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$w_{1+\infty}$ and Carrollian holography

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Abstract: In a $1+2$D Carrollian conformal field theory, the Ward identities of the two local fields S_0^+ and S_1^+, entirely built out of the Carrollian conformal stress-tensor, contain respectively up to the leading and the subleading positive helicity soft graviton theorems in the $1+3$D asymptotically flat space-time. This work investigates how the subsubleading soft graviton theorem can be encoded into the Ward identity of a Carrollian conformal field S_2^+. The operator product expansion (OPE) $S_2^+S_2^+$ is constructed using general Carrollian conformal symmetry principles and the OPE commutativity property, under the assumption that any time-independent, non-Identity field that is mutually local with S_0^+, S_1^+, S_2^+ has positive Carrollian scaling dimension. It is found that, for this OPE to be consistent, another local field S_3^+ must automatically exist in the theory. The presence of an infinite tower of local fields $S_{k\geq 3}^+$ is then revealed iteratively as a consistency condition for the $S_2^+S_{k-1}^+$ OPE. The general $S_k^+S_l^+$ OPE is similarly obtained and the symmetry algebra manifest in this OPE is found to be the Kac-Moody algebra of the wedge sub-algebra of $w_{1+\infty}$. The Carrollian time-coordinate plays the central role in this purely holographic construction. The 2D Celestial conformally soft graviton primary $H^k(z,\bar{z})$ is realized to be contained in the Carrollian conformal primary $S_{1-k}^+(t,z,\bar{z})$. Finally, the existence of the infinite tower of fields S_k^+ is shown to be directly related to an infinity of positive helicity soft graviton theorems.

Keywords: Conformal and W Symmetry, Field Theories in Lower Dimensions, Space-Time Symmetries, Scattering Amplitudes

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Contents

1 Introduction

The thriving research program to understand the holographic principle [1, 2] for the case of the $1 + 3$D asymptotically flat space-times (AFS) has been approached mainly via two seemingly different avenues. The currently most well-developed one is known as the Celestial holography [3] where the dual Celestial CFT [4] is thought to live on a Celestial sphere S^2 at the null-boundary of the AFS. The Celestial conformal fields depend on the two stereographic coordinates $(z, \bar{z})$ on the Celestial S^2 and are labeled by a Celestial conformal weight Δ_c which is a continuous parameter [5]; thus, these fields effectively are functions of three variables. In the other framework known as the Carrollian holography, gaining attention in the recent years [6–11], the dual Carrollian CFT (CarrCFT) resides on the three-dimensional null-infinity (with topology $\mathbb{R} \times S^2$) where the Carrollian conformal fields also depend on three variables: $(z, \bar{z})$ as well as the null-coordinate or the Carrollian time t.

The central idea of the Celestial holography [4, 5] is that the null-momentum space scattering amplitude of a mass-less scattering process in the $1 + 3$D bulk AFS can be re-expressed as a 2D Euclidean CFT correlation function, via a Mellin transformation that trades the energies $\{\omega\}$ of the external bulk scattering particles for the (continuous) Celestial conformal weights $\{\Delta_c\}$. Even before the explicit examples presented in [4, 5], that the holographic dual of the (quantum) gravity-theory in the $1 + 3$D AFS might be a 2D Euclidean CFT was partly motivated by the realization that the Weinberg (leading) soft graviton theorem [12] and the Cachazo-Strominger subleading soft graviton theorem [13] can be respectively recast into the 2D CFT U(1) Kac-Moody [14] and the 2D CFT energy-momentum (EM) tensor Ward identities [15]. These works were in turn inspired by linking up the following four observations: that the asymptotic symmetry group at the null-infinity of the $1+3$D AFS is the BMS_4 group [16, 17] in the presence of the gravitational radiation, that the Weinberg soft graviton theorem [12] is completely equivalent to the BMS_4 super-translation Ward identity [14, 18], that the Lorentz $\mathrm{SL}(2,\mathbb{C})$ subgroup of the original BMS_4 group should be infinitely enhanced to include the 2D singular conformal transformations (super-rotations) on the Celestial S^2 [19–21] and finally, that the Cachazo-Strominger subleading soft graviton theorem [13] implies a Virasoro symmetry of the $1+3$D AFS gravitational S-matrix [22].

On the other hand, the starting point of the Carrollian holography is the observation [23, 24] that the original BMS_4 group [16, 17] is isomorphic to the $1+2$D conformal Carroll group (at level 2) on a Carrollian manifold with topology $\mathbb{R}\times S^2$ (with S^2 equipped with the round metric). Inspired by this isomorphism, it was shown in [9] how the EM tensor Ward identities of a $1+2$D source-less CarrCFT on a Carrollian background with topology $\mathbb{R}\times S^2$ with assumed Weyl invariance can encode the $1+3$D bulk AFS leading [12, 18] and the subleading [13, 22] soft graviton theorems as well as the 2D Celestial CFT EM tensor [15] and the (BMS_4) super-translation Kac-Moody [14] Ward identities.

In addition to the leading [12] and subleading [13] soft graviton theorems, another soft graviton theorem at the subsubleading order was found in [13] explicitly for tree-level Einstein gravity in the $1+3$D bulk AFS. But, unlike the soft theorems at the two more leading orders, this subsubleading soft theorem is not universal [25, 26]. Equivalently in the Carrollian picture, the fact that the special Carrollian conformal fields S_0^+, S_1^+ and T, whose Ward identities contain the two universal AFS soft graviton theorems can be constructed purely out of the CarrCFT EM tensor [9], points towards an universality in the sense that every CarrCFT is expected to have an EM tensor. Consistently, the non-universal AFS subsubleading soft graviton theorems were not captured by those CarrCFT Ward identities.

In this work, we aim to understand how the $1+3$D bulk AFS subsubleading soft graviton theorem [13] can be encoded in the framework of the $1+2$D CarrCFT. Obviously, to provide a holographic description of a non-universal phenomenon in the bulk, the dual boundary theory also must possess some non-universal features. For this purpose, taking cue from the field contents of a 2D Euclidean CFT [27] or a $1+1$D CarrCFT [28] enjoying infinite additional symmetries, we postulate that, in a $1+2$D CarrCFT that carries the imprint of the positive-helicity subsubleading soft graviton theorem, there is a local quantum field S_2^+ in addition to the three universal local generators S_0^+, S_1^+ and T.

To avoid the potentially problematic hologram of the bulk ambiguity associated with the double soft-limits of opposite helicities [29], we do not attempt to simultaneously describe the negative-helicity subsubleading soft theorem in this work. By ignoring one helicity, we are not inviting inconsistency per se — we are focusing only on the holomorphic-sector of the $1+2$D CarrCFT; so, our holographic analysis will correspond only to the positive-helicity sectors of the bulk theories of gravitons. An exception is the MHV sector [30] of the tree-level Einstein gravity, where the opposite helicity soft sectors decouple and our conclusions are rendered applicable to both sectors (with trivial modifications).

Following the methods, elaborated in [9], to completely determine the singular parts of the mutual operator product expansions (OPEs) of the generators S_0^+ and S_1^+, we derive the mutual OPEs of S_2^+ and the two aforementioned generators. This algorithm only makes use of the general Carrollian symmetry principles and the OPE commutativity property under the assumption that there are no time-independent (in the OPE limit) local fields with negative Carrollian scaling dimension in the CarrCFT. Similar symmetry arguments (together with similar assumptions) fixed the singularities of the mutual OPEs of the symmetry generators in the cases of 2D Euclidean CFT in [27, 31] and of $1+1$D CarrCFT in [28, 32].

While trying to find the $S_2^+ S_2^+$ OPE, we realize that for this OPE to be consistent, another local Carrollian conformal field S_3^+ must automatically appear in the theory. Extending the algorithm, we find in general that for the CarrCFT OPE $S_2^+ S_{k\geq 2}^+$ to be consistent, there must already exist an infinite tower of local fields $S_{k+1}^+(t, z, \bar{z})$. The general CarrCFT OPE $S_k^+ S_l^+$ (4.21) is the main result of this work.

From this general $S_k^+(\mathbf{x}) S_l^+(\mathbf{x}_p)$ OPE (4.21), we immediately recover the 2D Celestial conformal OPE of two conformally soft [33, 34] primary gravitons $H^{1-k}(z,\bar{z}) H^{1-l}(z_p, \bar{z}_p)$ of [35], just by inspection. This conformally soft graviton OPE in [35] was directly obtained by taking the conformally soft limit of the general OPE between two general 2D Celestial conformal primary gravitons of arbitrary weights, derived in [36]. But the derivation of the general graviton primary OPE in [36] was explicitly for the tree-level (linearized) Einstein theory in the bulk AFS and required some hints from this specific bulk theory to fix the singular structure of the said OPE. In another method, this OPE was obtained via a Mellin transformation from the bulk graviton scattering amplitude in the collinear limit in [35, 36].

It is to be emphasized that we have not obtained the general Celestial conformal primary graviton OPE of [36] since that is a theory-specific result. Rather, we have given a completely holographic, Carrollian conformal symmetric derivation of the OPEs between the symmetry generators [35] that is expected to contain the asymptotic symmetry algebra of any (quantum) gravity theory in the $1+3$D bulk AFS. Another important difference is that while the above mentioned Celestial CFT OPEs in [35, 36] are valid only at tree-level Einstein theory (but exact for quantum self-dual gravity [37]), the Carrollian conformal derivation of the $S_k^+ S_l^+$ OPE (under the aforementioned assumption) involves no (Carrollian) perturbation theory analysis; its starting point is the CarrCFT Ward identities derived in [9] using a Carrollian path-integral formalism [38].

Using the $1+2$D CarrCFT OPE $\longleftrightarrow$ commutation-relation prescription developed in [9], we find that the (local) symmetry algebra manifest in the CarrCFT OPE (4.21) is the Kac-Moody algebra of the wedge subalgebra [39] of the $w_{1+\infty}$ algebra [40], in perfect agreement with the symmetry algebra derived in [41] from the Celestial conformally soft primary graviton OPE of [35].

Finally, we shed light on the direct connection between the existence of the infinite tower of Carrollian fields S_k^+ and an infinity of the soft graviton theorems discussed in [42, 43] in the context of the tree-level (linearized) Einstein theory. We find that the Ward identity of the Carrollian field S_2^+ does indeed encode up to the positive helicity subsubleading soft graviton theorems [13]. While it is hinted that the Ward identities of the other fields $S_{k>2}^+$ contain the soft theorems in more subleading orders, it is also demonstrated that the fields

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beyond S_2^+ does not generate any new (independent) global symmetries of the theory. This is the CarrCFT analogue of the fact that the Celestial conformally soft gravitons $H^{k<-1}$ do not impose any new constraints on the $1+3$D bulk AFS S-matrices [35].

The rest of the paper is organized as follows. In section 2, we review the main results of [9] on the universal features of a $1+2$D CarrCFT. We introduce the Carrollian conformal field S_2^+ whose Ward identity (supposedly) encodes the $1+3$D bulk AFS positive-helicity subsubleading soft graviton theorem, in section 3. We state the assumptions on the field content of the CarrCFT in section 3.1 and find that for the $S_2^+S_2^+$ OPE to be consistent, a local field S_3^+ must already exist in the theory. Our purely symmetry-based algorithm to find the OPEs reveal the automatic existence of the infinite tower of Carrollian fields $S_{k\geq 3}^+$ in section 4. We obtain the general $S_k^+S_l^+$ OPE in section 4.3, from which the quantum symmetry algebra is determined in section 5. Finally, in section 6 we relate the infinite tower of fields $S_{k\geq 2}^+$ with an infinite number of soft graviton theorems before concluding in section 7.

2 Review

In [9], it was shown how the EM tensor Ward identities of an honest (i.e. source-less) Carrollian CFT on a $1+2$D flat Carrollian background (with topology $\mathbb{R}\times S^2$ and the assumption of the Weyl invariance) can be recast into the forms resembling to the $1+3$D bulk AFS leading and subleading conformally soft [33, 34] graviton theorems [30, 44]. The $1+2$D Carrollian conformal fields $S_0^\pm$ and $S_1^\pm$ containing respectively the leading and the subleading conformally soft graviton primaries of the 2D Celestial CFT, as well as the fields T and $\bar{T}$ that contain respectively the holomorphic and the anti-holomorphic components of the 2D Celestial CFT EM tensor [15] were constructed purely out of the Carrollian EM tensor components $T^\mu{}_\nu$. Below we note the generic Ward identities[1] of the $1+2$D CarrCFT EM tensor, derived in [9] using the Carrollian path-integral formalism [38]:

$$\text{Translation:}\ \ \partial_\mu\langle T^\mu{}_\nu(\mathbf{x})X\rangle = -i\sum_{p=1}^{n}\ \partial_{\nu_p}\langle X\rangle\ \delta(t-t_p)\delta^2(\vec{x}-\vec{x}_p) \tag{2.1}$$

$$\text{Boost:}\ \ \langle T^i{}_t(\mathbf{x})X\rangle = -i\sum_{p=1}^{n}\ (\boldsymbol{\xi}_i)_p\cdot\langle X\rangle\ \delta(t-t_p)\delta^2(\vec{x}-\vec{x}_p) \tag{2.2}$$

$$\text{Dilation:}\ \ \langle T^\mu{}_\mu(\mathbf{x})X\rangle = -i\sum_{p=1}^{n}\ \Delta_p\langle X\rangle\ \delta(t-t_p)\delta^2(\vec{x}-\vec{x}_p) \tag{2.3}$$

$$\text{Rotation:}\ \ \langle T^z{}_z(\mathbf{x})X\rangle - \langle T^{\bar{z}}{}_{\bar{z}}(\mathbf{x})X\rangle = -i\sum_{p=1}^{n}\ s_p\langle X\rangle\ \delta(t-t_p)\delta^2(\vec{x}-\vec{x}_p) \tag{2.4}$$

where X is a string of Carrollian conformal primary multiplets; $(\boldsymbol{\xi}_i)_p$ is the Carrollian boost (in the i-th direction) representation-matrix, s_p is the spin (i.e. the eigenvalue of the spatial rotation) and Δ_p is the Carrollian conformal weight of the p-th primary field.

[1]All the correlators and OPEs considered in this work are implicitly covariant time-ordered (as defined in section 6.1.4. of [47]). Covariant time-ordering commutes with space-time differentiation and integration.

2.1 The fields $S_0^{\pm}$

Subtraction of the spatial divergence of (2.2) from $(2.1)_{\nu=t}$ leads to:

$$\partial_t\langle T^t{}_t(t,\vec{x})X\rangle = -i\sum_{p=1}^{n}\,\delta(t-t_p)\left[\delta^2(\vec{x}-\vec{x}_p)\partial_{t_p} - \left(\vec{\boldsymbol{\xi}}_p\cdot\boldsymbol{\nabla}\right)\delta^2(\vec{x}-\vec{x}_p)\right]\langle X\rangle \tag{2.5}$$

Choosing the following initial condition:

$$\langle T^t{}_t(t\to-\infty,\vec{x})X\rangle = 0 \tag{2.6}$$

the solution to the above temporal partial differential equation is obtained as:

$$\langle T^t{}_t(t,\vec{x})X\rangle = -i\sum_{p=1}^{n}\,\theta(t-t_p)\left[\delta^2(\vec{x}-\vec{x}_p)\partial_{t_p} - \left(\vec{\boldsymbol{\xi}}_p\cdot\boldsymbol{\nabla}\right)\delta^2(\vec{x}-\vec{x}_p)\right]\langle X\rangle \tag{2.7}$$

The S^2 contact-term singularities in (2.7) can be converted into pole singularities (but avoiding branch-cuts) if we note that:

$$\langle T^t{}_t(t,\vec{x})X\rangle = -\frac{i}{\pi}\sum_{p=1}^{n}\,\theta(t-t_p)\;\bar{\partial}^2\left[\frac{\bar{z}-\bar{z}_p}{z-z_p}\partial_{t_p} + \frac{\bar{z}-\bar{z}_p}{(z-z_p)^2}\boldsymbol{\xi}_p - \frac{\bar{\boldsymbol{\xi}}_p}{z-z_p}\right]\langle X\rangle \tag{2.8}$$

$$= -\frac{i}{\pi}\sum_{p=1}^{n}\,\theta(t-t_p)\;\partial^2\left[\frac{z-z_p}{\bar{z}-\bar{z}_p}\partial_{t_p} + \frac{z-z_p}{(\bar{z}-\bar{z}_p)^2}\bar{\boldsymbol{\xi}}_p - \frac{\boldsymbol{\xi}_p}{\bar{z}-\bar{z}_p}\right]\langle X\rangle \tag{2.9}$$

Inverting the $\bar{\partial}^2$ operator in (2.8) and the ∂^2 operator in (2.9), the Carrollian fields $S_0^{\pm}$ were respectively defined in [9] as:

$$S_0^+(t,z,\bar{z}) := \int_{S^2} d^2r'\;\frac{\bar{z}-\bar{z}'}{z-z'}\;T^t{}_t(t,\vec{x}') \quad\Longrightarrow\quad \bar{\partial}^2 S_0^+ = \bar{\partial}P = \pi T^t{}_t \tag{2.10}$$

$$S_0^-(t,z,\bar{z}) := \int_{S^2} d^2r'\;\frac{z-z'}{\bar{z}-\bar{z}'}\;T^t{}_t(t,\vec{x}') \quad\Longrightarrow\quad \partial^2 S_0^- = \partial\bar{P} = \pi T^t{}_t \tag{2.11}$$

where the descendant fields $P = \bar{\partial}S_0^+$ and $\bar{P} = \partial S_0^-$ consist respectively of the modes generating the holomorphic and the anti-holomorphic super-translations. The scaling dimension Δ and the spin m of the fields $S_0^{\pm}$ are $(\Delta, m) = (1, \pm 2)$. So, the defining relations (2.10) and (2.11) imply that $S_0^{\pm}$ are the 2D shadow-transformations (on S^2) of each other [45] by construction. Since, a field and its shadow (being a highly non-local integral transformation) can not both be treated as local fields in a theory [46], only one among $S_0^{\pm}$ is to be chosen as a local field while relegating the other merely to its non-local shadow.

