_s}\bar{\mathrm{D}}^2\Psi_{\alpha(s)\dot\alpha(s-1)} + \bar{\mathrm{D}}^{\dot\alpha_s}\mathrm{D}^2\bar\Psi_{\alpha(s-1)\alpha(s)}\right) \\ &\quad + c\mathrm{D}^\gamma\bar{\mathrm{D}}^2\mathrm{D}_\gamma V_{\alpha(s-1)\dot\alpha(s-1)} \qquad (4.7b) \end{aligned}$$

This is exactly the longitudinal-linear theory presented in [14] if we solve the superfield constraints and express their action in terms of the prepotential. Now, however we gain a different understanding of why the action has to be expressed in terms of a superfield like Ψ and why it has a gauge transformation as it does.

The work in [14] presented a second theory for integer super-helicity, the transverse-linear theory. That theory is most certainly consistent classically, but violates one of our assumptions in that some of its auxiliary fields possess spins greater than that carried by the gauge superfield. To our knowledge, no studies of the quantum behavior of these off-shell supersymmetrical and even free theories has been carried out. If is our suspicion that the presence of auxiliary superfields with a higher superspin than the main gauge superpotential is likely to have a more complicated ghost structure. It would be a very interesting investigation to test this idea.

We have managed to find a superspace action which is gauged invariant but still we haven't proved that this theory describes an integer super-helicity system. To do so, we must show that there is an object like $F_{\alpha(2s)}$, it is chiral and on-shell it satisfies the required by representation theory constraints.

Using the equations of motion we can now prove that a chiral superfield $F_{\alpha(2s)}$ exists and satisfies following Bianchi identity:

$$\begin{aligned} \bar{\mathrm{D}}^{\dot\alpha_{2s}}\bar F_{\dot\alpha(2s)} &= -\frac{i}{(2s-1)!c}\partial^{\alpha_s}{}_{(\dot\alpha_{2s-1}}\dots\partial^{\alpha_1}{}_{\dot\alpha_s}T_{\alpha(s)\dot\alpha(s-1))} \\ &\quad + \frac{B}{(2s-1)!}\bar{\mathrm{D}}^2\partial^{\alpha_{s-1}}{}_{(\dot\alpha_{2s-1}}\dots\partial^{\alpha_1}{}_{(\dot\alpha_{s+1}}\bar T_{\alpha(s-1)\dot\alpha(s))} \\ &\quad + \frac{1+2cB}{(2s-1)!2c}\bar{\mathrm{D}}_{(\dot\alpha_{2s-1}}\partial^{\alpha_{s-1}}{}_{\dot\alpha_{2s-2}}\dots\partial^{\alpha_1}{}_{\dot\alpha_s}G_{\alpha(s-1)\dot\alpha(s-1))} \\ &\quad + \frac{1}{(2s-1)!2c}\bar{\mathrm{D}}_{(\dot\alpha_{2s-1}}\mathrm{D}^{\alpha_s}\partial^{\alpha_{s-1}}{}_{\dot\alpha_{2s-2}}\dots\partial^{\alpha_1}{}_{\dot\alpha_s}T_{\alpha(s)\dot\alpha(s-1)} \qquad (4.8) \end{aligned}$$

where

$$\bar F_{\dot\alpha(2s)} = \frac{1}{(2s)!}\mathrm{D}^2\bar{\mathrm{D}}_{(\dot\alpha_{2s}}\partial^{\alpha_{s-1}}{}_{\dot\alpha_{2s-1}}\dots\partial^{\alpha_1}{}_{\dot\alpha_{s+1}}\bar\Psi_{\alpha(s-1)\dot\alpha(s))} \qquad (4.9)$$

and that shows that if $T_{\alpha(s)\dot\alpha(s-1)} = G_{\alpha(s-1)\dot\alpha(s-1))} = 0$, we obtain the desired constraints to describe a super-helicity $Y = s$ system, where B is a parameter determined by variations and definitions.

Before we start investigating the field spectrum of the above action, one more comment needs to be made. This specific action and superfield configuration is not unique but the simplest representative of a two parameter family of equivalent theories. To see that we

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can perform redefinitions of the superfields. Dimensionality and index structure allow us to make the following redefinition of Ψ

$$\Psi_{\alpha(s)\dot\alpha(s-1)} \to \Psi_{\alpha(s)\dot\alpha(s-1)} + \frac{z}{s!}\mathrm{D}_{(\alpha_s}V_{\alpha(s-1))\dot\alpha(s-1)} \tag{4.10}$$

where z is a complex parameter. This operation will generate an entire class of actions and transformation laws which all are related by the above redefinition. The action is

$$\begin{aligned} S = \int d^8w \Big\{ & -\frac{1}{2}c\ \Psi^{\alpha(s)\dot\alpha(s-1)}\bar{\mathrm{D}}^2\Psi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\ & +c\ \Psi^{\alpha(s)\dot\alpha(s-1)}\bar{\mathrm{D}}^{\dot\alpha_s}\mathrm{D}_{\alpha_s}\bar\Psi_{\alpha(s-1)\dot\alpha(s)} \\ & +c(z+\bar z-1)\ V^{\alpha(s-1)\dot\alpha(s-1)}\mathrm{D}^{\alpha_s}\bar{\mathrm{D}}^2\Psi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\ & +c\bar z\ V^{\alpha(s-1)\dot\alpha(s-1)}\bar{\mathrm{D}}^2\mathrm{D}^{\alpha_s}\Psi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\ & -\left[\frac{s-1}{s}\right]c\bar z\ V^{\alpha(s-1)\dot\alpha(s-1)}\bar{\mathrm{D}}_{\dot\alpha_{s-1}}\mathrm{D}^{\beta}\bar{\mathrm{D}}^{\dot\beta}\Psi_{\beta\alpha(s-1)\dot\beta\dot\alpha(s-2)} + c.c. \\ & +\frac{1}{2}c(z+\bar z-1)^2\ V^{\alpha(s-1)\dot\alpha(s-1)}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^2\mathrm{D}_{\gamma}V_{\alpha(s-1)\dot\alpha(s-1)} \\ & +\left[\frac{1}{s}\right]cz\bar z\ V^{\alpha(s-1)\dot\alpha(s-1)}\left\{\mathrm{D}^2,\bar{\mathrm{D}}^2\right\}V_{\alpha(s-1)\dot\alpha(s-1)} \\ & +\left[\frac{s-1}{2s}\right]cz(z+2\bar z-2)\ V^{\alpha(s-1)\dot\alpha(s-1)}\mathrm{D}_{\alpha_{s-1}}\bar{\mathrm{D}}^2\mathrm{D}^{\gamma}V_{\gamma\alpha(s-2)\dot\alpha(s-1)} + c.c. \\ & -\left[\frac{(s-1)^2}{2s^2}\right]cz\bar z\ V^{\alpha(s-1)\dot\alpha(s-1)}\mathrm{D}_{\alpha_{s-1}}\bar{\mathrm{D}}_{\dot\alpha_{s-1}}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^{\dot\gamma}V_{\gamma\alpha(s-2)\dot\gamma\dot\alpha(s-2)} + c.c. \Big\} \end{aligned} \tag{4.11}$$

and the transformation laws are

$$\begin{aligned} \delta_G\Psi_{\alpha(s)\dot\alpha(s-1)} &= (z-1)\,\mathrm{D}^2L_{\alpha(s)\dot\alpha(s-1)} - \frac{z}{s!}\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}^{\dot\alpha_s}\bar L_{\alpha(s-1))\dot\alpha(s)} \\ &\quad + \left[\frac{1}{(s-1)!}\right]\bar{\mathrm{D}}_{(\dot\alpha_{s-1}}\Lambda_{\alpha(s)\dot\alpha(s-2))} \end{aligned} \tag{4.12a}$$

$$\delta_G V_{\alpha(s-1)\dot\alpha(s-1)} = \mathrm{D}^{\alpha_s}L_{\alpha(s)\dot\alpha(s-1)} + \bar{\mathrm{D}}^{\dot\alpha_s}\bar L_{\alpha(s-1)\dot\alpha(s)} \tag{4.12b}$$

4.2 Projection and components

Although superspace was developed to describe supersymmetric theories in a more efficient, compact and clear way, there are still some reasons why we would like to study the off-shell component structure of the theory.

1. There are cases where two theories on-shell describe the same physical system. Therefore from the path integral point of view the theories are equivalent. Nevertheless the off-shell structure of the two theories might be completely different. Knowledge of the component formulation of the two theories will help us decide if they are different theories with the same on-shell description or they are the same theory and there is a 1-1 mapping between the two.

2. The off-shell component structure of a supersymmetric theory will give us clues about which theories can be used to realize higher $\mathcal{N}$ and higher D representations.

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For these reasons we would like to extract the component field content of the above superspace action, the number of degrees of freedom involved, their transformation law under supersymmetry and their gauge transformations.

Previous discussion to this use the Wess-Zumino and explicit θ-expansions. We propose a different technique that will illuminate a more natural way to define the component structure and make the entire process of finding the component action and SUSY-transformation laws efficiently.

The component action will depend on two kinds of fields, the dynamical ones that will give the dynamics of the various spin states and the auxiliary ones that exist so supersymmetry is preserved and they vanish on-shell. Exactly because the auxiliary fields will have to vanish on-shell, we should be able to make redefinitions such that they appear in the action in an algebraic way and specifically in quadratic monomials with each auxiliary field to appear in one and only one monomial. In this way their equations of motion will be the vanishing of these fields, making obvious their auxiliary status. That also means that in this configuration the auxiliary fields must be gauge invariant objects. That is because the full action is gauge invariant, the dynamical pieces are gauge invariant all together and the auxiliary fields appear in this special way.

Since we want the auxiliary fields of the final action to be gauge invariant it might be smart to define them using objects that are already gauge invariant. But the superspace action already provides us with two gauge invariant objects, the equations of motion:[5]

$$T_{\alpha(s)\dot{\alpha}(s-1)} = \frac{\delta S}{\delta \Psi^{\alpha(s)\dot{\alpha}(s-1)}}\,, \qquad \left[T_{\alpha(s)\dot{\alpha}(s-1)}\right] = 3/2 \tag{4.13}$$

$$G_{\alpha(s-1)\dot{\alpha}(s-1)} = \frac{\delta S}{\delta V^{\alpha(s-1)\dot{\alpha}(s-1)}}\,, \qquad \left[G_{\alpha(s-1)\dot{\alpha}(s-1)}\right] = 2 \tag{4.14}$$

$$G_{\alpha(s-1)\dot{\alpha}(s-1)} = \bar{G}_{\alpha(s-1)\dot{\alpha}(s-1)}$$

Because they are gauge invariant, if we expand them in components, each one of them will be gauge invariant. Furthermore because they vanish on-shell each one of these components will vanish as well. So it looks like the ideal place to look for the auxiliary component structure.

These superfields satisfy a set of equations that we will discover as we go along, but at the top of the list we have the Bianchi identities[6] and their consequences:

$$\mathrm{D}^2 T_{\alpha(s)\dot{\alpha}(s-1)} + \frac{1}{s!}\mathrm{D}_{(\alpha_s} G_{\alpha(s-1))\dot{\alpha}(s-1)} = 0 \rightsquigarrow \mathrm{D}^2 G_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \tag{4.15}$$

$$\bar{\mathrm{D}}^2 G_{\alpha(s-1)\dot{\alpha}(s-1)} = 0$$

$$\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}} T_{a(s)\dot{\alpha}(s-1)} = 0 \rightsquigarrow \quad \bar{\mathrm{D}}^2 T_{a(s)\dot{\alpha}(s-1)} = 0 \tag{4.16}$$

The results of these are that most of the components in the expansion of T and G vanish and we are left with very few that we can associate with auxiliary fields. For example, the bosonic auxiliary fields (dimensionality 2) have to be related to $\bar{\mathrm{D}}_{(\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s-1))}|$,

[5]There is also the superfield strength $F_{\alpha(2s)}$ but because of dimensionality reasons we can not write the action in terms of it.

[6]The Bianchi identities include the entire information about redundancy and therefore effectively they make everything that could have been gauged away, if we had followed the WZ-gauge path, disappear.

$D^{\alpha_s}T_{\alpha(s)\dot\alpha(s-1)}|$, $G_{\alpha(s-1)\dot\alpha(s-1)}|$ and the fermionic ones $(3/2,\ 5/2)$ will have to be related to $T_{\alpha(s)\dot\alpha(s-1)}|$, $\mathrm{D}^2 T_{\alpha(s)\dot\alpha(s-1)}|$. So by just looking at the Bianchi identities we find for free the spectrum of the auxiliary fields of the action and because they are gauge invariant we can do a straightforward counting of their degrees of freedom. For the dynamical fields, we can use the superfield strength $F_{\alpha(2s)}$ to connect them with some components of the superfields. Instead we will let the action, the equations of motion and their properties to guide us to their definition.

But if the equations of motion are the proper objects to define the components and we want to find the component action of the theory we must be able to express the action in terms of the equations of motion. That can be easily done by using the definitions of T and G to rewrite the action in the following form

$$\begin{aligned} S = & \int d^8z \left\{ \ \frac{1}{2}\Psi^{\alpha(s)\dot\alpha(s-1)}T_{\alpha(s)\dot\alpha(s-1)} + c.c. \right. \\ & \left. +\frac{1}{2}V^{\alpha(s-1)\dot\alpha(s-1)}G_{\alpha(s-1)\dot\alpha(s-1)} \right\} \\ = & \int d^4x \frac{1}{2}\mathrm{D}^2\bar{\mathrm{D}}^2\left(\Psi^{\alpha(s)\dot\alpha(s-1)}T_{\alpha(s)\dot\alpha(s-1)}\right) + c.c. \\ & +\frac{1}{2}\mathrm{D}^2\bar{\mathrm{D}}^2\left(V^{\alpha(s-1)\dot\alpha(s-1)}G_{\alpha(s-1)\dot\alpha(s-1)}\right) \end{aligned} \tag{4.17}$$

and now we distribute the covariant derivatives.

4.2.1 Fermions

Let us focus on the fermionic action first. After the distribution of D's and the usage of Bianchi identities we find for the fermionic Lagrangian:

$$\begin{aligned} \mathcal{L}_F = & \frac{1}{2}\mathrm{D}^2\bar{\mathrm{D}}^2\Psi^{\alpha(s)\dot\alpha(s-1)}|T_{\alpha(s)\dot\alpha(s-1)}| \\ & +\frac{1}{2}\left(\bar{\mathrm{D}}^2\Psi^{\alpha(s)\dot\alpha(s-1)} - \frac{1}{s!}\bar{\mathrm{D}}^2\mathrm{D}^{(\alpha_s}V^{\alpha(s-1))\dot\alpha(s-1)}\right)|\mathrm{D}^2T_{\alpha(s)\dot\alpha(s-1)}| \\ & -\frac{1}{2}\frac{1}{(s+1)!s!}\mathrm{D}^{(\alpha_{s+1}}\bar{\mathrm{D}}^{(\dot\alpha_s}\Psi^{\alpha(s))\dot\alpha(s-1))}|\frac{1}{(s+1)!s!}\mathrm{D}_{(\alpha_{s+1}}\bar{\mathrm{D}}_{(\dot\alpha_s}T_{\alpha(s))\dot\alpha(s-1))}| \\ & +\frac{1}{2}\frac{s}{s+1}\frac{1}{s!}\mathrm{D}_\gamma\bar{\mathrm{D}}^{(\dot\alpha_s}\Psi^{\gamma\alpha(s-1)\dot\alpha(s-1))}|\frac{1}{s!}\mathrm{D}^{\alpha_s}\bar{\mathrm{D}}_{(\dot\alpha_s}T_{\alpha(s)\dot\alpha(s-1))}| \\ & -\frac{s-1}{2s}\bar{\mathrm{D}}^2\mathrm{D}_\gamma V^{\gamma\alpha(s-2)\dot\alpha(s-1)}|\mathrm{D}^{\alpha_{s-1}}G_{\alpha(s-1)\dot\alpha(s-1)}| \\ & +c.c. \end{aligned} \tag{4.18}$$

At this point we can show that T and G satisfy a few more identities:

$$\begin{aligned} & \frac{1}{(s+1)!s!}\mathrm{D}_{(\alpha_{s+1}}\bar{\mathrm{D}}_{(\dot\alpha_s}T_{\alpha(s))\dot\alpha(s-1))} = \\ & = -\frac{ic}{(s+1)!}\partial_{(\alpha_{s+1}}{}^{\dot\alpha_{s+1}}\left[\frac{1}{(s+1)!s!}\bar{\mathrm{D}}_{(\dot\alpha_{s+1}}\mathrm{D}_{(\alpha_s}\bar\Psi_{\alpha(s-1)))\dot\alpha(s))}\right] \\ & \quad + \frac{ic}{(s+1)!s!}\frac{s}{s+1}\partial_{(\alpha_{s+1}(\dot\alpha_s}\left[\frac{1}{s!}\bar{\mathrm{D}}^{\dot\gamma}\mathrm{D}_{(\alpha_s}\bar\Psi_{\alpha(s-1)))\dot\gamma\dot\alpha(s-1))}\right] \end{aligned}$$

$$
\begin{aligned}
\frac{1}{s!}\mathrm{D}^{\alpha_s}\bar{\mathrm{D}}_{(\dot\alpha_s}T_{\alpha(s)\dot\alpha(s-1))} =& \frac{i}{s!}\frac{s+1}{s}\partial^{\alpha_s}{}_{(\dot\alpha_s}T_{\alpha(s)\dot\alpha(s-1))}\\
&+\frac{s+1}{s}\bar{\mathrm{D}}^2\bar{T}_{\alpha(s-1)\dot\alpha(s)}\\
&-\frac{ic}{s!(s+1)!}\partial^{\alpha_s\dot\alpha_{s+1}}\bar{\mathrm{D}}_{(\dot\alpha_{s+1}}\mathrm{D}_{(\alpha_s}\bar\Psi_{\alpha(s-1))\dot\alpha(s))}\\
&-\frac{ic}{s!s!}\frac{2s+1}{s(s+1)}\partial^{\alpha_s}{}_{(\dot\alpha_s}\bar{\mathrm{D}}^{\dot\gamma}\mathrm{D}_{(\alpha_s}\bar\Psi_{\alpha(s-1))\dot\gamma\dot\alpha(s-1))}\\
&-\frac{ic}{s!s!}\frac{s^2-1}{s}\partial_{(\alpha_{s-1}(\dot\alpha_s}\bar{\mathrm{D}}^2\mathrm{D}^\gamma V_{\gamma\alpha(s-2))\dot\alpha(s-1))}\\
\mathrm{D}^{\alpha_{s-1}}G_{\alpha(s-1)\dot\alpha(s-1)} =& i\partial^{\alpha_{s-1}\dot\alpha_s}\bar{T}_{\alpha(s-1)\dot\alpha(s)}\\
&-\frac{ic}{s!}\partial^{\alpha_{s-1}\dot\alpha_s}\mathrm{D}^\gamma\bar{\mathrm{D}}_{(\dot\alpha_s}\Psi_{\gamma\alpha(s-1)\dot\alpha(s-1))}\\
&-ic\frac{s-1}{s!}\partial^{\alpha_{s-1}}{}_{(\dot\alpha_{s-1}}\mathrm{D}^2\bar{\mathrm{D}}^{\dot\gamma}V_{\alpha(s-1)\dot\gamma\dot\alpha(s-2))}\\
\bar{\mathrm{D}}^2\bar{T}_{\alpha(s-1)\dot\alpha(s)}+\frac{i}{s!}\partial^{\alpha_s}{}_{(\dot\alpha_s}T_{\alpha(s)\dot\alpha(s-1))} =& \frac{ic}{s!s!}\partial^{\alpha_s}{}_{(\dot\alpha_s}\bar{\mathrm{D}}^{\dot\gamma}\mathrm{D}_{(\alpha_s}\bar\Psi_{\alpha(s-1))\dot\gamma\dot\alpha(s-1))}\\
&-c\bar{\mathrm{D}}^2\mathrm{D}^2\bar\Psi_{\alpha(s-1)\dot\alpha(s)}\\
&+ic\frac{(s-1)}{s!s!}\partial_{(\alpha_{s-1}(\dot\alpha_s}\bar{\mathrm{D}}^2\mathrm{D}^\gamma V_{\gamma\alpha(s-2))\dot\alpha(s-1))}
\end{aligned}
$$

We notice that in all the above there are some combinations that appear repeatedly. So let us define the following fields:

$$
\begin{aligned}
\frac{1}{s!(s+1)!}\mathrm{D}_{(\alpha_{s+1}}\bar{\mathrm{D}}_{(\dot\alpha_s}\Psi_{\alpha(s))\dot\alpha(s-1))}| &\equiv N_1\psi_{\alpha(s+1)\dot\alpha(s)}\\
\frac{1}{s!}\bar{\mathrm{D}}^{\dot\alpha_s}\mathrm{D}_{(\alpha_s}\bar\Psi_{\alpha(s-1))\dot\alpha(s)}| &\equiv N_2\psi_{\alpha(s)\dot\alpha(s-1)}\\
\mathrm{D}^2\bar{\mathrm{D}}^{\dot\alpha_{s-1}}V_{\alpha(s-1)\dot\alpha(s-1)}| &\equiv N_3\psi_{\alpha(s-1)\dot\alpha(s-2)}
\end{aligned}
$$

where $N_1,\ N_2,\ N_3,\ N_4$ are some overall normalization, to be fixed later as needed.

