Published for SISSA by Springer

Received: April 30, 2010
Accepted: June 16, 2010
Published: June 29, 2010

The non-compact elliptic genus: mock or modular

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Abstract: We analyze various perspectives on the elliptic genus of non-compact supersymmetric coset conformal field theories with central charge larger than three. We calculate the holomorphic part of the elliptic genus via a free field description of the model, and show that it agrees with algebraic expectations. The holomorphic part of the elliptic genus is directly related to an Appell-Lerch sum and behaves anomalously under modular transformations. We analyze the origin of the anomaly by calculating the elliptic genus through a path integral in a coset conformal field theory. The path integral codes both the holomorphic part of the elliptic genus, and a non-holomorphic remainder that finds its origin in the continuous spectrum of the non-compact model. The remainder term can be shown to agree with a function that mathematicians introduced to parameterize the difference between mock theta functions and Jacobi forms. The holomorphic part of the elliptic genus thus has a path integral completion which renders it non-holomorphic and modular.

Keywords: Conformal Field Models in String Theory, Extended Supersymmetry, Conformal and W Symmetry

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 doi:10.1007/JHEP06(2010)104

JHEP06(2010)104

Contents

1 Introduction

The elliptic genus [1–4] is a generalized index that codes information on the spectrum of $N = 2$ superconformal field theories in two dimensions. It has applications in the calculation of anomalies and threshold corrections in string theory, in algebraic geometry, in the theory of modular forms and in the microscopic calculation of black hole entropies.

A basic example of an elliptic genus is the elliptic genus of the compact $N = 2$ minimal superconformal field theories. These conformal field theories have a representation as the infrared fixed point of supersymmetric Landau-Ginzburg models [5, 6], or alternatively as a gauged Wess-Zumino-Witten model [7–9]. For these prototypical compact superconformal field theories, it is possible to calculate the elliptic genus in at least three ways. The elliptic genus of other compact models can then be computed from that basic building block for instance by taking orbifolded products. One can often compare the algebraic results to geometric calculations of elliptic genera.

The three ways to compute the elliptic genus of the $N = 2$ minimal models are as follows. The first is by identifying the spectrum of the model [7–11], and in particular the representations that appear in a modular invariant partition function [12, 13]. The elliptic genus is then the spectrum of left-movers (coded in a sum of irreducible characters) with the right-movers in a Ramond ground state. A second way to compute the elliptic genus is by a free field calculation [14] based on the fact that the $N = 2$ minimal models are the infrared fixed point of supersymmetric Landau-Ginzburg models [5, 6]. The calculation gives rise to an alternative expression for the elliptic genus which was shown to agree with the algebraic formula [15]. A third way of computing the elliptic genus is via the description of the model as a gauged Wess-Zumino-Witten model [16]. The fields of the model can be shown to acquire the same transformation properties as the free fields of the Landau-Ginzburg model, and moreover the path integral localizes, leading to an identical free field calculation as the one performed in the Landau-Ginzburg description. There are

JHEP06(2010)104

many applications to more complicated models based on basic building blocks of central charge smaller than three.

Our aim in this paper is to get a firmer grip on the elliptic genus[1] of a non-minimal $N = 2$ superconformal field theory which lies outside the above class of theories. We will deepen our understanding of the elliptic genus of building blocks with central charge larger than three from the three perspectives discussed above.

For now, the proposed elliptic genus is based on the calculation of the partition function of the bosonic $SL(2, \mathbb{R})$ coset model [17]. The supersymmetric generalization was performed in [18, 19]. This allows for a proposal for the elliptic genus as the algebraic sum of discrete characters in the Ramond sector [18, 20]. Only discrete characters are assumed to contribute, since only those allow for a Ramond sector ground state (for the right-movers). However, the regularization used in both [17] and [18, 19] is not modular invariant. In [19] is was also shown that keeping track of the multiplicities of descendent states necessitates a more covariant treatment. An unexploited idea is to regulate the volume divergence of the non-compact model covariantly by subtraction of the asymptotic linear dilaton spectrum. In the absence of such an analysis, the fact that the proposed elliptic genus transforms non-covariantly under modular transformation properties is not understood. Though it is known from the mathematics literature how to patch up this annoying feature of the holomorphic part of the elliptic genus [21–23], the physical origin of the patchwork is obscure.

In this paper, we first strengthen the motivation for the holomorphic part of the elliptic genus through a free field calculation. Secondly, we derive the holomorphic part of the elliptic genus from a path integral calculation, and identify its remainder, thus gaining insight into its modular properties. Indeed, we will identify a non-holomorphic part of the elliptic genus that is necessary to complete the elliptic genus into a Jacobi form.

This is a neat addition to our understanding of the modular properties of non-rational conformal field theories (see e.g. [24–26]), as well as the isolation of discrete states from the continuum in a chiral sector of conformal field theory [27]. Our analysis is also related to the modular properties of the characters of superconformal algebras [28], invariants of three-manifolds, and number theory [29], the entropy of Calabi-Yau manifolds [30], and to the entropy of black holes and the crossing of walls of marginal stability [31, 32].

2 Two perspectives on the holomorphic part of the elliptic genus

2.1 The free field perspective

Recall that the elliptic genus is the trace over the Hilbert space of an $N = 2$ superconformal field theory, weighted by the fermion number, the left $U(1)_R$ charge and the conformal dimensions of the states:

$$\mathrm{Tr}(-1)^F z^{J_0^R} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} . \tag{2.1}$$

[1] We will use the term elliptic genus for the twisted partition function of the theory as defined by a path integral twisted by U(1) R-charge. An alternative would be to reserve this term for the purely holomorphic part of the twisted partition function.

JHEP06(2010)104

Under suitable conditions on the Hilbert space, the trace projects onto right-moving ground states.

