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Generalized unitary evolution for symplectic scalar fermions

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Abstract: The theory of symplectic scalar fermion of LeClair and Neubert is studied. The theory evades the conventional spin-statistics theorem because its Hamiltonian is pseudo Hermitian. The definition of pseudo Hermiticity is examined in the interacting and the Heisenberg picture. For states that evolve under pseudo Hermitian Hamiltonians, we define the appropriate inner-product and matrix element of operators that preserve time translation symmetry. The resulting S-matrix is shown to satisfy the generalized unitarity relation. We clarify the derivation of the symplectic currents and charges. By demanding the currents and charges to be pseudo Hermitian, the global symmetry of the free Lagrangian density reduces from $\mathrm{Sp}(2,\mathbb{C})$ to $\mathrm{SU}(2)$. By explicit calculations, we show that the LeClair-Neubert model of N quartic self-interacting scalar fermions admits generalized unitary evolution.

Keywords: Space-Time Symmetries, Global Symmetries

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Contents

1 Introduction

In quantum mechanics and quantum field theory, physical operators such as the Hamiltonian, are postulated to be Hermitian because their eigenvalues, which are observables, are real. Transformations generated by Hermitian operators admit unitary evolution of physical systems which preserve transition probabilities. The synthesis of unitarity and Lorentz symmetry inevitably lead to the celebrated spin-statistics theorem [1–3].

The postulates in quantum mechanics and the spin-statistics theorem have played indispensable roles in our ongoing efforts to understand the fundamental constituents of matter. And yet, according to the ΛCDM model and measurements from the Planck Collaboration, elementary particles of the Standard Model (SM) accounts only up to 5% of the total energy and matter contents of the observed universe [4]. Therefore, it may be premature to suppose the present theoretical framework and their theorems are fixed. In fact, the development of physics from the beginning of the twentieth century has taught us that progresses are made by continual modifications and generalizations of the foundational axioms.[1]

In this spirit, the works on $\mathcal{PT}$ symmetric Hamiltonians by Bender and Boettcher [6, 7] and its subsequent generalization to pseudo Hermitian Hamiltonians by Mostafazadeh [8, 9], are of central importance. These works represent an expansive program to extend quantum mechanics and quantum field theory beyond the formalism of Hermitian operators. To be precise, a pseudo Hermitian Hamiltonian is defined by the following equation

$$H^{\#} \equiv \eta^{-1} H^{\dagger} \eta = H, \tag{1.1}$$

where η is an operator to be determined. While definition (1.1) was already known to Pauli [10], its physical implication was only realized after Mostafazadeh proved two important theorems concerning the spectrum of H and the generalized unitary evolution of states when

[1] In essence, we are paraphrasing Dirac's view on the development of physics which remains very much relevant even in the twenty-first century [5].

equipped with the appropriate inner-product. To facilitate ensuing discussions, we now present the two theorems of Mostafazadeh:

> Theorem 1. *Let* $|\alpha\rangle$ *be an eigenstate of* H *with eigenvalue* E_α*. From (1.1), the eigenvalue* E_α *is real when* $\eta_\alpha = \langle\alpha|\eta|\alpha\rangle \neq 0$*.*

> Theorem 2. *The invariant inner-product between two states is* $\langle\beta|\alpha\rangle_\eta \equiv \langle\beta|\eta|\alpha\rangle$*. The time translations are the canonical quantum mechanical transformations,* $|\alpha(t)\rangle = e^{-iHt}|\alpha\rangle$ *and* $\langle\beta(t)| = \langle\beta|e^{iH^\dagger t}$*, so that* $\langle\beta(t)|\alpha(t)\rangle_\eta = \langle\beta|\alpha\rangle_\eta$*.*

Physically well-defined free quantum field theories with pseudo Hermitian adjoints that evade the spin-statistics theorem have been constructed in the spin-zero and spin-half representations [11–15]. The author and collaborators have, from first principle, constructed spin-half bosonic as well as fermionic fields of mass dimension one and three-half [12–14]. Despite the use of pseudo Hermitian adjoints, the free theories are unitary. That is, the free Hamiltonians are Hermitian and positive-definite. In the presence of interactions, the Hamiltonians become pseudo Hermitian so it is necessary to use the η product to preserve time translation symmetry. An important problem for pseudo Hermitian theories is that the η is not positive-definite so further works are needed to establish consistency for the interacting theories. As we will demonstrate in this work, the indefinite product does not pose any difficulty for the model under investigation.

In this paper, we study the theory of symplectic scalar fermion constructed by LeClair and Neubert (LN) [11]. This theory is local, Lorentz-invariant, admits a positive-definite free Hamiltonian but furnishes fermionic statistics. The point of departure from the bosonic scalar field theory comes from the introduction of pseudo Hermitian adjoint. Due to the fermionic statistics, the theory is shown to have global symplectic symmetry and its β function, in the case of quartic self-interaction has non-trivial fix point [11]. The theory has found applications in conformal field theory [15] and dS/CFT correspondence [16–21].

The paper is organized as follows. In section 2, we review the theory of scalar fermion and the properties of pseudo Hermitian Hamiltonians. In section 3, we clarify the derivation of the global symplectic symmetry. In section 4, we show that for the model of N quartic self-interacting scalar fermions, the S-matrix satisfies the generalized unitarity relation.

2 Scalar fermions

Let ϕ be a complex scalar field and $\vec{\phi}$ be its adjoint. LeClair and Neubert made the crucial observation that $\vec{\phi}$ does not have to be the Hermitian conjugate of ϕ. In fact, the following expansions

$$\phi(x) = (2\pi)^{-3/2}\int \frac{d^3p}{\sqrt{2E}}\left[e^{-ip\cdot x}a(\boldsymbol{p}) + e^{ip\cdot x}b^\dagger(\boldsymbol{p})\right], \tag{2.1}$$

$$\vec{\phi}(x) = (2\pi)^{-3/2}\int \frac{d^3p}{\sqrt{2E}}\left[e^{ip\cdot x}a^\dagger(\boldsymbol{p}) - e^{-ip\cdot x}b(\boldsymbol{p})\right], \tag{2.2}$$

are also legitimate. The demand of locality, and the minus sign in (2.2) force ϕ and $\vec{\phi}$ to be fermionic rather than bosonic while respecting Lorentz symmetry. That is, at equal time, they anti-commute with each other $\{\phi(t,\boldsymbol{x}),\bar{\phi}(t,\boldsymbol{y})\}=0$. In fact, they also anti-commute at the same space-time point $\{\phi(x),\bar{\phi}(x)\}=0$. Given an arbitrary Lorentz transformation Λ, they transform as

$$U(\Lambda)\phi(x)U^{-1}(\Lambda)=\phi(\Lambda x), \tag{2.3}$$

$$U(\Lambda)\bar{\phi}(x)U^{-1}(\Lambda)=\bar{\phi}(\Lambda x), \tag{2.4}$$

where $U(\Lambda)$ is the unitary representation of Λ in the Hilbert space. Taking into account of the fermionic statistics, the free propagator and Lagrangian density are given by

$$S(x-y)=\frac{i}{(2\pi)^4}\int d^4q\left[e^{-iq\cdot(x-y)}\frac{1}{q^2-m^2+i\epsilon}\right], \tag{2.5}$$

and

$$\mathscr{L}=\partial^\mu\bar{\phi}\partial_\mu\phi-m^2\bar{\phi}\phi. \tag{2.6}$$

