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One-loop BPS amplitudes as BPS-state sums

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Abstract: Recently, we introduced a new procedure for computing a class of one-loop BPS-saturated amplitudes in String Theory, which expresses them as a sum of one-loop contributions of all perturbative BPS states in a manifestly T-duality invariant fashion. In this paper, we extend this procedure to *all* BPS-saturated amplitudes of the form $\int_{\mathcal{F}} \varGamma_{d+k,d}\, \varPhi$, with $\varPhi$ being a weak (almost) holomorphic modular form of weight $-k/2$. We use the fact that any such $\varPhi$ can be expressed as a linear combination of certain absolutely convergent Poincaré series, against which the fundamental domain $\mathcal{F}$ can be unfolded. The resulting BPS-state sum neatly exhibits the singularities of the amplitude at points of gauge symmetry enhancement, in a chamber-independent fashion. We illustrate our method with concrete examples of interest in heterotic string compactifications.

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Contents

1 Introduction

Scattering amplitudes in closed string theory involve, at h-th order in perturbation theory, an integral over the moduli space of conformal structures on genus h closed Riemann surfaces. The torus amplitude (corresponding to $h=1$) is particularly relevant, as it encodes the perturbative spectrum of excitations. Moreover, for special choices of vacua and of external states, corresponding to a special class of F-term interactions in the low energy effective action, the torus contribution exhausts the perturbative series, and thus can serve as a basis for quantitative tests of string dualities (see e.g. [1] and references therein).

The moduli space of conformal metrics on the torus is the Poincaré upper half plane $\mathbb{H}$, parameterised by the complex structure parameter $\tau = \tau_1 + \mathrm{i}\tau_2$, modulo the action of the modular group $\mathrm{SL}(2,\mathbb{Z})$. After performing the path integral over the world-sheet fields and over the location of the vertex-operator insertions, the relevant amplitude is then expressed as a modular integral

$$\int_{\mathcal{F}} \mathrm{d}\mu\, \mathcal{A}(\tau_1, \tau_2)\,, \tag{1.1}$$

where $\mathcal{F} = \{\tau \in \mathbb{H} \,|\, -\frac{1}{2} \leq \tau_1 < \frac{1}{2}, |\tau| \geq 1\}$ is the standard fundamental domain, $\mathrm{d}\mu = \tau_2^{-2}\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2$ is the $\mathrm{SL}(2,\mathbb{Z})$-invariant integration measure, and $\mathcal{A}$ is a modular-invariant function whose precise expression depends on the problem at hand. With this choice of integration domain, the imaginary part τ_2 can be identified with Schwinger's proper time, while the real part τ_1 is the Lagrange multiplier imposing the level-matching condition. Part of the difficulty in evaluating integrals of the form (1.1) is the unwieldy shape of $\mathcal{F}$, which intertwines the integrals over τ_1 and τ_2.

Depending on the function $\mathcal{A}(\tau_1, \tau_2)$ methods have been devised to overcome this problem. If $\mathcal{A}$ is a weak almost holomorphic function[1] of τ (or, alternatively, an anti-holomorphic function), the surface integral over $\mathcal{F}$ can be reduced by Stokes' theorem to a line-integral over its boundary $\partial\mathcal{F}$ that can be explicitly computed [2]. On the contrary, if $\mathcal{A}$ is a genuine non-holomorphic function, as is the case for the one-loop partition function of closed-oriented strings, no useful method is known to evaluate the integral, but one can use the Rankin-Selberg-Zagier transform [3] to connect the integral to the graded sum of physical degrees of freedom [4–7]. A frequently encountered intermediate case is that of modular integrals of the form

$$\int_{\mathcal{F}} \mathrm{d}\mu\, \varGamma_{d+k,d}(G, B, Y; \tau_1, \tau_2)\, \varPhi(\tau)\,, \tag{1.2}$$

where

$$\varGamma_{d+k,d}(G, B, Y; \tau_1, \tau_2) \equiv \tau_2^{d/2} \sum_{p_{\mathrm{L}}, p_{\mathrm{R}}} q^{\frac{1}{4}p_L^2}\, \bar{q}^{\frac{1}{4}p_R^2} \tag{1.3}$$

is the partition function of the Narain lattice of Lorentzian signature $(d+k, d)$, G, B, Y parameterise the Narain moduli space $\mathrm{SO}(d+k,d)/\mathrm{SO}(d+k) \times \mathrm{SO}(d)$, and $\varPhi(\tau)$ is a weak almost holomorphic modular form of negative weight $w = -k/2$, which we shall refer to as the *elliptic genus*. Such integrals occur in particular in one-loop corrections to certain BPS-saturated couplings in the low energy effective action of heterotic or type II superstrings.

The traditional approach in the physics literature for computing modular integrals of the form (1.2) has been the *orbit method*, which proceeds by unfolding the integration domain $\mathcal{F}$ against the lattice partition function $\varGamma_{d+k,d}$ [8–19]. While this procedure yields an infinite series expansion which is useful in certain limits in Narain moduli space, it does

[1] By *weak almost holomorphic* we mean an element in the graded polynomial ring generated by the holomorphic Eisenstein series E_4 and E_6, the almost holomorphic Eisenstein series $\hat{E}_2$ and the inverse of the discriminant $1/\varDelta$. Our notations for Eisenstein series and other modular forms are collected in appendix A.1. The adverb *weak* refers to the fact that the only singularity is, at most, a finite order pole at the cusp $q = 0$.

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not make manifest the invariance under the T-duality group $\mathrm{O}(d+k,d,\mathbb{Z})$ of the Narain lattice, nor does it clearly display the singularities of the amplitude at points of gauge symmetry enhancement.

In [20] we proposed a new method for dealing with modular integrals of the form (1.2), which relies on representing the elliptic genus Φ as a Poincaré series, and on unfolding the integration domain against *it* rather than against the lattice partition function. The advantage of this method is that T-duality remains manifest at all steps, and the result is valid in all chambers in Narain moduli space, unlike the conventional approach.[2] Moreover, the amplitude is expressed as a sum over all BPS states in the spectrum, thus generalising the constrained Eisenstein series constructed in [21].[3] Finally, the singularities of the amplitude at points of enhanced gauge symmetry can be immediately read-off from the contributions of those BPS states which become massless.

The main difficulty in implementing this strategy is due to the fact that the standard Poincaré series representation of a weak holomorphic modular form of weight $w \leq 0$ [28–30] is only conditionally convergent, and therefore unsuited for unfolding. In [20] we attempted to circumvent this problem by considering a class of non-holomorphic Poincaré series $E(s,\kappa,w)$ that provide a natural regularisation of the modular forms of interest by inserting a Kronecker-type convergence factor $\tau_2^{s-w/2}$ in the standard sum over images. Therefore, the resulting Poincaré series, originally studied in [31], converges absolutely for $\Re(s) > 1$, and the modular integral $\int_{\mathcal{F}} \Gamma_{d,d}\, E(s,\kappa,w)$ can be computed by unfolding $\mathcal{F}$ against it, at least for large s. The result should then be analytically continued to the desired value $s = \frac{w}{2}$, where $E(s,\kappa,w)$ becomes formally a holomorphic function of τ. This procedure would then allow to compute the modular integral (1.2) for any Φ which can be expressed as a linear combination of such $E(\frac{w}{2},\kappa,w)$'s, at least in principle. However, this strategy turned out to be quite difficult in practice, since this analytic continuation depends on the notoriously subtle analytic properties of the Kloosterman-Selberg zeta function which appears in the Fourier expansion of $E(s,\kappa,w)$. That is the reason why the analysis [20] was restricted to the case of zero modular weight, where the analytic continuation is fully under control.

In the present work, we overcome these difficulties by employing a different class of non-analytic Poincaré series introduced in the mathematics literature by Niebur [32] and Hejhal [33] and studied more recently by Bruinier, Ono and Bringmann [34–37]. Similarly to the Selberg-Poincaré series $E(s,\kappa,w)$, the Niebur-Poincaré series $\mathcal{F}(s,\kappa,w)$ converges absolutely for $\Re(s) > 1$, and formally becomes holomorphic in τ at the point $s = \frac{w}{2}$. However, the Niebur-Poincaré series can be specialised to the other interesting value $s = 1 - \frac{w}{2}$, which lies inside the domain of absolute convergence when the weight w is negative. Although at this value $\mathcal{F}(s,\kappa,w)$ belongs to the more general class of weak harmonic Maass forms,[4] that are typically non-holomorphic functions of τ, it has the important property

[2] See for instance [18] for a detailed discussion on chamber dependence of the traditional unfolding method.

[3] BPS states sums have appeared in earlier works [22–26]. In our approach these BPS sums follow directly from unfolding the fundamental domain against the elliptic genus, without any further assumption.

[4] A harmonic Maass form is an eigenmode of the weight-w Laplacian on $\mathbb{H}$ with the same eigenvalue as weak holomorphic modular forms. The positive frequency part of a weak harmonic Maass form is sometimes known as a Mock modular form. See section 2.3 for a more precise definition of weak harmonic Maass forms.

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that any linear combination of $\mathcal{F}(1-\frac{w}{2},\kappa,w)$, whose coefficients are determined by the principal part of a given weak holomorphic modular form Φ, is actually a weak holomorphic modular form, and equals Φ itself. Therefore, *given any weak holomorphic modular form Φ, the integral (1.2) can be computed by decomposing Φ into a sum of Niebur-Poincaré series, and by unfolding each of them against the integration domain.* Moreover, the same strategy works also for weak almost holomorphic modular forms (i.e. involving powers of $\hat{E}_2$), where now one has to specialise the Niebur-Poincaré series to the values $s=1-\frac{w}{2}+n$, with n a non-negative integer.

The outline of this work is as follows. In section 2, we introduce the Niebur-Poincaré series $\mathcal{F}(s,\kappa,w)$, discuss their main properties, present their Fourier coefficients and identify their limiting values at $s=1-\frac{w}{2}+n$. We conclude the section by showing the important result that *any* weak almost holomorphic modular form can be represented as a linear combination of them. In section 3 we evaluate the modular integral $\int_{\mathcal{F}} \Gamma_{d+k,d}\, \mathcal{F}\left(s,\kappa,-\frac{k}{2}\right)$ in terms of certain BPS-state sums and discuss their singularity structure. In section 4, we use this result to compute a sample of modular integrals of physical interest of the form (1.2). In appendix A, we define our notation for modular forms, we collect various definitions and properties of Whittaker and hypergeometric functions, and we introduce the Kloosterman sums and the Kloosterman-Selberg zeta function. Finally, in appendix B we briefly discuss the relation between the Selberg- and Niebur-Poincaré series, and between the "shifted constrained" Epstein zeta series and the above BPS-state sums. The reader interested only in physics applications may skip section 2 and proceed directly to section 3, which begins with an executive summary of the main properties of $\mathcal{F}(s,\kappa,w)$.

Note added. After having obtained most of the results in this paper, we became aware of ref. [34] where similar computations have been performed for general even lattices of signature $(d+k,d)$ with $d=0,1,2$, in particular reproducing Borcherds' automorphic products for $d=2$ [38]. Unlike [34], we restrict the analysis to even self-dual lattices (with $k=0 \bmod 8$), which suffices for our physics applications, but we allow for almost holomorphic modular forms and arbitrary dimension d.

2 Niebur-Poincaré series and almost holomorphic modular forms

In this section, we introduce the Niebur-Poincaré series $\mathcal{F}(s,\kappa,w)$, a modular invariant regularisation of the naïve Poincaré series of negative weight. We present its Fourier expansion for general values of s, and analyse its limit as $s\to 1-\frac{w}{2}+n$ where n is any non-negative integer. We explain how to represent any weak almost holomorphic modular form of negative weight as a suitable linear combinations of such Poincaré series.

2.1 Various Poincaré series

In order to motivate the construction of the Niebur-Poincaré series, let us start with a brief overview of Poincaré series in general. Let w be an even integer[5] and f a function on the

[5]In this paper we shall restrict to the case of even weight w in order to avoid complications with non-trivial multiplier systems, though the construction can be generalised to half-integer weights.

Poincaré upper half plane $\mathbb{H}$. The action of an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma = \mathrm{SL}(2,\mathbb{Z})$ on f is given by the Petersson slash operator

$$(f|_w\gamma)(\tau) = (c\tau+d)^{-w} f(\gamma\cdot\tau), \qquad \gamma\cdot\tau = \frac{a\tau+b}{c\tau+d}. \tag{2.1}$$

If f is invariant under $\Gamma_\infty = \left(\begin{smallmatrix} 1 & \star \\ 0 & 1 \end{smallmatrix}\right) \subset \Gamma$, the Poincaré series of *seed* f and weight w

$$P(f,w;\tau) \equiv P(f,w) = \tfrac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f|_w\gamma \tag{2.2}$$

defines an automorphic form of weight w on $\mathbb{H}$, which is absolutely convergent provided $f(\tau) \ll \tau_2^{1-\frac{w}{2}}$ as $\tau_2 \to 0$. As an example, the choice $f(\tau) = q^{-\kappa}$ with $w > 2$ leads to the usual holomorphic Poincaré series

$$P(\kappa,w) = \tfrac{1}{2} \sum_{(c,d)=1} (c\tau+d)^{-w}\, e^{-2\pi i\kappa\,\frac{a\tau+b}{c\tau+d}}, \tag{2.3}$$

where the pair (a,b) is determined modulo (c,d) by the condition $ad-bc=1$. Depending on the value of κ, eq. (2.3) describes different types of modular forms. For $\kappa=0$, $P(\kappa,w)$ is actually an Eisenstein series, while for $\kappa \le -1$ it is a cusp form, and must therefore vanish if $2 < w < 12$, an observation that will be important later. For $\kappa > 0$, eq. (2.3) represents instead a weak holomorphic modular form with a pole of order κ at $q=0$, $P(\kappa,w) = q^{-\kappa} + \mathcal{O}(q)$.

For $w \le 2$, the Poincaré series (2.3) is divergent and thus needs to be regularised. One possible regularisation scheme, introduced in the mathematical literature in [28, 29] and discussed in the physics literature in [30], is to consider the convergent sum

$$P(\kappa,w) = \tfrac{1}{2} \lim_{K\to\infty} \sum_{|c|\le K} \sum_{|d|<K;(c,d)=1} (c\tau+d)^{-w}\, e^{2\pi i\kappa\,\frac{a\tau+b}{c\tau+d}}\, R\left(\frac{2\pi i|\kappa|}{c(c\tau+d)}\right), \tag{2.4}$$

where R is a specific regulating factor such that $R(x) \sim x^{1-w}/\Gamma(2-w)$ as $x \to 0$ and approaches 1 as $x \to \infty$. While this regularisation preserves holomorphicity, it does not necessarily produce a modular form,[6] except for small $|w|$ where the modular anomaly can be shown to vanish. Moreover, the convergence of (2.4) is conditional, which makes it unsuitable for the unfolding procedure.

Another option, introduced by Selberg [31] and considered in our previous work [20], is to jettison holomorphicity and introduce a convergence factor *à la* Kronecker, thus considering the Poincaré-series

$$E(s,\kappa,w) \equiv \tfrac{1}{2} \sum_{(c,d)=1} \frac{\tau_2^{s-\frac{w}{2}}}{|c\tau+d|^{2s-w}} (c\tau+d)^{-w}\, e^{-2\pi i\kappa\,\frac{a\tau+b}{c\tau+d}} \tag{2.5}$$

[6] The holomorphic Poincaré series (2.4) is in general an Eichler integral, i.e. a function $F(\tau)$ which satisfies $F(\tau) - (F|_w\gamma)(\tau) = r_\gamma(\tau)$ where r_γ is a polynomial of degree $-w$ in τ, whose coefficients depend on a,b,c,d. We shall comment in section 2.3 on the modular completion of $P(\kappa,w)$.

associated to the seed $f(\tau) = \tau_2^{s-\frac{w}{2}}\, q^{-\kappa}$. We shall refer to (2.5) as the Selberg-Poincaré series. The series (2.5) converges absolutely for $\Re(s) > 1$ and becomes formally holomorphic at $s = \frac{w}{2}$. However, for $w \le 2$ this value lies outside the convergence domain, and the analytic continuation to $s = \frac{w}{2}$ depends on the analytic properties of the Kloosterman-Selberg Zeta function, defined in appendix B, which are notoriously subtle. In particular this analytic continuation generally leads to holomorphic anomalies. For this reason, in [20] we restricted the analysis to the case $w = 0$, where the analytic continuation is under control. Another drawback of the Selberg-Poincaré series (2.5) is that it fails to be an eigenmode of the Laplacian on $\mathbb{H}$, rather it satisfies [39]

$$\left[\Delta_w + \tfrac{1}{2}\, s(1-s) + \tfrac{1}{8}\, w(w+2)\right]\, E(s,\kappa,w) = 2\pi\kappa\, (s - \tfrac{w}{2})\, E(s+1,\kappa,w)\,, \tag{2.6}$$

where Δ_w is the weight-w hyperbolic Laplacian defined in (A.1). Since $E(s+1,\kappa,w)$ may in general have a pole at $s = \frac{w}{2}$, the analytic continuation of $E(s,\kappa,w)$ to this value is not even guaranteed to be harmonic.

