

Tri-unification: a separate SU(5) for each fermion family

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ABSTRACT: In this paper we discuss $SU(5)^3$ with cyclic symmetry as a possible grand unified theory (GUT). The basic idea of such a tri-unification is that there is a separate $SU(5)$ for each fermion family, with the light Higgs doublet(s) arising from the third family $SU(5)$, providing a basis for charged fermion mass hierarchies. $SU(5)^3$ tri-unification reconciles the idea of gauge non-universality with the idea of gauge coupling unification, opening the possibility to build consistent non-universal descriptions of Nature that are valid all the way up to the scale of grand unification. As a concrete example, we propose a grand unified embedding of the tri-hypercharge model $U(1)_Y^3$ based on an $SU(5)^3$ framework with cyclic symmetry. We discuss a minimal tri-hypercharge example which can account for all the quark and lepton (including neutrino) masses and mixing parameters. We show that it is possible to unify the many gauge couplings into a single gauge coupling associated with the cyclic $SU(5)^3$ gauge group, by assuming minimal multiplet splitting, together with a set of relatively light colour octet scalars. We also study proton decay in this example, and present the predictions for the proton lifetime in the dominant $e^+\pi^0$ channel.

KEYWORDS: Grand Unification, Theories of Flavour

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1 Introduction

The flavour problem remains one of the most intriguing puzzles of the Standard Model (SM), being responsible for most of its parameters. The origin of three families, which are identical under the SM gauge group, but differ greatly in mass, with the quark mixing being small while the lepton mixing is large, is not addressed, while the origin of CP-violation only adds to the mystery. It is quite common to address these puzzles by assuming that the fermions are distinguished by a new spontaneously broken family symmetry, however this is not the only way forwards.

Recently two of us proposed an embedding of the SM based on the existence of one local weak hypercharge associated to each fermion family [1],

$$SU(3)_c \times SU(2)_L \times U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}, \tag{1.1}$$

where each SM fermion family $i = 1, 2, 3$ is charged only under their corresponding $U(1)_{Y_i}$ factor. Such a framework avoids the family replication of the SM and is naturally anomaly-free. If the Higgs doublet(s) only carry third family hypercharge, then the third family is naturally heavier and the light families are massless in first approximation, providing a novel way of addressing the flavour problem.

In this paper we propose a grand unified embedding of the tri-hypercharge [1] model based on a non-supersymmetric $SU(5)^3$ framework with cyclic symmetry. This is a generalisation of $SU(5)$ grand unification [2] in which we assign a separate $SU(5)$ group to each fermion family, together with a cyclic symmetry to ensure gauge coupling unification.¹ We discuss a minimal example which can account for all the quark and lepton (including neutrino)

¹For related ideas see refs. [3, footnote 41], [4–9].

masses and mixing parameters. We show that it is possible to unify the gauge couplings into a single gauge coupling associated with the cyclic $SU(5)^3$ gauge group, by assuming minimalistic multiplet splitting, together with a set of relatively light colour octet scalars. We also study proton decay in this example, and present the predictions of the proton lifetime in the dominant $e^+\pi^0$ channel.

More generally, the $SU(5)^3$ framework proposed here may embed a broader class of gauge non-universal models, reconciling the ideas of gauge non-universality with gauge coupling unification at the GUT scale. In particular, $SU(5)^3$ may embed theories based on the family decomposition of the SM group, such as the $SU(2)_L^3$ model [10–15], the $SU(3)_c^3$ model [16] or the aforementioned tri-hypercharge model, as an alternative to the existing ultraviolet (UV) completions which are all based on (variations of) the Pati-Salam (PS) group [17–24]. We note however that while most of the previous papers explain the origin of the flavour structure of the SM, none of them provides a gauge unified framework. In general, gauge non-universal models can address the flavour puzzle at the price of complicating the gauge sector, which in such theories may contain up to nine arbitrary gauge couplings in the UV.

The layout of the remainder of the paper is as follows. In section 2 we discuss a general $SU(5)^3$ framework for model building. In rather lengthy section 3 we analyse an example $SU(5)^3$ unification model breaking to tri-hypercharge, including the charged fermion mass hierarchies and quark mixing, neutrino masses and mixing, gauge coupling unification and proton decay. Section 4 concludes the paper. In appendix A we detail the energy regimes, symmetries and particle content of the considered example. In appendix B we tabulate all possible hyperon embeddings in $SU(5)^3$ representations with dimension up to 45.