We treat S_0^+ as the local field and S_0^- as its shadow in this work. This corresponds to investigating the holomorphic sector of the 1 + 2D Carrollian CFT. The S_0^+ Ward identity, for a string X of n mutually local Carrollian conformal primary multiplet fields $\{\Phi_p(t_p, z_p, \bar{z}_p)\}$, reads [9]:

$$\langle S_0^+(t,z,\bar{z})X\rangle = -i\sum_{p=1}^{n}\,\theta(t-t_p)\left\{\frac{\bar{z}-\bar{z}_p}{z-z_p}\partial_{t_p} + \frac{\bar{z}-\bar{z}_p}{(z-z_p)^2}\boldsymbol{\xi}_p - \frac{\bar{\boldsymbol{\xi}}_p}{z-z_p}\right\}\langle X\rangle \tag{2.12}$$

$$\Longrightarrow\; \langle\bar{\partial}^2 S_0^+(t,z,\bar{z})X\rangle = -i\sum_{p=1}^{n}\,\theta(t-t_p)\;\left[\text{contact terms on } S^2\right]$$

$$\text{and}\quad \partial_t\langle S_0^+(t,z,\bar{z})X\rangle = [\text{temporal contact terms}]$$

where $\boldsymbol{\xi}, \bar{\boldsymbol{\xi}} := \boldsymbol{\xi}_x \pm i\boldsymbol{\xi}_y$ denote the matrix-representation of the classical Carrollian boost under which a Carrollian multiplet Φ transforms.[2]

The above correlator was derived from the (Carrollian) super-translation Ward identity in [9]. The temporal step-function appearing in this Carrollian correlator captures the essence of the $1+3$D bulk AFS super-translation memory effect [49, 51]. Consistently, temporal-Fourier transforming (2.12) to (positive) ω-space and then making the S_0^+ field energetically soft [49], one recovers the Weinberg (leading) positive-helicity soft graviton theorem [12, 18] as the residue of the leading $\frac{1}{\omega}$ pole when all of the primaries in X have $\boldsymbol{\xi} = \bar{\boldsymbol{\xi}} = 0$ [9]. Thus, explicitly at the $t \to \infty$ limit, the Ward identity (2.12) is same as the positive-helicity leading conformally soft graviton theorem [44] when all $\boldsymbol{\xi}_p = \bar{\boldsymbol{\xi}}_p = 0$.

The S^2 stereographic coordinates z and $\bar{z}$ are now treated as independent variables [52] so that terms like $\frac{(\bar{z}-\bar{z}_p)^r}{(z-z_p)^s}$ with $r \geq 0, s \geq 1$ have (meromorphic) pole singularity (avoiding phase ambiguity when $r = s$). Together with the form of the Ward identity (2.12), this suggests that, inside the correlator, S_0^+ can be decomposed as [30]:

$$
\begin{aligned}
& S_0^+(t,z,\bar{z}) = \bar{z}P_{-1}(t,z,\bar{z}) - P_0(t,z,\bar{z}) \qquad (2.13)\\
\text{with} \quad & \langle P_{-1}(t,z,\bar{z})X\rangle = -i\sum_{p=1}^{n} \theta(t-t_p)\left(\frac{\partial_{t_p}}{z-z_p} + \frac{\boldsymbol{\xi}_p}{(z-z_p)^2}\right)\langle X\rangle \\
\text{and} \quad & \langle P_0(t,z,\bar{z})X\rangle = -i\sum_{p=1}^{n} \theta(t-t_p)\left(\frac{\bar{z}_p\partial_{t_p} + \bar{\boldsymbol{\xi}}_p}{z-z_p} + \frac{\bar{z}_p\boldsymbol{\xi}_p}{(z-z_p)^2}\right)\langle X\rangle \\
\implies \quad & \langle \bar{\partial} P_i(t,z,\bar{z})X\rangle = -i\sum_{p=1}^{n} \theta(t-t_p)\ \left[\text{contact terms on } S^2\right]
\end{aligned}
$$

These relations reminisce holomorphic Kac-Moody like Ward identities in a 2D Euclidean CFT. Clearly, P_i and S_0^+ have the same holomorphic weight $h = \frac{3}{2}$.

Let us next discuss on the Carrollian conformal OPEs. As explained in [9, 32], it is convenient to convert the temporal step-function appearing in the correlators like (2.12) into a $j\epsilon$-prescription for this purpose, with j being a second complex unit. The starting point is to hyper-complexify the $(t,z,\bar{z})$ coordinates as below:

$$
\hat{z} := z + jt \quad ; \quad \hat{\bar{z}} := \bar{z} + jt \quad ; \quad \hat{t} := t
$$

While z is a complex number on the $x-y$ plane, $\hat{z}$ can be thought of as a complex number on a $y = ax + b$ plane of the 3D $t-x-y$ space. $t > 0$ is the upper-half of this plane.

It can be shown that all of $\frac{\partial \hat{z}}{\partial \hat{t}}, \frac{\partial \hat{\bar{z}}}{\partial \hat{t}}, \frac{\partial \hat{\bar{z}}}{\partial \hat{z}}$ vanish. So, $\hat{t}, \hat{z}, \hat{\bar{z}}$ can be treated as independent variables. Thus, in most cases, we choose the point of insertion of a Carrollian field to be at $(\hat{t}, \hat{z}, \hat{\bar{z}}) = (t, z, \bar{z})$. E.g. the $j\epsilon$-form of the Ward identity (2.12) is (with $\Delta\tilde{z}_p := z - z_p - j\epsilon(t-t_p)$):

$$
i\langle S_0^+(t,z,\bar{z})X\rangle = \lim_{\epsilon\to 0^+}\sum_{p=1}^{n}\left\{\frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\partial_{t_p} + \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)^2}\boldsymbol{\xi}_p - \frac{\bar{\boldsymbol{\xi}}_p}{(\Delta\tilde{z}_p)}\right\}\langle X\rangle
$$

[2]See [48] for a more general representation theory of the global Carrollian conformal algebra.

The main application of this $j\epsilon$-prescription is to establish the relation between the CarrCFT OPEs and the corresponding commutation relations while allowing for a straightforward utilization of the OPE commutativity property.

It is important to remember that $\frac{1}{\Delta\tilde{z}_p}$ reduces to $\frac{1}{z-z_p}$ only when $t-t_p>0$ and to 0 when $t-t_p<0$ in the sense of distributions [32] encoding the property of the temporal step-function. Thus, $\frac{1}{\Delta\tilde{z}_p}\equiv\frac{1}{z-z_p}$ when $t\to\infty$, and $\frac{1}{\Delta\tilde{z}_p}\equiv 0$ when $t\to-\infty$, thus providing a justification to the initial condition (2.6).

The OPE[3] of S_0^+ with a general (non-primary) Carrollian conformal field Φ (that is mutually local with S_0^+) is noted below [9]:

$$iS_0^+(t,z,\bar{z})\Phi(\mathbf{x}_p)\sim\lim_{\epsilon\to0^+}\left[(\bar{z}-\bar{z}_p)\left(\sum_{n\geq0}^{J}\frac{(P_{n,-1}\Phi)}{(\Delta\tilde{z}_p)^{n+2}}+\frac{\partial_{t_p}\Phi}{(\Delta\tilde{z}_p)}\right)-\sum_{n\geq-1}^{K}\frac{(P_{n,0}\Phi)}{(\Delta\tilde{z}_p)^{n+2}}\right](\mathbf{x}_p)\quad(2.14)$$

$$\implies\quad\bar{\partial}^2S_0^+(t,z,\bar{z})\Phi(\mathbf{x}_p)\sim0\quad(2.15)$$

$$\text{and}\quad\partial_tS_0^+(t,z,\bar{z})\Phi(\mathbf{x}_p)\sim0\quad(2.16)$$

where $\sim$ denotes 'modulo terms holomorphic (regular) in $\Delta\tilde{z}_p$ '. This OPE is a Laurent series in the holomorphic variable z (or $\tilde{z}=z-j\epsilon t$) but is (anti-)holomorphic (i.e. a Taylor series) in $\bar{z}$. Specially, a Carrollian conformal primary multiplet field Φ satisfies:

$$(P_{n+1,-1}\Phi)=0=(P_{n,0}\Phi)\qquad\text{for } n\geq0$$

while a Carrollian conformal quasi-primary (i.e. an only ISL$(2,\mathbb{C})$ covariant) multiplet only needs $(P_{0,0}\Phi)=0$.

2.2 The fields $S_1^{\pm}$

Next, we combine the CarrCFT EM tensor Ward identities (2.3), (2.4) and (2.7) into the following form:

$$\langle T^z{}_z(\mathbf{x})X\rangle+\frac{1}{2}\langle T^t{}_t(\mathbf{x})X\rangle=-i\sum_{p=1}^{n}h_p\langle X\rangle\,\delta(t-t_p)\delta^2(\vec{x}-\vec{x}_p)$$

$$\Rightarrow\; i\langle T^z{}_z(\mathbf{x})X\rangle=\sum_{p=1}^{n}\Big[\delta(t-t_p)\delta^2(\vec{x}-\vec{x}_p)h_p$$
$$-\frac{\theta(t-t_p)}{2}\left\{\delta^2(\vec{x}-\vec{x}_p)\partial_{t_p}-\left(\boldsymbol{\xi}_p\cdot\boldsymbol{\nabla}\right)\delta^2(\vec{x}-\vec{x}_p)\right\}\Big]\langle X\rangle$$

Thus, subtraction of $\partial_z\langle T^z{}_z(\mathbf{x})X\rangle$ from $(2.1)_{\nu=z}$ results into:

$$\partial_{\bar{z}}\langle T^{\bar{z}}{}_z(\mathbf{x})X\rangle+\partial_t\langle T^t{}_z(\mathbf{x})X\rangle=-i\sum_{p=1}^{n}\Big[\delta(t-t_p)\left\{\delta^2(\vec{x}-\vec{x}_p)\partial_{z_p}-h_p\partial_z\delta^2(\vec{x}-\vec{x}_p)\right\}$$
$$+\frac{\theta(t-t_p)}{2}\partial_z\left\{\delta^2(\vec{x}-\vec{x}_p)\partial_{t_p}-\left(\boldsymbol{\xi}_p\cdot\boldsymbol{\nabla}\right)\delta^2(\vec{x}-\vec{x}_p)\right\}\Big]\langle X\rangle$$

[3]The two Carrollian operators whose product is to be expanded are inserted at different spatial-positions as well as at different times [32].

Choosing an initial condition similar to (2.6), one obtains the following solution to the above temporal partial differential equation:

$$\langle T^t{}_z(\mathbf{x})X\rangle + \int_{-\infty}^{t} dt'\ \partial_{\bar z}\langle T^{\bar z}{}_z(t',\vec{x})X\rangle = -\,i\sum_{p=1}^{n}\theta(t-t_p)\Big[\delta^2(\vec{x}-\vec{x}_p)\partial_{z_p} - h_p\partial_z\delta^2(\vec{x}-\vec{x}_p) + \frac{t-t_p}{2}\partial_z\Big\{\delta^2(\vec{x}-\vec{x}_p)\partial_{t_p} - \Big(\bar{\boldsymbol{\xi}}_p\cdot\boldsymbol{\nabla}\Big)\delta^2(\vec{x}-\vec{x}_p)\Big\}\Big]\langle X\rangle \tag{2.17}$$

Its 'complex-conjugated' version can be obviously derived in an exactly similar way.

To convert the S^2 contact terms in (2.17) (or, its complex-conjugate) into pole singularities, we extract a ∂^3 (or, $\bar\partial^3$) -derivative from the R.H.S.; inverting these derivatives, the Carrollian conformal fields $S_1^{\pm}$ that contain the sub-leading conformally soft gravitons [33, 34] were defined in [9] as:

$$S_1^-(t,z,\bar z) = \int_{S^2} d^2r'\ \frac{(z-z')^2}{\bar z-\bar z'}\left[T^t{}_z(t,\vec{x}') + \int_{-\infty}^{t} dt'\partial_{\bar z'}T^{\bar z}{}_z(t',\vec{x}')\right] \tag{2.18}$$

$$S_1^+(t,z,\bar z) = \int_{S^2} d^2r'\ \frac{(\bar z-\bar z')^2}{z-z'}\left[T^t{}_{\bar z}(t,\vec{x}') + \int_{-\infty}^{t} dt'\partial_{z'}T^{z}{}_{\bar z}(t',\vec{x}')\right] \tag{2.19}$$

The dimensions of the fields $S_1^{\pm}$ are $(\Delta, m) = (0, \pm 2)$.

The S_1^+ Ward identity with a string of mutually local Carrollian conformal primaries was obtained as [9]:

$$i\langle S_1^+(t,z,\bar z)X\rangle = \sum_{p=1}^{n}\theta(t-t_p)\left[\frac{(\bar z-\bar z_p)^2}{z-z_p}\partial_{\bar z_p} - 2\bar h_p\frac{\bar z-\bar z_p}{z-z_p} + (t-t_p)\left(\frac{\bar z-\bar z_p}{z-z_p}\partial_{t_p} + \frac{\bar z-\bar z_p}{(z-z_p)^2}\boldsymbol{\xi}_p - \frac{\bar{\boldsymbol{\xi}}_p}{z-z_p}\right)\right]\langle X\rangle \tag{2.20}$$

$$\Longrightarrow\quad \langle\bar\partial^3 S_1^+(t,z,\bar z)X\rangle = -i\sum_{p=1}^{n}\ \theta(t-t_p)\ \Big[\text{contact terms on } S^2\Big]$$

$$\text{and}\quad \partial_t\langle S_1^+(t,z,\bar z)X\rangle - \langle S_0^+(t,z,\bar z)X\rangle = [\text{temporal contact terms}] \tag{2.21}$$

that for all $\boldsymbol{\xi}_p = \bar{\boldsymbol{\xi}}_p = 0$ resembles (in the limit $t\to\infty$) the positive-helicity subleading conformally soft graviton theorem as presented (but very differently interpreted) in [30]. More appropriately, the Ward identity (2.20) is the $1+2$D Carrollian conformal manifestation of the 2D Celestial subleading conformally soft graviton theorem.

This Ward identity was derived from the (Carrollian) super-rotation Ward identity in [9]. There, it was also shown that, upon temporal-Fourier transforming (2.20) and then taking the $\omega\to 0^+$ limit only for S_1^+ [50], one obtains a Laurent expansion around $\omega = 0$, the coefficient of the leading $\frac{1}{\omega^2}$ pole of which is the Weinberg positive-helicity soft-graviton theorem [12, 18] while the subleading $\frac{1}{\omega}$ pole's coefficient is recognized to be the Cachazo-Strominger subleading positive-helicity soft-graviton theorem [13, 22] when each of the primaries in X has $\boldsymbol{\xi} = \bar{\boldsymbol{\xi}} = 0$

and (Carrollian) scaling dimension $\Delta = 1$. Therefore, the temporal step-function in (2.20) is the Carrollian manifestation of the super-rotation memory effect [50, 51].

The last condition on the scaling dimension of a Carrollian conformal primary whose temporal-Fourier transformation can correspond to a $1 + 3$D bulk AFS null momentum-space field (see also [53, 54]) describing a mass-less external hard scattering particle, was discovered in [7, 8] by analyzing the radiative fall-off conditions of the bulk mass-less fields. The higher dimensional counterpart of this condition was obtained more recently in [11].

(2.21) and (2.14) motivate us to re-express the S_1^+ field inside the correlator as below [9]:

$$\begin{aligned} & S_1^+(t,z,\bar{z}) = S_{1e}^+(t,z,\bar{z}) + tS_0^+(t,z,\bar{z}) \\ \Longrightarrow \quad & \partial_t \langle S_{1e}^+(t,z,\bar{z})X\rangle = [\text{temporal contact terms}] \\ \text{and} \quad & \langle \bar{\partial}^3 S_{1e}^+(t,z,\bar{z})X\rangle = -i\sum_{p=1}^{n} \theta(t-t_p)\ \left[\text{contact terms on } S^2\right] \end{aligned} \tag{2.22}$$

It should be noted that S_{1e}^+ is not a local Carrollian field but merely a collection of the modes not appearing in the S_0^+ field. These modes generate the following local infinitesimal Carrollian diffeomorphisms:

$$z \to z' = z \ \ ; \ \ \bar{z} \to \bar{z}' = \bar{z} + \varepsilon \bar{z}^{q+1} f(z) \ \ ; \ \ t \to t' = t + \varepsilon \frac{t}{2}(q+1)\bar{z}^q f(z)$$

with $f(z)$ being a meromorphic function and $q = 0, \pm 1$. It is the 'Ward identity' $\langle S_{1e}^+(t \to \infty, z, \bar{z})X\rangle$ that directly gives rise to the Cachazo-Strominger subleading energetically soft graviton theorem [13, 22].

