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AdS pure spinor superstring in constant backgrounds

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Abstract: In this paper we study the pure spinor formulation of the superstring in $AdS_5 \times S^5$ around point particle solutions of the classical equations of motion. As a particular example we quantize the pure spinor string in the BMN background.

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Contents

1 Introduction

Although a lot of work has been done on strings on $AdS_5 \times S^5$, no complete quantization of the model has been performed. The classical integrability found in the Green-Schwarz [1] and pure spinor [2] formalisms, has not been extended to the full quantum theory. Interesting work in this direction was done by Benichou [3], where the pure spinor formalism was used to derive the Y-system equations [4] using world sheet techniques.

One particular advantage of the pure spinor formalism is that it can be used without fixing any gauge, so conformal field theory methods (and conformal perturbation theory) can be used [5, 6]. Quantum conformal invariance for any on shell classical background was proven in [7, 8]. Other quantum consistency checks for the AdS pure spinor string were done in [9, 10]. Furthermore, backgrounds that do not admit light-cone gauge fixing, the pure spinor description can be used. The formalism was recently used [11][1] to compute the energy of a particular string state conjectured to be dual to one element of the Konishi multiplet. The result agreed with the strong coupling computation using the Y-system [14]. Some aspects of semiclassical quantization of the AdS pure spinor superstring have been discussed in [15–17].

In this paper we use the background field method to study the pure spinor superstring in constant backgrounds. The method is the same as the one used in [11], although more details will be presented. The particular case of a BPS background [18] is given as an example. Unlike in [11], the BMN limit can be used and hence the full spectrum of the string

[1]For related work, see [12, 13].

can be computed. As expected, it agrees with light-cone GS formalism. The approach taken here is very different from a previous description of the pp-wave background using pure spinors [19]. In that work, the background is described by a contraction of the $\mathfrak{psu}(2,2|4)$ algebra and hence is nonlinear in the world-sheet variables. We will see that the background field method gives a much simpler description. We will also see that the part of the spectrum related to the supergravity multiplet has a very simple description, almost exactly as in the light-cone GS quantization [20]. This is a surprise because massless vertex operators for the pure spinor string in AdS are not known in general. For progress in this direction, see [21–24]. It would be interesting to study the spectrum we obtain away from the BMN limit.

In section 2 we review the construction of the string on a $AdS_5 \times S^5$ background using the pure spinor formalism. In section 3 we expand the world-sheet action around a bosonic solution of the classical equations of motion and verify how the BRST and conformal transformations are compensated because of a gauge fixing and such that the action remains invariant. Finally, in section 4 we study the special case of the BPS background, in particular, we study the spectrum in this case.

2 AdS pure spinor string

In this section we present a short review of the pure spinor formalism for the AdS background and describe the BPS background used in the rest of the paper.

It is well-known that the $AdS_5 \times S^5$ background is described by the coset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}$. Elements of this coset are described by $g \in \mathrm{PSU}(2,2|4)$ and two elements are identified if they differ by local right multiplication by en element of $\mathrm{SO}(1,4) \times \mathrm{SO}(5)$. Classical solutions in this background are elements of the coset $g(\tau,\sigma)$ satisfying the equations of motion that come from the action plus the classical Virasoro condition. In this section we use the notation $z = \tau - \sigma$, $\bar{z} = \tau + \sigma$ and σ is periodic with period 2π, and we use the world-sheet derivatives $\partial = \frac{1}{2}(\partial_\tau - \partial_\sigma), \overline{\partial} = \frac{1}{2}(\partial_\tau + \partial_\sigma)$.

The action in the pure spinor formalism is given by [6, 8]

$$S = \langle \frac{1}{2} J_2 \overline{J}_2 + \frac{3}{4} J_3 \overline{J}_1 + \frac{1}{4} J_1 \overline{J}_3 + \omega \overline{\nabla} \lambda + \overline{\omega} \nabla \overline{\lambda} - N \overline{N} \rangle, \tag{2.1}$$

where

$$\langle \cdots \rangle = \frac{1}{\pi \alpha'} \int d^2 z \mathrm{Tr}\,, \tag{2.2}$$

and, thanks to the $\mathbb{Z}_4$ grading of the $\mathfrak{psu}(2,2|4)$ Lie algebra,[2] the Metsaev-Tseytlin currents $J = g^{-1}\partial g$ have components

$$J_0 = (g^{-1}\partial g)^{[mn]} T_{[mn]}, \quad J_1 = (g^{-1}\partial g)^{\alpha} T_\alpha, \quad J_2 = (g^{-1}\partial g)^{m} T_m, \quad J_3 = (g^{-1}\partial g)^{\widehat{\alpha}} T_{\widehat{\alpha}}, \tag{2.3}$$

where $\{T_{[mn]}, T_m, T_\alpha, T_{\hat{\alpha}}\}$ are the algebra generators. Later we will use a more convenient basis. The non zero commutators can be found in the appendix. Similarly we define $\overline{J} = g^{-1}\overline{\partial} g$. The pure spinor variables are defined as

$$\omega = \omega_\alpha T_{\widehat{\alpha}} \delta^{\alpha\widehat{\alpha}}, \qquad \lambda = \lambda^\alpha T_\alpha, \qquad N = -\{\omega, \lambda\}, \tag{2.4}$$

[2] The Lie algebra $\mathcal{H}$ is decomposed as $\mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$. Note that the commutators in the Lie algebra map $\mathcal{H}_i \times \mathcal{H}_j$ into $\mathcal{H}_{i+j}$ mod 4.

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$$\overline{\omega} = \overline{\omega}_{\widehat{\alpha}} T_{\alpha} \delta^{\alpha\widehat{\alpha}}, \qquad \overline{\lambda} = \overline{\lambda}^{\widehat{\alpha}} T_{\widehat{\alpha}}, \qquad \overline{N} = -\{\bar{\omega}, \bar{\lambda}\}. \tag{2.5}$$

The world-sheet covariant derivatives in the action are defined as

$$\nabla = \partial + [J_0, \ ], \quad \overline{\nabla} = \overline{\partial} + [\overline{J}_0, \ ].$$

The BRST charge is given by

$$Q = \oint d\sigma \mathrm{Tr}\,[\lambda J_3 + \overline{\lambda}\bar{J}_1]. \tag{2.6}$$

We study classical solutions with vanishing BRST charge. A simple way to get this is with $J_3 = \overline{J}_1 = 0$ or with vanishing λ and $\overline{\lambda}$.