2 General $SU(5)^3$ framework for model building

The basic idea is to embed the SM gauge group into a semi-simple gauge group containing three $SU(5)$ factors,

$$SU(5)_1 \times SU(5)_2 \times SU(5)_3, \tag{2.1}$$

where each $SU(5)$ factor is associated to one family of chiral fermions $i = 1, 2, 3$. Moreover, we incorporate a cyclic permutation symmetry \mathbb{Z}_3 that relates the three $SU(5)$ factors, in the spirit of the trinification model [25]. This implies that at the high energy GUT scale where $SU(5)^3$ is broken (typically in excess of 10^{16} GeV) the gauge couplings of the three $SU(5)$ factors are equal by cyclic symmetry, such that the gauge sector is fundamentally described by one gauge coupling. Therefore, although $SU(5)^3$ is not a simple group, it may be regarded as a unified gauge theory.

The motivation for considering such an $SU(5)^3$ with cyclic symmetry is that it allows gauge non-universal theories of flavour to emerge at low energies² from a gauge universal theory, depending on the symmetry breaking chain. In the first step, $SU(5)^3$ may be³ broken

² $SU(5)^3$ tri-unification may provide a unified origin for many gauge non-universal theories proposed in the literature to address different questions beyond the flavour puzzle, see e.g. [26, 27].

³This first step of symmetry breaking is optional, but may be convenient to control the scale of gauge unification as discussed in section 3.3.

to three copies of the SM gauge group SM^3 . Then at lower energies, SM^3 is broken to some universal piece $G_{\text{universal}}$ consisting of some diagonal subgroups, together with some remaining family groups $G_1 \times G_2 \times G_3$. If the Higgs doublet(s) transform non-trivially under the third family group G_3 , but not under the first nor second, then third family fermions get natural masses at the electroweak scale, while first and second family fermions are massless in first approximation. Their small masses naturally arise from the breaking of the non-universal gauge group down to the SM, which is the diagonal subgroup, and an approximate $U(2)^5$ flavour symmetry emerges, which is known to provide an efficient suppression of the most dangerous flavour-violating effects for new physics [28, 29].

At still lower energies, the non-diagonal group factors $G_1 \times G_2 \times G_3$ are broken down to their diagonal subgroup, eventually leading to a flavour universal SM gauge group factor. This may happen in stages. It has been shown that the symmetry breaking pattern

$$G_1 \times G_2 \times G_3 \rightarrow G_{1+2} \times G_3 \rightarrow G_{1+2+3} \tag{2.2}$$

naturally explains the origin of fermion mass hierarchies and the smallness of quark mixing, while anarchic neutrino mixing may be incorporated via exotic variations of the type I seesaw mechanism [1, 20].

Minimal examples of this class of theories include the tri-hypercharge model [1], which we shall focus on in this paper, where the universal (diagonal) group consists of the non-Abelian SM gauge group factors $G_{\text{universal}} = SU(3)_c \times SU(2)_L$ while the remaining groups are the three gauge weak hypercharge factors $G_1 \times G_2 \times G_3 = U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}$. Another example is the $SU(2)_L^3$ model [10–15], where $G_{\text{universal}} = SU(3)_c \times U(1)_Y$ and $G_1 \times G_2 \times G_3 = SU(2)_{L1} \times SU(2)_{L2} \times SU(2)_{L3}$. There also exists the $SU(3)_c^3$ model [16] (which is only able to explain the smallness of quark mixing), where $G_{\text{universal}} = SU(2)_L \times U(1)_Y$ and $G_1 \times G_2 \times G_3 = SU(3)_{c1} \times SU(3)_{c2} \times SU(3)_{c3}$. Variations of these theories have been proposed in recent years, several of them assuming a possible embedding into (variations of) a Pati-Salam setup [17–24, 30].

All these theories share a common feature: they explain the origin of the flavour structure of the SM at the price of complicating the gauge sector, which may now contain up to nine arbitrary gauge couplings. We will motivate that $SU(5)^3$ as the embedding of general theories $G_{\text{universal}} \times G_1 \times G_2 \times G_3$ resolves this issue, by unifying the complicated gauge sector of these theories into a single gauge coupling. The main ingredients of our general setup are as follows:

- The presence of the \mathbb{Z}_3 symmetry, which is of fundamental importance to achieve gauge unification, imposes that the matter content of the model shall be invariant under cyclic permutations of the three $SU(5)$ factors. This enforces that each $SU(5)$ factor contains the same representations of fermions and scalars, i.e. if the representation $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is included, then $(\mathbf{C}, \mathbf{A}, \mathbf{B})$ and $(\mathbf{B}, \mathbf{C}, \mathbf{A})$ must be included too.
- Each family of chiral fermions i is embedded in the usual way into $\bar{\mathbf{5}}$ and $\mathbf{10}$ representations of their corresponding $SU(5)_i$ factor, that we denote as $F_i = (d_i^c, \ell_i) \sim \bar{\mathbf{5}}_i$ and $T_i = (q_i, u_i^c, e_i^c) \sim \mathbf{10}_i$ as shown in table 1. This choice is naturally consistent with the \mathbb{Z}_3 symmetry.