Since, z and $\bar{z}$ are treated independently, the form of (2.20) allows us to decompose S_{1e}^+ inside a correlator as [30, 55]:

$$\begin{aligned} & S_{1e}^+(t,z,\bar{z}) = j_e^{(+)}(t,z,\bar{z}) - 2\bar{z} j_e^{(0)}(t,z,\bar{z}) + \bar{z}^2 j_e^{(-)}(t,z,\bar{z}) \\ \Longrightarrow \quad & \langle \bar{\partial} j_e^a(t,z,\bar{z})X\rangle = -i\sum_{p=1}^{n} \theta(t-t_p)\ \left[\text{contact terms on } S^2\right] \end{aligned} \tag{2.23}$$

Since, all three j_e^a have holomorphic weight $h = 1$ like S_1^+, their Ward identities are effectively same as the holomorphic Kac-Moody Ward identities in a usual 2D CFT.

Finally, we note down the general OPE of the S_1^+ field in the $j\epsilon$-form [9]:

$$\begin{aligned} i\left\{S_1^+ - (t-t_p)S_0^+\right\}(t,z,\bar{z})\Phi(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \Bigg[& (\bar{z}-\bar{z}_p)^2 \left(\sum_{n\geq 1}^{L} \frac{\left(j_n^{(-)}\Phi\right)}{(\Delta\tilde{z}_p)^{n+1}} + \frac{\partial_{\bar{z}_p}\Phi}{\Delta\tilde{z}_p} \right) + \sum_{n\geq 0}^{M} \frac{\left(j_n^{(+)}\Phi\right)}{(\Delta\tilde{z}_p)^{n+1}} \\ & -2(\bar{z}-\bar{z}_p) \left(\sum_{n\geq 1}^{N} \frac{\left(j_n^{(0)}\Phi\right)}{(\Delta\tilde{z}_p)^{n+1}} + \frac{\bar{h}_p\Phi}{\Delta\tilde{z}_p} \right) \Bigg] (\mathbf{x}_p) \end{aligned} \tag{2.24}$$

$$\Longrightarrow \quad \bar{\partial}^3 S_1^+(t,z,\bar{z})\Phi(\mathbf{x}_p) \sim 0 \tag{2.25}$$

$$\text{and} \quad \left(\partial_t S_1^+ - S_0^+\right)(t,z,\bar{z})\Phi(\mathbf{x}_p) \sim 0 \tag{2.26}$$

For a Carrollian conformal primary multiplet Φ, we have:

$$\left(j_n^{(-)}\Phi\right) = \left(j_n^{(0)}\Phi\right) = \left(j_{n-1}^{(+)}\Phi\right) = 0 \qquad \text{for } n \geq 1$$

whereas an $\mathrm{SL}(2,\mathbb{C})$ or Lorentz covariant quasi-primary must satisfy: $\left(j_0^{(+)}\Phi\right) = 0$.

Obviously, the corresponding correlators and OPEs involving the S_1^- field (replacing S_1^+) are just the complex conjugations ($z \to \bar{z}, \bar{h} \to h, \boldsymbol{\xi} \to \bar{\boldsymbol{\xi}}$) of the above mentioned equations. But as shown in [9], following [46], the fields S_0^+ and S_1^- can not be simultaneously treated as local fields i.e. they are not mutually local. The reason is that while $\left(\bar{\partial}^2 S_0^+\right) S_1^- \sim 0$ respects (2.15), $S_1^- \bar{\partial}^2 S_0^+$ contains anti-meromorphic pole singularities, thus violating the OPE commutativity property. This is a Carrollian manifestation of the ordering ambiguity in the double soft limit involving two opposite helicity particles [29]. On the other hand, the OPE of S_0^+ with S_1^+ does not suffer from this problem; hence, S_0^+ and S_1^+ can both be simultaneously taken as local fields that we do.

As we shall see, the OPE conditions like (2.15), (2.16), (2.25) and (2.26) will play very crucial roles in this work.

2.3 The fields T and $\bar{T}$

All the S^2 contact terms in the Ward identity (2.17) (or, its conjugate version) can also be converted into pole singularities by extracting from the R.H.S. a $\bar{\partial}$ (or, ∂) -derivative. The Carrollian fields T and $\bar{T}$ were then constructed in [9] by inverting these derivatives as:

$$T(t,z,\bar{z}) = \int_{S^2} d^2r' \frac{T^t{}_z(t,\vec{x}')}{z-z'} + \pi \int_{-\infty}^{t} dt' T^{\bar{z}}{}_z(t',\vec{x}) \implies 2\bar{\partial}T = \partial^3 S_1^- \tag{2.27}$$

$$\bar{T}(t,z,\bar{z}) = \int_{S^2} d^2r' \frac{T^t{}_{\bar{z}}(t,\vec{x}')}{\bar{z}-\bar{z}'} + \pi \int_{-\infty}^{t} dt' T^{z}{}_{\bar{z}}(t',\vec{x}) \implies 2\partial\bar{T} = \bar{\partial}^3 S_1^+ \tag{2.28}$$

Since, the field T has dimensions $(\Delta, m) = (2, 2)$ and $\bar{T}$ has $(\Delta, m) = (2, -2)$, the above relations, together with the fact that $S_1^\pm$ have $(\Delta, m) = (0, \pm 2)$, imply that $(S_1^+, \bar{T})$ and (S_1^-, T) automatically are two shadow pairs (on S^2). Since, we have chosen S_1^+ as a local field, $\bar{T}$ now has to be treated as its non-local shadow.

With mutually local primaries, the T Ward identity was derived from the super-rotation Ward identity to be [9]:

$$\begin{aligned} & i\langle T(t,z,\bar{z})X\rangle \\ &= \sum_{p=1}^{n} \theta(t-t_p)\left[\frac{h_p}{(z-z_p)^2} + \frac{\partial_{z_p}}{z-z_p} - \frac{t-t_p}{2}\left\{\frac{\partial_{t_p}}{(z-z_p)^2} + \frac{2\boldsymbol{\xi}_p}{(z-z_p)^3} + \pi\bar{\boldsymbol{\xi}}_p \partial_z \delta^2(\vec{x}-\vec{x}_p)\right\}\right]\langle X\rangle \end{aligned} \tag{2.29}$$

$$\text{with} \quad \langle \bar{\partial} T(t,z,\bar{z})X\rangle = -i\sum_{p=1}^{n} \theta(t-t_p)\left[\text{contact terms on } S^2\right]$$

$$\text{and} \quad \partial_t \langle T(t,z,\bar{z})X\rangle - \frac{1}{2}\langle \partial\bar{\partial} S_0^+(t,z,\bar{z})X\rangle = [\text{temporal contact terms}]$$

Inspired by the last relation, we decompose the T field inside the correlator as [9]:

$$\begin{aligned} T(t,z,\bar{z}) &= \frac{t}{2}\partial\bar{\partial}S_0^+(t,z,\bar{z}) + T_e(t,z,\bar{z}) \\ \implies \partial_t \langle T_e(t,z,\bar{z})X\rangle &= [\text{temporal contact terms}] \end{aligned} \tag{2.30}$$

where T_e is not a local Carrollian field but contains the modes generating the holomorphic super-rotations. It is the object $T_e(t \to \infty, z, \bar{z})$ that corresponds to the 2D Celestial stress-tensor [15]; by construction, it is the 2D shadow transformation of the negative-helicity energetically soft graviton $S^-_{1e}(t \to \infty, z, \bar{z})$. The $\langle T_e(t, z, \bar{z}) X \rangle$ 'Ward identity' is the Carrollian conformal analogue of that of the 2D Celestial holomorphic stress-tensor [56, 57] when all $\boldsymbol{\xi}_p = \bar{\boldsymbol{\xi}}_p = 0$.

The generic OPE involving the T field is given in the $j\epsilon$-form as [9]:

$$i \left(T - \frac{t - t_p}{2}\, \partial\bar{\partial} S^+_0 \right)(t, z, \bar{z}) \Phi(\mathbf{x}_p) \sim \lim_{\epsilon \to 0^+} \left[\sum_{n \geq 1}^{P} \frac{(L_n \Phi)}{(\Delta \tilde{z}_p)^{n+2}} + \frac{h_p \Phi}{(\Delta \tilde{z}_p)^2} + \frac{\partial_{z_p} \Phi}{\Delta \tilde{z}_p} \right] (\mathbf{x}_p) \tag{2.31}$$

$$\implies \quad \bar{\partial} T(t, z, \bar{z}) \Phi(\mathbf{x}_p) \sim 0 \quad \text{and} \quad \left(\partial_t T - \frac{1}{2} \partial\bar{\partial} S^+_0 \right)(t, z, \bar{z}) \Phi(\mathbf{x}_p) \sim 0 \tag{2.32}$$

For a Carrollian conformal primary, $(L_n \Phi) = 0$ for $n \geq 1$. An $\mathrm{SL}(2, \mathbb{C})$ quasi-primary on the other hand needs to satisfy only $(L_1 \Phi) = 0$.

In this work, we shall simultaneously treat the three Carrollian fields S^+_0, S^+_1 and T as $\mathrm{SL}(2, \mathbb{C})$ quasi-primary (non-descendant) local fields, as was shown to be allowed in [9].

3 The Carrollian conformal field S^+_2

The features reviewed in the previous section are the generic properties of any (Weyl invariant) $1 + 2$D Carrollian CFT on flat (Carrollian) background. The Carrollian conformal Ward identities that were shown [7–9] to contain the equivalent information as the bulk AFS leading [12] and subleading [13] soft graviton theorems were obtained as a consequence of only the Poincaré, the super-translation and the Weyl invariance in [9]. In view of the putative AFS/CarrCFT duality [23], this is in perfect agreement with the conclusions of [26] that in a generic theory of quantum gravity, only the leading and the subleading soft graviton theorems are universal. Consistently in [66], only these two soft graviton theorems were reached via a 'large AdS radius' limit from the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. The non-universality of the subsubleading soft graviton [13] theorem was discussed in [25, 26].

To probe the non-universal subsubleading soft graviton theorem that occurs, e.g. at the tree-level (linearized) Einstein gravity [13], it then naturally appears that additional Carrollian conformal fields beyond the usual Carrollian generators $S^\pm_0$, $S^\pm_1$, T and $\bar{T}$ are required to be present in the CarrCFT. This situation is similar in spirit to those considered in [27, 28] where, besides the conformal EM tensor, extra symmetry generators were postulated to exist in the 2D theory.

As we reviewed, the Carrollian fields $S^\pm_1$ encode both the $\pm$ve helicity leading and the subleading energetically soft graviton theorems while $S^\pm_0$ account for only the leading ones. The OPEs of the same spin fields among them are related by e.g. (2.26) while the $S^\pm_0$ OPEs satisfy e.g. (2.16). Moreover, it is the temporal step-function (and its time-integrals) appearing in the Ward identities, from which the energetically-soft pole structures are arising. Inspired by these observations, *we assume that, in the theory, there exists a Carrollian* *conformal field S_2^+ that, in the OPE limit, satisfies*:

$$\left(\partial_t S_2^+ - S_1^+\right)(t,z,\bar{z})\Phi(\mathbf{x}_p) \sim 0 \tag{3.1}$$

Since S_2^+ is postulated to be a Carrollian field, it must contain some new modes. In view of (2.22), this suggests that inside a correlator, S_2^+ can be decomposed as:

$$\begin{aligned} S_2^+(t,z,\bar{z}) &= S_{2e}^+(t,z,\bar{z}) + tS_{1e}^+(t,z,\bar{z}) + \frac{t^2}{2}S_0^+(t,z,\bar{z}) \\ \implies \quad \partial_t\langle S_{2e}^+(t,z,\bar{z})X\rangle &= [\text{temporal contact terms}] \end{aligned} \tag{3.2}$$

where the S_{2e}^+ part consists of the new modes. Thus, the dimensions of the field S_2^+ are $(\Delta, m) = (-1, 2)$.

Clearly, an analogously introduced Carrollian field S_2^- with spin $m = -2$ can not be treated as a mutually local field with S_0^+, S_1^+ and T, following the argument presented in [9, 46]. While its shadow or light transformations (on S^2) may be fine in this regard, we leave this possibility for a future work. Thus, in this work, we refrain from introducing the S_2^- field or its integral transformations.

It is also evident that the shadow or the light transformations of the S_2^+ field can not be mutually local with S_0^+, S_1^+ and T. So, we would like S_2^+ itself to fit in as a local field in the holomorphic sector of the 1+2D CarrCFT. It will be possible only if the $S_2^+\Phi$ OPEs have the similar singularity structure as those of the above three generators, i.e. having meromorphic pole singularities while being anti-analytic. We shall proceed by assuming this to be true.

3.1 The OPEs of S_2^+ with the generators

In [9], the singular parts of mutual (self and cross) OPEs between the three generators S_0^+, S_1^+ and T were completely determined by demanding the OPE commutativity property to hold, after making appropriate ansatz respecting the general forms (2.14), (2.24) and (2.31) and truncating those ansatz by assuming that:

1. no local field in the theory has negative scaling dimension $\Delta < 0$, following the 2D Euclidean [27, 31] and Carrollian [28, 32] CFT cases.

2. the fields S_0^+ and S_1^+ are Lorentz quasi-primaries, following the Celestial CFT case [52].

The first assumption is clearly not respected by the field S_2^+ with $\Delta = -1$. So, it needs to be relaxed into a weaker one that is stated below:

- no *time-independent* local field in the theory has negative scaling dimension; moreover, the time-independent local field with $\Delta = 0 = m$ is unique and it is the identity operator.

(We will see that there are several time-dependent fields with $\Delta = 0 = m$.)

Fortunately, the modified assumption does not change any of the results obtained in [9] for the mutual OPEs between S_0^+, S_1^+ and T. This statement can be verified by using the restrictions (2.16) and (2.26), then repeating the steps to derive those mutual OPEs as elaborated in [9] and finally, keeping in mind that the global space-time translation invariance must remain unbroken. Below we collect the mutual OPEs between S_0^+ and S_1^+ in the $j\epsilon$-form [9]:

$$
\begin{aligned}
&iS_0^+(\mathbf{x})S_0^+(\mathbf{x}_p)\sim 0\,; \quad iS_0^+(\mathbf{x})S_1^+(\mathbf{x}_p)\sim \lim_{\epsilon\to 0^+}\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,S_0^+(\mathbf{x}_p)\\
&iS_1^+(\mathbf{x})S_0^+(\mathbf{x}_p)\sim \lim_{\epsilon\to 0^+}\left[\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\,\partial_{\bar z_p}S_0^+ + \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,S_0^+\right](\mathbf{x}_p) \qquad (3.3)\\
&i\left\{S_1^+-(t-t_p)S_0^+\right\}(\mathbf{x})S_1^+(\mathbf{x}_p)\sim \lim_{\epsilon\to 0^+}\left[\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)^2}K+\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\,\partial_{\bar z_p}S_1^+ + \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,2S_1^+\right](\mathbf{x}_p)
\end{aligned}
$$

where K is a constant not fixed by symmetry.

We now proceed to find the mutual OPEs involving the S_2^+ field. The general form of the S_0^+ OPE (2.14), the modified first assumption, the relation (3.1) and the form of the $S_0^+S_1^+$ OPE together completely fix the singular part of the $S_0^+S_2^+$ OPE to be:

$$
iS_0^+(\mathbf{x})S_2^+(\mathbf{x}_p)\sim \lim_{\epsilon\to 0^+}\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,S_1^+(\mathbf{x}_p) \qquad (3.4)
$$

Using the OPE (bosonic) commutativity property, the $S_2^+S_0^+$ OPE is readily obtained as:

$$
\begin{aligned}
iS_2^+(\mathbf{x})S_0^+(\mathbf{x}_p)\sim \lim_{\epsilon\to 0^+}\Bigg[&\frac{(\bar z-\bar z_p)^3}{(\Delta\tilde z_p)}\,\frac{1}{2}\bar\partial_p^2S_1^+ + \frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\,\bar\partial_pS_1^+ + \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,S_1^+\\
&+(t-t_p)\left\{\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\,\bar\partial_pS_0^+ + \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,S_0^+\right\}\Bigg](\mathbf{x}_p) \qquad (3.5)
\end{aligned}
$$

where the OPE gets truncated by virtue of the restrictions (2.15), (2.16), (2.25) and (2.26).