The theory is also constrained by the conformal symmetry which is generated by the stress tensor. The classical stress tensor is given by

$$T = \frac{1}{\alpha'}\mathrm{Tr}\,[J_2 J_2 + 2 J_1 J_3 + \omega\nabla\lambda], \quad \overline{T} = \frac{1}{\alpha'}\mathrm{Tr}\,[\overline{J}_2\overline{J}_2 + 2\overline{J}_1\overline{J}_3 + \overline{\omega}\overline{\nabla}\overline{\lambda}]. \tag{2.7}$$

Using the world-sheet equations of motion, it can be shown that T is holomorphic and $\overline{T}$ is antiholomorphic. The conformal transformation of the group element g is

$$\delta g = \varepsilon\partial g + \overline{\varepsilon}\overline{\partial} g, \tag{2.8}$$

where ε is a holomorphic parameter and $\overline{\varepsilon}$ is an antiholomorphic parameter. The pure spinor variables transform as

$$\begin{aligned} \delta\lambda &= \varepsilon\partial\lambda + \overline{\varepsilon}\overline{\partial}\lambda, & \delta\omega &= \partial(\varepsilon\omega) + \overline{\varepsilon}\overline{\partial}\omega \\ \delta\widehat{\lambda} &= \varepsilon\partial\widehat{\lambda} + \overline{\varepsilon}\overline{\partial}\widehat{\lambda}, & \delta\widehat{\omega} &= \overline{\partial}(\overline{\varepsilon}\widehat{\omega}) + \varepsilon\partial\widehat{\omega}. \end{aligned} \tag{2.9}$$

It is possible to show that, under (2.8) and (2.9), the action (2.1) is invariant. A similar calculation shows that, using Noether theorem, (2.7) are the conserved quantities determined by this symmetry.

3 Background field expansion

We are interested in σ independent classical solutions which are not necessarily BPS. For instance, the action (2.1) has a classical non BPS bosonic solution with energy $\mathcal{E}$ and angular momentum $\mathcal{J}$ in one direction of S^5. The solution is given by the group element

$$g = e^{\alpha'\tau(-\mathcal{E}T + \mathcal{J}J)}, \tag{3.1}$$

where τ is the time coordinate of the world-sheet. Here T is the translation along the time-like direction of AdS_5 and J is a translation along a great equator of S^5. Note that both T and J belong to $\mathcal{H}_2$. The pure spinor variables are zero for this solution.

To show that $\mathcal{E}$ and $\mathcal{J}$ are the energy and the angular momentum of the classical solution, we first find the conserved current due to the global PSU$(2,2|4)$ symmetry of the action (2.1). Infinitesimally, we have $\delta g = \varepsilon g$ and we follow the Noether procedure to find

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the conserved current. That is, we assume ε to be non-constant such that the action varies with a term linear in the derivative of ε. Varying the action (2.1) we obtain

$$\delta S = \langle \partial\varepsilon\bar{j} + \bar{\partial}\varepsilon j\rangle, \tag{3.2}$$

where

$$j = g\left(\frac{1}{2}J_2 + \frac{3}{4}J_3 + \frac{1}{4}J_1 + N\right)g^{-1}, \quad \bar{j} = g\left(\frac{1}{2}\bar{J}_1 + \frac{1}{4}\bar{J}_3 + \frac{3}{4}\bar{J}_1 + \widehat{N}\right)g^{-1}. \tag{3.3}$$

For the solution (3.1) the non-zero currents are $J_2 = \bar{J}_2 = (-\mathcal{E}T + \mathcal{J}J)$. We have that the energy and the angular momentum of the classical string configuration are given by

$$\begin{aligned}\mathcal{E} &= \frac{1}{2\pi\alpha'}\oint d\sigma \mathrm{Str}(Tj_\tau), \\ \mathcal{J} &= \frac{1}{2\pi\alpha'}\oint d\sigma \mathrm{Str}(Jj_\tau).\end{aligned} \tag{3.4}$$

Note that the classical Virasoro constraint is violated if $\mathcal{E} \neq |\mathcal{J}|$

$$T = \bar{T} = \frac{\alpha'}{4}(-\mathcal{E}^2 + \mathcal{J}^2).$$

We now expand around the classical solution $g = g_0 e^{\sqrt{\alpha'}X}$, where X belongs to $\mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$. This is a gauge choice. This gauge choice does not preserve the other classical symmetries. Consider a BRST transformation

$$Q_B g = g(\lambda + \widehat{\lambda}).$$

From here, we obtain

$$e^{-\sqrt{\alpha'}X} Q_B e^{\sqrt{\alpha'}X} = \lambda + \widehat{\lambda}, \tag{3.5}$$

and expanding the left hand side,

$$\sqrt{\alpha'}Q_B X + \alpha'\frac{1}{2}[Q_B X, X] + \cdots = \lambda + \widehat{\lambda}. \tag{3.6}$$

We try to determine the BRST transformation of X as an expansion in powers of X, that is

$$Q_B X = \frac{1}{\sqrt{\alpha'}}\sum_{n=0}(Q_B)_n X, \tag{3.7}$$

where $(Q_B)_n X$ contains n powers of $\sqrt{\alpha'}X$. Plugging this expansion in (3.5) we obtain that

$$Q_B X = \frac{1}{\sqrt{\alpha'}}(\lambda + \widehat{\lambda}) - \frac{1}{2}[(\lambda + \widehat{\lambda}), X] + \cdots. \tag{3.8}$$

Since the original X does not have an $\mathcal{H}_0$ component, and $[\lambda, X_3] + [\widehat{\lambda}, X_1]$ belongs to $\mathcal{H}_0$, this BRST transformation does not preserve the form of g. Note that this analysis does not depend on the form of the background. To preserve our initial gauge choice we have to perform a compensating $\mathrm{SO}(1,4) \times \mathrm{SO}(5)$ gauge transformation. The $\mathrm{SO}(1,4) \times \mathrm{SO}(5)$

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gauge transformations for X_0 is given by $\delta X_0 = \frac{1}{\sqrt{\alpha'}}\Lambda$, where $\Lambda \in \mathrm{SO}(1,4)\times\mathrm{SO}(5)$. The $\mathcal{H}_0$ component in (3.8) can be eliminated if we choose $\Lambda = \frac{\sqrt{\alpha'}}{2}[\lambda, X_3] + \frac{\sqrt{\alpha'}}{2}[\widehat{\lambda}, X_1] + \cdots$. This compensating gauge transformation will affect all fields, since now the BRST transformation should come together with it. In particular, the pure spinor ghosts now have a non vanishing BRST transformation.

A similar problem happens when we consider the conformal transformation of the gauge group element (2.8) which implies

$$e^{-\sqrt{\alpha'}X}\delta e^{\sqrt{\alpha'}X} = \varepsilon(e^{-\sqrt{\alpha'}X}\partial e^{\sqrt{\alpha'}X} + e^{-\sqrt{\alpha'}X}J_{(0)}e^{\sqrt{\alpha'}X}) + \bar{\varepsilon}(e^{-\sqrt{\alpha'}X}\overline{\partial} e^{\sqrt{\alpha'}X} + e^{-\sqrt{\alpha'}X}\overline{J}_{(0)}e^{\sqrt{\alpha'}X}), \tag{3.9}$$

where $J_{(0)}$ and $\overline{J}_{(0)}$ are the components of the Metsaev-Tseytlin currents evaluated in the background (3.1). Note that these currents have values in $\mathcal{H}_2$ only. Expanding the exponentials in (3.9) we have

$$\sqrt{\alpha'}\delta X + \alpha'\frac{1}{2}[\sqrt{\alpha'}dX, X] + \cdots = \varepsilon(J_{(0)} + \sqrt{\alpha'}\partial X + \sqrt{\alpha'}[J_{(0)}, X] + \cdots) + \bar{\varepsilon}(J_{(0)} + \sqrt{\alpha'}\overline{\partial} X + \sqrt{\alpha'}[J_{(0)}, X] + \cdots). \tag{3.10}$$