In this section we give a free field derivation of (the holomorphic part of) the elliptic genus of a superconformal field theory with central charge c greater than three and of the form $c = 3 + \frac{6}{k}$ where k is an integer. The analysis is similar to the compact case [14], so we mainly concentrate on the differences. We specify the superconformal field theory in terms of a supersymmetric action with a Liouville potential term. From the potential term, we derive the necessary properties of the free field that allow us to then compute the elliptic genus while neglecting the interactions due to the potential.

We consider a two-dimensional $N = (2, 2)$ supersymmetric quantum field theory containing a free chiral superfield Φ with standard kinetic term, supplemented with a superpotential F-term:

$$S_{pot} = \mu \int d^2\sigma\, d^2\theta\, e^{\sqrt{\frac{k}{2}}\Phi}. \tag{2.2}$$

There is also a supersymmetrized coupling to the background worldsheet Ricci scalar curvature which corresponds to a linear dilaton in the direction Φ with slope $Q = \sqrt{2/k}$. It renders the superpotential term S_{pot} marginal. The bosonic superfield Φ contains a complex dynamical bosonic field $\phi = -\rho + i\theta$, a complex auxiliary field F and two Weyl fermions $\psi_\pm$. We choose the field θ to have radius $R = \sqrt{2k}$.[2] For an $N = (2, 2)$ supersymmetric model in two dimensions with a chiral superfield Φ, the supersymmetry transformation rules read:[3]

$$\begin{aligned}
\delta\phi &= \sqrt{2}(\epsilon_+\psi_- - \epsilon_-\psi_+)\\
\delta\psi_+ &= i\sqrt{2}(\partial_0 + \partial_1)\phi\bar{\epsilon}_- - \sqrt{2}\epsilon_+\sqrt{\frac{k}{2}}e^{\sqrt{\frac{k}{2}}\phi^*}\\
\delta\psi_- &= -i\sqrt{2}(\partial_0 - \partial_1)\phi\bar{\epsilon}_+ - \sqrt{2}\epsilon_-\sqrt{\frac{k}{2}}e^{\sqrt{\frac{k}{2}}\phi^*}.
\end{aligned} \tag{2.3}$$

The four supersymmetry parameters are $\epsilon_\pm, \bar{\epsilon}_\pm$. The derivatives with respect to the coordinates $\sigma^{0,1}$ are denoted $\partial_{0,1}$. To compute the elliptic genus, we must determine the charges of the fields under the left-moving $\mathrm{U}(1)_R$ group. We assign charge 0 to the right-moving supersymmetry transformation parameter ϵ_- and charge $+1$ to the supersymmetry parameter ϵ_+. From the supersymmetry transformation rules and the fact that:

$$\delta e^{\sqrt{\frac{k}{2}}\phi} = \sqrt{k}e^{\sqrt{\frac{k}{2}}\phi}(\epsilon_+\psi_- - \epsilon_-\psi_+), \tag{2.4}$$

[2] We work in units where the self-dual radius is $\sqrt{2}$. Note that any integer multiple of the radius $R_{\min} = \sqrt{2/k}$ is a consistent choice. In particular, for our purposes here the radius $R_{\min} = \sqrt{2/k}$ would be an equivalent choice, up to the interchange of winding and momentum in the following discussion.

[3] We use the conventions and notations of [33] where the transformation rules of [34] were reduced to two dimensions.

we conclude that the left-moving $\mathrm{U}(1)_R$ acts on the fields as follows:

$$\begin{aligned} \psi_- &\rightarrow e^{-i\beta}\psi_- \\ e^{\sqrt{\frac{k}{2}}\phi} &\rightarrow e^{+i\beta}e^{\sqrt{\frac{k}{2}}\phi} \\ \psi_+ &\rightarrow \psi_+. \end{aligned} \tag{2.5}$$

We draw our first conclusions:

- At radius $R = \sqrt{2k}$ we have left-moving momenta $p_{\theta,L} = \sqrt{\frac{1}{2k}}(n - kw)$ (and right-moving momenta $p_{\theta,R} = \sqrt{\frac{1}{2k}}(n + kw)$) where n and w are integer.
- The left-moving fermion has charge -1. The right-moving fermion has zero charge.
- The field $e^{i\sqrt{\frac{1}{2k}}\theta}$ (which for k integer has the minimal left-moving momentum) has charge $+\frac{1}{k}$. The charge is carried by the zero mode of the angular component of the complex boson.

Remark. To remain formally closer to the compact case, we could have written the superpotential term as

$$W = Z^{-k} \tag{2.6}$$

where $Z = e^{-\frac{1}{\sqrt{2k}}\Phi}$. A crucial difference with the compact case is that there is a singular region for the potential near $Z = 0$. Moreover, the potential leaves the field Z free to fluctuate at large values in field space. The first difference prompts us to choose the exponential variable which is better behaved near the origin, along with a standard kinetic term. This has the crucial consequence that the configuration space is punctured, and that we can have non-trivial winding configurations around the origin of configuration space. Moreover, it is now only the zero-mode of the complex boson that carries charge. The second difference forces us to specify a linear dilaton behaviour at infinity (see e.g. [35]), in order to obtain the right central charge in the non-compact, free region of configuration space. Finally, we note that the exponential form of the potential makes it manifest that strictly speaking, one cannot continuously dial the coupling constant to zero.

