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Einstein gravity with generalized cosmological term from five-dimensional AdS-Maxwell-Chern-Simons gravity

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Abstract: Some time ago, the standard geometric framework of Einstein gravity was extended by gauging the Maxwell algebra as well as the so called AdS-Maxwell algebra. In this paper it is shown that the actions for these four-dimensional extended Einstein gravities can be obtained from the five-dimensional Chern-Simons gravities actions by using the Randall-Sundrum compactification procedure. It is found that the Inönü-Wigner contraction procedure, in the Weimar-Woods sense, can be used both to obtain the Maxwell-Chern-Simons action from the AdS-Maxwell-Chern-Simons action and to obtain the Maxwell extension of Einstein gravity in $4D$ from the four-dimensional extended AdS-Maxwell-Einstein-Hilbert action. It is also shown that the extended four-dimensional gravities belongs to the Horndeski family of scalar-tensor theories.

Keywords: Chern-Simons Theories, Classical Theories of Gravity

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Contents

1 Introduction

The problem of adding a cosmological term, including abelian gauge fields, to Einstein's field equations was treated in refs. [1, 2] by gauging the $D = 4$ Maxwell algebra [3, 4], which can also be obtained from (A)dS algebra using the Lie algebra expansion procedure developed in refs. [5–8].[1] This method allows also to derive the so called AdS-Maxwell algebra [9–11], whose generators satisfy the following commutation relations,[2]

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac},\\
[J_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac},\\
[Z_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac},\\
[J_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b, \qquad\qquad [P_a, P_b] = Z_{ab},\\
[Z_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b.
\end{aligned} \tag{1.1}$$

[1] An expansion is, in general, an algebra dimension-changing process, i.e., is a way to obtain new algebras of increasingly higher dimensions from a given one. A physical motivation for increasing the dimension of Lie algebras is that increasing the number of generators of an algebra is a non-trivial way of enlarging spacetime symmetries. Examples of this can be found in refs. [12, 13], where applications of Maxwell's algebra in gravity were studied (this algebra, also known as $\mathfrak{B}_4$ algebra, is a modification to the Poincaré symmetries and can be obtained, via S-expansion, from the anti-de Sitter (AdS) algebra). Another interesting modification to the Poincaré symmetries are the so-called generalized Poincaré algebras [11] of which the $\mathcal{M}$ algebra is an example.

[2] This algebra was also reobtained in ref. [13] from Maxwell algebra through a procedure known as deformation.

This algebra was found in [9, 10], where was called "semisimple extended Poincaré algebra". In ref. [13] this algebra was obtained from Maxwell algebra through of the deformation procedure and called "AdS-Maxwell algebra" and in ref. [11] it was obtained from the expansion procedure and called "AdS-Lorentz (AdS$\mathcal{L}_N$) algebra".

The Chern-Simons gravity has been extensively investigated within several theoretical frameworks. In three-dimensional spacetime, the Chern-Simons gravity invariant under (A)dS algebra is equivalent to the Einstein-Hilbert action with a cosmological constant [14]. Furthermore, in the context of higher dimensions, the (A)dS-Chern-Simons gravity can be obtained by properly selecting the coefficients in the Lovelock theory [15, 16]. These results have also been generalized for symmetries that are given by expansions and contractions of the (A)dS algebra [17–20]. Nevertheless, the formulation of Chern-Simons gravity is limited to odd dimensions. On the other hand, it was shown in ref. [21] that several terms of ($D = 4$) Horndeski action [22] emerged from the Kaluza-Klein dimensional reduction of Lovelock theory. Consequently, it is interesting to investigate the effective theories derived from dimensional reductions of extended Chern-Simons gravity, which incorporate non-abelian fields [23–27].

From the commutation relations (1.1) we can note that the set $\mathfrak{I} = (P_a, Z_{ab})$ satisfies the conditions $[\mathfrak{I}, \mathfrak{I}] \subset \mathfrak{I}$, $[so(3,1), \mathfrak{I}] \subset \mathfrak{I}$, i.e. $\mathfrak{I}$ is an ideal of the AdS-Maxwell algebra, which means that the AdS-Maxwell$= so(3,1) \uplus \mathfrak{I}$.

The main purpose of this article is to show that the four-dimensional extended Einstein gravity with a cosmological term including non-abelian gauge fields, found in [26, 27], may derive from five-dimensional AdS-Maxwell-Chern-Simons gravity. This can be achieved by replacing a Randall-Sundrum type metric in the five-dimensional Chern-Simons action for AdS-Maxwell algebra. The same procedure is used to obtain the four-dimensional extended Einstein gravity, with a cosmological term including Abelian gauge fields, found in refs. [1, 2], from the five-dimensional Maxwell-Chern-Simons gravity action.

This paper is organized as follows: in section 2 we consider a brief review of the construction of the AdS-Maxwell-Chern-Simons Lagrangian gravity and then we obtain the Maxwell-Chern-Simons Lagrangian using the In önü-Wigner contraction procedure in the Weimar-Woods sense. In Section 3 we apply the so called Randall-Sundrum compactification procedure to the AdS-Maxwell-Chern-Simons Lagrangian gravity to obtain the extended four-dimensional Einstein-Hilbert action with a cosmological term including non-abelian gauge fields. This procedure is also applied to the Maxwell-Chern-Simons gravity action to obtain an action for the extended four-dimensional Einstein gravity, which coincides, except for some coefficients, with the action obtained some years ago in references [1, 2]. Finally, it is shown in section 4 that the four dimensional actions obtained in section 3 belong to the family of Horndeski actions. Three appendices and concluding remarks end this article.

2 Maxwell-Chern-Simons action from AdS-Maxwell-Chern-Simons gravity

In this section we use the dual S-expansion procedure [8] to find the five-dimensional Chern-Simons Lagrangian invariant under the AdS-Maxwell algebra [11], and then using the Inönü-Wigner contraction procedure we find the Chern-Simons Lagrangian for the Maxwell algebra.

2.1 AdS-Maxwell-Chern-Simons gravity action

In order to write down an AdS-Maxwell-Chern-Simons Lagrangian, we start from the AdS-Maxwell algebra valued one-form gauge connection

$$A = \frac{1}{l}e^a P_a + \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{2}k^{ab} Z_{ab},$$

where $a, b = 0, 1, 2, 3, 4$ are tangent space indices raised and lowered with the Minkowski metric η_{ab}, and where

$$e^a = e^a_\mu dx^\mu, \quad \omega^{ab} = \omega^{ab}_\mu dx^\mu, \quad k^{ab} = k^{ab}_\mu dx^\mu,$$

are the e^a_μ fúnfbein, the ω^{ab}_μ spin connection and the k^{ab}_μ new non-abelian gauge fields. The corresponding associated curvature 2-form, is given by

$$F = \frac{1}{l}\mathcal{T}^a P_a + \frac{1}{2}R^{ab} J_{ab} + \frac{1}{2}F^{ab} Z_{ab}, \tag{2.1}$$

with

$$\begin{aligned} \mathcal{T}^a &= T^a + k^a_{\ c} e^c, \\ R^{ab} &= d\omega^{ab} + \omega^a_{\ c}\omega^{cb}, \\ F^{ab} &= Dk^{ab} + k^{[a}_{\ \ c} k^{c|b]} + \frac{1}{l^2} e^a e^b. \end{aligned} \tag{2.2}$$

In this point, it might be of interest to remember that: (i) clearly l could be eliminated by absorbing it in the definition of the vielbein, but then the space-time metric $g_{\mu\nu}$ would no longer be related to e^a through the relation $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$; (ii) the interpretation of the l parameter as a parameter related to the radius of curvature of the AdS space-time, is inherited for the space-time whose symmetries are described by the Maxwell algebra.