One can readily verify that the fields and their conjugate momenta satisfy the canonical equal-time anti-commutation relations and that the free Hamiltonian is positive-definite after normal-ordering [11]

$$H_0=\int d^3p\sqrt{|\boldsymbol{p}|^2+m^2}\left[a^\dagger(\boldsymbol{p})a(\boldsymbol{p})+b^\dagger(\boldsymbol{p})b(\boldsymbol{p})\right]. \tag{2.7}$$

Because ϕ and $\bar{\phi}$ furnish fermionic statistics, we *cannot* add a Hermitian conjugate term $\mathscr{L}^\dagger$ to (2.6) to make the Lagrangian density Hermitian as it would lead to non-locality and non-unitarity because ϕ does not anti-commute with $\phi^\dagger$. Similarly, given a pseudo Hermitian interacting potential, we cannot add a Hermitian conjugate to it. Therefore, the kinematics and dynamics of scalar fermions can only be described by $\bar{\phi}$ and ϕ. Because $\bar{\phi}\neq\phi^\dagger$, the free Lagrangian density is non-Hermitian. Nevertheless, the free propagator remains the scalar propagator and the free Hamiltonian is Hermitian. Therefore, the states have unitary evolution under H_0 and the fields evolve via

$$\bar{\phi}(t,\boldsymbol{x})=e^{iH_0t}\bar{\phi}(0,\boldsymbol{x})e^{-iH_0t}, \tag{2.8}$$

$$\phi(t,\boldsymbol{x})=e^{iH_0t}\phi(0,\boldsymbol{x})e^{-iH_0t}. \tag{2.9}$$

Similarly, the free momentum operators are also Hermitian so the states and fields are unitary under spatial translation.

But in the interacting picture, the full Hamiltonians that are functions of $\bar{\phi}\phi$ and its generalizations are non-Hermitian. To deal with non-Hermiticity, we first review and elaborate on the observation made by LN. As for how states evolve under such Hamiltonians, we defer the discussions to section 4. LeClair and Neubert noted that while $\bar{\phi}\phi$ is non-Hermitian, it is pseudo Hermitian [11]. There exists a Hermitian operator η such that

$$\left[\bar{\phi}(0,\boldsymbol{x})\phi(0,\boldsymbol{x})\right]^{\#}\equiv\eta^{-1}\left[\bar{\phi}(0,\boldsymbol{x})\phi(0,\boldsymbol{x})\right]^\dagger\eta=\bar{\phi}(0,\boldsymbol{x})\phi(0,\boldsymbol{x}). \tag{2.10}$$

Since $[H_0,\eta]=O$, (2.10) holds at all times under the evolution of H_0. We require η to be Hermitian so that $(\bar{\phi}\phi)^{\#\#}=\bar{\phi}\phi$. To find η, we take $\bar{\phi}$ to be the pseudo Hermitian conjugate of ϕ in the sense that [11]

$$\bar{\phi}(x)=\eta^{-1}\phi^\dagger(x)\eta. \tag{2.11}$$

Expanding (2.11) using (2.1)–(2.2) yields

$$\eta^{-1}a(\boldsymbol{p})\eta = a(\boldsymbol{p}), \quad \eta^{-1}b^\dagger(\boldsymbol{p})\eta = -b^\dagger(\boldsymbol{p}), \tag{2.12}$$

which are equivalent to

$$\eta^\dagger|a(\boldsymbol{p})\rangle = |a(\boldsymbol{p})\rangle, \quad \eta|b(\boldsymbol{p})\rangle = -|b(\boldsymbol{p})\rangle, \tag{2.13}$$

where $a^\dagger|0\rangle = |a\rangle$, $b^\dagger|0\rangle = |b\rangle$ and we have demanded that the vacuum to be invariant under the action of η. We note, if the scalar field is taken to be real, then there is no non-trivial η. This tells us that a consistent theory of scalar fermion *cannot* be formulated in terms of real scalar fields. Doing so inevitably leads to non-locality and non-unitarity.

Equation (2.13) can now be solved by the ansatz $\eta = \exp(i\theta\chi)$ where $\chi \equiv \int d^3p[b^\dagger(\boldsymbol{p})b(\boldsymbol{p})]$ and $\theta \in \mathbb{R}$ is a phase to be determined. Acting η on the particle and anti-particle state using the ansatz, we obtain $\eta^\dagger|a\rangle = |a\rangle$ and $\eta|b\rangle = e^{i\theta}|b\rangle$. Choosing $\theta = -\pi$, we obtain [22]

$$\eta = \exp\left[-i\pi\int d^3p\, b^\dagger(\boldsymbol{p})b(\boldsymbol{p})\right], \tag{2.14}$$

so the operator is unitary. Next, act η on an arbitrary state $|\alpha\rangle$ successively, we obtain

$$\eta^2|\alpha\rangle = e^{-2in_\alpha\pi}|\alpha\rangle = |\alpha\rangle \tag{2.15}$$

where n_α is the number of anti-particle states in $|\alpha\rangle$. Therefore, $\eta^2 = \eta^\dagger\eta = I$ so η is Hermitian. Using (2.12), we find $[H_0, \eta] = O$. Therefore, the free Lagrangian density is pseudo Hermitian

$$\mathscr{L}_0^\# \equiv \eta^{-1}\mathscr{L}_0^\dagger\eta = \mathscr{L}_0. \tag{2.16}$$

Similarly, interactions that are functions of $\vec{\phi}\phi$ and its generalizations are pseudo Hermitian at all times. If we perform transformations on the fields $\phi' = U\phi$, $\vec{\phi}' = \vec{\phi}U^\#$, we find that $\vec{\phi}'\phi'$ remains pseudo Hermitian provided that U satisfies $U^\# U = I$.

2.1 Pseudo Hermitian Hamiltonians

We study the time evolutions of the scalar fermionic fields in the Heisenberg picture to demonstrate that the associated definition of pseudo Hermiticity is consistent with its counterpart in the interacting picture. Pseudo Hermitian Hamiltonian is defined as

$$H^\# \equiv \eta^{-1}H^\dagger\eta = H \tag{2.17}$$

where $[\eta, H] \neq O$ so that $H^\dagger \neq H$. In the Heisenberg picture, this means that η has a time dependence. Taking η to be at time $t = 0$, we obtain

$$\eta_H(T) \equiv e^{iHT}\eta e^{-iHT} \neq \eta. \tag{2.18}$$

If we instead define pseudo Hermiticity using $\eta_H(T)$ where $T \neq 0$, it is equivalent to (2.17) since

$$\eta_H^{-1}(T)H^\dagger\eta_H(T) = \eta^{-1}H^\dagger\eta = H. \tag{2.19}$$

Equations (2.17) and (2.19) show that the time translation on η induces a non-uniqueness in the definition of pseudo Hermiticity. Also, any similarity transformations $\eta \to \eta' = U\eta U^{-1}$ where $[U, H] = O$ satisfies (2.17) and (2.19).

If we only consider the definition of pseudo Hermiticity of the Hamiltonian, then η is unique up to similarity transformation and time translation. However, these considerations do not take into account of the fact that Hamiltonians are constructed from quantum fields. The demand of pseudo Hermiticity on quantum fields imposes additional constraint on η. For scalar fermionic fields, the relation $\bar{\phi} = \eta^{-1}\phi^\dagger\eta$ uniquely fix η to be (2.13) thus removing the non-uniqueness.