Free field proposal. We compute the free field elliptic genus in two steps. First, let us consider the subsector of the Hilbert space in which the left- and right-moving momentum of the compact boson θ are equal. In other words, we are in the zero winding sector. In the sector of the Hilbert space where we only allow for real Liouville momenta p_ρ, the elliptic genus will be zero. When we allow for imaginary momenta as well, then we can compensate the right-moving conformal dimension of the operators $e^{in\sqrt{\frac{1}{2k}}\theta}$ by the Liouville momentum contribution, and obtain a right-moving ground state. In the left-moving sector, due to the diagonal spectrum for the Liouville mode, as well as the diagonal subsector we are in, these modes then act as zero-modes with $\mathrm{U}(1)_R$ charge $+n/k$. To compute the partition function, we must regulate the contribution from these left-moving zero-modes (for instance by assigning consistently a slightly non-zero conformal dimension to one of the zero-modes and taking into account only the modes with positive conformal dimension).[4] Thus, in the diagonal subsector, we have one fermion of charge -1, and one bosonic zero-mode of charge $+1/k$. The diagonal partition function weighted on the right with the fermion number and on the left with the $\mathrm{U}(1)_R$ charge coded by the power of $z = e^{2\pi i \alpha}$ is:[5]

$$\chi_{free,off-diag} = \frac{i\theta_{11}(q,z)}{\eta^3} \frac{1}{1-z^{1/k}}. \tag{2.7}$$

Note that we evaluated the elliptic genus without regard to the presence of the superpotential term. In a second step, we re-introduce the winding sectors of the model.

Go with the flow. We introduce the winding sectors for the compact boson as follows. The integer left- and right-moving momenta differ by a multiple of $2k$. The winding sectors can be taken into account by summing independently over an extra integer m on the left that subtracts $2km$ units of left-moving momentum. We now want to implement that extra sum in the partition function. To understand an easy way to implement the sum, it is convenient to temporarily consider the expressions for the asymptotic $N = 2$ superconformal algebra (at large values of ρ, where the potential is negligible). The asymptotic $N = 2$ superconformal algebra is (see e.g. [37]):

$$\begin{aligned} T_{as} &= -\frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(\partial\theta)^2 - \frac{1}{2}(\psi_\rho\partial\psi_\rho + \psi_\theta\partial\psi_\theta) - \frac{1}{2}Q\partial^2\rho \\ G^{\pm}_{as} &= \frac{i}{\sqrt{2}}(\psi_\rho \pm i\psi_\theta)\partial(\rho \mp i\theta) + \frac{i}{\sqrt{2}}Q\partial(\psi_\rho \pm i\psi_\theta) \\ J^R_{as} &= iQ\partial\theta - i\psi_\rho\psi_\theta, \end{aligned} \tag{2.8}$$

with slope $Q = \sqrt{2/k}$ and central charge $c = 3 + 6/k$. The real boson ρ parameterizes the asymptotic linear dilaton direction, and the real field θ the angular direction at infinity. We also define the bosonized complexified fermions $e^{\pm iH} = \frac{1}{\sqrt{2}}(\psi_\rho \pm i\psi_\theta)$ and find the algebra in terms of these variables:

$$\begin{aligned} T_{as} &= -\frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(\partial\theta)^2 - \frac{1}{2}(\partial H)^2 - \frac{1}{2}Q\partial^2\rho \\ G^{\pm}_{as} &= ie^{\pm iH}\partial(\rho \mp i\theta) + iQ\partial e^{\pm iH} \\ &= ie^{\pm iH}\partial(\rho \mp i(\theta - QH)) \\ J^R_{as} &= iQ\partial\theta + i\partial H. \end{aligned} \tag{2.9}$$

We introduce the extra left-moving momentum $-2km$ in the field θ by performing a local $\mathrm{U}(1)_R$ transformation (since this is consistent with the $N = 2$ superconformal algebra and makes it easy to write down the partition sum in the new sector). This is equivalent to the action of spectral flow on the $N = 2$ superconformal algebra. Explicitly, we see that introducing the quantum number m on the left is equivalent to mapping $\theta \to \theta - im\sqrt{2k}z$

[4]This is as in the compact case [14].
[5]We use the conventions of [36] for the η- and θ-functions.

and $H \to H - imkz$ where z is the complexified periodic worldsheet coordinate on the torus. The new asymptotic algebra then reads:

$$\begin{aligned}
T_{as} &\to T_{as} + \frac{c}{6}(km)^2 + km(iQ\partial\theta + i\partial H) \\
G^{\pm}_{as} &\to e^{\mp kmz} G^{\pm}_{as} \\
J_{as} &\to J_{as} + \frac{c}{3}km.
\end{aligned} \tag{2.10}$$

The latter transformation is spectral flow by km units. Therefore, we can restore the off-diagonal sectors by performing the spectral flow operation on the holomorphic partition sum (while keeping track of the fact that also the fermion number shifted by km) and we find:

$$\chi_{hol} = \sum_{m\in\mathbb{Z}} \frac{i\theta_{11}(q,z)}{\eta^3} z^{2m} q^{km^2} \frac{1}{1 - z^{1/k} q^m}. \tag{2.11}$$

The interpretation is clear. The right-moving momentum of the angular mode is still compensated by a diagonal Liouville momentum, but in the process every unit left-moving momentum mode picks up a conformal dimension m in sector m. The vacuum has obtained additional left-moving R-charge and it acquired an extra contribution to its conformal dimension from the extra left-moving momenta.