On the another hand, it could also be interesting to observe that J_{ab} are still Lorentz generators, but P_a are no longer AdS boosts. In fact, $[P_a, P_b] = Z_{ab}$. However e^a still transforms as a vector under Lorentz transformations, as it must, in order to recover gravity in this scheme.

A Chern-Simons Lagrangian in $D = 5$ dimensions is defined to be the following local function of a one-form gauge connection A:

$$\mathcal{L}^{(5D)}_{\rm ChS}(A) = \left\langle AF^2 - \frac{1}{2}A^3 F + \frac{1}{10}A^5 \right\rangle, \tag{2.3}$$

where $\langle \cdots \rangle$ denotes an invariant tensor for the corresponding Lie algebra, $F = dA + AA$ is the corresponding two-form curvature [28].

Using theorem VII.2 of ref. [8], it is possible to show that the only non-vanishing components of an invariant tensor for the AdS-Maxwell algebra are given by

$$\begin{aligned} \langle J_{ab} J_{cd} P_e \rangle &= \frac{4}{3}\alpha_1 l^3 \varepsilon_{abcde}, \\ \langle Z_{ab} Z_{cd} P_e \rangle &= \frac{4}{3}\alpha_1 l^3 \varepsilon_{abcde}, \\ \langle J_{ab} Z_{cd} P_e \rangle &= \frac{4}{3}\alpha_1 l^3 \varepsilon_{abcde}, \end{aligned}$$

where α_1 is an arbitrary constant of dimensions $[\text{length}]^{-3}$.

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Using the dual S-expansion procedure in terms of Maurer-Cartan forms [8], we find that the five-dimensional Chern-Simons Lagrangian invariant under the AdS-Maxwell algebra is given by [11]

$$\begin{aligned}\mathcal{L}_{\mathrm{ChS}}^{(\mathrm{AdS}\mathcal{M})} = \alpha_1 \varepsilon_{abcde} \Big\{ & l^2 R^{ab} R^{cd} e^e + l^2 \left(Dk^{ab}\right)\left(Dk^{cd}\right) e^e \\ & + l^2 k^a_f k^{fb} k^c_g k^{gd} e^e + \frac{1}{5l^2} e^a e^b e^c e^d e^e + 2l^2 R^{ab} k^c_f k^{fd} e^e \\ & + \frac{2}{3}\left(Dk^{ab}\right) e^c e^d e^e + 2l^2 R^{ab}\left(D_\omega k^{cd}\right) e^e \\ & + 2l^2 \left(Dk^{ab}\right) k^c_f k^{fd} e^e + \frac{2}{3} R^{ab} e^c e^d e^e + \frac{2}{3} k^a_f k^{fb} e^c e^d e^e \Big\}, \end{aligned} \tag{2.4}$$

where α_1 is a parameter of the theory, $R^{ab} = \mathrm{d}\omega^{ab} + \omega^a_c \omega^{cb}$ correspond to the curvature 2-form in the first-order formalism related to the spin connection 1-form, e^a is the vielbein 1-form, and k^{ab} 1-form are others gauge fields presents in the theory.

2.2 Maxwell Chern-Simons action

Keeping in mind that the Maxwell algebra can be obtained from AdS-Maxwell algebra by a generalized Inönü-Wigner contraction [11, 29], the natural question is how to obtain the corresponding Lagrangian for the Maxwell algebra from the Lagrangian for the AdS-Maxwell algebra? We find that it is also possible to obtain this relation using the same procedure which was applied to the algebras. In fact, carrying out the rescaling of the generators $P_a \to \xi P_a$, $Z_{ab} \to \xi^2 Z_{ab}$ and of the fields $e^a \to \xi^{-1} e^a$, $k^{ab} \to \xi^{-2} k^{ab}$ in the Lagrangian (2.4) we obtain

$$\begin{aligned} L_{\mathrm{ChS}}^{(\mathcal{M})} = {} & \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e + \frac{2}{3}\alpha_1 \varepsilon_{abcde} R^{ab} e^c e^d e^e \\ & + \frac{\alpha_1}{5l^2} \varepsilon_{abcde} e^a e^b e^c e^d e^e + \frac{2}{3}\alpha_1 \varepsilon_{abcde}\left(D_\omega k^{ab}\right) e^c e^d e^e \\ & + \alpha_1 l^2 \varepsilon_{abcde}\left(D_\omega k^{ab}\right)\left(D_\omega k^{cd}\right) e^e + 2\alpha_1 \varepsilon_{abcde} R^{ab}\left(D_\omega k^{cd}\right) e^e, \end{aligned} \tag{2.5}$$

which corresponds to the five-dimensional Chern-Simons Lagrangian for the Maxwell algebra.

3 Extended four-dimensional Einstein-Hilbert action from AdS-Maxwell-Chern-Simons gravity action

In order to obtain an action for a 4-dimensional gravity theory from the Chern-Simons action for AdS-Maxwell algebra we will consider the following 5-dimensional Randall Sundrum type metric [30–33]

$$\begin{aligned} ds^2 &= e^{2f(\phi)} \tilde{g}_{\mu\nu}(\tilde{x}) d\tilde{x}^\mu d\tilde{x}^\nu + r_c^2 d\phi^2 \\ &= e^{2f(\phi)} \tilde{\eta}_{mn} \tilde{e}^m \tilde{e}^n + r_c^2 d\phi^2, \end{aligned} \tag{3.1}$$

where $e^{2f(\phi)}$ is the so-called "warp factor", and r_c is the so-called "compactification radius" of the extra dimension, which is associated with the coordinate $0 \leqslant \phi < 2\pi$. The symbol $\sim$

denotes 4-dimensional quantities related to the space-time Σ_4. We will use the usual notation,

$$\begin{aligned} & x^\alpha = (\tilde{x}^\mu, \phi)\,; \qquad \alpha, \beta = 0, \dots, 4; \qquad a, b = 0, \dots, 4; \\ & \mu, \nu = 0, \dots, 3; \qquad m, n = 0, \dots, 3; \\ & \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1, 1); \qquad \tilde{\eta}_{mn} = \mathrm{diag}(-1, 1, 1, 1). \end{aligned} \tag{3.2}$$

This allows us, for example, to write

$$\begin{aligned} e^m(\phi, \tilde{x}) &= e^{f(\phi)} \tilde{e}^m(\tilde{x}) = e^{f(\phi)} \tilde{e}^m_\mu(\tilde{x}) d\tilde{x}^\mu; \qquad e^4(\phi) = r_c d\phi. \\ k^{mn}(\phi, \tilde{x}) &= \tilde{k}^{mn}(\tilde{x}), k^{m4} = k^{4m} = 0, \end{aligned} \tag{3.3}$$

where, following Randall and Sundrum [30, 31] matter fields are null in the fifth dimension.

From the vanishing torsion condition

$$T^a = de^a + \omega^a_b e^b = 0, \tag{3.4}$$

we obtain

$$\omega^a_{b\alpha} = -e^\beta_b \left(\partial_\alpha e^a_\beta - \Gamma^\gamma_{\alpha\beta} e^a_\gamma\right), \tag{3.5}$$

where $\Gamma^\gamma_{\alpha\beta}$ is the Christoffel symbol.