Similarly, the general form of the S_1^+ OPE (2.24) and the modified first assumption together with the relation (3.1), the $S_1^+S_1^+$ and the $S_0^+S_2^+$ OPEs determine the singular structure of the $S_1^+S_2^+$ OPE as (S_2^+ has $\bar h=-\frac{3}{2}$):

$$
iS_1^+(\mathbf{x})S_2^+(\mathbf{x}_p)\sim \lim_{\epsilon\to 0^+}\left[\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)^2}Kt+\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\,\bar\partial_pS_2^+ + \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,3S_2^+ + (t-t_p)\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,S_1^+\right](\mathbf{x}_p)
$$

It is now evident that the $\langle S_1^+(\mathbf{x})S_2^+(\mathbf{x}_p)\rangle$ correlator can not be time-translation invariant unless $K=0$; so, the final form of the above OPE reduces to:

$$
iS_1^+(\mathbf{x})S_2^+(\mathbf{x}_p)\sim \lim_{\epsilon\to 0^+}\left[\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\,\bar\partial_pS_2^+ + \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,3S_2^+ + (t-t_p)\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,S_1^+\right](\mathbf{x}_p) \qquad (3.6)
$$

To find the $S_2^+S_1^+$ OPE using the OPE commutativity property from (3.6), we first need to know if the resultant Taylor series gets truncated into a polynomial in $(\bar z-\bar z_p)$. Recalling (2.25), we observe that the OPE (3.4) satisfies:

$$
S_0^+\bar\partial^{3+N}S_2^+\sim 0 \quad (N\in\mathbb{N}) \qquad \Longrightarrow \qquad \left(\bar\partial^{3+N}S_2^+\right)S_0^+\sim 0
$$

Now, we consider an arbitrary operator product $\bar\partial_1^{3+N}S_2^+(\mathbf{x}_1)S_0^+(\mathbf{x}_2)\Phi(\mathbf{x}_3)$ (where the field Φ is mutually local with both S_2^+ and S_0^+) and apply the OPE associativity property[4]

[4]Following the 2D Euclidean CFT [31], we assume the associativity property as a consistency condition for the Carrollian conformal OPEs in this work.

in two different ways such that both the resulting series are convergent for the following ordering of the flat Carrollian norms: $|\vec{x}_1 - \vec{x}_2| < |\vec{x}_2 - \vec{x}_3| < |\vec{x}_1 - \vec{x}_3|$. Treating $(z_2 - z_3)$ and $(z_1 - z_3)$ as the two independent variables, we easily reach the following restriction on S_2^+, analogous to (2.15) and (2.25):

$$\bar{\partial}^4 S_2^+(t, z, \bar{z}) \Phi(\mathbf{x}_p) \sim 0 \tag{3.7}$$

implying that an $S_2^+\Phi$ OPE will be at most a cubic order polynomial in $(\bar{z} - \bar{z}_p)$. As a consequence and remembering that we have also postulated that the $S_2^+\Phi$ OPEs have only meromorphic pole singularities while being anti-analytic so that S_2^+, S_1^+, S_0^+ are mutually local and that $z, \bar{z}$ are treated independently, inside a correlator the S_{2e}^+ part can be decomposed as:

$$S_{2e}^+(t, z, \bar{z}) = -k_e^{(+2)}(t, z, \bar{z}) + 3\bar{z} k_e^{(+1)}(t, z, \bar{z}) - 3\bar{z}^2 k_e^{(0)}(t, z, \bar{z}) + \bar{z}^3 k_e^{(-1)}(t, z, \bar{z}) \tag{3.8}$$

$$\text{such that} \quad \langle \bar{\partial} k_e^a(t, z, \bar{z}) X \rangle = -i \sum_{p=1}^{n} \theta(t - t_p) \; \Big[\text{contact terms on } S^2\Big]$$

just like (2.13) and (2.23). The holomorphic weight of all four k_e^a as well as the field S_2^+ is $h = \frac{1}{2}$.

With the restriction (3.7) in mind, we can immediately write the $S_2^+ S_1^+$ OPE from (3.6) using the OPE commutativity property as below:

$$\begin{aligned} iS_2^+(\mathbf{x}) S_1^+(\mathbf{x}_p) \sim \lim_{\epsilon \to 0^+} \Bigg[&\frac{(\bar{z}-\bar{z}_p)^3}{(\Delta \tilde{z}_p)} \, \frac{1}{2} \bar{\partial}_p^2 S_2^+ + \frac{(\bar{z}-\bar{z}_p)^2}{(\Delta \tilde{z}_p)} \, 2\bar{\partial}_p S_2^+ + \frac{\bar{z}-\bar{z}_p}{(\Delta \tilde{z}_p)} \, 3S_2^+ \\ &+ (t-t_p) \left\{ \frac{(\bar{z}-\bar{z}_p)^2}{(\Delta \tilde{z}_p)} \, \bar{\partial}_p S_1^+ + \frac{\bar{z}-\bar{z}_p}{(\Delta \tilde{z}_p)} \, 2S_1^+ \right\} + \frac{1}{2}(t-t_p)^2 \, \frac{\bar{z}-\bar{z}_p}{(\Delta \tilde{z}_p)} \, S_0^+ \Bigg] (\mathbf{x}_p) \end{aligned} \tag{3.9}$$

which is also seen to be consistent with the decompositions (3.2) and (3.8).

Finally, we proceed to find the $S_2^+ S_2^+$ OPE with all the above information at our disposal.

3.2 The $S_2^+ S_2^+$ OPE

We first make the following ansatz directly in the $j\epsilon$-form for the $S_2^+ S_2^+$ OPE that is consistent with the decomposition (3.2) and the restriction (3.7):

$$\begin{aligned} iS_2^+(\mathbf{x}) S_2^+(\mathbf{x}_p) \sim \lim_{\epsilon \to 0^+} \Bigg[&\sum_{r=0}^{3} \sum_{s \geq 1} \frac{(\bar{z} - \bar{z}_p)^r}{(\Delta \tilde{z}_p)^s} \, A_{r,s} + (t - t_p) \left\{ \frac{(\bar{z} - \bar{z}_p)^2}{(\Delta \tilde{z}_p)} \, \bar{\partial}_p S_2^+ + \frac{\bar{z} - \bar{z}_p}{(\Delta \tilde{z}_p)} \, 3S_2^+ \right\} \\ &+ \frac{1}{2}(t - t_p)^2 \, \frac{\bar{z} - \bar{z}_p}{(\Delta \tilde{z}_p)} \, S_1^+ \Bigg] (\mathbf{x}_p) \end{aligned} \tag{3.10}$$

where $A_{r,s}(\mathbf{x}_p)$ are yet undetermined fields mutually local with S_2^+, S_1^+, S_0^+. Now, due to the restriction (3.1), singular part of $S_2^+(\mathbf{x}) \partial_{t_p} S_2^+(\mathbf{x}_p)$ obtained from this ansatz must completely match with that of the $S_2^+(\mathbf{x}) S_1^+(\mathbf{x}_p)$ OPE given by (3.9). Only the $\mathcal{O}\left((t - t_p)^0\right)$

terms, while comparing, give non-trivial constraints that are listed below (with $\dot{}$ representing time-derivative):

$$\begin{aligned} &\dot{A}_{r,s\geq 2} \sim 0\,, \qquad\qquad \dot{A}_{0,1} \sim 0 \\ &\dot{A}_{3,1} \sim \frac{1}{2}\bar{\partial}_p^2 S_2^+\,, \quad \dot{A}_{2,1} \sim 3\bar{\partial}_p S_2^+\,, \quad \dot{A}_{1,1} \sim 6S_2^+ \end{aligned} \tag{3.11}$$

The first line says that all the local fields $A_{r,s\geq 2}$ and $A_{0,1}$ are time-independent (in the OPE limit). Moreover, since the l.h.s. of the OPE (3.10) has a total scaling dimension $\Delta = -2$, all of these fields have negative scaling dimensions. Hence, by the modified first assumption, we set all of them zero.

More interestingly, the requirements (3.11) reveal that, for the $S_2^+ S_2^+$ OPE to be consistent, there must exist a local field $A_{1,1}$ that, in the OPE limit, satisfies $\dot{A}_{1,1} \sim 6S_2^+$ which is a condition analogous to (3.1). Let us denote the local field as S_3^+ that obeys:

$$\left(\partial_t S_3^+ - S_2^+\right)(t, z, \bar{z})\Phi(\mathbf{x}_p) \sim 0 \tag{3.12}$$

The dimensions of the field S_3^+ then are $(\Delta, m) = (-2, 2)$.

It needs to be emphasized that, unlike S_2^+, we did not need to postulate the existence of the local field S_3^+. Rather its existence is automatically demanded if the $S_2^+ S_2^+$ OPE is to be consistent. This can be interpreted as the non-closure of the mode-algebra of the three fields S_2^+, S_1^+, S_0^+ alone.

Finally, we write down the $S_2^+ S_2^+$ OPE below:

$$\begin{aligned} iS_2^+(\mathbf{x})S_2^+(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \Bigg[&\frac{(\bar{z}-\bar{z}_p)^3}{(\Delta\tilde{z}_p)} \left(K_1 + \frac{1}{2}\bar{\partial}_p^2 S_3^+\right) + \frac{(\bar{z}-\bar{z}_p)^2}{(\Delta\tilde{z}_p)}\, 3\bar{\partial}_p S_3^+ + \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, 6S_3^+ \\ &+ (t-t_p)\left\{\frac{(\bar{z}-\bar{z}_p)^2}{(\Delta\tilde{z}_p)}\, \bar{\partial}_p S_2^+ + \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, 3S_2^+\right\} + \frac{1}{2}(t-t_p)^2\, \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, S_1^+\Bigg](\mathbf{x}_p) \end{aligned} \tag{3.13}$$

with K_1 being a constant not yet fixed. It is clear from this OPE that the commutators of only the modes contained in S_{2e}^+ will involve new modes appearing in S_{3e}^+ that is defined below analogously to (3.2) and consistently with (3.12):

$$\begin{aligned} &S_3^+(t, z, \bar{z}) = S_{3e}^+(t, z, \bar{z}) + tS_{2e}^+(t, z, \bar{z}) + \frac{t^2}{2}S_{1e}^+(t, z, \bar{z}) + \frac{t^3}{3!}S_0^+(t, z, \bar{z}) \\ \Longrightarrow \quad &\partial_t\langle S_{3e}^+(t, z, \bar{z})X\rangle = [\text{temporal contact terms}] \end{aligned} \tag{3.14}$$

We shall now investigate on the properties of the local field S_3^+ and discover that a tower of local fields S_k^+ $(k \geq 4)$ will be required to automatically exist for the consistency of the OPEs.

4 The tower of fields S_k^+

We shall begin by looking into the mutual OPEs of the four local fields S_k^+ with $0 \leq k \leq 3$ and observe that, for the OPE $S_2^+ S_3^+$ to be consistent, a new local field S_4^+ must exist. Similarly, for the consistency of the $S_2^+ S_4^+$ OPE, another new field S_5^+ is required to automatically exist. As is anticipated, this sort of argument will recursively generate the whole tower of the fields S_k^+ $(k \geq 4)$.

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4.1 The field S_3^+

Following the steps taken (and remembering the modified first assumption) in the previous section to construct the OPEs involving S_2^+, we directly write down the following OPEs for S_3^+ (it has $\bar{h} = -2$):

$$iS_0^+(\mathbf{x})S_3^+(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, S_2^+(\mathbf{x}_p) \tag{4.1}$$

$$iS_1^+(\mathbf{x})S_3^+(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \left[\frac{(\bar{z}-\bar{z}_p)^2}{(\Delta\tilde{z}_p)}\, \bar{\partial}_p S_3^+ + \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, 4S_3^+ + (t-t_p)\frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, S_2^+\right](\mathbf{x}_p) \tag{4.2}$$

Together with the restriction (3.7), the first OPE implies, similarly as derived for S_2^+, that:

$$\bar{\partial}^5 S_3^+(t,z,\bar{z})\Phi(\mathbf{x}_p) \sim 0 \tag{4.3}$$

which means that an $S_3^+\Phi$ OPE will be at most a quartic polynomial in $(\bar{z}-\bar{z}_p)$. Since, the singularity structure of such an OPE must be the same as that of the $S_0^+\Phi$ and $S_1^+\Phi$ OPEs, the condition (4.3) allows us to decompose S_{3e}^+ into five objects $l_e^a(t,z,\bar{z})$ whose correlators are both time-independent and holomorphic in the OPE limit, analogous to (3.8).

Now, to find the $S_2^+S_3^+$ OPE, let us first construct the following ansatz consistent with the conditions (3.2) and (3.7):

$$\begin{aligned} iS_2^+(\mathbf{x})S_3^+(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \Bigg[\sum_{r=0}^{3}\sum_{s\geq 1} \frac{(\bar{z}-\bar{z}_p)^r}{(\Delta\tilde{z}_p)^s}\, B_{r,s} + (t-t_p)\left\{\frac{(\bar{z}-\bar{z}_p)^2}{(\Delta\tilde{z}_p)}\, \bar{\partial}_p S_3^+ + \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, 4S_3^+\right\} \\ + \frac{1}{2}(t-t_p)^2\, \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, S_2^+\Bigg](\mathbf{x}_p) \end{aligned}$$

where $B_{r,s}(\mathbf{x}_p)$ are yet undetermined local fields. Now, obeying the restriction (3.12), we match the singular part of $S_2^+(\mathbf{x})\partial_{t_p}S_3^+(\mathbf{x}_p)$ obtained from this ansatz with that of the $S_2^+(\mathbf{x})S_2^+(\mathbf{x}_p)$ OPE given by (3.13) to find that, again, only the $\mathcal{O}\left((t-t_p)^0\right)$ terms give non-trivial constraints collected below:

$$\begin{aligned} &\dot{B}_{r,s\geq 2} \sim 0\,, && \dot{B}_{0,1} \sim 0 \\ &\dot{B}_{3,1} \sim \frac{1}{2}\bar{\partial}_p^2 S_3^+ + K_1\,, && \dot{B}_{2,1} \sim 4\bar{\partial}_p S_3^+\,, \quad \dot{B}_{1,1} \sim 10 S_3^+ \end{aligned} \tag{4.4}$$

Clearly, all of the time-independent local fields $B_{r,s\geq 2}$ and $B_{0,1}$ have negative scaling dimensions; hence, all of them are set zero by the modified first assumption. But, the conditions (4.4) demand that there must exist a local field S_4^+ such that:

$$\left(\partial_t S_4^+ - S_3^+\right)(t,z,\bar{z})\Phi(\mathbf{x}_p) \sim 0 \tag{4.5}$$

The dimensions of S_4^+ hence must be $(\Delta, m) = (-3, 2)$.

The $S_2^+S_3^+$ OPE then finally is:

$$\begin{aligned} iS_2^+(\mathbf{x})S_3^+(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \Bigg[\frac{(\bar{z}-\bar{z}_p)^3}{(\Delta\tilde{z}_p)}\, \frac{1}{2}\bar{\partial}_p^2 S_4^+ + \frac{(\bar{z}-\bar{z}_p)^2}{(\Delta\tilde{z}_p)}\, 4\bar{\partial}_p S_4^+ + \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, 10 S_4^+ \\ + (t-t_p)\left\{\frac{(\bar{z}-\bar{z}_p)^2}{(\Delta\tilde{z}_p)}\, \bar{\partial}_p S_3^+ + \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, 4S_3^+\right\} + \frac{1}{2}(t-t_p)^2\, \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, S_2^+\Bigg](\mathbf{x}_p) \end{aligned} \tag{4.6}$$

with $K_1 = 0$ also, to keep the time-invariance of the $\langle S_2^+(\mathbf{x})S_3^+(\mathbf{x}_p)\rangle$ correlator intact.

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It is by now apparent that this procedure will recursively reveal the automatic existence of a tower of local fields S^+_{k+1} ($k \geq 2$) just from the requirement of the consistency of the $S^+_2 S^+_k$ OPE such that:

$$\left(\partial_t S^+_{k+1} - S^+_k\right)(t,z,\bar{z})\Phi(\mathbf{x}_p) \sim 0 \tag{4.7}$$

if we merely postulate that the $1+2$D CarrCFT contains the field S^+_2 in the first place.

We shall now provide a recursive construction of the general $S^+_2 S^+_k$ OPE.

4.2 The fields S^+_k

The relation (4.7) between the fields S^+_k and S^+_{k+1} leads to:

$$\left(\partial^k_t S^+_k - S^+_0\right)(t,z,\bar{z})\Phi(\mathbf{x}_p) \sim 0$$

that implies that the field S^+_k must have dimensions $(\Delta, m) = (1-k, 2)$. So, the following OPEs are immediately constructed:

$$iS^+_0(\mathbf{x})S^+_{k+1}(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, S^+_k(\mathbf{x}_p) \tag{4.8}$$

$$iS^+_1(\mathbf{x})S^+_{k+1}(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \left[\frac{(\bar{z}-\bar{z}_p)^2}{(\Delta\tilde{z}_p)}\,\bar{\partial}_p S^+_{k+1} + \frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\,(k+2)S^+_{k+1} + (t-t_p)\frac{\bar{z}-\bar{z}_p}{(\Delta\tilde{z}_p)}\, S^+_k\right](\mathbf{x}_p) \tag{4.9}$$

As can be deduced by induction from the knowledge of (3.7), the first OPE implies that:

$$\bar{\partial}^{k+2} S^+_k(t,z,\bar{z})\Phi(\mathbf{x}_p) \sim 0 \tag{4.10}$$

which translates into the fact that an $S^+_k\Phi$ OPE will be at most a $(k+1)$-th order polynomial in $(\bar{z}-\bar{z}_p)$.

The relation (4.7) allows for the following decomposition of S^+_k inside a correlator:

$$S^+_k(t,z,\bar{z}) = \sum_{r=1}^{k} \frac{t^{k-r}}{(k-r)!}\, S^+_{r(e)}(t,z,\bar{z}) + \frac{t^k}{k!} S^+_0(t,z,\bar{z}) \tag{4.11}$$

$$\implies \quad \partial_t\langle S^+_{r(e)}(t,z,\bar{z})X\rangle = [\text{temporal contact terms}]$$

while the condition (4.10) permits further decomposition of $S^+_{k(e)}$ inside a correlator as shown below:

$$S^+_{k(e)}(t,z,\bar{z}) = \frac{1}{(k+1)!}\sum_{s=0}^{k+1}\binom{k+1}{s}(-)^{k+1-s}\bar{z}^s\, H^k_{\frac{k+1}{2}-s}(t,z,\bar{z}) \tag{4.12}$$

$$S^+_0(t,z,\bar{z}) = \bar{z}H^0_{-\frac{1}{2}}(t,z,\bar{z}) - H^0_{\frac{1}{2}}(t,z,\bar{z})$$

$$\text{such that } \langle\bar{\partial}H^k_{\frac{k+1}{2}-s}(t,z,\bar{z})X\rangle = -i\sum_{p=1}^{n}\,\theta(t-t_p)\,\left[\text{contact terms on } S^2\right]$$

Such a decomposition is possible because of the holomorphic singularity structure and $z, \bar{z}$ being independent. As before, $S^+_{k(e)}$ are local Carrollian fields but are merely collections of modes. The holomorphic weight of $H^k_{\frac{k+1}{2}-s}$ as well as the field S^+_k is $h = \frac{3-k}{2}$.