To circumvent these problems, following [32–34] we introduce a different regularisation of the Poincaré series (2.3) for negative weight, which is both modular invariant and annihilated by the operator on the l.h.s. of (2.6). Namely, we choose the seed in (2.2) to be $f(\tau) = \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}$ where

$$\mathcal{M}_{s,w}(y) = |4\pi y|^{-\frac{w}{2}}\, M_{\frac{w}{2}\mathrm{sgn}(y),\, s-\frac{1}{2}}\, (4\pi |y|) \tag{2.7}$$

is expressed in terms of the Whittaker function[7] $M_{\lambda,\mu}(z)$. We thus define the Niebur-Poincaré series

$$\begin{aligned} \mathcal{F}(s,\kappa,w) &= \tfrac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}\, |_w\, \gamma \\ &= \tfrac{1}{2} \sum_{(c,d)=1} (c\tau+d)^{-w}\, \mathcal{M}_{s,w}\left(-\frac{\kappa\,\tau_2}{|c\tau+d|^2}\right)\, \exp\left\{-2i\pi\kappa \left(\frac{a}{c} - \frac{c\tau_1+d}{c|c\tau+d|^2}\right)\right\}. \end{aligned} \tag{2.8}$$

Since $\mathcal{M}_{s,w}(y) \sim y^{\Re(s)-\frac{w}{2}}$ as $y\to 0$, eq. (2.8) converges absolutely for $\Re(s) > 1$, independently of w and κ. Moreover, for $\kappa > 0$, the case of main interest in this work, the seed behaves as

$$\mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1} \sim \frac{\Gamma(2s)}{\Gamma(s+\frac{w}{2})}\, q^{-\kappa} \qquad \text{as} \quad \tau_2 \to \infty\,, \tag{2.9}$$

so that $\mathcal{F}(s,\kappa,w)$ can indeed be viewed as a regulated version of the naïve Poincaré series $P(q^{-\kappa}, w)$, up to an overall normalisation. By construction it is an eigenmode of the weight-w Laplacian on $\mathbb{H}$,

$$\left[\Delta_w + \tfrac{1}{2}\, s(1-s) + \tfrac{1}{8}\, w(w+2)\right]\, \mathcal{F}(s,\kappa,w) = 0\,, \tag{2.10}$$

for all values of s,κ,w. We shall denote by $\mathcal{H}(s,w) = \mathcal{H}(1-s,w)$ the space of real-analytic solutions to (2.10) which transform with modular weight w under Γ.

[7] For a definition of Whittaker functions and some of their properties see appendix A.2.

The raising and lowering operators D_w, $\bar{D}_w$ defined in (A.2), map $\mathcal{H}(s,w)$ into $\mathcal{H}(s,w\pm 2)$, and have a simple action on the Niebur-Poincaré series

$$\begin{aligned} D_w \cdot \mathcal{F}(s,\kappa,w) &= 2\kappa\,\left(s+\tfrac{w}{2}\right)\,\mathcal{F}(s,\kappa,w+2)\,,\\ \bar{D}_w \cdot \mathcal{F}(s,\kappa,w) &= \frac{1}{8\kappa}\left(s-\tfrac{w}{2}\right)\,\mathcal{F}(s,\kappa,w-2)\,. \end{aligned} \tag{2.11}$$

Furthermore, under the action of the Hecke operator (A.7) $\mathcal{F}(s,\kappa,w)$ transforms as

$$T_{\kappa'}\cdot\mathcal{F}(s,\kappa,w) = \sum_{d|(\kappa,\kappa')} d^{1-w}\,\mathcal{F}(s,\kappa\kappa'/d^2,w)\ . \tag{2.12}$$

In particular, setting $\kappa = 1$, the series $\mathcal{F}(s,\kappa',w)$ is obtained by acting with $T_{\kappa'}$ on $\mathcal{F}(s,1,w)$.

While the Poincaré series (2.8) converges absolutely only for $\Re(s) > 1$, it is known to have a meromorphic continuation to the complex s-plane, holomorphic in the region $\Re(s) > \frac{1}{2}$ [32, 40], but with poles on the lines $s\in\frac{1}{2}+\mathrm{i}\mathbb{R}$ and $s\in\frac{1}{4}+\mathrm{i}\mathbb{R}$. Moreover, the 'completed' series

$$\mathcal{F}^\star(s,\kappa,w) = \frac{\Gamma(1-2s)}{\Gamma\left(1-s+\frac{w}{2}\,\mathrm{sgn}(\kappa)\right)}\mathcal{F}(s,\kappa,w) \tag{2.13}$$

is known to be invariant under $s\mapsto 1-s$, up to an additive contribution proportional to the non-holomorphic Eisenstein series $E^\star(s,w)$ [32, 40]. In this work however we shall only consider $\mathcal{F}(s,\kappa,w)$ in its domain of convergence $\Re(s)>1$, except for $w=0$ where we allow $s=1$.

2.2 Fourier expansion of the Niebur-Poincaré series

The Fourier expansion of $\mathcal{F}(s,\kappa,w)$ can be obtained following the standard procedure of extracting the contribution from $c=0, d=1$, setting $d=d'+mc$ in the remaining sum, and Poisson resumming over m. The result is [34, 36]

$$\mathcal{F}(s,\kappa,w) = \mathcal{M}_{s,w}(-\kappa\tau_2)\,e^{-2\pi\mathrm{i}\kappa\tau_1} + \sum_{m\in\mathbb{Z}}\tilde{\mathcal{F}}_m(s,\kappa,w)\,e^{2\pi\mathrm{i}m\tau_1}\,, \tag{2.14}$$

where, for zero frequency

$$\tilde{\mathcal{F}}_0(s,\kappa,w) = \frac{2^{2-w}\,\mathrm{i}^{-w}\,\pi^{1+s-\frac{w}{2}}\,\kappa^{s-\frac{w}{2}}\,\Gamma(2s-1)\,\sigma_{1-2s}(\kappa)}{\Gamma\left(s-\frac{w}{2}\right)\,\Gamma\left(s+\frac{w}{2}\right)\,\zeta(2s)}\,\tau_2^{1-s-\frac{w}{2}}\,, \tag{2.15}$$

while for non-vanishing integer frequencies[8]

$$\tilde{\mathcal{F}}_m(s,\kappa,w) = \frac{4\pi\,\kappa\mathrm{i}^{-w}\,\Gamma(2s)}{\Gamma\left(s+\frac{w}{2}\,\mathrm{sgn}(m)\right)}\left|\frac{m}{\kappa}\right|^{\frac{w}{2}}\,\mathcal{Z}_s(m,-\kappa)\,\mathcal{W}_{s,w}(m\tau_2)\,. \tag{2.16}$$

In these expressions, $\sigma_s(k) = \sum_{d|k} d^s$ is the divisor function and $\mathcal{Z}_s(m,-\kappa)$ is the Kloosterman-Selberg zeta function (A.37), a number-theoretical function which plays a

[8]Note that $\tilde{\mathcal{F}}_{-\kappa<0}$ does *not* include the contribution from the first term in (2.14).

central rôle in the theory of Poincaré series. The function $\mathcal{W}_{s,w}$ is expressed in terms of the Whittaker W-function as

$$\mathcal{W}_{s,w}(y) = |4\pi y|^{-\frac{w}{2}}\, W_{\frac{w}{2}\mathrm{sgn}(y),s-\frac{1}{2}}\,(4\pi|y|)\;, \tag{2.17}$$

and is determined uniquely by the requirement that $\mathcal{W}_{s,w}(n\tau_2)\, e^{2\pi i m \tau_1}$ be annihilated by the Laplace operator on the l.h.s. of (2.10), and be exponentially suppressed as $\tau_2 \to \infty$.

Using the properties (A.35) and (A.36), it is straightforward to check that all Fourier modes transform according to (2.11) under the raising and lowering operators $D_w, \bar{D}_w$. Moreover, using the action (A.9) of the Hecke operators on the Fourier coefficients, and the Selberg identity (A.39) satisfied by the Kloosterman sums, one can show that

$$T_\kappa \cdot \mathcal{F}(s,1,w) = \mathcal{F}(s,\kappa,w)\,. \tag{2.18}$$

Eq. (2.12) follows then from this equation and from the Hecke algebra (A.8).

2.3 Harmonic Maass forms from Niebur-Poincaré series

Let us focus on the Niebur-Poincaré series $\mathcal{F}(s,\kappa,w)$ at the point $s = 1 - \frac{w}{2}$. To motivate this value, we recall that any weak holomorphic modular form is an eigenmode of Δ_w with eigenvalue $-\frac{w}{2}$, and therefore belongs to $\mathcal{H}(s,w)$ for $s = 1 - \frac{w}{2}$ (or equivalently, $s = \frac{w}{2}$). However, weak holomorphic modular forms are not the only eigenmodes of Δ_w with this eigenvalue. In fact, the space $\mathcal{H}(1-\frac{w}{2}, w)$ is known as the space of *weak harmonic Maass forms* of weight w, of which weak holomorphic modular forms are only a proper subspace.

The Fourier expansion of a general weak harmonic Maass form Φ of weight w is given by [41]

$$\Phi = \sum_{m=-\infty}^{-1} (-m)^{w-1}\, \bar{b}_{-m}\, \Gamma(1-w, -4\pi m \tau_2)\, q^m + \frac{\bar{b}_0\,(4\pi\tau_2)^{1-w}}{w-1} + \sum_{m=-\kappa}^{\infty} a_m\, q^m\,, \tag{2.19}$$

where $\Gamma(s,x)$ is the incomplete Gamma function and a_m, b_m are coefficients constrained by modular invariance. As a result, a generic weak harmonic Maass form has an infinite number of negative frequency components, which are non-holomorphic functions of τ. A harmonic Maass form splits into the sum $\Phi = \Phi_a + \Phi_b$ of a holomorphic part $\Phi_a = \sum_{m=-\kappa}^{\infty} a_m\, q^m$, sometimes called a Mock modular form, and a non-holomorphic part Φ_b. The non-holomorphic and holomorphic parts can be extracted using the lowering operator $\bar{D}_w$ and the iterated raising operator D_w^{1-w}. Indeed,

- the operator $\bar{D}_w$ annihilates the holomorphic part, and produces, up to powers of τ_2, the complex conjugate of a holomorphic modular form Ψ of weight $2-w$,

$$\bar{D}_w \cdot \Phi = \bar{D}_w \cdot \Phi_b = -2^{1-2w}\,(\pi\tau_2)^{2-w}\,\overline{\Psi}\,, \qquad \Psi(\tau) = \sum_{m=0}^{\infty} b_m\, q^m\,, \tag{2.20}$$

  sometimes known as the *shadow*.

- the iterated raising operator D_w^{1-w}, also known in the physics literature as the Farey transform [42], annihilates the non-holomorphic part, and produces a weak holomorphic modular form $\varXi$ of weight $2-w$,

$$D_w^{1-w}\cdot\varPhi = D_w^{1-w}\cdot\varPhi_a = \varXi\,,\qquad \varXi \equiv \sum_{m=-\kappa}^{\infty} (-2m)^{1-w}\, a_m\, q^m\,, \tag{2.21}$$

that we shall call the *ghost*. The ghost encodes the holomorphic part of the harmonic Maass form (modulo an additive constant).[9]

Returning to the Niebur-Poincaré series, we see that by construction the series $\mathcal{F}(s,\kappa,w)$ at the special point $s=1-\frac{w}{2}$ — which, for $w<0$, *belongs to the convergence domain* — is a weak harmonic Maass form of weight w. Indeed, using (A.30) we find that its Fourier expansion (2.14) reduces to

$$\mathcal{F}(1-\tfrac{w}{2},\kappa,w) = \mathcal{M}_{1-\frac{w}{2},w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1} + \sum_{m\in\mathbb{Z}} \tilde{\mathcal{F}}_m(1-\tfrac{w}{2},\kappa,w)\, e^{2i\pi m\tau_1}\,, \tag{2.22}$$

where the seed simplifies to a finite sum

$$\begin{aligned}\mathcal{M}_{1-\frac{w}{2},w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1} &= \varGamma(2-w)\left(q^{-\kappa} - \bar{q}^{\kappa}\sum_{\ell=0}^{-w}\frac{(4\pi\kappa\tau_2)^\ell}{\ell!}\right)\\ &= \left[\varGamma(2-w) + (1-w)\,\varGamma(1-w;4\pi\kappa\tau_2)\right] q^{-\kappa}\,,\end{aligned} \tag{2.23}$$

and the remaining Fourier coefficients reduce to

$$\begin{aligned}\tilde{\mathcal{F}}_{m>0}(1-\tfrac{w}{2},\kappa,w) &= 4\pi\, i^{-w}\,\kappa\,\varGamma(2-w)\,\left(\frac{m}{\kappa}\right)^{\frac{w}{2}}\,\mathcal{Z}_{1-\frac{w}{2}}(m,-\kappa)\, e^{-2\pi m\tau_2}\,,\\ \tilde{\mathcal{F}}_{m<0}(1-\tfrac{w}{2},\kappa,w) &= 4\pi\, i^{-w}\,\kappa\,(1-w)\left(\frac{-m}{\kappa}\right)^{\frac{w}{2}}\mathcal{Z}_{1-\frac{w}{2}}(m,-\kappa)\,\varGamma(1-w,-4\pi m\tau_2)\, e^{-2\pi m\tau_2}\,,\\ \tilde{\mathcal{F}}_{m=0}(1-\tfrac{w}{2},\kappa,w) &= \frac{4\pi^2\,\kappa}{(2\pi i)^w}\,\frac{\sigma_{w-1}(\kappa)}{\zeta(2-w)}\,.\end{aligned} \tag{2.24}$$

One thus recognises an expansion of the form (2.19) with coefficients

$$\begin{aligned} a_{-\kappa} &= \varGamma(2-w)\,,\\ a_{-\kappa<m<0} &= 0\,,\\ a_0 &= \frac{4\pi^2\,\kappa}{(2\pi i)^w}\,\frac{\sigma_{w-1}(\kappa)}{\zeta(2-w)}\,,\\ a_{m>0} &= 4\pi\, i^{-w}\,\kappa\,\varGamma(2-w)\,\left(\frac{m}{\kappa}\right)^{\frac{w}{2}}\,\mathcal{Z}_{1-\frac{w}{2}}(m,-\kappa)\,,\\ b_0 &= 0\,,\\ b_{m>0} &= (1-w)\,\kappa^{1-w}\,\delta_{m,\kappa} + 4\pi\, i^w\,(1-w)\,(m\,\kappa)^{1-\frac{w}{2}}\,\mathcal{Z}_{1-\frac{w}{2}}(m,\kappa)\,.\end{aligned} \tag{2.25}$$

[9] Notice that the ghost is only defined for integer weight w, unlike the shadow, which extends to the case of half-integer weight Mock theta series.

In particular, $b_0=0$, so that the shadow of $\mathcal{F}\left(1-\frac{w}{2},\kappa,w\right)$ is a cusp form of weight $2-w$, proportional to the holomorphic Poincaré series $P(-\kappa,2-w)$. Indeed, using (2.11) we find

$$
\begin{aligned}
\bar{D}_w\cdot\mathcal{F}(1-\tfrac{w}{2},\kappa,w) =&\frac{1-w}{8\kappa}\,\mathcal{F}(1-\tfrac{w}{2},\kappa,w-2)\\
=&\frac{1-w}{8\kappa}\,(4\pi\,\kappa\,\tau_2)^{2-w}\,\overline{P(-\kappa,2-w)}\,,
\end{aligned}
\tag{2.26}
$$

where in the second line we have recognised the Fourier expansion of the standard holomorphic Poincaré series of weight greater than 2. Similarly, using (2.11) the ghost of $\mathcal{F}(1-\frac{w}{2},\kappa,w)$ is

$$
D_w^{1-w}\cdot\mathcal{F}(1-\tfrac{w}{2},\kappa,w)=(2\kappa)^{1-w}\,\Gamma(2-w)\,\mathcal{F}(1-\tfrac{w}{2},\kappa,2-w)\,, \tag{2.27}
$$

and corresponds to the Niebur-Poincaré series $\mathcal{F}\left(\frac{w'}{2},\kappa,w'\right)$, with $w'=2-w>2$ *within the convergence domain*. Moreover, the Fourier expansion of the latter reproduces that of the Poincaré series $P(\kappa,w)$ of positive weight

$$
\mathcal{F}(\tfrac{w'}{2},\kappa,w')=q^{-\kappa}+2\pi\,\mathrm{i}^{-w'}\sum_{m>0}\left(\frac{m}{\kappa}\right)^{\frac{w'-1}{2}}\sum_{c>0}\frac{S(m,-\kappa;c)}{c}\,I_{w'-1}\left(\frac{4\pi}{c}\sqrt{\kappa m}\right)q^m\,. \tag{2.28}
$$

As an aside, we note that the holomorphic part $\mathcal{F}_a\left(1-\frac{w}{2},\kappa,w\right)$ of the Niebur-Poincaré series $\mathcal{F}(s,\kappa,w)$ at $s=1-\frac{w}{2}$ reproduces the Fourier expansion of the Poincaré series $\Gamma(2-w)\,P(\kappa,w)$ defined by holomorphic regularisation as in (2.4) and worked out in [28, 29]. Therefore, the non-holomorphic part $\mathcal{F}_b\left(1-\frac{w}{2},\kappa,w\right)$ of the same Niebur-Poincaré series provides the modular completion of the Eichler integral $P(\kappa,w)$ — a clear advantage of modular-invariant regularisation over holomorphic regularisation.