A remark on the individual contributions. It is natural to rewrite the holomorphic partition sum in terms of a higher level Appell function K_l:

$$K_l(q,z,y) = \sum_{m\in Z} \frac{q^{lm^2/2} z^{ml}}{1 - yzq^m}. \tag{2.12}$$

We have:

$$\chi_{hol} = \frac{i\theta_{11}(q,z)}{\eta^3} K_{2k}(q, z^{1/k}, 1). \tag{2.13}$$

Above we introduced the winding sectors via a spectral flow operation. It is interesting to ask how the regularization of the bosonic zero-mode carries over to the non-diagonal sectors through the spectral flow operation. Given the choice $|q| < |z^{1/k}| < 1$, we can expand the elliptic genus, and identify the individual state contributions that we have allowed:

$$\begin{aligned}
\chi_{hol} &= \frac{i\theta_{11}(q,z)}{\eta^3} \sum_{m\in\mathbb{Z}} \frac{z^{2m} q^{km^2}}{1 - z^{1/k} q^m} \\
&= \frac{i\theta_{11}(q,z)}{\eta^3} \left(\sum_{m\geq 0, p\geq 0} - \sum_{m\leq -1, p\leq -1} \right) q^{km^2+pm} z^{2m} z^{p/k} \\
&= \frac{i\theta_{11}(q,z)}{\eta^3} \left(\sum_{w\leq 0, n+kw\geq 0} - \sum_{w\geq 1, n+kw\leq -1} \right) q^{-nw} z^{(n-kw)/k}.
\end{aligned} \tag{2.14}$$

For zero winding, we recuperate the fact that we allowed positive momentum only. The generalization is that for negative winding, we allow positive right-moving (angular) momenta, and for strictly positive winding, we allow strictly negative right-moving momenta. It is possible to understand this as follows. We fix the difference of left- and right-moving momenta in each sector labelled by w. We then allow for diagonal operators only to act, and this includes a dimension zero operator with tuned Liouville momentum adapted to the diagonal angular momentum. We pick a right-moving ground state and can act with the diagonal operator to create new right-moving ground states. It is the conformal dimension of the diagonal operators that is regularized in each sector, and therefore, the sign of the right-moving momentum that determines our regularization scheme. The left-moving momentum, on the other hand, can be non-zero since the left-movers are not necessarily in the ground state. It is set by the diagonal operator, and the sector label w, for a total momentum of $n - kw$ on the left.

2.2 The algebraic perspective

We want to compare the final expression we obtained in the free field analysis with a proposal [18] that has an algebraic origin [17, 18]. The higher level Appell function is a sum over extended twisted Ramond sector characters [18] for an $N = 2$ superconformal theory with central charge of the form $c = 3 + \frac{6}{k}$ with k a positive integer. Recall that the extended twisted R-sector characters are [38]:

$$\begin{aligned} Ch_d^{\tilde{R}}(j, r'; q, z) &= \sum_m \frac{i\theta_{11}(q,z)}{\eta^3} \frac{1}{1 - zq^{k(m+(2r'+1)/2k)}} \\ &\qquad z^{\frac{2j-1}{k}} q^{k(m+\frac{2r'+1}{2k})(2j-1)/k} z^{2(m+\frac{2r'+1}{2k})} q^{k(m+\frac{2r'+1}{2k})^2}. \end{aligned} \tag{2.15}$$

When we evaluate the character for a representation built on a ground state with $r' = -1/2$ (which gives rise to zero conformal dimension for the value $m = 0$ in the infinite sum), and we sum over spins j that lie in the set $\{1/2, 1, \ldots, k/2\}$ and which appear in a decomposition of the partition function [17–19], we find the following result [18]:

$$\begin{aligned} \sum_{2j-1=0}^{k-1} Ch_d^{\tilde{R}}(j, -1/2; q, z) &= \frac{i\theta_{11}(q,z)}{\eta^3} \sum_m z^{2m} q^{km^2} \frac{1}{1 - zq^{km}} \sum_{2j-1=0}^{k-1} z^{\frac{2j-1}{k}} q^{m(2j-1)} \\ &= \frac{i\theta_{11}(q,z)}{\eta^3} \sum_m z^{2m} q^{km^2} \frac{1}{1 - zq^{km}} \frac{1 - zq^{km}}{1 - z^{1/k} q^m} \\ &= \chi_{hol}. \end{aligned} \tag{2.16}$$

It is gratifying that the simple free field derivation agrees with the algebraic perspective. The algebraic result is based on the idea that only a definite range of spins contribute to the elliptic genus, and that the elliptic genus is made of extended characters, i.e. that all spectrally flowed Hilbert spaces must be taken into account. This is consistent with the analysis of the partition function [17–19]. Moreover, there are various string theory inspired consistency checks on these assumptions ranging from consistency with linear dilaton holography [39] to the physics of strings puffing up in AdS_3 [40].

Quandary. The free field and the algebraic perspective on the holomorphic part of the elliptic genus are well-motivated and give the same result, yet they cannot tell us the whole story. The modular transformation properties of the higher level Appell function supplemented with the θ- and η-functions (see e.g. [28]) do not agree with the expectation that the elliptic genus be a Jacobi form. The anomaly is associated to the non-compactness of target space which renders the above derivations approximate (though they were exact in the compact case). We will more carefully regularize the volume divergence to gain further insight into the modular, or holomorphic anomaly.

3 The path integral and modularity

Since the path integral (or Lagrangian) formulation of the elliptic genus manifestly codes its modular transformation properties, it will be interesting to compare the path integral calculation to the holomorphic perspectives given in the previous section. We will perform the full path integral (in contrast to the compact case, where it was computed through localization, which reduces the calculation to the free field calculation [16]). Most of the details of the derivation of the path integral expression are identical to those provided in [17]. Furthermore, we draw upon the analysis in the compact case [16] to properly implement the twisting by the left $\mathrm{U}(1)_R$ charge.