From equations (3.3) and (3.4), we find

$$\omega^m_4 = \frac{e^f f'}{r_c} \tilde{e}^m, \qquad \text{with} \quad f' = \frac{\partial f}{\partial \phi}, \tag{3.6}$$

and the 4-dimensional vanishing torsion condition

$$\tilde{T}^m = \tilde{d}\tilde{e}^m + \tilde{\omega}^m_n \tilde{e}^n = 0, \qquad \text{with} \quad \tilde{\omega}^m_n = \omega^m_n \quad \text{and} \quad \tilde{d} = d\tilde{x}^\mu \frac{\partial}{\partial \tilde{x}^\mu}. \tag{3.7}$$

From (3.6), (3.7) and the Cartan's second structural equation, $R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}$, we obtain the components of the 2-form curvature

$$R^{m4} = \frac{e^f}{r_c} \left(f'^2 - f''\right) d\phi \tilde{e}^m, \qquad R^{mn} = \tilde{R}^{mn} - \left(\frac{e^f f'}{r_c}\right)^2 \tilde{e}^m \tilde{e}^n, \tag{3.8}$$

where the 4-dimensional 2-form curvature is given by

$$\tilde{R}^{mn} = \tilde{d}\tilde{\omega}^{mn} + \tilde{\omega}^m_p \tilde{\omega}^{pn}. \tag{3.9}$$

From equation (2.4) we can see that the Lagrangian contains ten terms that we will denote as $L_1, L_2, \cdots, L_{10}$, where L_1 corresponds to the Gauss-Bonnet term, L_4 corresponds to the cosmological term, L_9 correspond to the Einstein-Hilbert term. In fact, following refs. [32, 33] we replace (3.3) and (3.8) in (2.4), and using $\tilde{\varepsilon}_{mnpq} = \varepsilon_{mnpq4}$, we obtain

$$
\begin{aligned}
L_1 &= \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e \\
&= \alpha_1 l^2 r_c d\phi \left\{ \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{R}^{pq} - \left(\frac{2e^{2f}}{r_c^2} \right) \left(3f'^2 + 2f'' \right) \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q \right. \\
&\qquad \left. + \left(\frac{e^{4f}}{r_c^4} f'^2 \right) \left(5f'^2 + 4f'' \right) \tilde{\varepsilon}_{mnpq} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q \right\}, \qquad (3.10) \\
L_2 &= \alpha_1 l^2 r_c d\phi \tilde{\varepsilon}_{mnpq} Dk^{mn} Dk^{pq}, \qquad (3.11) \\
L_3 &= \alpha_1 \ell^2 r_c d\phi \tilde{\epsilon}_{mnpq} \tilde{k}^m_{\ f} \tilde{k}^{fn} \tilde{k}^p_g \tilde{k}^{gq}, \qquad (3.12) \\
L_4 &= \frac{\alpha_1}{l^2} r_c d\phi e^{4f} \tilde{\varepsilon}_{mnpq} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q, \qquad (3.13) \\
L_5 &= 2\alpha_1 \ell^2 r_c d\phi \left[\tilde{\epsilon}_{mnpq} \tilde{R}^{mn} \tilde{k}^p_f \tilde{k}^{fq} - \frac{e^{2f(\phi)}}{r_c^2} \left(2f'' + 3f'^2 \right) \tilde{\epsilon}_{mnpq} \tilde{k}^m_f \tilde{k}^{fn} \tilde{e}^p \tilde{e}^q \right], \\
L_6 &= 2\alpha_1 r_c d\phi e^{2f(\phi)} \tilde{\epsilon}_{mnpq} \left(D_\omega \tilde{k}^{mn} \right) \tilde{e}^p \tilde{e}^q, \qquad (3.14) \\
L_7 &= 2\alpha_1 l^2 r_c d\phi \tilde{\varepsilon}_{mnpq} \left\{ \tilde{R}^{mn} Dk^{pq} - \frac{e^{2f}}{r_c^2} \left(3f'^2 + 2f'' \right) Dk^{mn} \tilde{e}^p \tilde{e}^q \right\}, \qquad (3.15) \\
L_8 &= 2\alpha_1 \ell^2 r_c d\phi \tilde{\epsilon}_{mnpq} \left(D_\omega \tilde{k}^{mn} \right) \tilde{k}^p_f \tilde{k}^{fq}, \qquad (3.16) \\
L_9 &= \frac{2}{3} \alpha_1 r_c d\phi \left\{ \left(3e^{2f} \right) \tilde{\varepsilon}_{mnpq} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q - \left(\frac{e^{4f}}{r_c^2} \right) \left(5f'^2 + 2f'' \right) \tilde{\varepsilon}_{mnpq} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q \right\}, \\
&\qquad (3.17) \\
L_{10} &= 2\alpha_1 r_c d\phi e^{2f(\phi)} \tilde{\varepsilon}_{mnpq} \tilde{k}^m_f \tilde{k}^{fn} \tilde{e}^p \tilde{e}^q. \qquad (3.18)
\end{aligned}
$$

By replacing (3.10)–(3.18) in (2.4) and integrating over the fifth dimension we find

$$
\begin{aligned}
S_{4D}^{\mathrm{AdS}\mathcal{M}} = \int_{\Sigma_4} & A \tilde{\varepsilon}_{mnpq} \left[\tilde{R}^{mn} \tilde{e}^p \tilde{e}^q + \tilde{k}^m_f \tilde{k}^{fn} \tilde{e}^p \tilde{e}^q + D\tilde{k}^{mn} \tilde{e}^p \tilde{e}^q \right] \\
& + B \tilde{\varepsilon}_{mnpq} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{e}^q + C \tilde{\varepsilon}_{mnpq} \left[D\tilde{k}^{mn} D\tilde{k}^{pq} + \tilde{k}^m_f \tilde{k}^{fn} \tilde{k}^p_g \tilde{k}^{gq} \right. \\
& \left. + 2\tilde{R}^{mn} \tilde{k}^p_f \tilde{k}^{fq} + 2D\tilde{k}^{mn} \tilde{k}^p_f \tilde{k}^{fq} \right] + \text{sourface terms}, \qquad (3.19)
\end{aligned}
$$

where,

$$
\begin{aligned}
A &= 2\alpha_1 r_c \int_0^{2\pi} e^{2f(\phi)} \left[1 - \frac{\ell^2}{r_c^2} \left(2f'' + 3f'^2 \right) \right] d\phi, \qquad (3.20) \\
&= \frac{2\pi\alpha_1 \left(\ell^2 + r_c^2 \right)}{r_c} \qquad (3.21) \\
B &= \alpha_1 r_c \int_0^{2\pi} e^{4f(\phi)} \left[\frac{\ell^2}{r_c^4} \left(4f'' + 5f'^2 \right) f'^2 + \frac{1}{\ell^2} - \frac{2}{3r_c^2} \left(2f'' + 5f'^2 \right) \right] d\phi \\
&= \frac{\pi\alpha_1}{4\ell^2 r_c^3} \left[3r_c^4 + 2\ell^2 r_c^2 - \ell^4 \right], \qquad (3.22) \\
C &= 2\pi\alpha_1 \ell^2 r_c \qquad (3.23)
\end{aligned}
$$

with $f(\phi)$ an arbitrary and continuously differentiable function. Since we are working with a cylindrical variety, we have chosen (non-unique choice) $f(\phi) = \ln(\sin\phi)$.

From the action (3.19) we see that it includes non-Abelian fields $\tilde{k}_\mu^{mn}$, which could be interpreted as non-Abelian gauge field that driven inflation (see e.g. [34–38]).