As before, we try to solve this equation in powers of X such that

$$\delta X = \frac{1}{\sqrt{\alpha'}}\sum_{n=0}\delta_n X, \tag{3.11}$$

where $\delta_n X$ contains n powers of $\sqrt{\alpha'}X$. Then, up to first order in X,

$$\delta X = \varepsilon\left(\frac{1}{\sqrt{\alpha'}}J_{(0)} + \partial X + \frac{1}{2}[J_{(0)}, X] + \cdots\right) + \bar{\varepsilon}\left(\frac{1}{\sqrt{\alpha'}}J_{(0)} + \overline{\partial} X + \frac{1}{2}[J_{(0)}, X] + \cdots\right). \tag{3.12}$$

Since the original X does not have an $\mathcal{H}_0$ component, and $[J_{(0)}, X_2]$ belongs to $\mathcal{H}_0$, this conformal transformation does not preserve the gauge choice for X. The compensating gauge transformation to restore the gauge choice for X in this case is $\Lambda = -\sqrt{\alpha'}\frac{\varepsilon}{2}[J_{(0)}, X_2] - \sqrt{\alpha'}\frac{\bar{\varepsilon}}{2}[J_{(0)}, X_2] + \cdots$, where $\cdots$ are higher order terms in α'. Another way to obtain these transformations is to use canonical quantization. In this case, the momenta conjugate to the fields have higher order corrections and these corrections are used to replace time derivatives of X.

3.1 Expansion of the action

Before expanding the action, we need to determine up to what order we will expand it to check BRST and conformal invariance. Generically, the action is expanded as

$$S = S_0 + S_1 + S_2 + S_3 + \cdots, \tag{3.13}$$

where S_n is of order n in $(X, \lambda, \omega, \widehat{\lambda}, \widehat{\omega})$. Note that S_0 is the value of the action for the background and S_1 vanishes because the background satisfies the classical equations of motion.

Consider the BRST symmetry first. The transformation for the world-sheet fields are given by (3.8). Note that they start with a term linear in $(X, \lambda, \omega, \widehat{\lambda}, \widehat{\omega})$, then $Q_B = (Q_B)_0 + (Q_B)_1 + \cdots$. When we act the BRST generator on the action, we can expand $Q_B S$ as an expansion in powers of $(X, \lambda, \omega, \widehat{\lambda}, \widehat{\omega})$ as

$$Q_B S = (Q_B)_0 S_2 + ((Q_B)_0 S_3 + (Q_B)_1 S_2) + \cdots. \tag{3.14}$$

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The invariance of the action order by order implies

$$(Q_B)_0 S_2 = 0, \quad ((Q_B)_0 S_3 + (Q_B)_1 S_2) = 0. \tag{3.15}$$

Then, if we want to test BRST invariance up to the the order shown above, then the action has to be expanded up to order three in $(X, \lambda, \omega, \widehat{\lambda}, \widehat{\omega})$.

Consider now the conformal symmetry. The transformation for the world-sheet fields are given by (2.9) and (3.12). Note that they start with a term independent of X in (3.12), then $\delta = \delta_0 + \delta_1 + \cdots$. When we vary the action under the conformal transformations, δS can be written as an expansion in powers of $(X, \lambda, \omega, \widehat{\lambda}, \widehat{\omega})$ as

$$\delta S = \delta_0 S_2 + (\delta_1 S_2 + \delta_0 S_3) + \cdots . \tag{3.16}$$

The action is invariant order by order implies

$$\delta_0 S_2 = 0, \quad (\delta_1 S_2 + \delta_0 S_3) = 0. \tag{3.17}$$

Then, if we want to test conformal invariance up to the the order shown above, we are again led to the conclusion that the action the action has to be expanded up to order three in $(X, \lambda, \omega, \widehat{\lambda}, \widehat{\omega})$.

We now expand the action up to order three in the quantum fields $(X, \lambda, \omega, \widehat{\lambda}, \widehat{\omega})$. We expand the action (2.1) around the background given in (3.1), that is $g = g_0 e^{\sqrt{\alpha'} X}$ with g_0 given as in (3.1) and $X = X_1 + X_2 + X_3$. This solution is such that $J_{(0)} = \overline{J}_{(0)}$. The Metsaev-Tseytlin current becomes

$$J = g^{-1} dg = e^{-\sqrt{\alpha'} X} J_{(0)} e^{\sqrt{\alpha'} X} + e^{-\sqrt{\alpha'} X} d e^{\sqrt{\alpha'} X}, \tag{3.18}$$

where d represents ∂ and $\overline{\partial}$. Expanding, we obtain

$$J = J_{(0)} + \sqrt{\alpha'} J_{(1)} + \frac{\alpha'}{2} J_{(2)} + \frac{\alpha'^{\frac{3}{2}}}{6} J_{(3)} + \cdots , \tag{3.19}$$

where

$$J_{(1)} = dX + [J_{(0)}, X], \quad J_{(2)} = [J_{(1)}, X], \quad J_{(3)} = [J_{(2)}, X], \cdots .$$

We plug these expansions into (2.1) and determine S_2 to be

$$\begin{aligned} S_2 = & \langle \frac{1}{2} \partial X_2 \overline{\partial} X_2 + \partial X_1 \overline{\partial} X_3 + \frac{1}{2} J_{(0)} ([X_1, \overline{\partial} X_1] + [X_3, \partial X_3]) \rangle \\ & + \langle \frac{1}{2} J_{(0)} \left[[J_{(0)}, X_2], X_2 \right] + \omega \overline{\partial} \lambda + \widehat{\omega} \partial \widehat{\lambda} \rangle . \end{aligned} \tag{3.20}$$

To verify conformal invariance, we need the action up to order 3 because of (3.17). Note that $\delta_1 S_2$ contains quantum fluctuations of order 2, then we have to search for terms of order 2 in $\delta_0 S_3$. Since, the S_3 is of order 3, its variation is of the form $\langle X X \delta X \rangle$. Then, to get terms of order 2 here, we need variations of X which are independent of the quantum fluctuation X. The only one are X_2. Therefore, we determine the terms involving X_2 only in S_3. They are

$$\frac{1}{\sqrt{\alpha'}} S_3 = \langle -\frac{1}{8} X_1 ([\partial X_1, \overline{\partial} X_2] + [\partial X_2, \overline{\partial} X_1]) + \frac{1}{4} X_2 ([\partial X_3, \overline{\partial} X_3] - [\partial X_1, \overline{\partial} X_1]) \tag{3.21}$$