Putting everything together we find the fermionic terms of the Lagrangian

$$
\begin{aligned}
\mathcal{L}_F =& -\frac{1}{2c}T^{\alpha(s)\dot\alpha(s-1)}|\left(2\mathrm{D}^2T_{\alpha(s)\dot\alpha(s-1)}+\frac{i}{s!}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar{T}_{\alpha(s-1))\dot\alpha(s)}\right)|+c.c.\\
&-ic|N_1|^2\ \bar\psi^{\alpha(s)\dot\alpha(s+1)}\partial^{\alpha_{s+1}}{}_{\dot\alpha_{s+1}}\psi_{\alpha(s+1)\dot\alpha(s)}\\
&-ic\frac{s}{s+1}N_1N_2\ \psi^{\alpha(s+1)\dot\alpha(s)}\partial_{\alpha_{s+1}\dot\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1)}+c.c.\\
&+ic\frac{2s+1}{(s+1)^2}|N_2|^2\ \bar\psi^{\alpha(s-1)\dot\alpha(s)}\partial^{\alpha_s}{}_{\dot\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1)}\\
&+ic\frac{s-1}{s}N_2N_3\ \psi^{\alpha(s)\dot\alpha(s-1)}\partial_{\alpha_s\dot\alpha_{s-1}}\psi_{\alpha(s-1)\dot\alpha(s-2)}+c.c.\\
&+ic\left(\frac{s-1}{s}\right)^2|N_3|^2\ \bar\psi^{\alpha(s-2)\dot\alpha(s-1)}\partial^{\alpha_{s-1}}{}_{\dot\alpha_{s-1}}\psi_{\alpha(s-1)\dot\alpha(s-2)}
\end{aligned}
$$

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The first term in the Lagrangian is the algebraic term of two auxiliary fields and the rest of the terms have exactly the structure of a theory that describes helicity $h = s+1/2$ [23].[7] For an exact match we choose coefficients

$$
\begin{aligned}
c &= -1\,, & N_2 &= 1 \\
N_1 &= 1\,, & N_3 &= -\frac{s}{s-1}
\end{aligned}
$$

So the fields that appear in the fermionic action are defined as:

$$
\begin{aligned}
\rho_{\alpha(s)\dot\alpha(s-1)} &\equiv T_{\alpha(s)\dot\alpha(s-1)}| \\
\beta_{\alpha(s)\dot\alpha(s-1)} &\equiv \mathrm{D}^2 T_{\alpha(s)\dot\alpha(s-1)}| + \frac{i}{2s!}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar T_{\alpha(s-1))\dot\alpha(s)}| \\
\psi_{\alpha(s+1)\dot\alpha(s)} &\equiv \frac{1}{s!(s+1)!}\mathrm{D}_{(\alpha_{s+1}}\bar{\mathrm{D}}_{(\dot\alpha_s}\Psi_{\alpha(s))\dot\alpha(s-1))}| \\
\psi_{\alpha(s)\dot\alpha(s-1)} &\equiv \frac{1}{s!}\bar{\mathrm{D}}^{\dot\alpha_s}\mathrm{D}_{(\alpha_s}\bar\Psi_{\alpha(s-1))\dot\alpha(s)}| \\
\psi_{\alpha(s-1)\dot\alpha(s-2)} &\equiv -\frac{s-1}{s}\mathrm{D}^2\bar{\mathrm{D}}^{\dot\alpha_{s-1}}V_{\alpha(s-1)\dot\alpha(s-1)}|
\end{aligned}
\tag{4.19}
$$

The Lagrangian is

$$
\begin{aligned}
\mathcal{L}_F = \ & \rho^{\alpha(s)\dot\alpha(s-1)}\beta_{\alpha(s)\dot\alpha(s-1)} + c.c. \\
& +i\ \bar\psi^{\alpha(s)\dot\alpha(s+1)}\partial^{\alpha_{s+1}}{}_{\dot\alpha_{s+1}}\psi_{\alpha(s+1)\dot\alpha(s)} \\
& +i\left[\frac{s}{s+1}\right]\ \psi^{\alpha(s+1)\dot\alpha(s)}\partial_{\alpha_{s+1}\dot\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\
& -i\left[\frac{2s+1}{(s+1)^2}\right]\ \bar\psi^{\alpha(s-1)\dot\alpha(s)}\partial^{\alpha_s}{}_{\dot\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1)} \\
& +i\ \psi^{\alpha(s)\dot\alpha(s-1)}\partial_{\alpha_s\dot\alpha_{s-1}}\psi_{\alpha(s-1)\dot\alpha(s-2)} + c.c. \\
& -i\ \bar\psi^{\alpha(s-2)\dot\alpha(s-1)}\partial^{\alpha_{s-1}}{}_{\dot\alpha_{s-1}}\psi_{\alpha(s-1)\dot\alpha(s-2)}
\end{aligned}
\tag{4.20}
$$

and the gauge transformations of the fields are

$$
\begin{aligned}
\delta_G\rho_{\alpha(s)\dot\alpha(s-1)} &= 0\,, & \delta_G\psi_{\alpha(s+1)\dot\alpha(s)} &= \frac{1}{s!(s+1)!}\partial_{(\alpha_{s+1}(\dot\alpha_s}\xi_{\alpha(s))\dot\alpha(s-1))} \\
\delta_G\beta_{\alpha(s)\dot\alpha(s-1)} &= 0\,, & \delta_G\psi_{\alpha(s)\dot\alpha(s-1)} &= -\frac{1}{s!}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar\xi_{\alpha(s-1))\dot\alpha(s)} \\
& & \delta_G\psi_{\alpha(s-1)\dot\alpha(s-2)} &= \frac{s-1}{s}\partial^{\alpha_s\dot\alpha_{s-1}}\xi_{\alpha(s)\dot\alpha(s-1)} \\
& & \text{with } \xi_{\alpha(s)\dot\alpha(s-1)} &= -i\mathrm{D}^2 L_{\alpha(s)\dot\alpha(s-1)}|
\end{aligned}
\tag{4.21}
$$

[7]We are following the conventions of [24] which differ from the conventions used in [23].

4.2.2 Bosons

For the bosonic action we follow exactly the same procedure as was presented for the fermionic sector. The fields that appear in the action are defined as:

$$\begin{aligned}
U_{\alpha(s+1)\dot\alpha(s-1)} &\equiv \frac{1}{(s+1)!}\mathrm{D}_{(\alpha(s+1)}T_{\alpha(s))\dot\alpha(s-1)}|\\
u_{\alpha(s)\dot\alpha(s)} &\equiv \frac{1}{2s!}\left\{\bar{\mathrm{D}}_{(\dot\alpha_s}T_{\alpha(s)\dot\alpha(s-1))} - \mathrm{D}_{(\alpha_s}\bar{T}_{\alpha(s-1))\dot\alpha(s)}\right\}|\\
v_{\alpha(s)\dot\alpha(s)} &\equiv -\frac{i}{2s!}\left\{\bar{\mathrm{D}}_{(\dot\alpha_s}T_{\alpha(s)\dot\alpha(s-1))} + \mathrm{D}_{(\alpha_s}\bar{T}_{\alpha(s-1))\dot\alpha(s)}\right\}|\\
A_{\alpha(s-1)\dot\alpha(s-1)} &\equiv G_{\alpha(s-1)\dot\alpha(s-1)}| - \frac{s}{2s+1}\left(\mathrm{D}^{\alpha_s}T_{\alpha(s)\dot\alpha(s-1)} + \bar{\mathrm{D}}^{\dot\alpha_s}\bar{T}_{\alpha(s)\dot\alpha(s-1)}\right)|\\
S_{\alpha(s-1)\dot\alpha(s-1)} &\equiv \frac{1}{2}\left\{\mathrm{D}^{\alpha_s}T_{\alpha(s)\dot\alpha(s-1)} + \bar{\mathrm{D}}^{\dot\alpha_s}\bar{T}_{\alpha(s)\dot\alpha(s-1)}\right\}|\\
P_{\alpha(s-1)\dot\alpha(s-1)} &\equiv -\frac{i}{2}\left\{\mathrm{D}^{\alpha_s}T_{\alpha(s)\dot\alpha(s-1)} - \bar{\mathrm{D}}^{\dot\alpha_s}\bar{T}_{\alpha(s)\dot\alpha(s-1)}\right\}|\\
h_{\alpha(s)\dot\alpha(s)} &\equiv \frac{1}{\sqrt{2}}\left\{\frac{1}{s!}\mathrm{D}_{(\alpha_s}\bar{\Psi}_{\alpha(s-1))\dot\alpha(s)} - \frac{1}{s!}\bar{\mathrm{D}}_{(\dot\alpha_s}\Psi_{\alpha(s)\dot\alpha(s-1))}\right.\\
&\quad\left. - \frac{1}{2s!s!}\left[\mathrm{D}_{(\alpha_s},\bar{\mathrm{D}}_{(\dot\alpha_s}\right]V_{\alpha(s-1))\dot\alpha(s-1))}\right\}|\\
h_{\alpha(s-2)\dot\alpha(s-2)} &\equiv -\frac{1}{2\sqrt{2}}\frac{s-1}{s^2}\left[\mathrm{D}^{\alpha_{s-1}},\bar{\mathrm{D}}^{\dot\alpha_{s-1}}\right]V_{\alpha(s-1)\dot\alpha(s-1)}|
\end{aligned} \tag{4.22}$$

the gauge transformations are

$$\begin{aligned}
\delta_G U_{\alpha(s+1)\dot\alpha(s-1)} &= 0, & \delta_G A_{\alpha(s-1)\dot\alpha(s-1)} &= 0\\
\delta_G u_{\alpha(s)\dot\alpha(s)} &= 0, & \delta_G S_{\alpha(s-1)\dot\alpha(s-1)} &= 0\\
\delta_G v_{\alpha(s)\dot\alpha(s)} &= 0, & \delta_G P_{\alpha(s-1)\dot\alpha(s-1)} &= 0\\
\delta_G h_{\alpha(s)\dot\alpha(s)} &= \frac{1}{s!s!}\partial_{(\alpha_s(\dot\alpha_s}\zeta_{\alpha(s-1))\dot\alpha(s-1))}\\
\delta_G h_{\alpha(s-2)\dot\alpha(s-2)} &= \frac{s-1}{s^2}\partial^{\alpha_{s-1}\dot\alpha_{s-1}}\zeta_{\alpha(s-1)\dot\alpha(s-1)}
\end{aligned} \tag{4.23}$$

where

$$\zeta_{\alpha(s-1)\dot\alpha(s-1)} = \frac{i}{2\sqrt{2}}\left(\mathrm{D}^{\alpha_s}L_{\alpha(s)\dot\alpha(s-1)} - \bar{\mathrm{D}}^{\dot\alpha_s}\bar{L}_{\alpha(s-1)\dot\alpha(s)}\right)$$

and the Lagrangian is

$$\begin{aligned}
\mathcal{L}_B = &-\frac{1}{2}\,U^{\alpha(s+1)\dot\alpha(s-1)}U_{\alpha(s+1)\dot\alpha(s-1)} + c.c.\\
&+ u^{\alpha(s)\dot\alpha(s)}u_{\alpha(s)\dot\alpha(s)}\\
&+ v^{\alpha(s)\dot\alpha(s)}v_{\alpha(s)\dot\alpha(s)}\\
&- \left[\frac{2s+1}{4s}\right]A^{\alpha(s-1)\dot\alpha(s-1)}A_{\alpha(s-1)\dot\alpha(s-1)}\\
&- \left[\frac{s^2}{(2s+1)(s+1)}\right]S^{\alpha(s-1)\dot\alpha(s-1)}S_{\alpha(s-1)\dot\alpha(s-1)}\\
&- \left[\frac{s^2}{s+1}\right]P^{\alpha(s-1)\dot\alpha(s-1)}P_{\alpha(s-1)\dot\alpha(s-1)}
\end{aligned}$$

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$$
\begin{aligned}
&+ h^{\alpha(s)\dot\alpha(s)} \Box h_{\alpha(s)\dot\alpha(s)} \\
&- \frac{s}{2}\; h^{\alpha(s)\dot\alpha(s)} \partial_{\alpha_s\dot\alpha_s} \partial^{\gamma\dot\gamma} h_{\gamma\alpha(s-1)\dot\gamma\dot\alpha(s-1)} \\
&+ s(s-1)\; h^{\alpha(s)\dot\alpha(s)} \partial_{\alpha_s\dot\alpha_s} \partial_{\alpha_{s-1}\dot\alpha_{s-1}} h_{\alpha(s-2)\dot\alpha(s-2)} \\
&- s(2s-1)\; h^{\alpha(s-2)\dot\alpha(s-2)} \Box h_{\alpha(s-2)\dot\alpha(s-2)} \\
&- \left[\frac{s(s-2)^2}{2}\right] h^{\alpha(s-2)\dot\alpha(s-2)} \partial_{\alpha_{s-2}\dot\alpha_{s-2}} \partial^{\gamma\dot\gamma} h_{\gamma\alpha(s-3)\dot\gamma\dot\alpha(s-3)}
\end{aligned}
$$

4.2.3 Off-shell degrees of freedom

Let us count the bosonic degrees of freedom of the theory:

fields	*d.o.f*	*redundancy*	*net*
$h_{\alpha(s)\dot\alpha(s)}$	$(s+1)^2$	s^2	s^2+2
$h_{\alpha(s-2)\dot\alpha(s-2)}$	$(s-1)^2$		
$u_{\alpha(s)\dot\alpha(s)}$	$(s+1)^2$	0	$(s+1)^2$
$v_{\alpha(s)\dot\alpha(s)}$	$(s+1)^2$	0	$(s+1)^2$
$A_{\alpha(s-1)\dot\alpha(s-1)}$	s^2	0	s^2
$U_{\alpha(s+1)\dot\alpha(s-1)}$	$2(s+2)s$	0	$2(s+2)s$
$S_{\alpha(s-1)\dot\alpha(s-1)}$	s^2	0	s^2
$P_{\alpha(s-1)\dot\alpha(s-1)}$	s^2	0	s^2
		Total	$8s^2+8s+4$

and the same counting for the Fermionic degrees of freedom:

fields	*d.o.f*	*redundancy*	*net*
$\psi_{\alpha(s+1)\dot\alpha(s)}$	$2(s+2)(s+1)$	$2(s+1)s$	$4s^2+4s+4$
$\psi_{\alpha(s)\dot\alpha(s-1)}$	$2(s+1)s$		
$\psi_{\alpha(s-1)\dot\alpha(s-2)}$	$2s(s-1)$		
$\rho_{\alpha(s)\dot\alpha(s-1)}$	$2(s+1)s$	0	$2(s+1)s$
$\beta_{\alpha(s)\dot\alpha(s-1)}$	$2(s+1)s$	0	$2(s+1)s$
		Total	$8s^2+8s+4$

4.2.4 SUSY-transformation laws

The last thing left to do is to find explicit expressions for the SUSY-transformation laws of the fields. The transformation under susy can be easily calculated by the action of the SUSY-generators on the specific component. In terms of the covariant derivatives $\mathrm{D}(\bar{\mathrm{D}})$ we see that

$$
\delta_S \text{Component} = -\left(\epsilon^\beta \mathrm{D}_\beta + \bar\epsilon^{\dot\beta} \bar{\mathrm{D}}_{\dot\beta}\right) \text{Component}| \,.
$$

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But not all the fields are on equal footing. The dynamical ones ($\in \mathcal{D}$) are treated as equivalence classes, in other words they have a gauge transformation of the form $\{\mathcal{D}\} \sim \{\mathcal{D}\} + \partial\,(\zeta)$. Hence when we apply the susy transformation they will possess an extra term in the gauge parameter space

$$\delta_S\{\mathcal{D}\} \sim \delta_S\{\mathcal{D}\} + \partial\,(\delta_S\zeta)$$

This says that we must identify these two classes as well, therefore we can ignore any terms in the transformation law of the dynamical fields that have the same structure as their gauge transformation.

With all that in mind we find for the transformation of the fermionic fields:

$$\begin{aligned}
\delta_S\rho_{\alpha(s)\dot\alpha(s-1)} =& -\epsilon^{\alpha_{s+1}}U_{\alpha(s+1)\dot\alpha(s-1)} \\
&+\frac{s}{(s+1)!}\epsilon_{(\alpha_s}\left[S_{\alpha(s-1))\dot\alpha(s-1)}+iP_{\alpha(s-1))\dot\alpha(s-1)}\right] \\
&-\bar\epsilon^{\dot\alpha_s}\left[u_{\alpha(s)\dot\alpha(s)}+iv_{\alpha(s)\dot\alpha(s)}\right] \qquad (4.24)
\end{aligned}$$

$$\begin{aligned}
\delta_S\beta_{\alpha(s)\dot\alpha(s-1)} =& -i\bar\epsilon^{\dot\beta}\partial^{\alpha_{s+1}}{}_{\dot\beta}U_{\alpha(s+1)\dot\alpha(s-1)} \\
&-\frac{i}{2s!}\bar\epsilon^{\dot\alpha_{s+1}}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar U_{\alpha(s-1))\dot\alpha(s+1)} \\
&+\frac{i}{2s!}\epsilon^{\beta}\partial_{(\alpha_s}{}^{\dot\alpha_s}\left[u_{\beta\alpha(s-1))\dot\alpha(s)}-iv_{\beta\alpha(s-1))\dot\alpha(s)}\right] \\
&+\frac{i}{2}\frac{1}{s!s!}\bar\epsilon^{\dot\alpha_s}\partial_{(\alpha_s(\dot\alpha_s}A_{\alpha(s-1))\dot\alpha(s-1))} \\
&+\frac{i}{2}\left[\frac{2s^2-1}{(s+1)(2s+1)}\right]\frac{1}{s!s!}\bar\epsilon^{\dot\alpha_s}\partial_{(\alpha_s(\dot\alpha_s}S_{\alpha(s-1))\dot\alpha(s-1))} \\
&+\frac{1}{2}\left[\frac{2s^2-2s-1}{s+1}\right]\frac{1}{s!s!}\bar\epsilon^{\dot\alpha_s}\partial_{(\alpha_s(\dot\alpha_s}P_{\alpha(s-1))\dot\alpha(s-1))} \\
&-\frac{i}{2}\left[\frac{(s-1)^2}{s(s+1)}\right]\frac{1}{s!(s-1)!}\bar\epsilon_{(\dot\alpha_{s-1}}\partial_{(\alpha_s}{}^{\dot\gamma}S_{\alpha(s-1))\dot\gamma\dot\alpha(s-2))} \\
&+\frac{1}{2}\left[\frac{(s-1)(3s+1)}{s(s+1)}\right]\frac{1}{s!(s-1)!}\bar\epsilon_{(\dot\alpha_{s-1}}\partial_{(\alpha_s}{}^{\dot\gamma}P_{\alpha(s-1))\dot\gamma\dot\alpha(s-2))} \\
&-\sqrt{2}\bar\epsilon^{\dot\alpha_s}\Box h_{\alpha(s)\dot\alpha(s)} \\
&+\frac{s}{\sqrt{2}}\frac{1}{s!s!}\bar\epsilon^{\dot\alpha_s}\partial_{(\alpha_s(\dot\alpha_s}\partial^{\gamma\dot\gamma}h_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-1))} \\
&-\frac{s(s-1)}{\sqrt{2}}\frac{1}{s!s!}\bar\epsilon^{\dot\alpha_s}\partial_{(\alpha_s(\dot\alpha_s}\partial_{\alpha_{s-1}\dot\alpha_{s-1}}h_{\alpha(s-2))\dot\alpha(s-2))} \qquad (4.25)
\end{aligned}$$

$$\begin{aligned}
\delta_S\psi_{\alpha(s+1)\dot\alpha(s)} =& -\frac{1}{s!}\bar\epsilon_{(\dot\alpha_s}U_{\alpha(s+1)\dot\alpha(s-1))} \\
&-\frac{1}{(s+1)!}\epsilon_{(\alpha_{s+1}}\left[u_{\alpha(s))\dot\alpha(s)}-iv_{\alpha(s))\dot\alpha(s)}\right] \\
&+\frac{i\sqrt{2}}{(s+1)!}\bar\epsilon^{\dot\beta}\partial_{(\alpha_{s+1}\dot\beta}h_{\alpha(s))\dot\alpha(s)} \qquad (4.26)
\end{aligned}$$

$$
\begin{aligned}
\delta_S \psi_{\alpha(s)\dot\alpha(s-1)} = & \bar\epsilon^{\dot\alpha_s}\left[u_{\alpha(s)\dot\alpha(s)} + i v_{\alpha(s)\dot\alpha(s)}\right] \\
& - \frac{1}{s!}\frac{s}{2s+1}\epsilon_{(\alpha_s} S_{\alpha(s-1))\dot\alpha(s-1)} \\
& - \frac{is}{s!}\epsilon_{(\alpha_s} P_{\alpha(s-1))\dot\alpha_(s-1)} \\
& + \frac{1}{s!}\frac{s+1}{2s}\epsilon_{(\alpha_s} A_{\alpha(s-1))\dot\alpha(s-1)} \\
& + i\frac{s-1}{\sqrt{2}}\epsilon^{\beta}\partial_{\beta}{}^{\dot\alpha_s} h_{\alpha(s)\dot\alpha(s)} \\
& + i\frac{(s+1)s(s-1)}{\sqrt{2}s!s!}\epsilon_{(\alpha_s}\partial_{\alpha_{s-1}(\dot\alpha_{s-1}} h_{\alpha(s-2))\dot\alpha(s-2))}
\end{aligned} \tag{4.27}
$$

$$
\begin{aligned}
\delta_S \psi_{\alpha(s-1)\dot\alpha(s-2)} = & \frac{1}{2}\frac{(s-1)(2s+1)}{s^2}\bar\epsilon^{\dot\alpha_{s-1}} A_{\alpha(s-1)\dot\alpha(s-1)} \\
& + \frac{i}{\sqrt{2}}\frac{(s-1)^2}{s}\frac{1}{(s-1)!^2}\bar\epsilon^{\dot\alpha_{s-1}}\partial_{(\alpha_{s-1}(\dot\alpha_{s-1}} h_{\alpha(s-2))\dot\alpha(s-2))} \\
& - i\sqrt{2}\frac{(s-1)^2}{s}\frac{1}{(s-1)!^2}\partial_{(\alpha_{s-1}}{}^{\dot\alpha_{s-1}}\bar\epsilon_{(\dot\alpha_{s-1}} h_{\alpha(s-2))\dot\alpha(s-2))}
\end{aligned} \tag{4.28}
$$