3.1 Maxwell Einstein gravity from Maxwell Chern-Simons action

In order to obtain an action for a 4-dimensional gravity theory from the Chern-Simons action for Maxwell álgebra we follow the same procedure used in the previous section. In fact, replacing (3.3) and (3.8) in (2.5), and using $\tilde{\varepsilon}_{mnpq} = \varepsilon_{mnpq4}$,

$$S_{\mathcal{M}}^{\mathrm{GEH}} = \int_{\Sigma_4} A\varepsilon_{mnpq}\left[\tilde{R}^{mn}\tilde{e}^p\tilde{e}^q + D\tilde{k}^{mn}\tilde{e}^p\tilde{e}^q\right] + B\varepsilon_{mnpq}\tilde{e}^m\tilde{e}^n\tilde{e}^p\tilde{e}^q + 2\pi\alpha_1\ell^2 r_c\varepsilon_{mnpq}D\tilde{k}^{mn}D\tilde{k}^{pq}, \tag{3.24}$$

where the coeficients A and B are given by eqs. (3.21), (3.22). This action matches with the action (A.2), which correspond to the equation (29) of reference [1], as long as $A = -1/2\kappa = \lambda/2\kappa\Lambda$, $B = \lambda/4\kappa$, $2\pi\alpha_1\ell^2 r_c = \lambda/2\kappa\Lambda^2$. This means that the action (3.24) and the action (A.2) coincide only if $\Lambda = 1/l^2$, that is if $\lambda = -1/l^2$.

It is of interest to note that the action (3.24) can be obtained from the action (3.19) using the generalized Inönü-Wigner contraction, namely, carrying out the rescaling of the generators $P_a \to \xi P_a$, $Z_{ab} \to \xi^2 Z_{ab}$ and of the fields $e^a \to \xi^{-1}e^a$, $k^{ab} \to \xi^{-2}k^{ab}$ in the Lagrangian (3.24).

4 From AdS-Maxwell gravity to scalar-tensor theory

In this section it is found that the four-dimensional actions obtained from Chern-Simons gravity actions invariants under the so called generalized (A)dS-Maxwell symmetries belongs to a larges class of theories known as Horndeski theories (see appendix B).

The non-abelian gauge field in four-dimensional spacetime is a rank-three tensor with two anti-symmetric indices, $\tilde{k}_{[mn]p}$. This means that it has 24 degrees of freedom (d.o.f.). We can decompose this field with respect to the Lorentz group into three irreducible tensors [41, 42], namely

$$\tilde{k}_{[mn]p} = -\frac{1}{3}\left(\tilde{k}_m\eta_{np} - \tilde{k}_n\eta_{mp}\right) - \frac{1}{6}\varepsilon_{mnpq}S^q + q_{mnp}, \tag{4.1}$$

where, the trace vector $\tilde{k}_m \equiv \tilde{k}^n_{mn}$ has 4 d.o.f.. The axial vector S^q possesses 4 d.o.f., while q_{mnp} exhibits 16 d.o.f., representing the traceless and non-totally anti-symmetric component of the tensor. In order to obtain an effective scalar-tensor theory from AdS-Maxwell gravity, it is necessary to consider a single additional degree of freedom; accordingly, we set a value of zero to both the axial vector and the q tensor. Furthermore, we choose the trace vector as $\tilde{k}_m = \tilde{e}^\mu_m D_\mu\varphi$ that is, we postulate that $\tilde{k}_m$ is the gradient of a scalar field, therefore depending completely on one degree of freedom. Thus, we have that

$$\tilde{k}^{mn} \equiv \tilde{k}^{mn}_p\tilde{e}^p = -\frac{1}{3}\left(\tilde{k}^m\tilde{e}^n - \tilde{k}^n\tilde{e}^m\right). \tag{4.2}$$

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Taking this ansatz into account we have that the eight terms of (3.19) take the form (see appendix C)

$$A\tilde{\epsilon}_{mnpq}\tilde{R}^{mn}\tilde{e}^p\tilde{e}^q = 4A\sqrt{-\tilde{g}}\tilde{R}d^4\tilde{x}. \tag{4.3}$$

$$A\tilde{\epsilon}_{mnpq}\tilde{k}^m_f\tilde{k}^{fn}\tilde{e}^p\tilde{e}^q = -\frac{4}{3}AD_\mu\varphi D^\mu\varphi\sqrt{\tilde{g}}d^4x. \tag{4.4}$$

$$A\tilde{\epsilon}_{mnpq}D\tilde{k}^{mn}\tilde{e}^p\tilde{e}^q = -4AD_\mu D^\mu\varphi\sqrt{-\tilde{g}}d^4\tilde{x} \tag{4.5}$$

$$B\tilde{\epsilon}_{mnpq}\tilde{e}^m\tilde{e}^n\tilde{e}^p\tilde{e}^q = 24B\sqrt{-\tilde{g}}d^4\tilde{x}. \tag{4.6}$$

$$C\tilde{\varepsilon}_{mnpq}D\tilde{k}^{mn}D\tilde{k}^{pq} = \frac{8C}{9}\left\{(D_\alpha D^\alpha\varphi)^2 - D_\gamma D^\nu\varphi D_\nu D^\gamma\varphi\right\}\sqrt{-\tilde{g}}d^4\tilde{x} \tag{4.7}$$

$$C\tilde{\varepsilon}_{mnpq}\tilde{k}^m_f\tilde{k}^{fn}\tilde{k}^p{}_g\tilde{k}^{gq} = 0 \tag{4.8}$$

$$2C\tilde{\varepsilon}_{mnpq}\tilde{R}^{mn}\tilde{k}^p_f\tilde{k}^{fq} = \frac{-8C}{3^2}\sqrt{-\tilde{g}}d^4\tilde{x}D^\alpha\varphi D^\beta\varphi\tilde{G}_{\alpha\beta} - \frac{8C}{3^2}\sqrt{-\tilde{g}}d^4\tilde{x}D_\nu\varphi D^\nu\varphi\tilde{R} \tag{4.9}$$

$$2C\tilde{\varepsilon}_{mnpq}D\tilde{k}^{mn}\tilde{k}^p_f\tilde{k}^{fq} = \sqrt{-\tilde{g}}d^4\tilde{x}\frac{8C}{3^3}D_\gamma A^\gamma, \qquad \text{with} \quad A^\gamma = D_\alpha\varphi D^\alpha\varphi D^\gamma\varphi. \tag{4.10}$$

Replacing (4.3), (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), (4.10) in (3.19), we find:

$$\begin{aligned} S_{4D}^{\text{AdS}\mathcal{M}} = \int_{\Sigma_4} d^4\tilde{x}\sqrt{-\tilde{g}}\Big\{&\left(4A - \frac{8C}{9}D_\nu\varphi D^\nu\varphi\right)\tilde{R} - \frac{4}{3}AD_\mu\varphi D^\mu\varphi - 4AD_\mu D^\mu\varphi \\ &+ 24B + \frac{8C}{9}\left[(D_\alpha D^\alpha\varphi)^2 - D_\gamma D_\nu\varphi D^\nu D^\gamma\varphi\right] \\ &+ \frac{8C}{9}\phi D^\alpha D^\beta\varphi\tilde{G}_{\alpha\beta} + \text{surface terms}\Big\}, \end{aligned} \tag{4.11}$$

where, we have used the followin identities

$$\begin{aligned} D^\alpha\varphi D^\beta\varphi\tilde{G}_{\alpha\beta} &= D^\alpha\left[\varphi D^\beta\varphi\tilde{G}_{\alpha\beta}\right] - \varphi D^\alpha D^\beta\varphi\tilde{G}_{\alpha\beta}, \\ D^\alpha\tilde{G}_{\alpha\beta} &= 0. \end{aligned} \tag{4.12}$$