Now, we make an ansatz for the $S_2^+ S_k^+$ OPE following the same steps as before but omitting the terms that will be eventually set zero by the modified first assumption:

$$\begin{aligned}iS_2^+(\mathbf{x})S_k^+(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+}\Bigg[&\frac{(\bar z-\bar z_p)^3}{(\Delta\tilde z_p)}\,A_3^{(k)}+\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\,A_2^{(k)}+\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,A_1^{(k)}\\&+(t-t_p)\left\{\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\,\bar\partial_p S_k^+ +\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,(k+1)S_k^+\right\}+\frac{1}{2}(t-t_p)^2\,\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,S_{k-1}^+\Bigg](\mathbf{x}_p)\end{aligned}$$

with $A_p^{(k)}$ being some local fields. The $S_2^+(\mathbf{x})\dot S_k^+(\mathbf{x}_p)$ OPE obtained from this ansatz must be the same as the ansatz for the $S_2^+(\mathbf{x})S_{k-1}^+(\mathbf{x}_p)$ OPE due to the relation (4.7). Again, the non-trivial constraints come only from the $\mathcal{O}\left((t-t_p)^0\right)$ terms. These give rise to the following recursive system for the local fields $A_p^{(k)}$:

$$\begin{aligned}&\dot A_3^{(k)}\sim A_3^{(k-1)}, &&\dot A_2^{(k)}-\bar\partial_p S_k^+\sim A_2^{(k-1)}, &&\dot A_1^{(k)}-(k+1)S_k^+\sim A_1^{(k-1)}\\ \text{with seeds}\quad &A_3^{(2)}=\frac{1}{2}\bar\partial^2 S_3^+, &&A_2^{(2)}=3\bar\partial S_3^+, &&A_1^{(2)}=6S_3^+\end{aligned}$$

We demonstrate the solution of the A_1 recursive system below:

$$\begin{aligned}\dot A_1^{(k)}\sim A_1^{(k-1)}+(k+1)S_k^+ \;&\Rightarrow\; \partial_t^2 A_1^{(k)}\sim A_1^{(k-2)}+(k+1)S_{k-1}^+ + kS_{k-1}^+\\&\Rightarrow\; \partial_t^{k-2}A_1^{(k)}\sim A_1^{(2)}+[(k+1)+k+(k-1)+\ldots+4]S_3^+\\&\Rightarrow\; \partial_t^{k-2}A_1^{(k)}\sim\frac{1}{2}(k+1)(k+2)S_3^+\\&\Rightarrow\; A_1^{(k)}\sim\frac{1}{2}(k+1)(k+2)S_{k+1}^+\end{aligned}$$

The inversion of the time-derivatives are unique because of the severe restrictions imposed by the modified first assumption. This can also be motivated from the specific examples studied above.

The unique solutions to all the three recursions are similarly obtained to be:

$$A_3^{(k)}\sim\frac{1}{2}\bar\partial^2 S_{k+1}^+\,,\qquad A_2^{(k)}\sim(k+1)\bar\partial S_{k+1}^+\,,\qquad A_1^{(k)}\sim\frac{1}{2}(k+1)(k+2)S_{k+1}^+$$

consistent with all the previously considered specific cases. The $S_2^+S_k^+$ OPE is finally noted below:

$$\begin{aligned}iS_2^+(\mathbf{x})S_k^+(\mathbf{x}_p)\sim\lim_{\epsilon\to 0^+}\Bigg[&\frac{(\bar z-\bar z_p)^3}{(\Delta\tilde z_p)}\,\frac{1}{2}\bar\partial_p^2 S_{k+1}^+ +\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\,(k+1)\bar\partial_p S_{k+1}^+ +\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,\frac{1}{2}(k+1)(k+2)S_{k+1}^+\\&+(t-t_p)\left\{\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\,\bar\partial_p S_k^+ +\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,(k+1)S_k^+\right\}+\frac{1}{2}(t-t_p)^2\,\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\,S_{k-1}^+\Bigg](\mathbf{x}_p)\end{aligned}\tag{4.13}$$

With the above derivation as a warm-up, we finally attempt to find the most general $S_k^+S_l^+$ OPE.

4.3 The general $S_k^+S_l^+$ OPE

We shall find the general $S_k^+S_l^+$ OPE via a recursive method analogous to the one demonstrated above. To find the seeds of the recursions, we first note down the three cases with $l=0,1,2$.

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This can be achieved using the OPE commutativity property from the known OPEs $S_l^+ S_k^+$ with $l = 0, 1, 2$, i.e. from the OPEs (4.8), (4.9) and (4.13) respectively. While employing the OPE commutativity property, we need to recall that an $S_k^+ \Phi$ OPE is a $(k+1)$-th order polynomial in $(\bar{z} - \bar{z}_p)$ and a k-th order polynomial in $(t - t_p)$, due to respectively (4.10) and (4.11). Also, since in an $S_k^+ S_l^+$ OPE only the $\mathcal{O}\left((t-t_p)^0\right)$ terms are new in the sense that the $\mathcal{O}\left((t-t_p)^r\right)$ (with $r \in \mathbb{N}$) terms' coefficients have already been the $\mathcal{O}\left((t-t_p)^0\right)$ term in the $S_{k-r}^+ S_l^+$ OPE, we shall only explicitly write the $\mathcal{O}\left((t-t_p)^0\right)$ term from now on in a general OPE. The results are (with h.o.t. denoting the $\mathcal{O}\left((t-t_p)^r\right)$ terms with $r \geq 1$):

$$iS_k^+(\mathbf{x}) S_0^+(\mathbf{x}_p) \sim \lim_{\epsilon \to 0^+} \sum_{m=0}^{k} \frac{(\bar{z} - \bar{z}_p)^{m+1}}{(\Delta \tilde{z}_p)} \frac{1}{m!} \bar{\partial}_p^m S_{k-1}^+(\mathbf{x}_p) + (\text{h.o.t.}) \tag{4.14}$$

$$iS_k^+(\mathbf{x}) S_1^+(\mathbf{x}_p) \sim \lim_{\epsilon \to 0^+} \sum_{m=0}^{k} \frac{(\bar{z} - \bar{z}_p)^{m+1}}{(\Delta \tilde{z}_p)} \frac{k+1-m}{m!} \bar{\partial}_p^m S_k^+(\mathbf{x}_p) + (\text{h.o.t.}) \tag{4.15}$$

$$iS_k^+(\mathbf{x}) S_2^+(\mathbf{x}_p) \sim \lim_{\epsilon \to 0^+} \sum_{m=0}^{k} \frac{(\bar{z} - \bar{z}_p)^{m+1}}{(\Delta \tilde{z}_p)} \frac{(k+1-m)(k+2-m)}{2 \cdot m!} \bar{\partial}_p^m S_{k+1}^+(\mathbf{x}_p) + (\text{h.o.t.}) \tag{4.16}$$

With this knowledge, we shall now find the $S_k^+ S_3^+$ OPE in exactly the same way we derived the $S_2^+ S_3^+$ OPE exploiting the $S_2^+ S_2^+$ OPE. Let the ansatz for the $S_k^+ S_3^+$ OPE be:

$$iS_k^+(\mathbf{x}) S_3^+(\mathbf{x}_p) \sim \lim_{\epsilon \to 0^+} \left[\sum_{m=0}^{k} \frac{(\bar{z}-\bar{z}_p)^{m+1}}{(\Delta \tilde{z}_p)} B_m^{(k)} + (t-t_p) \sum_{m=0}^{k-1} \frac{(\bar{z}-\bar{z}_p)^{m+1}}{(\Delta \tilde{z}_p)} B_m^{(k-1)} \right] (\mathbf{x}_p) + (\text{h.o.t.})'$$

where $B_m^{(r)}(\mathbf{x}_p)$ are the local fields to be determined via recursion. Now, due to the relation (3.12), the $iS_k^+(\mathbf{x}) \dot{S}_3^+(\mathbf{x}_p)$ OPE derived from this ansatz must be the same as the OPE (4.16). This gives rise to the following recursion relation:

$$\begin{aligned}
&\dot{B}_m^{(k)} \sim B_m^{(k-1)} \frac{(k+1-m)(k+2-m)}{2 \cdot m!} \bar{\partial}_p^m S_{k+1}^+ \\
\Rightarrow\ &\partial_t^2 B_m^{(k)} \sim B_m^{(k-2)} + \frac{(k+1-m)(k+2-m)}{2 \cdot m!} \bar{\partial}_p^m S_k^+ + \frac{(k-m)(k+1-m)}{2 \cdot m!} \bar{\partial}_p^m S_k^+ \\
\Rightarrow\ &\partial_t^{k-m+1} B_m^{(k)} \sim B_m^{(m-1)} + \frac{1}{2 \cdot m!} \left[\sum_{n=1}^{k+1-m} n(n+1) \right] \bar{\partial}_p^m S_{m+1}^+
\end{aligned}$$

To solve this recursion, we need a seed. Since we are comparing the coefficients of the $(t-t_p)^0 \frac{(\bar{z}-\bar{z}_p)^{m+1}}{(\Delta \tilde{z}_p)}$ terms, we need to remember that such terms only occurs in the $iS_r^+ S_3^+$ OPEs with $r \geq m$. This means that $B_m^{(m-1)} = 0$ and that is the seed of the recursion. Thus, we have:

$$\begin{aligned}
&\partial_t^{k-m+1} B_m^{(k)} \sim \frac{(k+1-m)(k+2-m)(k+3-m)}{6 \cdot m!} \bar{\partial}_p^m S_{m+1}^+ \\
\Rightarrow\ &B_m^{(k)} \sim \frac{(k+1-m)(k+2-m)(k+3-m)}{6 \cdot m!} \bar{\partial}_p^m S_{k+2}^+
\end{aligned}$$

So, the $S_k^+ S_3^+$ OPE is given by:

$$iS_k^+(\mathbf{x}) S_3^+(\mathbf{x}_p) \sim \lim_{\epsilon \to 0^+} \sum_{m=0}^{k} \frac{(\bar{z} - \bar{z}_p)^{m+1}}{(\Delta \tilde{z}_p)} \frac{(k+1-m)_3}{6 \cdot m!} \bar{\partial}_p^m S_{k+2}^+(\mathbf{x}_p) + (\text{h.o.t.}) \tag{4.17}$$

where $(\ldots)_3$ is an upward Pochhammer symbol.

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In the exactly similar way, we find the $S_k^+ S_4^+$ OPE from the knowledge of the $S_k^+ S_3^+$ OPE to be:

$$iS_k^+(\mathbf{x})S_4^+(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \sum_{m=0}^{k} \frac{(\bar{z}-\bar{z}_p)^{m+1}}{(\Delta\tilde{z}_p)} \frac{(k+1-m)_4}{24\cdot m!}\, \bar{\partial}_p^m S_{k+3}^+(\mathbf{x}_p) + (\text{h.o.t.}) \tag{4.18}$$

and the $S_k^+ S_5^+$ OPE from the $S_k^+ S_4^+$ OPE as:

$$iS_k^+(\mathbf{x})S_5^+(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \sum_{m=0}^{k} \frac{(\bar{z}-\bar{z}_p)^{m+1}}{(\Delta\tilde{z}_p)} \frac{(k+1-m)_5}{120\cdot m!}\, \bar{\partial}_p^m S_{k+4}^+(\mathbf{x}_p) + (\text{h.o.t.}) \tag{4.19}$$

The form of the explicit examples (4.14)–(4.19) inspires the following obvious guess for the general OPE $S_k^+ S_l^+$:

$$iS_k^+(\mathbf{x})S_l^+(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \sum_{m=0}^{k} \frac{(\bar{z}-\bar{z}_p)^{m+1}}{(\Delta\tilde{z}_p)} \frac{(k+1-m)_l}{l!\cdot m!}\, \bar{\partial}_p^m S_{k+l-1}^+(\mathbf{x}_p) + (\text{h.o.t.}) \tag{4.20}$$

That this is indeed true can be quickly verified by noticing that the OPE $S_k^+ \dot{S}_l^+$ is exactly the same as the OPE $S_k^+ S_{l-1}^+$, as must hold because of the relation (4.7).

For the sake of completeness, the full form of the above OPE, consistent with the decomposition (4.11), is written below:

$$iS_k^+(\mathbf{x})S_l^+(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \sum_{r=0}^{k} \frac{(t-t_p)^{k-r}}{(k-r)!} \sum_{m=0}^{r} \frac{(\bar{z}-\bar{z}_p)^{m+1}}{(\Delta\tilde{z}_p)} \frac{(r+1-m)_l}{l!\cdot m!}\, \bar{\partial}_p^m S_{r+l-1}^+(\mathbf{x}_p) \tag{4.21}$$

Having derived the general $S_k^+ S_l^+$ OPE, we now take on the final goal of this paper: uncovering the symmetry algebra manifest in the mutual OPEs of the tower of fields $\left\{S_k^+\right\}$.

5 The symmetry algebra

To find the quantum symmetry algebra from the OPEs, one needs to first identify the contributions of the modes to the OPEs. As is implied by the decomposition (4.11), the new modes in a field S_k^+ that do not appear in the fields S_{k-r}^+ with $1 \leq r \leq k$ are all contained in the part $S_{k(e)}^+$. In this sense, the part $S_{k(e)}^+$ is the unique signature of the field S_k^+. Similarly, the 'new' information content of the $S_k^+ S_l^+$ OPE is described by the $S_{k(e)}^+ S_{l(e)}^+$ part.

We recall that the part $S_{k(e)}^+$ is not a Carrollian conformal field. So, strictly speaking, the $S_{k(e)}^+ S_{l(e)}^+$ part is not a Carrollian conformal OPE. Nevertheless, we shall call it an 'OPE' from now on. It can be readily extracted from the $S_k^+ S_l^+$ OPE (4.21) by comparing the $\mathcal{O}(t^0 t_p^0)$ terms from both sides to be:

$$iS_{k(e)}^+(t,z,\bar{z})S_{l(e)}^+(t_p,z_p,\bar{z}_p) \sim \lim_{\epsilon\to 0^+} \sum_{m=0}^{k} \frac{(\bar{z}-\bar{z}_p)^{m+1}}{(\Delta\tilde{z}_p)} \frac{(k+1-m)_l}{l!\cdot m!}\, \bar{\partial}_p^m S_{k+l-1(e)}^+(t_p,z_p,\bar{z}_p) \tag{5.1}$$

In the explicit $t \to \infty$ limit where $\frac{1}{\Delta\tilde{z}_p} \equiv \frac{1}{z-z_p}$ (and then recalling that any $S_{r(e)}^+$ is time-independent in the OPE limit), this 'OPE' is exactly the same as the OPE of the positive-helicity conformally soft gravitons at the tree level of the Einstein gravity in the bulk AFS [35]. There, it was obtained by taking the conformally soft limit from the general OPE of two Celestial conformal primary gravitons derived in [36] that needed some crucial inputs from the explicit bulk gravity theory. Here, we have reached the conformally soft graviton OPE directly by instead exploiting only the constraints imposed by the Carrollian conformal symmetry along with the two assumptions stated in section 3.1. No hint from the underlying bulk theory was needed in our field theory analysis. The Carrollian time-coordinate t played the central role in our construction.

Thus, the conformally soft graviton operator $H^k(z,\bar z)$ of [35] that is a primary field in 2D Celestial CFT is the two-dimensional object $iS^+_{1-k(e)}(\infty,z,\bar z)$ appearing in the Carrollian conformal field $iS^+_{1-k}(t\to\infty,z,\bar z)$.

Moving back to our goal of finding the symmetry algebra, we now substitute the anti-holomorphic mode-expansion (4.12) into the 'OPE' (5.1) and match the powers of both $\bar z$ and $\bar z_p$ from both hand sides to find an 'OPE' $H^k_{\frac{k+1}{2}-r}(z)H^l_{\frac{l+1}{2}-s}(z_p)$. The dependence on the time- and the anti-holomorphic coordinates are removed since, by (4.11) and (4.12), the Carrollian 'modes' $H^k_{s'}(t,z,\bar z)$ are independent of respectively t and $\bar z$ in the OPE limit, thus making them effectively holomorphic inside an OPE. While it is straightforward to match the powers of $\bar z$, to do the same with $\bar z_p$ we must first insert the anti-holomorphic mode-expansion for $\bar\partial_p^m S^+_{k+l-1(e)}(z_p,\bar z_p)$ on the r.h.s.. Doing these, one finds the following 'OPE' (with $\Delta\tilde z_p = z - z_p - j\epsilon 0^+$ from now on):

$$
\begin{aligned}
&-i\frac{\binom{k+1}{r}}{(k+1)!}\frac{\binom{l+1}{s}}{(l+1)!}H^k_{\frac{k+1}{2}-r}(z)H^l_{\frac{l+1}{2}-s}(z_p)\\
&\sim \lim_{\epsilon\to 0^+}\sum_{m=\max\{0,r-1\}}^{k}\frac{(-1)^{m+1-r}}{(\Delta\tilde z_p)}\binom{m+1}{r}\binom{s+r-1}{m}\frac{(k+1-m)_l}{l!\cdot m!}\frac{\binom{k+l}{r+s-1}}{(k+l)!}H^{k+l-1}_{\frac{k+l}{2}-r-s+1}(z_p)
\end{aligned}
$$

The summation over m at the r.h.s. of this 'OPE' can be done as below (for $r\geq 1$):

$$
\begin{aligned}
&\sum_{m=r-1}^{k}(-1)^{m+1-r}\binom{m+1}{r}\binom{s+r-1}{m}\frac{(k+1-m)_l}{l!\cdot m!}\\
&= (-)^{k+1-r}\frac{(s+r-1)!}{s!r!}\sum_{m=0}^{k+1-r}(m+r)\binom{s}{m}\frac{(l+1)_{k+1-r-m}}{(k+1-r-m)!}(-)^{k+1-r-m}\\
&= (-)^{k+1-r}\frac{(s+r-1)!}{s!r!}\times\left[\text{Coeff. of } x^{k+1-r} \text{ in } [r+x(r+s)](1+x)^{s-l-2}\right]\\
&= \frac{(s+r-1)!}{s!r!}\frac{(k+l-s-r+1)!}{(k+1-r)!(l+1-s)!}\left[r(l+1)-s(k+1)\right]
\end{aligned}
$$

that also holds true for $r=0$.