The SUSY-transformation laws for the bosonic fields are:

$$
\begin{aligned}
\delta_S U_{\alpha(s+1)\dot\alpha(s-1)} = & \frac{1}{(s+1)!}\epsilon_{(\alpha_{s+1}}\beta_{\alpha(s))\dot\alpha(s-1)} \\
& - \frac{i}{2}\frac{1}{(s+1)!}\epsilon_{(\alpha_{s+1}}\partial_{\alpha_s}{}^{\dot\alpha_s}\bar\rho_{\alpha(s-1))\dot\alpha(s)} \\
& - \frac{i}{(s+1)!}\bar\epsilon^{\dot\beta}\partial_{(\alpha_{s+1}\dot\beta}\rho_{\alpha(s))\dot\alpha(s-1)} \\
& - \frac{i}{(s+1)!}\bar\epsilon^{\dot\alpha_s}\partial_{(\alpha_{s+1}}{}^{\dot\alpha_{s+1}}\bar\psi_{\alpha(s))\dot\alpha(s+1)} \\
& - i\frac{s}{s+1}\frac{1}{(s+1)!s!}\bar\epsilon^{\dot\alpha_s}\partial_{(\alpha_{s+1}(\dot\alpha_s}\psi_{\alpha(s))\dot\alpha(s-1))}
\end{aligned} \tag{4.29}
$$

$$
\begin{aligned}
\delta_S\left(u_{\alpha(s)\dot\alpha(s)} + i v_{\alpha(s)\dot\alpha(s)}\right) = & \frac{i}{(s+1)!}\epsilon^{\alpha_{s+1}}\partial_{(\alpha_{s+1}}{}^{\dot\alpha_{s+1}}\bar\psi_{\alpha(s))\dot\alpha(s+1)} \\
& - i\frac{s}{s+1}\frac{1}{s!}\epsilon_{(\alpha_s}\partial^{\gamma\dot\alpha_{s+1}}\bar\psi_{\gamma\alpha(s-1))\dot\alpha(s+1)} \\
& + i\frac{s}{s+1}\frac{1}{(s+1)!s!}\epsilon^{\alpha_{s+1}}\partial_{(\alpha_{s+1}(\dot\alpha_s}\psi_{\alpha(s))\dot\alpha(s-1))} \\
& + i\frac{2s+1}{(s+1)^2}\frac{1}{s!s!}\epsilon_{(\alpha_s}\partial^{\gamma}{}_{(\dot\alpha_s}\psi_{\gamma\alpha(s-1))\dot\alpha(s-1))} \\
& + i\frac{1}{s!s!}\epsilon_{(\alpha_s}\partial_{\alpha_{s-1}(\dot\alpha_s}\bar\psi_{\alpha(s-2))\dot\alpha(s-1))} \\
& + \frac{1}{s!}\epsilon_{(\alpha_s}\bar\beta_{\alpha(s-1))\dot\alpha(s)} \\
& + \frac{i}{2}\frac{1}{s!s!}\epsilon_{(\alpha_s}\partial^{\gamma}{}_{(\dot\alpha_s}\rho_{\gamma\alpha(s-1))\dot\alpha(s-1))}
\end{aligned} \tag{4.30}
$$

$$\begin{aligned}
\delta_S A_{\alpha(s-1)\dot\alpha(s-1)} =& -\frac{i}{2s+1}\frac{1}{s!}\bar\epsilon^{\dot\alpha_s}\partial^{\alpha_s}{}_{(\dot\alpha_s}\rho_{\alpha(s)\dot\alpha(s-1)}+c.c.\\
&+i\frac{(s-1)(s+1)}{s(2s+1)}\frac{1}{(s-1)!}\bar\epsilon_{(\dot\alpha_{s-1}}\partial^{\alpha_s\dot\gamma}\rho_{\alpha(s)\dot\gamma\dot\alpha(s-2))}+c.c.\\
&+i\frac{s}{2s+1}\bar\epsilon^{\dot\alpha_s}\partial^{\alpha_s\dot\alpha_{s+1}}\bar\psi_{\alpha(s)\dot\alpha(s+1)}+c.c.\\
&-\frac{i}{s+1}\frac{1}{s!}\bar\epsilon^{\dot\alpha_s}\partial^{\alpha_s}{}_{(\dot\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1))}+c.c.\\
&+i\frac{s-1}{s!}\epsilon_{(\alpha_{s-1}}\partial^{\gamma\dot\alpha_s}\bar\psi_{\gamma\alpha(s-2))\dot\alpha(s)}+c.c.\\
&+i\frac{s+1}{2s+1}\frac{1}{(s-1)!s!}\bar\epsilon^{\dot\alpha_s}\partial_{(\alpha_{s-1}(\dot\alpha_s}\bar\psi_{\alpha(s-2))\dot\alpha(s-1))}+c.c.\\
&-i\frac{s-1}{s!(s-1)!}\epsilon_{(\alpha_{s-1}}\partial^{\gamma}{}_{(\dot\alpha_{s-1}}\psi_{\gamma\alpha(s-2))\dot\alpha(s-2))}+c.c.
\end{aligned}\tag{4.31}$$

$$\begin{aligned}
\delta_S\big(S_{\alpha(s-1)\dot\alpha(s-1)} &+iP_{\alpha(s-1)\dot\alpha(s-1)}\big)=\\
&=\epsilon^{\alpha_s}\beta_{\alpha(s)\dot\alpha(s-1)}\\
&+\frac{s+1}{s}\bar\epsilon^{\dot\alpha_s}\bar\beta_{\alpha(s-1)\dot\alpha(s)}\\
&-\frac{i}{2s!}\epsilon^{\alpha_s}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar\rho_{\alpha(s-1)\dot\alpha(s)}\\
&-i\frac{s-1}{2s}\frac{1}{s!}\bar\epsilon^{\dot\alpha_s}\partial^{\alpha_s}{}_{(\dot\alpha_s}\rho_{\alpha(s)\dot\alpha(s-1))}\\
&+i\frac{s-1}{s!}\bar\epsilon_{(\dot\alpha_{s-1}}\partial^{\alpha_s\dot\gamma}\rho_{\alpha(s)\dot\gamma\dot\alpha(s-2))}\\
&-i\bar\epsilon^{\dot\alpha_s}\partial^{\alpha_s\dot\alpha_{s+1}}\bar\psi_{\alpha(s)\dot\alpha(s+1)}\\
&+i\frac{2s+1}{s(s+1)}\frac{1}{s!}\bar\epsilon^{\dot\alpha_s}\partial^{\alpha_s}{}_{(\dot\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1)}\\
&+i\frac{s+1}{s}\frac{1}{(s-1)!s!}\bar\epsilon^{\dot\alpha_s}\partial_{(\alpha_{s-1}(\dot\alpha_s}\bar\psi_{\alpha(s-2))\dot\alpha(s-1))}
\end{aligned}\tag{4.32}$$

$$\begin{aligned}
\delta_S h_{\alpha(s)\dot\alpha(s)} =&\frac{1}{\sqrt2 s!}\epsilon_{(\alpha_s}\bar\rho_{\alpha(s-1))\dot\alpha(s)}+c.c.\\
&+\frac{1}{\sqrt2}\bar\epsilon^{\dot\alpha_{s+1}}\bar\psi_{\alpha(s)\dot\alpha(s+1)}+c.c.\\
&-\frac{1}{\sqrt2(s+1)}\frac{1}{s!}\bar\epsilon_{(\dot\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1))}+c.c.
\end{aligned}\tag{4.33}$$

$$\delta_S h_{\alpha(s-2)\dot\alpha(s-2)} = -\frac{1}{\sqrt2 s}\epsilon^{\alpha_{s-1}}\psi_{\alpha(s-1)\dot\alpha(s-2)}+c.c.\tag{4.34}$$

5 Half odd integer superhelicity theories

Now we repeat the entire procedure for the half odd integer superhelicity irreducible representations. Unlike the integer case we will see that there are two different classes of theories that describe this type of physical systems. For this case the action will be constructed by equivalence class of $[H_{\alpha(s)\dot\alpha(s)}]$.

5.1 The superspace action

Superfield H must have mass dimension zero[8] and the action must involve four covariant derivatives. The most general action we can write is

$$
\begin{aligned}
S = \int d^8z a_1 & H^{\alpha(s)\dot\alpha(s)} \mathrm{D}^\gamma \bar{\mathrm{D}}^2 \mathrm{D}_\gamma H_{\alpha(s)\dot\alpha(s)} \\
& + a_2 H^{\alpha(s)\dot\alpha(s)} \left\{\mathrm{D}^2, \bar{\mathrm{D}}^2\right\} H_{\alpha(s)\dot\alpha(s)} \\
& + a_3 H^{\alpha(s)\dot\alpha(s)} \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}^2 \mathrm{D}^\gamma H_{\gamma\alpha(s-1)\dot\alpha(s)} + c.c. \\
& + a_4 H^{\alpha(s)\dot\alpha(s)} \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}_{\dot\alpha_s} \mathrm{D}^\gamma \bar{\mathrm{D}}^{\dot\gamma} H_{\gamma\alpha(s-1)\dot\gamma\dot\alpha(s-1)} + c.c.
\end{aligned} \tag{5.1}
$$

The deformation of the action under the redundancy is:

$$
\begin{aligned}
\delta_G S = \int d^8z \Bigg[& \Big(-2a_1 + 2\frac{s+1}{s} a_3 + 2a_4 \Big) \mathrm{D}^2 \bar{\mathrm{D}}_{\dot\alpha_s} H^{\alpha(s)\dot\alpha(s)} \\
& + (-2a_3 - \frac{s+1}{s} a_4) \mathrm{D}^{\alpha_s} \bar{\mathrm{D}}_{\dot\gamma} \mathrm{D}_\gamma H^{\gamma\alpha(s-1)\dot\gamma\dot\alpha(s-1)} \Bigg] \big(\bar{\mathrm{D}}^2 L_{\alpha(s)\dot\alpha(s-1)} \\
& + \mathrm{D}^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot\alpha(s-1)} \big) \\
& + 2a_2 H^{\alpha(s)\dot\alpha(s)} \mathrm{D}^2 \bar{\mathrm{D}}^2 \mathrm{D}_{\alpha_s} \bar{L}_{\alpha(s-1)\dot\alpha(s)} \\
& - 2a_4 \bar{\mathrm{D}}_{\dot\beta} \mathrm{D}_\gamma \bar{\mathrm{D}}_{\dot\gamma} H^{\gamma\alpha(s-1)\dot\beta\dot\gamma\dot\alpha(s-2)} \Bigg[\bar{\mathrm{D}}^{\dot\alpha_{s-1}} \mathrm{D}^{\alpha_s} L_{\alpha(s)\dot\alpha(s-1)} \\
& + \frac{s-1}{s} \mathrm{D}^{\alpha_s} \bar{\mathrm{D}}^{\dot\alpha_{s-1}} L_{a(s)\dot\alpha(s-1)} \\
& + \bar{\mathrm{D}}_{\dot\alpha_{s-2}} J_{\alpha(s-1)\dot\alpha(s-3)} \Bigg] \\
& + c.c.
\end{aligned} \tag{5.2}
$$

Notice that because of the D-algebra we have the freedom to add terms like $\mathrm{D}^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot\alpha(s-1)}$ and $\bar{\mathrm{D}}_{\dot\alpha_{s-2}} J_{\alpha(s-1)\dot\alpha(s-3)}$ which identically vanish and they don't effect the result.

Obviously we can not set the variation of the action to zero just by picking values for the a's without setting them all to zero, but we can introduce compensators with proper mass dimensionality and index structure. There are two different ways to do that

- (I) Choose coefficients to kill the last two terms ($a_2 = a_4 = 0$) and introduce a compensator that cancels the first term
- (II) Choose coefficients to kill the first two terms ($-2a_1 + 2\frac{s+1}{s}a_3 + 2a_4 = 0$, $-2a_3 - \frac{s+1}{s}a_4$, $a_2 = 0$) and introduce a compensator to cancel the last term

These two different approaches will lead to the two different formulations of half-integer super-helicity, mentioned above.

[8]Its highest spin component is a propagating boson.

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5.1.1 Case (I) — transverse theory

For case (I) we find

$$
\begin{aligned}
a_2 &= a_4 = 0 \\
\delta_G S &= \int d^8 z \left[\left(-2a_1 + 2\frac{s+1}{s} a_3 \right) \mathrm{D}^2 \bar{\mathrm{D}}_{\dot{\alpha}_s} H^{\alpha(s)\dot{\alpha}(s)} \right. \\
&\quad \left. + -2a_3 \mathrm{D}^{\alpha_s} \bar{\mathrm{D}}_{\dot{\gamma}} \mathrm{D}_\gamma H^{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \right] \left(\bar{\mathrm{D}}^2 L_{\alpha(s)\dot{\alpha}(s-1)} \right. \\
&\quad \left. + \mathrm{D}^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)} \right)
\end{aligned} \tag{5.3}
$$

This suggests us to introduce a fermionic compensator $\chi_{\alpha(s)\dot{\alpha}(s-1)}$ which transforms like $\delta_G \chi_{\alpha(s)\dot{\alpha}(s-1)} = \bar{\mathrm{D}}^2 L_{\alpha(s)\dot{\alpha}(s-1)} + \mathrm{D}^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}$. So in order to obtain invariance we add to the action two new pieces: the coupling term of H with χ and the kinetic energy terms for χ. The full action takes the form

$$
\begin{aligned}
S = \int & d^8 z a_1 H^{\alpha(s)\dot{\alpha}(s)} \mathrm{D}^\gamma \bar{\mathrm{D}}^2 \mathrm{D}_\gamma H_{\alpha(s)\dot{\alpha}(s)} \\
& + a_3 H^{\alpha(s)\dot{\alpha}(s)} \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}^2 \mathrm{D}^\gamma H_{\gamma\alpha(s-1)\dot{\alpha}(s)} + c.c. \\
& - \left(2a_1 - 2\frac{s+1}{s} a_3 \right) H^{\alpha(s)\dot{\alpha}(s)} \bar{\mathrm{D}}_{\dot{\alpha}_s} \mathrm{D}^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + 2a_3 H^{\alpha(s)\dot{\alpha}(s)} \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}_{\dot{\alpha}_s} \mathrm{D}^\gamma \chi_{\gamma\alpha(s-1)\dot{\alpha}(s-1)} + c.c. \\
& + b_1 \chi^{\alpha(s)\dot{\alpha}(s-1)} \mathrm{D}^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + b_2 \chi^{\alpha(s)\dot{\alpha}(s-1)} \bar{\mathrm{D}}^2 \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + b_3 \chi^{\alpha(s)\dot{\alpha}(s-1)} \bar{\mathrm{D}}^{\dot{\alpha}_s} \mathrm{D}_{\alpha_s} \bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + b_4 \chi^{\alpha(s)\dot{\alpha}(s-1)} \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}^{\dot{\alpha}_s} \bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)}
\end{aligned} \tag{5.4}
$$

and it has to be invariant under

$$
\delta_G H_{\alpha(s)\dot{\alpha}(s)} = \frac{1}{s!} \mathrm{D}_{(\alpha_s} \bar{L}_{\alpha(s-1))\dot{\alpha}(s)} - \frac{1}{s!} \bar{\mathrm{D}}_{(\dot{\alpha}_s} L_{\alpha(s)\dot{\alpha}(s-1))} \tag{5.5a}
$$

$$
\delta_G \chi_{\alpha(s)\dot{\alpha}(s-1)} = \bar{\mathrm{D}}^2 L_{\alpha(s)\dot{\alpha}(s-1)} + \mathrm{D}^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)} \tag{5.5b}
$$

The equations of motion of the superfields are the variation of the action with respect the superfield

$$
T_{\alpha(s)\dot{\alpha}(s)} = \frac{\delta S}{\delta H^{\alpha(s)\dot{\alpha}(s)}}, \quad G_{\alpha(s)\dot{\alpha}(s-1)} = \frac{\delta S}{\delta \chi^{\alpha(s)\dot{\alpha}(s-1)}} \tag{5.6}
$$

and the invariance of the action gives the following Bianchi Identities

$$
\bar{\mathrm{D}}^{\dot{\alpha}_s} T_{\alpha(s)\dot{\alpha}(s)} - \bar{\mathrm{D}}^2 G_{\alpha(s))\dot{\alpha}(s-1)} = 0 \tag{5.7a}
$$

$$
\frac{1}{(s+1)!} \mathrm{D}_{(\alpha_{s+1}} G_{a(s))\dot{\alpha}(s-1)} = 0 \tag{5.7b}
$$

The Bianchi identities fix all the coefficients

$$
\begin{aligned}
a_3 &= 0, & b_3 &= 0 \\
b_1 &= -\frac{s+1}{s} a_1, & b_4 &= 2a_1 \\
b_2 &= 0
\end{aligned}
$$

and the final form of the action is:

$$
\begin{aligned}
S = \int d^8 z \Big\{ & c\ H^{\alpha(s)\dot\alpha(s)} \mathrm{D}^\gamma \bar{\mathrm{D}}^2 \mathrm{D}_\gamma H_{\alpha(s)\dot\alpha(s)} \\
& -2c\ H^{\alpha(s)\dot\alpha(s)} \bar{\mathrm{D}}_{\dot\alpha_s} \mathrm{D}^2 \chi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\
& -\frac{s+1}{s} c\ \chi^{\alpha(s)\dot\alpha(s-1)} \mathrm{D}^2 \chi_{\alpha(s)\dot\alpha(s-1)} + c.c. \\
& + 2c\ \chi^{\alpha(s)\dot\alpha(s-1)} \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}^{\dot\alpha_s} \bar\chi_{\alpha(s-1)\dot\alpha(s)} \Big\}
\end{aligned} \tag{5.8}
$$

The expressions for the equations of motion are:

$$
\begin{aligned}
T_{\alpha(s)\dot\alpha(s)} &= 2c \mathrm{D}^\gamma \bar{\mathrm{D}}^2 \mathrm{D}_\gamma H_{\alpha(s)\dot\alpha(s)} \\
&\quad + \frac{2c}{s!} \left(\mathrm{D}_{(\alpha_s} \bar{\mathrm{D}}^2 \bar\chi_{\alpha(s-1))\dot\alpha(s)} - \bar{\mathrm{D}}_{(\dot\alpha_s} \mathrm{D}^2 \chi_{\alpha(s)\dot\alpha(s-1))} \right) && (5.9a) \\
G_{\alpha(s)\dot\alpha(s-1)} &= -2c \mathrm{D}^2 \bar{\mathrm{D}}^{\dot\alpha_s} H_{\alpha(s)\dot\alpha(s)} - 2c \frac{s+1}{s} \mathrm{D}^2 \chi_{\alpha(s)\dot\alpha(s-1)} \\
&\quad + \frac{2c}{s!} \mathrm{D}_{(\alpha_s} \bar{\mathrm{D}}^{\dot\alpha_s} \bar\chi_{\alpha(s-1))\dot\alpha(s)} && (5.9b)
\end{aligned}
$$

where c is a free overall parameter that can be absorbed in the definition of the superfields but for the moment we'll leave it as it is and fix it later when we define the components.

The above action is the same as the transversely-linear theory presented in [22] if we solve the constraints and express it in terms of the prepotential, but now we have an alternate understanding why we have to consider these types of superfields in order to construct the action and why they have these gauge transformation.

To prove that indeed this action describes the desired representation, using the equations of motion we can now show that a chiral superfield $F_{\alpha(2s+1)}$ exists and satisfies the following Bianchi identity

$$
\begin{aligned}
\mathrm{D}^{\alpha_{2s+1}} F_{\alpha(2s+1)} = & \frac{1}{2c} \frac{1}{(2s)!} \partial_{(\alpha_{2s}}{}^{\dot\alpha_s} \dots \partial_{\alpha_{s+1}}{}^{\dot\alpha_1} T_{\alpha(s))\dot\alpha(s)} \\
& + \frac{i}{2c} \frac{s}{2s+1} \frac{B}{B+\Delta} \frac{1}{(2s)!} \mathrm{D}_{(\alpha_{2s}} \bar{\mathrm{D}}^2 \partial_{\alpha_{2s-1}}{}^{\dot\alpha_{s-1}} \dots \partial_{\alpha_{s+1}}{}^{\dot\alpha_1} G_{\alpha(s))\dot\alpha(s-1)} \\
& + \frac{1}{2c} \frac{s}{2s+1} \frac{1}{(2s)!} \mathrm{D}_{(\alpha_{2s}} \partial_{\alpha_{2s-1}}{}^{\dot\alpha_s} \dots \partial_{\alpha_s}{}^{\dot\alpha_1} \bar{G}_{\alpha(s-1))\dot\alpha(s)} \\
& + \frac{i}{2c} \frac{s}{2s+1} \frac{\Delta}{B+\Delta} \frac{1}{(2s)!} \mathrm{D}_{(\alpha_{2s}} \bar{\mathrm{D}}^{\dot\alpha_s} \partial_{\alpha_{2s-1}}{}^{\dot\alpha_{s-1}} \dots \partial_{\alpha_{s+1}}{}^{\dot\alpha_1} T_{\alpha(s))\dot\alpha(s)}
\end{aligned} \tag{5.10}
$$

where

$$
F_{\alpha(2s+1)} = \frac{1}{(2s+1)!} \bar{\mathrm{D}}^2 \mathrm{D}_{(\alpha_{2s+1}} \partial_{\alpha_{2s}}{}^{\dot\alpha_s} \dots \partial_{\alpha_{s+1}}{}^{\dot\alpha_1} H_{\alpha(s))\dot\alpha(s)}
$$

On-shell where $T_{\alpha(s)\dot\alpha(s)} = G_{\alpha(s)\dot\alpha(s-1)} = 0$, we find the desired constraints to describe a super-helicity $Y = s + 1/2$ system. The constants B and Δ are only constrained by $B + \Delta \neq 0$.