To make this discussion less abstract, we shall now exhibit the harmonic Maass form $\mathcal{F}\left(1-\frac{w}{2},\kappa,w\right)$, its shadow and its ghost for the two cases $w=-10$ and $w=-14$ (lower values of $|w|$ will be discussed in the next subsection) and $\kappa=1$. Evaluating the Fourier coefficients numerically, we find:

- For $w=-10$,

$$
\mathcal{F}(6,1,-10)=\mathcal{F}_b(6,1,-10)+11!\left[q^{-1}-\frac{65520}{691}-1842.89\,q-23274.08\,q^2+\dots\right] \tag{2.29}
$$

where $\mathcal{F}_b(6,1,-10)$ is the non-holomorphic component. The shadow of (2.29) reads

$$
\mathcal{F}(6,1,-12)=(4\pi\tau_2)^{12}\,\overline{P(-1,12)}\,,\qquad P(-1,12)=\beta_{12}\,\Delta\,, \tag{2.30}
$$

where the modular discriminant Δ generates the space of cusp forms of weight 12. The ghost, obtained by acting with D^{11} on the holomorphic part, can be written as

$$
\begin{aligned}
\mathcal{F}(6,1,12)&=\frac{83E_4^3E_6^2-11E_6^4}{72\Delta}+\alpha_{12}\,\frac{(E_4^3-E_6^2)^2}{\Delta}\\
&=q^{-1}+1842.89\,q+47665306.53\,q^2+\dots\,,
\end{aligned}
\tag{2.31}
$$

where $\alpha_{12}=0.201029508104\dots,\beta_{12}=2.840287517\dots$ are irrational numbers. The coefficients 1842.89, 23274.08, 47665306.53 are two-digit approximations of the exact values $324(9216\,\alpha_{12}-1847)$, $(60617-69984\,\alpha_{12})/2$, $1024(60617-69984\,\alpha_{12})$, respectively. This example was discussed in detail in [37].

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- Similarly, for $w=-14$,

$$\mathcal{F}(8,1,-14) = \mathcal{F}_b(8,1,-14) + 15!\left[q^{-1} - \frac{16320}{3617} - 45.67\,q - 366.47\,q^2 + \dots\right] \quad (2.32)$$

where $\mathcal{F}_b(8,1,-14)$ is the non-holomorphic component. The shadow of (2.32) reads

$$\mathcal{F}(8,1,-16) = (4\pi\tau_2)^{16}\,\overline{P(-1,16)}\,, \qquad P(-1,16) = \beta_{16}\,E_4\,\Delta \quad (2.33)$$

where $E_4\Delta$ generate the space of cusp forms of weight 16. The ghost, obtained by acting with D^{15} on the holomorphic part, can be written as

$$\begin{aligned}\mathcal{F}(8,1,16) &= \frac{73E_4^4E_6^2 - E_4E_6^4}{72\Delta} + \alpha_{16}\,\frac{E_4^7 - 2E_4^4E_6^2 + E_4E_6^4}{\Delta} \\ &= q^{-1} + 45.67\,q + 12008361.57\,q^2 + \dots\,,\end{aligned} \quad (2.34)$$

where $\alpha_{16} = 0.137975847804\dots$ and $\beta_{16} = 1.3061364711\dots$ are irrational numbers. The coefficients 45.67, 366.47 in eq. (2.32) are two-digit approximations of the exact values $36(82944\,\alpha_{16} - 11443)$, $(314928\,\alpha_{16} - 37589)/16$, respectively.

These two examples illustrate the fact that Fourier coefficients of harmonic Maass forms are in general irrational numbers.

2.4 Weak holomorphic modular forms from Niebur-Poincaré series

We now come to our main goal, i.e. to find an absolutely convergent Poincaré series representation of any weak holomorphic modular form Φ_w of weight $w \leq 0$ and κ-order pole at the cusp, with given principal part

$$\Phi_w^-(\tau) = \sum_{-\kappa\leq m<0} a_m\,q^m\,. \quad (2.35)$$

As we shall see, any such Φ_w can be expressed as a linear combination of the Niebur-Poincaré series $\mathcal{F}(s,\kappa,w)$.

We have noted in the previous subsection that the eigenvalue of a weak holomorphic modular form under the hyperbolic Laplacian Δ_w coincides with the eigenvalue of the Niebur-Poincaré series whenever $s = 1-\frac{w}{2}$. At this value, however, $\mathcal{F}\left(1-\frac{w}{2},\kappa,w\right)$ is a weak harmonic Maass form, in general not holomorphic. Exceptions to this statement occur at the special values $w \in \{-2,-4,-6,-8,-12\}$, where the space of holomorphic cusp forms of weight $2-w$ is empty, and $\mathcal{F}\left(1-\frac{w}{2},\kappa,w\right)$ can be recognised as an element of the ring of weak holomorphic modular forms by matching the principal part of their expansions. For $\kappa=1$ the exact identification is reported in table 1, while for $\kappa>1$, the proper identification of $\mathcal{F}\left(1-\frac{w}{2},\kappa,w\right)$ can be obtained by acting on $\mathcal{F}\left(1-\frac{w}{2},1,w\right)$ with the Hecke operator T_κ, as given by eq. (2.18).

For $w\leq 0$ outside the list above, the space of cusp forms of weight $2-w$ is not empty, and $\mathcal{F}(1-\frac{w}{2},\kappa,w)$ is indeed a genuine harmonic Maass form, with non-vanishing shadow. Nevertheless, it can be shown [34] that the linear combination

$$\mathcal{G}(s,w) \equiv \frac{1}{\Gamma(2-w)}\sum_{-\kappa\leq m<0} a_m\,\mathcal{F}(s,m,w)\,, \quad (2.36)$$

w	$\mathcal{F}\big(1-\frac{w}{2},1,w\big)$	$\mathcal{F}\big(1-\frac{w}{2},1,2-w\big)$
0	$j+24$	$E_4^2 E_6\,\Delta^{-1}$
-2	$3!\,E_4 E_6\,\Delta^{-1}$	$E_4(j-240)$
-4	$5!\,E_4^2\,\Delta^{-1}$	$E_6(j+204)$
-6	$7!\,E_6\,\Delta^{-1}$	$E_4^2(j-480)$
-8	$9!\,E_4\,\Delta^{-1}$	$E_4E_6(j+264)$
-12	$13!\,\Delta^{-1}$	$E_4^2E_6(j+24)$

Table 1. Weak holomorphic modular forms obtained as the limit $s\to 1-\frac{w}{2}$ of $\mathcal{F}(s,1,w)$, for the values $w\in\{0,-2,-4,-6,-8,-12\}$. For w negative and outside this range, the limit yields a weak harmonic Maass form. The second line shows the ghost, which is a weak holomorphic modular form of weight $2-w$ with vanishing constant term (aside from the case $w=0$).

with coefficients a_m determined by the principal part

$$\Phi_w^- = \sum_{-\kappa\le m<0} a_m\, q^{-m} \tag{2.37}$$

of any weak holomorphic form Φ_w of negative weight w, reduces to a weak holomorphic modular form for $s=1-\frac{w}{2}$, namely Φ_w itself. Said differently, the shadows of the weak harmonic Maass forms $\mathcal{F}\left(1-\frac{w}{2},\kappa,w\right)$ cancel in the linear combination (2.36). As a result, any Φ_w can be represented as the linear combination

$$\Phi_w = \frac{1}{\Gamma(2-w)} \sum_{-\kappa\le m<0} a_m\,\mathcal{F}\left(1-\tfrac{w}{2},m,w\right)\,, \tag{2.38}$$

or equivalently, using (2.23), as an absolutely convergent Poincaré sum

$$\Phi_w = \tfrac12 \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left(\Phi_w^- - \sum_{-\kappa\le m<0}\sum_{\ell=0}^{-w} a_m\,\bar q^{m}\,\frac{(4\pi\kappa\tau_2)^\ell}{\ell!}\right)\Bigg|_w \gamma\,, \tag{2.39}$$

where the subtraction in the bracket ensures that the seed is $\mathcal{O}(\tau_2^{1-\frac{w}{2}})$ as $\tau_2\to 0$. We stress that, unlike the holomorphic regularisation in (2.4), the expression (2.39) is manifestly modular covariant and absolutely convergent.

To illustrate the power of eq. (2.38), let us reconsider the two examples of the previous subsection, now allowing for $\kappa=1,2$.

- For $w=-10$, $\mathcal{F}(6,2,-10)$ and $\mathcal{F}(6,1,-10)$ are separately weak harmonic Maass forms with irrational coefficients, but the linear combination

$$\mathcal{F}(6,2,-10)+24\,\mathcal{F}(6,1,-10) = 11!\,\frac{E_4^2E_6}{\Delta^2} = 11!\,(q^{-2}+24\,q^{-1}-196560+\dots) \tag{2.40}$$

produces (up to an overall normalisation) the unique weak holomorphic form[10] of weight -10 with a double pole at $q=0$;

[10] Compare the simplicity of our expression to the corresponding equation in Sec 4.1 of [37].

- Similarly, for $w=-14$, $\mathcal{F}(8,2,-14)$ and $\mathcal{F}(8,1,-14)$ are separately weak harmonic Maass forms with irrational coefficients, but the linear combination

$$\mathcal{F}(8,2,-14)-216\,\mathcal{F}(8,1,-14)=15!\,\frac{E_4E_6}{\Delta^2}=15!\,(q^{-2}-216\,q^{-1}-146880+\dots) \quad (2.41)$$

produces (up to an overall normalisation) the unique weak holomorphic form of weight -14 with a double pole at $q=0$.

Similar relations occur for higher negative weight $w<-14$ and higher order κ of the pole at $q=0$.

2.5 Weak almost holomorphic modular forms from Niebur-Poincaré series

For physics applications it is important to extend our previous analysis to the case of weak *almost* holomorphic modular forms, i.e. elements of the ring generated by the almost holomorphic Eisenstein series $\hat{E}_2$ and the ordinary weak holomorphic modular forms, or equivalently, by the modular derivatives $D^n\Phi$ of ordinary weak holomorphic modular forms.

To this end, it is important to note that for any integer $n\geq 0$, it follows from (2.11) that the Niebur-Poincaré series $\mathcal{F}(s,\kappa,w)$ evaluated at the point $s=1-\frac{w}{2}+n$ can be expressed as

$$\mathcal{F}\left(1-\tfrac{w}{2}+n,\kappa,w\right)=\frac{1}{(2\kappa)^n\,n!}\,D^n\,\mathcal{F}\left(1-\tfrac{w}{2}+n,\kappa,w-2n\right)\,, \quad (2.42)$$

where D^n is the iterated modular derivative (A.6). The Niebur-Poincaré series $\mathcal{F}(s',\kappa,w')$ appearing on the r.h.s. satisfies $s'=1-\frac{w'}{2}$, and thus is a harmonic Maass form.

As a result, provided that the coefficients a_m in the linear combination (2.36) are chosen such that

$$\Phi^-_{w-2n}\equiv\sum_{-\kappa\leq m<0}\frac{a_m}{(2m)^n\,n!}\,q^m \quad (2.43)$$

is the principal part of a weak holomorphic modular form Φ_{w-2n} of weight $w-2n$, then the linear combination $\mathcal{G}(s,w)$ in (2.36) evaluated at the point $s=1-\frac{w}{2}+n$ reproduces an almost holomorphic modular form of weight w,

$$\mathcal{G}\left(1-\tfrac{w}{2}+n,w\right)=\frac{1}{\Gamma(2-w)}\sum_{-\kappa\leq m<0}a_m\,\mathcal{F}\left(1-\tfrac{w}{2}+n,m,w\right)=D^n\,\Phi_{w-2n}\,. \quad (2.44)$$

More generally, we refer to the space $\bigoplus_{n\geq 0}\mathcal{H}\left(1-\frac{w}{2}+n,w\right)$ as the space of "weak almost harmonic Maass forms", of which almost holomorphic modular forms are only a subspace. The general Fourier expansion of such forms can be obtained by taking the limit $s=1-\frac{w}{2}+n$ in eqs. (2.14) and (2.16), and by using the identities (A.33) and (A.34). Similarly, the series $\mathcal{F}(s,\kappa,w)$ at the point $s=1-\frac{w}{2}+n$ for $n\leq -2$ may be obtained from $\mathcal{F}(s',\kappa,w')$ at the point $s'=-\frac{w'}{2}$ by using the lowering operator $\bar{D}_w$.

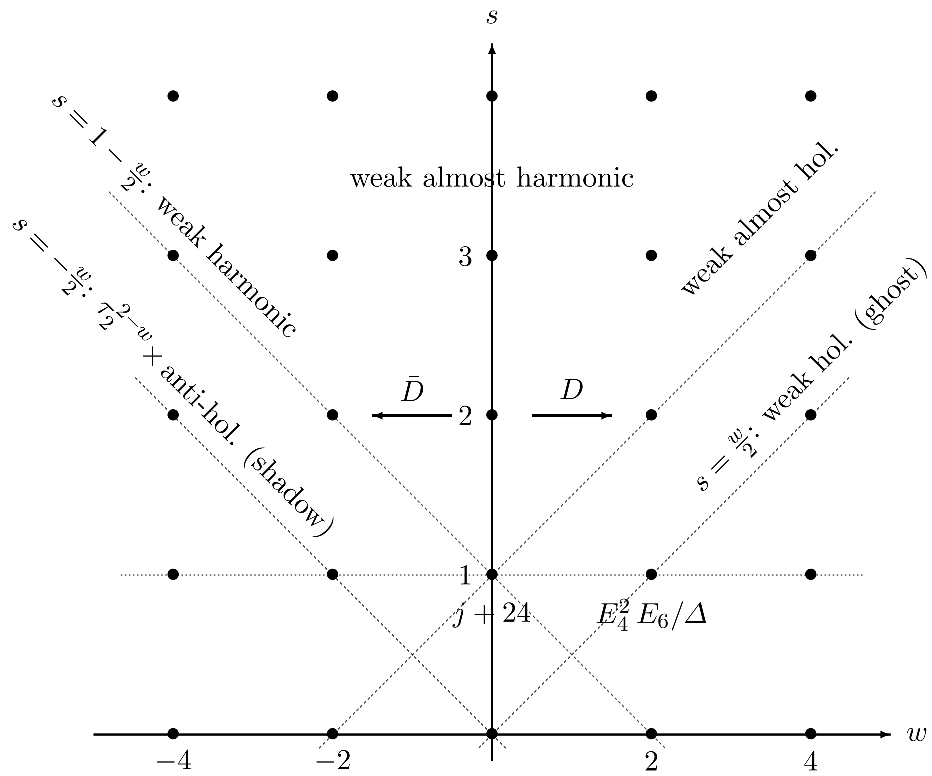

Figure 1. Phase diagram for the Niebur-Poincaré series $\mathcal{F}(s, \kappa, w)$ for integer values of $\left(\frac{w}{2}, s\right)$ with $s \geq 1$. For low negative values of w, $\mathcal{F}(s, \kappa, w)$ reduces to an ordinary weak almost holomorphic Maass form, see table 2.

2.6 Summary

To summarise this discussion, it is useful to consider the plane of the variables $\left(\frac{w}{2}, s\right)$ as in figure 1. The Niebur-Poincaré series $\mathcal{F}(s, \kappa, w)$ converges absolutely for $s > 1$. For integer values of s, it is generally a weak almost harmonic Maass form, and on the line $s = 1 - \frac{w}{2}$ (and $w < 0$), $\mathcal{F}(s, \kappa, w)$ becomes a weak harmonic Maass form. On the line $s = -\frac{w}{2}$, obtained from the former by acting with the lowering operator $\overline{D}$, $\mathcal{F}(s, \kappa, w)$ reduces, up to an overall multiplicative factor τ_2^{-w}, to the complex conjugate of a cusp form of weight $2 - w$, known as the shadow of the harmonic Maass form $\mathcal{F}(s, \kappa, w + 2)$. On the line $s = \frac{w}{2}$, $\mathcal{F}(s, \kappa, w)$ is instead a weak holomorphic modular form. It is connected to its expression on the line $s = 1 - \frac{w}{2}$ by the action of the iterated raising operator D^{1-w}, and thus we refer to it as the 'ghost' of the harmonic Maass form $\mathcal{F}(s, \kappa, 2 - w)$. In the quadrant $w > 2, s > 1$, $\mathcal{F}(s, \kappa, w)$ is more generally a weak almost holomorphic modular form. For low negative values of w and s integer, $\mathcal{F}(s, \kappa, w)$ is in fact always a weak almost holomorphic modular form, as displayed in table 2. Genuine harmonic Maass forms start appearing at $s = 6$ and $s \geq 8$.

$s\backslash w$	-10	-8	-6	-4	-2	0	2	4	6	8	10
5	0	$9!\frac{E_4}{\Delta}$	$\frac{9!}{2}D\frac{E_4}{\Delta}$	$\frac{9!}{8}D^2\frac{E_4}{\Delta}$	$\frac{9!}{2^3\,3!}D^3\frac{E_4}{\Delta}$	$\frac{9!}{2^4\,4!}D^4\frac{E_4}{\Delta}$	$\frac{9!}{2^5\,5!}D^5\frac{E_4}{\Delta}$	$\frac{9!}{2^6\,6!}D^6\frac{E_4}{\Delta}$	$\frac{9!}{2^7\,7!}D^7\frac{E_4}{\Delta}$	$\frac{9!}{2^7\,8!}D^8\frac{E_4}{\Delta}$	$E_4E_6(j+264)$
4	0	0	$7!\frac{E_6}{\Delta}$	$\frac{7!}{2}D\frac{E_6}{\Delta}$	$\frac{7!}{8}D^2\frac{E_6}{\Delta}$	$\frac{7!}{2^3\,3!}D^3\frac{E_6}{\Delta}$	$\frac{7!}{2^4\,4!}D^4\frac{E_6}{\Delta}$	$\frac{7!}{2^5\,5!}D^5\frac{E_6}{\Delta}$	$\frac{7!}{2^6\,6!}D^6\frac{E_6}{\Delta}$	$E_4^2(j-480)$	$\frac{7!}{2^8\,8!}D^8\frac{E_6}{\Delta}$
3	0	0	0	$5!\frac{E_4^2}{\Delta}$	$\frac{5!}{2}D\frac{E_4^2}{\Delta}$	$\frac{5!}{8}D^2\frac{E_4^2}{\Delta}$	$\frac{5!}{2^3 3!}D^3\frac{E_4^2}{\Delta}$	$\frac{5!}{2^4 4!}D^4\frac{E_4^2}{\Delta}$	$E_6(j+504)$	$\frac{5!}{2^6 6!}D^6\frac{E_4^2}{\Delta}$	$\frac{5!}{2^7 7!}D^7\frac{E_4^2}{\Delta}$
2	0	0	0	0	$3!\frac{E_4E_6}{\Delta}$	$3D\frac{E_4E_6}{\Delta}$	$\frac{3}{4}D^2\frac{E_4E_6}{\Delta}$	$E_4(j-240)$	$\frac{3!}{2^4 4!}D^4\frac{E_4E_6}{\Delta}$	$\frac{3!}{2^5 5!}D^5\frac{E_4E_6}{\Delta}$	$\frac{3!}{2^6 6!}D^6\frac{E_4E_6}{\Delta}$
1	0	0	0	0	0	$j+24$	$\frac{E_4^2E_6}{\Delta}$	$\frac{1}{2^2 2!}D^2 j$	$\frac{1}{2^3 3!}D^3 j$	$\frac{1}{2^4 4!}D^4 j$	$\frac{1}{2^5 5!}D^5 j$

Table 2. Niebur-Poincaré series $\mathcal{F}(s,1,w)$ at the special values $s=1-\frac{w}{2}+n$ with n integer, for low negative values of w.