A careful discussion of the calculation of the elliptic genus in the Lagrangian formulation of compact $N=2$ superconformal Wess-Zumino-Witten models was given in [16]. The analysis is mostly valid for non-compact target space groups as well. If we introduce an $\mathrm{SL}(2,\mathbb{R})$ group valued field g, two (right- and left-moving) fermions $\chi^{\pm}$ which take values in the algebra $sl(2,\mathbb{R})$ (mod $u(1)$), and a generator U of the $\mathrm{U}(1)$ gauge group, then the fields are argued to transform under the left-moving $\mathrm{U}(1)_R$ symmetry with parameter γ as:

$$\begin{aligned}
\delta\chi_+ &= -\frac{i\gamma}{k}\chi_+ \\
\delta\chi_- &= \frac{i\gamma(k+1)}{k}\chi_- \\
\delta g &= \frac{i\gamma}{k}(Ug-gU).
\end{aligned} \tag{3.1}$$

We will not review the derivation of these formulas, based on the chiral anomalies of the two-dimensional model, but we indicate the main differences with the compact case. When we compare to equation (25) of [16], we have made the following changes. By convention, we work with the supersymmetric level k of the model, namely the level that takes into account both bosons and fermions.[6] Moreover we work with a non-compact model which formally corresponds to changing the sign of the total level k. When we take those changes into account, we find the formulas quoted above from the analysis of [16] adapted to an axially gauged non-compact model. We conclude that in some suitable normalizations the charges of the right-moving fermion χ_+ and of the (non-Cartan) complex boson agree (and are $1/k$), while the charge of the other fermion is bigger by a factor $k+1$.

[6] Therefore for a compact model our level $k^{rmSU(2)}$ would be related to the level k_{Hen} as $k^{rmSU(2)} = k_{Hen}+2$.

The other ingredient we use is the path integral with ordinary boundary conditions in the NSNS sector of the theory as computed in [17–19]. It consists of a contribution from the non-compact coset, from the fermions, and from Wilson lines $s_{1,2}$ for the gauge field on the torus. However, in comparison to [17–19], we make the following changes. It will be more intuitive to gauge the direction T-dual to the one gauged in those papers. We must also take into account the fact that the right-moving boson and fermion oscillators cancel out against each other. The result of the path integral calculation of the elliptic genus (which in the coordinate choice found in [42] reduces to the evaluation of a Ray-Singer torsion, a twisted fermion partition function, and zero modes) is then:

$$\chi = \sum_{m,w} \int_0^1 ds_1 ds_2 \frac{\theta_{11}(\tau, s_1\tau + s_2 - \frac{k+1}{k}\alpha)}{\theta_{11}(\tau, s_1\tau + s_2 - \frac{1}{k}\alpha)} e^{2\pi i \alpha w/k} e^{-\frac{\pi}{k\tau_2}|(m+ks_2)+\tau(w+ks_1)|^2}. \quad (3.2)$$

The θ_{11} functions have a twisted argument that depends on the holonomies of the gauge field on the torus, and on the twist by the $\mathrm{U}(1)_R$ charges, with a weight determined by the charges of the left-moving bosons and fermions (analyzed above). The last factor codes the coupling of the oscillators to the bosonic zero-modes at radius $R = \sqrt{2/k}$ via the holonomies $s_{1,2}$. The twist of the bosonic zero-modes also introduces and extra phase factor $e^{2\pi i \alpha w/k}$.

Note that the result is not holomorphic. This possibility is opened up by the cancellation of bosonic and fermionic oscillators on the right, in the presence of further zero modes. The cancellation is one between a right-moving fermionic zero-mode and a volume divergence. Moreover, the expression is formal at this stage, since at non-zero twist α, there are poles in the theta-function in the denominator that are not compensated by zeroes in the numerator (in contrast to the case $\alpha = 0$, where the calculation gives a Witten index equal to k). Thus, the integral and the result are divergent. We will provide a regularization.

A first look at modularity. In a first step though we check the modular transformation properties of the formal integral expression for the elliptic genus. We show that the path integral result satisfies the expected modular covariance properties. We have a Jacobi form of weight 0 and index $k(k+2)/2 = k^2c/6$.[7]

The expected modular covariance properties are [41], from the boundary conditions on the path integral and the factorization of the $\mathrm{U}(1)_R$ current algebra:

$$\begin{aligned} \chi(\tau+1,\alpha) &= \chi(\tau,\alpha) \\ \chi(-\frac{1}{\tau},\frac{\alpha}{\tau}) &= e^{2\pi i \frac{c}{6}\alpha^2/\tau}\chi(\tau,\alpha). \end{aligned} \quad (3.3)$$

If all $\mathrm{U}(1)_R$ charges in the NS sector are multiples of $1/k$, then we expect, with $\mu \in k\mathbb{Z}$:

$$\chi(\tau,\alpha+\mu) = (-1)^{\frac{c}{3}\mu}\chi(\tau,\alpha) \quad (3.4)$$

and from mapping the Ramond sector states into the Ramond sector states after an integer number of spectral flows (with $\lambda \in k\mathbb{Z}$):

$$\chi(\tau,\alpha+\lambda\tau) = (-1)^{\frac{c}{3}\lambda} e^{-2\pi i \frac{c}{6}(\lambda^2\tau + 2\lambda\alpha)}\chi(\tau,\alpha). \quad (3.5)$$

[7]Strictly speaking, for odd level k, the index is $2k(k+2)$.