Like in the integer super-helicity case, this action and superfield configuration are not unique, but a simple representative of a two parameter family of equivalent theories. To see that we perform redefinitions of the superfields. Dimensionality and index structure allow us to do the following redefinition of χ

$$\chi_{\alpha(s)\dot{\alpha}(s-1)} \to \chi_{\alpha(s)\dot{\alpha}(s-1)} + z\bar{\mathrm{D}}^{\dot{\alpha}_s} H_{\alpha(s)\dot{\alpha}(s)} \tag{5.11}$$

where z is a complex parameter. This operation will generate an entire class of actions and transformation laws which all are related by the above redefinition.

The generalized action is

$$\begin{aligned}
S = \int d^8wc\; & H^{\alpha(s)\dot{\alpha}(s)}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^2\mathrm{D}_{\gamma}H_{\alpha(s)\dot{\alpha}(s)} \\
& -2c\left[1+\frac{s+1}{s}z\right]\; H^{\alpha(s)\dot{\alpha}(s)}\bar{\mathrm{D}}_{\dot{\alpha}_s}\mathrm{D}^2\chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& -2c\bar{z}\; H^{\alpha(s)\dot{\alpha}(s)}\mathrm{D}_{\alpha_s}\bar{\mathrm{D}}_{\dot{\alpha}_s}\mathrm{D}^{\gamma}\chi_{\gamma\alpha(s-1)\dot{\alpha}(s-1)} + c.c. \\
& -2c\bar{z}\left[1+\frac{s+1}{s}\bar{z}\right]\; H^{\alpha(s)\dot{\alpha}(s)}\mathrm{D}_{\alpha_s}\bar{\mathrm{D}}^2\mathrm{D}^{\gamma}H_{\gamma\alpha(s-1)\dot{\alpha}(s)} + c.c. \\
& -c|z|^2\; H^{\alpha(s)\dot{\alpha}(s)}\mathrm{D}_{\alpha_s}\bar{\mathrm{D}}_{\dot{\alpha}_s}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^{\dot{\gamma}}H_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} + c.c. \\
& -\frac{s+1}{s}c\; \chi^{\alpha(s)\dot{\alpha}(s-1)}\mathrm{D}^2\chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& +2c\; \chi^{\alpha(s)\dot{\alpha}(s-1)}\mathrm{D}_{\alpha_s}\bar{\mathrm{D}}^{\dot{\alpha}_s}\bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)}
\end{aligned} \tag{5.12}$$

and the generalized transformation laws are

$$\delta_G H_{\alpha(s)\dot{\alpha}(s)} = \frac{1}{s!}\mathrm{D}_{(\alpha_s}\bar{L}_{\alpha(s-1))\dot{\alpha}(s)} - \frac{1}{s!}\bar{\mathrm{D}}_{(\dot{\alpha}_s}L_{\alpha(s)\dot{\alpha}(s-1))} \tag{5.13a}$$

$$\begin{aligned}
\delta_G\chi_{\alpha(s)\dot{\alpha}(s-1)} = & \left[1+\frac{s+1}{s}z\right]\bar{\mathrm{D}}^2L_{\alpha(s)\dot{\alpha}(s-1)} - \frac{z}{s!}\bar{\mathrm{D}}^{\dot{\alpha}_s}\mathrm{D}_{(\alpha_s}\bar{L}_{\alpha(s-1))\dot{\alpha}(s)} \\
& +\mathrm{D}^{\alpha_{s+1}}\Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}
\end{aligned} \tag{5.13b}$$

5.1.2 Case (II) — longitudinal theory

For case (II) we obtain the conditions

$$a_1 = c, \qquad a_2 = 0$$

$$a_3 = \frac{s(s+1)}{2s+1}c, \qquad a_4 = -\frac{s^2}{2s+1}c$$

and we have to introduce a fermionic compensator $\chi_{\alpha(s-1)\dot{\alpha}(s-2)}$ which transforms like

$$\begin{aligned}
\delta_G\chi_{\alpha(s-1)\dot{\alpha}(s-2)} = & \bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\mathrm{D}^{\alpha_s}L_{\alpha(s)\dot{\alpha}(s-1)} + \frac{s-1}{s}\mathrm{D}^{\alpha_s}\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}L_{a(s)\dot{\alpha}(s-1)} \\
& + \bar{\mathrm{D}}_{\dot{\alpha}_{s-2}}J_{\alpha(s-1)\dot{\alpha}(s-3)}
\end{aligned}$$

and couples with the term $\bar{\mathrm{D}}^{\dot{\beta}}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^{\dot{\gamma}}H_{\gamma\alpha(s-1)\dot{\beta}\dot{\gamma}\dot{\alpha}(s-2)}$

So in order to achieve invariance we add to the action two new pieces, the coupling term of H with χ and the kinetic energy terms for χ. The full action takes the form

$$
\begin{aligned}
S = \int & d^8z c \; H^{\alpha(s)\dot{\alpha}(s)}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^2\mathrm{D}_{\gamma}H_{\alpha(s)\dot{\alpha}(s)} \\
&+\frac{s(s+1)}{2s+1}c \; H^{\alpha(s)\dot{\alpha}(s)}\mathrm{D}_{\alpha_s}\bar{\mathrm{D}}^2\mathrm{D}^{\gamma}H_{\gamma\alpha(s-1)\dot{\alpha}(s)} + c.c. \\
&-\frac{s^2}{2s+1}c \; H^{\alpha(s)\dot{\alpha}(s)}\mathrm{D}_{\alpha_s}\bar{\mathrm{D}}_{\dot{\alpha}_s}\mathrm{D}^{\gamma}\bar{\mathrm{D}}^{\dot{\gamma}}H_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} + c.c. \\
&-\frac{2s^2}{2s+1}c \; H^{\alpha(s)\dot{\alpha}(s)}\bar{\mathrm{D}}_{\dot{\alpha}_s}\mathrm{D}_{\alpha_s}\bar{\mathrm{D}}_{\dot{\alpha}_{s-1}}\chi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\
&+b_1\chi^{\alpha(s-1)\dot{\alpha}(s-2)}\mathrm{D}^2\chi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\
&+b_2\chi^{\alpha(s-1)\dot{\alpha}(s-2)}\bar{\mathrm{D}}^2\chi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\
&+b_3\chi^{\alpha(s-1)\dot{\alpha}(s-2)}\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\mathrm{D}_{\alpha_{s-1}}\bar{\chi}_{\alpha(s-2)\dot{\alpha}(s-1)} \\
&+b_4\chi^{\alpha(s-1)\dot{\alpha}(s-2)}\mathrm{D}_{\alpha_{s-1}}\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\bar{\chi}_{\alpha(s-2)\dot{\alpha}(s-1)}
\end{aligned}
\tag{5.14}
$$

and it has to be invariant under

$$
\delta_G H_{\alpha(s)\dot{\alpha}(s)} = \frac{1}{s!}\mathrm{D}_{(\alpha_s}\bar{L}_{\alpha(s-1))\dot{\alpha}(s)} - \frac{1}{s!}\bar{\mathrm{D}}_{(\dot{\alpha}_s}L_{\alpha(s)\dot{\alpha}(s-1))} \tag{5.15a}
$$

$$
\begin{aligned}
\delta_G\chi_{\alpha(s-1)\dot{\alpha}(s-2)} = {} & \bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\mathrm{D}^{\alpha_s}L_{\alpha(s)\dot{\alpha}(s-1)} + \frac{s-1}{s}\mathrm{D}^{\alpha_s}\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}L_{a(s)\dot{\alpha}(s-1)} \\
& + \bar{\mathrm{D}}_{\dot{\alpha}_{s-2}}J_{\alpha(s-1)\dot{\alpha}(s-3)}
\end{aligned}
\tag{5.15b}
$$

The equations of motion of the superfields are

$$
T_{\alpha(s)\dot{\alpha}(s)} = \frac{\delta S}{\delta H^{\alpha(s)\dot{\alpha}(s)}}, \quad G_{\alpha(s-1)\dot{\alpha}(s-2)} = \frac{\delta S}{\delta\chi^{\alpha(s-1)\dot{\alpha}(s-2)}} \tag{5.16}
$$

and satisfy the Bianchi Identities

$$
\begin{aligned}
\bar{\mathrm{D}}^{\dot{\alpha}_s}T_{\alpha(s)\dot{\alpha}(s)} & + \frac{1}{s!(s-1)!}\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}_{(\dot{\alpha}_{s-1}}G_{\alpha(s-1)))\dot{\alpha}(s-2))} \\
& + \left[\frac{s-1}{s}\right]\frac{1}{s!(s-1)!}\bar{\mathrm{D}}_{(\dot{\alpha}_{s-1}}\mathrm{D}_{(\alpha_s}G_{\alpha(s-1))\dot{\alpha}(s-2))} = 0
\end{aligned}
\tag{5.17a}
$$

$$
\bar{\mathrm{D}}^{\dot{\alpha}_{s-2}}G_{a(s-1)\dot{\alpha}(s-2)} = 0 \;\rightsquigarrow\; \bar{\mathrm{D}}^2G_{a(s-1)\dot{\alpha}(s-2)} = 0 \tag{5.17b}
$$

which fix all free coefficients to the following values:

$$
b_1 = 0, \qquad b_2 = \frac{s^2(s+1)}{(2s+1)(s-1)}c
$$

$$
b_4 = 0, \qquad b_3 = \frac{2s^2}{2s+1}c
$$

The superspace action takes the final form

$$
\begin{aligned}
S = \int d^8z c\ & H^{\alpha(s)\dot\alpha(s)} \mathrm{D}^\gamma \bar{\mathrm{D}}^2 \mathrm{D}_\gamma H_{\alpha(s)\dot\alpha(s)} \\
&+ \left[\frac{s(s+1)}{2s+1}\right] c\ H^{\alpha(s)\dot\alpha(s)} \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}^2 \mathrm{D}^\gamma H_{\gamma\alpha(s-1)\dot\alpha(s)} + c.c. \\
&- \left[\frac{s^2}{2s+1}\right] c\ H^{\alpha(s)\dot\alpha(s)} \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}_{\dot\alpha_s} \mathrm{D}^\gamma \bar{\mathrm{D}}^{\dot\gamma} H_{\gamma\alpha(s-1)\dot\gamma\dot\alpha(s-1)} + c.c. \\
&- \left[\frac{2s^2}{2s+1}\right] c\ H^{\alpha(s)\dot\alpha(s)} \bar{\mathrm{D}}_{\dot\alpha_s} \mathrm{D}_{\alpha_s} \bar{\mathrm{D}}_{\dot\alpha_{s-1}} \chi_{\alpha(s-1)\dot\alpha(s-2)} + c.c. \\
&+ \left[\frac{s^2(s+1)}{(2s+1)(s-1)}\right] c\ \chi^{\alpha(s-1)\dot\alpha(s-2)} \bar{\mathrm{D}}^2 \chi_{\alpha(s-1)\dot\alpha(s-2)} + c.c. \\
&+ \left[\frac{2s^2}{2s+1}\right] c\ \chi^{\alpha(s-1)\dot\alpha(s-2)} \bar{\mathrm{D}}^{\dot\alpha_{s-1}} \mathrm{D}_{\alpha_{s-1}} \bar\chi_{\alpha(s-2)\dot\alpha(s-1)}
\end{aligned} \tag{5.18}
$$

and the equations of motion are

$$
\begin{aligned}
T_{\alpha(s)\dot\alpha(s)} =& 2c \mathrm{D}^\gamma \bar{\mathrm{D}}^2 \mathrm{D}_\gamma H_{\alpha(s)\dot\alpha(s)} \\
&+ \frac{2c}{s!}\left[\frac{s(s+1)}{2s+1}\right] \mathrm{D}_{(\alpha_s} \bar{\mathrm{D}}^2 \mathrm{D}^\gamma H_{\gamma\alpha(s-1)\dot\alpha(s)} \\
&+ \frac{2c}{s!}\left[\frac{s(s+1)}{2s+1}\right] \bar{\mathrm{D}}_{(\dot\alpha_s} \mathrm{D}^2 \bar{\mathrm{D}}^\gamma H_{\alpha(s)\dot\gamma\dot\alpha(s-1))} \\
&- \frac{2c}{s!s!}\left[\frac{s^2}{2s+1}\right] \mathrm{D}_{(\alpha_s} \bar{\mathrm{D}}_{(\dot\alpha_s} \mathrm{D}^\gamma \bar{\mathrm{D}}^{\dot\gamma} H_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-1))} \\
&- \frac{2c}{s!s!}\left[\frac{s^2}{2s+1}\right] \bar{\mathrm{D}}_{(\dot\alpha_s} \mathrm{D}_{(\alpha_s} \bar{\mathrm{D}}^{\dot\gamma} \mathrm{D}^\gamma H_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-1))} \\
&- \frac{2c}{s!s!}\left[\frac{s^2}{2s+1}\right] \bar{\mathrm{D}}_{(\dot\alpha_s} \mathrm{D}_{(\alpha_s} \bar{\mathrm{D}}_{\dot\alpha_{s-1}} \chi_{\alpha(s-1))\dot\alpha(s-2))} \\
&- \frac{2c}{s!s!}\left[\frac{s^2}{2s+1}\right] \mathrm{D}_{(\alpha_s} \bar{\mathrm{D}}_{(\dot\alpha_s} \mathrm{D}_{\alpha_{s-1}} \bar\chi_{\alpha(s-2))\dot\alpha(s-1))}
\end{aligned} \tag{5.19}
$$

$$
\begin{aligned}
G_{\alpha(s-1)\dot\alpha(s-2)} =& 2c\left[\frac{s^2}{2s+1}\right] \bar{\mathrm{D}}^{\dot\alpha_{s-1}} \mathrm{D}^{\alpha_s} \bar{\mathrm{D}}^{\dot\alpha_s} H_{\alpha(s)\dot\alpha(s)} \\
&+ 2c\left[\frac{s^2(s+1)}{(2s+1)(s-1)}\right] \bar{\mathrm{D}}^2 \chi_{\alpha(s-1)\dot\alpha(s-2)} \\
&+ \frac{2c}{(s-1)!}\left[\frac{s^2}{2s+1}\right] \bar{\mathrm{D}}^{\dot\alpha_{s-1}} \mathrm{D}_{(\alpha_{s-1}} \bar\chi_{\alpha(s-2))\dot\alpha(s-1)}
\end{aligned} \tag{5.20}
$$

Using the equations of motion we can now prove that a chiral superfield $F_{\alpha(2s+1)}$ exist and satisfies the following identity

$$
\mathrm{D}^{\alpha_{2s+1}} F_{\alpha(2s+1)} = \tfrac{1}{2c}\tfrac{1}{(2s)!} \partial_{(\alpha_{2s}}{}^{\dot\alpha_s} \dots \partial_{\alpha_{s+1}}{}^{\dot\alpha_1} T_{\alpha(s))\dot\alpha(s)} \tag{5.21}
$$

where

$$
F_{\alpha(2s+1)} = \frac{1}{(2s+1)!} \bar{\mathrm{D}}^2 \mathrm{D}_{(\alpha_{2s+1}} \partial_{\alpha_{2s}}{}^{\dot\alpha_s} \dots \partial_{\alpha_{s+1}}{}^{\dot\alpha_1} H_{\alpha(s))\dot\alpha(s)}
$$

and on-shell theory we obtain the desired constraints to describe a super-helicity $Y = s{+}1/2$ system.

Unlike the previous theories of half-integer and integer super-helicity, we can not perform any local redefinitions of the superfields because of the difference in their index structure. So the above action is unique.

5.2 Component structure of transverse theories (I)

The superspace actions derived above in terms of unconstrained objects will be the starting point for our component discussion. We will use the method described before in order to derive the field structure of the theory, the component action and their SUSY-transformations laws. We start with the component structure of transverse theories.

The two superfields $T_{\alpha(s)\dot\alpha(s)}, G_{\alpha(s)\dot\alpha(s-1)}$ in (5.9a) have mass dimensionality $\left[T_{\alpha(s)\dot\alpha(s)}\right] = 2$, $\left[G_{\alpha(s)\dot\alpha(s-1)}\right] = 3/2$ and satisfy the Bianchi identities and their consequences:

$$\begin{aligned}
\bar{\mathrm{D}}^{\dot\alpha_s}T_{\alpha(s)\dot\alpha(s)} - \bar{\mathrm{D}}^2 G_{\alpha(s))\dot\alpha(s-1)} = 0 \rightsquigarrow \bar{\mathrm{D}}^2 T_{\alpha(s)\dot\alpha(s)} &= 0 \qquad (5.22)\\
\mathrm{D}^2 T_{\alpha(s)\dot\alpha(s)} &= 0 \quad \text{reality}\\
\frac{1}{(s+1)!}\mathrm{D}_{(\alpha_{s+1}}G_{a(s))\dot\alpha(s-1)} = 0 \rightsquigarrow \mathrm{D}^2 G_{\alpha(s)\dot\alpha(s-1)} &= 0 \qquad (5.23)
\end{aligned}$$

These identities constrained must of the components of superfields T and G and only few of them remain to play the role of off-shell auxiliary components. So just by looking at them we immediately see the structure of auxiliary fields:

$$\begin{aligned}
\bar{\mathrm{D}}^{\dot\alpha_{s-1}}G_{\alpha(s))\dot\alpha(s-1)}|,\ \bar{\mathrm{D}}_{(\dot\alpha_s}G_{\alpha(s)\dot\alpha(s-1))}|,\ T_{\alpha(s)\dot\alpha(s)}|,\ \mathrm{D}^{\alpha_s}G_{\alpha(s)\dot\alpha(s-1)}| &\quad \text{for bosons}\\
G_{\alpha(s)\dot\alpha(s-1)}|,\ \mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}^{\dot\alpha_s}\bar{G}_{\alpha(s-1))\dot\alpha(s)}| &\quad \text{for fermions}
\end{aligned}$$

The next step is to express the action in terms of T and G

$$\begin{aligned}
S = \int d^8z \Big\{ &\frac{1}{2}H^{\alpha(s)\dot\alpha(s)}T_{\alpha(s)\dot\alpha(s)}\\
&+\frac{1}{2}\chi^{\alpha(s)\dot\alpha(s-1)}G_{\alpha(s)\dot\alpha(s-1)} + c.c.\Big\}\\
= \int d^4x &\frac{1}{2}\bar{\mathrm{D}}^2\mathrm{D}^2\left(H^{\alpha(s)\dot\alpha(s)}T_{\alpha(s)\dot\alpha(s)}\right)\\
&+\frac{1}{2}\bar{\mathrm{D}}^2\mathrm{D}^2\left(\chi^{\alpha(s)\dot\alpha(s-1)}G_{\alpha(s)\dot\alpha(s-1)}\right) + c.c. \qquad (5.24)
\end{aligned}$$

and then to distribute the covariant derivatives.