3 A new road to one-loop modular integrals

We are interested in the evaluation of one-loop modular integrals of the form (1.2), while keeping manifest at all steps the automorphisms of the Narain lattice, i.e. T-duality. Such integrals encode, for instance, threshold corrections to the running of gauge and gravitational couplings. The function Φ, related to the elliptic genus and dependent on the vacuum under consideration, is in general a weak almost holomorphic modular form of non-positive weight. For example, in $\mathcal{N}=4$ compactifications of the SO(32) heterotic string (with vanishing Wilson lines) one finds a linear combination of zero-weight weak almost holomorphic modular forms [2]

$$\begin{aligned}\Phi(\tau) =& \, t_8\,\mathrm{tr}F^4 + \frac{1}{2^7\,3^2\,5}\,\frac{E_4^3}{\Delta}\,t_8\,\mathrm{tr}R^4 + \frac{1}{2^9\,3^2}\,\frac{\hat{E}_2^2\,E_4^2}{\Delta}\,t_8(\mathrm{tr}R^2)^2 \\ & + \frac{1}{2^8\,3^2}\left(\frac{\hat{E}_2\,E_4\,E_6}{\Delta} - \frac{\hat{E}_2^2\,E_4^2}{\Delta}\right)t_8\,\mathrm{tr}F^2\,\mathrm{tr}\,R^2 \\ & + \frac{1}{2^9\,3^2}\left(\frac{E_4^3}{\Delta} + \frac{\hat{E}_2^2\,E_4^2}{\Delta} - 2\,\frac{\hat{E}_2\,E_4\,E_6}{\Delta} - 2^7\,3^2\right)t_8\,(\mathrm{tr}\,F^2)^2\,,\end{aligned} \tag{3.1}$$

where t_8 is the familiar tensor appearing in four-point amplitudes of the heterotic string, and F and R are the gauge field strength and curvature two-form. A similar expression arises for gauge and gravitational couplings in the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string.

While the traditional procedure for evaluating integrals of the form (1.2) has been to unfold the integration domain $\mathcal{F}$ against the lattice partition function $\Gamma_{d+k,d}$, in [20] we instead proposed to represent Φ as a Poincaré series of the form (2.2), which is then amenable to the unfolding procedure. The advantage of this approach is that T-duality is kept manifest at all steps and the final result is expressed as a sum over BPS states which is manifestly invariant under $\mathrm{O}(d+k,d;\mathbb{Z})$. Moreover, singularities associated to states becoming massless at special points in the Narain moduli space are easily read off from this representation.

3.1 Niebur-Poincaré series in a nutshell

In order to implement this strategy, it is essential to represent Φ as an *absolutely convergent* Poincaré series, so that the unfolding of the fundamental domain is justified. Fortunately, as discussed in detail in section 2 and summarised in the following, any weak almost holomorphic modular form Φ_w of weight $w\le 0$ can be written as a linear combination of Niebur-Poincaré series, defined as

$$\begin{aligned}\mathcal{F}(s,\kappa,w) &= \tfrac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathcal{M}_{s,w}(-\kappa\tau_2)\,e^{-2\pi\mathrm{i}\kappa\tau_1}\,|_w\,\gamma \\ &= \tfrac{1}{2}\sum_{(c,d)=1}(c\tau+d)^{-w}\,\mathcal{M}_{s,w}\left(\frac{-\kappa\tau_2}{|c\tau+d|^2}\right)\exp\left\{-2\mathrm{i}\pi\kappa\left(\frac{a}{c}-\frac{c\tau_1+d}{c|c\tau+d|^2}\right)\right\}.\end{aligned} \tag{3.2}$$

Here $\mathcal{M}_{s,w}$ is related to the Whittaker M-function via

$$\mathcal{M}_{s,w}(-y) = (4\pi y)^{-w/2}\,M_{-\frac{w}{2},s-\frac{1}{2}}(4\pi y)\,, \tag{3.3}$$

and s is a complex parameter, the real part of which must be larger than 1 for absolute convergence. The choice of the Whittaker function in (3.2) is dictated by the requirement that $\mathcal{F}(s,\kappa,w)$ be an eigenmode of the hyperbolic Laplacian Δ_w (see eq. (2.10)), and behave as $q^{-\kappa}$ at the cusp $q \equiv e^{2\pi i\tau} = 0$ (see eq. (2.9)), thus reproducing, for $\kappa = 1$, the simple pole associated to the unphysical tachyon of the heterotic string. The set of Niebur-Poincaré series $\mathcal{F}(s,\kappa,w)$ is closed under the action of the derivative operators D_w and $\bar{D}_w$ defined in (A.2), which, according to (2.11), act by raising or lowering the weight w by two units while keeping s fixed.

At the special point $s = 1 - \frac{w}{2}$, which for $w < 0$ lies within the domain of absolute convergence, the Niebur-Poincaré series $\mathcal{F}(s,\kappa,w)$ becomes a weak harmonic Maass form.[11] In particular, unless w takes one of the special values listed in table 1, it is in general not holomorphic. Although the values listed in the table essentially exhaust all the cases of interest in string theory, it is a remarkable fact that linear combinations of Niebur-Poincaré series, with coefficients determined by the principal part of a weak holomorphic modular form $\mathit{\Phi}_w$, are in fact weakly holomorphic, and reproduce $\mathit{\Phi}_w$ itself [34]:

$$\mathit{\Phi}_w^- = \sum_{-\kappa\leq m<0} a_m\, q^{-m} \quad \Rightarrow \quad \mathit{\Phi}_w = \frac{1}{\Gamma(2-w)} \sum_{-\kappa\leq m<0} a_m\, \mathcal{F}(1-\tfrac{w}{2}, m, w)\ . \tag{3.4}$$

Moreover, upon using (2.11) one can also relate weak almost holomorphic modular forms involving (up to) n powers of $\hat{E}_2$ — or equivalently, obtained by acting up to n times with the derivative operator D_w on a weak holomorphic modular form — to linear combinations of $\mathcal{F}(s,\kappa,w)$ evaluated at the special points $s = 1 - \frac{w}{2} + n'$, with $0 \leq n' \leq n$.

In the cases relevant to heterotic string threshold corrections, the elliptic genus $\mathit{\Phi}_w$ has a simple pole at $q = 0$, corresponding to the unphysical tachyon, and therefore the expansion (3.4) includes only one term, with $\kappa = m = 1$ (modulo an additive constant in the case $w = 0$). Moreover, the weight w is related to the signature $(d+k,d)$ of the Narain lattice by $w = -k/2$. Since string theory restricts the Narain lattice to be even and self-dual, so that $\mathit{\Gamma}_{d+k,d}$ is covariant under the full modular group $\mathit{\Gamma} = \mathrm{SL}(2,\mathbb{Z})$, the possible values of w are $w = 0$ (corresponding to the point of unbroken $\mathrm{E}_8\times\mathrm{E}_8$ or $\mathrm{SO}(32)$ gauge symmetry), $w = -4$ (corresponding to the point of unbroken E_8 symmetry, with arbitrary Wilson lines for the other E_8 factor), or $w = -8$ (corresponding to generic values of the Wilson lines in $\mathrm{E}_8\times\mathrm{E}_8$ or $\mathrm{SO}(32)$). The complete list of weak almost holomorphic modular forms with a simple pole at $q = 0$ and modular weights $w = 0, -2, -4, -6, -8, -10$, together with their expressions as linear combinations of Niebur-Poincaré series, can be found in table 3. Although string-theory applications only require $\kappa = 1$, our methods apply equally well for arbitrary positive integer values of κ, which we therefore keep general until section 3.5.

3.2 One-loop BPS amplitudes as BPS-state sums

Since any weak almost holomorphic modular form of negative weight can be represented as a linear combination of Niebur-Poincaré series, for the purpose of computing integrals

[11] For a definition of weak harmonic Maass forms see section 2.3.

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$w=0$
$\frac{\hat{E}_2 E_4 E_6}{\Delta} = \mathcal{F}(2,1,0) - 5\,\mathcal{F}(1,1,0) - 144$
$\frac{\hat{E}_2^2 E_4^2}{\Delta} = \frac{1}{5}\,\mathcal{F}(3,1,0) - 4\mathcal{F}(2,1,0) + 13\,\mathcal{F}(1,1,0) + 144$
$\frac{\hat{E}_2^3 E_6}{\Delta} = \frac{3}{175}\,\mathcal{F}(4,1,0) - \frac{3}{5}\,\mathcal{F}(3,1,0) + \frac{33}{5}\,\mathcal{F}(2,1,0) - 17\,\mathcal{F}(1,1,0) - 144$
$\frac{\hat{E}_2^4 E_4}{\Delta} = \frac{1}{1225}\,\mathcal{F}(5,1,0) - \frac{6}{175}\,\mathcal{F}(4,1,0) + \frac{18}{35}\,\mathcal{F}(3,1,0) - \frac{16}{5}\,\mathcal{F}(2,1,0) + \frac{29}{5}\,\mathcal{F}(1,1,0) + \frac{144}{5}$
$\frac{\hat{E}_2^6}{\Delta} = \frac{1}{1926925}\mathcal{F}(7,1,0) - \frac{3}{2695}\mathcal{F}(5,1,0) + \frac{6}{175}\mathcal{F}(4,1,0) - \frac{3}{7}\mathcal{F}(3,1,0) + \frac{12}{5}\mathcal{F}(2,1,0) - \frac{29}{7}\mathcal{F}(1,1,0) - \frac{144}{7}$
$w=-2$
$\frac{\hat{E}_2 E_4^2}{\Delta} = \frac{1}{40}\mathcal{F}(3,1,-2) - \frac{1}{3}\mathcal{F}(2,1,-2)$
$\frac{\hat{E}_2^2 E_6}{\Delta} = \frac{1}{525}\mathcal{F}(4,1,-2) - \frac{1}{20}\mathcal{F}(3,1,-2) + \frac{11}{30}\mathcal{F}(2,1,-2)$
$\frac{\hat{E}_2^3 E_4}{\Delta} = \frac{1}{11760}\mathcal{F}(5,1,-2) - \frac{1}{350}\mathcal{F}(4,1,-2) + \frac{9}{280}\mathcal{F}(3,1,-2) - \frac{2}{15}\mathcal{F}(2,1,-2)$
$\frac{\hat{E}_2^5}{\Delta} = \frac{1}{19819800}\mathcal{F}(7,1,-2) - \frac{1}{12936}\mathcal{F}(5,1,-2) + \frac{1}{525}\mathcal{F}(4,1,-2) - \frac{1}{56}\mathcal{F}(3,1,-2) + \frac{1}{15}\mathcal{F}(2,1,-2)$
$w=-4$
$\frac{\hat{E}_2 E_6}{\Delta} = \frac{1}{2520}\mathcal{F}(4,1,-4) - \frac{1}{120}\mathcal{F}(3,1,-4)$
$\frac{\hat{E}_2^2 E_4}{\Delta} = \frac{1}{70560}\mathcal{F}(5,1,-4) - \frac{1}{2520}\mathcal{F}(4,1,-4) + \frac{1}{280}\mathcal{F}(3,1,-4)$
$\frac{\hat{E}_2^4}{\Delta} = \frac{1}{148648500}\mathcal{F}(7,1,-4) - \frac{1}{129360}\mathcal{F}(5,1,-4) + \frac{1}{6300}\mathcal{F}(4,1,-4) - \frac{1}{840}\mathcal{F}(3,1,-4)$
$w=-6$
$\frac{\hat{E}_2 E_4}{\Delta} = \frac{1}{241920}\mathcal{F}(5,1,-6) - \frac{1}{10080}\mathcal{F}(4,1,-6)$
$\frac{\hat{E}_2^3}{\Delta} = \frac{1}{792792000}\mathcal{F}(7,1,-6) - \frac{1}{887040}\mathcal{F}(5,1,-6) + \frac{1}{50400}\mathcal{F}(4,1,-6)$
$w=-8$
$\frac{\hat{E}_2^2}{\Delta} = \frac{1}{2854051200}\mathcal{F}(7,1,-8) - \frac{1}{3991680}\mathcal{F}(5,1,-8)$
$w=-10$
$\frac{\hat{E}_2}{\Delta} = \frac{1}{13!}\mathcal{F}(7,1,-10)$

Table 3. List of all weak almost holomorphic modular forms of negative weight with a simple pole at $q=0$, as linear combination of Niebur-Poincaré series $\mathcal{F}(1-\frac{w}{2}+n,1,w)$ (the holomorphic ones appear in the first column of table 1).

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of the form (1.2) it suffices to consider the basic integral

$$\mathcal{I}_{d+k,d}(G,B,Y;s,\kappa;\mathcal{T}) \equiv \mathcal{I}_{d+k,d}(s,\kappa;\mathcal{T}) = \int_{\mathcal{F}_{\mathcal{T}}} \mathrm{d}\mu\, \varGamma_{d+k,d}(G,B,Y)\, \mathcal{F}\left(s,\kappa,-\tfrac{k}{2}\right) , \tag{3.5}$$

where the modular weight $w = -k/2$ of the Niebur-Poincaré series is determined, via modular invariance, by the signature of the Narain lattice. In order to regulate potential infrared divergences, associated to massless string states, we have introduced in (3.5) an infrared cut-off $\mathcal{T}$, which we shall eventually take to infinity.

According to the unfolding procedure, extended in the presence of a hard cut-off $\mathcal{T}$ in [3], the truncated fundamental domain $\mathcal{F}_{\mathcal{T}}$ can be extended to the truncated strip $\left\{0 < \tau_2 < \mathcal{T}, -\frac{1}{2} \leq \tau_1 < \frac{1}{2}\right\}$ at the expense of restricting the sum over images in the Niebur-Poincaré series to the trivial coset, and subtracting the contribution of the non-trivial ones integrated over the complement $\mathcal{F} - \mathcal{F}_{\mathcal{T}}$. In equations

$$\begin{aligned} \mathcal{I}_{d+k,d}(s,\kappa,\mathcal{T}) = & \int_0^{\mathcal{T}} \frac{\mathrm{d}\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} \mathrm{d}\tau_1\, \varGamma_{d+k,d}\, \mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\, e^{-2\mathrm{i}\pi\kappa\tau_1} \\ & - \int_{\mathcal{F}-\mathcal{F}_{\mathcal{T}}} \mathrm{d}\mu\, \varGamma_{d+k,d} \left(\mathcal{F}\left(s,\kappa,-\tfrac{k}{2}\right) - \mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\, e^{-2\mathrm{i}\pi\kappa\tau_1}\right) . \end{aligned} \tag{3.6}$$

Using the asymptotic behaviours

$$\mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2) \sim \tau_2^{s+\frac{k}{4}} , \quad \varGamma_{d+k,d} \sim \tau_2^{-\frac{d+k}{2}} , \qquad \text{as} \quad \tau_2 \to 0 , \tag{3.7}$$

and

$$\varGamma_{d+k,d} \sim \tau_2^{\frac{d}{2}} \qquad \text{as} \quad \tau_2 \to \infty , \tag{3.8}$$

together with the Fourier expansion (2.14), one can show that the second integral in (3.6) converges for $\Re(s) > \frac{1}{4}(2d+k)$, while the first integral in (3.6) converges for $\Re(s) > 1 + \frac{1}{4}(2d+k)$. For $\Re(s)$ in this range, one may then remove the IR cut-off and extend, in the first integral, the τ_2 range to the full $\mathbb{R}_+$. Moreover, the τ_1 integral vanishes unless the lattice vector satisfies the level-matching constraint

$$p_{\mathrm{L}}^2 - p_{\mathrm{R}}^2 = 4\kappa . \tag{3.9}$$

In heterotic string vacua (with $\kappa = 1$) this condition selects the contributions of the half-BPS states in the perturbative spectrum, and thus the first integral in (3.6) can be written as a BPS-state sum

$$\mathcal{I}_{d+k,d}(s,\kappa) \equiv \sum_{\mathrm{BPS}} \int_0^{\infty} \frac{\mathrm{d}\tau_2}{\tau_2^2}\, \mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\, \tau_2^{d/2}\, e^{-\pi\tau_2(p_{\mathrm{L}}^2+p_{\mathrm{R}}^2)/2} . \tag{3.10}$$

Here we have introduced the short-hand notation

$$\sum_{\mathrm{BPS}} \equiv \sum_{p_{\mathrm{L}}\,,\,p_{\mathrm{R}}} \delta(p_{\mathrm{L}}^2 - p_{\mathrm{R}}^2 - 4\kappa) \tag{3.11}$$

to denote the sum over those lattice vectors satisfying the level-matching condition (3.9), and corresponding to half-BPS states if $\kappa = 1$. By the previous estimates, this sum is absolutely convergent for $\Re(s) > 1 + \frac{1}{4}(2d+k)$, and thus defines an analytic function of s in this range.