Now we consider pseudo Hermiticity in the Heisenberg picture. Let ϕ_H and $\bar{\phi}_H$ be the scalar fermionic fields in the Heisenberg picture. At $t = 0$, we have

$$\phi_H(0, \boldsymbol{x}) = \phi(0, \boldsymbol{x}), \tag{2.20}$$

$$\bar{\phi}_H(0, \boldsymbol{x}) = \bar{\phi}(0, \boldsymbol{x}). \tag{2.21}$$

We take their time evolutions to be

$$\phi_H(x) \equiv e^{iHt}\phi(0, \boldsymbol{x})e^{-iHt}, \tag{2.22}$$

$$\bar{\phi}_H(x) \equiv e^{iHt}\bar{\phi}(0, \boldsymbol{x})e^{-iHt}. \tag{2.23}$$

Because the Hamiltonian is pseudo Hermitian, the demand that both ϕ_H and $\bar{\phi}_H$ have the same time evolution requires further justification. Specifically, we need to demonstrate that no inconsistencies arise from (2.20)–(2.23). Towards this end, we rewrite (2.21) as $\phi^\dagger = \eta\bar{\phi}\eta^{-1}$ and evolve $\phi^\dagger$ to obtain

$$\begin{aligned}\phi_H^\dagger(x) &= e^{iH^\dagger t}\phi^\dagger(0, \boldsymbol{x})e^{-iH^\dagger t} \\ &= e^{iH^\dagger t}\left[\eta\bar{\phi}(0, \boldsymbol{x})\eta^{-1}\right]e^{-iH^\dagger t}.\end{aligned} \tag{2.24}$$

Since $H^\dagger \neq H$, the time evolution of $\phi_H^\dagger$ is different from ϕ_H. Using the identity

$$e^{iH^\dagger t}\eta e^{-iHt} = \eta, \tag{2.25}$$

we obtain

$$e^{iHt}\bar{\phi}(0, \boldsymbol{x})e^{-iHt} = \eta^{-1}\phi_H^\dagger(x)\eta. \tag{2.26}$$

By comparing (2.26) with (2.23), consistency requires $\bar{\phi}_H(x) = \eta^{-1}\phi_H^\dagger(x)\eta$. Therefore, both fields ϕ_H and $\bar{\phi}_H$ must have the same time evolution. Their product transforms as

$$\bar{\phi}_H(x)\phi_H(x) = e^{iHt}\bar{\phi}(0, \boldsymbol{x})\phi(0, \boldsymbol{x})e^{-iHt}. \tag{2.27}$$

Since $\bar{\phi}\phi$ is pseudo Hermitian, using (2.25), we find

$$e^{iHt}\left[\bar{\phi}(0, \boldsymbol{x})\phi(0, \boldsymbol{x})\right]e^{-iHt} = \eta^{-1}\left[\bar{\phi}_H(x)\phi_H(x)\right]^\dagger\eta \tag{2.28}$$

which is equivalent to

$$\bar{\phi}_H(x)\phi_H(x) = \left[\bar{\phi}_H(x)\phi_H(x)\right]^\#. \tag{2.29}$$

The above analysis of pseudo Hermitian conjugation on scalar fermionic fields also apply to general operators. Given an operator A and its pseudo Hermitian conjugate $A^{\#}$ at $t = 0$, they both have the same time evolution in the Heisenberg picture. If an operator is pseudo Hermitian at $t = 0$, then it is pseudo Hermitian at all times.

3 Symplectic symmetry

Apart from the global U(1) symmetry, the Lagrangian density also has a global symplectic symmetry $\mathrm{Sp}(2, \mathbb{C})$ [11]. Since $\{\phi(x), \bar{\phi}(x)\} = 0$, the product $\bar{\phi}(x)\phi(x)$ can be written as

$$\bar{\phi}(x)\phi(x) = \frac{1}{2}\Phi^{\mathrm{T}}(x)\Omega\Phi(x) \tag{3.1}$$

where

$$\Phi(x) = \begin{bmatrix} \bar{\phi}(x) \\ \phi(x) \end{bmatrix}, \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.2}$$

Therefore, the Lagrangian density is invariant under the symplectic transformations[2]

$$\Phi(x) \to M\Phi(x), \quad \Phi^{\mathrm{T}}(x) \to \Phi^{\mathrm{T}}(x)M^{\mathrm{T}} \tag{3.3}$$

where M is a 2×2 complex matrix satisfying

$$M^{\mathrm{T}}\Omega M = \Omega \tag{3.4}$$

with M^{T} being the transposition of M. Equation (3.4) is satisfied for all complex matrices of unit determinant. Therefore, they are continuous transformations and can be generated via $M = e^X$. Expand M near the identity

$$\left[I + X^{\mathrm{T}} + O({X^{\mathrm{T}}}^2)\right]\Omega\left[I + X + O(X^2)\right] = \Omega, \tag{3.5}$$

we obtain

$$X^{\mathrm{T}}\Omega = -\Omega X. \tag{3.6}$$

Solving (3.6), we find the general solution

$$X = \begin{pmatrix} \theta_1 + i\theta_2 & \theta_3 + i\theta_4 \\ \theta_5 + i\theta_6 & -\theta_1 - i\theta_2 \end{pmatrix}, \quad \theta_i \in \mathbb{R} \tag{3.7}$$

Substituting (3.7) into M, the generators are given by

$$X_i = \left.\frac{dM}{d\theta_i}\right|_{\theta_i = 0} \tag{3.8}$$

from which we obtain

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.9}$$

$$X_4 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}. \tag{3.10}$$

[2]If we perform the same analysis for complex bosonic scalar field, we would have obtained the conserved currents and charges associated with the U(1) symmetry.

JHEP05(2024)181

Under the infinitesimal transformation

$$\Phi_n \to \Phi_n + \theta_i (X_i)_{nm} \Phi_m, \tag{3.11}$$

the conserved currents are given by

$$J_i^\mu = N\left[\frac{\partial \mathscr{L}_0}{\partial(\partial_\mu \Phi_n)}(X_i)_{nm}\Phi_m\right]. \tag{3.12}$$

Due to fermionic statistics, there is an ambiguity in the ordering of $\bar{\phi}$ and ϕ. To deal with this issue, we introduce the operation N to order $\bar{\phi}$ to the left of ϕ. The results are

$$J_1^\mu = \bar{\phi}(\partial^\mu \phi) - (\partial^\mu \bar{\phi})\phi, \tag{3.13}$$
$$J_2^\mu = i\left[\bar{\phi}(\partial^\mu \phi) - (\partial^\mu \bar{\phi})\phi\right], \tag{3.14}$$
$$J_3^\mu = (\partial^\mu \phi)\phi = -\phi(\partial^\mu \phi), \tag{3.15}$$
$$J_4^\mu = i(\partial^\mu \phi)\phi, \tag{3.16}$$
$$J_5^\mu = (\partial^\mu \bar{\phi})\bar{\phi} = -\bar{\phi}(\partial^\mu \bar{\phi}), \tag{3.17}$$
$$J_6^\mu = i(\partial^\mu \bar{\phi})\bar{\phi} = -i\bar{\phi}(\partial^\mu \bar{\phi}). \tag{3.18}$$