5.2.1 Fermions

After the distribution of D's and the usage of Bianchi identities we derive for the fermionic Lagrangian:

$$
\begin{aligned}
\mathcal{L}_F =& \frac{1}{2}\mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}_{s+1}}H^{\alpha(s)\dot{\alpha}(s)}|\frac{1}{(s+1)!}\bar{\mathrm{D}}_{(\dot{\alpha}_{s+1}}T_{\alpha(s)\dot{\alpha}(s))}| \\
&+\frac{1}{2}\left(-\frac{s}{s+1}\mathrm{D}^2\bar{\mathrm{D}}_{\dot{\gamma}}H^{\alpha(s)\dot{\gamma}\dot{\alpha}(s-1)}+\mathrm{D}^2\chi^{\alpha(s)\dot{\alpha}(s-1)}\right)|\bar{\mathrm{D}}^{\dot{\alpha}_s}T_{\alpha(s)\dot{\alpha}(s)}| \\
&+\frac{1}{2}\frac{s}{s+1}\bar{\mathrm{D}}^{\dot{\alpha}_s}\mathrm{D}_\gamma\chi^{\gamma\alpha(s-1)\dot{\alpha}(s-1)}|\frac{1}{s!}\bar{\mathrm{D}}_{(\dot{\alpha}_s}\mathrm{D}^{\alpha_s}G_{\alpha(s)\dot{\alpha}(s-1))}| \\
&-\frac{1}{2}\frac{s-1}{s+1}\bar{\mathrm{D}}_{\dot{\gamma}}\mathrm{D}_\gamma\chi^{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-2)}|\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\mathrm{D}^{\alpha_s}G_{\alpha(s)\dot{\alpha}(s-1)}| \\
&+\frac{1}{2}\bar{\mathrm{D}}^2\mathrm{D}^2\chi^{\alpha(s)\dot{\alpha}(s-1)}|G_{\alpha(s)\dot{\alpha}(s-1)}| \\
&+c.c.
\end{aligned}
\tag{5.25}
$$

T and G satisfy a few more identities:

$$
\begin{aligned}
\frac{1}{(s+1)!}\bar{\mathrm{D}}_{(\dot{\alpha}_{s+1}}T_{\alpha(s)\dot{\alpha}(s))} =& \frac{2ic}{(s+1)!^2}\partial^{\alpha_{s+1}}{}_{(\dot{\alpha}_{s+1}}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{s+1}}H_{\alpha(s))\dot{\alpha}(s))} \\
&-\frac{2ic}{(s+1)!s!}\frac{s}{s+1}\partial_{(\alpha_s(\dot{\alpha}_{s+1}}\left[\bar{\mathrm{D}}^2\mathrm{D}^\gamma H_{\gamma\alpha(s-1))\dot{\alpha}(s))}-\frac{s+1}{s}\bar{\mathrm{D}}^2\bar{\chi}_{\alpha(s-1)\dot{\alpha}(s)}\right]
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\mathrm{D}^{\alpha_s}G_{\alpha(s)\dot{\alpha}(s-1)} =& i\frac{s+1}{s}\partial^{\alpha_s\dot{\alpha}_{s-1}}\left[G_{\alpha(s)\dot{\alpha}(s-1))}+2c\mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}_s}H_{\alpha(s)\dot{\alpha}(s)}\right. \\
&\left.+2c\frac{s+1}{s}\mathrm{D}^2\chi_{\alpha(s)\dot{\alpha}(s-1)}\right] \\
&-\frac{2ic}{(s-1)!}\frac{s^2-1}{s^2}\partial_{(\alpha_{s-1}}{}^{\dot{\alpha}_{s-1}}\mathrm{D}^\gamma\bar{\mathrm{D}}^{\dot{\gamma}}\bar{\chi}_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-1)}
\end{aligned}
\tag{5.27}
$$

$$
\begin{aligned}
\bar{\mathrm{D}}^{\dot{\alpha}_s}T_{\alpha(s)\dot{\alpha}(s)} =& \frac{2ic}{(s+1)!}\partial^{\alpha_{s+1}\dot{\alpha}_s}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{s+1}}H_{\alpha(s))\dot{\alpha}(s)} \\
&+\frac{2ic}{s!}\frac{2s+1}{s(s+1)}\partial_{(\alpha_s}{}^{\dot{\alpha}_s}\left[\bar{\mathrm{D}}^2\mathrm{D}^\gamma H_{\gamma\alpha(s-1))\dot{\alpha}(s)}-\frac{s+1}{s}\bar{\mathrm{D}}^2\bar{\chi}_{\alpha(s-1))\dot{\alpha}(s)}\right] \\
&+\frac{2ic}{s!(s-1)!}\frac{s^2-1}{s^2}\partial_{(\alpha_s(\dot{\alpha}_{s-1}}\bar{\mathrm{D}}^{\dot{\gamma}}\mathrm{D}^\gamma\chi_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-2))} \\
&+\frac{1}{s!}\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}^{\dot{\alpha}_s}\bar{G}_{\alpha(s-1))\dot{\alpha}(s)} \\
&-\frac{i}{s!}\frac{s+1}{s}\partial_{(\alpha_s}{}^{\dot{\alpha}_s}\bar{G}_{\alpha(s-1))\dot{\alpha}(s)}
\end{aligned}
\tag{5.28}
$$

We observe that in all the above expressions and in the fermionic Lagrangian there are some specific combinations that appear repeatedly. So let us define

$$
\begin{aligned}
\frac{1}{(s+1)!}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{s+1}}H_{\alpha(s))\dot{\alpha}(s)}| &\equiv N_1\psi_{\alpha(s+1)\dot{\alpha}(s)} \\
\left\{\mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}_s}H_{\alpha(s)\dot{\alpha}(s)}+\frac{s+1}{s}\mathrm{D}^2\chi_{\alpha(s)\dot{\alpha}(s-1)}\right\}| &\equiv N_2\psi_{\alpha(s)\dot{\alpha}(s-1)} \\
\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\mathrm{D}^{\alpha_s}\chi_{\alpha(s)\dot{\alpha}(s-1)}| &\equiv N_3\psi_{\alpha(s-1)\dot{\alpha}(s-2)}
\end{aligned}
\tag{5.29}
$$

where $N_1,\ N_2,\ N_3$ are normalization constants to be fixed later. Putting everything together we have for the Lagrangian

$$\begin{aligned}\mathcal{L}_F =\ & G^{\alpha(s)\dot\alpha(s-1)}| \left(-\frac{1}{2c}\frac{s}{s+1}\frac{1}{s!}\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}^{\dot\alpha_s}\bar{G}_{\alpha(s-1))\dot\alpha(s)}\right.\\
&\left.+\frac{i}{4c}\frac{1}{s!}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar{G}_{\alpha(s-1))\dot\alpha(s)}\right)| + c.c.\\
&+2ic|N_1|^2\bar\psi^{\alpha(s)\dot\alpha(s+1)}\partial^{\alpha_{s+1}}{}_{\dot\alpha_{s+1}}\psi_{\alpha(s+1)\dot\alpha(s)}\\
&-2ic\frac{s}{s+1}N_1N_2\psi^{\alpha(s+1)\dot\alpha(s)}\partial_{\alpha_{s+1}\dot\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1)} + c.c.\\
&-2ic\frac{2s+1}{(s+1)^2}|N_2|^2\bar\psi^{\alpha(s-1)\dot\alpha(s)}\partial^{\alpha_s}{}_{\dot\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1)}\\
&+2ic\frac{s-1}{s}N_2N_3\psi^{\alpha(s)\dot\alpha(s-1)}\partial_{\alpha_s\dot\alpha_{s-1}}\psi_{\alpha(s-1)\dot\alpha(s-2)} + c.c.\\
&-2ic\left(\frac{s-1}{s}\right)^2|N_3|^2\bar\psi^{\alpha(s-2)\dot\alpha(s-1)}\partial^{\alpha_{s-1}}{}_{\dot\alpha_{s-1}}\psi_{\alpha(s-1)\dot\alpha(s-2)}\end{aligned}\tag{5.30}$$

The first term in the Lagrangian is the algebraic kinetic energy term of two auxiliary fields and the rest of the terms are exactly the structure of a theory that describes helicity $h = s + 1/2$. To have an exact match we choose coefficients

$$c = 1\,,\qquad N_2 = -\frac{1}{\sqrt{2}}$$
$$N_1 = \frac{1}{\sqrt{2}}\,,\qquad N_3 = -\frac{1}{\sqrt{2}}\frac{s}{s-1}$$

So the fields that appear in the fermionic action are defined as:

$$\begin{aligned}\rho_{\alpha(s)\dot\alpha(s-1)} &\equiv G_{\alpha(s)\dot\alpha(s-1)}|\\
\beta_{\alpha(s)\dot\alpha(s-1)} &\equiv -\frac{1}{2s!}\left\{\frac{s}{s+1}\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}^{\dot\alpha_s}\bar{G}_{\alpha(s-1))\dot\alpha(s)} - \frac{i}{2}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar{G}_{\alpha(s-1))\dot\alpha(s)}\right\}|\\
\psi_{\alpha(s+1)\dot\alpha(s)} &\equiv \frac{\sqrt{2}}{(s+1)!}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{s+1}}H_{\alpha(s))\dot\alpha(s)}|\\
\psi_{\alpha(s)\dot\alpha(s-1)} &\equiv -\sqrt{2}\left\{\mathrm{D}^2\bar{\mathrm{D}}^{\dot\alpha_s}H_{\alpha(s)\dot\alpha(s)} + \frac{s+1}{s}\mathrm{D}^2\chi_{\alpha(s)\dot\alpha(s-1)}\right\}|\\
\psi_{\alpha(s-1)\dot\alpha(s-2)} &\equiv -\sqrt{2}\frac{(s-1)}{s}\bar{\mathrm{D}}^{\dot\alpha_{s-1}}\mathrm{D}^{\alpha_s}\chi_{\alpha(s)\dot\alpha(s-1)}|\end{aligned}\tag{5.31}$$

The Lagrangian is

$$\begin{aligned}\mathcal{L}_F =\ & \rho^{\alpha(s)\dot\alpha(s-1)}\beta_{\alpha(s)\dot\alpha(s-1)} + c.c.\\
&+i\ \bar\psi^{\alpha(s)\dot\alpha(s+1)}\partial^{\alpha_{s+1}}{}_{\dot\alpha_{s+1}}\psi_{\alpha(s+1)\dot\alpha(s)}\\
&+i\left[\frac{s}{s+1}\right]\ \psi^{\alpha(s+1)\dot\alpha(s)}\partial_{\alpha_{s+1}\dot\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1)} + c.c.\\
&-i\left[\frac{2s+1}{(s+1)^2}\right]\ \bar\psi^{\alpha(s-1)\dot\alpha(s)}\partial^{\alpha_s}{}_{\dot\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1)}\\
&+i\ \psi^{\alpha(s)\dot\alpha(s-1)}\partial_{\alpha_s\dot\alpha_{s-1}}\psi_{\alpha(s-1)\dot\alpha(s-2)} + c.c.\\
&-i\ \bar\psi^{\alpha(s-2)\dot\alpha(s-1)}\partial^{\alpha_{s-1}}{}_{\dot\alpha_{s-1}}\psi_{\alpha(s-1)\dot\alpha(s-2)}\end{aligned}\tag{5.32}$$

and the gauge transformations of the fields are

$$
\begin{aligned}
\delta_G \rho_{\alpha(s)\dot\alpha(s-1)} &= 0\,, & \delta_G \psi_{\alpha(s+1)\dot\alpha(s)} &= \frac{1}{s!(s+1)!}\partial_{(\alpha_{s+1}(\dot\alpha_s}\xi_{\alpha(s))\dot\alpha(s-1))} \\
\delta_G \beta_{\alpha(s)\dot\alpha(s-1)} &= 0\,, & \delta_G \psi_{\alpha(s)\dot\alpha(s-1)} &= -\frac{1}{s!}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar\xi_{\alpha(s-1))\dot\alpha(s)} \\
& & \delta_G \psi_{\alpha(s-1)\dot\alpha(s-2)} &= \frac{s-1}{s}\partial^{\alpha_s\dot\alpha_{s-1}}\xi_{\alpha(s)\dot\alpha(s-1)} \\
& & \text{with } \xi_{\alpha(s)\dot\alpha(s-1)} &= -i\sqrt{2}\,\bar{\mathrm{D}}^2 L_{\alpha(s)\dot\alpha(s-1)}|
\end{aligned}
\tag{5.33}
$$

5.2.2 Bosons

For the bosonic action we follow exactly the same procedure. The fields that appear in the action are defined as:

$$
\begin{aligned}
U_{\alpha(s)\dot\alpha(s-2)} &\equiv \bar{\mathrm{D}}^{\dot\alpha_{s-1}} G_{\alpha(s))\dot\alpha(s-1)}| \\
u_{\alpha(s)\dot\alpha(s)} &\equiv \frac{1}{2s!}\left\{\mathrm{D}_{(\alpha_s}\bar G_{\alpha(s-1))\dot\alpha(s)} - \bar{\mathrm{D}}_{(\dot\alpha_s}G_{\alpha(s)\dot\alpha(s-1))}\right\}| \\
v_{\alpha(s)\dot\alpha(s)} &\equiv -\frac{i}{2s!}\left\{\mathrm{D}_{(\alpha_s}\bar G_{\alpha(s-1))\dot\alpha(s)} + \bar{\mathrm{D}}_{(\dot\alpha_s}G_{\alpha(s)\dot\alpha(s-1))}\right\}| \\
A_{\alpha(s)\dot\alpha(s)} &\equiv T_{\alpha(s)\dot\alpha(s)}| + \frac{s}{2s+1}\frac{1}{s!}\left(\mathrm{D}_{(\alpha_s}\bar G_{\alpha(s-1))\dot\alpha(s)} - \bar{\mathrm{D}}_{(\dot\alpha_s}G_{\alpha(s)\dot\alpha(s-1))}\right)| \\
S_{\alpha(s-1)\dot\alpha(s-1)} &\equiv \frac{1}{2}\left\{\mathrm{D}^{\alpha_s}G_{\alpha(s)\dot\alpha(s-1)} + \bar{\mathrm{D}}^{\dot\alpha_s}\bar G_{\alpha(s)\dot\alpha(s-1)}\right\}| \\
P_{\alpha(s-1)\dot\alpha(s-1)} &\equiv -\frac{i}{2}\left\{\mathrm{D}^{\alpha_s}G_{\alpha(s)\dot\alpha(s-1)} - \bar{\mathrm{D}}^{\dot\alpha_s}\bar G_{\alpha(s)\dot\alpha(s-1)}\right\}| \\
h_{\alpha(s+1)\dot\alpha(s+1)} &\equiv \frac{1}{2}\frac{1}{(s+1)!^2}\left[\mathrm{D}_{(\alpha_{s+1}},\mathrm{D}_{(\dot\alpha_{s+1}}\right] H_{\alpha(s))\dot\alpha(s))}| \\
h_{\alpha(s-1)\dot\alpha(s-1)} &\equiv \frac{1}{2}\frac{s}{(s+1)^2}\left[\mathrm{D}^{\alpha_s},\bar{\mathrm{D}}^{\dot\alpha_s}\right] H_{\alpha(s)\dot\alpha(s)}| \\
&\quad + \frac{1}{s+1}\left(\mathrm{D}^{\alpha_s}\chi_{\alpha(s)\dot\alpha(s-1)} + \bar{\mathrm{D}}^{\dot\alpha_s}\bar\chi_{\alpha(s-1)\dot\alpha(s)}\right)|
\end{aligned}
\tag{5.34}
$$

the gauge transformations are

$$
\begin{aligned}
\delta_G U_{\alpha(s)\dot\alpha(s-2)} &= 0, & \delta_G A_{\alpha(s)\dot\alpha(s)} &= 0 \\
\delta_G u_{\alpha(s)\dot\alpha(s)} &= 0, & \delta_G S_{\alpha(s-1)\dot\alpha(s-1)} &= 0 \\
\delta_G v_{\alpha(s)\dot\alpha(s)} &= 0, & \delta_G P_{\alpha(s-1)\dot\alpha(s-1)} &= 0 \\
\delta_G h_{\alpha(s+1)\dot\alpha(s+1)} &= \frac{1}{(s+1)!^2}\partial_{(\alpha_{s+1}(\dot\alpha_{s+1}}\zeta_{\alpha(s))\dot\alpha(s))} \\
\delta_G h_{\alpha(s-1)\dot\alpha(s-1)} &= \frac{s}{(s+1)^2}\partial^{\alpha_s\dot\alpha_s}\zeta_{\alpha(s)\dot\alpha(s)}
\end{aligned}
\tag{5.35}
$$

where

$$
\zeta_{\alpha(s)\dot\alpha(s)} = \frac{i}{2s!}\left(\mathrm{D}_{(\alpha_s}\bar L_{\alpha(s-1))\dot\alpha(s)} + \bar{\mathrm{D}}_{(\dot\alpha_s}L_{\alpha(s)\dot\alpha(s-1))}\right)|
$$

and the Lagrangian

$$
\begin{aligned}
\mathcal{L}_B = &\frac{1}{4}\left[\frac{s-1}{s+1}\right]\ U^{\alpha(s)\dot\alpha(s-2)}U_{\alpha(s)\dot\alpha(s-2)} + c.c. \\
&+\left[\frac{s}{2}\right]\ u^{\alpha(s)\dot\alpha(s)}u_{\alpha(s)\dot\alpha(s)} \\
&-\frac{1}{2}\left[\frac{s}{2s+1}\right]\ v^{\alpha(s)\dot\alpha(s)}v_{\alpha(s)\dot\alpha(s)} \\
&+\frac{1}{8}\left[\frac{2s+1}{s+1}\right]\ A^{\alpha(s)\dot\alpha(s)}A_{\alpha(s)\dot\alpha(s)} \\
&-\frac{1}{2}\left[\frac{s^2}{(s+1)^2}\right]\ S^{\alpha(s-1)\dot\alpha(s-1)}S_{\alpha(s-1)\dot\alpha(s-1)} \\
&-\frac{1}{2}\left[\frac{s^2}{(s+1)^2}\right]\ P^{\alpha(s-1)\dot\alpha(s-1)}P_{\alpha(s-1)\dot\alpha(s-1)} \\
&+\ h^{\alpha(s+1)\dot\alpha(s+1)}\Box h_{\alpha(s+1)\dot\alpha(s+1)} \\
&-\left[\frac{s+1}{2}\right]\ h^{\alpha(s+1)\dot\alpha(s+1)}\partial_{\alpha_{s+1}\dot\alpha_{s+1}}\partial^{\gamma\dot\gamma}h_{\gamma\alpha(s)\dot\gamma\dot\alpha(s)} \\
&+[s(s+1)]\ h^{\alpha(s+1)\dot\alpha(s+1)}\partial_{\alpha_{s+1}\dot\alpha_{s+1}}\partial_{\alpha_s\dot\alpha_s}h_{\alpha(s-1)\dot\alpha(s-1)} \\
&-[(s+1)(2s+1)]\ h^{\alpha(s-1)\dot\alpha(s-1)}\Box h_{\alpha(s-1)\dot\alpha(s-1)} \\
&-\left[\frac{(s+1)(s-1)^2}{2}\right]\ h^{\alpha(s-1)\dot\alpha(s-1)}\partial_{\alpha_{s-1}\dot\alpha_{s-1}}\partial^{\gamma\dot\gamma}h_{\gamma\alpha(s-2)\dot\gamma\dot\alpha(s-2)}
\end{aligned}
\tag{5.36}
$$

gives rise to the theory of helicity $h = s+1$ as expected.

5.2.3 Off-shell degrees of freedom

The bosonic degrees of freedom are:

fields	*d.o.f*	*redundancy*	*net*
$h_{\alpha(s+1)\dot\alpha(s+1)}$	$(s+2)^2$	$(s+1)^2$	s^2+2s+3
$h_{\alpha(s-1)\dot\alpha(s-1)}$	s^2		
$u_{\alpha(s)\dot\alpha(s)}$	$(s+1)^2$	0	$(s+1)^2$
$v_{\alpha(s)\dot\alpha(s)}$	$(s+1)^2$	0	$(s+1)^2$
$A_{\alpha(s)\dot\alpha(s)}$	$(s+1)^2$	0	$(s+1)^2$
$U_{\alpha(s)\dot\alpha(s-2)}$	$2(s+1)(s-1)$	0	$2(s+1)(s-1)$
$S_{\alpha(s-1)\dot\alpha(s-1)}$	s^2	0	s^2
$P_{\alpha(s-1)\dot\alpha(s-1)}$	s^2	0	s^2
		Total	$8s^2+8s+4$

and the fermionic degrees of freedom are:

fields	*d.o.f*	*redundancy*	*net*
$\psi_{\alpha(s+1)\dot\alpha(s)}$	$2(s+2)(s+1)$	$2(s+1)s$	$4s^2+4s+4$
$\psi_{\alpha(s)\dot\alpha(s-1)}$	$2(s+1)s$		
$\psi_{\alpha(s-1)\dot\alpha(s-2)}$	$2s(s-1)$		
$\rho_{\alpha(s)\dot\alpha(s-1)}$	$2(s+1)s$	0	$2(s+1)s$
$\beta_{\alpha(s)\dot\alpha(s-1)}$	$2(s+1)s$	0	$2(s+1)s$
		Total	$8s^2+8s+4$

5.2.4 SUSY-transformation laws

The SUSY-transformation laws for the fermionic fields are:

$$\begin{aligned}\delta_S\rho_{\alpha(s)\dot\alpha(s-1)} =&\frac{s}{s+1}\frac{1}{s!}\epsilon_{(\alpha_s}\left[S_{\alpha(s-1))\dot\alpha(s-1)}+iP_{\alpha(s-1))\dot\alpha(s-1)}\right]\\&+\bar\epsilon^{\dot\alpha_s}\left[u_{\alpha(s)\dot\alpha(s)}-iv_{\alpha(s)\dot\alpha(s)}\right]\\&+\frac{s-1}{s}\frac{1}{(s-1)!}\bar\epsilon_{(\dot\alpha_{s-1}}U_{\alpha(s)\dot\alpha(s-2))}\end{aligned}\tag{5.37}$$

$$\begin{aligned}\delta_S\beta_{\alpha(s)\dot\alpha(s-1)} =&\frac{s}{s+1}\frac{1}{s!}\epsilon_{(\alpha_s}\partial^{\gamma\dot\gamma}A_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-1)}\\&\frac{2s}{(s+1)^2}\frac{i}{s!}\epsilon_{(\alpha_s}\partial^{\gamma\dot\gamma}u_{\gamma\alpha(s-1)\dot\gamma\dot\alpha(s-1)}\\&\frac{2s}{s!}\epsilon_{(\alpha_s}\partial^{\gamma\dot\gamma}v_{\gamma\alpha(s-1)\dot\gamma\dot\alpha(s-1)}\\&-\frac{i}{s!}\epsilon^{\gamma}\partial_{(\alpha_s}{}^{\dot\gamma}u_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-1)}\\&+\frac{1}{s!}\epsilon^{\gamma}\partial_{(\alpha_s}{}^{\dot\gamma}v_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-1)}\\&+\frac{i}{s!}\frac{2s-1}{s+1}\bar\epsilon^{\dot\gamma}\partial_{(\alpha_s\dot\gamma}\left[S_{\alpha(s-1))\dot\alpha(s-1)}-iP_{\alpha(s-1))\dot\alpha(s-1)}\right]\\&+\frac{i}{s!(s-1)!}\frac{s-1}{s+1}\bar\epsilon_{(\dot\alpha_{s-1}}\partial_{(\alpha_s}{}^{\dot\gamma}\left[S_{\alpha(s-1))\dot\gamma\dot\alpha(s-2))}\right.\\&\left.-iP_{\alpha(s-1))\dot\gamma\dot\alpha(s-2))}\right]\\&+\frac{2i}{s!(s-1)!}\frac{s-1}{s+1}\epsilon_{(\alpha_s}\partial^{\gamma}{}_{(\dot\alpha_{s-1}}U_{\gamma\alpha(s-1))\dot\alpha(s-2))}\\&+\frac{i}{s!}\frac{s-1}{s}\epsilon_{(\alpha_s}\partial_{\alpha_{s-1}}{}^{\dot\gamma}\bar U_{\alpha(s-2))\dot\gamma\dot\alpha(s-1)}\\&+\frac{2s}{s!}\epsilon_{(\alpha_s}\partial^{\gamma\dot\gamma}\partial^{\beta\dot\beta}h_{\beta\gamma\alpha(s-1))\dot\beta\dot\gamma\dot\alpha(s-1)}\\&-\frac{2(s-1)^2}{s!(s-1)!}\epsilon_{(\alpha_s}\partial_{\alpha_{s-1}(\dot\alpha_{s-1}}\partial^{\gamma\dot\gamma}h_{\gamma\alpha(s-2))\dot\gamma\dot\alpha(s-2))}\end{aligned}\tag{5.38}$$