$$
\begin{aligned}
&+\frac{1}{8}X_3([\partial X_2,\overline{\partial}X_3]+[\partial X_3,\overline{\partial}X_2])\rangle\\
&+\langle J_{(0)}\left(\frac{1}{3}\left[[\partial X_2+\overline{\partial}X_2,X_2],X_2\right]+\frac{1}{24}\left[[5\partial X_1+11\overline{\partial}X_1,X_3],X_2\right]\right)\rangle\\
&+\langle J_{(0)}\left(\frac{1}{24}\left[[11\partial X_3+5\overline{\partial}X_3,X_1],X_2\right]+\frac{1}{24}\left[[5\partial X_2-\overline{\partial}X_2,X_3],X_1\right]\right)\rangle\\
&+\langle J_{(0)}\left(\frac{1}{6}\left[[2\partial X_3-\overline{\partial}X_3,X_2],X_1\right]-\frac{1}{6}\left[[\partial X_1-2\overline{\partial}X_1,X_2],X_3\right]\right)\rangle\\
&-\langle J_{(0)}\left(\frac{1}{24}\left[[\partial X_2-5\overline{\partial}X_2,X_1],X_3\right]\right)\rangle\\
&+\langle\frac{1}{3}[J_{(0)},X_1]\left(\left[[J_{(0)},X_2],X_1\right]+\left[[J_{(0)},X_1],X_2\right]\right)\rangle\\
&-\langle\frac{1}{6}[J_{(0)},X_2]\left(\left[[J_{(0)},X_3],X_3\right]+\left[[J_{(0)},X_1],X_1\right]\right)\rangle\\
&+\langle\frac{1}{3}[J_{(0)},X_3]\left(\left[[J_{(0)},X_2],X_3\right]+\left[[J_{(0)},X_3],X_2\right]\right)\rangle\\
&+\langle J_{(0)}[X_2,N+\widehat{N}]\rangle.
\end{aligned}
$$

Note that the first two lines in (3.21) becomes, after integrating by parts,

$$\langle\frac{1}{2}X_2([\partial X_3,\overline{\partial}X_3]-[\partial X_1,\overline{\partial}X_1])\rangle.$$

Also note that

$$\langle[J_{(0)},X_1]\left[[J_{(0)},X_1],X_2\right]\rangle=\langle[J_{(0)},X_3]\left[[J_{(0)},X_3],X_2\right]\rangle=0.$$

Apart from being important to the symmetries of the action, these cubic terms are necessary to compute the canonical momenta for some variables.

4 A BPS background

The background around which we choose to quantize the string is given by

$$g_0=e^{\alpha'\mathcal{E}\tau T}e^{\alpha'\mathcal{J}\tau J}. \tag{4.1}$$

It describes a point-like string rotating along an equator in S^5. The only non vanishing left invariant current is

$$J_\tau=g_0^{-1}\partial_\tau g=\alpha'\mathcal{E}T+\alpha'\mathcal{J}J. \tag{4.2}$$

One can see immediately see that such classical configurations satisfy the equations of motion. The classical Virasoro constraint for such configuration reads

$$T+\bar{T}=\frac{1}{2\alpha'}\mathrm{Str}\,\tilde{J}_\tau\tilde{J}_\tau=\frac{\alpha'}{2}(-\mathcal{E}^2+\mathcal{J}^2)=0. \tag{4.3}$$

and it implies that $\mathcal{E}=|\mathcal{J}|$, which is the usual BPS condition. Later we will see that there are quantum corrections to this constraint. We can also calculate the value of the conserved current

$$j_\tau=\frac{1}{2\pi\alpha'}gJ_\tau g^{-1}=\frac{1}{2\pi}(\mathcal{E}T+\mathcal{J}J), \tag{4.4}$$

the charges are given by

$$\mathsf{E} = -\oint d\sigma \mathrm{Tr}\left[Tj_\tau\right] = \frac{1}{2\pi}\oint d\sigma \mathcal{E} = \mathcal{E}, \quad \mathsf{J} = \oint d\sigma \mathrm{Tr}\left[\mathsf{J}j_\tau\right] = \frac{1}{2\pi}\oint d\sigma \mathcal{J} = \mathcal{J}. \tag{4.5}$$

which confirms the claim that this is a BPS solution, i.e. $\mathsf{E}-\mathsf{J}=0$, when $\mathcal{E}=\mathcal{J}$. From now on we will set $\mathcal{E}=\mathcal{J}$. The BMN limit is

$$\alpha' \to 0\,,\ \mathcal{J}\to\infty\,,\ \alpha'\mathcal{J} = \text{finite}.$$

4.1 Expansion of the action

Now that we know the background, we would like to know what is the spectrum of quantum fluctuations around it. In order to do that, we use the background field quantization. The quantum coset element is

$$g = g_0 e^{\sqrt{\alpha'}X}, \quad \text{where} \quad X = tT + \phi J + X^A P_A + X^I P_I + \Theta^a Q_a + \Theta^{\dot a} Q_{\dot a} + \hat\Theta^a \hat Q_a + \hat\Theta^{\dot a}\hat Q_{\dot a}. \tag{4.6}$$

Using (3.20) and the conventions in appendix A we get the pure spinor action for the superstring on a pp-wave background

$$\begin{aligned} S = \frac{1}{\pi}\int d^2 z\Bigg\{&\frac{1}{2}\left(-\partial t\bar\partial t + \partial\phi\bar\partial\phi + \partial X^i\bar\partial X^i - \frac{\alpha'^2\mathcal{J}^2}{4}X^iX^i\right) + \\ &+\partial\hat\Theta^{\hat a}\bar\partial\Theta^b\Pi_{\hat a b} - i\frac{\mathcal{J}\alpha'}{4}\left(\Theta^a\delta_{ab}\bar\partial\Theta^b + \hat\Theta^{\hat a}\delta_{\hat a\hat b}\partial\hat\Theta^{\hat b}\right) + \omega_a\bar\partial\lambda^a + \bar\omega_{\hat a}\partial\bar\lambda^{\hat a} + \\ &+\partial\hat\Psi^{\dot a}\bar\partial\Psi^{\dot b}\Pi_{\dot{\hat a}\dot b} + \omega_{\dot a}\bar\partial\lambda^{\dot a} + \bar\omega_{\dot{\hat a}}\partial\bar\lambda^{\dot{\hat a}}\Bigg\} \end{aligned} \tag{4.7}$$

where $i=(A,I)$ and we have renamed $\Theta^{\dot a}\to\Psi^{\dot a}$ and $\hat\Theta^{\dot{\hat a}}\to\hat\Psi^{\dot a}$. Note that the ghost action is just the same as in flat space. This is so because we are keeping only second order terms, we don't have any contribution from $[J_0,\lambda]$ or $N\bar N$. These are the leading terms for the action at the BMN limit. However, we are going to see that the cubic action (3.21) still contributes when one computes the conjugate momenta.