To relate the BPS-state sum to the modular integral of interest, we note that upon using (3.6) and rearranging terms, eq. (3.10) may be rewritten as

$$\begin{aligned}\mathcal{I}_{d+k,d}(s,\kappa) =& \mathcal{I}_{d+k,d}(s,\kappa,\mathcal{T}) \\ &+ \int_{\mathcal{F}-\mathcal{F}_\mathcal{T}} \mathrm{d}\mu\, \Gamma_{d+k,d} \left(\mathcal{F}\left(s,\kappa,-\tfrac{k}{2}\right) - \mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\, e^{-2\mathrm{i}\pi\kappa\tau_1} - f_0(s)\, \tau_2^{1-s+\frac{k}{4}} \right) \\ &+ \int_{\mathcal{F}-\mathcal{F}_\mathcal{T}} \mathrm{d}\mu\, (\Gamma_{d+k,d} - \tau_2^{\frac{d}{2}}) \left(\mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\, e^{-2\mathrm{i}\pi\kappa\tau_1} + f_0(s)\, \tau_2^{1-s+\frac{k}{4}} \right) \\ &+ \int_{\mathcal{F}-\mathcal{F}_\mathcal{T}} \mathrm{d}\mu\, \tau_2^{\frac{d}{2}} \left(\mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\, e^{-2\mathrm{i}\pi\kappa\tau_1} + f_0(s)\, \tau_2^{1-s+\frac{k}{4}} \right), \end{aligned} \tag{3.12}$$

where

$$f_0(s) = \frac{(4\pi)^{1+\frac{k}{4}} \pi^s\, \mathrm{i}^{\frac{k}{2}}\, \Gamma(2s-1)\, \kappa^{s+\frac{k}{4}}\, \sigma_{1-2s}(\kappa)}{\Gamma\left(s+\frac{k}{4}\right)\, \Gamma\left(s-\frac{k}{4}\right)\, \zeta(2s)} \tag{3.13}$$

is the coefficient of the zero-frequency Fourier mode (2.15), and the r.h.s. of (3.12) is independent of $\mathcal{T}$. The first three lines in (3.12) are analytic functions of s for $\Re(s) > 1$, since $\mathcal{I}_{d+k,d}(s,\kappa,\mathcal{T})$ is integrated over the compact domain $\mathcal{F}_\mathcal{T}$, while the integrands in the second and third line are exponentially suppressed as $\tau_2 \to \infty$, away from the points of enhanced gauge symmetry. The fourth line, however, evaluates to

$$f_0(s) \int_\mathcal{T}^\infty \mathrm{d}\tau_2\, \tau_2^{-1-s+\frac{2d+k}{4}} = f_0(s)\, \frac{\mathcal{T}^{\frac{2d+k}{4}-s}}{s-\frac{2d+k}{4}} \tag{3.14}$$

and is therefore analytic in s, except for a simple pole at $s = \frac{1}{4}(2d+k)$. We thus conclude that the BPS state sum (3.10) admits a meromorphic continuation to $\Re(s) > 1$, with a simple pole at $s = \frac{2d+k}{4}$ with residue $f_0\left(\frac{2d+k}{4}\right)$. Moreover, taking the limit $\mathcal{T} \to \infty$ in (3.12), we find that the BPS-state sum (3.10) is actually equal to the renormalised integral

$$\begin{aligned}\text{R.N.} \int_\mathcal{F} \mathrm{d}\mu\, \Gamma_{d+k,d}\, \mathcal{F}\left(s,\kappa,-\tfrac{k}{2}\right) &= \lim_{\mathcal{T}\to\infty} \left[\mathcal{I}_{d+k,d}(s,\kappa,\mathcal{T}) + f_0(s) \frac{\mathcal{T}^{\frac{2d+k}{4}-s}}{s-\frac{2d+k}{4}} \right] \\ &= \mathcal{I}_{d+k,d}(s,\kappa)\end{aligned} \tag{3.15}$$

for generic values of $s \neq \frac{2d+k}{4}$. At the point $s = \frac{2d+k}{4}$, the renormalised integral is instead equal to the constant term in the Laurent expansion of $\mathcal{I}_{d+k,d}(s,\kappa)$ around $s = \frac{2d+k}{4}$,

$$\begin{aligned}\text{R.N.} \int_\mathcal{F} \mathrm{d}\mu\, \Gamma_{d+k,d}\, \mathcal{F}(s,\kappa,-\tfrac{k}{2}) &= \lim_{\mathcal{T}\to\infty} \left[\mathcal{I}_{d+k,d}\left(\tfrac{2d+k}{4},\kappa,\mathcal{T}\right) - f_0\left(\tfrac{2d+k}{4}\right) \log\mathcal{T} + f_0'\left(\tfrac{2d+k}{4}\right) \right] \\ &= \hat{\mathcal{I}}_{d+k,d}\left(\tfrac{2d+k}{4},\kappa\right),\end{aligned} \tag{3.16}$$

where $f_0'(s) = \mathrm{d}f_0/\mathrm{d}s$, and the r.h.s. is defined as the limit of $\mathcal{I}_{d+k,d}(s,\kappa)$ after the pole is properly subtracted,

$$\hat{\mathcal{I}}_{d+k,d}\left(\frac{2d+k}{4},\kappa\right) \equiv \lim_{s\to\frac{2d+k}{4}} \left[\mathcal{I}_{d+k,d}(s,\kappa) - \frac{f_0\left(\frac{2d+k}{4}\right)}{s-\frac{2d+k}{4}} \right]. \tag{3.17}$$

Eqs. (3.15) and (3.16) relate the renormalised integral to the BPS state sum (3.10), or to its analytic continuation whenever $\Re(s) > 1$. We note that this renormalisation prescription amounts to subtracting only the infrared divergent contribution of the massless states, unlike other schemes used in the literature where the full contribution of the massless states is subtracted. Of course, any two renormalisation schemes differ by an additive constant independent of the moduli.

Having discussed the analytic properties of the BPS-state sum (3.10), and its relation to the regulated integral (3.5), let us now evaluate the integral in (3.10). Using the relation (A.16) between the Whittaker M-function and the confluent hypergeometric function ${}_1F_1$, as well as the identity

$$\int_0^\infty dt\, t^{a-1}\, e^{-z\,t}\, {}_1F_1(b;c;t) = z^{-a}\, \Gamma(a)\, {}_2F_1(a,b;c;z^{-1})\,, \tag{3.18}$$

we arrive at our main result

$$\begin{aligned}\mathcal{I}_{d+k,d}(s,\kappa) =& (4\pi\kappa)^{1-\frac{d}{2}}\, \Gamma(s+\tfrac{2d+k}{4}-1) \\ &\times \sum_{\rm BPS} {}_2F_1\left(s-\frac{k}{4}\,,\, s+\frac{2d+k}{4}-1\,;\, 2s\,;\, \frac{4\kappa}{p_{\rm L}^2}\right) \left(\frac{p_{\rm L}^2}{4\kappa}\right)^{1-s-\frac{2d+k}{4}}\,.\end{aligned} \tag{3.19}$$

The sum in (3.19) converges absolutely for $\Re(s) > \frac{2d+k}{4}$ and can be analytically continued to a meromorphic function on $\Re(s) > 1$ with a simple pole at $s = \frac{2d+k}{4}$ [34]. Again, for $\kappa = 1$ the sum in (3.19) can be physically interpreted as a sum of the one-loop contributions of all physical BPS states satisfying the level-matching condition (3.9). This expression is manifestly invariant under T-duality, independent of any choice of chamber, and generalises the constrained Epstein zeta series considered in [20, 21] to the case of a non-trivial elliptic genus. We would like to stress that these properties follow directly from our approach, as opposed to the conventional unfolding method, which depends on a choice of chamber to ensure convergence.

Moreover, using the fact that the lattice partition function satisfies the differential equation [21]

$$\left[\Delta_{{\rm SO}(d+k,d)} - 2\,\Delta_{-k/2} + \tfrac{1}{4}\, d(d+k-2)\right]\, \Gamma_{d+k,d} = 0\,, \tag{3.20}$$

we find that the BPS state sum (3.19) is an eigenmode of the Laplacian $\Delta_{{\rm SO}(d+k,d)}$ on the Narain moduli space

$$\left[\Delta_{{\rm SO}(d,d+k)} + \tfrac{1}{16}\,(2d+k-4s)(2d+k+4s-4)\right]\, \mathcal{I}_{d+k,d}(s,\kappa) = 0\,. \tag{3.21}$$

For $s = \frac{2d+k}{4}$, the eigenvalue vanishes but the BPS state sum $\mathcal{I}_{d+k,d}(s,\kappa)$ has a pole. After subtracting the pole, one finds that the renormalised BPS state sum is an almost harmonic function on the Narain moduli space, namely its image under the Laplacian is a constant

$$\Delta_{{\rm SO}(d,d+k)}\, \hat{\mathcal{I}}_{d+k,d}\left(\frac{2d+k}{4},\kappa\right) = (1-d-\tfrac{k}{2})\, f_0\left(\frac{2d+k}{4}\right)\,. \tag{3.22}$$

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3.3 One-loop BPS amplitudes with momentum insertions

Our method carries over straightforwardly to cases where insertions of left-moving or right-moving momenta appear in the lattice sum, i.e. to modular integrals of the type

$$\int_{\mathcal{F}} \mathrm{d}\mu \left[\tau_2^{-\lambda/2} \sum_{p_{\mathrm{L}},p_{\mathrm{R}}} \rho\left(p_{\mathrm{L}}\sqrt{\tau_2}, p_{\mathrm{R}}\sqrt{\tau_2}\right) q^{\frac14 p_{\mathrm{L}}^2}\, \bar q^{\frac14 p_{\mathrm{R}}^2} \right] \varPhi(\tau)\,, \tag{3.23}$$

considered for example in [26, 38]. The term in the square bracket is a modular form of weight $(\lambda + d + \frac{k}{2}, 0)$, provided that the function $\rho(x_{\mathrm{L}}\,,\, x_{\mathrm{R}})$ satisfies

$$\left[\partial_{x_{\mathrm{L}}}^2 - \partial_{x_{\mathrm{R}}}^2 - 2\pi\left(x_{\mathrm{L}}\partial_{x_{\mathrm{L}}} - x_{\mathrm{R}}\,\partial_{x_{\mathrm{R}}} - \lambda - d\right)\right] \rho(x_{\mathrm{L}}\,,\, x_{\mathrm{R}}) = 0\,, \tag{3.24}$$

and that $\rho(x_{\mathrm{L}},\, x_{\mathrm{R}})\, e^{-\pi\,(x_{\mathrm{L}}^2 + x_{\mathrm{R}}^2)}$ should decay sufficiently fast at infinity [27].[12] For example, upon choosing $\rho = \tau_2\, p_{\mathrm{L}}^2 - \frac{d+k}{2\pi}$ (respectively $\rho = \tau_2\, p_{\mathrm{R}}^2 - \frac{d}{2\pi}$), it is proportional to the modular derivative $D\cdot\varGamma_{d+k,d}$ (respectively, $\overline{D}\cdot\varGamma_{d+k,d}$) of the usual Narain lattice partition function. The integrand in (3.23) is then modular invariant provided $\lambda + d + \frac{k}{2} = -w$.

As usual, expressing the elliptic genus as a linear combination of Niebur Poincaré series, one is left to consider integrals of the form

$$\int_{\mathcal{F}} \mathrm{d}\mu\, \tau_2^{-\lambda/2} \sum_{p_{\mathrm{L}},p_{\mathrm{R}}} \rho\left(p_L\sqrt{\tau_2}, p_R\sqrt{\tau_2}\right) q^{\frac14 p_{\mathrm{L}}^2}\, \bar q^{\frac14 p_{\mathrm{R}}^2}\, \mathcal{F}(s,\kappa,w)\,, \tag{3.25}$$

and following similar steps as in the previous subsection, one finds the result

$$(4\pi\kappa)^{1+\frac{\lambda}{2}} \sum_{\mathrm{BPS}} \int_0^\infty dt\, t^{s+\frac{2d+k}{4}-2}\, {}_1F_1\!\left(s - \frac{2\lambda+2d+k}{4}; 2s; t\right) \rho\!\left(\frac{p_{\mathrm{L}}}{\sqrt{4\pi\kappa}}\sqrt{t}, \frac{p_{\mathrm{R}}}{\sqrt{4\pi\kappa}}\sqrt{t}\right) e^{-t\,p_{\mathrm{L}}^2/4\kappa}\,. \tag{3.26}$$

In most applications, ρ is a polynomial in $p_{\mathrm{L}}^a, p_{\mathrm{R}}^b$, and the integral can be evaluated using (3.18). As a result, each monomial can be evaluated to

$$\begin{aligned} \int_{\mathcal{F}} \mathrm{d}\mu\, \tau_2^{\delta} \sum_{p_{\mathrm{L}},p_{\mathrm{R}}} p_{\mathrm{L}}^{a_1}\cdots p_{\mathrm{L}}^{a_\alpha}\, p_{\mathrm{R}}^{b_1}\cdots p_{\mathrm{R}}^{b_\beta}\, q^{\frac14 p_{\mathrm{L}}^2} \bar q^{\frac14 p_{\mathrm{R}}^2}\, \mathcal{F}(s,\kappa,w) &\rightrightarrows (4\pi\kappa)^{1-\delta}\, \varGamma(s + \tfrac{|w|}{2} + \delta - 1) \\ \times \sum_{\mathrm{BPS}} p_{\mathrm{L}}^{a_1}\cdots p_{\mathrm{L}}^{a_\alpha}\, p_{\mathrm{R}}^{b_1}\cdots p_{\mathrm{R}}^{b_\beta}\, {}_2F_1\left(s - \frac{|w|}{2}, s + \frac{|w|}{2} + \delta - 1; 2s; \frac{4\kappa}{p_L^2}\right) &\left(\frac{p_L^2}{4\kappa}\right)^{1-s-\frac{|w|}{2}-\delta}\,, \end{aligned} \tag{3.27}$$

with $\delta = (\alpha + \beta - \lambda)/2$. Clearly, this result is meaningful only when the various monomials are combined into a solution of (3.24), as required by modular invariance.

3.4 BPS-state sum for integer s

For special values of s and w, the hypergeometric function ${}_2F_1$ appearing in the BPS-state sum (3.19) can actually be expressed in terms of elementary functions. For example, for

[12] We are grateful to J. Manschot for pointing out this reference.

$d=1$ and $w=0$, ${}_2F_1(s,s-\frac{1}{2},2s;z)=2^{2s-1}(1+\sqrt{1-z})^{1-2s}$, and thus

$$\begin{aligned}\mathcal{I}_{1,1}(1+n,\kappa)&=\sqrt{4\pi\kappa}\,2^{1+2n}\,\Gamma\left(n+\tfrac{1}{2}\right)\sum_{\rm BPS}\left(\sqrt{\frac{p_{\rm L}^2}{4\kappa}}+\sqrt{\frac{p_{\rm R}^2}{4\kappa}}\right)^{-1-2n}\\&=\tfrac{1}{2}\sqrt{\pi}\,(16\,\kappa)^{1+n}\,\Gamma\left(n+\tfrac{1}{2}\right)\sum_{\substack{p,q\in\mathbb{Z}\\pq=\kappa}}\left(\left|p\,R+q\,R^{-1}\right|+\left|p\,R-q\,R^{-1}\right|\right)^{-1-2n},\end{aligned}\tag{3.28}$$

with $s=1+n$. For $n=0$, this agrees with the expression derived in [20] using the Selberg-Poincaré series $\mathcal{E}(s,\kappa,w)$ at $s=0$.