To couple the currents to gauge fields, they must be pseudo Hermitian. Given an arbitrary operator $\mathcal{O}$, it can be made pseudo Hermitian by the linear combination $e^{i\vartheta}\mathcal{O} + e^{-i\vartheta}\mathcal{O}^\#$ where $\vartheta \in \mathbb{R}$. Applying this procedure to the above currents, we find

$$J_1^\mu \to 2i\sin\vartheta_1\left[\bar{\phi}(\partial^\mu \phi) - (\partial^\mu \bar{\phi})\phi\right], \tag{3.19}$$
$$J_2^\mu \to 2i\cos\vartheta_2\left[\bar{\phi}(\partial^\mu \phi) - (\partial^\mu \bar{\phi})\phi\right], \tag{3.20}$$
$$J_3^\mu \to e^{i\vartheta_3}(\partial^\mu \phi)\phi + e^{-i\vartheta_3}\bar{\phi}(\partial^\mu \bar{\phi}), \tag{3.21}$$
$$J_4^\mu \to e^{i\vartheta_4}i(\partial^\mu \phi)\phi - e^{-i\vartheta_4}i\bar{\phi}(\partial^\mu \bar{\phi}), \tag{3.22}$$
$$J_5^\mu \to e^{i\vartheta_5}(\partial^\mu \bar{\phi})\bar{\phi} + e^{-i\vartheta_5}\phi(\partial^\mu \phi), \tag{3.23}$$
$$J_6^\mu \to e^{i\vartheta_6}i(\partial^\mu \bar{\phi})\bar{\phi} - e^{-i\vartheta_6}i\phi(\partial^\mu \phi), \tag{3.24}$$

for $\vartheta_i \in \mathbb{R}$. We find the following currents, namely, J_1^μ, J_3^μ, and J_4^μ to be linearly-dependent on J_2^μ, J_5^μ, and J_6^μ respectively. Therefore, there are three linearly-independent currents

$$K_1^\mu = J_3^\mu = i\sin\vartheta_1\left[\bar{\phi}(\partial^\mu \phi) - (\partial^\mu \bar{\phi})\phi\right], \tag{3.25}$$
$$K_2^\mu = e^{i\vartheta_3}(\partial^\mu \phi)\phi + e^{-i\vartheta_3}\bar{\phi}(\partial^\mu \bar{\phi}), \tag{3.26}$$
$$K_3^\mu = e^{i\vartheta_4}i(\partial^\mu \phi)\phi - e^{-i\vartheta_4}i\bar{\phi}(\partial^\mu \bar{\phi}). \tag{3.27}$$

The corresponding generators are

$$Y_1 = \sin\vartheta_1\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & e^{i\vartheta_3} \\ -e^{-i\vartheta_3} & 0\end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & ie^{i\vartheta_4} \\ ie^{-i\vartheta_4} & 0\end{pmatrix}. \tag{3.28}$$

Choosing the phases to be

$$\sin\vartheta_1 = 1, \quad e^{i\vartheta_3} = e^{i\vartheta_4} = e^{i\vartheta}, \tag{3.29}$$

the resulting generators satisfy the su(2) algebra

$$[Y_i, Y_j] = 2\epsilon_{ijk} Y_k. \tag{3.30}$$

The currents become

$$K_1^\mu = i\left[\bar{\phi}(\partial^\mu \phi) - (\partial^\mu \bar{\phi})\phi\right], \tag{3.31}$$

$$K_2^\mu = e^{i\vartheta}(\partial^\mu \phi)\phi + e^{-i\vartheta}\bar{\phi}(\partial^\mu \bar{\phi}), \tag{3.32}$$

$$K_3^\mu = i\left[e^{i\vartheta}(\partial^\mu \phi)\phi - e^{-i\vartheta}\bar{\phi}(\partial^\mu \bar{\phi})\right], \tag{3.33}$$

and the conserved charges are given by

$$Q_1 = i\int d^3x \left[\bar{\phi}(\partial_t \phi) - (\partial_t \bar{\phi})\phi\right], \tag{3.34}$$

$$Q_2 = \int d^3x \left[e^{i\vartheta}(\partial_t \phi)\phi + e^{-i\vartheta}\bar{\phi}\partial_t \bar{\phi}\right], \tag{3.35}$$

$$Q_3 = i\int d^3x \left[e^{i\vartheta}(\partial_t \phi)\phi - e^{-i\vartheta}\bar{\phi}(\partial_t \bar{\phi})\right]. \tag{3.36}$$

Substituting (2.1)–(2.2) into (3.34)–(3.36), we obtain the normal-ordered charges

$$Q_1 = \int d^3p \left[a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) - b^\dagger(\boldsymbol{p})b(\boldsymbol{p})\right], \tag{3.37}$$

$$Q_2 = i\int d^3p \left[e^{-i\vartheta}a^\dagger(\boldsymbol{p})b(\boldsymbol{p}) + e^{i\vartheta}b^\dagger(\boldsymbol{p})a(\boldsymbol{p})\right], \tag{3.38}$$

$$Q_3 = \int d^3p \left[e^{-i\vartheta}a^\dagger(\boldsymbol{p})b(\boldsymbol{p}) - e^{i\vartheta}b^\dagger(\boldsymbol{p})a(\boldsymbol{p})\right]. \tag{3.39}$$

All three charges are pseudo Hermitian. Additionally, Q_1 and $iQ_{2,3}$ are Hermitian, satisfying the su(2) algebra

$$[Q_1, (iQ_2)] = 2i(iQ_3), \tag{3.40}$$

$$[(iQ_2), (iQ_3)] = 2iQ_1, \tag{3.41}$$

$$[Q_1, (iQ_3)] = -2i(iQ_2). \tag{3.42}$$

After normal ordering Q_1, it defines the charges of the particle and anti-particle, namely,

$$Q_1|a\rangle = +|a\rangle, \quad Q_1|b\rangle = -|b\rangle. \tag{3.43}$$

For $Q_{2,3}$, they map the particle state to the anti-particle state and vice versa[3]

$$Q_2|a\rangle = ie^{i\vartheta}|b\rangle, \quad Q_2|b\rangle = ie^{-i\vartheta}|a\rangle, \tag{3.44}$$

$$Q_3|a\rangle = -e^{i\vartheta}|b\rangle, \quad Q_3|b\rangle = e^{-i\vartheta}|a\rangle. \tag{3.45}$$

[3] LeClair and Neubert asserts that due to the relation $\mathrm{Sp}(2) \cong \mathrm{SO}(3)$, the states are of spin-half. We believe their interpretation to be incorrect. This is because the spin of states are determined by their eigenvalues with respect to one of the Casimir invariant operators of the Poincaré group, namely $s = -m^2 j(j+1)$ with $j = 0, \frac{1}{2}, \cdots$. Since the scalar fermionic fields are constructed from the scalar representation of the Poincaré group, we have $j = 0$ and hence $s = 0$.

JHEP05(2024)181

By demanding the currents to be pseudo Hermitian, the global $\mathrm{Sp}(2, \mathbf{C})$ symmetry becomes SU(2). Therefore, the Lagrangian density, including interacting potentials constructed from $\bar{\phi}$ and ϕ has a global $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry. Because gauge group is semi-simple and compact, by treating $\bar{\phi}$ and ϕ as doublet (3.2), we can couple them to non-Abelian gauge fields that resemble the electroweak sector of the SM. However, we should also note that the doublet structure presented here is fundamentally different from the SM fermionic doublet because it is bosonic and contains only of one specie of particle. It would be interesting to see if the symmetry is preserved in presence of quantum corrections. We leave this task for future investigations.