Thus, the above 'OPE' reduces to the following simple form:

$$
iH^k_a(z)H^l_b(z_p)\sim\lim_{\epsilon\to 0^+}\left[a(l+1)-b(k+1)\right]\frac{H^{k+l-1}_{a+b}(z_p)}{z-z_p-j\epsilon 0^+} \tag{5.2}
$$

with $-\frac{k+1}{2}\leq a\leq\frac{k+1}{2}$ and $-\frac{l+1}{2}\leq b\leq\frac{l+1}{2}$ now ($2a,2b\in\mathbb{Z}$). This has precisely the same appearance as a Kac-Moody current OPE in a holomorphic 2D CFT (after explicitly putting

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$\epsilon \to 0^+$). But the infinite-dimensional Lie algebra underlying such a Kac-Moody current algebra is not standard in the sense that it has no name! Fortunately, as we see below, this Lie algebra can be recast into a standard form.

For this purpose, we relabel the 'modes' H^k_a as:

$$w^p_a \equiv \frac{i}{2} H^{2p-3}_a \qquad \Longrightarrow \qquad p \in \frac{\mathbb{N}}{2} + 1 \ \text{ and } \ 1 - p \le a \le p - 1 \tag{5.3}$$

and get the following 'OPE' for the modes w^p_a from (5.2):

$$w^p_a(z) w^q_b(z_p) \sim \lim_{\epsilon \to 0^+} \left[a(q-1) - b(p-1)\right] \frac{w^{p+q-2}_{a+b}(z_p)}{z - z_p - j\epsilon 0^+} \tag{5.4}$$

that resembles a 2D CFT Kac-Moody current OPE with the Lie algebra being (a sub-algebra of) the $w_{1+\infty}$ algebra [40]. The underlying Lie algebra would actually be the 'wedge sub-algebra' defined by the restriction (5.3) on a [39] of the full $w_{1+\infty}$ algebra with unconstrained $a \in \mathbb{Z}$ [40].

In the context of 2D Celestial CFT, the algebra of conformally soft gravitons was re-expressed as this same Kac-Moody algebra in [41]. In this work, the central term w^1_0 of [41] is set zero to respect the time-translation invariance. But more importantly, we do not interpret the mode-redefinition (5.3) as a (discrete) light-transformation as opposed to the descriptions presented in [41, 59].

We shall now explicitly show that the 'OPE' (5.3) actually gives rise to the above said current-algebra symmetry in the CarrCFT (rather than the suggestive resemblance) as the algebra of the modes. This check is important because the CarrCFT technology has some differences with those in the usual 2D CFT. In particular, the mode-commutation relation is shown in [9, 32] to be related with the corresponding OPE via a complex-contour integral in CarrCFT without the need to perform any radial-quantization. It is the temporal step-function- or the $j\epsilon$- prescriptions of the CarrCFT OPEs instead that play the crucial role in establishing such a relation.

First we notice that the objects $H^k_a(z)$ can be holomorphic mode-expanded inside a correlator as below, by remembering that its holomorphic weight is $h = \frac{3-k}{2}$:

$$H^k_a(z) = \sum_{n \in \mathbb{Z} + \{\frac{k+1}{2}\}} H^k_{a;n} z^{-n-\frac{3-k}{2}} \qquad\qquad \{p\} \equiv \text{frac}(p) \tag{5.5}$$

facilitated by the property that H^k_a is independent of $\bar{z}$ (and t) in the OPE limit. This mode-expansion is inverted to recover the modes as following [9]:

$$H^k_{a;n} = \frac{1}{2\pi j} \oint_{C'_u} d\hat{z} \ \hat{z}^{n+\frac{1-k}{2}} \ H^k_a(t, \hat{z}, \hat{\bar{z}}) \tag{5.6}$$

where C'_u is the counterclockwise contour on a $y = ax + b$ plane in the $1+2$D Carrollian space-time that encloses the entire upper half plane $t > 0$ as well as the line $t = 0$.

Finally, we shall now find the algebra of the modes $H^k_{a;n}$. This will be (part of) the symmetry algebra manifest at the level of the OPEs of a CarrCFT that contains the field

S_2^+. For that, we first compute the following commutator:

$$
\begin{aligned}
\left[H^k_{a;n}\ ,\ H^l_b(\mathbf{x}_p)\right] &= \left[\frac{1}{2\pi j}\oint\limits_{C'_u} d\hat{z}\ \hat{z}^{n+\frac{1-k}{2}}\ H^k_a(t,\hat{z},\hat{\bar{z}})\ ,\ H^l_b(\mathbf{x}_p)\right] \\
&= \frac{1}{2\pi j}\oint\limits_{C_u} d\hat{z}\ \hat{z}^{n+\frac{1-k}{2}}\ \hat{\mathcal{T}}\left[H^k_a(t_p^+,\hat{z},\hat{\bar{z}})H^l_b(t_p,\vec{x}_p) - H^k_a(t_p^-,\hat{z},\hat{\bar{z}})H^l_b(t_p,\vec{x}_p)\right] \\
&= \lim_{\epsilon\to 0^+} -i\left[a(l+1)-b(k+1)\right]\frac{1}{2\pi j}\oint\limits_{C_u} d\hat{z}\ \hat{z}^{n+\frac{1-k}{2}}\ \frac{H^{k+l-1}_{a+b}(\mathbf{x}_p)}{\hat{z}-z_p-j\epsilon 0^+} - 0 \\
&= -i\left[a(l+1)-b(k+1)\right] z_p^{n+\frac{1-k}{2}}\ H^{k+l-1}_{a+b}(\mathbf{x}_p)
\end{aligned}
$$

where the contour C_u encloses only the upper half plane $t>0$ but not the $t=0$ line; passing from the C'_u to the C_u is valid only in the $\epsilon\to 0^+$ limit. Clearly, the C_u contour does not enclose the singularity at $\hat{z}=z_p+j\epsilon 0^-$ coming from the $H^k_a(t_p^-,\hat{z})H^l_b(t_p,z_p)$ term; so it has no contribution to the above commutator.

Next, using the mode-expansions (5.5) for $H^l_b(z_p)$ and $H^{k+l-1}_{a+b}(z_p)$ and comparing the powers of z_p on both h.s. of the above commutator, we get the following mode-algebra:

$$
i\left[H^k_{a;n}\ ,\ H^l_{b;m}\right] = \left[a(l+1)-b(k+1)\right] H^{k+l-1}_{a+b;n+m} \tag{5.7}
$$

or, in terms of the relabeled modes from (5.3): $w^p_{a;n} \equiv \frac{i}{2}H^{2p-3}_{a;n}$ with $p+n\in\mathbb{Z}$ as:

$$
\left[w^p_{a;n}\ ,\ w^q_{b;m}\right] = \left[a(q-1)-b(p-1)\right] w^{p+q-2}_{a+b;n+m} \tag{5.8}
$$

confirming that the (wedge sub-algebra of) $w_{1+\infty}$ Kac-Moody algebra indeed arises as the algebra of the modes from the OPEs of a CarrCFT containing the field S_2^+.

Let us have a closer look into the algebra (5.7) or equivalently the 'OPE' (5.2). As discussed earlier, the existence of the fields S_1^+ and S_0^+ is a universal feature of any $1+2$D CarrCFT since they are constructed purely in terms of the Carrollian EM tensor. The 'modes' H^1_a (or w^2_a) and H^0_b (or $w^{\frac{3}{2}}_b$) are their respective unique signatures. In [9], from the mutual 'OPE's (that remains unchanged even under the modified first assumption) of these five 'modes', the symmetry algebra was derived to be the Kac-Moody extension of the $\overline{\mathrm{sl}(2,\mathbb{R})}$ algebra with an abelian super-translation ideal. This is thus the 'universal' sub-algebra of the symmetry algebra (5.7). The Carrollian conformal modes $H^1_{a;n}$ generate the $\widehat{\overline{\mathrm{sl}(2,\mathbb{R})}}$ sub-algebra and the modes $H^0_{\pm\frac{1}{2};m}$, the ideal. On the space of the Carrollian quantum fields, the three zero-modes $H^1_{a;0}$ generate the three $\overline{\mathrm{sl}(2,\mathbb{R})}$ Lorentz transformations while the four modes $H^0_{\pm\frac{1}{2};\pm\frac{1}{2}}$ inflict the $\mathrm{isl}(2,\mathbb{C})$ Poincaré translations.

The special 'OPE's involving the 'universal modes' H^1_a contain some information on representation theoretic properties of the tower of fields S_k^+ or, rather, their signature 'modes' H^k_a. From the general 'OPE' (5.2), we readily find that:

$$
iH^1_{\pm 1}H^k_b \sim \lim_{\epsilon\to 0^+}\left[\pm(k+1)-2b\right]\frac{H^k_{b\pm 1}}{z-z_p-j\epsilon 0^+}\quad ;\quad iH^1_0 H^k_b \sim \lim_{\epsilon\to 0^+} -2b\ \frac{H^k_b}{z-z_p-j\epsilon 0^+} \tag{5.9}
$$

implying that the $(k+2)$ 'modes' H^k_b transform under the $(k+2)$-dimensional representation of the group $\overline{\mathrm{SL}(2,\mathbb{R})}$. Consistently, the $\overline{\mathrm{SL}(2,\mathbb{R})}$ generator 'modes' H^1_a transform under the 3-dimensional adjoint-representation of the said group.

We recall that there is another 'universal' Carrollian conformal field T, built out of the Carrollian EM tensor, that can be treated as a local field simultaneously with S^+_0 and S^+_1 [9]. Since the $T\Phi$ OPEs (2.31) have only meromorphic pole-singularities but are anti-analytic, these have the same singularity structures as that of all the $S^+_k\Phi$ OPEs. So, the field T and the tower of fields S^+_k are mutually local.

Even with the modified first assumption, the OPEs TS^+_0 and TS^+_1 remain unchanged from the following, as derived in [9]:

$$iT(t,z,\bar{z})S^+_0(t_p,z_p,\bar{z}_p) \sim \lim_{\epsilon\to 0^+}\left[\frac{\frac{3}{2}S^+_0}{(\Delta\tilde{z}_p)^2}+\frac{\partial_p S^+_0}{\Delta\tilde{z}_p}\right](t_p,z_p,\bar{z}_p) \tag{5.10}$$

$$iT(t,z,\bar{z})S^+_1(t_p,z_p,\bar{z}_p) \sim \lim_{\epsilon\to 0^+}\left[\frac{S^+_1}{(\Delta\tilde{z}_p)^2}+\frac{\partial_p S^+_1}{\Delta\tilde{z}_p}-\frac{t-t_p}{2}\,\frac{S^+_0}{(\Delta\tilde{z}_p)^2}\right](t_p,z_p,\bar{z}_p)$$

while the S^+_0T and the S^+_1T OPEs were readily obtained from the above by using the OPE commutativity property and remembering the conditions (2.15), (2.16), (2.25), (2.26) and (2.32) to be:

$$iS^+_0(\mathbf{x})T(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+}\left[(\bar{z}-\bar{z}_p)\left(\frac{\frac{1}{2}\partial_p\bar{\partial}_p S^+_0}{(\Delta\tilde{z}_p)}+\frac{\frac{3}{2}\bar{\partial}_p S^+_0}{(\Delta\tilde{z}_p)^2}\right)+\frac{\frac{3}{2}S^+_0}{(\Delta\tilde{z}_p)^2}+\frac{\frac{1}{2}\partial_p S^+_0}{(\Delta\tilde{z}_p)}\right](\mathbf{x}_p) \tag{5.11}$$

$$i\left\{S^+_1-(t-t_p)S^+_0\right\}(\mathbf{x})T(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+}\left[(\bar{z}-\bar{z}_p)^2\,\frac{\frac{1}{2}\bar{\partial}^2_p S^+_1}{(\Delta\tilde{z}_p)^2}+(\bar{z}-\bar{z}_p)\,\frac{\bar{\partial}_p S^+_1}{(\Delta\tilde{z}_p)^2}+\frac{S^+_1}{(\Delta\tilde{z}_p)^2}\right](\mathbf{x}_p)$$

On the other hand, the TT OPE was derived to be [9]:

$$iT(\mathbf{x})T(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+}\left[\frac{-i\frac{c}{2}}{(\Delta\tilde{z}_p)^4}+\frac{2T}{(\Delta\tilde{z}_p)^2}+\frac{\partial_{z_p}T}{\Delta\tilde{z}_p}-\frac{t-t_p}{2}\left\{\frac{\frac{1}{2}\partial_p\bar{\partial}_p S^+_0}{(\Delta\tilde{z}_p)^2}+\frac{3\bar{\partial}_p S^+_0}{(\Delta\tilde{z}_p)^3}\right\}\right](\mathbf{x}_p) \tag{5.12}$$

where c is a constant not fixed by symmetry arguments alone.

The (singular parts of the) general TS^+_k OPEs can be completely determined by remembering the general form (2.31) of the $T\Phi$ OPEs and the modified first assumption to be:

$$iT(t,z,\bar{z})S^+_k(t_p,z_p,\bar{z}_p) \sim \lim_{\epsilon\to 0^+}\left[\frac{\frac{3-k}{2}S^+_k}{(\Delta\tilde{z}_p)^2}+\frac{\partial_p S^+_k}{\Delta\tilde{z}_p}-\frac{t-t_p}{2}\,\frac{S^+_{k-1}}{(\Delta\tilde{z}_p)^2}\right](t_p,z_p,\bar{z}_p) \tag{5.13}$$

that is easily checked to be consistent with the relation (4.7) and the restriction (4.10). The S^+_kT OPEs can be immediately derived from this, using the OPE commutativity property and applying these two conditions.

Thus, together with the $S^+_0S^+_k$ and the $S^+_1S^+_k$ OPEs (4.8), (4.9), the TS^+_k OPE (5.13) implies that:

$$\textit{each } S^+_k \textit{ is a Carrollian conformal primary field with } \left(\bar{\boldsymbol{\xi}}\cdot S^+_k\right)=0=\left(\boldsymbol{\xi}\cdot S^+_k\right).$$

But, this is not the case with the field T. From the S^+_0T, S^+_1T and the TT OPEs above, we can immediately conclude that T is not even an $\mathrm{ISL}(2,\mathbb{C})$ quasi-primary field; it is only an $\mathrm{SL}(2,\mathbb{C})$ or Lorentz quasi-primary [9]. Moreover, both $(\boldsymbol{\xi}\cdot T)$ and $\left(\bar{\boldsymbol{\xi}}\cdot T\right)$ are non-zero.

We now proceed to find the mode-algebra from the TS^+_k OPE (5.13). Just as the 'modes' H^k_a, contained in the $S^+_{k(e)}$ part of the field S^+_k is its unique signature, the object T_e introduced in (2.30), that corresponds to the 2D Celestial CFT EM tensor [15], is the unique signature of the field T. So, from (5.13), one can easily find the following 'OPE':

$$iT_e(t,z,\bar{z})S^+_{k(e)}(t_p,z_p,\bar{z}_p) \sim \lim_{\epsilon\to 0^+}\left[\frac{\frac{3-k}{2}S^+_{k(e)}}{(\Delta\tilde{z}_p)^2}+\frac{\partial_p S^+_{k(e)}}{\Delta\tilde{z}_p}\right](t_p,z_p,\bar{z}_p) \tag{5.14}$$

In the limit $t\to t_p^+$ and $t_p\to\infty$, this is actually a 2D Celestial CFT OPE saying that the Celestial conformally soft graviton field $S^+_{k(e)}(\infty,z,\bar{z})$ is a Celestial conformal primary [35].

We then note the following holomorphic mode-expansion for T_e in a CarrCFT OPE [9]:

$$iT_e(z)=\sum_{n\in\mathbb{Z}}L_n z^{-n-2} \qquad \Longrightarrow \qquad L_n=\frac{1}{2\pi j}\oint_{C'_u} d\hat{z}\;\hat{z}^{n+1}\;iT_e(t,\hat{z},\hat{\bar{z}}) \tag{5.15}$$

Using this and then, first the anti-holomorphic decomposition (4.12) for $S^+_{k(e)}$ and next the holomorphic mode-expansion (5.5), we find the following commutator from the 'OPE' (5.14) in a manner similar to the derivation of the algebra (5.7):

$$\left[L_n\;,\;H^k_{a;m}\right]=\left(n\frac{1-k}{2}-m\right)H^k_{a;n+m} \quad\equiv\quad \left[L_n\;,\;w^p_{a;m}\right]=\left[n(2-p)-m\right]w^p_{a;n+m} \tag{5.16}$$

We also note the $[L_n,L_m]$ commutator derived in [9] from the T_eT_e 'OPE':

$$[L_n\;,\;L_m]=(n-m)\,L_{m+n}+\frac{c}{12}\left(n^3-n\right)\delta_{n+m,0}$$

which is the holomorphic Virasoro algebra Vir.