4.2 Equations of Motion and Quantization

The equations of motion that come from the action (4.7) are

$$\partial\bar\partial t = \partial\bar\partial\phi = \left(4\partial\bar\partial + \mathcal{J}^2\alpha'^2\right)X^I = \partial\bar\partial\Psi^{\dot a} = \partial\bar\partial\hat\Psi^{\dot a} = 0 \tag{4.8}$$

$$\partial\bar\partial\Theta + i\frac{\mathcal{J}\alpha'}{2}\Pi\partial\hat\Theta = \partial\bar\partial\hat\Theta - i\frac{\mathcal{J}\alpha'}{2}\Pi\bar\partial\Theta = 0 \tag{4.9}$$

The solutions for the bosonic coordinates are the standard ones

$$t = t_0 + p_t\tau + \sum_{n\neq 0}(t_n f_n + \bar t_n\bar f_n), \quad \phi = \phi_0 + p_\phi\tau + \sum_{n\neq 0}(\phi_n f_n + \bar\phi_n \bar f_n). \tag{4.10}$$

$$\Psi = \psi - \Pi p_{\hat\psi}\tau + \sum_{n\neq 0}(\psi_n f_n + \bar\psi_n\bar f_n), \quad \hat\Psi = \hat\psi + \Pi p_\psi\tau + \sum_{n\neq 0}(\hat\psi_n f_n + \bar{\hat\psi}_n\bar f_n) \tag{4.11}$$

$$X^i = \cos(J\alpha'\tau)(a_+^i + a_-^i) + i\frac{1}{J\alpha'}\sin(J\alpha'\tau)(a_+^i - a_-^i) + \sum_{n\neq 0}(x_n^i g_n + \bar{x}_n^i \bar{g}_n) \tag{4.12}$$

$$\Theta = \cos(J\alpha'\tau)\theta_0 - \sin(J\alpha'\tau)\Pi\hat{\theta}_0 + \sum_{n\neq 0}\left(\theta_n g_n - \frac{i}{J\alpha'}(\omega_n - n)\Pi\hat{\theta}_n\bar{g}_n\right) + \sum_n \vartheta_n f_n \tag{4.13}$$

$$\hat{\Theta} = \cos(J\alpha'\tau)\hat{\theta}_0 + \sin(J\alpha'\tau)\Pi\theta_0 + \sum_{n\neq 0}\left(\hat{\theta}_n\bar{g}_n + \frac{i}{J\alpha'}(\omega_n - n)\Pi\theta_n g_n\right) + \sum_n \hat{\vartheta}_n\bar{f}_n \tag{4.14}$$

$$\lambda = \sum_n \lambda_n f_n, \quad \omega = \sum_n \omega_n f_n, \quad \bar{\lambda} = \sum_n \bar{\lambda}_n \bar{f}_n, \quad \bar{\omega} = \sum_n \bar{\omega}_n \bar{f}_n. \tag{4.15}$$

where

$$f_n = e^{-in(\tau-\sigma)}\ ,\ \bar{f}_n = e^{-in(\tau+\sigma)}\ ,\ g_n = e^{-i(\omega_n\tau - n\sigma)}\ ,\ \bar{g}_n = e^{-i(\omega_n\tau + n\sigma)}\ ,\ \omega_n = \pm\sqrt{(\mathcal{J}\alpha')^2 + n^2}.$$

The conjugate momenta for the quantum fluctuations are computed using the standard definition in terms of the variation of the action with respect to their time derivatives.[3] For the transverse fields and spinors we have

$$P^i = \partial_\tau X^i\ ,\ P_\Psi = \partial_\tau\hat{\Psi}\Pi\ ,\ P_{\hat{\Psi}} = -\partial_\tau\Psi\Pi, \quad P_\Theta = \partial_\tau\hat{\Theta}\Pi - \frac{J\alpha'}{2}\Theta, \quad P_{\hat{\Theta}} = -\partial_\tau\Theta\Pi - \frac{J\alpha'}{2}\hat{\Theta} \tag{4.16}$$

The system is quantized using standard equal-time commutation relations

$$[P_i, (\tau,\sigma), X^j(\tau,\sigma')] = -i\delta_i^j\delta(\sigma-\sigma'), \qquad [P_t, (\tau,\sigma), t(\tau,\sigma')] = -i\delta(\sigma-\sigma'), \tag{4.17}$$

$$[P_\phi, (\tau,\sigma), \phi(\tau,\sigma')] = -i\delta(\sigma-\sigma'), \qquad \{P_\psi, (\tau,\sigma), \Psi(\tau,\sigma')\} = i\delta(\sigma-\sigma'), \tag{4.18}$$

$$\{P_\Theta, (\tau,\sigma), \Theta(\tau,\sigma')\} = i\delta(\sigma-\sigma'), \qquad \{P_{\hat{\Theta}}, (\tau,\sigma), \hat{\Theta}(\tau,\sigma')\} = i\delta(\sigma-\sigma'). \tag{4.19}$$

The only subtle point is that, as we will see shortly, the stress-energy tensor has a liner term in ∂t and $\partial\phi$. Because of this, when computing their conjugate momenta, we have to go to cubic order of quantum fields (3.21) . These higher order corrections are interpreted as corrections to the time derivatives of t and ϕ when we replace them with the momenta. From the cubic part of the action we find

$$P_t = -\partial_\tau t + \sqrt{\alpha'}\Big(\alpha'\mathcal{J}X_A X_A - \frac{i}{2}(\Theta_a\partial_\sigma\Theta_a + \Psi_{\dot{a}}\partial_\sigma\Psi_{\dot{a}} - \hat{\Theta}_a\partial_\sigma\hat{\Theta}_a - \hat{\Theta}_{\dot{a}}\partial_\sigma\hat{\Theta}_{\dot{a}}) + \frac{i}{2}\alpha'\mathcal{J}\Pi_{\dot{a}\dot{b}}\Theta_{\dot{a}}\hat{\Theta}_{\dot{b}}\Big), \tag{4.20}$$

$$P_\phi = \partial_\tau\phi - \sqrt{\alpha'}\Big(\alpha'\mathcal{J}X_I X_I - \frac{i}{2}(\Theta_a\partial_\sigma\Theta_a - \Psi_{\dot{a}}\partial_\sigma\Psi_{\dot{a}} - \hat{\Theta}_a\partial_\sigma\hat{\Theta}_a + \hat{\Theta}_{\dot{a}}\partial_\sigma\hat{\Theta}_{\dot{a}}) - \frac{i}{2}\alpha'\mathcal{J}\Pi_{\dot{a}\dot{b}}\Theta_{\dot{a}}\hat{\Theta}_{\dot{b}}\Big). \tag{4.21}$$

These equations we be used to replace τ derivatives, and are interpreted as corrections to τ derivatives, not the momenta.

The quantum conserved charges are calculated from

$$j_\tau = \frac{1}{2\pi\alpha'}g[(J_2)_\tau + (J_1)_\tau + (J_3)_\tau + N + \bar{N}]g^{-1}$$

using the classical solution and the quantum fluctuations. The charge of a particular generator T_A is given by

$$Q_A = \frac{1}{2\pi\alpha'}\oint d\sigma \mathrm{Tr}\left[T_A g[(J_2)_\tau + (J_1)_\tau + (J_3)_\tau + N + \bar{N}]g^{-1}\right] \tag{4.22}$$

[3]For fermions the definition is with the right derivative.