4 Generalized unitary evolution

Interacting Hamiltonians constructed as functions of $\bar{\phi}$ and ϕ are pseudo Hermitian and hence complex. One may therefore suspect the resulting S-matrix to be non-unitary. This concern was partially addressed in [13]. There, a formalism to compute scattering amplitudes with pseudo Hermitian Hamiltonians and the definition of generalized unitarity relation were proposed. Here we review the formalism and discuss problems to be addressed.

Let $|\alpha\rangle$ and $\langle\beta|$ to be states that evolve under the pseudo Hermitian Hamiltonian

$$|\alpha(t)\rangle = e^{-iHt}|\alpha\rangle, \quad \langle\beta(t)| = \langle\alpha|e^{iH^\dagger t}. \tag{4.1}$$

To preserve time translation symmetry, we use the η-product [8]

$$\langle\beta|\alpha\rangle_\eta \equiv \langle\beta|\eta|\alpha\rangle. \tag{4.2}$$

Using (2.25), we find $\langle\beta(t)|\alpha(t)\rangle_\eta = \langle\beta|\alpha\rangle_\eta$. The matrix element of operator must now be defined as

$$A^{(\eta)}_{\beta\alpha} \equiv \langle\beta|\eta A|\alpha\rangle \tag{4.3}$$

so that it is invariant under time translation. To see this, we take $U(t) \equiv e^{iHt}$ and find

$$\begin{aligned} A^{(\eta)}_{\beta\alpha} &= \langle\beta|U^\dagger(t)[U^{\dagger-1}(t)\eta U^{-1}(t)][U(t)AU^{-1}(t)]U(t)|\alpha\rangle \\ &= \langle\beta(t)|\eta A_H(t)|\alpha(t)\rangle \\ &= \left[A^{(\eta)}_H(t)\right]_{\beta\alpha}. \end{aligned} \tag{4.4}$$

From (4.3), we define the expectation value to be $\langle A\rangle_\eta \equiv A^{(\eta)}_{\alpha\alpha}$. When A is pseudo Hermitian, the expectation value is real.

In the scattering process $\alpha \to \beta$ described by pseudo Hermitian Hamiltonian, the S-matrix is given by

$$S_{\beta\alpha} \equiv \langle\beta_+|\alpha_-\rangle_\eta. \tag{4.5}$$

where $|\alpha_-\rangle$ and $|\beta_+\rangle$ are the 'in' and 'out' states. They are related to the free states by

$$|\alpha_-\rangle = \Omega_-|\alpha_0\rangle, \tag{4.6}$$

$$|\beta_+\rangle = \Omega_+|\beta_0\rangle, \tag{4.7}$$

where

$$\Omega(\tau) = e^{iH\tau}e^{-iH_0\tau}, \quad \Omega_\pm = \lim_{\tau\to\pm\infty}\Omega(\tau). \tag{4.8}$$

Using $\Omega_\pm$, we write (4.5) as

$$\begin{aligned} S_{\beta\alpha} &= \langle\beta_0|\Omega_+^\dagger\eta\Omega_-|\alpha_0\rangle \\ &\equiv \langle\beta_0|S|\alpha_0\rangle \end{aligned} \tag{4.9}$$

where $S = \Omega_+^\dagger\eta\Omega_-$. Because H is pseudo Hermitian, both Ω and the S-matrix are *non-unitary*. Instead, their inverses are obtained via pseudo Hermitian conjugation

$$\Omega^{-1} = \Omega^{\#}, \quad S^{-1} = S^{\#}. \tag{4.10}$$

The matrix component of S^{-1} is given by

$$\begin{aligned} S^{-1}_{\gamma\beta} = S^{\#}_{\gamma\beta} &\equiv \langle\gamma_0|S^{\#}|\beta_0\rangle \\ &= \langle\gamma_0|\eta^{-1}S^\dagger\eta|\beta_0\rangle \\ &= \langle\gamma_0|\Omega_-^{-1}\Omega_+\eta^{-1}|\beta_0\rangle. \end{aligned} \tag{4.11}$$

In obtaining (4.11), we have used $\eta^\dagger = \eta$ and $\eta = \eta^{-1}$. Now, we recall that the free states evolve under the free Hamiltonian which is Hermitian. These states admit a Hermitian inner-product that is positive-definite and invariant under time translation

$$\langle\beta_0|\alpha_0\rangle = \delta(\beta - \alpha). \tag{4.12}$$

Therefore, they satisfy the completeness relation the completeness relation

$$\int d\beta|\beta_0\rangle\langle\beta_0| = I \tag{4.13}$$

and the S-matrix satisfies the identity

$$\int d\beta S^{\#}_{\gamma\beta}S_{\beta\alpha} = \delta(\gamma - \alpha). \tag{4.14}$$

This is the *generalized unitarity relation*. For any S-matrices defined in terms of the η-product with pseudo Hermitian Hamiltonians, they will satisfy (4.14).

The completeness relation for the free states holds at all times because the free Hamiltonian is Hermitian so that $|\alpha_0, t\rangle\langle\alpha_0, t| = |\alpha_0\rangle\langle\alpha_0|$. As for the in and out states that evolve under pseudo Hermitian Hamiltonians, their completeness relation cannot take the form of (4.13) since $|\alpha_\pm, t\rangle\langle\alpha_\pm, t| \neq |\alpha_\pm\rangle\langle\alpha_\pm|$. To derive the completeness relation for the in and out states, we use the fact that their η-product is invariant under time translation and that

$$\langle\beta_\pm|\alpha_\pm\rangle_\eta = \langle\beta_0|\alpha_0\rangle_\eta. \tag{4.15}$$

Using (2.14), we find $\eta|\alpha_0\rangle = (-1)^{n_\alpha}|\alpha_0\rangle$ where n_α is the total number of anti-particles contained in $|\alpha_0\rangle$. Therefore,

$$\langle\beta_\pm|\alpha_\pm\rangle_\eta = (-1)^{n_\alpha}\delta(\beta - \alpha). \tag{4.16}$$

JHEP05(2024)181

The completeness relation then reads

$$\int d\beta (-1)^{n_\beta} |\beta_\pm\rangle\langle\beta_\pm|\eta = I, \tag{4.17}$$

where

$$\int d\beta (-1)^{n_\beta} |\beta_\pm\rangle\langle\beta_\pm|\eta|\alpha_\pm\rangle = |\alpha_\pm\rangle, \tag{4.18}$$

and

$$\int d\beta (-1)^{n_\beta} \langle\alpha_\pm|\eta|\beta_\pm\rangle\langle\beta_\pm|\eta = \langle\alpha_\pm|\eta. \tag{4.19}$$

The completeness relation (4.17) holds at all times since $|\alpha_\pm, t\rangle\langle\alpha_\pm, t|\eta = |\alpha_\pm\rangle\langle\alpha_\pm|\eta$. The fact that the η product is not positive-definite has important implications for pseudo Hermitian theories to be discussed below and in section 5.