More generally, similar simplifications also take place for $s=1-\frac{w}{2}+n=1+\frac{k}{4}+n$, with n a positive integer, which are the special values relevant for representing weak almost holomorphic modular forms, and are thus of interest for our physical applications. While it is cumbersome to express ${}_2F_1$ directly in terms of elementary functions, it is simpler to notice that the Whittaker M-function appearing in (3.10) reduces to the finite sum (A.33). As a result, the integral (3.10) reduces to

$$\begin{aligned}\mathcal{I}_{d+k,d}(1+\tfrac{k}{4}+n,\kappa)&=\sum_{\rm BPS}\int_0^\infty {\rm d}\tau_2\,\tau_2^{\frac{d}{2}-2+\alpha}\,\mathcal{M}_{1+\frac{k}{4}+n,-\frac{k}{2}}(-\kappa\tau_2)\,e^{-\pi\tau_2(p_{\rm L}^2+p_{\rm R}^2)/2}\,,\\&=(4\pi\kappa)^{1-\frac{d}{2}}\,\frac{\Gamma\left(2(n+1)+\frac{k}{2}\right)\Gamma\left(n+\frac{d+k}{2}\right)}{n!}\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m}{\Gamma\left(n-m+\frac{d+k}{2}\right)}\\&\quad\times\sum_{\rm BPS}\left(\frac{p_{\rm L}^2}{4\kappa}\right)^{n-m}\int_0^\infty{\rm d}z\;z^{\frac{d}{2}-m-2}\left(e^{-z\,p_{\rm R}^2/4\kappa}-e^{-z\,p_{\rm L}^2/4\kappa}\sum_{\ell=0}^{2n+\frac{k}{2}}\frac{z^\ell}{\ell!}\right),\end{aligned}\tag{3.29}$$

where, in going from the first to the second line we have set $z=4\pi\kappa\tau_2$, and have integrated by parts n times, and we used the fact that the boundary terms vanish. Although the full integrand vanishes rapidly enough as $z\to 0$, so that the integral exists, this is not true of each individual term, unless $\frac{d}{2}-n-1>0$. To regulate these unphysical divergences, we introduce a convergence factor τ_2^α in the integrand, and evaluate each integral in (3.29) for large enough α. The desired result is then expressed as the limit

$$\begin{aligned}\mathcal{I}_{d+k,d}(1+\tfrac{k}{4}+n,\kappa)=&(4\pi\kappa)^{1-\frac{d}{2}}\,\frac{\Gamma\left(2(n+1)+\frac{k}{2}\right)\Gamma(n+\frac{d+k}{2})}{n!}\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m}{\Gamma\left(n-m+\frac{d+k}{2}\right)}\\&\times\sum_{\rm BPS}\left(\frac{p_{\rm L}^2}{4\kappa}\right)^{n-m}\lim_{\alpha\to0}\left[\Gamma\left(\tfrac{d}{2}-m-1+\alpha\right)\left(\frac{p_{\rm R}^2}{4\kappa}\right)^{m+1-\frac{d}{2}-\alpha}\right.\\&\left.-\sum_{\ell=0}^{2n+k/2}\frac{\Gamma\left(\frac{d}{2}-m-1+\ell+\alpha\right)}{\ell!}\left(\frac{p_{\rm L}^2}{4\kappa}\right)^{1+m-\frac{d}{2}-\ell-\alpha}\right].\end{aligned}\tag{3.30}$$

The series (3.30) converges absolutely for $n>\frac{d}{2}-1$, as a result of the finiteness of the original modular integral. For $n\le\frac{d}{2}-1$, it is a formal (divergent) sum over BPS states, which nevertheless captures the singularities of the amplitude at points of gauge symmetry enhancement.

For $n < \frac{d}{2} - 1$, or whenever d is odd, independently of n, the limit $\alpha \to 0$ is trivial, leading to $\mathcal{I}_{d+k,d}(s,\kappa) = \mathcal{I}^{(1)}_{d+k,d}(s,\kappa)$ where

$$\begin{aligned}\mathcal{I}^{(1)}_{d+k,d}(1+\tfrac{k}{4}+n,\kappa) = (4\pi\kappa)^{1-\frac{d}{2}}\, &\frac{\Gamma(2(n+1)+\frac{k}{2})\,\Gamma(n+\frac{d+k}{2})}{n!} \\ &\times \sum_{m=0}^{d/2-2} \binom{n}{m} \frac{(-1)^m}{\Gamma(n-m+\frac{d+k}{2})} \sum_{\text{BPS}} \left(\frac{p_{\rm L}^2}{4\kappa}\right)^{n-m} \\ &\times \left[\Gamma\left(\tfrac{d}{2}-m-1\right) \left(\frac{p_{\rm R}^2}{4\kappa}\right)^{m+1-\frac{d}{2}} \right. \\ &\qquad \left. - \sum_{\ell=0}^{2n+k/2} \frac{\Gamma\left(\frac{d}{2}-m-1+\ell\right)}{\ell!} \left(\frac{p_{\rm L}^2}{4\kappa}\right)^{1+m-\frac{d}{2}-\ell} \right].\end{aligned} \tag{3.31}$$

If d is even and $n \geq \frac{d}{2} - 1$ one finds $\mathcal{I}_{d+k,d}(s,\kappa) = \mathcal{I}^{(1)}_{d+k,d}(s,\kappa) + \mathcal{I}^{(2)}_{d+k,d}(s,\kappa)$ where the first term is still given by (3.31) and the second term is

$$\begin{aligned}\mathcal{I}^{(2)}_{d+k,d}(1+\tfrac{k}{4}+n,\kappa) = (4\pi\kappa)^{1-\frac{d}{2}}\, &\frac{\Gamma\left(2(n+1)+\frac{k}{2}\right)\,\Gamma\left(n+\frac{d+k}{2}\right)}{n!} \\ &\times \sum_{\text{BPS}} \sum_{m=d/2-1}^{n} \binom{n}{m} \frac{(-1)^m}{\Gamma\left(n-m+\frac{d+k}{2}\right)} \left(\frac{p_{\rm L}^2}{4\kappa}\right)^{n-m} \\ &\times \left\{ - \sum_{\ell=m+2-d/2}^{2n+k/2} \frac{\Gamma\left(\frac{d}{2}-m-1+\ell\right)}{\ell!} \left(\frac{p_{\rm L}^2}{4\kappa}\right)^{1+m-\frac{d}{2}-\ell} \right. \\ &+ \frac{(-1)^{m+1-\frac{d}{2}}}{\Gamma(m+2-\frac{d}{2})} \left(\frac{p_{\rm R}^2}{4\kappa}\right)^{m+1-\frac{d}{2}} \left[H_{m+1-\frac{d}{2}} - \log\left(\frac{p_{\rm R}^2}{p_{\rm L}^2}\right)\right] \\ &\left. - \frac{1}{\Gamma\left(m+2-\frac{d}{2}\right)} \sum_{\ell=0}^{m+1-d/2} \binom{m+1-\frac{d}{2}}{\ell} \left(-\frac{p_{\rm L}^2}{4\kappa}\right)^{m+1-\frac{d}{2}-\ell} H_{m+1-\frac{d}{2}-\ell} \right\},\end{aligned} \tag{3.32}$$

where $H_N = \sum_{k=1}^{N} k^{-1}$ is the N-th harmonic number. The combination (3.31) vanishes for $d = 2$, the sum over m being void. The results (3.31) and (3.32) allow to write any integral of the type (1.2) as a formal sum over physical BPS states (which converges absolutely for $n > \frac{d}{2} - 1$). In particular, the result is manifestly invariant under the T-duality group $\mathrm{O}(d+k,d;\mathbb{Z})$

We conclude this subsection with some simple examples for special values of n and k. For $n = 0$ the sum over m in (3.29) is void and only few terms contribute to the integral, corresponding to the various terms in (2.23). When $d \neq 2$ the limit $\alpha \to 0$ is trivial and one arrives at the simple expression

$$\begin{aligned}\mathcal{I}_{d+k,d}(1+\tfrac{k}{4},\kappa) = (4\pi\kappa)^{1-\frac{d}{2}}\,\Gamma(2+\tfrac{k}{2}) \sum_{\text{BPS}} &\left[\Gamma(\tfrac{d}{2}-1) \left(\frac{p_{\rm R}^2}{4\kappa}\right)^{1-\frac{d}{2}} \right. \\ &\left. - \sum_{\ell=0}^{k/2} \frac{\Gamma(\frac{d}{2}+\ell-1)}{\ell!} \left(\frac{p_{\rm L}^2}{4\kappa}\right)^{1-\frac{d}{2}-\ell} \right].\end{aligned} \tag{3.33}$$

When $d=2$, the limit $\alpha\to 0$ is subtler and leads to logarithmic contributions. One obtains, for $n=0$, any k,

$$\mathcal{I}_{2+k,2}(1+\tfrac{k}{4},\kappa)=-\Gamma(2+\tfrac{k}{2})\sum_{\rm BPS}\left[\log\left(\frac{p_{\rm R}^2}{p_{\rm L}^2}\right)+\sum_{\ell=1}^{k/2}\frac{1}{\ell}\left(\frac{p_{\rm L}^2}{4\kappa}\right)^{-\ell}\right]\,, \tag{3.34}$$

and for $k=0$, any n,

$$\begin{aligned}\mathcal{I}_{2,2}(1+n,\kappa)=\frac{(2n+1)!}{n!}\sum_{\rm BPS}\left(\frac{p_{\rm L}^2}{4\kappa}\right)^n\sum_{m=0}^{n}\binom{n}{m}^2\Bigg\{\left(\frac{p_{\rm R}^2}{p_{\rm L}^2}\right)^m\left[H_m-\log\left(\frac{p_{\rm R}^2}{p_{\rm L}^2}\right)\right]\\ -\sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}\left(\frac{p_{\rm L}^2}{4\kappa}\right)^{-\ell}H_{m-\ell}-(-1)^m\sum_{\ell=m+1}^{2n}\frac{\Gamma(\ell-m)\,m!}{\ell!}\left(\frac{p_{\rm L}^2}{4\kappa}\right)^{-\ell}\Bigg\}.\end{aligned} \tag{3.35}$$

As usual, the left and right-handed momenta are defined by

$$\begin{aligned}p_{{\rm L},I}&=(m_i+Y_i^aQ^a+\tfrac12 Y_i^aY_j^an^j+(G+B)_{ij}n^j\,,\ Q^a+Y_j^an^j)\,,\\ p_{{\rm R},i}&=m_i+Y_i^aQ^a+\tfrac12 Y_i^aY_j^an^j-(G-B)_{ij}n^j\,,\end{aligned} \tag{3.36}$$

with m_i and n^i the Kaluza-Klein and winding numbers and Q^a are the charge vectors. In the $d=2$ case, and in the absence of Wilson lines, it is often convenient to express them in terms of the Kähler modulus T and of the complex structure modulus U as

$$\begin{aligned}p_{\rm L}^2&=\frac{1}{T_2\,U_2}\left|m_2-U\,m_1+\bar T(n^1+U\,n^2)\right|^2\,,\\ p_{\rm R}^2&=\frac{1}{T_2\,U_2}\left|m_2-U\,m_1+T(n^1+U\,n^2)\right|^2\,.\end{aligned} \tag{3.37}$$

The relation between these results (for $k=0$) and the 'shifted constrained Epstein zeta series' of [20] is discussed in appendix B.

3.5 Singularities at points of gauge symmetry enhancement

In addition to keeping T-duality manifest, another advantage of this approach for the evaluation of one-loop modular integrals is that it allows to easily read-off the singularity structure of the amplitudes at point of enhanced gauge symmetry. These points are characterised by the appearance of extra massless states with $p_{\rm R}=0$. Depending on the dimension of the Narain lattice, as well as on the value of n, the amplitude may diverge (we refer to this case as real singularity) or one of its derivatives can be discontinuous (we refer to this case as conical singularity).

For odd dimension d, the modular integral $\mathcal{I}_{d+k,d}(s,\kappa)$ always develops *conical* singularities, as exemplified in the one-dimensional case by eq. (3.28). In addition, for $d\geq 3$ real singularities appear from terms with $m<\frac{d}{2}-1$ in (3.30).

For even dimension real singularities always appear. They are are power-like in $\mathcal{I}^{(1)}$ whenever $d\geq 4$ and logarithmic in $\mathcal{I}^{(2)}$ for any even $d\leq 2n+2$. Moreover, conical singularities do not appear.

Notice that for $d = 2$ the singularities cancel in the combination

$$\mathcal{I}_{2,2}(1+n,1) - \frac{(2n+1)!}{n!}\,\hat{\mathcal{I}}_{2,2}(1,1)\,, \tag{3.38}$$

which is therefore a continuous function over the Narain moduli space, including at points of enhanced gauge symmetry. Since, using the results in [10, 18] and the fact that $\mathcal{F}(1,1,0) = j + 24$,

$$\hat{\mathcal{I}}_{2,2}(1,1) = -\log|j(T) - j(U)|^4 - 24\log\left[T_2 U_2 |\eta(T)\,\eta(U)|^4\right] + \text{const}\,, \tag{3.39}$$

we conclude that all integrals $\mathcal{I}_{2,2}(1+n,1)$ exhibit the same universal singular behaviour up to an overall normalisation

$$\mathcal{I}_{2,2}(1+n,1) \sim -\frac{(2n+1)!}{n!}\,\log|j(T) - j(U)|^4\,. \tag{3.40}$$

This expression can be generalised as in [18] if Wilson lines are turned on.

4 Some examples from string threshold computations

In this section we evaluate a sample of modular integrals that enter in threshold corrections to gauge and gravitational couplings in heterotic string vacua using the method developed in the previous section. We express the elliptic genus as a linear combination of Niebur-Poincaré series, and we evaluate the modular integral in terms of the BPS-state sums $\mathcal{I}_{d+k,d}(s,\kappa)$ defined in eqs. (3.19) and (3.30).

4.1 A gravitational coupling in maximally supersymmetric heterotic vacua

Let us start with the example of toroidally compactified SO(32) heterotic string, for which the elliptic genus takes the form (3.1). Using table 3 and the relation $E_4^3\,\Delta^{-1} = j + 744$, this can be conveniently expressed in terms of the Niebur-Poincaré series as

$$\begin{aligned}
\varPhi(\tau) =\, & t_8 \,\mathrm{tr}\, F^4 + \frac{1}{2^7\,3^2\,5}\left[\mathcal{F}(1,1,0) + 720\right]\, t_8\,\mathrm{tr}\, R^4 \\
& + \frac{1}{2^9\,3^2}\left[\tfrac{1}{5}\mathcal{F}(3,1,0) - 4\mathcal{F}(2,1,0) + 13\mathcal{F}(1,1,0) + 144\right]\, t_8\,(\mathrm{tr}\, R^2)^2 \\
& + \frac{1}{2^8\,3^2}\left[-\tfrac{1}{5}\mathcal{F}(3,1,0) + 5\mathcal{F}(2,1,0) - 18\mathcal{F}(1,1,0) + 288\right]\, t_8\,\mathrm{tr}\, R^2\,\mathrm{tr}\, F^2 \\
& + \frac{1}{2^9\,3^2}\left[\tfrac{1}{5}\mathcal{F}(3,1,0) - 6\mathcal{F}(2,1,0) + 24\mathcal{F}(1,1,0) - 576\right]\, t_8\,(\mathrm{tr}\, F^2)^2\,.
\end{aligned} \tag{4.1}$$

Therefore, using the results in the previous section, the renormalised modular integral (1.2) can be expressed as the linear combination

$$\begin{aligned}
\text{R.N.}\int_{\mathcal{F}} \mathrm{d}\mu\, \varGamma_{d,d}\,\varPhi =\, & \mathcal{I}_{d,d}\, t_8\,\mathrm{tr}\, F^4 + \frac{1}{2^7\,3^2\,5}\left[\mathcal{I}_{d,d}(1,1) + 720\,\mathcal{I}_{d,d}\right]\, t_8\,\mathrm{tr}\, R^4 \\
& + \frac{1}{2^9\,3^2}\left[\tfrac{1}{5}\,\mathcal{I}_{d,d}(3,1) - 4\,\mathcal{I}_{d,d}(2,1) + 13\,\mathcal{I}_{d,d}(1,1) + 144\,\mathcal{I}_{d,d}\right]\, t_8\,(\mathrm{tr}\, R^2)^2 \\
& + \frac{1}{2^8\,3^2}\left[-\tfrac{1}{5}\,\mathcal{I}_{d,d}(3,1) + 5\,\mathcal{I}_{d,d}(2,1) - 18\,\mathcal{I}_{d,d}(1,1) + 288\,\mathcal{I}_{d,d}\right]\, t_8\,\mathrm{tr} R^2\,\mathrm{tr} F^2 \\
& + \frac{1}{2^9\,3^2}\left[\tfrac{1}{5}\,\mathcal{I}_{d,d}(3,1) - 6\,\mathcal{I}_{d,d}(2,1) + 24\,\mathcal{I}_{d,d}(1,1) - 576\,\mathcal{I}_{d,d}\right]\, t_8(\mathrm{tr} F^2)^2\,,
\end{aligned} \tag{4.2}$$

where, as computed in [20, 21],

$$\mathcal{I}_{d,d} \equiv \text{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{d,d}(G,B) = \frac{\Gamma\left(\frac{d}{2}-1\right)}{\pi^{\frac{d}{2}-1}}\, \mathcal{E}^d_V\left(G,B;\tfrac{d}{2}-1\right)\,, \tag{4.3}$$

with $\mathcal{E}^d_V$ being the constrained Epstein zeta function defined in [20, 21]. In this expression, any time $n=\frac{d}{2}-1$ the BPS-state sum $\mathcal{I}_{d,d}(1+n,\kappa)$ should be replaced by $\hat{\mathcal{I}}_{d,d}(1+n,\kappa)$, as explained in section 3.2.

In the one-dimensional case the constrained sums can be easily evaluated, leading to

$$\begin{aligned}\int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{1,1}\, \Phi =& \frac{\pi}{3}(R+R^{-1})\, t_8 \,\mathrm{tr}\, F^4 + \frac{\pi}{2^3\, 3^2\, 5}\left(15R+16R^{-1}\right)\, t_8\,\mathrm{tr}\, R^4\\ &+\frac{\pi}{2^5\, 3^2}\left(3R+16R^{-1}-24R^{-3}+12R^{-5}\right)\, t_8\,(\mathrm{tr}\, R^2)^2\\ &+\frac{\pi}{2^3\, 3}\left(R-2R^{-1}+5R^{-3}-2R^{-5}\right)\, t_8\,\mathrm{tr}\, R^2\,\mathrm{tr}\, F^2\\ &-\frac{\pi}{2^3\, 3}\left(R-R^{-1}+3R^{-3}-R^{-5}\right)\, t_8\,(\mathrm{tr}\, F^2)^2\,,\end{aligned} \tag{4.4}$$

for $R>1$. The expression for $R<1$ can be obtained by replacing in the previous expression $R\mapsto R^{-1}$. Notice that, aside from the threshold correction to $t_8\,\mathrm{tr}\, F^4$, all other terms develop a conical singularity at the self-dual radius $R=1$.