JHEP06(2010)104

Finally, we have

$$\chi(\tau,\alpha) = \chi(\tau,-\alpha) \tag{3.6}$$

for a charge conjugation invariant Ramond spectrum. To check the modular properties, it is convenient to first doubly Poisson resum the path integral result. We obtain:

$$\chi = k\sum_{m,w}\int_0^1 ds_1 ds_2 \frac{\theta_{11}(\tau, s_1\tau + s_2 - \frac{k+1}{k}\alpha)}{\theta_{11}(\tau, s_1\tau + s_2 - \frac{1}{k}\alpha)} e^{-2\pi i s_2 k w} e^{2\pi i s_1(km-\alpha)} e^{-\frac{\pi k}{\tau_2}|(m-\frac{\alpha}{k})+\tau w|^2}. \tag{3.7}$$

Under the simultaneous transformations (implied by following the holonomies and winding numbers under the modular action):

$$\tau \to \tau + 1 \qquad s_2 \to s_2 - s_1 \qquad m \to m - w, \tag{3.8}$$

the elliptic genus is invariant. Under the transformations:

$$\begin{aligned} &\tau \to -\frac{1}{\tau} && s_1 \to -s_2 && w \to -m \\ &\alpha \to \frac{\alpha}{\tau} && s_2 \to s_1 && m \to w, \end{aligned} \tag{3.9}$$

we pick up the expected factor:

$$e^{\pi i((s_1\tau+s_2-\frac{k+1}{k}\alpha)^2-(s_1\tau+s_2-\frac{\alpha}{k})^2)/\tau} e^{2\pi i\alpha s_2/\tau} e^{2\pi i s_1\alpha} = e^{\pi i/\tau \alpha^2(1+\frac{2}{k})}. \tag{3.10}$$

One similarly verifies that the elliptic genus satisfies the periodicity and parity requirements as well. These properties make the elliptic genus χ a Jacobi form of weight zero and index $k^2c/6 = (k+2)k/2$, at least formally.

Distinguishing two contributions. The path integral result can be related to the holomorphic perspective through an analysis similar to [17–19] while keeping track of the degeneracies of descendent states as in [19]. We perform the following manipulations on the path integral result. We singly Poisson resum on m in equation (3.7) to find:

$$\chi = \sqrt{k\tau_2}\sum_{n,w}\int_0^1 ds_1 \int_0^1 ds_2 \frac{\theta_{11}(\tau, s_1\tau+s_2-\frac{k+1}{k}\alpha)}{\theta_{11}(\tau, s_1\tau+s_2-\frac{1}{k}\alpha)} e^{2\pi i\alpha\frac{n}{k}-2\pi i s_2 k w} q^{(kw-(n+ks_1))^2/4k}\bar{q}^{(kw+(n+ks_1))^2/4k}.$$

We now regularize the path integral by assuming that $1 > |q^{s_1}z^{-\frac{1}{k}}| > |q|$. We take $|z|$ very close to one, and cut off the boundaries of the integration over the holonomy s_1 such as to satisfy the above equation. This regularization allows us to expand the θ_{11} function in the denominator in terms of the special functions $S_r(\tau)$ which are known to code the degeneracies of descendant states in the discrete characters of the $SL(2,R)/U(1)$ coset conformal field theory [43–46]. The definition of the series S_r (which are related to Hecke indefinite modular forms) is:

$$S_r(\tau) = \sum_{n=0}^{+\infty}(-1)^n q^{\frac{n(n+2r+1)}{2}}. \tag{3.11}$$

After the expansion, we obtain the expression:

$$\chi = -\sqrt{k\tau_2}\frac{1}{\eta^3}\sum_{m,n,w,r}\int ds_1\int ds_2(-1)^m q^{(m-\frac{1}{2})^2/2}(z^{1+\frac{1}{k}}e^{-2\pi i s_2}q^{-s_1})^{m-1/2}(e^{2\pi i s_2}q^{s_1}z^{-\frac{1}{k}})^{r+\frac{1}{2}}S_r$$
$$e^{-2\pi i s_2 kw}z^{\frac{n}{k}}q^{(kw-(n+ks_1))^2/4k}\bar{q}^{(kw+(n+ks_1))^2/4k}. \tag{3.12}$$

The integral over s_2 implies that we must have $m-r-1+kw=0$ to get a non-zero contribution. We shift the summation variable n to $v=n+kw$ and find:

$$\chi = -\sqrt{k\tau_2}\frac{1}{\eta^3}\sum_{m,v,w,r}\int ds_1\int ds_2(-1)^m q^{(m-\frac{1}{2})^2/2}z^{m-\frac{1}{2}}S_r$$
$$z^{-2w+\frac{v}{k}}e^{-2\pi i s_2(m-r-1+kw)}q^{kw^2-vw}(q\bar{q})^{(v+ks_1)^2/4k}. \tag{3.13}$$

We can then introduce an integral over continuous momenta s to make the exponent linear in s_1:

$$\chi = -2\tau_2\frac{1}{\eta^3}\sum_{m,v,w,r}\int ds_1\int ds_2\int_{-\infty}^{+\infty}ds(-1)^m q^{(m-\frac{1}{2})^2/2}z^{m-\frac{1}{2}}S_r z^{-2w+\frac{v}{k}}q^{kw^2-vw}$$
$$e^{2\pi i s_2(r-kw-m+1)}(q\bar{q})^{s_1(is+\frac{v}{2})+\frac{s^2}{k}+\frac{v^2}{4k}}, \tag{3.14}$$

which allows us to easily perform the integrations over both holonomies $s_{1,2}$:

$$\chi = \frac{1}{\pi}\frac{1}{\eta^3}\sum_{m,v,w}\int_{\mathbb{R}-i\epsilon}\frac{ds}{2is+v}(-1)^m q^{(m-\frac{1}{2})^2/2}z^{m-\frac{1}{2}}(q\bar{q})^{\frac{s^2}{k}+\frac{v^2}{4k}}z^{-2w+\frac{v}{k}}q^{kw^2-vw}$$
$$S_{kw+m-1}((q\bar{q})^{is+\frac{v}{2}}-1). \tag{3.15}$$