$$
\begin{aligned}
\delta_S \psi_{\alpha(s+1)\dot{\alpha}(s)} =& \frac{\sqrt{2}i}{(s+1)!}\epsilon^{\gamma}\partial_{(\alpha_{s+1}}{}^{\dot{\gamma}} h_{\gamma\alpha(s))\dot{\gamma}\dot{\alpha}(s)} \\
& - \frac{i}{\sqrt{2}(s+1)!}\epsilon_{(\alpha_{s+1}}\partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s))\dot{\gamma}\dot{\alpha}(s)} \\
& + \frac{1}{2\sqrt{2}}\frac{2s+1}{s+1}\frac{1}{(s+1)!}\epsilon_{(\alpha_{s+1}}A_{\alpha(s))\dot{\alpha}(s)}
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
\delta_S \psi_{\alpha(s)\dot{\alpha}(s-1)} =& -\frac{1}{2\sqrt{2}}\frac{s}{s+1}\bar{\epsilon}^{\dot{\alpha}_s}A_{\alpha(s)\dot{\alpha}(s)} \\
& + \frac{1}{\sqrt{2}}\frac{s+1}{2s+1}\bar{\epsilon}^{\dot{\alpha}_s}u_{\alpha(s)\dot{\alpha}(s)} \\
& - i\frac{s+1}{\sqrt{2}}\bar{\epsilon}^{\dot{\alpha}_s}v_{\alpha(s)\dot{\alpha}(s)} \\
& + \frac{1}{\sqrt{2}}\frac{s-1}{s!}\bar{\epsilon}_{(\dot{\alpha}_{s-1}}U_{\alpha(s)\dot{\alpha}(s-2))} \\
& - \frac{is}{\sqrt{2}}\bar{\epsilon}^{\dot{\alpha}_s}\partial^{\gamma\dot{\gamma}}h_{\gamma\alpha(s)\dot{\gamma}\dot{\alpha}(s)} \\
& \frac{i}{s!^2}\frac{s(s+2)}{\sqrt{2}}\bar{\epsilon}^{\dot{\alpha}_s}\partial_{(\alpha_s(\dot{\alpha}_s}h_{\alpha(s-1))\dot{\alpha}(s-1))}
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
\delta_S \psi_{\alpha(s-1)\dot{\alpha}(s-2)} =& -\frac{1}{\sqrt{2}}\frac{s-1}{s+1}\epsilon^{\alpha_s}U_{\alpha(s)\dot{\alpha}(s-2)} \\
& - \frac{1}{\sqrt{2}}\frac{s-1}{s+1}\bar{\epsilon}^{\dot{\alpha}_{s-1}}\left[S_{\alpha(s-1)\dot{\alpha}(s-1)} - iP_{\alpha(s-1)\dot{\alpha}(s-1)}\right] \\
& + i\sqrt{2}\frac{s-1}{s!}\epsilon^{\alpha_s}\partial_{(\alpha_s}{}^{\dot{\alpha}_{s-1}}h_{\alpha(s-1))\dot{\alpha}(s-1)}
\end{aligned}
\tag{5.41}
$$

and the SUSY-transformation laws for the bosonic fields are:

$$
\begin{aligned}
\delta_S A_{\alpha(s)\dot{\alpha}(s)} =& -\frac{i\sqrt{2}}{(s+1)!}\bar{\epsilon}^{\dot{\alpha}_{s+1}}\partial^{\alpha_{s+1}}{}_{(\dot{\alpha}_{s+1}}\psi_{\alpha(s+1)\dot{\alpha}(s))} + c.c. \\
& + \frac{i\sqrt{2}}{s!}\frac{s^2}{(2s+1)(s+1)}\bar{\epsilon}_{(\dot{\alpha}_s}\partial^{\gamma\dot{\gamma}}\psi_{\gamma\alpha(s)\dot{\gamma}\dot{\alpha}(s-1))} + c.c. \\
& - \frac{i}{s!(s+1)!}\frac{s}{2s+1}\bar{\epsilon}^{\dot{\alpha}_{s+1}}\partial_{(\alpha_s(\dot{\alpha}_{s+1}}\bar{\rho}_{\alpha(s-1))\dot{\alpha}(s))} + c.c. \\
& - \frac{i}{s!^2}\frac{s}{(2s+1)(s+1)}\bar{\epsilon}_{(\dot{\alpha}_s}\partial_{(\alpha_s}{}^{\dot{\gamma}}\bar{\rho}_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1))} + c.c. \\
& + \frac{i\sqrt{2}}{s!(s+1)!}\frac{s}{s+1}\bar{\epsilon}^{\dot{\alpha}_{s+1}}\partial_{(\alpha_s(\dot{\alpha}_{s+1}}\bar{\psi}_{\alpha(s-1))\dot{\alpha}(s)} + c.c. \\
& + \frac{i\sqrt{2}}{s!^2}\frac{s}{(s+1)^2}\bar{\epsilon}_{(\dot{\alpha}_s}\partial_{(\alpha_s}{}^{\dot{\gamma}}\bar{\psi}_{\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1))} + c.c. \\
& - \frac{i\sqrt{2}}{s!^2}\frac{s}{2s+1}\bar{\epsilon}_{(\dot{\alpha}_s}\partial_{(\alpha_s\dot{\alpha}_{s-1}}\psi_{\alpha(s-1))\dot{\alpha}(s-2))} + c.c.
\end{aligned}
\tag{5.42}
$$

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$$
\begin{aligned}
\delta_S\left(u_{\alpha(s)\dot\alpha(s)}+iv_{\alpha(s)\dot\alpha(s)}\right)=&-\frac{i\sqrt{2}}{s!}\epsilon_{(\alpha_s}\partial^{\gamma\dot\gamma}\bar\psi_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s)}\\
&+\frac{i\sqrt{2}}{s!^2}\frac{2s+1}{s(s+1)}\epsilon_{(\alpha_s}\partial^{\gamma}{}_{(\dot\alpha_s}\psi_{\gamma\alpha(s-1))\dot\alpha(s-1))}\\
&+\frac{i\sqrt{2}}{s!^2}\frac{s+1}{s}\epsilon_{(\alpha_s}\partial_{\alpha_{s-1}(\dot\alpha_s}\bar\psi_{\alpha(s-2))\dot\alpha(s-1))}\\
&-\frac{2}{s!}\frac{s+1}{s}\epsilon_{(\alpha_s}\bar\beta_{\alpha(s-1))\dot\alpha(s)}\\
&+\frac{2}{s!}\bar\epsilon_{(\dot\alpha_s}\beta_{\alpha(s)\dot\alpha(s-1))}\\
&-\frac{i}{s!^2}\frac{s+1}{2s}\epsilon_{(\alpha_s}\partial^{\gamma}{}_{\dot\alpha_s}\rho_{\gamma\alpha(s-1))\dot\alpha(s-1))}\\
&-\frac{i}{s!(s+1)!}\bar\epsilon^{\dot\alpha_{s+1}}\partial_{(\alpha_s(\dot\alpha_{s+1}}\bar\rho_{\alpha(s-1))\dot\alpha(s)}\\
&+\frac{i}{s!^2}\frac{s-1}{2(s+1)}\bar\epsilon_{(\dot\alpha_s}\partial_{(\alpha_s}{}^{\dot\gamma}\bar\rho_{\alpha(s-1))\dot\gamma\dot\alpha(s-1))} \qquad (5.43)\\
\delta_S U_{\alpha(s)\dot\alpha(s-2)}=&i\sqrt{2}\bar\epsilon^{\dot\alpha_{s-1}}\partial^{\alpha_{s+1}\dot\alpha_s}\psi_{\alpha(s+1)\dot\alpha(s)}\\
&+\frac{i\sqrt{2}}{s!}\frac{2s+1}{s(s+1)}\bar\epsilon^{\dot\alpha_{s-1}}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar\psi_{\alpha(s-1))\dot\alpha(s)}\\
&+\frac{i\sqrt{2}}{s!}\epsilon_{(\alpha_s}\partial^{\gamma\dot\gamma}\psi_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-2)}\\
&+\frac{i\sqrt{2}}{s!}\epsilon_{(\alpha_s}\partial_{\alpha_{s-1}}{}^{\dot\alpha_{s-1}}\bar\psi_{\alpha(s-2))\dot\alpha(s-1)}\\
&-\frac{i\sqrt{2}}{s!(s-1)!}\frac{s+1}{s}\bar\epsilon^{\dot\alpha_{s-1}}\partial_{(\alpha_s(\dot\alpha_{s-1}}\psi_{\alpha(s-1))\dot\alpha(s-2))}\\
&-\frac{i}{s!}\frac{1}{s+1}\epsilon_{(\alpha_s}\partial^{\gamma\dot\gamma}\rho_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-2)}\\
&-\frac{i}{(s+1)!}\epsilon^{\alpha_{s+1}}\partial_{(\alpha_{s+1}}{}^{\dot\alpha_{s-1}}\rho_{\alpha(s))\dot\alpha(s-1)}\\
&-\frac{i}{s!}\frac{s+1}{2s}\bar\epsilon^{\dot\alpha_{s-1}}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar\rho_{\alpha(s-1))\dot\alpha(s)}\\
&-2\frac{s+1}{s}\bar\epsilon^{\dot\alpha_{s-1}}\beta_{\alpha(s)\dot\alpha(s-1)} \qquad (5.44)\\
\delta_S\left(S_{\alpha(s-1)\dot\alpha(s-1)}+iP_{\alpha(s-1)\dot\alpha(s-1)}\right)=&2\frac{s+1}{s}\bar\epsilon^{\dot\alpha_s}\bar\beta_{\alpha(s-1)\dot\alpha(s)}\\
&-\frac{i}{s!}\frac{s+1}{2s}\bar\epsilon^{\dot\alpha_s}\partial^{\alpha_s}{}_{(\dot\alpha_s}\rho_{\alpha(s)\dot\alpha(s-1))} \qquad (5.45)\\
&+\frac{i}{(s-1)!}\frac{(s-1)(s+1)}{s^2}\bar\epsilon_{(\dot\alpha_{s-1}}\partial^{\gamma\dot\gamma}\rho_{\gamma\alpha(s-1)\dot\gamma\dot\alpha(s-2)}\\
&-\frac{i\sqrt{2}}{(s-1)!}\frac{(s-1)(s+1)}{s^2}\bar\epsilon_{(\dot\alpha_{s-1}}\partial^{\gamma\dot\gamma}\psi_{\gamma\alpha(s-1)\dot\gamma\dot\alpha(s-2))}\\
&-\frac{i\sqrt{2}}{(s-1)!^2}\frac{(s-1)(s+1)}{s^2}\bar\epsilon_{(\dot\alpha_{s-1}}\partial_{(\alpha_{s-1}}{}^{\dot\gamma}\bar\psi_{\alpha(s-2))\dot\gamma\dot\alpha(s-2))}\\
\delta_S h_{\alpha(s+1)\dot\alpha(s+1)}=&\frac{1}{\sqrt{2}(s+1)!}\epsilon_{(\alpha_{s+1}}\bar\psi_{\alpha(s))\dot\alpha(s+1)}+c.c. \qquad (5.46)
\end{aligned}
$$

$$\delta_S h_{\alpha(s-2)\dot\alpha(s-2)} = \frac{1}{\sqrt{2}(s+1)^2}\epsilon^{\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1)} + c.c.$$
$$- \frac{1}{2(s+1)}\epsilon^{\alpha_s}\rho_{\alpha(s)\dot\alpha(s-1)} + c.c.$$
$$- \frac{1}{\sqrt{2}(s+1)}\frac{1}{(s-1)!}\bar\epsilon_{(\dot\alpha_{s-1}}\psi_{\alpha(s-1)\dot\alpha(s-2))} \tag{5.47}$$

5.3 Component structure for longitudinal theories (II)

We repeat the same steps for the second formulation of half-integer super-helicity theories. The superspace action (5.18) can be expressed like

$$\begin{aligned} S = &\int d^8z \left\{ \frac{1}{2}H^{\alpha(s)\dot\alpha(s)}T_{\alpha(s)\dot\alpha(s)} \right. \\ &\left. +\frac{1}{2}\chi^{\alpha(s-1)\dot\alpha(s-2)}G_{\alpha(s-1)\dot\alpha(s-2)} + c.c. \right\} \\ = &\int d^4x \frac{1}{2}\mathrm{D}^2\bar{\mathrm{D}}^2\left(H^{\alpha(s)\dot\alpha(s)}T_{\alpha(s)\dot\alpha(s)}\right) \\ &+\frac{1}{2}\mathrm{D}^2\bar{\mathrm{D}}^2\left(\chi^{\alpha(s-1)\dot\alpha(s-2)}G_{\alpha(s-1)\dot\alpha(s-2)}\right) + c.c. \end{aligned} \tag{5.48}$$

where $T,\ G$ are defined by (5.20)

5.3.1 Fermions

For the fermionic Lagrangian we have

$$\begin{aligned} \mathcal{L}_F = &\frac{1}{2}\frac{1}{(s+1)!}\mathrm{D}^2\bar{\mathrm{D}}^{(\dot\alpha_{s+1}}H^{\alpha(s)\dot\alpha(s))}|\frac{1}{(s+1)!}\bar{\mathrm{D}}_{(\dot\alpha_{s+1}}T_{\alpha(s)\dot\alpha(s))}| \\ &+\left(\frac{1}{2}\frac{1}{s+1}\mathrm{D}^2\bar{\mathrm{D}}_{\dot\gamma}H^{\alpha(s)\dot\gamma\dot\alpha(s-1)} - \frac{1}{2}\frac{1}{s!(s-1)!}\mathrm{D}^{(\alpha_s}\bar{\mathrm{D}}^{(\dot\alpha_{s-1}}\chi^{\alpha(s-1))\dot\alpha(s-2))}\right. \\ &\left.-\frac{i}{2}\frac{1}{s!}\mathrm{D}^{(\alpha_s}\partial_{\gamma\dot\gamma}H^{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-1)}\right)|\frac{1}{s!(s-1)!}\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}_{(\dot\alpha_{s-1}}G_{\alpha(s-1))\dot\alpha(s-2))}| \\ &+\left(\frac{1}{2}\frac{s-1}{s}\frac{1}{(s-1)!}\mathrm{D}_\gamma\bar{\mathrm{D}}^{(\dot\alpha_{s-1}}\chi^{\gamma\alpha(s-2)\dot\alpha(s-2))}\right. \\ &\left.+\frac{i}{2}\frac{s-1}{s}\mathrm{D}_\beta\partial_{\gamma\dot\gamma}H^{\beta\gamma\alpha(s-2)\dot\gamma\dot\alpha(s-1)}\right)|\frac{1}{(s-1)!}\mathrm{D}^{\alpha_{s-1}}\bar{\mathrm{D}}_{(\dot\alpha_{s-1}}G_{\alpha(s-1)\dot\alpha(s-2))}| \\ &+\left(-\frac{i}{2}\frac{s-1}{s+1}\partial_{\alpha_s\dot\alpha_{s-1}}\mathrm{D}^2\bar{\mathrm{D}}_{\dot\alpha_s}H_{\alpha(s)\dot\alpha(s)}\right. \\ &\left.+\frac{1}{2}\mathrm{D}^2\bar{\mathrm{D}}^2\chi^{\alpha(s-1)\dot\alpha(s-2)}\right)|G_{\alpha(s-1)\dot\alpha(s-2)}| \\ &+\frac{1}{2}\bar{\mathrm{D}}^2\chi^{\alpha(s-1)\dot\alpha(s-2)}|\mathrm{D}^2G_{\alpha(s-1)\dot\alpha(s-2)}| \\ &+c.c. \end{aligned} \tag{5.49}$$

We can prove the following identities for T and G:

$$
\begin{aligned}
\frac{1}{(s+1)!}\bar{\mathrm{D}}_{(\dot\alpha_{s+1}}T_{\alpha(s)\dot\alpha(s))} = {} & \frac{2ic}{(s+1)!}\partial^{\alpha_{s+1}}{}_{(\dot\alpha_{s+1}}\left\{\frac{1}{(s+1)!}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{s+1}}H_{\alpha(s))\dot\alpha(s))}\right\} \\
& - \frac{2ic}{(s+1)!s!}\frac{s^2}{(2s+1)(s+1)}\partial_{(\alpha_s(\dot\alpha_{s+1}}\Big\{\bar{\mathrm{D}}^2\mathrm{D}^\gamma H_{\gamma\alpha(s-1))\dot\alpha(s))} \\
& + \frac{i(s+1)}{s!}\bar{\mathrm{D}}_{\dot\alpha_s}\partial^{\gamma\dot\gamma}H_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-1)))} \\
& + \frac{s+1}{s!(s-1)!}\bar{\mathrm{D}}_{(\dot\alpha_s}\mathrm{D}_{(\alpha_{s-1}}\bar\chi_{\alpha(s-2)))\dot\alpha(s-1)))}\Big\}
\end{aligned} \tag{5.50}
$$

$$
\begin{aligned}
& \frac{1}{s!(s-1)!}\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}_{(\dot\alpha_{s-1}}G_{\alpha(s-1))\dot\alpha(s-2))} = \\
& = - \frac{2ic}{s!}\frac{s^2}{(2s+1)(s+1)}\partial_{(\alpha_s}{}^{\dot\alpha_s}\Big\{\bar{\mathrm{D}}^2\mathrm{D}^\gamma H_{\gamma\alpha(s-1))\dot\alpha(s)} \\
& \quad + \frac{i(s+1)}{s!}\bar{\mathrm{D}}_{(\dot\alpha_s}\partial^{\gamma\dot\gamma}H_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-1))} \\
& \quad + \frac{s+1}{s!(s-1)!}\bar{\mathrm{D}}_{(\dot\alpha_s}\mathrm{D}_{(\alpha_{s-1}}\bar\chi_{\alpha(s-2)))\dot\alpha(s-1))}\Big\} \\
& \quad - 2ic\frac{s^2}{2s+1}\partial^{\alpha_{s+1}\dot\alpha_s}\left\{\frac{1}{(s+1)!}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{s+1}}H_{\alpha(s))\dot\alpha(s)}\right\} \\
& \quad + \frac{2ic}{s!(s-1)!}\frac{s(s-1)}{2s+1}\partial_{(\alpha_s(\dot\alpha_{s-1}}\Big\{i\bar{\mathrm{D}}^{\dot\beta}\partial^{\gamma\dot\gamma}H_{\gamma\alpha(s-1))\dot\beta\dot\gamma\dot\alpha(s-2))} \\
& \quad + \frac{1}{(s-1)!}\bar{\mathrm{D}}^{\dot\gamma}\mathrm{D}_{(\alpha_{s-1}}\bar\chi_{\alpha(s-2)))\dot\gamma\dot\alpha(s-2))}\Big\}
\end{aligned} \tag{5.51}
$$

$$
\begin{aligned}
& \frac{1}{(s-1)!}\bar{\mathrm{D}}^{\dot\alpha_{s-1}}\mathrm{D}_{(\alpha_{s-1}}\bar G_{\alpha(s-2))\dot\alpha(s-1)} = \\
& = - \frac{s}{s+1}\mathrm{D}^2 G_{\alpha(s-1)\dot\alpha(s-2)} \\
& \quad + \frac{i}{(s-1)!}\frac{s(s-1)}{(s+1)^2}\partial_{(\alpha_{s-1}}{}^{\dot\alpha_{s-1}}\bar G_{\alpha(s-2))\dot\alpha(s-1)} \\
& \quad - 2ic\frac{s^2}{(2s+1)(s+1)}\partial^{\alpha_s\dot\alpha_{s-1}}\Big\{\mathrm{D}^2\bar{\mathrm{D}}^{\dot\gamma}H_{\alpha(s)\dot\gamma\dot\alpha(s-1)} \\
& \quad + \frac{i(s+1)}{s!}\mathrm{D}_{(\alpha_s}\partial^{\gamma\dot\gamma}H_{\gamma\alpha(s-1))\dot\gamma\dot\alpha(s-1)} \\
& \quad + \frac{s+1}{s!(s-1)!}\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}_{(\dot\alpha_{s-1}}\chi_{\alpha(s-1))\dot\alpha(s-2))}\Big\} \\
& \quad + 2ic\frac{s(s-1)}{(s+1)^2}\frac{1}{(s-1)!}\partial_{(\alpha_{s-1}}{}^{\dot\alpha_{s-1}}\Big\{i\mathrm{D}^\beta\partial^{\gamma\dot\gamma}H_{\beta\gamma\alpha(s-2))\dot\gamma\dot\alpha(s-1)} \\
& \quad + \frac{1}{(s-1)!}\mathrm{D}^\gamma\bar{\mathrm{D}}_{(\dot\alpha_{s-1}}\chi_{\gamma\alpha(s-2))\dot\alpha(s-2))}\Big\}
\end{aligned} \tag{5.52}
$$

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Let us define the following fields

$$\frac{1}{(s+1)!}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{s+1}}H_{\alpha(s))\dot{\alpha}(s))}| \equiv N_1\ \psi_{\alpha(s+1)\dot{\alpha}(s)}$$

$$\Big\{\mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}_s}H_{\alpha(s)\dot{\alpha}(s)} + \frac{i(s+1)}{s!}\mathrm{D}_{(\alpha_s}\partial^{\gamma\dot{\gamma}}H_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)}$$
$$+\frac{s+1}{s!(s-1)!}\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}_{(\dot{\alpha}_{s-1}}\chi_{\alpha(s-1))\dot{\alpha}(s-2))}\Big\}| \equiv N_2\ \psi_{\alpha(s)\dot{\alpha}(s-1)}$$

$$\Big\{i\bar{\mathrm{D}}^{\dot{\beta}}\partial^{\gamma\dot{\gamma}}H_{\gamma\alpha(s-1)\dot{\beta}\dot{\gamma}\dot{\alpha}(s-2)}$$
$$+\frac{1}{(s-1)!}\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\mathrm{D}_{(\alpha_{s-1}}\bar{\chi}_{\alpha(s-2))\dot{\alpha}(s-1)}\Big\}| \equiv N_3\ \psi_{\alpha(s-1)\dot{\alpha}(s-2)}$$