4.2 Gauge-thresholds in $\mathcal{N}=2$ heterotic vacua with/without Wilson lines

Let us turn now to $\mathcal{N}=2$ heterotic vacua in the orbifold limit $T^2\times T^4/\mathbb{Z}_2$, with a standard embedding on the gauge sector. At the orbifold point, the gauge group is broken to

$$\mathrm{E}_8\times\mathrm{E}_8\to\mathrm{E}_8\times\mathrm{E}_7\times\mathrm{SU}(2)\,, \tag{4.5}$$

and, in the absence of Wilson lines, gauge threshold corrections read

$$\begin{aligned}\Delta_{\mathrm{E}_8} &= -\frac{1}{12}\int_{\mathcal{F}}\mathrm{d}\mu\,\Gamma_{2,2}\,\frac{\hat{E}_2\, E_4\, E_6 - E_6^2}{\Delta}\,,\\ \Delta_{\mathrm{E}_7} &= -\frac{1}{12}\int_{\mathcal{F}}\mathrm{d}\mu\,\Gamma_{2,2}\,\frac{\hat{E}_2\, E_4\, E_6 - E_4^3}{\Delta}\,.\end{aligned} \tag{4.6}$$

From table 3 one can read that

$$\begin{aligned}\frac{\hat{E}_2\, E_4\, E_6 - E_6^2}{\Delta} &= \mathcal{F}(2,1,0)-6\,\mathcal{F}(1,1,0)+864\,,\\ \frac{\hat{E}_2\, E_4\, E_6 - E_4^3}{\Delta} &= \mathcal{F}(2,1,0)-6\,\mathcal{F}(1,1,0)-864\,,\end{aligned} \tag{4.7}$$

and thus

$$\begin{aligned}\Delta_{\mathrm{E}_8} &= \sum_{\mathrm{BPS}}\left[1+\frac{p_{\mathrm{R}}^2}{4}\,\log\left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2}\right)\right]+72\,\log\left(T_2\,U_2\,|\eta(T)\,\eta(U)|^4\right)+\text{const}\,,\\ \Delta_{\mathrm{E}_7} &= \sum_{\mathrm{BPS}}\left[1+\frac{p_{\mathrm{R}}^2}{4}\,\log\left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2}\right)\right]-72\,\log\left(T_2\,U_2\,|\eta(T)\,\eta(U)|^4\right)+\text{const}\end{aligned} \tag{4.8}$$

Notice that the combination $\mathcal{I}_2(2,1,0)-6\,\mathcal{I}_2(1,1,0)$ is regular at any point in moduli space (and in any chamber), as expected since the unphysical tachyon is neutral and therefore does not contribute to the running of the non-Abelian gauge couplings.

Turning on Wilson lines on the E_8 group factor along the spectator T^2, yields

$$\varDelta_{\mathrm{E}_7} = -\frac{1}{12}\int_{\mathcal{F}} \mathrm{d}\mu\, \varGamma_{2,10}\, \frac{\hat{E}_2\, E_6 - E_4^2}{\varDelta}\,. \tag{4.9}$$

Using table 3, one easily finds

$$\frac{\hat{E}_2\, E_6 - E_4^2}{\varDelta} = \frac{2}{7!}\mathcal{F}(4,1,-4) - \frac{2}{5!}\,\mathcal{F}(3,1,-4)\,, \tag{4.10}$$

and thus

$$\begin{aligned} \varDelta_{\mathrm{E}_7} &= -\frac{1}{720}\left[\frac{1}{42}\,\mathcal{I}_{10,2}(4,1) - \mathcal{I}_{10,2}(3,1)\right] \\ &= \sum_{\mathrm{BPS}}\left[1 + \frac{p_{\mathrm{R}}^2}{4}\,\log\left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2}\right) - \frac{2}{p_{\mathrm{L}}^2} - \frac{8}{3\,p_{\mathrm{L}}^4} - \frac{16}{3\,p_{\mathrm{L}}^6} - \frac{64}{5\,p_{\mathrm{L}}^8}\right]. \end{aligned} \tag{4.11}$$

In this expression $p_{\mathrm{L,R}}$ depend also on the Wilson lines, and the constraint in the BPS-sum now reads

$$p_{\mathrm{L}}^2 - p_{\mathrm{R}}^2 = 4 \quad\Rightarrow\quad m^{\mathrm{T}} n + \tfrac{1}{2}\, Q^{\mathrm{T}} Q = 1\,, \tag{4.12}$$

where Q is the U(1)-charge vector in the Cartan sub-algebra of E_8.

4.3 Kähler metric corrections in $\mathcal{N}=2$ heterotic vacua

Our procedure can also be used to compute loop corrections to Kähler metric and other terms in the low-energy effective action. For instance, in $\mathcal{N}=2$ heterotic vacua at the orbifold point, the one-loop correction to the Kähler metric for the T modulus reads

$$\begin{aligned} K_{T\bar{T}}\Big|_{1-\mathrm{loop}} &= \frac{\mathrm{i}}{12\pi\, T_2^2}\int_{\mathcal{F}} \mathrm{d}\mu\, \frac{E_4\, E_6}{\varDelta}\, \partial_\tau\, \varGamma_{2,2} \\ &= \frac{\mathrm{i}}{72\pi\, T_2^2}\int_{\mathcal{F}} \mathrm{d}\mu\, \mathcal{F}(2,1,-2)\, \partial_\tau\, \varGamma_{2,2}\,, \end{aligned} \tag{4.13}$$

where we have used the relation between $E_4 E_6 \varDelta^{-1}$ and $\mathcal{F}(s,\kappa,w)$ from table 1. Integrating by parts, and using the action of the modular derivative on the Niebur-Poincaré series, one immediately finds

$$\begin{aligned} K_{T\bar{T}}\Big|_{1-\mathrm{loop}} &= -\frac{\mathrm{i}}{72\pi\, T_2^2}\int_{\mathcal{F}} \mathrm{d}\mu\, \varGamma_{2,2}\, D_{-2}\mathcal{F}(2,1,-2) \\ &= \frac{1}{36\, T_2^2}\int_{\mathcal{F}} \mathrm{d}\mu\, \varGamma_{2,2}\, \mathcal{F}(2,1,0) \\ &= \frac{1}{36\, T_2^2}\,\mathcal{I}_{2,2}(2,1)\,. \end{aligned} \tag{4.14}$$

Similar results can be obtained for higher-derivative couplings in $\mathcal{N}=4$ vacua.

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4.4 An example from non-compact heterotic vacua

In some heterotic constructions on ALE spaces and in the presence of background NS5 branes, gauge threshold corrections include a contribution of the (finite) integral [45]

$$L = \int_{\mathcal{F}} \mathrm{d}\mu\, (\sqrt{\tau_2}\,\eta\,\bar\eta)^3\, \frac{\hat E_2\, E_4 \left(\hat E_2\, E_4 - 2E_6\right)}{\Delta}\,. \tag{4.15}$$

Despite its apparent complexity, this integral can be easily computed using our techniques. In fact, the relation

$$\vartheta_1'(0|\tau) = 2\pi\eta^3\,, \tag{4.16}$$

and the standard bosonisation formulae, allow one to write

$$(\sqrt{\tau_2}\,\eta\bar\eta)^3 = -\frac{1}{8\pi}\,\frac{\partial}{\partial R}\left[\frac{1}{R}\left(\Gamma_{1,1}(2R) - \Gamma_{1,1}(R)\right)\right]_{R=1/\sqrt{2}}\,. \tag{4.17}$$

Combining this observation with table 3, eq. (3.28), and with the standard result $\int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{1,1}(R) = \frac{\pi}{3}(R+R^{-1})$, the integral reduces to

$$\begin{aligned}
L =& -\frac{1}{8\pi}\,\frac{\partial}{\partial R}\Bigg[\frac{1}{R}\int_{\mathcal{F}} \mathrm{d}\mu\, \left(\Gamma_{1,1}(2R) - \Gamma_{1,1}(R)\right)\\
&\times \left(\tfrac{1}{5}\mathcal{F}(3,1,0) - 6\mathcal{F}(2,1,0) + 23\,\mathcal{F}(1,1,0) + 432\right)\Bigg]_{R=1/\sqrt{2}}\\
=& -\frac{1}{8\pi}\,\frac{\partial}{\partial R}\Bigg[\frac{1}{R}\Big(\tfrac{1}{5}\,\mathcal{I}_{1,1}(2R;3,1) - 6\,\mathcal{I}_{1,1}(2R;2,1) + 23\,\mathcal{I}_{1,1}(2R;1,1)\\
&- \left(\tfrac{1}{5}\,\mathcal{I}_{1,1}(R;3,1) - 6\,\mathcal{I}_{1,1}(R;2,1) + 23\,\mathcal{I}_{1,1}(R;1,1)\right) + 144\,\pi\left(R - \tfrac{1}{2}\,R^{-1}\right)\Big)\Bigg]_{R=1/\sqrt{2}}\\
=& -20\,\sqrt{2}\,.
\end{aligned} \tag{4.18}$$

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A Notations and useful identities

In this appendix we collect various definitions, notations and formulae used in the text.

A.1 Operators acting on modular forms

The hyperbolic Laplacian acts on modular forms of weight w via[13]

$$\Delta_w = 2\,\tau_2^2\,\partial_{\bar\tau}\left(\partial_\tau - \frac{\mathrm{i}w}{2\tau_2}\right)\,. \tag{A.1}$$

We denote by $\mathcal{H}(s,w)$ the eigenspace of Δ_w with eigenvalue $\frac{1}{2}s(s-1)-\frac{1}{8}w(w+2)$, in the space of real analytic functions of modular weight w under $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. The raising and lowering operators D_w, $\bar D_w$ defined by

$$D_w = \frac{\mathrm{i}}{\pi}\left(\partial_\tau - \frac{\mathrm{i}w}{2\tau_2}\right)\,, \qquad \bar D_w = -\mathrm{i}\pi\,\tau_2^2\partial_{\bar\tau}\,, \tag{A.2}$$

map $\mathcal{H}(s,w)$ to $\mathcal{H}(s,w\pm 2)$,

$$\mathcal{H}(s,w-2) \xleftarrow{\bar D_w} \mathcal{H}(s,w) \xrightarrow{D_w} \mathcal{H}(s,w+2)\,, \tag{A.3}$$

and satisfy the commutation identity

$$D_{w-2}\cdot\bar D_w - \bar D_{w+2}\cdot D_w = \frac{w}{4}\,. \tag{A.4}$$

The operator D_w (and of course, $\bar D_w$) satisfies the Leibniz rule

$$D_{w+w'}\,(f_w\,f_{w'}) = (D_w\,f_w)\,f_{w'} + f_w\,(D_{w'}\,f_{w'})\,, \tag{A.5}$$

where f_w is a modular form of weight w. We denote by $D^r_w\,f_w$ (or simply $D^r f$) the iterated derivative $D_{w+2r-2}\cdot\ldots\cdot D_{w+2}\cdot D_w\cdot f_w$, a modular form of weight $w+2r$. One has

$$D^r_w = \left(\frac{\mathrm{i}}{\pi}\right)^r \sum_{j=0}^{r} \frac{r!}{j!\,(r-j)!}\,\frac{\Gamma(w+r)}{\Gamma(w+j)}\,(2\mathrm{i}\tau_2)^{j-r}\,\partial^j_\tau\,. \tag{A.6}$$

For $w\le 0$, the operator D^{1-w}_w simplifies to $(\mathrm{i}/\pi)^{1-w}\partial^{1-w}_\tau$ (Bol's identity), and is known in the physics literature as the Farey transform [42].

The Hecke operators T_κ are defined by

$$(T_\kappa\cdot\Phi)(\tau) = \sum_{a,d>0,ad=\kappa}\ \sum_{b\ \mathrm{mod}\ d} d^{-w}\,\Phi\left(\frac{a\tau+b}{d}\right)\,, \tag{A.7}$$

and satisfy the commutative algebra

$$T_\kappa\,T_{\kappa'} = \sum_{d|(\kappa,\kappa')} d^{1-w}\,T_{\kappa\kappa'/d^2}\,, \tag{A.8}$$

If $\Phi = \sum_{n\in\mathbb{Z}}\Phi(n,\tau_2)\,e^{2\pi n\tau_1}$ is a modular form of weight w, then the Fourier coefficients of $T_\kappa\cdot\Phi$ are

$$(T_\kappa\cdot\Phi)(n,\tau_2) = \kappa^{1-w}\sum_{d|(n,\kappa)} d^{w-1}\,\Phi\left(n\kappa/d^2, d^2\tau_2/\kappa\right)\,. \tag{A.9}$$

[13]Our Laplacian is related to the one used *e.g* in [36] via $\Delta_w = -\frac{1}{2}\Delta_{w;\mathrm{BO}} - \frac{w}{2}$.

The generators of the ring of holomorphic modular forms are the normalised Eisenstein series

$$E_4 = 1 + 240 \sum_{n=0}^{\infty} \frac{n^3 q^n}{1-q^n} \qquad \text{and} \qquad E_6 = 1 - 504 \sum_{n=0}^{\infty} \frac{n^5 q^n}{1-q^n}\,, \tag{A.10}$$

with modular weight 4 and 6, respectively. The discriminant function is the weight 12 cusp form

$$\mathit{\Delta} = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \tfrac{1}{1728}(E_4^3 - E_6^2)\,. \tag{A.11}$$

The generators of the ring of weak holomorphic modular forms are E_4, E_6 and $1/\mathit{\Delta}$. The modular j-invariant is the unique weak holomorphic modular form of weight zero with $j = 1/q + \mathcal{O}(q)$,

$$j = \frac{E_4^3}{\mathit{\Delta}} - 744 = \frac{E_6^2}{\mathit{\Delta}} + 984\,. \tag{A.12}$$

The ring of weak almost holomorphic modular forms is obtained by adding to $E_4, E_6, 1/\mathit{\Delta}$ the almost holomorphic Eisenstein series

$$\hat{E}_2 = E_2 - \frac{3}{\pi\tau_2} = 1 - 24 \sum_{n=0}^{\infty} \frac{n\, q^n}{1-q^n} - \frac{3}{\pi\tau_2}\,. \tag{A.13}$$

Under the raising operator D_w one has

$$D\hat{E}_2 = \tfrac{1}{6}(E_4 - \hat{E}_2^2)\,, \quad DE_4 = \tfrac{2}{3}(E_6 - \hat{E}_2 E_4)\,, \quad DE_6 = E_4^2 - \hat{E}_2 E_6\,, \quad D(1/\mathit{\Delta}) = 2\hat{E}_2/\mathit{\Delta}\,, \tag{A.14}$$

where, for simplicity, we have left implicit the specification of the weight in D. Using the Leibniz rule (A.5), this allows to compute the action of D on any weak almost holomorphic modular form.

Finally, the operator T_κ maps the weak holomorphic modular form $\mathit{\Phi} = 1/q + \mathcal{O}(q)$ to $T_\kappa \mathit{\Phi} = 1/q^\kappa + \mathcal{O}(q)$.

A.2 Whittaker and hypergeometric functions

Whittaker functions and hypergeometric functions, more in general, are central in the analysis of the Niebur-Poincaré series and the evaluation of one-loop modular integrals. We summarise here their definitions and some of their main properties.

Whittaker functions are solutions of the second-order differential equation

$$u'' + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) u = 0\,. \tag{A.15}$$

For 2μ not integer the two independent solutions are given by the Whittaker M-functions

$$M_{\lambda,\pm\mu}(z) = e^{-z/2}\, z^{\pm\mu+\frac{1}{2}}\, {}_1F_1\left(\pm\mu - \lambda + \tfrac{1}{2}; 1 \pm 2\mu; z\right)\,, \tag{A.16}$$

and are expressed in terms of the confluent hypergeometric function

$$ {}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)} \frac{z^n}{n!} \,. \tag{A.17} $$

When 2μ is an integer, however, the second solution is not defined anymore, and thus it is useful to introduce a second Whittaker function defined by

$$ W_{\lambda,\mu}(z) = -\frac{1}{2\pi \mathrm{i}} \Gamma(\lambda+\tfrac{1}{2}-\mu)\, e^{-z/2}\, z^{\lambda} \int_{\infty}^{(0+)} (-t)^{-\lambda-\frac{1}{2}+\mu} \left(1+\frac{t}{z}\right)^{\lambda-\frac{1}{2}+\mu} e^{-t}\, dt \,, \tag{A.18} $$

where $|\arg(-t)| \leq \pi$, and the contour does not contain the point $t=-z$ and circles the origin counter-clockwise. The functions $W_{\lambda,\mu}(z)$ and $W_{-\lambda,\mu}(-z)$ are then the two independent solutions of the differential equation (A.15), and have the asymptotic behaviour

$$ \mathcal{W}_{s,w}(y) \sim |4\pi y|^{\frac{w}{2}(\mathrm{sgn}(y)-1)}\, e^{-2\pi|y|} \qquad \text{as} \quad |y| \to \infty \,. \tag{A.19} $$

The Whittaker M-function can then be expressed as the linear combination

$$ M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} e^{\mathrm{i}\pi\lambda}\, W_{-\lambda,\mu}(e^{\mathrm{i}\pi} z) + \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+\lambda+\frac{1}{2}\right)} e^{\mathrm{i}\pi(\lambda-\mu-\frac{1}{2})}\, W_{\lambda,\mu}(z) \,. \tag{A.20} $$

Using the symmetry of the W-functions, $W_{\lambda,\mu}(z) = W_{\lambda,-\mu}(z)$, one can invert the previous relation and write

$$ W_{\lambda,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\lambda\right)} M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\lambda\right)} M_{\lambda,-\mu}(z) \,. \tag{A.21} $$

This implies that the functions $\mathcal{M}_{s,w}$ and $\mathcal{W}_{s,w}$ obey

$$ \mathcal{W}_{s,w}(y) = \frac{\Gamma(1-2s)}{\Gamma\left(1-s-\frac{w}{2}\,\mathrm{sgn}(y)\right)} \mathcal{M}_{s,w}(y) + \frac{\Gamma(2s-1)}{\Gamma\left(s-\frac{w}{2}\,\mathrm{sgn}(y)\right)} \mathcal{M}_{1-s,w}(y) \,. \tag{A.22} $$

For special values of λ and μ, the Whittaker functions reduce to elementary functions or to other special functions. This derives from the properties of the hypergeometric functions, for instance

$$ \begin{aligned} {}_1F_1(a;a;z) &= e^{z} \,, \\ {}_1F_1(1,a,z) &= (a-1)\, z^{1-a}\, e^{z}\, \gamma(a-1,z) \,, \\ {}_1F_1(a,a+1,z) &= a(-z)^{-a}\, \gamma(a,-z) \,, \\ {}_1F_1(a;2a;z) &= e^{z/2} \left(\tfrac{1}{4} z\right)^{\frac{1}{2}-a} \Gamma\left(a+\tfrac{1}{2}\right) I_{a-\frac{1}{2}}(z/2) \,, \end{aligned} \tag{A.23} $$

where

$$ \gamma(a,z) = \int_0^z e^{-t}\, t^{a-1} dt = \Gamma(a) - \Gamma(a,z) \tag{A.24} $$

is the incomplete Gamma function, and

$$ I_\nu(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)} = \mathrm{i}^{-\nu}\, J_\nu(\mathrm{i}z) \tag{A.25} $$

is the modified Bessel function and $J_\nu(z)$ is the Bessel function of the first kind.