The charges E and J are straightforward to compute up to second order in quantum fields

$$\mathsf{E} = \mathcal{J} + \frac{1}{2\pi\alpha'}\oint d\sigma\Big[\sqrt{\alpha'}\partial_\tau X^0 + \alpha'\delta_{ab}(\Theta^a\partial_\tau\Theta^b + \widehat{\Theta}^a\partial_\tau\widehat{\Theta}^b) + \alpha'\delta_{\dot a\dot b}(\Psi^{\dot a}\partial_\tau\Psi^{\dot b} + \widehat{\Psi}^{\dot a}\partial_\tau\widehat{\Psi}^{\dot b})\Big], \quad (4.23)$$

$$\mathsf{J} = \mathcal{J} + \frac{1}{2\pi\alpha'}\oint d\sigma\Big[\sqrt{\alpha'}\partial_\tau X^5 + \alpha'\delta_{ab}(\Theta^a\partial_\tau\Theta^b + \widehat{\Theta}^a\partial_\tau\widehat{\Theta}^b) - \alpha'\delta_{\dot a\dot b}(\Psi^{\dot a}\partial_\tau\Psi^{\dot b} + \widehat{\Psi}^{\dot a}\partial_\tau\widehat{\Psi}^{\dot b})\Big]$$

In the BMN limit only the constant and first terms contribute to these charges. So the leading contribution to $\mathsf{E}-\mathsf{J}$ is

$$\mathsf{E}-\mathsf{J} = \frac{1}{2\pi\sqrt{\alpha'}}\oint d\sigma[P_t - P_\phi] = \frac{1}{\sqrt{\alpha'}}[p_t - p_\phi]. \quad (4.24)$$

4.3 Spectrum

In the BMN limit we have a free field theory with massless and massive excitations. To know the spectrum, we have to impose BRST and Virasoro conditions. The stress-energy tensor up to quadratic order in quantum fields in this background is given by:

$$\begin{aligned} T &= \partial X^m\partial X^n\eta_{mn} - \frac{1}{\sqrt{\alpha'}}(\mathcal{J}\alpha')\partial(t-\phi) - \frac{(\mathcal{J}\alpha')^2}{4}X^iX^i - \quad (4.25)\\ &-2\partial\Psi^{\dot a}\partial\hat{\Psi}^{\dot b}\Pi_{\dot a\dot b} - 2\partial\Theta^a\partial\hat{\Theta}^b\Pi_{ab} - i\frac{\mathcal{J}\alpha'}{2}\Theta^a\delta_{ab}\partial\Theta^b - i\frac{\mathcal{J}\alpha'}{2}\hat{\Theta}^a\delta_{ab}\partial\hat{\Theta}^b\\ \bar{T} &= \bar\partial X^m\bar\partial X^n\eta_{mn} - \frac{1}{\sqrt{\alpha'}}(\mathcal{J}\alpha')\bar\partial(t-\phi) - \frac{(\mathcal{J}\alpha')^2}{4}X^iX^i - \quad (4.26)\\ &-2\bar\partial\Psi^{\dot a}\bar\partial\hat{\Psi}^{\dot b}\Pi_{\dot a\dot b} - 2\bar\partial\Theta^a\bar\partial\hat{\Theta}^b\Pi_{ab} - i\frac{\mathcal{J}\alpha'}{2}\Theta^a\delta_{ab}\bar\partial\Theta^b - i\frac{\mathcal{J}\alpha'}{2}\hat{\Theta}^a\delta_{ab}\bar\partial\hat{\Theta}^b \end{aligned}$$

Using (4.20) and (4.21) the replace time derivatives of t and ϕ and the mode expansion, the zero mode of the Virasoro constraint is given by

$$\begin{aligned} L_0 + \bar{L}_0 &= \oint d\sigma(T+\bar{T}) = 4\mathcal{J}\alpha' - \frac{1}{2}p_t^2 + \frac{1}{2}p_\phi^2 + \mathcal{J}\sqrt{\alpha'}(-p_t+p_\phi) + p_\psi p_{\hat\psi} + \quad (4.27)\\ &+\theta_0\Pi\hat{\theta}_0 + \mathcal{J}\alpha' a_+^i a_-^i + \sum_{n>0}\omega_n(x_{-n}^i x_n^i + \bar{x}_{-n}^i\bar{x}_n^i + \theta_{-n}^a\theta_n^a + \hat{\theta}_{-n}^a\hat{\theta}_n^a) +\\ &+\sum_{n>0} n(t_{-n}t_n + \bar{t}_{-n}\bar{t}_n + \phi_{-n}\phi_n + \bar\phi_{-n}\bar\phi_n + \psi_{-n}\hat\psi_n + \hat\psi_{-n}\psi_n) \end{aligned}$$

We see that the energy momentum tensor just counts the energy and the number of modes on some particular state. The normal ordering constant $4J\alpha'$ comes from the eight massive bosons. The massive spinors do not contribute. As usual, physical states have to be annihilated by both T and $\bar{T}$. Another requirement for physical states is that they are annihilated by the BRST charge, which up to quadratic order is

$$\begin{aligned} Q &= \oint d\sigma\mathrm{Tr}\,[\lambda J_3 + \bar\lambda\bar{J}_1] = \oint d\sigma\Big[\Pi_{\dot a\dot b}\lambda^{\dot a}\partial\Theta^{\dot b} + \Pi_{\dot a\dot b}\hat\lambda^{\dot a}\bar\partial\hat\Theta^{\dot b} + \quad (4.28)\\ &+\Pi_{ab}\lambda^a\left(\partial\Theta^b + \frac{J\alpha'}{2}\Pi^b{}_c\hat\Theta^c\right) + \Pi_{ab}\hat\lambda^a\left(\bar\partial\hat\Theta^b - \frac{J\alpha'}{2}\Pi^b{}_c\Theta^c\right)\Big]. \end{aligned}$$

Inserting the mode expansion into the BRST charge we see that the massive fermionic modes decouple, and only the massless fermions remain. This means that the all massive modes create physical states and the massless ones decouple from the spectrum.[4] At this order in quantum fields the bosonic fluctuations do not appear in the BRST charge. The restriction imposed on these fields will come from the Virasoro condition.

We define the vacuum $|0\rangle$ to be annihilated by all positive modes. Excited states are created by acting with other modes. Physical states should have ghost number $(1,1)$. The simplest one was introduced by Berkovits [25] and describes the radius modulus. In the notation of the previous section this state is given by $(\lambda^a\bar\lambda^b\Pi_{ab} + \lambda^{\dot a}\bar\lambda^{\dot b}\Pi_{\dot a\dot b})|0\rangle$ and we will denote it by $|\lambda\bar\lambda\rangle$. Acting with the charges we have $\mathsf{E}|\lambda\bar\lambda\rangle = J|\lambda\bar\lambda\rangle$ and $\mathsf{J}|\lambda\bar\lambda\rangle = J|\lambda\bar\lambda\rangle$. The normal ordering constant in $T+\bar T$ means that this state is not annihilated by it. To fix this we consider another state

$$|\Omega\rangle = e^{-2i\sqrt{\alpha'}(x^0-x^5)}|\lambda\bar\lambda\rangle, \tag{4.29}$$

computing its $\mathsf{E}-\mathsf{J}$ we find it is 4. This is in fact consistent with the identification made in [25]. This string state corresponds to the SYM operator