Comparing these results with the equations (B.1) of appendix B, we can see that the Lagrangian (4.11) can be written as

$$\begin{aligned} S_{4D}^{\text{AdS}\mathcal{M}} = \int_{\Sigma_4} d^4\tilde{x}\sqrt{-\tilde{g}}\Big\{&G_2(\varphi, X) + G_3(\varphi, X)D_\mu D^\mu\varphi + G_4(\varphi, X)\tilde{R} \\ &- 2G_{4X}(\varphi, X)\left[(D_\alpha D^\alpha\varphi)^2 - (D^\mu D^\nu\varphi)(D_\mu D_\nu\varphi)\right] \\ &+ G_5(\varphi, X)\tilde{G}_{\mu\upsilon}D^\mu D^\nu\varphi + \text{surface terms}\Big\}, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} G_2(\varphi, X) &= -\frac{4}{3}AD_\mu\varphi D^\mu\varphi + 24B \\ G_3(\varphi, X) &= -4A \\ G_4(\varphi, X) &= 4A - \frac{8C}{9}D_\nu\varphi D^\nu\varphi \\ G_{4X}(\varphi, X) &= -\frac{4C}{9} \\ G_5(\varphi, X) &= \frac{8C}{9}\varphi. \end{aligned} \tag{4.14}$$

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This action explicitly includes Einstein-Hilbert gravity with a cosmological constant, along with various additional components of Horndeski theory; therefore, this is a second-order scalar-tensor theory. On the other hand, by considering all the degrees of freedom of the non-abelian gauge field, we will obtain an extended theory with a sector corresponding to a Horndeski family, as well as new terms whose interpretation will depend on the physical quantities introduced. We will investigate this theory in future works.

It should be noted that an analogous result can be obtained in the case of the Maxwell action (3.24), where it is straightforward to see that the four terms of the Lagrangian corresponding to the action (3.24) are given by (4.3), (4.4), (4.5), (4.6), (4.7).

5 Concluding remarks

In this article we have obtained a four-dimensional extended Einstein gravity with a cosmological term, including non-abelian gauge fields, from five-dimensional AdS-Maxwell-Chern-Simons gravity. This gravity in $4D$ includes non-Abelian fields $\tilde{k}^{mn}$, which could be interpreted as gauge fields that driven inflation. This was achieved by making use of the Randall-Sundrum compactification procedure, which is also used to re-obtain, from the Maxwell Chern-Simons gravity action, the extended Einstein gravity in $4D$ with a cosmological term, which including abelian gauge fields of refs. [1, 2].

The Inönü-Wigner contraction procedure in the Weimar-Woods sense is used both to obtain the Maxwell-Chern-Simons action from the Chern-Simon saction for AdS-Maxwell algebra and to obtain the Maxwell extension of Einstein gravity in $4D$ from AdS-Maxwell-Einstein-Hilbert action.

It might be of interest to note that both extensions of Einstein's gravity with cosmological terms (which includes Abelian and non-Abelian gauge fields respectively) are not invariant under the respective local transformations but only under local Lorentz transformations. Here, we have shown that is possible to obtain this generalized four-dimensional Einstein-Hilbert actions from the genuinely invariant five-dimensional Chern-Simons gravities. This seems to indicate that the compactification procedure breaks the original symmetries of the Chern-Simons actions (Maxwell and AdS-Maxwell) to the Lorentz symmetry.

We have also shown that the four-dimensional actions obtained from Chern-Simons gravity actions invariants under the so called generalized (A)dS-Maxwell symmetries as well as under the Maxwell symmetries belongs to the family actions for the Horndeski theory. This result allows us to conjecture that the compactification of Chern-Simons gravities corresponding to groups with symmetries greater than those presented here will lead to Lagrangians that involve more Lagrangians of the basis $\mathcal{L}_i\left[G_i\right]$, $i = 1, 2, 3, 4, 5, \cdots$ (see appendix B and ref. [40]).

A The Maxwell algebra

The generators of Maxwell algebra (P_a, J_{ab}, Z_{ab}) satisfying the following commutation relations [3, 4]

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \\
[J_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b, \qquad\qquad [P_a, P_b] = Z_{ab},
\end{aligned}$$

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$$
\begin{aligned}
&[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac}, \\
&[Z_{ab}, Z_{cd}] = 0, \qquad\qquad [Z_{ab}, P_c] = 0.
\end{aligned} \tag{A.1}
$$

From (A.1) we see that the set $I = (P_a, Z_{ab})$ is an ideal of Maxwell's algebra because $[I, I] \subset I$, $[so(3,1), I] \subset I$ [1]. This means that the Maxwell algebra $\mathcal{M}$ is the semidirect sum of the Lorentz algebra $so(3,1)$ and the ideal I, that is $\mathcal{M} = so(3,1) \uplus I$ [1].

The Abelan four-vector fields k_μ^{ab} associated with their Abelian tensorial generators Z_{ab} and the set of curvatures associated to this gauge potentials denoted by $F_{\mu\nu}^{ab}$ allow to construct a generalization of the Einstein-Hilbert Lagrangian given by

$$
\begin{aligned}
\mathcal{L} = -\ & \frac{1}{2\kappa}\varepsilon_{abcd} R^{ab} e^c e^d + \frac{\lambda}{4\kappa}\varepsilon_{abcd} e^a e^b e^c e^d + \frac{\mu}{2\kappa}\varepsilon_{abcd} D k^{ab} e^c e^d \\
& + \frac{\mu^2}{4\kappa\lambda}\varepsilon_{abcd} D k^{ab} D k^{cd}, \qquad \text{with} \quad \mu = \lambda/\Lambda \quad \text{and} \quad \Lambda = 1/l^2
\end{aligned} \tag{A.2}
$$

which corresponds to equation (29) of reference [1], which is not invariant under local Maxwell transformations but only under local Lorentz transformations. This action contain geometric Abelian gauge fields, k^{ab}, playing the role of vectorial inflatons [34], which contribute to a generalization of the cosmological term.

On the other hand, actions containing cosmological terms that describe vector inflations by means of geometric non-Abelian gauge fields remain as an open problem. The idea that dark energy could be understood by non-abelian vector fields, that is, that non-abelian gauge fields could be responsible for the accelerated expansion of the universe, was postulated in references [35–38].

B Horndeski theories

Although Horndeski's theories have Lagrange functions that contain at most second derivatives of a scalar field, they correspond to the more general tenso-scalar theory that leads to second-order equations of motion. These theories can be written as a linear combination of the following Lagrange functions ([39, 40])

$$
\begin{aligned}
\mathcal{L}_2\left[G_2\right] &\equiv G_2\left(\phi, X\right) \\
\mathcal{L}_3\left[G_3\right] &\equiv G_3\left(\phi, X\right) D_\mu D^\mu \phi \\
\mathcal{L}_4\left[G_4\right] &\equiv G_4\left(\phi, X\right) \tilde{R} - 2G_{4X}\left(\phi, X\right)\left[\left(D_\alpha D^\alpha \phi\right)^2 - \left(D^\mu D^\nu \phi\right)\left(D_\mu D_\nu \phi\right)\right] \\
\mathcal{L}_5\left[G_5\right] &\equiv G_5\left(\phi, X\right) \tilde{G}_{\mu v} D^\mu D^\nu \phi + \frac{1}{3} G_5\left(\phi, X\right)\left[\left(D_\mu D^\mu \phi\right)^3 - 3\left(D_\mu D^\mu \phi\right)\left(D^\gamma D^\nu \phi\right)\left(D_\gamma D_\nu \phi\right)\right] \\
&\quad + 2\left(D_\mu D_\nu \phi\right)\left(D^\sigma D^\nu \phi\right)\left(D_\sigma D^\mu \phi\right).
\end{aligned} \tag{B.1}
$$

C Lagrangian terms (3.19) in tensor language

We consider the tensor form of the eight terms of the Lagrangian corresponding to teh action (3.19). To do this we will use the equation (4.2), with $\tilde{k}_m = \tilde{e}_m^\mu \partial_\mu \phi$.