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Given the definitions (A.16), (2.7) and

$$K_\nu(z) = \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu}\,, \tag{A.26}$$

one can then show that ($y > 0$)

$$\begin{aligned} \mathcal{M}_{s,0}(\pm y) &= 2^{2s-1}\,\Gamma(s+\tfrac{1}{2})\,(4\pi|y|)^{\frac{1}{2}}\,I_{s-\frac{1}{2}}(2\pi|y|)\,,\\ \mathcal{W}_{s,0}(\pm y) &= 2\,|y|^{\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi|y|)\,, \end{aligned} \tag{A.27}$$

$$\begin{aligned} \mathcal{M}_{\frac{w}{2},w}(-y) &= e^{2\pi y}\,,\\ \mathcal{W}_{\frac{w}{2},w}(-y) &= \Gamma(1-w,4\pi y)\,e^{2\pi y}\,,\\ \mathcal{W}_{\frac{w}{2},w}(y) &= e^{-2\pi y}\,, \end{aligned} \tag{A.28}$$

$$\begin{aligned} \mathcal{M}_{-\frac{w}{2},w}(-y) &= (4\pi y)^{w}\,e^{-2\pi y}\,,\\ \mathcal{W}_{-\frac{w}{2},w}(-y) &= (4\pi y)^{-w}\,e^{-2\pi y}\,,\\ \mathcal{W}_{-\frac{w}{2},w}(y) &= (4\pi y)^{-w}\,\Gamma(1+w,4\pi y)\,e^{2\pi y}\,, \end{aligned} \tag{A.29}$$

$$\begin{aligned} \mathcal{M}_{1-\frac{w}{2},w}(-y) &= (1-w)\,\gamma(1-w,4\pi y)\,e^{2\pi y}\,,\\ \mathcal{W}_{1-\frac{w}{2},w}(-y) &= \Gamma(1-w,4\pi y)\,e^{2\pi y}\,,\\ \mathcal{W}_{1-\frac{w}{2},w}(y) &= e^{-2\pi y}\,. \end{aligned} \tag{A.30}$$

For integer values of the arguments, the confluent hypergeometric function, and thus the Whittaker functions, take a particularly simple expression

$$\begin{aligned} {}_1F_1(n+1,a,z) &= \frac{\Gamma(a)}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[z^{n+1-a}\left(e^z - \sum_{k=0}^{a-2}\frac{z^k}{k!}\right)\right]\\ &= \Gamma(a)\,z^{1-a}\left[e^z\,L_n^{(1-a)}(-z) - L_{a-2-n}^{(1-a)}(z)\right]\,, \end{aligned} \tag{A.31}$$

where

$$L_n^{(k)}(x) = \frac{x^{-k}\,e^x}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\left[x^{n+k}\,e^{-x}\right] \tag{A.32}$$

are the associated Laguerre polynomials.

As a result, the seed function that enters in the definition of the Niebur-Poincaré series involves only a finite number of terms when $s = 1 - \frac{w}{2} + n$, and reads

$$\begin{aligned} \mathcal{M}_{1-\frac{w}{2}+n,w}(-y) =& \,(4\pi y)^{1-w+n} e^{-2\pi y}\,\frac{(2n+1-w)!}{n!}\\ &\times \frac{\mathrm{d}^n}{\mathrm{d}(4\pi y)^n}\left[(4\pi y)^{w-n-1}\left(e^{4\pi y} - \sum_{k=0}^{2n-w}\frac{(4\pi y)^k}{k!}\right)\right]\\ =& \,\Gamma(2n+2-w)\,(4\pi y)^{-n}\\ &\times\left[e^{2\pi y}\,L_n^{(-1-2n+w)}(-4\pi y) - e^{-2\pi y}\,L_{n-w}^{(-1-2n+w)}(4\pi y)\right]\,. \end{aligned} \tag{A.33}$$

Similarly,

$$\begin{aligned} \mathcal{W}_{1-\frac{w}{2}+n,w}(y) &= (-1)^n\, n!\,(4\pi y)^{-n}\, e^{-2\pi y}\, L_n^{(-1-2n+w)}(4\pi y)\,, \\ \mathcal{W}_{1-\frac{w}{2}+n,w}(-y) &= (-1)^{n-w}\, \Gamma(n-w+1)\,(4\pi y)^{-n}\, e^{2\pi y}\, L_{n-w}^{(-1-2n+w)}(-4\pi y)\,. \end{aligned} \tag{A.34}$$

The modular derivatives (A.2) have a natural action on the Whittaker functions, so that

$$\begin{aligned} D_w\cdot\left[\mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}\right] &= 2\kappa(s+\tfrac{w}{2})\,\mathcal{M}_{s,w+2}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}\,, \\ \bar{D}_w\cdot\left[\mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}\right] &= \frac{1}{8\kappa}(s-\tfrac{w}{2})\,\mathcal{M}_{s,w-2}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}\,, \end{aligned} \tag{A.35}$$

and

$$\begin{aligned} D_w\cdot\left[\mathcal{W}_{s,w}(n\tau_2)\, e^{2\pi i n\tau_1}\right] &= \mathcal{W}_{s,w+2}(n\tau_2)\, e^{2\pi i n\tau_1}\times\begin{cases} -2n\,, & n>0\,, \\ 2n\,(s+\frac{w}{2})\,(s-\frac{w}{2}-1)\,, & n<0\,, \end{cases} \\ \bar{D}_w\cdot\left[\mathcal{W}_{s,w}(n\tau_2)\, e^{2\pi i n\tau_1}\right] &= \mathcal{W}_{s,w-2}(n\tau_2)\, e^{2\pi i n\tau_1}\times\begin{cases} \frac{1}{8n} & n<0\,, \\ -\frac{1}{8n}\,(s-\frac{w}{2})\,(s+\frac{w}{2}-1) & n>0\,. \end{cases} \end{aligned} \tag{A.36}$$

A.3 Kloosterman-Selberg zeta function

The Kloosterman-Selberg zeta function entering in the expression (2.16) of the Fourier coefficients of the Niebur-Poincaré series is defined as [43]

$$\mathcal{Z}_s(a,b) = \frac{1}{2\sqrt{|ab|}}\sum_{c>0}\frac{S(a,b;c)}{c}\times\begin{cases} J_{2s-1}\left(\frac{4\pi}{c}\sqrt{ab}\right) & \text{if}\quad a\,b>0\,, \\ I_{2s-1}\left(\frac{4\pi}{c}\sqrt{-ab}\right) & \text{if}\quad a\,b<0\,, \end{cases} \tag{A.37}$$

where $I_s(x)$ and $J_s(x)$ are the Bessel I and J functions, and $S(a,b;c)$ are the classical Kloosterman sums for the modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$,

$$S(a,b;c) = \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*}\exp\left[\frac{2\pi i}{c}(a\,d+b\,d^{-1})\right]. \tag{A.38}$$

Here a, b and c are integers, and d^{-1} is the inverse of d mod c. $S(a,b;c)$ is clearly symmetric under the exchange of a and b. Less evidently, it satisfies the Selberg identity

$$S(a,b;c) = \sum_{d|\gcd(a,b,c)} d\, S(ab/d^2,1;c/d)\,. \tag{A.39}$$

In the special case $a\neq 0$, $b=0$, the Kloosterman sum reduces to the Ramanujan sum

$$S(a,0;c) = S(0,a;c) = \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*}\exp\left(\frac{2\pi i}{c}a\,d\right) = \sum_{d|\gcd(c,a)} d\,\mu(c/d)\,, \tag{A.40}$$

with $\mu(n)$ the Möbius function. For $a=b=0$, $S(a,b;c)$ reduces instead to the Euler totient function $\phi(c)$, and one can verify that

$$\sum_{c>0}\frac{S(0,0;c)}{c^{2s}} = \frac{\zeta(2s-1)}{\zeta(2s)}\,,\qquad \sum_{c>0}\frac{S(0,\pm\kappa;c)}{c^{2s}} = \frac{\sigma_{1-2s}(\kappa)}{\zeta(2s)}\qquad(\kappa\neq 0)\,, \tag{A.41}$$

with $\sigma_x(n)$ the divisor function. Under complex conjugation, $\mathcal{Z}_s(a,b)$ transforms as

$$\overline{\mathcal{Z}_s(a,b)} = \mathcal{Z}_{\bar{s}}(-a,-b) \ . \tag{A.42}$$

The Kloosterman-Selberg zeta function defined in (A.37) is related to the zeta function

$$Z(a,b;s) \equiv \sum_{c>0} \frac{S(a,b;c)}{c^{2s}} \tag{A.43}$$

originally considered in [31] and used in [20] via

$$\mathcal{Z}_s(a,b) = \pi\,(4\pi^2|a\,b|)^{s-1} \sum_{m=0}^{\infty} \frac{(-4\pi^2 a\,b)^m}{m!\,\Gamma(2s+m)}\, Z(a,b;s+m)\,. \tag{A.44}$$

B Selberg-Poincaré series vs. Niebur-Poincaré series

In this section, we briefly discuss the relation between the Niebur-Poincaré series (2.8) and the Selberg-Poincaré series (2.5), considered in our previous work [20] in the special case $w=0$, as well as the relation between the BPS-state sum (3.19) and the "shifted constrained Epstein zeta series" considered in [20].

Comparing the differential equations (2.10) and (2.6), it is easily seen that a set of solutions of one can be converted into a set of solutions of the other by considering the linear combinations [33, 44]

$$\begin{aligned}
\mathcal{F}(s,\kappa,w) &= \sum_{m\geq 0} a(s,\kappa,w,m)\, E(s+m,\kappa,w)\,,\\
E(s,\kappa,w) &= \sum_{m\geq 0} b(s,\kappa,w,m)\, \mathcal{F}(s+m,\kappa,w)\,,
\end{aligned} \tag{B.1}$$

such that the coefficients satisfy the recursion relations

$$\frac{a(s,\kappa,w,m+1)}{a(s,\kappa,w,m)} = -\frac{4\pi\kappa\left(s+m-\frac{w}{2}\right)}{(m+1)(m+2s)}\,, \qquad \frac{b(s,\kappa,w,m+1)}{b(s+1,\kappa,w,m)} = \frac{4\pi\kappa\left(s-\frac{w}{2}\right)}{(m+1)(m+2s)}\,. \tag{B.2}$$

Comparing also the constant term (2.15) of the Niebur-Poincaré series and the constant term

$$\tilde{E}_0(s,\kappa,w) = \sum_{m=0}^{\infty} \frac{2^{2(1-s)}\,\pi\,\mathrm{i}^{-w}\,(\pi\kappa)^m\,\Gamma(2s+m-1)\,\sigma_{1-2s-2m}(\kappa)}{m!\,\Gamma\left(\frac{w}{2}+s+m\right)\,\Gamma\left(s-\frac{w}{2}\right)\,\zeta(2s+2m)}\,\tau_2^{\,1-s-m-\frac{w}{2}}\,, \tag{B.3}$$

of the Selberg-Poincaré series, we find that the coefficients are given by

$$\begin{aligned}
a(s,\kappa,w,m) &= (-1)^m\,\frac{2^{2s+2m-w}\,(\pi\kappa)^{s-\frac{w}{2}+m}\,\Gamma(2s)\,\Gamma\left(s+m-\frac{w}{2}\right)}{m!\,\Gamma(2s+m)\,\Gamma\left(s-\frac{w}{2}\right)}\,,\\
b(s,\kappa,w,m) &= \frac{2^{w-2s}\,(\pi\kappa)^{-s+\frac{w}{2}}\,\Gamma(2s+m-1)\,\Gamma\left(s+m-\frac{w}{2}\right)}{m!\,\Gamma(2s+2m-1)\,\Gamma\left(s-\frac{w}{2}\right)}\,.
\end{aligned} \tag{B.4}$$

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In particular, in the limit $s \to \frac{w}{2}$ where the summand of the Selberg-Poincaré series (2.5) becomes holomorphic, one finds $E(s,\kappa,w) = \mathcal{F}(\frac{w}{2},\kappa,w)$ for $w \geq 2$, but

$$E\left(\tfrac{w}{2},\kappa,w\right) = \mathcal{F}\left(\tfrac{w}{2},\kappa,w\right) + \sum_{m=1}^{-\frac{w}{2}-1} b'_m \,\mathrm{Res}_{s=\frac{w}{2}+m}\,\mathcal{F}(s,\kappa,w) + \sum_{m=-\frac{w}{2}+1}^{1-w} b_m\,\mathcal{F}\left(\tfrac{w}{2}+m,\kappa,w\right) \tag{B.5}$$

for $w \leq 0$, where $b_m \equiv \lim_{s\to\frac{w}{2}} b(s,\kappa,w,m)$ and $b'_m \equiv \lim_{s\to\frac{w}{2}} \frac{\mathrm{d}}{\mathrm{d}s} b(s,\kappa,w,m)$. In writing (B.5), we have assumed that the singularities of $\mathcal{F}(s,\kappa,w)$ on the real s-axis can be read off from the constant term (2.15), namely that $\mathcal{F}(s,\kappa,w)$ is regular at integer values of s provided $s \geq 0$ or $s \leq -|w|/2$, and has simple poles for integer values of s such that $-\frac{|w|}{2} < s < 0$. In particular, $E(\frac{w}{2},\kappa,w)$ receives a contribution (for $m = 1-w$) proportional to the harmonic Maass form $\mathcal{F}(1-\frac{w}{2},\kappa,w)$, which is the main object of interest in the present work, but is contaminated by other Niebur-Poincaré series lying outside the convergence domain $\Re(s) > 1$. For example, for $w = 0$, we find

$$E(0,\kappa,0) = \mathcal{F}(0,\kappa,0) + \tfrac{1}{2}\mathcal{F}(1,\kappa,0)\,, \tag{B.6}$$

consistently with the identifications

$$E(0,\kappa,0) = T_\kappa\, j + 12\sigma(\kappa)\,, \quad \mathcal{F}(1,\kappa,0) = T_\kappa\, j + 24\sigma(\kappa)\,, \quad \mathcal{F}(0,\kappa,0) = \tfrac{1}{2} T_\kappa\, j\,. \tag{B.7}$$

Moreover, the relation (B.1) between the Niebur-Poincaré and Selberg-Poincaré series implies a similar relation between the BPS-state sum (3.19) and the "shifted constrained Epstein zeta series"

$$\mathcal{E}^d_V(G,B,Y;s,\kappa) \equiv 2^s \sum_{\mathrm{BPS}} (p_L^2 + p_R^2 - 4\,\kappa)^{-s} = \sum_{\mathrm{BPS}} (p_{\mathrm{R}}^2)^{-s} \tag{B.8}$$

generalising the constructio in [20] to the case $k \neq 0$. Namely, using the same techniques as in our previous paper, one may show that (B.8) arises from the modular integral

$$\lim_{\mathcal{T}\to\infty} \int_{\mathcal{F}_\mathcal{T}} \mathrm{d}\mu\, \Gamma_{d+k,d}(G,B,Y)\, E\left(s,\kappa,-\tfrac{k}{2}\right) = \frac{\Gamma\left(s+\frac{2d+k}{4}-1\right)}{\pi^{s+\frac{2d+k}{4}-1}}\, \mathcal{E}^d_V\left(G,B,Y;s+\tfrac{2s+k}{4}-1,\kappa\right). \tag{B.9}$$

Since the BPS-state sum $\mathcal{I}_{d+k,d}(s,\kappa)$ arises from the limit $\mathcal{T}\to\infty$ of the integral (3.15), from (B.1) we conclude that

$$\mathcal{E}^d_V\left(G,B,Y;s+\tfrac{2d+k}{4}-1,\kappa\right) = \frac{\pi^{s+\frac{2d+k}{4}-1}}{\Gamma\left(s+\frac{2d+k}{4}-1\right)} \sum_{m\geq 0} b(s,\kappa,w,m)\,\mathcal{I}_{d+k,d}(s+m,\kappa)\,, \tag{B.10}$$

for large $\Re(s)$.

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