$$Tr[F^2Z^J]$$

Another was to compensate the Virasoro constraint it to add zero modes of the massive spinors. For instance, consider the state

$$|\mathbf{70}\rangle = \theta_0^a\theta_0^b\theta_0^c\theta_0^d|\lambda\bar\lambda\rangle,$$

it corresponds to a KK mode with J charge J of the graviton and self-dual four form. Breaking the SO(8) spinors to SO(4) $\times$ SO(4) spinors, one element of the above multiplet corresponds to $\mathrm{Tr}\,[Z^{J+2}]$. Similarly, we can construct all $\mathbf{128}+\mathbf{128}$ lowest states expected. Since all massive modes of Θ decouple from the BRST charge, these are all physical states. The massless spinors are not BRST-closed in the scalar ghost vacuum. It is interesting to note that they are all constructed on the scalar combination of the pure spinor ghosts. This is a sharp contrast with the flat space spectrum. But it should be pointed out that the $J\alpha'\to 0$ limit do not correspond to flat space. In summary, the lowest states are

$$\begin{array}{lrl}
\mathsf{E}-\mathsf{J}=4: & e^{-2i\sqrt{\alpha'}(x^0-x^5)}|\lambda\bar\lambda\rangle & \mathbf{1}\\
\mathsf{E}-\mathsf{J}=3: & \theta_0^a e^{-\frac{3}{2}i\sqrt{\alpha'}(x^0-x^5)}|\lambda\bar\lambda\rangle & \mathbf{8}\\
\mathsf{E}-\mathsf{J}=2: & \theta_0^a\theta_0^b e^{-i\sqrt{\alpha'}(x^0-x^5)}|\lambda\bar\lambda\rangle & \mathbf{28}\\
\mathsf{E}-\mathsf{J}=1: & \theta_0^a\theta_0^b\theta_0^c e^{-\frac{1}{2}i\sqrt{\alpha'}(x^0-x^5)}|\lambda\bar\lambda\rangle & \mathbf{56}\\
\mathsf{E}-\mathsf{J}=0: & \theta_0^a\theta_0^b\theta_0^c\theta_0^d|\lambda\bar\lambda\rangle & \mathbf{70}
\end{array} \tag{4.30}$$

States with more θ's are complex conjugates of the ones above. Further states can be obtained acting with the zero modes of the transverse directions. The first type of excited state is the one that changes the values of E and J in such way that $\mathsf{E}-\mathsf{J}=0$. This states is

$$|\Omega_n\rangle = \theta_0^a\theta_0^b\theta_0^c\theta_0^d e^{-in\sqrt{\alpha'}(x^0+x^5)}|\lambda\bar\lambda\rangle, \tag{4.31}$$

[4] We will see bellow that the full spectrum can be constructed using only a scalar combination of ghost vacuum.

where $n = 1, 2, \ldots$ is quantized because x^5 is an angle variable. This state satisfy both Virasoro and BRST conditions. With the quantum conserved charges we can verify that

$$\mathsf{E}|\Omega_n\rangle = (\mathcal{J} + n)|\Omega\rangle, \quad \mathsf{J}|\Omega\rangle = (\mathcal{J} + n)|\Omega_n\rangle. \tag{4.32}$$

One of its components corresponds to

$$Tr[Z^{J+2+n}].$$

In the state above, n can be positive or negative. However, it cannot be of order $1/\alpha'$, since higher order terms in the action would be needed to study it. This is not surprising, since a state with such large negative n would correspond to a state with finite J charge. It is precisely this feature the differentiates the pure spinor quantization with light-cone GS. In GS, the J charge is fixed to be the string length. Since the pure spinor case is conformal, it is natural to expect that the we can change J by finite amounts.

Higher stringy states are created by the non zero modes of the massive world sheet fields satisfying level match condition. For example, consider

$$x^I_{-n}\bar{x}^J_{-n}\theta^a_0\theta^b_0\theta^c_0\theta^d_0|\lambda\bar{\lambda}\rangle,$$

however, this state does not satisfy the Virasoro condition. The same was as above, we fix this adding an exponential

$$x^I_{-n}\bar{x}^J_{-n}\theta^a_0\theta^b_0\theta^c_0\theta^d_0 e^{-i\frac{\Delta}{2}\sqrt{\alpha'}(x^0-x^5)}|\lambda\bar{\lambda}\rangle,$$

where

$$\Delta = \frac{2}{\mathcal{J}\alpha'}\sqrt{(\mathcal{J}\alpha')^2 + n^2} = 2\sqrt{1 + \frac{n^2}{(\mathcal{J}\alpha')^2}}$$

Computing $\mathsf{E} - \mathsf{J}$, we find that is it indeed Δ. Thus, we derive the full sting spectrum on a pp-wave background. In [11], the physical state considered was of this form, however, there the background value of E was left as a parameter to be fixed using conformal invariance. One could repeat the analysis of that paper using the same strategy as the one used here.

5 Conclusion

We have studied the application of the background field method to the $AdS_5 \times S^5$ pure spinor superstring around classical solution describing point-like strings. In the BMN limit the action is quadratic in the quantum fields and the spectrum can be computed exactly. The pure spinor description of the BMN limit given here resembles the standard Green-Schwarz description, as opposed to the description given in [19]. There the BMN limit is described by a contraction of the $\mathfrak{psu}(2,2|4)$ algebra and the resulting action is non linear. Although the action given here is quadratic, the world-sheet and space-time symmetries act non-linearly on the fields.

The main motivation for the present work was to understand the symmetries of these constant backgrounds in order to further understand the model proposed to compute the

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energy of the Konishi state in [11]. A key assumption in [11] is that conformal invariance is preserved at quantum level. Although a general argument for quantum conformal invariance was given in [8], an explicit computation for that case remains to be done. Another possible application is to use this formulation to compute scattering amplitudes in the BMN background. As pointed out in [19], such amplitudes would be easier to compute using the pure spinor formulation.

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A Conventions

We will use an $so(4)\times so(4)$ decomposition of the $\mathfrak{psu}(2,2|4)$ algebra. The generators will be denoted by:

$$\mathcal{H}_0=(M_{AB},M_A,M_{IJ},M_I),\ \mathcal{H}_2=(T,P_A,J,P_I),\ \mathcal{H}_1=(Q_a,Q_{\dot a}),\ \mathcal{H}_3=(\widehat{Q}_a,\widehat{Q}_{\dot a}). \tag{A.1}$$