The S-matrix and its inverse admit the following expansions [13]

$$S_{\beta\alpha} = \eta_{\beta\alpha} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \langle\beta_0|\eta T\left[\mathscr{H}(x_1)\cdots\mathscr{H}(x_n)\right]|\alpha_0\rangle, \tag{4.20}$$

$$S^{\#}_{\gamma\beta} = \eta_{\gamma\beta} + \sum_{n=1}^{\infty} \frac{(+i)^n}{n!} \int d^4x_1 \cdots d^4x_n \langle\gamma_0|\eta T\left[\mathscr{H}^\dagger(x_n)\cdots\mathscr{H}^\dagger(x_1)\right]|\beta_0\rangle, \tag{4.21}$$

where $\eta_{\beta\alpha} = \langle\beta_0|\eta|\alpha_0\rangle$. Normalizing the S-matrix as

$$S_{\beta\alpha} \equiv \eta_{\beta\alpha} - 2\pi i M_{\beta\alpha}\delta^4(p_\beta - p_\alpha), \tag{4.22}$$

$$S^{\#}_{\gamma\beta} \equiv \eta_{\gamma\beta} + 2\pi i M^{\#}_{\gamma\beta}\delta^4(p_\gamma - p_\beta), \tag{4.23}$$

where $M^{\#}_{\gamma\beta} = \langle\gamma_0|\eta^{-1}M^\dagger\eta|\beta_0\rangle$, we obtain the generalized optical theorem

$$\begin{aligned} &i\int d\beta \left[\eta_{\gamma\beta}M_{\beta\alpha}\delta^4(p_\beta - p_\alpha) - M^{\#}_{\gamma\beta}\eta_{\beta\alpha}\delta^4(p_\gamma - p_\beta)\right] \\ &= 2\pi \int d\beta \left[\delta^4(p_\beta - p_\gamma)\delta^4(p_\beta - p_\alpha)M^{\#}_{\gamma\beta}M_{\beta\alpha}\right]. \end{aligned} \tag{4.24}$$

Setting $\gamma = \alpha$ yields

$$i\int d\beta \left(\eta_{\alpha\beta}M_{\beta\alpha} - M^{\#}_{\alpha\beta}\eta_{\beta\alpha}\right) = 2\pi \int d\beta \left[\delta^4(p_\beta - p_\alpha)M^{\#}_{\alpha\beta}M_{\beta\alpha}\right]. \tag{4.25}$$

The generalized unitarity relation with pseudo Hermitian Hamiltonians is a generalization to unitary quantum mechanics with Hermitian Hamiltonians. For the generalization to be consistent, there must be a prescription to compute transition probabilities. Two criteria are required to ensure consistency. Firstly, the transition probability $P(\alpha \to \beta)$ for any process $\alpha \to \beta$ must be positive-definite. Secondly, we need $\sum_\beta P(\alpha \to \beta) = 1$. When the S-matrix is unitary, both criteria are equivalent and are trivially satisfied.

The important question is: *How do we compute transition probabilities for pseudo Hermitian theories?* The generalized unitarity relation of the S-matrix suggests that the transition probability for $\alpha \to \beta$ ought to be proportional to $M^{\#}_{\alpha\beta}M_{\beta\alpha}$ but this quantity is

JHEP05(2024)181

not positive-definite. So instead, it was proposed in [13] that we multiply it by a phase $\wp_{\beta\alpha}$ to obtain $\wp_{\beta\alpha} M^{\#}_{\alpha\beta} M_{\beta\alpha} \geq 0$ and interpret this quantity to be proportional to the transition probability. However, this proposal is unsatisfactory for the following reason. If we replace the term $M^{\#}_{\alpha\beta} M_{\beta\alpha}$ in (4.25) by $\wp_{\beta\alpha} M^{\#}_{\alpha\beta} M_{\beta\alpha}$ for processes where $\wp_{\beta\alpha} \neq 1$, the generalized optical theorem is no longer satisfied. What this means is that had we adopted this prescription, then we would end up with $\sum_\beta P(\alpha \to \beta) \neq 1$ which is unacceptable for any physical theories.

Defining the correct transition probabilities is an important problem to be addressed. But fortunately for us, in the LN model which involves quartic self-interacting scalar fermions to be studied, the observed difficulty does not arise. Because even though the interaction of the LN model is pseudo Hermitian, the generalized unitarity relation reduces to the unitarity relation. To see how it occurs, let us examine $M^{\#}_{\alpha\beta}$. Using the definition of pseudo Hermitian conjugation and the solution of η given in (2.14), we find

$$M^{\#}_{\alpha\beta} = \langle \alpha_0 | \eta^{-1} M^\dagger \eta | \beta_0 \rangle = e^{-i\pi(n_\alpha - n_\beta)} M^\dagger_{\alpha\beta} \tag{4.26}$$

where n_α and n_β are the number of anti-particles in states $|\alpha\rangle$ and $|\beta\rangle$ respectively. When $n_\alpha = n_\beta$ (or up to a integer multiples of 2π), we have $M^{\#}_{\alpha\beta} = M^\dagger_{\alpha\beta}$. This is precisely what happens in the LN model. Using this formalism, we show that the LN-model [11] admits (generalized) unitary evolution up to one-loop in perturbation theory.

The LN-model. We now consider the LN-model [11]

$$\mathscr{L} = \sum_{i=1}^{N} \left(\partial^\mu \bar{\phi}_i \partial_\mu \phi_i - m^2 \bar{\phi}_i \phi_i \right) - \frac{g}{2} \left(\sum_{i=1}^{N} \bar{\phi}_i \phi_i \right)^2 \tag{4.27}$$

where all the fields have equal mass and anti-commute with each other

$$\{\phi_i(x), \phi_j(x)\} = \left\{\phi_i(x), \bar{\phi}_j(x)\right\} = \left\{\bar{\phi}_i(x), \bar{\phi}_j(x)\right\} = 0. \tag{4.28}$$

Due to fermionic statistics, $\bar{\phi}_i^2(x) = \phi_i^2(x) = 0$. The interacting density simplifies to[4]

$$\mathscr{H} = g \sum_{i<j}^{N} \left[\bar{\phi}_i(x) \phi_i(x) \bar{\phi}_j(x) \phi_j(x) \right]. \tag{4.29}$$

The model is pseudo Hermitian because there exists an η given by

$$\eta = \prod_{i=1}^{N} \eta_i, \quad \eta_i = \exp\left[-i\pi \int d^3p\, b_i^\dagger(\boldsymbol{p}) b_i(\boldsymbol{p}) \right], \tag{4.30}$$

such that $\mathscr{L}^{\#} = \mathscr{L}$, $\mathscr{H}^{\#} = \mathscr{H}$. From (4.29), the following two-body scattering processes are allowed

$$ij \to i'j', \tag{4.31}$$
$$i\bar{j} \to i'\bar{j}', \tag{4.32}$$
$$i\bar{i} \to j'\bar{j}', \quad \text{for all } j \neq i, \tag{4.33}$$

[4]We can expand the interaction $\int d^3x \mathscr{H}$ to show that it is non-Hermitian.

where i, j and $\bar{i}, \bar{j}$ denote particle and anti-particle states created by the ith and jth fields respectively. We will now verify that the S-matrix for these processes satisfy the generalized unitarity relation up to one-loop.