Field	SU(5) ₁	SU(5) ₂	SU(5) ₃
F_1	$\bar{\mathbf{5}}$	$\mathbf{1}$	$\mathbf{1}$
F_2	$\mathbf{1}$	$\bar{\mathbf{5}}$	$\mathbf{1}$
F_3	$\mathbf{1}$	$\mathbf{1}$	$\bar{\mathbf{5}}$
T_1	$\mathbf{10}$	$\mathbf{1}$	$\mathbf{1}$
T_2	$\mathbf{1}$	$\mathbf{10}$	$\mathbf{1}$
T_3	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{10}$
Ω_1	$\mathbf{24}$	$\mathbf{1}$	$\mathbf{1}$
Ω_2	$\mathbf{1}$	$\mathbf{24}$	$\mathbf{1}$
Ω_3	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{24}$
H_1	$\mathbf{5}$	$\mathbf{1}$	$\mathbf{1}$
H_2	$\mathbf{1}$	$\mathbf{5}$	$\mathbf{1}$
H_3	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{5}$

Table 1. Minimal content for the general SU(5)³ setup. Due to the cyclic symmetry, there are only four independent representations, one for each of the fermions F_i , T_i and the scalars Ω_i , H_i .

- In a similar manner, three Higgs doublets H_1 , H_2 and H_3 are embedded into $\mathbf{5}$ representations, one for each SU(5)_{*i*} factor. Notice that in non-universal theories of flavour it is commonly assumed the existence of only one Higgs doublet H_3 , which transforms only under the third site in order to explain the heaviness of the third family. This way, the SU(5)³ framework involves the restriction of having three Higgses rather than only one, but we will argue that if the \mathbb{Z}_3 symmetry is broken below the GUT scale, then only the third family Higgs H_3 may be light and perform electroweak symmetry breaking, while H_1 and H_2 are heavier and may play the role of heavy messengers for the effective Yukawa couplings of the light families.
- Higgs scalars in bi-representations connecting the different sites may be needed to generate the SM flavour structure at the level of the $G_{\text{universal}} \times G_1 \times G_2 \times G_3$ theory, e.g. $(\mathbf{2}, \bar{\mathbf{2}})$ scalars in SU(2)_L³ or $(Y, -Y)$ scalars in tri-hypercharge (the so-called hyperons). These can be embedded in the associated bi-representations of SU(5)³, e.g. $(\mathbf{5}, \bar{\mathbf{5}})$ scalars, $(\mathbf{10}, \bar{\mathbf{10}})$ scalars and so on. In appendix B we tabulate all such scalars from SU(5)³ representations with dimension up to $\mathbf{45}$, along with the hyperons that they generate at low energies.
- Finally, three scalar fields in the adjoint representation of each SU(5), $\Omega_i \sim \mathbf{24}_i$, spontaneously break each SU(5)_{*i*}. We argue that the three Ω_i are enough to perform *both* horizontal and vertical breaking of the SU(5) groups, thanks to the \mathbb{Z}_3 symmetry. Indeed, thanks to the \mathbb{Z}_3 symmetry, the three adjoint scalars can get naturally degenerate VEVs, in such a way that the three SU(5) groups can be simultaneously broken at the

GUT scale down to the non-universal gauge group $G_{\text{universal}} \times G_1 \times G_2 \times G_3$ of choice that later explains the flavour structure of the SM (e.g. tri-hypercharge or $SU(2)_L^3$). Another possibility that we will explore is breaking $SU(5)^3$ first to three copies of the SM (one for each family) and then to $G_{\text{universal}} \times G_1 \times G_2 \times G_3$ in a second step.

To summarise, the general pattern of symmetry breaking we assume is as follows,⁴

$$SU(5)^3 \rightarrow SM_1 \times SM_2 \times SM_3 \tag{2.3}$$

$$\rightarrow G_{\text{universal}} \times G_1 \times G_2 \times G_3 \tag{2.4}$$

$$\rightarrow G_{\text{universal}} \times G_{1+2} \times G_3 \tag{2.5}$$

$$\rightarrow SM_{1+2+3}, \tag{2.6}$$

where the SM^3 step is optional but may be convenient to achieve unification. In particular, the first step of symmetry breaking makes use of the three SM singlets contained in $\Omega_i \sim \mathbf{24}_i$, while the second step may be performed via the remaining degrees of freedom in Ω_i , depending on the details of the low energy gauge theory that survives. The two final breaking steps are performed by Higgs scalars connecting the different sites that need to be specified for each particular model.

Beyond the general considerations listed in this section, when building a specific model one needs to choose the symmetry group $G_{\text{universal}} \times G_1 \times G_2 \times G_3$, and add explicit scalars and/or fermion messengers that mediate the effective Yukawa couplings of light fermions.