Note that we slightly shifted the integration over the momentum s off the real axis for future convenience. Drawing inspiration from [17] on how to disentangle contributions with continuous radial momentum from discrete contributions, we now distinguish between two parts in the path integral. One which we will call the remainder term χ_{rem}, and a second one which will turn out to be holomorphic:

$$\begin{aligned}
\chi &= \chi_{hol}+\chi_{rem}\\
\chi_{hol} &= \frac{1}{\pi}\frac{1}{\eta^3}\sum_{m,v,w}\int_{\mathbb{R}-i\epsilon}\frac{ds}{2is+v}(-1)^m q^{(m-\frac{1}{2})^2/2}z^{m-\frac{1}{2}}(q\bar{q})^{\frac{s^2}{k}+\frac{v^2}{4k}}z^{-2w+\frac{v}{k}}q^{kw^2-vw}\\
&\quad (1-S_{kw+m-1}(1-(q\bar{q})^{is+\frac{v}{2}})\\
\chi_{rem} &= -\frac{1}{\pi}\frac{1}{\eta^3}\sum_{m,v,w}\int_{\mathbb{R}-i\epsilon}\frac{ds}{2is+v}(-1)^m q^{(m-\frac{1}{2})^2/2}z^{m-\frac{1}{2}}(q\bar{q})^{\frac{s^2}{k}+\frac{v^2}{4k}}z^{-2w+\frac{v}{k}}q^{kw^2-vw}.
\end{aligned} \tag{3.16}$$

We first massage the holomorphic part, using the property $S_r=S_{-r-1}-1$ and shifting the summation variable m by one:

$$\chi_{hol} = \frac{1}{\pi}\frac{1}{\eta^3}\sum_{m,v,w}\int_{\mathbb{R}-i\epsilon}\frac{ds}{2is+v}(-1)^m q^{(m-\frac{1}{2})^2/2}z^{m-\frac{1}{2}}(q\bar{q})^{\frac{s^2}{k}+\frac{v^2}{4k}}z^{-2w+\frac{v}{k}}q^{kw^2-vw}$$
$$(S_{-m-kw}-zq^{m+is+\frac{v}{2}}\bar{q}^{is+\frac{v}{2}}S_{kw+m}) \tag{3.17}$$

and using furthermore the formula $q^r S_r = S_{-r}$ we find:

$$\chi_{hol} = \frac{1}{\pi}\frac{1}{\eta^3}\sum_{m,v,w}\int_{\mathbb{R}-i\epsilon}\frac{ds}{2is+v}(-1)^m q^{(m-\frac{1}{2})^2/2} z^{m-\frac{1}{2}} (q\bar{q})^{\frac{s^2}{k}+\frac{v^2}{4k}} z^{-2w+\frac{v}{k}} q^{kw^2-vw} S_{-m-kw}$$
$$(1 - zq^{-kw+is+\frac{v}{2}}\bar{q}^{is+\frac{v}{2}}). \tag{3.18}$$

The neat property of the second term in this expression is that it can be rewritten as an integral over the real axis, shifted upward by $k/2$, if we simultaneously shift the momentum v:

$$\chi_{hol} = \frac{1}{\pi}\frac{1}{\eta^3}\sum_{m,v,w}\left(\int_{\mathbb{R}-i\epsilon} - \int_{\mathbb{R}+i\frac{k}{2}-i\epsilon}\right)\frac{ds}{2is+v}(-1)^m q^{(m-\frac{1}{2})^2/2} z^{m-\frac{1}{2}}(q\bar{q})^{\frac{s^2}{k}+\frac{v^2}{4k}} z^{-2w+\frac{v}{k}} q^{kw^2-vw} S_{-m-kw}.$$

Splitting the integral, and performing the spectral flow operation has disentangled discrete characters from the continuum. Indeed, the good convergence properties at infinity allow us to interpret our formula for χ_{hol} as a contour integral that picks up the poles in $2is + v$ that lie between $-i\epsilon$ and $ik/2 - i\epsilon$. The poles lie at Liouville momenta that give rise to right-moving ground states. After some further algebraic operations, this gives the announced holomorphic result:

$$\begin{aligned}\chi_{hol} &= \frac{1}{\eta^3}\sum_{2j-1=0}^{k-1}\sum_{w,m}(-1)^m q^{(m-\frac{1}{2})^2/2} z^{m-\frac{1}{2}} z^{-2w} q^{kw^2} S_{-m-kw}(z^{-\frac{1}{k}}q^w)^{2j-1} \\ &= \frac{1}{\eta^3}\sum_{m\in\mathbb{Z}}\frac{i\theta_{11}(q,z)}{1-z^{\frac{1}{k}}q^m} z^{2m} q^{km^2}.\end{aligned} \tag{3.19}$$

Thus we see that the path integral does partially agree with the intuitive derivation of the (holomorphic part of the) elliptic genus from the free field (see equation (2.11)) and algebraic perspectives (see equation (2.16)).

The remainder term: from mock theta-functions to Jacobi forms. We do also have the remainder term χ_{rem}. The expression (3.16) for the remainder term can be read as a trace over the asymptotic continuum of states with weighting by the left $\mathrm{U}(1)_R$-charge and the conformal weights, in which the right-moving oscillators of fermions and bosons have cancelled out. The cancellation has left a measure factor which is given by one over the total right-moving momentum, which is formally the ratio of the right-moving fermionic zero mode and a right-moving supercharge (as can be seen from the analogue of equation (2.9) for the right-moving superconformal algebra). The measure is such that the contributions which localize in momentum space are holomorphic. However, the integral over the continuum associated to the non-compactness of the target space is non-zero and non-holomorphic. The expression provides a Hamiltonian viewpoint on the holomorphic anomaly. We evaluate the integral over the continuum of momenta s to obtain:

$$\chi_{rem} = -\frac{i\theta_{11}(q,z)}{2\eta^3}\sum_{v,w} z^{-2w+\frac{v}{k}} q^{kw^2-vw}\left(\mathrm{sgn}(v+\epsilon) - \mathrm{Erf}\left(v\sqrt{\frac{\pi\tau_2}{k}}\right)\right), \tag{3.20}$$