For all three processes, as there are equal number of particles and anti-particles in the initial and final states, we have $M^{\#}_{\alpha\beta} = M^{\dagger}_{\alpha\beta}$. Therefore, the optical theorem (4.25) can be rewritten in terms of two-body cross sections

$$i \int d\beta \left(\eta_{\alpha\beta} M_{\beta\alpha} - M^{\dagger}_{\alpha\beta}\eta_{\beta\alpha}\right) = \frac{u_\alpha}{8\pi^3}\sigma_\alpha \tag{4.34}$$

where

$$\sigma_\alpha = \frac{(2\pi)^4}{u_\alpha} \int d\beta \left[\delta^4(p_\beta - p_\alpha)|M_{\beta\alpha}|^2\right] \tag{4.35}$$

and

$$u_\alpha = \frac{\sqrt{(p_1 \cdot p_2)^2 - m^4}}{E_1 E_2}. \tag{4.36}$$

In the center of mass frame $p_1 = (E, \boldsymbol{p})$ and $p_2 = (E, -\boldsymbol{p})$, so that $u_\alpha = 4|\boldsymbol{p}|/E_{\rm CM}$ with $E_{\rm CM} = 2E$. The cross-section simplifies to ([3], section 3.4)

$$\sigma_\alpha = \frac{(2\pi)^4 E^2_{\rm CM}}{16} \int d\Omega |M_{\beta\alpha}|^2 \tag{4.37}$$

where $d\Omega$ is the differential solid angle of the final particle states. For these processes, the matrix elements of $\eta_{\beta\alpha}$ are given by

$$\eta_{(i'j')(ij)} = \eta_{(ij)(i'j')} = +\delta[(i'j') - (ij)], \tag{4.38}$$

$$\eta_{(i'\bar{j}')(i\bar{j})} = \eta_{(ij)(i'\bar{j}')} = -\delta[(i'\bar{j}') - (i\bar{j})], \tag{4.39}$$

$$\eta_{(j'\bar{j}')(i\bar{i})} = \eta_{(i\bar{i})(j'\bar{j}')} = -\delta[(j'\bar{j}') - (i\bar{i})], \tag{4.40}$$

so we obtain

$$\mathrm{Im}\left[M_{(ij)(ij)}\right] = -\frac{1}{8\pi^3}\left[1 - \frac{4m^2}{E^2_{\rm CM}}\right]^{1/2} \sigma_{ij}, \tag{4.41}$$

$$\mathrm{Im}\left[M_{(i\bar{j})(i\bar{j})}\right] = +\frac{1}{8\pi^3}\left[1 - \frac{4m^2}{E^2_{\rm CM}}\right]^{1/2} \sigma_{i\bar{j}}, \tag{4.42}$$

$$\mathrm{Im}\left[M_{(i\bar{i})(i\bar{i})}\right] = +\frac{1}{8\pi^3}\left[1 - \frac{4m^2}{E^2_{\rm CM}}\right]^{1/2} \sigma_{i\bar{i}}, \tag{4.43}$$

where we have used $|\boldsymbol{p}| = \left(1 - 4m^2/E^2_{\rm CM}\right)^{1/2}$. At tree-level, the cross-sections are given by

$$\sigma_{ij} = \sigma_{i\bar{j}} = \frac{g^2}{16\pi E^2_{\rm CM}}, \tag{4.44}$$

$$\sigma_{i\bar{i}} = \frac{g^2}{16\pi E^2_{\rm CM}}(N-1). \tag{4.45}$$

Substituting the cross-sections into the right-hand side of (4.41)–(4.42) and compare them with the imaginary part of the amplitudes given by (A.15)–(A.17), we find that the generalized optical theorem is satisfied.

JHEP05(2024)181

5 Conclusions

According to the conventional spin-statistics theorem, scalar fields must furnish bosonic statistics. When phrased as a no-go theorem, it means that anti-commuting scalar fields violate locality and unitarity. The theory of scalar fermion constructed by LN showed that once the Hamiltonian is allowed to be pseudo Hermitian, the no-go theorem no longer applies. We believe this construct to be of fundamental importance because it represents an extension to the spin-statistics theorem. We have shown in this paper, not only does the theory respect Lorentz symmetry, it also admit generalized unitary evolution. It would be satisfying, if scalar fermion is to find applications in elementary particles physics. We shall now list, what is in our opinion, topics worthy of future investigations.

The LN model is a special case where $M^{\#}_{\alpha\beta}M_{\beta\alpha}$ is positive-definite and generalized unitarity redcues to unitarity. However, this is in general not true. For instance, let us consider the interaction $g\overline{\phi}\phi\varphi^2$ where φ is a real scalar field. At tree-level, for the process $\varphi\varphi \to \overline{\phi}\phi$, we find $M^{\#}_{(\overline{\phi}\phi)(\varphi\varphi)}M_{(\overline{\phi}\phi)(\varphi\varphi)} < 0$. This issue can be traced to the fact that the η-norm is not positive-definite. Theories with indefinite norms are usually discarded as being pathological. Nevertheless, since the LN model is consistent, we believe that there is a consistent formalism to compute transition probabilities and physical observables for pseudo Hermitian Hamiltonians circumventing the problem. We leave this important task for future investigations.

An interesting feature of scalar fermion is that when the currents are pseudo Hermitian, the Lagrangian density has a global $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry. This allows us to investigate their interactions with non-Abelian gauge fields having similar structure to the electroweak sector of the SM. Of course, this similarity may be purely coincidental as the bosonic doublet considered here is fundamentally different from the lepton doublet in the SM and cannot be associated with the notion of weak isopin and hypercharge. Regardless of its relevance to the SM, whether the theory is free of quantum anomalies should to be ascertained. If there are anomalies, one can attempt to look for possible ways to cancel them. Should anomalies be absent or can be cancelled, then in addition to the $\mathrm{U}(1)$ charge, it would be interesting to study how the symplectic charges contribute to the partition function at finite-temperature.

Besides possible gauge and gravitational interactions with the SM particles, interactions between the SM fermions and the scalar fermions are limited. As the scalar fermionic fields are complex, direct interactions with Dirac fermions would be of the form $\overline{\psi}\psi\overline{\phi}\phi$ which has mass dimension five and is therefore suppressed. If we consider the LN model as an extension to the SM, it would potentially be a model of self-interacting fermionic dark matter of spin-zero.

Concerning the quartic self-interaction involving N scalar fermions, this feature is reminiscent of the Gross-Neveu [23] and the Nambu-Jona Lasinio model [24]. Except here, the scalar fermions have renormalizable quartic self-interaction in four space-time dimensions. It would be interesting to study the massless LN model in the large N limit and investigate the possibility of dynamical mass generation.

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A Amplitudes

We compute the one-loop amplitudes and their imaginary part that are relevant for verifying the generalized optical theorem in section 4. Because of the fermionic statistics and the non-trivial adjoint, caution must be exercised when contracting the fields among themselves and with the states. The amplitudes are given by

$$\begin{aligned}
S(ij\to i'j') &= \frac{(-ig)^2}{2}\int d^4x d^4y \langle j'i'|\eta T\left[(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_x(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_y\right]|ij\rangle\\
&= \frac{(-ig)^2}{2}\int d^4x d^4y \langle j'i'|T\left[(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_x(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_y\right]|ij\rangle\\
&= -\frac{(-ig)^2}{2}\int d^4x d^4y \langle j'i'|T\left[(\overset{\neg}{\phi}_i\overset{\neg}{\phi}_j\phi_i\phi_j)_x(\overset{\neg}{\phi}_i\overset{\neg}{\phi}_j\phi_j\phi_i)_y\right]|ij\rangle\\
&= -(-ig)^2\int d^4x d^4y \langle j'i'|(\overset{\neg}{\phi}_i\overset{\neg}{\phi}_j\phi_i\phi_j)_x(\overset{\neg}{\phi}_i\overset{\neg}{\phi}_j\phi_j\phi_i)_y|ij\rangle+\cdots\\
&= (-ig)^2\int d^4x d^4y \langle j'i'|(\overset{\neg}{\phi}_i\overset{\neg}{\phi}_j)_x S_i(x-y)S_j(x-y)(\phi_j\phi_i)_y|ij\rangle+\cdots
\end{aligned}\tag{A.1}$$