The non zero commutators are given by

$$[M_{AB},M_{CD}]=-\delta_{A[C}M_{D]B}+\delta_{A[C}M_{D]B},\ [M_{AB},M_C]=\delta_{C[A}M_{B]},\ [M_A,M_B]=-M_{AB}, \tag{A.2}$$

$$[M_{IJ},M_{KL}]=-\delta_{I[K}M_{L]J}+\delta_{J[K}M_{L]I},\ [M_{IJ},M_K]=\delta_{K[I}M_{J]},\ [M_I,M_J]=M_{IJ}. \tag{A.3}$$

$$[T,P_A]=-M_A,\quad [P_A,P_B]=-M_{AB},\quad [J,P_I]=M_I,\quad [P_I,P_J]=M_{IJ}. \tag{A.4}$$

$$[M_{AB},T]=0,\quad [M_{AB},P_C]=\delta_{C[A}P_{B]},\quad [M_A,T]=-P_A,\quad [M_A,P_B]=-\delta_{AB}T, \tag{A.5}$$

$$[M_{IJ},J]=0,\quad [M_{IJ},P_K]=\delta_{K[I}P_{J]},\quad [M_I,J]=P_I,\quad [M_I,P_J]=-\delta_{IJ}J. \tag{A.6}$$

$$[M_{AB},Q_a]=\frac{1}{2}(\sigma_{AB})_{ab}Q_b,\quad [M_{AB},Q_{\dot a}]=\frac{1}{2}(\sigma_{AB})_{\dot a\dot b}Q_{\dot b}, \tag{A.7}$$

$$[M_A,Q_a]=\frac{1}{2}(\sigma_A)_{a\dot b}Q_{\dot b},\quad [M_A,Q_{\dot a}]=\frac{1}{2}(\sigma_A)_{\dot a b}Q_b, \tag{A.8}$$

$$[M_{IJ},Q_a]=\frac{1}{2}(\sigma_{IJ})_{ab}Q_b,\quad [M_{IJ},Q_{\dot a}]=\frac{1}{2}(\sigma_{IJ})_{\dot a\dot b}Q_{\dot b}, \tag{A.9}$$

$$[M_I,Q_a]=\frac{1}{2}(\sigma_I)_{a\dot b}Q_{\dot b},\quad [M_I,Q_{\dot a}]=-\frac{1}{2}(\sigma_I)_{\dot a b}Q_b. \tag{A.10}$$

$$[M_{AB},\widehat{Q}_a]=\frac{1}{2}(\sigma_{AB})_{ab}\widehat{Q}_b,\quad [M_{AB},\widehat{Q}_{\dot a}]=\frac{1}{2}(\sigma_{AB})_{\dot a\dot b}\widehat{Q}_{\dot b}, \tag{A.11}$$

$$[M_A,\widehat{Q}_a]=\frac{1}{2}(\sigma_A)_{a\dot b}\widehat{Q}_{\dot b},\quad [M_A,\widehat{Q}_{\dot a}]=\frac{1}{2}(\sigma_A)_{\dot a b}\widehat{Q}_b, \tag{A.12}$$

$$[M_{IJ},\widehat{Q}_a]=\frac{1}{2}(\sigma_{IJ})_{ab}\widehat{Q}_b,\quad [M_{IJ},\widehat{Q}_{\dot a}]=\frac{1}{2}(\sigma_{IJ})_{\dot a\dot b}\widehat{Q}_{\dot b}, \tag{A.13}$$

$$[M_I,\widehat{Q}_a]=\frac{1}{2}(\sigma_I)_{a\dot b}\widehat{Q}_{\dot b},\quad [M_I,\widehat{Q}_{\dot a}]=-\frac{1}{2}(\sigma_I)_{\dot a b}\widehat{Q}_b. \tag{A.14}$$

$$[T,Q_a]=[J,Q_a]=\frac{1}{2}\Pi_{ab}\widehat{Q}_b,\quad [T,Q_{\dot a}]=-[J,Q_{\dot a}]=\frac{1}{2}\Pi_{\dot a\dot b}\widehat{Q}_{\dot b}, \tag{A.15}$$

$$[P_A,Q_a]=\frac{1}{2}(\sigma_A)_{a\dot b}\Pi_{\dot b\dot c}\widehat{Q}_{\dot c},\quad [P_A,Q_{\dot a}]=\frac{1}{2}(\sigma_A)_{\dot a b}\Pi_{bc}\widehat{Q}_c, \tag{A.16}$$

$$[P_I,Q_a]=\frac{1}{2}(\sigma_I)_{a\dot b}\Pi_{\dot b\dot c}\widehat{Q}_{\dot c},\quad [P_I,Q_{\dot a}]=\frac{1}{2}(\sigma_I)_{\dot a b}\Pi_{bc}\widehat{Q}_c. \tag{A.17}$$

$$[T,\widehat{Q}_a]=[J,\widehat{Q}_a]=-\frac{1}{2}\Pi_{ab}Q_b,\quad [T,\widehat{Q}_{\dot a}]=-[J,\widehat{Q}_{\dot a}]=-\frac{1}{2}\Pi_{\dot a\dot b}Q_{\dot b}, \tag{A.18}$$

$$[P_A,\widehat{Q}_a]=-\frac{1}{2}(\sigma_A)_{a\dot b}\Pi_{\dot b\dot c}Q_{\dot c},\quad [P_A,\widehat{Q}_{\dot a}]=-\frac{1}{2}(\sigma_A)_{\dot a b}\Pi_{bc}Q_c, \tag{A.19}$$

$$[P_I,\widehat{Q}_a]=-\frac{1}{2}(\sigma_I)_{a\dot b}\Pi_{\dot b\dot c}Q_{\dot c},\quad [P_I,\widehat{Q}_{\dot a}]=-\frac{1}{2}(\sigma_I)_{\dot a b}\Pi_{bc}Q_c. \tag{A.20}$$

$$\{Q_a,Q_b\}=-i\delta_{ab}(T-J),\ \{Q_a,Q_{\dot b}\}=i\sigma^A_{a\dot b}P_A+i\sigma^I_{a\dot b}P_I,\ \{Q_{\dot a},Q_{\dot b}\}=-i\delta_{\dot a\dot b}(T+J). \tag{A.21}$$

$$\{\widehat{Q}_a,\widehat{Q}_b\}=-i\delta_{ab}(T-J),\ \{\widehat{Q}_a,\widehat{Q}_{\dot b}\}=i\sigma^A_{a\dot b}P_A+i\sigma^I_{a\dot b}P_I,\ \{\widehat{Q}_{\dot a},\widehat{Q}_{\dot b}\}=-i\delta_{\dot a\dot b}(T+J). \tag{A.22}$$

$$\{Q_a,\widehat{Q}_b\}=\frac{i}{2}\left((\sigma^{AB})_{ac}\Pi_{cb}M_{AB}-(\sigma^{IJ})_{ac}\Pi_{cb}M_{IJ}\right), \tag{A.23}$$

$$\{Q_a,\widehat{Q}_{\dot b}\}=-i\left((\sigma^{A})_{a\dot c}\Pi_{\dot c\dot b}M_{A}+(\sigma^{I})_{a\dot c}\Pi_{\dot c\dot b}M_{I}\right), \tag{A.24}$$

$$\{Q_{\dot a},\widehat{Q}_{\dot b}\}=\frac{i}{2}\left((\sigma^{AB})_{\dot a\dot c}\Pi_{\dot c\dot b}M_{AB}-(\sigma^{IJ})_{\dot a\dot c}\Pi_{\dot c\dot b}M_{IJ}\right), \tag{A.25}$$

$$\{Q_{\dot a},\widehat{Q}_b\}=-i\left((\sigma^{A})_{\dot a c}\Pi_{cb}M_{A}-(\sigma^{I})_{\dot a c}\Pi_{cb}M_{I}\right). \tag{A.26}$$

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