where Erf is the error function. The role of the remainder term in the path integral is to render the elliptic genus modular covariant. Indeed, we can now make our analysis of the modular transformation properties rigorous by comparing our results to techniques developed in the mathematics literature in the context of the definition and analysis of mock theta functions. See e.g. [29] for a review and [30] for a recent application to the entropy of Calabi-Yau manifolds. Here we closely follow the presentation of [21, 22] where the definition of the level l Appell-Lerch sums is taken to be:

$$A_l(u,v;\tau) = a^{l/2}\sum_{n\in\mathbb{Z}} \frac{(-1)^{ln}q^{ln(n+1)/2}b^n}{1-aq^n} \tag{3.21}$$

where $a = e^{2\pi iu}$ and $b = e^{2\pi iv}$. It is known that there exists a correction term for the Appell function that completes it into a Jacobi form of weight 1 and index $\begin{pmatrix} -l & 1 \\ 1 & 0 \end{pmatrix}$. The Jacobi form is the Appell sum plus a remainder term:

$$\hat{A}_l(u,v;\tau) = A_l(u,v;\tau)+\frac{i}{2l}a^{(l-1)/2}\sum_{m\,\mathrm{mod}\,l}\theta_{11}\left(\frac{v+m}{l}+\frac{l-1}{2l}\tau;\frac{\tau}{l}\right)R\left(u-\frac{v+m}{l}-\frac{l-1}{2l}\tau;\frac{\tau}{l}\right) \tag{3.22}$$

where

$$R(u;\tau) = \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} (\mathrm{sgn}(\nu)-\mathrm{Erf}(\sqrt{2\pi\tau_2}(\nu+\mathrm{Im}u/\tau_2))(-1)^{\nu-\frac{1}{2}}a^{-\nu}q^{-\frac{\nu^2}{2}}. \tag{3.23}$$

We first express the holomorphic part of the elliptic genus as proportional to an Appell-Lerch sum:

$$\chi_{hol} = z^{-1}\frac{i\theta_{11}(q,z)}{\eta^3}A_{2k}(z^{\frac{1}{k}},z^2q^{-k};q). \tag{3.24}$$

It is now a matter of straightforward calculation to show that our remainder term χ_{rem} precisely agrees with the correction term of [21, 22] when we evaluate the latter at $l = 2k$, $a = z^{1/k}$, $b = q^{-k}z^2$ and multiply by $z^{-1}i\theta_{11}/\eta^3$. The fact that $\hat{A}_{2k}$ is a Jacobi form was rigorously proven. We have the modular properties [21, 22]:

$$\begin{aligned}
\hat{A}_l(u+1,v) &= (-1)^l\hat{A}_l(u,v) & \hat{A}_l(u,v+1) &= \hat{A}_l(u,v)\\
\hat{A}_l(u+\tau,v) &= (-1)^l a^l b^{-1} q^{\frac{l}{2}}\hat{A}_l(u,v) & \hat{A}_l(u,v+\tau) &= a^{-1}\hat{A}_l(u,v)\\
\hat{A}_l(u,v;\tau+1) &= \hat{A}_l(u,v;\tau) & \hat{A}_l\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right) &= \tau e^{\pi i(2v-lu)u/\tau}\hat{A}_l(u,v;\tau).
\end{aligned} \tag{3.25}$$

Thus the chosen regularization of the path integral is indeed modular covariant. The non-holomorphic elliptic genus as coded in the regularized path integral is a Jacobi form with weight zero and index $k(k+2)/2$. Multiplication by the θ- and η-function indeed gives the expected weight and index to the elliptic genus. To check this it is useful to use the periodicity of the generalized Appell-Lerch sum and write:

$$\chi = \frac{i\theta_{11}(q,z)}{\eta^3}\hat{A}_{2k}(z^{\frac{1}{k}},z^2;q), \tag{3.26}$$

and then use the modular covariance properties listed above to complete the proof of modularity.

4 Conclusions

In this paper we gave a free field derivation of the holomorphic part of the elliptic genus of a basic $N = 2$ superconformal field theory with central charge c larger than three and equal to $c = 3 + \frac{6}{k}$ with k integer. The holomorphic part of the elliptic genus is directly related to an Appell-Lerch sum. We also explicitly evaluated the path integral over the coset conformal field theory that corresponds to the non-compact Landau-Ginzburg model. We showed that it contains a holomorphic part which agrees with the free field and the algebraic analysis. The full path integral result though is non-holomorphic and modular. The holomorphic anomaly satisfied by the elliptic genus finds its origin in the non-compactness of target space. The path integral provides a physical origin for the remainder term postulated in the mathematics literature to complete mock theta-functions into Jacobi forms. Our result provides a modular covariant regularization to a problem plagued by a volume divergence that cannot be factored out as the volume of a symmetry group. It gives rise to an expression that can consistently be integrated over moduli space.

There are many directions that open up thanks to this elementary result. We mention only three. The derivation can be extended to $N = 4$ models, to combinations of compact and non-compact $N = 2$ or $N = 4$ models, and orbifolds thereof. One may also attempt to extend our analysis to fractional values of the level, where in analogy with the conformal field theory of compact bosons, we may expect a relation to the Appell sums at level equal to the product of denominator and numerator. Secondly, we can analyze further the holomorphic anomaly differential equation that is satisfied by the remainder term (when we derive with respect to $\bar{\tau}$). Thirdly, we can apply the insight we gained into modular covariant regularization to the bulk modular invariant partition function (of the bosonic or supersymmetric coset model).

Acknowledgments

We would like to thank Sujay Ashok, Costas Bachas, Raphael Benichou, Tohru Eguchi, Dan Israel, Ari Pakman and Giuseppe Policastro for useful discussions. Our work was supported in part by the grant ANR-09-BLAN-0157-02.

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JHEP06(2010)104

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