Putting everything together, the component Lagrangian takes the form

$$\begin{aligned}
\mathcal{L}_F =\ & 2ic|N_1|^2\ \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)}\partial^{\alpha_{s+1}}{}_{\dot{\alpha}_{s+1}}\psi_{\alpha(s+1)\dot{\alpha}(s)} \\
& -2ic\frac{s^2}{(2s+1)(s+1)}N_1N_2\ \psi^{\alpha(s+1)\dot{\alpha}(s)}\partial_{\alpha_{s+1}\dot{\alpha}_s}\psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& -2ic\frac{s^2}{(2s+1)(s+1)^2}|N_2|^2\ \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)}\partial^{\alpha_s}{}_{\dot{\alpha}_s}\psi_{\alpha(s)\dot{\alpha}(s-1)} \\
& -2ic\frac{s(s-1)}{(2s+1)(s+1)}N_2N_3\ \psi^{\alpha(s)\dot{\alpha}(s-1)}\partial_{\alpha_s\dot{\alpha}_{s-1}}\psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\
& -2ic\left(\frac{s-1}{s+1}\right)^2|N_3|^2\ \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)}\partial^{\alpha_{s-1}}{}_{\dot{\alpha}_{s-1}}\psi_{\alpha(s-1)\dot{\alpha}(s-2)} \\
& +\frac{1}{2c}\frac{(2s+1)(s-1)}{s^2(s+1)^2}G^{\alpha(s)\dot{\alpha}(s-1)}|\Big(\mathrm{D}^2G_{\alpha(s-1)\dot{\alpha}(s-2)} \\
& -\frac{i}{2}\frac{s-1}{s+1}\frac{1}{(s-1)!}\partial_{(\alpha_{s-1}}{}^{\dot{\alpha}_{s-1}}\bar{G}_{\alpha(s-2))\dot{\alpha}(s-1)}\Big)| + c.c
\end{aligned} \tag{5.53}$$

The last term in the Lagrangian is the algebraic kinetic energy term of two auxiliary fields and the rest of the terms are exactly the struc- ture of a theory that describes helicity $h = s + 1/2$. To have an exact match we choose coefficients

$$c = 1\,, \qquad N_2 = -\frac{1}{\sqrt{2}}\frac{2s+1}{s}$$
$$N_1 = \frac{1}{\sqrt{2}}\,, \qquad N_3 = \frac{1}{\sqrt{2}}\frac{s+1}{s-1}$$

So the fields that appear in the fermionic action are defined as:

$$\begin{aligned}
\rho_{\alpha(s-1)\dot{\alpha}(s-2)} &\equiv G_{\alpha(s-1)\dot{\alpha}(s-2)}| \\
\beta_{\alpha(s-1)\dot{\alpha}(s-2)} &\equiv \Big\{\mathrm{D}^2G_{\alpha(s-1)\dot{\alpha}(s-2)} \\
&\qquad -\frac{i}{2}\frac{s-1}{s+1}\frac{1}{(s-1)!}\partial_{(\alpha_{s-1}}{}^{\dot{\alpha}_{s-1}}\bar{G}_{\alpha(s-2))\dot{\alpha}(s-1)}\Big\}| \\
\psi_{a(s+1)\dot{\alpha}(s)} &\equiv \frac{\sqrt{2}}{(s+1)!}\bar{\mathrm{D}}^2\mathrm{D}_{(\alpha_{s+1}}H_{\alpha(s))\dot{\alpha}(s)}|
\end{aligned}$$

$$\psi_{\alpha(s)\dot{\alpha}(s-1)} \equiv -\sqrt{2}\,\frac{s}{2s+1}\Big\{\mathrm{D}^2\bar{\mathrm{D}}^{\dot{\alpha}_s}H_{\alpha(s)\dot{\alpha}(s)}$$
$$+\frac{i(s+1)}{s!}\mathrm{D}_{(\alpha_s}\partial^{\gamma\dot{\gamma}}H_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)}$$
$$+\frac{s+1}{s!(s-1)!}\mathrm{D}_{(\alpha_s}\bar{\mathrm{D}}_{(\dot{\alpha}_{s-1}}\chi_{\alpha(s-1))\dot{\alpha}(s-2))}\Big\}\Big|$$
$$\psi_{\alpha(s-1)\dot{\alpha}(s-2)} \equiv \sqrt{2}\,\frac{s-1}{s+1}\Big\{i\bar{\mathrm{D}}^{\dot{\beta}}\partial^{\gamma\dot{\gamma}}H_{\gamma\alpha(s-1)\dot{\beta}\dot{\gamma}\dot{\alpha}(s-2)}$$
$$+\frac{1}{(s-1)!}\bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\mathrm{D}_{(\alpha_{s-1}}\bar{\chi}_{\alpha(s-2))\dot{\alpha}(s-1)}\Big\}\Big| \tag{5.54}$$

The Lagrangian is

$$\begin{aligned}
\mathcal{L}_F =\ & \rho^{\alpha(s)\dot{\alpha}(s-1)}\beta_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& +i\ \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)}\partial^{\alpha_{s+1}}{}_{\dot{\alpha}_{s+1}}\psi_{\alpha(s+1)\dot{\alpha}(s)} \\
& +i\left[\frac{s}{s+1}\right]\ \psi^{\alpha(s+1)\dot{\alpha}(s)}\partial_{\alpha_{s+1}\dot{\alpha}_s}\psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& -i\left[\frac{2s+1}{(s+1)^2}\right]\ \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)}\partial^{\alpha_s}{}_{\dot{\alpha}_s}\psi_{\alpha(s)\dot{\alpha}(s-1)} \\
& +i\ \psi^{\alpha(s)\dot{\alpha}(s-1)}\partial_{\alpha_s\dot{\alpha}_{s-1}}\psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\
& -i\ \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)}\partial^{\alpha_{s-1}}{}_{\dot{\alpha}_{s-1}}\psi_{\alpha(s-1)\dot{\alpha}(s-2)}
\end{aligned} \tag{5.55}$$

and the gauge transformations of the fields are

$$\begin{aligned}
\delta_G\rho_{\alpha(s)\dot{\alpha}(s-1)} &= 0\,, & \delta_G\psi_{\alpha(s+1)\dot{\alpha}(s)} &= \frac{1}{s!(s+1)!}\partial_{(\alpha_{s+1}(\dot{\alpha}_s}\xi_{\alpha(s))\dot{\alpha}(s-1))} \\
\delta_G\beta_{\alpha(s)\dot{\alpha}(s-1)} &= 0\,, & \delta_G\psi_{\alpha(s)\dot{\alpha}(s-1)} &= -\frac{1}{s!}\partial_{(\alpha_s}{}^{\dot{\alpha}_s}\bar{\xi}_{\alpha(s-1))\dot{\alpha}(s)} \\
& & \delta_G\psi_{\alpha(s-1)\dot{\alpha}(s-2)} &= \frac{s-1}{s}\partial^{\alpha_s\dot{\alpha}_{s-1}}\xi_{\alpha(s)\dot{\alpha}(s-1)} \\
& & \text{with } \xi_{\alpha(s)\dot{\alpha}(s-1)} &= -i\sqrt{2}\ \bar{\mathrm{D}}^2 L_{\alpha(s)\dot{\alpha}(s-1)}\big|
\end{aligned} \tag{5.56}$$

5.3.2 Bosons

For the bosonic Lagrangian we do the same. The fields that appear in the action are defined as:

$$\begin{aligned}
A_{\alpha(s)\dot{\alpha}(s)} &\equiv T_{\alpha(s))\dot{\alpha}(s)}\big| \\
U_{\alpha(s)\dot{\alpha}(s-2)} &\equiv \frac{1}{s!}\mathrm{D}_{(\alpha_s}G_{\alpha(s-1))\dot{\alpha}(s-2)}\big| \\
u_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv \frac{1}{2(s-1)!}\left\{\bar{\mathrm{D}}_{(\dot{\alpha}_{s-1}}G_{\alpha(s-1)\dot{\alpha}(s-2))} - \mathrm{D}_{(\alpha_{s-1}}\bar{G}_{\alpha(s-2))\dot{\alpha}(s-1)}\right\}\big| \\
v_{\alpha(s-1)\dot{\alpha}(s-1)} &\equiv -\frac{i}{2(s-1)!}\left\{\bar{\mathrm{D}}_{(\dot{\alpha}_{s-1}}G_{\alpha(s-1)\dot{\alpha}(s-2))} + \mathrm{D}_{(\alpha_{s-1}}\bar{G}_{\alpha(s-2))\dot{\alpha}(s-1)}\right\}\big| \\
S_{\alpha(s-2)\dot{\alpha}(s-2)} &\equiv \frac{1}{2}\left\{\mathrm{D}^{\alpha_{s-1}}G_{\alpha(s-1)\dot{\alpha}(s-2)} + \bar{\mathrm{D}}^{\dot{\alpha}_{s-1}}\bar{G}_{\alpha(s-2)\dot{\alpha}(s-1)}\right\}\big|
\end{aligned}$$

$$
\begin{aligned}
P_{\alpha(s-2)\dot\alpha(s-2)} &\equiv -\frac{i}{2}\left\{\mathrm{D}^{\alpha_{s-1}}G_{\alpha(s-1)\dot\alpha(s-2)} - \bar{\mathrm{D}}^{\dot\alpha_{s-1}}\bar{G}_{\alpha(s-2)\dot\alpha(s-1)}\right\}| \\
h_{\alpha(s+1)\dot\alpha(s+1)} &\equiv \frac{1}{2}\frac{1}{(s+1)!^2}\left[\mathrm{D}_{(\alpha_{s+1}},\bar{\mathrm{D}}_{(\dot\alpha_{s+1}}\right]H_{\alpha(s))\dot\alpha(s))}| \\
h_{\alpha(s-1)\dot\alpha(s-1)} &\equiv -\frac{1}{2}\frac{s}{(2s+1)(s+1)^2}\left[\mathrm{D}^{\alpha_s},\bar{\mathrm{D}}^{\dot\alpha_s}\right]H_{\alpha(s)\dot\alpha(s)}| \\
&\quad -\frac{s}{(2s+1)(s+1)}\frac{1}{(s-1)!}\Big(\mathrm{D}_{(\alpha_{s-1}}\bar\chi_{\alpha(s-2))\dot\alpha(s-1)} \\
&\quad -\bar{\mathrm{D}}_{(\dot\alpha_{s-1}}\chi_{\alpha(s-1)\dot\alpha(s-2))}\Big)|
\end{aligned} \tag{5.57}
$$

the gauge transformations are

$$
\begin{aligned}
\delta_G U_{\alpha(s)\dot\alpha(s-2)} &= 0, \qquad \delta_G A_{\alpha(s)\dot\alpha(s)} = 0 \\
\delta_G u_{\alpha(s-1)\dot\alpha(s-1)} &= 0, \quad \delta_G S_{\alpha(s-2)\dot\alpha(s-2)} = 0 \\
\delta_G v_{\alpha(s-1)\dot\alpha(s-1)} &= 0, \quad \delta_G P_{\alpha(s-2)\dot\alpha(s-2)} = 0 \\
\delta_G h_{\alpha(s+1)\dot\alpha(s+1)} &= \frac{1}{(s+1)!^2}\partial_{(\alpha_{s+1}(\dot\alpha_{s+1}}\zeta_{\alpha(s))\dot\alpha(s))} \\
\delta_G h_{\alpha(s-1)\dot\alpha(s-1)} &= \frac{s}{(s+1)^2}\partial^{\alpha_s\dot\alpha_s}\zeta_{\alpha(s)\dot\alpha(s)}
\end{aligned} \tag{5.58}
$$

where

$$
\zeta_{\alpha(s)\dot\alpha(s)} = \frac{i}{2s!}\left(\mathrm{D}_{(\alpha_s}\bar{L}_{\alpha(s-1))\dot\alpha(s)} + \bar{\mathrm{D}}_{(\dot\alpha_s}L_{\alpha(s)\dot\alpha(s-1))}\right)|
$$

and the bosonic Lagrangian is

$$
\begin{aligned}
\mathcal{L}_B = &-\frac{1}{4}\left[\frac{(2s+1)(s-1)}{s^2(s+1)}\right] U^{\alpha(s)\dot\alpha(s-2)}U_{\alpha(s)\dot\alpha(s-2)} + c.c. \\
&+\frac{1}{8}\left[\frac{2s+1}{s+1}\right] A^{\alpha(s)\dot\alpha(s)}A_{\alpha(s)\dot\alpha(s)} \\
&-\frac{1}{2}\left[\frac{2s+1}{s^2}\right] u^{\alpha(s-1)\dot\alpha(s-1)}u_{\alpha(s-1)\dot\alpha(s-1)} \\
&-\frac{1}{2}\left[\frac{2s+1}{s^2}\right] v^{\alpha(s-1)\dot\alpha(s-1)}v_{\alpha(s-1)\dot\alpha(s-1)} \\
&-\frac{1}{2}\left[\frac{(2s+1)(s-1)^2}{s^3}\right] S^{\alpha(s-2)\dot\alpha(s-2)}S_{\alpha(s-2)\dot\alpha(s-2)} \\
&+\frac{1}{2}\left[\frac{(s-1)^2}{s^3}\right] P^{\alpha(s-2)\dot\alpha(s-2)}P_{\alpha(s-2)\dot\alpha(s-2)} \\
&+ h^{\alpha(s+1)\dot\alpha(s+1)}\Box h_{\alpha(s+1)\dot\alpha(s+1)} \\
&-\left[\frac{s+1}{2}\right] h^{\alpha(s+1)\dot\alpha(s+1)}\partial_{\alpha_{s+1}\dot\alpha_{s+1}}\partial^{\gamma\dot\gamma}h_{\gamma\alpha(s)\dot\gamma\dot\alpha(s)} \\
&+[s(s+1)]\ h^{\alpha(s+1)\dot\alpha(s+1)}\partial_{\alpha_{s+1}\dot\alpha_{s+1}}\partial_{\alpha_s\dot\alpha_s}h_{\alpha(s-1)\dot\alpha(s-1)} \\
&-[(s+1)(2s+1)]\ h^{\alpha(s-1)\dot\alpha(s-1)}\Box h_{\alpha(s-1)\dot\alpha(s-1)} \\
&-\left[\frac{(s+1)(s-1)^2}{2}\right] h^{\alpha(s-1)\dot\alpha(s-1)}\partial_{\alpha_{s-1}\dot\alpha_{s-1}}\partial^{\gamma\dot\gamma}h_{\gamma\alpha(s-2)\dot\gamma\dot\alpha(s-2)}
\end{aligned}
$$

and gives rise to the theory of helicity $h = s + 1$ as expected.

5.3.3 Off-shell degrees of freedom

Let us count the bosonic degrees of freedom

fields	*d.o.f*	*redundancy*	*net*
$h_{\alpha(s+1)\dot\alpha(s+1)}$	$(s+2)^2$	$(s+1)^2$	s^2+2s+3
$h_{\alpha(s-1)\dot\alpha(s-1)}$	s^2		
$u_{\alpha(s-1)\dot\alpha(s-1)}$	s^2	0	s^2
$v_{\alpha(s-1)\dot\alpha(s-1)}$	s^2	0	s^2
$A_{\alpha(s)\dot\alpha(s)}$	$(s+1)^2$	0	$(s+1)^2$
$U_{\alpha(s)\dot\alpha(s-2)}$	$2(s+1)(s-1)$	0	$2(s+1)(s-1)$
$S_{\alpha(s-2)\dot\alpha(s-2)}$	$(s-1)^2$	0	$(s-1)^2$
$P_{\alpha(s-2)\dot\alpha(s-2)}$	$(s-1)^2$	0	$(s-1)^2$
		Total	$8s^2+4$

and the same counting for the fermionic degrees of freedom

fields	*d.o.f*	*redundancy*	*net*
$\psi_{\alpha(s+1)\dot\alpha(s)}$	$2(s+2)(s+1)$	$2(s+1)s$	$4s^2+4s+4$
$\psi_{\alpha(s)\dot\alpha(s-1)}$	$2(s+1)s$		
$\psi_{\alpha(s-1)\dot\alpha(s-2)}$	$2s(s-1)$		
$\rho_{\alpha(s-1)\dot\alpha(s-2)}$	$2(s-1)s$	0	$2(s-1)s$
$\beta_{\alpha(s-1)\dot\alpha(s-2)}$	$2(s-1)s$	0	$2(s-)s$
		Total	$8s^2+4$

5.3.4 SUSY-transformation laws

The explicit expressions for the SUSY-transformation laws of the fields can be found in the same way as for case (I). For the fermionic fields:

$$\begin{aligned}
\delta_S\rho_{\alpha(s-1)\dot\alpha(s-2)} =& -\epsilon^{\alpha_s}U_{\alpha(s)\dot\alpha(s-2)} \\
&+\left[\frac{s-1}{s}\right]\frac{1}{(s-1)!}\epsilon_{(\alpha_{s-1}}\left[S_{\alpha(s-2))\dot\alpha(s-2)}+iP_{\alpha(s-2))\dot\alpha(s-2)}\right] \\
&-\bar\epsilon^{\dot\alpha_{s-1}}\left[u_{\alpha(s-1)\dot\alpha(s-1)}+iv_{\alpha(s-1)\dot\alpha(s-1)}\right] \qquad (5.59)\\
\delta_S\beta_{\alpha(s-1)\dot\alpha(s-2)} =& \frac{i}{2}\frac{s^2}{s+1}\bar\epsilon^{\dot\alpha_{s-1}}\partial^{\alpha_s\dot\alpha_s}A_{\alpha(s)\dot\alpha(s)} \\
&+\frac{s^2}{2s+1}\bar\epsilon^{\dot\alpha_{s-1}}\partial^{\alpha_{s+1}\dot\alpha_{s+1}}\partial^{\alpha_s\dot\alpha_s}h_{\alpha(s+1)\dot\alpha(s+1)} \\
&-2s\bar\epsilon^{\dot\alpha_{s-1}}\Box h_{\alpha(s-1)\dot\alpha(s-2)} \\
&-\frac{s(s-1)^2}{2s+1}\frac{1}{(s-1)!^2}\bar\epsilon^{\dot\alpha_{s-1}}\partial_{(\alpha_{s-1}(\dot\alpha_{s-1}}\partial^{\beta\dot\beta}h_{\beta\alpha(s-2))\dot\beta\dot\alpha(s-2))} \\
&-\frac{i}{(s-1)!}\bar\epsilon^{\dot\alpha_{s-1}}\partial^{\alpha_s}{}_{(\dot\alpha_{s-1}}U_{\alpha(s)\dot\alpha(s-2))}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{s-2}{s-1}\frac{i}{(s-2)!}\bar{\epsilon}_{(\dot{\alpha}_{s-2}}\partial^{\beta\dot{\beta}}U_{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-3))}\\
&+\frac{1}{2}\frac{s-1}{s+1}\frac{i}{(s-1)!}\bar{\epsilon}^{\dot{\alpha}_{s-1}}\partial_{(\alpha_{s-1}}{}^{\dot{\alpha}_s}\bar{U}_{\alpha(s-2))\dot{\alpha}(s)}\\
&+\frac{(s-1)(2s^2+2s+1)}{2s(s+1)}\frac{i}{(s-1)!^2}\bar{\epsilon}^{\dot{\alpha}_{s-1}}\partial_{(\alpha_{s-1}(\dot{\alpha}_{s-1}}S_{\alpha(s-2))\dot{\alpha}(s-2))}\\
&-\frac{(s-1)(2s^2+4s+3)}{2s(s+1)(2s+1)}\frac{1}{(s-1)!^2}\bar{\epsilon}^{\dot{\alpha}_{s-1}}\partial_{(\alpha_{s-1}(\dot{\alpha}_{s-1}}P_{\alpha(s-2))\dot{\alpha}(s-2))}\\
&-\frac{(s-2)(3s+2)}{2s(s+1)}\frac{i}{(s-2)!(s-1)!}\bar{\epsilon}_{(\dot{\alpha}_{s-2}}\partial_{(\alpha_{s-1}}{}^{\dot{\beta}}S_{\alpha(s-2))\dot{\beta}\dot{\alpha}(s-3))}\\
&+\frac{(s-2)(s+2)}{2s(s+1)}\frac{1}{(s-2)!(s-1)!}\bar{\epsilon}_{(\dot{\alpha}_{s-2}}\partial_{(\alpha_{s-1}}{}^{\dot{\beta}}P_{\alpha(s-2))\dot{\beta}\dot{\alpha}(s-3))}\\
&-\frac{1}{2}\frac{s-1}{s+1}\frac{i}{(s-1)!}\epsilon^{\beta}\partial_{(\alpha_{s-1}}{}^{\dot{\alpha}_{s-1}}u_{\beta\alpha(s-2))\dot{\alpha}(s-1)}\\
&-\frac{1}{2}\frac{s-1}{s+1}\frac{1}{(s-1)!}\epsilon^{\beta}\partial_{(\alpha_{s-1}}{}^{\dot{\alpha}_{s-1}}v_{\beta\alpha(s-2))\dot{\alpha}(s-1)}
\end{aligned}
\tag{5.60}
$$