Thus, the complete symmetry algebra manifest in the OPEs of a $1+2$D CarrCFT that contains a local field S^+_2 obeying the relation (3.1) is the semi-direct product of Vir and the wedge sub-algebra [39] of $\hat{w}_{1+\infty}$ with the semi-direct product structure given by (5.16). The 'universal' sub-algebra of this algebra, i.e. Vir $\ltimes \widehat{\overline{\mathrm{sl}(2,\mathbb{R})}}$ with an abelian super-translation ideal, is the OPE-level symmetry (in the holomorphic sector) of any $1+2$D CarrCFT [9].

Unlike the Celestial CFT literature [30, 35, 41, 58, 59], we reached the above conclusion solely from the Carrollian conformal symmetry arguments and the general properties of OPEs, under the two assumptions stated in section 3.1, without requiring any hint from the explicit (quantum) theory of gravitation in the $1+3$D bulk AFS. Thus, our analysis is purely holographic in nature.

6 An infinity of soft theorems

We shall now uncover, in the current framework of the $1+2$D CarrCFT, the direct connection between the existence of the infinite tower of conformally soft graviton fields, as described in [35] in the context of the 2D Celestial CFT and in section 4 of this work and an infinite number of soft graviton theorems manifest as the Ward identities of large diffeomorphisms, as presented in [42, 43].

As suggested in section 3, for a Carrollian conformal field S_2^+ to encode the subsubleading energetically soft graviton theorem [13] in its Ward identity, it should obey a relation like (3.1). As we have seen in sections 4 and 5, recursive iteration of this suggestion reproduces the correct Carrollian conformal OPEs containing the conformally soft graviton OPEs [35] of 2D Celestial CFT. Providing a direct justification to this suggestion is the first step towards the goal of this section. We shall closely follow the argument presented in [9] that showed the relation between the source-less 1 + 2D CarrCFT EM tensor Ward identities and the leading [12] and the subleading [13] soft graviton theorems.

We begin by noting down the $S_2^+ S_k^+$ OPE (4.13) below:

$$iS_2^+(\mathbf{x})S_k^+(\mathbf{x}_p) \sim \lim_{\epsilon\to0^+}\Bigg[\frac{(\bar z-\bar z_p)^3}{(\Delta\tilde z_p)}\;\frac{1}{2}\bar\partial_p^2 S_{k+1}^+ + \frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\;(k+1)\bar\partial_p S_{k+1}^+ + \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\;\frac{1}{2}(k+1)(k+2)S_{k+1}^+ \\ +(t-t_p)\left\{\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\;\bar\partial_p S_k^+ + \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\;(k+1)S_k^+\right\} + \frac{1}{2}(t-t_p)^2\;\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\;S_{k-1}^+\Bigg](\mathbf{x}_p)$$

and recall that all the fields S_k^+ are Carrollian conformal primaries with $\left(\bar{\boldsymbol\xi}\cdot S_k^+\right) = 0 = \left(\boldsymbol\xi\cdot S_k^+\right)$ with dimensions $(h,\bar h) = \left(\frac{3-k}{2}, -\frac{k+1}{2}\right)$. Since valid for an infinite number of fields S_k^+, we postulate that the field S_2^+ has the following OPE with a special Carrollian conformal primary[5] Φ with dimensions $(h,\bar h)$ and $\left(\bar{\boldsymbol\xi}\cdot\Phi\right) = 0 = (\boldsymbol\xi\cdot\Phi)$:

$$iS_2^+(\mathbf{x})\Phi(\mathbf{x}_p) \sim \lim_{\epsilon\to0^+}\Bigg[\frac{(\bar z-\bar z_p)^3}{(\Delta\tilde z_p)}\;\frac{1}{2}\bar\partial_p^2\Phi_1 - \frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\;2\bar h\bar\partial_p\Phi_1 + \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\;\frac{1}{2}(2\bar h)(2\bar h-1)\Phi_1 \\ +(t-t_p)\left\{\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\;\bar\partial_p\Phi - \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\;2\bar h\Phi\right\} + \frac{1}{2}(t-t_p)^2\;\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\;\dot\Phi\Bigg](\mathbf{x}_p) \quad (6.1)$$

with the unique local Carrollian field Φ_1 satisfying $\dot\Phi_1 \sim \Phi$. It is important to note that the OPE of the local field Φ_1 must not be completely determinable in terms of that of the field Φ; more precisely, it should hold that:

$$\Phi_1(t,z,\bar z) \sim \int_{-\infty}^{t} dt'\;\Phi(t',z,\bar z) + (\text{terms with temporal step-function factor})$$

as happens for the fields S_k^+.

Remembering the decomposition (4.11), we first collect the $S_{2(e)}^+\Phi$ 'OPE' as:

$$iS_{2(e)}^+(t,z,\bar z)\Phi(\mathbf{x}_p) \sim \lim_{\epsilon\to0^+}\Bigg[\frac{(\bar z-\bar z_p)^3}{(\Delta\tilde z_p)}\;\frac{1}{2}\bar\partial_p^2\Phi_1 - \frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\;2\bar h\bar\partial_p\Phi_1 + \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\;\frac{1}{2}(2\bar h)(2\bar h-1)\Phi_1 \\ -t_p\left\{\frac{(\bar z-\bar z_p)^2}{(\Delta\tilde z_p)}\;\bar\partial_p\Phi - \frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\;2\bar h\Phi\right\} + \frac{t_p^2}{2}\;\frac{\bar z-\bar z_p}{(\Delta\tilde z_p)}\;\dot\Phi\Bigg](\mathbf{x}_p)$$

[5]We expect the Carrollian conformal weight Δ to be discrete unlike its Celestial counterpart Δ_c. This expectation stems from the fact that a 1 + 2D Carrollian conformal primary field that corresponds to a 4D bulk field describing a mass-less (hard) scattering particle must possess $\Delta = 1$. Clearly, all the descendants of such a field have integer Carrollian weights.

that, is the Carrollian conformal counterpart of the subsubleading conformally soft graviton OPE with a 2D Celestial conformal primary as in [60], since clearly, $\Delta_{\Phi_1} = \Delta_\Phi - 1$ and $m_{\Phi_1} = m_\Phi$. This observation is consistent with the earlier interpretation in section 5 of the object $S^+_{2(e)}(\infty, z, \bar{z})$ as the Celestial conformally soft graviton field $H^{-1}(z, \bar{z})$ in [35]. This 'OPE' is further decomposed according to the anti-holomorphic 'mode-expansion' (4.12) as:

$$\begin{aligned}
iH^2_{-\frac{3}{2}}(z)\Phi(\mathbf{x}_p) &\sim \lim_{\epsilon\to 0^+} \frac{3\bar{\partial}_p^2\Phi_1}{(\Delta\tilde{z}_p)}; \quad iH^2_{-\frac{1}{2}}(z)\Phi(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+} \frac{3\bar{z}_p\bar{\partial}_p^2\Phi_1 + 4\bar{h}\bar{\partial}_p\Phi_1 + 2t_p\bar{\partial}_p\Phi}{(\Delta\tilde{z}_p)} \\
iH^2_{\frac{1}{2}}(z)\Phi(\mathbf{x}_p) &\sim \lim_{\epsilon\to 0^+} \frac{3\bar{z}_p^2\bar{\partial}_p^2\Phi_1 + 8\bar{h}\bar{z}_p\bar{\partial}_p\Phi_1 + 2\bar{h}(2\bar{h}-1)\Phi_1 + 4\bar{z}_p t_p\bar{\partial}_p\Phi + 4\bar{h}t_p\Phi + t_p^2\dot{\Phi}}{(\Delta\tilde{z}_p)} \qquad (6.2) \\
iH^2_{\frac{3}{2}}(z)\Phi(\mathbf{x}_p) &\sim \lim_{\epsilon\to 0^+} 3\frac{\bar{z}_p^3\bar{\partial}_p^2\Phi_1 + 4\bar{h}\bar{z}_p^2\bar{\partial}_p\Phi_1 + 2\bar{h}(2\bar{h}-1)\bar{z}_p\Phi_1 + 2\bar{z}_p^2 t_p\bar{\partial}_p\Phi + 4\bar{h}t_p\bar{z}_p\Phi + t_p^2\bar{z}_p\dot{\Phi}}{(\Delta\tilde{z}_p)}
\end{aligned}$$

where it is understood that the fields on the r.h.s. are at $\mathbf{x}_p$. Finally, the holomorphic mode-expansion of the 'modes' H^2_k are given by (5.5) as:

$$H^2_a(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} H^2_{a;n} z^{-n-\frac{1}{2}}$$

that implies the following transformation generated by, e.g. the modes $H^2_{-\frac{1}{2};n}$, from (6.2):

$$i\left[H^2_{-\frac{1}{2};n}\ ,\ \Phi(\mathbf{x}_p)\right] = z_p^{n-\frac{1}{2}}\left[3\bar{z}_p\bar{\partial}_p^2\Phi_1 + 4\bar{h}\bar{\partial}_p\Phi_1 + 2t_p\bar{\partial}_p\Phi\right](\mathbf{x}_p) \qquad (6.3)$$

derived using the CarrCFT OPE $\longleftrightarrow$ commutator prescription developed in [9].

Thus, the four Carrollian conformal modes $H^2_{a;\frac{1}{2}}$ generate four global transformations on the quantum field Φ, that can be read off of the numerators of the 'OPE's (6.2). These are the four global transformations generated by the Carrollian conformal field S^+_2. Consequently, the existence of the field S^+_2 in the theory demands that the CarrCFT correlators must be invariant under these four global transformations, in addition to the ten global Poincaré constraints imposed by the three 'universal' generators S^+_1, S^+_0 and T.

The most striking feature of the transformations generated by S^+_2 is their non-locality in time (but locality in space): they involve a time-integral of the original primary field. This is consistent with the conclusion of an Einstein-gravity analysis in [61] that the subsubleading soft graviton theorem arises as a consequence of conservation of a spin-2 charge generating a non-local space-time symmetry at null infinity; these symmetry transformations are also non-local only in (retarded-)time.

That in the $\mathcal{O}\left((t-t_p)^0\right)$ terms of the $S^+_2\Phi$ OPE (6.1), it is indeed $\bar{h}$ that appears instead of h can be checked by considering e.g., a Jacobi identity involving the primary Φ and the two Carrollian conformal modes $H^2_{-\frac{1}{2};\frac{1}{2}}$ and $H^0_{\frac{1}{2};-\frac{1}{2}}$. There is no doubt about the terms linear and quadratic in $(t-t_p)$ that are already fixed by the restriction (3.1). Also, by considering another Jacobi identity with $H^0_{\frac{1}{2};-\frac{1}{2}}$ replaced by $H^0_{\frac{1}{2};\frac{1}{2}}$, it becomes apparent that the form of the $\mathcal{O}\left((t-t_p)^0\right)$ terms in the OPE (6.1) must be modified when the primary Φ has non-zero $\left(\bar{\boldsymbol{\xi}}\cdot\Phi\right)$ and/or $(\boldsymbol{\xi}\cdot\Phi)$ or Φ is a non-primary. An important example of the second case is the $S^+_2 T$ OPE whose $\mathcal{O}\left((t-t_p)^0\right)$ terms are very different from those in (6.1). In this work, we do not consider the case of the general primaries because only the Carrollian primaries with $\left(\bar{\boldsymbol{\xi}}\cdot\Phi\right)=0=(\boldsymbol{\xi}\cdot\Phi)$ can describe mass-less scattering in the bulk AFS [9].

Restoring the temporal step-function factor, the S_2^+ Ward identity corresponding to the OPE (6.1) is given by:

$$\begin{aligned} i\langle S_2^+(t,z,\bar{z})X\rangle = \sum_{p=1}^{n}\theta(t-t_p)\Bigg[&\frac{(\bar{z}-\bar{z}_p)^3}{z-z_p}\,\frac{1}{2}\bar{\partial}_p^2\partial_{t_p}^{-1}-\frac{(\bar{z}-\bar{z}_p)^2}{z-z_p}\,2\bar{h}_p\bar{\partial}_p\partial_{t_p}^{-1}+\frac{\bar{z}-\bar{z}_p}{z-z_p}\,\bar{h}_p(2\bar{h}_p-1)\partial_{t_p}^{-1}\\ &+(t-t_p)\left\{\frac{(\bar{z}-\bar{z}_p)^2}{z-z_p}\bar{\partial}_p-2\bar{h}_p\frac{\bar{z}-\bar{z}_p}{z-z_p}\right\}+\frac{(t-t_p)^2}{2}\,\frac{\bar{z}-\bar{z}_p}{z-z_p}\partial_{t_p}\Bigg]\langle X\rangle \end{aligned} \tag{6.4}$$

with X being a string of Carrollian conformal primaries, all with $\left(\bar{\boldsymbol{\xi}}\cdot\Phi\right)=0=(\boldsymbol{\xi}\cdot\Phi)$. The terms regular in $z-z_p$ all vanish because we demand that the $\langle S_2^+(t,z,\bar{z})X\rangle$ correlator be finite whenever $z\neq z_p$; in particular, this correlator must be finite when $z=\infty$ (remembering that the CarrCFT primary S_2^+ has $h>0$). The complete argument is analogous to the ones elaborated in [9] for the finite-ness of the correlators $\langle S_1^+X\rangle$, $\langle S_0^+X\rangle$ and $\langle TX\rangle$. Since the integral operator $\partial_{t_p}^{-1}$ clearly changes $(\Delta,m)\to(\Delta-1,m)$, the $S_{2(e)}^+$ part of this Ward identity is the Carrollian conformal version of the positive-helicity subsubleading conformally soft graviton Ward identity in [36].

In [9], it was shown how to recover the 1+3D bulk AFS leading [12] and the subleading [13] energetically soft graviton theorems by simply taking a temporal Fourier transformation [49–51] of the S_1^+ Ward identity (2.20) and then imposing an energetically soft limit only for the field S_1^+. To perform the temporal Fourier transformation, opposite phase conventions were required [8] for the Carrollian conformal primaries with $\Delta=1$ that were to describe the outgoing or incoming bulk AFS mass-less particles in null-momentum space [7, 8, 11]; this convention was chosen in [9] as (with $\omega\geq 0$):

$$\tilde{\Phi}_{\text{out}}(\omega,z,\bar{z})=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\;e^{-i\omega t}\Phi(t,z,\bar{z})\qquad\text{and}\qquad\tilde{\Phi}_{\text{in}}(\omega,z,\bar{z})=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\;e^{i\omega t}\Phi(t,z,\bar{z}) \tag{6.5}$$

We now temporal Fourier transform the S_2^+ Ward identity (6.4) according to the above convention while choosing the outgoing convention for S_2^+, complexify the energy ω of the field S_2^+, set all $\Delta_p=1$, $m_p=s_p$ (the helicity of the bulk mass-less particle) and finally, take the energetically soft $\omega\to 0$ limit to obtain:[6]

$$\begin{aligned} \lim_{\omega\to 0} i\left\langle\tilde{S}_2^+(\omega,z,\bar{z})\tilde{X}_{\text{out}}\tilde{X}_{\text{in}}\right\rangle = \lim_{\omega\to 0}-\sum_{p\in\text{all}}\Bigg\{&\frac{\omega_p}{\omega^3}\frac{\bar{z}-\bar{z}_p}{z-z_p}\epsilon_p+\frac{1}{\omega^2}\frac{(\bar{z}-\bar{z}_p)^2\bar{\partial}_p+(\bar{z}-\bar{z}_p)\left(s_p+\omega_p\partial_{\omega_p}\right)}{z-z_p}\\ &+\frac{1}{\omega}\frac{\epsilon_p}{2\omega_p}\frac{\left[(\bar{z}-\bar{z}_p)^2\bar{\partial}_p+(\bar{z}-\bar{z}_p)\left(s_p+\omega_p\partial_{\omega_p}\right)\right]^2}{(z-z_p)(\bar{z}-\bar{z}_p)}+\mathcal{O}(\omega^0)\Bigg\}\left\langle\tilde{X}_{\text{out}}\tilde{X}_{\text{in}}\right\rangle \end{aligned} \tag{6.6}$$

with $\epsilon_p=+1$ for outgoing and $\epsilon_p=-1$ for incoming fields. As expected, the r.h.s. is recognized as the soft factorization and energetically (holomorphic[7] [13]) soft expansion of

[6] Modulo the zero-mode problem of $\partial_{t_p}^{-1}$.

[7] Inside a Carrollian correlator, the relation $\partial_t^2 S_2^+=S_0^+$ is translated into $\omega^2\tilde{S}_2^+=-\tilde{S}_0^+$ in the ω-space; so, it is consistent with the fact that insertion of $\lim_{\omega\to 0}i\tilde{S}_0^+(\omega,z,\bar{z})$ leads to the 'natural' soft-factorization [13].

the soft factor of a $1+3$D bulk AFS mass-less scattering amplitude involving an outgoing soft external graviton up to the subsubleading order. The leading $\mathcal{O}(\frac{1}{\omega^3})$ term is the Weinberg universal soft factor [12, 18], the subleading $\mathcal{O}(\frac{1}{\omega^2})$ term is the Cachazo-Strominger universal soft factor [13, 22] and the subsubleading $\mathcal{O}(\frac{1}{\omega})$ is a non-universal [25, 26] soft factor; its form agrees with the one in [62] explicitly calculated for the example of tree-level Einstein-gravity [13]. Within the Carrollian context, this subsubleading soft factor is non-universal since it is present only in those $1+2$D CarrCFTs that contain the field S_2^+, unlike the two in the more leading orders [9].