First term: in this case it is direct to see that

$$A\tilde{\epsilon}_{mnpq}\tilde{R}^{mn}\tilde{e}^p\tilde{e}^q = 4A\sqrt{-\tilde{g}}\tilde{R}d^4\tilde{x}. \tag{C.1}$$

Second term: writing the second term in the form,

$$\begin{aligned} A\tilde{\epsilon}_{mnpq}\tilde{k}^m_f\tilde{k}^{fn}\tilde{e}^p\tilde{e}^q &= \frac{A}{9}\tilde{\epsilon}_{mnpq}\left(\tilde{k}^m\tilde{e}_f - \tilde{k}_f\tilde{e}^m\right)\left(\tilde{k}^f\tilde{e}^n - \tilde{k}^n\tilde{e}^f\right)\tilde{e}^p\tilde{e}^q \\ &= \frac{A}{9}\tilde{\epsilon}_{mnpq}\left(\tilde{k}^m\tilde{e}_f\tilde{k}^f\tilde{e}^n - \tilde{k}_f\tilde{k}^f\tilde{e}^m\tilde{e}^n - \tilde{k}^m\tilde{k}^n\tilde{e}_f\tilde{e}^f - \tilde{k}_f\tilde{e}^f\tilde{e}^m\tilde{k}^n\right)\tilde{e}^p\tilde{e}^q, \end{aligned} \tag{C.2}$$

and considering that $\tilde{e}^f\tilde{k}_f = \left(\tilde{e}^f_\nu dx^\nu\right)\left(\tilde{e}^\mu_f\partial_\mu\phi\right) = \delta^\mu_\nu dx^\nu\partial_\mu\phi = d\phi$, we have

$$A\tilde{\epsilon}_{mnpq}\tilde{k}^m_f\tilde{k}^{fn}\tilde{e}^p\tilde{e}^q = \frac{2A}{9}\tilde{\epsilon}_{mnpq}d\varphi\tilde{k}^m\tilde{e}^n\tilde{e}^p\tilde{e}^q - \frac{A}{9}\tilde{\epsilon}_{mnpq}\tilde{k}^2\tilde{e}^m\tilde{e}^n\tilde{e}^p\tilde{e}^q, \tag{C.3}$$

where,

$$\begin{aligned} \tilde{\epsilon}_{mnpq}\tilde{k}^2\tilde{e}^m\tilde{e}^n\tilde{e}^p\tilde{e}^q &= \tilde{\epsilon}_{mnpq}e^m_\alpha e^n_\beta\tilde{e}^p_\gamma\tilde{e}^q_\delta dx^\alpha dx^\beta dx^\gamma dx^\delta D_\mu\varphi D^\mu\varphi \\ &= \sqrt{\tilde{g}}d^4x\tilde{\epsilon}_{\alpha\beta\gamma\delta}\tilde{\epsilon}^{\alpha\beta\gamma\delta}D_\mu\varphi D^\mu\varphi = \sqrt{\tilde{g}}D_\mu\varphi D^\mu\varphi 4!d^4x \\ 2\tilde{\epsilon}_{mnpq}d\varphi\tilde{k}^m\tilde{e}^n\tilde{e}^p\tilde{e}^q &= 2\tilde{\epsilon}_{mnpq}e^m_\alpha e^n_\beta\tilde{e}^p_\gamma\tilde{e}^q_\delta dx^\mu dx^\beta dx^\gamma dx^\delta\partial_\mu\varphi D^\alpha\varphi \\ &= 2\sqrt{\tilde{g}}d^4x\tilde{\epsilon}_{\alpha\beta\gamma\delta}\tilde{\epsilon}^{\mu\beta\gamma\delta}D_\mu\varphi D^\alpha\varphi \\ &= 2\cdot 3!D_\mu\varphi D^\mu\varphi\sqrt{\tilde{g}}d^4x \end{aligned} \tag{C.4}$$

so that,

$$\begin{aligned} A\tilde{\epsilon}_{mnpq}\tilde{k}^m_f\tilde{k}^{fn}\tilde{e}^p\tilde{e}^q &= \frac{A}{9}\left(2\cdot 3!D_\mu\varphi D^\mu\varphi - 4!D_\mu\varphi D^\mu\varphi\right)\sqrt{\tilde{g}}d^4x \\ &= -\frac{4}{3}AD_\mu\varphi D^\mu\varphi\sqrt{\tilde{g}}d^4x\,. \end{aligned} \tag{C.5}$$

Third term: for the third term, we have

$$A\tilde{\epsilon}_{mnpq}D\tilde{k}^{mn}\tilde{e}^p\tilde{e}^q = -\frac{2}{3}A\tilde{\epsilon}_{mnpq}D\tilde{k}^m\tilde{e}^n\tilde{e}^p\tilde{e}^q, \tag{C.6}$$

where we have taken into account that the absence of four-dimensional torsion, that is, $\tilde{T}^m = D\tilde{e}^m = 0$. In tensor language C.6 takes the form,

$$\begin{aligned} A\tilde{\epsilon}_{mnpq}D\tilde{k}^{mn}\tilde{e}^p\tilde{e}^q &= -\frac{2}{3}A\tilde{\epsilon}_{mnpq}e^m_\alpha\tilde{e}^n_\beta\tilde{e}^p_\gamma\tilde{e}^q_\delta D_\mu D^\alpha\varphi dx^\mu dx^\beta dx^\gamma dx^\delta \\ &= -\frac{2}{3}A\tilde{\epsilon}_{\alpha\beta\gamma\delta}\tilde{\epsilon}^{\mu\beta\gamma\delta}D_\mu D^\alpha\varphi\sqrt{-\tilde{g}}d^4\tilde{x} \\ &= -4AD_\mu D^\mu\varphi\sqrt{-\tilde{g}}d^4\tilde{x}. \end{aligned} \tag{C.7}$$

Fourth term: in this case it is direct to see that

$$B\tilde{\epsilon}_{mnpq}\tilde{e}^m\tilde{e}^n\tilde{e}^p\tilde{e}^q = 24B\sqrt{-\tilde{g}}d^4\tilde{x}. \tag{C.8}$$

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Fifth term: in this case we can write

$$C\tilde{\varepsilon}_{mnpq}D\tilde{k}^{mn}D\tilde{k}^{pq} = \frac{1}{9}C\tilde{\varepsilon}_{mnpq}\left(D\tilde{k}^{m}\tilde{e}^{n} - D\tilde{k}^{n}\tilde{e}^{m}\right)\left(D\tilde{k}^{p}\tilde{e}^{q} - D\tilde{k}^{q}\tilde{e}^{p}\right)$$
$$= \frac{4}{9}C\tilde{\varepsilon}_{mnpq}D\tilde{k}^{m}\tilde{e}^{n}D\tilde{k}^{p}\tilde{e}^{q}, \tag{C.9}$$

where we have taken into account the absence of four-dimensional torsion. So we have,

$$C\tilde{\varepsilon}_{mnpq}D\tilde{k}^{mn}D\tilde{k}^{pq} = \frac{4C}{9}\tilde{\varepsilon}_{mnpq}\tilde{e}^{m}_{\alpha}\tilde{e}^{n}_{\beta}\tilde{e}^{p}_{\nu}\tilde{e}^{q}_{\delta}dx^{\mu}dx^{\beta}dx^{\gamma}dx^{\delta}D_{\mu}D^{\alpha}\varphi D_{\gamma}D^{\nu}\varphi$$
$$= \frac{4C}{9}\sqrt{-\tilde{g}}d^{4}\tilde{x}\tilde{\varepsilon}_{\alpha\beta\gamma\delta}\varepsilon^{\mu\beta\nu\delta}D_{\mu}D^{\alpha}\varphi D_{\nu}D^{\gamma}\varphi$$
$$= \frac{8C}{9}\left\{(D_{\alpha}D^{\alpha}\varphi)^{2} - D_{\gamma}D^{\nu}\varphi D_{\nu}D^{\gamma}\varphi\right\}\sqrt{-\tilde{g}}d^{4}\tilde{x}\,. \tag{C.10}$$