$$
\begin{aligned}
\delta_S\psi_{\alpha(s+1)\dot{\alpha}(s)}=&\frac{\sqrt{2}i}{(s+2)!}\epsilon^{\alpha_{s+2}}\partial_{(\alpha_{s+2}}{}^{\dot{\alpha}_{s+1}}h_{\alpha(s+1))\dot{\alpha}(s+1)}\\
&-\frac{1}{\sqrt{2}}\frac{s}{s+2}\frac{i}{(s+1)!}\epsilon_{(\alpha_{s+1}}\partial^{\gamma\dot{\gamma}}h_{\gamma\alpha(s))\dot{\gamma}\dot{\alpha}(s)}\\
&+\frac{1}{2\sqrt{2}}\frac{2s+1}{s+1}\frac{1}{(s+1)!}\epsilon_{(\alpha_{s+1}}A_{\alpha(s))\dot{\alpha}(s)}
\end{aligned}
\tag{5.61}
$$

$$
\begin{aligned}
\delta_S\psi_{\alpha(s)\dot{\alpha}(s-1)}=&\frac{1}{\sqrt{2}}\frac{s+1}{s}\frac{1}{s!}\epsilon_{(\alpha_s}\big[-u_{\alpha(s-1))\dot{\alpha}(s-1)+iv_{\alpha(s-1))\dot{\alpha}(s-1)}}\big]\\
&+\frac{1}{\sqrt{2}}\frac{s-1}{s}\frac{1}{(s-1)!}\bar{\epsilon}_{(\dot{\alpha}_{s-1}}U_{\alpha(s)\dot{\alpha}(s-2))}\\
&-\frac{1}{2\sqrt{2}}\frac{s}{s+1}\bar{\epsilon}^{\dot{\alpha}_s}A_{\alpha(s)\dot{\alpha}(s)}\\
&-\frac{is}{\sqrt{2}}\bar{\epsilon}^{\dot{\alpha}_s}\partial^{\alpha_{s+1}\dot{\alpha}_{s+1}}h_{\alpha(s+1)\dot{\alpha}(s+1)}\\
&+\frac{is(s+2)}{\sqrt{2}s!s!}\bar{\epsilon}^{\dot{\alpha}_s}\partial_{(\alpha_s(\dot{\alpha}_s}h_{\alpha(s-1))\dot{\alpha}(s-1))}
\end{aligned}
\tag{5.62}
$$

$$
\begin{aligned}
\delta_S\psi_{\alpha(s-1)\dot{\alpha}(s-2)}=&-\frac{1}{\sqrt{2}}\frac{(2s+1)(s-1)}{s^2(s+1)}\bar{\epsilon}^{\dot{\alpha}_{s-1}}u_{\alpha(s-1)\dot{\alpha}(s-1)}\\
&-\frac{i}{\sqrt{2}}\frac{(2s+1)(s-1)}{s^2(s+1)}\bar{\epsilon}^{\dot{\alpha}_{s-1}}v_{\alpha(s-1)\dot{\alpha}(s-1)}\\
&-\frac{1}{\sqrt{2}}\frac{(s-1)^2(2s+1)}{s^2(s+1)}\frac{1}{(s-1)!}\epsilon_{\alpha_{s-1}}S_{\alpha(s-2))\dot{\alpha}(s-2)}\\
&+\frac{i}{\sqrt{2}}\frac{(s-1)^2}{s^2(s+1)}\frac{1}{(s-1)!}\epsilon_{\alpha_{s-1}}P_{\alpha(s-2))\dot{\alpha}(s-2)}\\
&+i\sqrt{2}\frac{(s-1)^2(s+1)}{s}\frac{1}{(s-1)!}\epsilon_{(\alpha_{s-1}}\partial^{\gamma\dot{\gamma}}h_{\gamma\alpha(s-2))\dot{\gamma}\dot{\alpha}(s-2)}
\end{aligned}
\tag{5.63}
$$

and the SUSY-transformation laws for the bosonic fields are:

$$\begin{aligned}
\delta_S A_{\alpha(s)\dot\alpha(s)} =& -\frac{i\sqrt{2}}{(s+1)!}\bar\epsilon^{\dot\alpha_{s+1}}\partial^{\alpha_{s+1}}{}_{(\dot\alpha_{s+1}}\psi_{\alpha(s+1)\dot\alpha(s))}+c.c.\\
&+\frac{i\sqrt{2}}{s!}\frac{s^2}{(s+1)(2s+1)}\bar\epsilon_{(\dot\alpha_s}\partial^{\gamma\dot\gamma}\psi_{\gamma\alpha(s)\dot\gamma\dot\alpha(s-1))}+c.c.\\
&+\frac{i\sqrt{2}}{(s+1)!s!}\frac{s}{s+1}\bar\epsilon^{\dot\alpha_{s+1}}\partial_{(\alpha_s(\dot\alpha_{s+1}}\bar\psi_{\alpha(s-1))\dot\alpha(s))}+c.c.\\
&+\frac{i\sqrt{2}}{s!s!}\frac{s}{(s+1)^2}\bar\epsilon_{(\dot\alpha_s}\partial_{(\alpha_s}{}^{\dot\gamma}\bar\psi_{\alpha(s-1))\dot\gamma\dot\alpha(s-1))}+c.c.\\
&-\frac{i\sqrt{2}}{s!s!}\frac{s}{2s+1}\bar\epsilon_{(\dot\alpha_s}\partial_{(\alpha_s\dot\alpha_{s-1}}\psi_{\alpha(s-1))\dot\alpha(s-2))}+c.c.\\
&-\frac{i}{s!s!}\frac{s-1}{s+1}\bar\epsilon_{(\dot\alpha_s}\partial_{(\alpha_s\dot\alpha_{s-1}}\rho_{\alpha(s-1))\dot\alpha(s-2))}+c.c.
\end{aligned}\tag{5.64}$$

$$\begin{aligned}
\delta_S U_{\alpha(s)\dot\alpha(s-2)} =& \frac{1}{s!}\epsilon_{(\alpha_s}\beta_{\alpha(s-1))\dot\alpha(s-2)}\\
&-\frac{i}{s!(s-1)!}\bar\epsilon^{\dot\alpha_{s-1}}\partial_{(\alpha_s(\dot\alpha_{s-1}}\rho_{\alpha(s-1))\dot\alpha(s-2))}\\
&+\frac{i}{s!(s-2)!}\frac{s-2}{s-1}\bar\epsilon_{(\dot\alpha_{s-2}}\partial_{(\alpha_s}{}^{\dot\gamma}\rho_{\alpha(s-1))\dot\gamma\dot\alpha(s-3))}\\
&+\frac{i}{s!}\frac{s-1}{2(s+1)}\epsilon_{(\alpha_s}\partial_{\alpha_{s-1}}{}^{\dot\alpha_{s-1}}\bar\rho_{\alpha(s-2))\dot\alpha(s-1)}\\
&-i\sqrt{2}\frac{s^2}{2s+1}\bar\epsilon^{\dot\alpha_{s-1}}\partial^{\alpha_{s+1}\dot\alpha_s}\psi_{\alpha(s+1)\dot\alpha(s)}\\
&-\frac{i\sqrt{2}}{s!}\frac{s}{s+1}\bar\epsilon^{\dot\alpha_{s-1}}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar\psi_{\alpha(s-1))\dot\alpha(s)}\\
&+\frac{i\sqrt{2}}{s!(s-1)!}\frac{s(s+1)}{2s+1}\bar\epsilon^{\dot\alpha_{s-1}}\partial_{(\alpha_s(\dot\alpha_{s-1}}\psi_{\alpha(s-1))\dot\alpha(s-2))}
\end{aligned}\tag{5.65}$$

$$\begin{aligned}
&\delta_S\left(u_{\alpha(s-1)\dot\alpha(s-1)}+iv_{\alpha(s-1)\dot\alpha(s-1)}\right)=\\
&=i\sqrt{2}\frac{s^2}{2s+1}\epsilon^{\alpha_s}\partial^{\alpha_{s+1}\dot\alpha_s}\psi_{\alpha(s+1)\dot\alpha(s)}\\
&\quad+i\sqrt{2}\frac{s}{s+1}\frac{1}{s!}\epsilon^{\alpha_s}\partial_{(\alpha_s}{}^{\dot\alpha_s}\bar\psi_{\alpha(s-1))\dot\alpha(s)}\\
&\quad-i\sqrt{2}\frac{s(s+1)}{2s+1}\frac{1}{s!(s-1)!}\epsilon^{\alpha_s}\partial_{(\alpha_s(\dot\alpha_{s-1}}\psi_{\alpha(s-1))\dot\alpha(s-2)))}\\
&\quad-\frac{s^2}{(s+1)(s-1)}\frac{1}{(s-1)!}\epsilon_{(\alpha_{s-1}}\bar\beta_{\alpha(s-2))\dot\alpha(s-1)}\\
&\quad-i\sqrt{2}\frac{s^2}{(s+1)(s-1)}\frac{1}{(s-1)!}\epsilon_{(\alpha_{s-1}}\partial^{\gamma\dot\alpha_s}\bar\psi_{\gamma\alpha(s-2))\dot\alpha(s)}\\
&\quad+i\sqrt{2}\frac{s^2}{(s+1)(s-1)}\frac{1}{(s-1)!}\epsilon_{(\alpha_{s-1}}\partial^{\gamma}{}_{(\dot\alpha_{s-1}}\psi_{\gamma\alpha(s-2))\dot\alpha(s-2))}
\end{aligned}\tag{5.66}$$

$$
\begin{aligned}
&\delta_S\left(S_{\alpha(s-2)\dot\alpha(s-2)}+iP_{\alpha(s-2)\dot\alpha(s-2)}\right)=\\
&=\epsilon^{\alpha_{s-1}}\beta_{\alpha(s-1)\dot\alpha(s-2)}\\
&\quad+\frac{i}{2}\frac{s-1}{s+1}\frac{1}{(s-1)!}\epsilon^{\alpha_{s-1}}\partial_{(\alpha_{s-1}}{}^{\dot\alpha_{s-1}}\bar\rho_{\alpha(s-2))\dot\alpha_{s-1}}\\
&\quad-\frac{i}{(s-1)!}\bar\epsilon^{\dot\alpha_{s-1}}\partial^{\alpha_{s-1}}{}_{(\dot\alpha_{s-1}}\rho_{\alpha(s-1)\dot\alpha(s-2))}\\
&\quad+\frac{i}{(s-2)!}\frac{s-2}{s-1}\bar\epsilon_{(\dot\alpha_{s-2}}\partial^{\gamma\dot\gamma}\rho_{\gamma\alpha(s-2)\dot\gamma\dot\alpha(s-3))}\\
&\quad-\frac{s}{s+1}\bar\epsilon^{\dot\alpha_{s-1}}\bar\beta_{\alpha(s-2)\dot\alpha(s-1)}\\
&\quad-i\sqrt{2}\frac{s}{s+1}\bar\epsilon^{\dot\alpha_{s-1}}\partial^{\alpha_{s-1}\dot\alpha_s}\bar\psi_{\alpha(s-1)\dot\alpha(s)}\\
&\quad+\frac{i\sqrt{2}}{(s-1)!}\frac{s}{s+1}\bar\epsilon^{\dot\alpha_{s-1}}\partial^{\alpha_{s-1}}{}_{(\dot\alpha_{s-1}}\psi_{\alpha(s-1)\dot\alpha(s-2))}
\end{aligned}
\tag{5.67}
$$

$$
\delta_S h_{\alpha(s+1)\dot\alpha(s+1)}=\frac{1}{\sqrt{2}(s+1)!}\epsilon_{(\alpha_{s+1}}\bar\psi_{\alpha(s))\dot\alpha(s+1)}+c.c. \tag{5.68}
$$

$$
\begin{aligned}
\delta_S h_{\alpha(s-1)\dot\alpha(s-1)}=&\frac{1}{\sqrt{2}}\frac{1}{(s+1)^2}\epsilon^{\alpha_s}\psi_{\alpha(s)\dot\alpha(s-1)}+c.c.\\
&+\frac{1}{\sqrt{2}}\frac{1}{(s+1)}\frac{1}{(s-1)!}\epsilon_{(\alpha_{s-1}}\bar\psi_{\alpha(s-2))\dot\alpha(s-1)}+c.c.\\
&-\frac{1}{2}\frac{s-1}{s(s+1)^2}\frac{1}{(s-1)!}\epsilon_{(\alpha_{s-1}}\bar\rho_{\alpha(s-2))\dot\alpha(s-1)}+c.c.
\end{aligned}
\tag{5.69}
$$

6 Map of superhelicity theories and hints for $\mathcal{N}=2$

To summarize the results, the landscape of the massless irreducible representations that describe the highest superhelicity supermultiplets is shown in figure 1.

There are three infinite towers of theories, one for the integer superhelicity and two for the half-integer superhelicity. A solid line represents the corresponding theory for that value of s. The corresponding theory for integer and half odd integer (I) is a two parameter family of actions, but for half odd integer (II) it is a unique action. At the bottom, the superfield structure of the action and the number of degrees of freedom involved are being displayed. For the $s=0$ case of the integer tower, there is a gap. The reason is that for $s=0$ there is no superfield Ψ and the tower starts from $s=1$. The $Y=s=0$ theory is being generated by a chiral superfield Φ. Similarly the $s=1$ case of the half odd integer (II) theories, where its place takes a triplet of theories, the old minimal, the new minimal and the new-new minimal. The dash line represent theories that don't fall in the pattern. These are low helicity 'accidents' that don't generalize to arbitrary s.

A very intriguing observation is that the number of off-shell degrees of freedom for integer superhelicity and half odd integer superhelicity (I) theories is the same. That means that for every boson in one theory there is a fermion in the other. So if we add

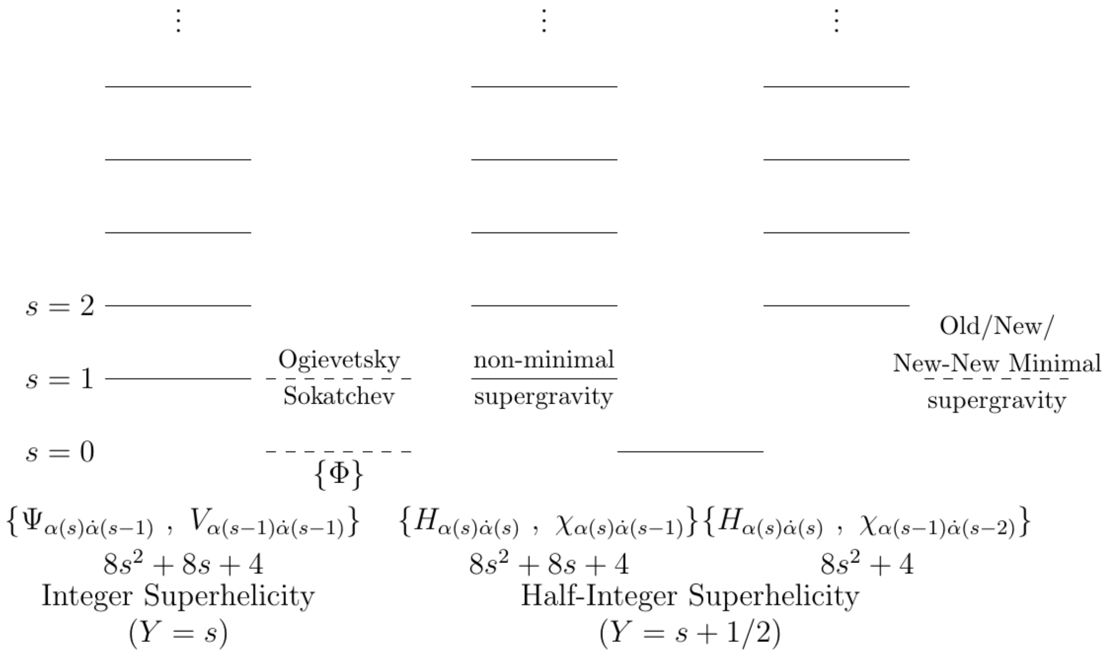

Figure 1. Free massless highest superhelicity irreducible representations.

these two theories together we can have a second direction of supersymmetry. Therefore we can construct an irreducible representation of $4D$, $\mathcal{N} = 2$ Super-Poincaré group. Also in the same manner we can understand why a possible pair of integer theory with half odd integer (II) theories can never work. This has been confirmed by trial an error in [29] where the second supersymmetry was realized non-linearly in terms of the covariant derivatives of the first supersymmetry.

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References

[1] P.A.M. Dirac, *Relativistic Wave Equations*, *Proc. Roy. Soc. Lond.* **A 155** (1936) 447.

[2] M. Fierz and W. Pauli, *On relativistic wave equations for particles of arbitrary spin in an electromagnetic field*, *Proc. Roy. Soc. Lond.* **A 173** (1939) 211 [INSPIRE].

[3] L.P.S. Singh and C.R. Hagen, *Lagrangian formulation for arbitrary spin. 1. The boson case*, *Phys. Rev.* **D 9** (1974) 898 [INSPIRE].

JHEP06(2014)098

[4] L.P.S. Singh and C.R. Hagen, *Lagrangian formulation for arbitrary spin. 2. The fermion case*, Phys. Rev. **D 9** (1974) 910 [INSPIRE].

[5] C. Fronsdal, *Massless Fields with Integer Spin*, Phys. Rev. **D 18** (1978) 3624 [INSPIRE].

[6] J. Fang and C. Fronsdal, *Massless Fields with Half Integral Spin*, Phys. Rev. **D 18** (1978) 3630 [INSPIRE].

[7] M.A. Vasiliev, *Higher spin gauge theories in various dimensions*, Fortsch. Phys. **52** (2004) 702 [hep-th/0401177] [INSPIRE].

[8] I.L. Buchbinder and V.A. Krykhtin, *Gauge invariant Lagrangian construction for massive bosonic higher spin fields in D dimensions*, Nucl. Phys. **B 727** (2005) 537 [hep-th/0505092] [INSPIRE].

[9] S.D. Rindani and M. Sivakumar, *Gauge-invariant description of massive higher-spin particles by dimensional reduction*, Phys. Rev. **D 32** (1985) 3238.

[10] I. Buchbinder and V.A. Krykhtin, *BRST approach to higher spin field theories*, hep-th/0511276 [INSPIRE].

[11] T. Curtright, *Massless Field Supermultiplets With Arbitrary Spin*, Phys. Lett. **B 85** (1979) 219 [INSPIRE].

[12] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace Or One Thousand and One Lessons in Supersymmetry*, Benjamin/Cummings, Reading U.S.A. (1983) [hep-th/0108200] [INSPIRE].

[13] S.M. Kuzenko, A.G. Sibiryakov and V.V. Postnikov, *Massless gauge superfields of higher half integer superspins*, *JETP Lett.* **57** (1993) 534 [*Pisma Zh. Eksp. Teor. Fiz.* **57** (1993) 521] [INSPIRE].

[14] S.M. Kuzenko and A.G. Sibiryakov, *Massless gauge superfields of higher integer superspins*, *JETP Lett.* **57** (1993) 539 [*Pisma Zh. Eksp. Teor. Fiz.* **57** (1993) 526] [INSPIRE].

[15] I.L. Buchbinder, S.M. Kuzenko and A.G. Sibiryakov, *Quantization of higher spin superfields in the anti-de Sitter superspace*, Phys. Lett. **B 352** (1995) 29 [hep-th/9502148] [INSPIRE].

[16] S.J. Gates Jr. and S.M. Kuzenko, *4D, $N = 1$ higher spin gauge superfields and quantized twistors*, JHEP **10** (2005) 008 [hep-th/0506255] [INSPIRE].

[17] S.M. Kuzenko, *Comments on the gauge models for massless higher spin off-shell supermultiplets*, arXiv:1103.5584 [INSPIRE].

[18] S.J. Gates Jr. and K. Koutrolikos, *A Codicil To Massless Gauge Superfields of Higher Half-Odd Integer Superspins*, arXiv:1103.3564 [INSPIRE].

[19] S.J. Gates Jr. and K. Koutrolikos, *A Codicil To Massless Gauge Superfields of Higher Integer Superspins*, arXiv:1103.3565 [INSPIRE].

[20] S.J. Gates Jr. and W. Siegel, *(3/2, 1) Superfield of $O(2)$ Supergravity*, Nucl. Phys. **B 164** (1980) 484 [INSPIRE].

[21] S.J. Gates Jr., I. Linch, William Divine and J. Phillips, *Field strengths of linearized $5 - D$, $N = 1$ superfield supergravity on a three-brane*, JHEP **02** (2005) 036 [hep-th/0311153] [INSPIRE].

[22] S.M. Kuzenko, A.G. Sibiryakov and V.V. Postnikov, *Massless gauge superfields of higher half integer superspins*, *JETP Lett.* **57** (1993) 534 [INSPIRE].

[23] I.L. Buchbinder and S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity: Or A Walk Through Superspace*, IOP press, Bristol U.K. (1998).

[24] S.J. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, *Superspace Or One Thousand and One Lessons in Supersymmetry*, *Front. Phys.* **58** (1983) 1.

[25] I.L. Buchbinder, S.J. Gates Jr., I. Linch, William Divine and J. Phillips, *New* $4-D$, $N=1$ *superfield theory: Model of free massive superspin* 3/2 *multiplet*, *Phys. Lett.* **B 535** (2002) 280 [hep-th/0201096] [INSPIRE].

[26] I.L. Buchbinder, S.J. Gates Jr., I. Linch, William Divine and J. Phillips, *Dynamical superfield theory of free massive superspin-1 multiplet*, *Phys. Lett.* **B 549** (2002) 229 [hep-th/0207243] [INSPIRE].

[27] S.J. Gates Jr., S.M. Kuzenko and G. Tartaglino-Mazzucchelli, *New massive supergravity multiplets*, *JHEP* **02** (2007) 052 [hep-th/0610333] [INSPIRE].

[28] S.J. Gates Jr. and K. Koutrolikos, *A Dynamical Theory for Free Massive Superspin* 3/2, UMDEPP-013-017.

[29] S.J. Gates Jr., S.M. Kuzenko and A.G. Sibiryakov, $N=2$ *supersymmetry of higher superspin massless theories*, *Phys. Lett.* **B 412** (1997) 059 [hep-th/9609141].

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