Thus, we have shown that the Ward identity of the Carrollian conformal field S_2^+ obeying the relation (3.1), indeed encodes the bulk AFS soft graviton theorems up to the subsubleading order. In section 5, it was also concluded that this S_2^+ field contains the positive-helicity subsubleading conformally soft graviton primary [35, 36] of the 2D Celestial CFT.

Inspired by the above conclusions for the field S_2^+, one may get tempted to draw a blind analogy for the whole infinite tower of fields S_k^+ containing, as shown in section 5, the whole family of the 2D Celestial conformal primary soft gravitons of [35]. We shall find out that while this infinite tower of fields S_k^+ indeed encodes, as Ward identities, an infinity of energetically soft theorems described in [42, 43], the fields $S_{k>2}^+$ do not generate any additional global symmetries and hence, impose no new constraints on the CarrCFT correlators, consistent with [35].

We begin by rewriting the general $S_k^+S_l^+$ OPE (4.21) as below:

$$iS_k^+(\mathbf{x})S_l^+(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+}\sum_{r=0}^{k}\frac{(t-t_p)^{k-r}}{(k-r)!}\sum_{m=0}^{r}\frac{(\bar{z}-\bar{z}_p)^{m+1}}{(\Delta\tilde{z}_p)}\frac{(l+1)_{r-m}}{(r-m)!\cdot m!}\,\bar{\partial}_p^m S_{r+l-1}^+(\mathbf{x}_p)$$

and recall that dimensions of the field S_l^+ are $(h,\bar{h}) = \left(\frac{3-l}{2}, -\frac{1+l}{2}\right)$. In exact similarity with the case of S_2^+, we postulate the following $S_k^+\Phi$ OPE:

$$iS_k^+(\mathbf{x})\Phi(\mathbf{x}_p) \sim \lim_{\epsilon\to 0^+}\sum_{r=0}^{k}\frac{(t-t_p)^{k-r}}{(k-r)!}\sum_{m=0}^{r}\frac{(\bar{z}-\bar{z}_p)^{m+1}}{(\Delta\tilde{z}_p)}\frac{(-2\bar{h})_{r-m}}{(r-m)!\cdot m!}\,\bar{\partial}_p^m\Phi_{r-1}(\mathbf{x}_p) \tag{6.7}$$

for a special Carrollian conformal primary Φ with dimensions $(h,\bar{h})$ and with $\left(\bar{\boldsymbol{\xi}}\cdot\Phi\right) = 0 = (\boldsymbol{\xi}\cdot\Phi)$. The unique local fields $\{\Phi_r\}$ satisfy:

$$\partial_t^r\Phi_r \sim \Phi\ (r\geq 1)\,,\qquad \Phi_0 = \Phi \quad\text{and}\quad \Phi_{-1} = \dot{\Phi}$$

Following the discussion on S_2^+, we can extract the 'OPE's $H_a^k(z)\Phi(\mathbf{x}_p)$ analogous to (6.2), by appealing to the decomposition (4.11) and (4.12) and from there, derive the space-time transformations inflicted by the Carrollian conformal modes $H_{a;n}^k$ on the quantum fields, just like (6.3). It is clear that all of these transformations are non-local in time but spatially local as they involve time-integrals of various order of the original primary field. For a discussion on the Einstein-gravity dual of these transformations, see [63].

We are now in a position to conclude that the infinite number of $S_k^+\Phi$ OPEs will thus generate an infinite number of global space-time symmetry transformations under all of which the CarrCFT correlators must be invariant, just as we did for the S_2^+ case. But that is not the case!

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To show this, let us start with the case of S_3^+. The transformations that its modes generate is derived as $\left[H^3_{a;n}\ ,\ \Phi(\mathbf{x}_p)\right]$ via the OPE $\longleftrightarrow$ commutator prescription starting from the $S_3^+\Phi$ OPE. But, from the symmetry algebra (5.7), we see that:

$$i\left[H^2_{a;n}\ ,\ H^2_{b;m}\right] = 3\,(a-b)\,H^3_{a+b;n+m}$$

Thus, the transformation $\left[H^3_{a;n}\ ,\ \Phi(\mathbf{x}_p)\right]$ can be directly found out using a Jacobi identity involving the field Φ and two appropriate modes $H^2_{r;l}$ and $H^2_{s;m}$ and the knowledge of the transformations $\left[H^2_{b;k}\ ,\ \Phi(\mathbf{x}_p)\right]$, without any need to learn the $S_3^+\Phi$ OPE at all. We can iterate this process to extract the transformations generated by the modes of the field $S^+_{k>2}$ from the knowledge of the transformations inflicted by the modes of S_2^+ and S^+_{k-1} without ever appealing to the $S_k^+\Phi$ OPE. Hence by induction, all we require to find the transformations generated by the tower of fields $S^+_{k>2}$ is the knowledge of only the transformations that S_2^+ generate. Thus, the seemingly infinite number of global symmetries are not independent at all from the four generated by S_2^+. Thus, the correlators in such a CarrCFT are subject to a merely four additional global symmetry constraints in addition to the ten 'universal' Poincaré constraints. This conclusion resonates with the observation in section 4 that in a CarrCFT containing a field S_2^+ obeying (3.1), an infinite tower of fields $S^+_{k\geq 3}$ are required to automatically exist to render the $S_2^+S^+_{k-1}$ OPEs consistent.

Since the temporal Fourier transformation of a $1+2$D position-space CarrCFT correlator of primaries all with $\left(\bar{\boldsymbol{\xi}}\cdot\Phi\right) = 0 = (\boldsymbol{\xi}\cdot\Phi)$ [9] and $\Delta = 1$ [7, 8, 11] gives the $1+3$D bulk AFS null-momentum space S-matrix [8], the above discussion implies that no new global symmetry constraint besides the ten 'universal' Poincaré plus the four generated by the field S_2^+ is imposed on this S-matrix by the tower of fields $S^+_{k\geq 3}$. The 2D Celestial CFT counterpart of this statement is proved in [35].

We shall now finally establish that the infinite tower of Carrollian conformal primaries S_k^+ does indeed imply the existence of an infinity of projected (energetically) soft graviton theorems [42, 43] as their Ward identities.

Restoring the temporal step-function, the Ward identity corresponding to the CarrCFT OPE (6.7) is given as the following:

$$i\langle S_k^+(t,z,\bar{z})X\rangle = \sum_{p=1}^{n}\theta(t-t_p)\sum_{r=0}^{k}\frac{(t-t_p)^{k-r}}{(k-r)!}\sum_{m=0}^{r}\frac{(\bar{z}-\bar{z}_p)^{m+1}}{z-z_p}\,\frac{(-2\bar{h}_p)_{r-m}}{(r-m)!\cdot m!}\,\bar{\partial}_p^m\partial_{t_p}^{1-r}\langle X\rangle \tag{6.8}$$

up to a $(k-3)$-th degree polynomial in z that can not be fixed by symmetry considerations alone. Obviously, it will now receive the same treatment as the S_2^+ Ward identity (6.4). For that, we first note the following identity:

$$\int_{-\infty}^{\infty} dt\ e^{-i\omega t}\theta(t-t_p)\,\frac{(t-t_p)^s}{s!} = \lim_{a\to 0^+}\lim_{b\to 0^+}\frac{e^{-i\omega t_p}}{[i\omega+(b-ia)]^{s+1}} \tag{6.9}$$

where ω is a complex quantity.

Following the convention (6.5), we now temporal Fourier transform the S_k^+ Ward identity (6.8) choosing the outgoing convention for S_k^+, set all $\Delta_p = 1$, $m_p = s_p$, use the identity (6.9) (and explicitly put the limits for a, b) and finally impose the energetically soft $\omega \to 0$ limit to obtain the following schematic Laurent series around $\omega = 0$:

$$\lim_{\omega\to 0} i^{k+1} \left\langle \tilde{S}_k^+(\omega, z, \bar{z})\tilde{X}_{\text{out}}\tilde{X}_{\text{in}}\right\rangle = \lim_{\omega\to 0}\left[\frac{F^{(0)}}{\omega^{k+1}} + \frac{F^{(1)}}{\omega^k} + \frac{F^{(2)}}{\omega^{k-1}} + \ldots + \frac{F^{(k)}}{\omega} + \mathcal{O}(\omega^0)\right]\left\langle \tilde{X}_{\text{out}}\tilde{X}_{\text{in}}\right\rangle$$

where $F^{(0)}$ is the Weinberg leading soft factor [12, 18], $F^{(1)}$ is the Cachazo-Strominger subleading soft factor [13, 22] and $F^{(2)}$ is the subsubleading soft factor [13, 62] that appear in (6.6); $F^{(k\geq 3)}$ are the next order soft factors. A soft factor $F^{(k)}$ has $(z - z_p)$ in the denominator and a $(k+1)$-th degree polynomial in $(\bar{z} - \bar{z}_p)$ in the numerator. It is possible (but tedious) to derive its explicit form following our derivation of $F^{(2)}$. From the explicit examples of the S_0^+, S_1^+, S_2^+ cases, it is clear that the soft factor $F^{(k)}$ first appears from the Ward identity of the field S_k^+ at the order $\mathcal{O}(\frac{1}{\omega})$. In this manner, an infinite number of projected energetically soft graviton theorems [42, 43] arises from the Ward identities of the infinite number of Carrollian conformal primaries S_k^+ that contain the 2D Celestial conformal primary soft gravitons H^{1-k} of [35]. Furthermore, the undetermined terms polynomial in z in the $\langle S_k^+ X\rangle$ Ward identity correspond to the homogeneous part of the graviton amplitude that are projected out to obtain the infinite-order soft factorization [42, 43].

7 Discussion

Building on the direct relation between the EM tensor Ward identities of a $1+2$D source-less CarrCFT on a flat Carrollian background (with $\mathbb{R}\times S^2$ topology) and the universal leading [12] and the subleading [13] soft graviton theorems that was uncovered in [9], in this work we investigated how the non-universal [25, 26] subsubleading soft graviton theorem [13] can be holographically encoded into a CarrCFT if at all.

We found out that in addition to the following three universal local generator fields in any generic $1+2$D CarrCFT: S_0^+ (and its non-local shadow S_0^-) whose Ward identity contains the leading soft graviton theorems and S_1^+ and T encoding the subleading ones (of both helicities) [9], a local Carrollian conformal field S_2^+ must be postulated to exist in the theory such that it obeys the relation (3.1), for capturing the positive-helicity subsubleading soft graviton theorem. To avoid the ambiguity associated with double soft limits of opposite helicities [29], here we have refrained from looking into the case of the negative-helicity non-universal subsubleading soft theorem.

After introducing the local field S_2^+ in the theory, we attempted to construct the mutual OPEs of the three fields S_0^+, S_1^+, S_2^+ using only the general forms of the CarrCFT OPEs (2.14) and (2.24), derived completely from Carrollian symmetry arguments in [9], and the OPE commutativity property. For this method to work, we needed to assume that there is no time-independent local field with negative scaling dimension Δ in the theory with the Identity being the unique time-independent field with dimensions $(\Delta, m) = (0, 0)$.

In [31] and in [27], it was assumed that no local field with negative holomorphic weight h exists in the 2D Euclidean (chiral) CFT, with the unique field having $h = 0$ being the Identity, to completely determine the singular parts of respectively the TT OPE and the JJ OPEs using only general conformal symmetry principles and the OPE commutativity property. T

is the holomorphic EM tensor and J represents an additional symmetry generating primary with $h \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$. The above mentioned CarrCFT assumption is similar in spirit to this one. Since, $h > 0$ is the unitary bound in a 2D chiral CFT, we wonder if the non-existence of time-independent local fields with $\Delta < 0$ is analogously a unitarity condition for a $1+2$D CarrCFT. It will be very interesting to establish or dismiss this speculation.

Proceeding to construct the OPEs using only the general symmetry arguments under this assumption, we discovered that for the $S_2^+ S_2^+$ OPE to be consistent, there must automatically exist another local primary S_3^+ in the theory, obeying the relation (3.12). Similarly, it was found that another local field S_4^+, satisfying (4.5) must automatically exist to render the $S_2^+ S_3^+$ OPE consistent. Iterating the algorithm, it could be seen that an OPE $S_2^+ S_k^+$ is consistent if a local field S_{k+1}^+ is automatically present in the theory, that obeys the condition (4.7). Thus, the local primary S_2^+, if present in the CarrCFT, generates an infinite tower of Carrollian conformal primaries $S_{k\geq 3}^+$. Each of these primaries satisfies a null-state condition (4.10).

The general Carrollian conformal OPE $S_k^+(\mathbf{x}) S_l^+(\mathbf{x}_p)$ was obtained as (4.21). In this derivation, the Carrollian time coordinate played the central role via the condition (4.7). Comparing the $\mathcal{O}(t^0 t_p^0)$ terms on both sides of this OPE and then imposing the $t \to \infty$ limit, we recovered the following OPE of two conformally soft graviton primaries: $H^{1-k}(z, \bar{z}) H^{1-l}(z_p, \bar{z}_p)$ of the 2D Celestial CFT [35]. This Celestial OPE was obtained in [35] by taking the conformally soft limit of the OPE of two conformal primary gravitons with arbitrary (Celestial) weights that was itself derived in [36] for the specific case of the bulk tree-level (linearized) Einstein gravity. On the contrary, the Carrollian derivation presented here needed no input from the explicit theory of quantum gravity in the bulk AFS and hence, is purely holographic.

Performing the Carrollian conformal mode-expansion according to the decomposition (4.11) and (4.12) of each primary S_k^+ with holomorphic weight $h = \frac{3-k}{2}$, we found, using the OPE $\longleftrightarrow$ commutator map developed in [9], that the symmetry algebra manifest in the CarrCFT OPEs (4.21) is the (wedge sub-algebra [39] of the) $w_{1+\infty}$ Kac-Moody algebra, in perfect agreement with the conclusion reached in [41] in the context of 2D Celestial CFT. Thus, the complete symmetry algebra at the level of the OPEs of a CarrCFT containing the field S_2^+ is the semi-direct product of the (chiral) Virasoro algebra and the wedge sub-algebra of $\hat{w}_{1+\infty}$; the semi-direct product structure is given by (5.16) arising from the $T S_k^+$ OPEs (5.13).

It is important to recall that we did not rely on a (Carrollian) perturbative analysis to obtain the OPEs (under the assumptions mentioned in section 3.1 though) from which the above symmetry algebra was derived. The starting point of the construction of these OPEs rather was a (Carrollian) path-integral derivation of the CarrCFT EM tensor Ward identities and, from there, the Ward identities of the universal generators S_0^+, S_1^+, T in [9]. We speculate if this implies, in the absence of the operators that modify the subsubleading soft graviton theorem at the tree-level in the effective field theory [25], that the semi-direct product of the (chiral) Virasoro algebra and the wedge sub-algebra of $\hat{w}_{1+\infty}$ is an exact quantum symmetry of the positive-helicity sector of any gravity-theory in $1+3$D bulk AFS, just like the specific case of the quantum self-dual gravity [37].

In [64], it was shown that there is a discrete infinite family of 2D Celestial CFTs possessing the (wedge sub-algebra of) $w_{1+\infty}$ symmetry. Two known examples of such theories are the

MHV gravitons [30, 60] and the quantum self-dual gravity [37]. Since any $1+2$D CarrCFT containing the local field S_2^+ will enjoy the above said symmetry at the level of the OPEs, the conclusion of [64] suggests that there also exists an infinite number of $1+2$D CarrCFT of this type. It will be very interesting to construct such an explicit CarrCFT example.

Finally, we showed that the CarrCFT Ward identity of the field S_2^+ with a special class of primaries [7–9, 11] does indeed encode up to the subsubleading energetically soft graviton theorem [13, 62]. Following this method, the infinite number of soft graviton theorems of [42, 43] were then directly interpreted as the Ward identities of the members of the infinite tower of Carrollian conformal primaries S_k^+.

In [9], it was found that the three universal CarrCFT generators S_0^+, S_1^+, T inflict the ten global ISL$(2,\mathbb{C})$ Poincaré transformations on the quantum fields. Here, we showed that S_2^+ generates four additional global symmetry transformations that, unlike the Poincaré ones, are non-local in time (but local in space). The Einstein-gravity analogue of this result is described in [61]. We further clarified that the other primaries $S_{k\geq 3}^+$ do not generate any further independent global symmetries. Recalling that the global symmetries constrain the correlators of a theory and the CarrCFT correlators can be mapped to the bulk AFS null-momentum space S-matrices [8], our results provided a Carrollian justification of the statement [35] that the infinite number of soft graviton theorems of [42, 43] beyond the subsubleading order does not impose any additional constraints on the bulk AFS mass-less S-matrices.

An obvious future direction that can be pursued following the methodology presented in [9] and in this work would be to figure out how the soft theorems of the gauge theories [42, 43, 65] in the $1+3$D bulk AFS and the tower of conformally soft gluons of 2D Celestial CFT [35] can arise in the framework of $1+2$D CarrCFT. By now it is apparent that some additional Carrollian conformal field(s) besides the three universal generators S_0^+, S_1^+, T must be postulated to exist in the theory, the Ward identities of which would encode the soft gluon theorems. Similar situations have been considered in [27] for 2D chiral CFTs and in [28] for $1+1$D CarrCFTs.

More important is to try to find a resolution to the problem of the double soft limits of opposite helicities [29] within the CarrCFT framework. Extending the current work, one needs to start by assuming the existence of an opposite-spin counterpart of S_2^+. But as discussed in section 3, this field S_2^- can not be simultaneously treated as mutually local with S_0^+, S_1^+, S_2^+, T. The findings of the work [46] in the context of Celestial CFT are expected to play a very crucial role in this endeavour. We hope to report on this in a very near future.

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