$$\begin{aligned}
S(i\bar{i}\to i'\bar{i}') &= \sum_{j\neq i}^{N}\frac{(-ig)^2}{2!}\int d^4x d^4y \langle \bar{i}'i'|\eta T\left[(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_x(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_y\right]|i\bar{i}\rangle\\
&= -\sum_{j\neq i}^{N}\frac{(-ig)^2}{2!}\int d^4x d^4y \langle \bar{i}'i'|T\left[(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_x(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_y\right]|i\bar{i}\rangle\\
&= -\sum_{j\neq i}^{N}\frac{(-ig)^2}{2!}\int d^4x d^4y \langle \bar{i}'i'|T\left[(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_x(\overset{\neg}{\phi}_j\phi_j\overset{\neg}{\phi}_i\phi_i)_y\right]|i\bar{i}\rangle\\
&= -(-ig)^2\int d^4x d^4y \sum_{j\neq i}^{N}\langle \bar{i}'i'|(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_x(\overset{\neg}{\phi}_j\phi_j\overset{\neg}{\phi}_i\phi_i)_y|i\bar{i}\rangle+\cdots\\
&= (-ig)^2\int d^4x d^4y \sum_{j\neq i}^{N}\langle \bar{i}'i'|(\overset{\neg}{\phi}_i\phi_i S_j(x-y)S_j(y-x)\overset{\neg}{\phi}_i\phi_i)_y|i\bar{i}\rangle+\cdots
\end{aligned}\tag{A.2}$$

$$\begin{aligned}
S(i\bar{j}\to i'\bar{j}') &= \frac{(-ig)^2}{2!}\int d^4x d^4y \langle \bar{j}'i'|\eta T\left[(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_x(\overset{\neg}{\phi}_i\phi_i\overset{\neg}{\phi}_j\phi_j)_y\right]|i\bar{j}\rangle\\
&= -\frac{(-ig)^2}{2!}\int d^4x d^4y \langle \bar{j}'i'|T\left[(\overset{\neg}{\phi}_i\phi_j\phi_i\overset{\neg}{\phi}_j)_x(\overset{\neg}{\phi}_i\phi_j\phi_i\overset{\neg}{\phi}_j)_y\right]|i\bar{j}\rangle\\
&= \frac{(-ig)^2}{2!}\int d^4x d^4y \langle \bar{j}'i'|T\left[(\overset{\neg}{\phi}_i\phi_j\phi_i\overset{\neg}{\phi}_j)_x(\overset{\neg}{\phi}_i\phi_j\overset{\neg}{\phi}_j\phi_i)_y\right]|i\bar{j}\rangle
\end{aligned}$$

JHEP05(2024)181

$$= (-ig)^2 \int d^4x d^4y \langle \bar{j}' i' | (\bar{\phi}_i \phi_j \phi_i \bar{\phi}_j)_x (\bar{\phi}_i \phi_j \bar{\phi}_j \phi_i)_y | i\bar{j} \rangle + \cdots$$

$$= (-ig)^2 \int d^4x d^4y \langle \bar{j}' i' | (\bar{\phi}_i \phi_j S_i(x-y) S_j(y-x) \bar{\phi}_j \phi_i)_y | i\bar{j} \rangle + \cdots \tag{A.3}$$

For the above expressions, we have made explicit the contributions from the s-channel as they are the only terms that have non-vanishing imaginary parts. Since all the particles have equal masses, in the center of mass frame, using the expansions of the field and its adjoint given by (2.1)–(2.2), the contractions of the fields to the initial and final particle states are given by

$$(\phi_j \phi_i)_y | ij \rangle = +e^{-i(p_i+p_j)\cdot y} \left[(2\pi)^3 E_{\mathrm{CM}} \right]^{-1}, \tag{A.4}$$

$$\langle j' i' | (\bar{\phi}_j \bar{\phi}_i)_x = +e^{i(p_{i'}+p_{j'})\cdot x} \left[(2\pi)^3 E_{\mathrm{CM}} \right]^{-1}, \tag{A.5}$$

$$(\bar{\phi}_i \phi_i)_y | i\bar{i} \rangle = -e^{-i(p_i+p_{\bar{i}})\cdot y} \left[(2\pi)^3 E_{\mathrm{CM}} \right]^{-1}, \tag{A.6}$$

$$\langle \bar{i}' i' | (\bar{\phi}_i \phi_i)_x = +e^{i(p_{i'}+p_{\bar{i}'})\cdot x} \left[(2\pi)^3 E_{\mathrm{CM}} \right]^{-1}, \tag{A.7}$$

$$(\bar{\phi}_j \phi_i)_y | i\bar{j} \rangle = -e^{-i(p_i+p_{\bar{j}})\cdot y} \left[(2\pi)^3 E_{\mathrm{CM}} \right]^{-1}, \tag{A.8}$$

$$\langle \bar{j}' i' | (\bar{\phi}_i \phi_j)_x = +e^{i(p_{i'}+p_{\bar{j}'})\cdot x} \left[(2\pi)^3 E_{\mathrm{CM}} \right]^{-1}. \tag{A.9}$$

In the limit where the initial and final momenta coincide, the s-channel amplitudes are given by

$$M_{(ij)(ij)}(s) = +\frac{i}{(2\pi)^7 E_{\mathrm{CM}}^2} \left[-ig(2\pi)^4 \right]^2 V(s), \tag{A.10}$$

$$M_{(i\bar{i})(i\bar{i})}(s) = -\frac{i}{(2\pi)^7 E_{\mathrm{CM}}^2} \left[-ig(2\pi)^4 \right]^2 (N-1) V(s), \tag{A.11}$$

$$M_{(i\bar{j})(i\bar{j})}(s) = -\frac{i}{(2\pi)^7 E_{\mathrm{CM}}^2} \left[-ig(2\pi)^4 \right]^2 V(s), \tag{A.12}$$

where

$$V(s) = \left[\frac{i}{(2\pi)^4} \right]^2 \int d^4k \left(\frac{1}{k^2 - m^2 + i\epsilon} \right) \left[\frac{1}{(k+p)^2 - m^2 + i\epsilon} \right]. \tag{A.13}$$

By dimensional regularization, we obtain

$$V(s) = -i\pi^2 \left[\frac{i}{(2\pi)^4} \right]^2 \int_0^1 dx \left\{ \ln \left[m^2 - s(1-x)x \right] + \frac{2}{\epsilon} + \gamma_E \right\}, \tag{A.14}$$

where $\epsilon = (d-4)$ and γ_E is the Euler constant. The imaginary part of the amplitudes are given by

$$\mathrm{Im} \left[M_{(ij)(ij)} \right] = -\frac{g^2}{128\pi^4 E_{\mathrm{CM}}^2} \left[1 - \frac{4m^2}{E_{\mathrm{CM}}^2} \right]^{1/2}, \tag{A.15}$$

$$\mathrm{Im} \left[M_{(i\bar{i})(i\bar{i})} \right] = +\frac{g^2}{128\pi^4 E_{\mathrm{CM}}^2} (N-1) \left[1 - \frac{4m^2}{E_{\mathrm{CM}}^2} \right]^{1/2}, \tag{A.16}$$

$$\mathrm{Im} \left[M_{(i\bar{j})(i\bar{j})} \right] = +\frac{g^2}{128\pi^4 E_{\mathrm{CM}}^2} \left[1 - \frac{4m^2}{E_{\mathrm{CM}}^2} \right]^{1/2}. \tag{A.17}$$

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JHEP05(2024)181