Sixth term: using the previous result,

$$\tilde{k}^{m}_{f}\tilde{k}^{fn} = \frac{1}{3^{2}}\left(2d\varphi\tilde{k}^{m}\tilde{e}^{n} - \tilde{k}^{2}\tilde{e}^{m}\tilde{e}^{n}\right),$$

we have,

$$C\tilde{\varepsilon}_{mnpq}\tilde{k}^{m}_{f}\tilde{k}^{fn}\tilde{k}^{p}_{\ g}\tilde{k}^{gq} = \frac{C}{3^{4}}\tilde{\varepsilon}_{mnpq}\left(2d\varphi\tilde{k}^{m}\tilde{e}^{n} - \tilde{k}^{2}\tilde{e}^{m}\tilde{e}^{n}\right)\left(2d\varphi\tilde{k}^{p}\tilde{e}^{q} - \tilde{k}^{2}\tilde{e}^{p}\tilde{e}^{q}\right)$$
$$= \frac{C}{3^{4}}\tilde{\varepsilon}_{mnpq}\left\{4\left(d\varphi\right)^{2}\tilde{k}^{m}\tilde{e}^{n}\tilde{k}^{p}\tilde{e}^{q} - 4d\varphi\tilde{k}^{2}\tilde{k}^{m}\tilde{e}^{n}\tilde{e}^{p}\tilde{e}^{q} + \tilde{k}^{4}\tilde{e}^{m}\tilde{e}^{n}\tilde{e}^{p}\tilde{e}^{q}\right\}. \tag{C.11}$$

To obtain this result in tensor language, let's analyze each term separately. Indeed,

$$\frac{4C}{3^{4}}\tilde{\varepsilon}_{mnpq}\left(d\varphi\right)^{2}\tilde{k}^{m}\tilde{e}^{n}\tilde{k}^{p}\tilde{e}^{q} = \frac{4C}{3^{4}}\tilde{\varepsilon}_{mnpq}\tilde{e}^{m}_{\alpha}\tilde{e}^{n}_{\beta}\tilde{e}^{p}_{\gamma}\tilde{e}^{q}_{\delta}dx^{\mu}dx^{\nu}dx^{\beta}dx^{\delta}\partial_{\mu}\varphi\partial_{\nu}\phi D^{\alpha}\varphi D^{\gamma}\varphi$$
$$= -\frac{4C}{3^{4}}\sqrt{-\tilde{g}}d^{4}\tilde{x}\tilde{\varepsilon}_{\alpha\gamma\beta\delta}\tilde{\varepsilon}^{\mu\nu\beta\delta}\partial_{\mu}\varphi\partial_{\nu}\varphi D^{\alpha}\varphi D^{\gamma}\varphi$$
$$= -\frac{8C}{3^{4}}\sqrt{-\tilde{g}}d^{4}\tilde{x}\left(D_{\alpha}\varphi D^{\alpha}\varphi D_{\gamma}\varphi D^{\gamma}\varphi\right.$$
$$\left.-D_{\alpha}\varphi D^{\alpha}\varphi D_{\gamma}\varphi D^{\gamma}\varphi\right) = 0 \tag{C.12}$$

$$-\frac{4C}{3^{4}}\tilde{\varepsilon}_{mnpq}d\varphi\tilde{k}^{2}\tilde{k}^{m}\tilde{e}^{n}\tilde{e}^{p}\tilde{e}^{q} = -\frac{4C}{3^{4}}\tilde{\varepsilon}_{mnpq}\tilde{e}^{m}_{\alpha}\tilde{e}^{n}_{\beta}\tilde{e}^{p}_{\gamma}\tilde{e}^{q}_{\delta}dx^{\mu}dx^{\beta}dx^{\gamma}dx^{\delta}\partial_{\mu}\varphi D^{\alpha}\varphi\tilde{k}^{2}$$
$$= -\frac{4C}{3^{4}}\sqrt{-\tilde{g}}d^{4}\tilde{x}\tilde{\varepsilon}_{\alpha\beta\gamma\delta}\tilde{\varepsilon}^{\mu\beta\gamma\delta}D_{\mu}\varphi D^{\alpha}\varphi\tilde{k}^{2}$$
$$= -\frac{8C}{3^{3}}\sqrt{-\tilde{g}}d^{4}\tilde{x}D_{\alpha}\varphi D^{\alpha}\varphi D_{\nu}\varphi D^{\nu}\varphi$$
$$= -\frac{8C}{3^{3}}\sqrt{-\tilde{g}}d^{4}\tilde{x}\left(D_{\alpha}\varphi\right)^{4} \tag{C.13}$$

$$\frac{C}{3^{4}}\tilde{\varepsilon}_{mnpq}\tilde{k}^{4}\tilde{e}^{m}\tilde{e}^{n}\tilde{e}^{p}\tilde{e}^{q} = \frac{C}{3^{4}}\tilde{\varepsilon}_{mnpq}\tilde{e}^{m}_{\alpha}\tilde{e}^{n}_{\beta}\tilde{e}^{p}_{\gamma}\tilde{e}^{q}_{\delta}dx^{\alpha}dx^{\beta}dx^{\gamma}dx^{\delta}D_{\mu}\varphi D^{\mu}\varphi D_{\nu}\varphi D^{\nu}\varphi$$
$$= \frac{C}{3^{4}}\sqrt{-\tilde{g}}d^{4}\tilde{x}\tilde{\varepsilon}_{\alpha\beta\gamma\delta}\tilde{\varepsilon}^{\alpha\beta\gamma\delta}D_{\mu}\varphi D^{\mu}\varphi D_{\nu}\varphi D^{\nu}\varphi$$
$$= \frac{8C}{3^{3}}\sqrt{-\tilde{g}}d^{4}\tilde{x}\left(D_{\mu}\varphi\right)^{4}. \tag{C.14}$$

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Introducing (C.12), (C.13), (C.14) into (C.12), we find

$$C\tilde{\varepsilon}_{mnpq}\tilde{k}^m_f\tilde{k}^{fn}\tilde{k}^p{}_g\tilde{k}^{gq} = 0, \tag{C.15}$$

which proves that the sixth term is null.

Seventh term: in this case we can write,

$$\begin{aligned} 2C\tilde{\varepsilon}_{mnpq}\tilde{R}^{mn}\tilde{k}^p_f\tilde{k}^{fq} &= \frac{2C}{3^2}\tilde{\varepsilon}_{mnpq}\tilde{R}^{mn}\left(2d\varphi\tilde{k}^p\tilde{e}^q - \tilde{k}^2\tilde{e}^p\tilde{e}^q\right) \\ &= \frac{4C}{3^2}d\varphi\tilde{\varepsilon}_{mnpq}\tilde{R}^{mn}\tilde{k}^p\tilde{e}^q - \frac{2C}{3^2}\tilde{k}^2\tilde{\varepsilon}_{mnpq}\tilde{R}^{mn}\tilde{e}^p\tilde{e}^q, \end{aligned} \tag{C.16}$$

where,

$$\begin{aligned} \frac{4C}{3^2}d\phi\tilde{\varepsilon}_{mnpq}\tilde{R}^{mn}\tilde{k}^p\tilde{e}^q &= \frac{2C}{3^2}D_\mu\phi\tilde{\varepsilon}_{mnpq}\tilde{e}^m_\alpha\tilde{e}^n_\beta\tilde{e}^p_\gamma\tilde{e}^q_\delta dx^\mu dx^\lambda dx^\rho dx^\delta\tilde{R}^{\alpha\beta}_{\lambda\rho}D^\gamma\varphi \\ &= \frac{2C}{3^2}\sqrt{-\tilde{g}}d^4\tilde{x}\tilde{\varepsilon}_{\alpha\beta\gamma\delta}\tilde{\varepsilon}^{\mu\lambda\rho\delta}D_\mu\varphi D^\gamma\varphi\tilde{R}^{\alpha\beta}_{\lambda\rho} \\ &= \frac{2C}{3^2}\sqrt{-\tilde{g}}d^4\tilde{x}\delta^{\mu\lambda\rho}_{\alpha\beta\gamma}D_\mu\varphi D^\gamma\varphi\tilde{R}^{\alpha\beta}_{\lambda\rho}, \\ &= \frac{-8C}{3^2}\sqrt{-\tilde{g}}d^4\tilde{x}D^\alpha\varphi D^\beta\varphi\left\{\tilde{R}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\tilde{R}\right\} \end{aligned} \tag{C.17}$$

$$-\frac{2C}{3^2}\tilde{k}^2\tilde{\varepsilon}_{mnpq}\tilde{R}^{mn}\tilde{e}^p\tilde{e}^q = -\frac{8C}{3^2}\sqrt{-\tilde{g}}d^4\tilde{x}D_\nu\varphi D^\nu\varphi\tilde{R}. \tag{C.18}$$

Introducing (C.17), (C.18) in (C.16), we have

$$2C\tilde{\varepsilon}_{mnpq}\tilde{R}^{mn}\tilde{k}^p_f\tilde{k}^{fq} = \frac{-8C}{3^2}\sqrt{-\tilde{g}}d^4\tilde{x}D^\alpha\varphi D^\beta\varphi\tilde{G}_{\alpha\beta} - \frac{8C}{3^2}\sqrt{-\tilde{g}}d^4\tilde{x}D_\nu\varphi D^\nu\varphi\tilde{R} \tag{C.19}$$

Eighth term: as in the previous cases we write

$$\begin{aligned} 2C\tilde{\varepsilon}_{mnpq}D\tilde{k}^{mn}\tilde{k}^p_f\tilde{k}^{fq} &= -\frac{2}{3\cdot 3^2}C\tilde{\varepsilon}_{mnpq}D\left(\tilde{k}^m\tilde{e}^n - \tilde{k}^n\tilde{e}^m\right)\left(2d\phi\tilde{k}^p\tilde{e}^q - \tilde{k}^2\tilde{e}^p\tilde{e}^q\right) \\ &= -\frac{8C}{3^3}d\phi\tilde{\varepsilon}_{mnpq}D\tilde{k}^m\tilde{e}^n\tilde{k}^p\tilde{e}^q + \frac{4C}{3^3}\tilde{k}^2\tilde{\varepsilon}_{mnpq}D\tilde{k}^m\tilde{e}^n\tilde{e}^p\tilde{e}^q. \end{aligned} \tag{C.20}$$

Analyzing each term separately, we have,

$$\begin{aligned} -\frac{8C}{3^3}d\varphi\tilde{\varepsilon}_{mnpq}D\tilde{k}^m\tilde{e}^n\tilde{k}^p\tilde{e}^q &= -\frac{8C}{3^3}\tilde{\varepsilon}_{mnpq}\tilde{e}^m_\alpha\tilde{e}^n_\beta\tilde{e}^p_\gamma\tilde{e}^q_\delta dx^\mu dx^\nu dx^\beta dx^\delta D_\mu\varphi D_\nu D^\alpha\varphi D^\gamma\varphi \\ &= \frac{8C}{3^3}\sqrt{-\tilde{g}}d^4\tilde{x}\tilde{\varepsilon}_{\alpha\gamma\beta\delta}\tilde{\varepsilon}^{\mu\nu\beta\delta}D_\mu\varphi D_\nu D^\alpha\varphi D^\gamma\varphi \\ &= \frac{16C}{3^3}\sqrt{-\tilde{g}}d^4\tilde{x}\left\{D_\alpha\varphi\left(D_\gamma D^\alpha\varphi\right)D^\gamma\varphi - D_\gamma\varphi D^\gamma\varphi\left(D_\alpha D^\alpha\varphi\right)\right\} \\ & \frac{16C}{3^3}\sqrt{-\tilde{g}}d^4\tilde{x}\left\{\frac{1}{2}D_\gamma\left[D_\alpha\varphi D^\alpha\varphi D^\gamma\varphi\right] - \frac{3}{2}D_\alpha\varphi D^\alpha\varphi\left(D_\gamma D^\gamma\varphi\right)\right\}, \end{aligned} \tag{C.21}$$

where we have used

$$\left(D_\gamma D_\alpha\varphi\right)D^\alpha\varphi D^\gamma\varphi = \frac{1}{2}D_\gamma\left[D_\alpha\varphi D^\alpha\varphi D^\gamma\varphi\right] - \frac{1}{2}D_\alpha\varphi D^\alpha\varphi D_\gamma D^\gamma\varphi.$$

For the second term of (C.20), it is found,

$$
\begin{aligned}
\frac{4C}{3^3}\tilde{k}^2\tilde{\varepsilon}_{mnpq}D\tilde{k}^m\tilde{e}^n\tilde{e}^p\tilde{e}^q &= \frac{4C}{3^3}\tilde{\varepsilon}_{mnpq}\tilde{e}^m_\alpha\tilde{e}^n_\beta\tilde{e}^p_\gamma\tilde{e}^q_\delta dx^\nu dx^\beta dx^\gamma dx^\delta D_\mu\varphi D^\mu\varphi D_\nu D^\alpha\varphi \\
&= \frac{4C}{3^3}\sqrt{-\tilde{g}}d^4\tilde{x}\tilde{\varepsilon}_{\alpha\beta\gamma\delta}\tilde{\varepsilon}^{\nu\beta\gamma\delta}D_\mu\varphi D^\mu\varphi D_\nu D^\alpha\varphi \\
&= \frac{8C}{3^2}\sqrt{-\tilde{g}}d^4\tilde{x}D_\mu\varphi D^\mu\varphi D_\alpha D^\alpha\varphi. \qquad \text{(C.22)}
\end{aligned}
$$

Introducing (C.21), (C.22) into (C.20), we find that the eighth term is given by,

$$
\begin{aligned}
2C\tilde{\varepsilon}_{mnpq}D\tilde{k}^{mn}\tilde{k}^p_f\tilde{k}^{fq} &= \frac{16C}{3^3}\sqrt{-\tilde{g}}d^4\tilde{x}\left\{\frac{1}{2}D_\gamma\left[D_\alpha\varphi D^\alpha\varphi D^\gamma\varphi\right] - \frac{3}{2}D_\alpha\varphi D^\alpha\varphi\left(D_\gamma D^\gamma\varphi\right)\right\} \\
&= \sqrt{-\tilde{g}}d^4\tilde{x}\frac{8C}{3^3}D_\gamma A^\gamma, \text{ con } A^\gamma = D_\alpha\varphi D^\alpha\varphi D^\gamma\varphi\,. \qquad \text{(C.23)}
\end{aligned}
$$

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