Finally, one needs to study the Renormalization Group Equations (RGEs) of the various gauge couplings at the different steps all the way up to the $SU(5)^3$ scale where all gauge couplings need to unify. This is not a simple task, but we shall see that the relatively light messengers required to generate the effective Yukawa couplings, along with the presence of the approximate \mathbb{Z}_3 symmetry at low energies, may naturally help to achieve unification. In the following, we shall illustrate this by describing a working example of the $SU(5)^3$ framework based on tri-hypercharge [1], where the various gauge couplings of the tri-hypercharge model unify at the GUT scale into a single gauge coupling.

3 An example $SU(5)^3$ unification model breaking to tri-hypercharge

We now turn to the main example of interest, namely $G_{\text{universal}} = SU(3)_c \times SU(2)_L$ (the diagonal non-Abelian SM gauge group factors) with $G_1 \times G_2 \times G_3 = U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}$ (the tri-hypercharge model). In this example, the basic idea is that $SU(5)^3$ breaks, via a sequence of scales, to the low energy (well below the GUT scale) tri-hypercharge gauge group with a separate gauged weak hypercharge for each fermion family,

$$SU(5)^3 \rightarrow \dots \rightarrow SU(3)_c \times SU(2)_L \times U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}. \tag{3.1}$$

⁴One should note that none of the individual groups, $SU(3)$, $SU(2)$ or $U(1)$, in each SM_i group correspond to the SM's $SU(3)_c$, $SU(2)_L$ or $U(1)_Y$. The latter emerge after symmetry breaking from the diagonal sub-groups of the former. Nevertheless, we will denote each $(SU(3) \times SU(2) \times U(1))_i$ as SM_i and the total $(SU(3) \times SU(2) \times U(1))^3$ group as SM^3 for the sake of brevity.

In [1] it was shown that the low energy tri-hypercharge model can naturally generate the flavour structure of the SM if spontaneously broken to SM hypercharge in a convenient way. The minimal setup involves the vacuum expectation values (VEVs) of the new Higgs “hyperons”

$$\phi_{q12} \sim (\mathbf{1}, \mathbf{1})_{(-1/6, 1/6, 0)}, \quad \phi_{q23} \sim (\mathbf{1}, \mathbf{1})_{(0, -1/6, 1/6)}, \quad \phi_{\ell 23} \sim (\mathbf{1}, \mathbf{1})_{(0, 1/2, -1/2)}. \quad (3.2)$$

At the GUT scale, the hyperons are embedded into bi- $\bar{\mathbf{5}}$ and bi- $\mathbf{10}$ representations of $SU(5)^3$ expressed as $\Phi_{ij}^{T,F}$, which must preserve the cyclic symmetry, as shown in table 2. Although this involves the appearance of many hyperons (and other scalars) beyond the minimal set of hyperons that we need, we shall assume that only the desired hyperons get a VEV (and the rest of scalars may remain very heavy). Moreover, the $SU(5)^3$ framework also poses constraints on the possible family hypercharges of the hyperons, as collected in appendix B. For the $SU(5)^3$ setup, it is convenient to add

$$\phi_{q13} \sim (\mathbf{1}, \mathbf{1})_{(-1/6, 0, 1/6)}, \quad \phi_{\ell 13} \sim (\mathbf{1}, \mathbf{1})_{(1/2, 0, -1/2)}, \quad (3.3)$$

which are anyway required by the cyclic symmetry, to the set of hyperons which get a VEV.

The hyperons allow to write a set of non-renormalisable operators that provide effective Yukawa couplings for light fermions, as described in [1] by working in an effective field theory (EFT) framework. However, in our unified model, we need to introduce heavy messengers that mediate such effective operators in order to obtain a UV complete setup. For this, we add one set of vector-like fermions transforming in the $\mathbf{10}$ representation for each $SU(5)$ factor, i.e. $\chi_i \sim \mathbf{10}_i$ and $\bar{\chi}_i \sim \bar{\mathbf{10}}_i$. We shall assume that only the quark doublets $Q_i \sim (\mathbf{3}, \mathbf{2})_{1/6_i}$ and $\bar{Q}_i \sim (\bar{\mathbf{3}}, \mathbf{2})_{-1/6_i}$ are relatively light and play a role in the effective Yukawa couplings, while the remaining degrees of freedom in χ_i and $\bar{\chi}_i$ remain very heavy,

$$\chi_i \rightarrow Q_i \sim (\mathbf{3}, \mathbf{2})_{1/6_i}, \quad \bar{\chi}_i \rightarrow \bar{Q}_i \sim (\bar{\mathbf{3}}, \mathbf{2})_{-1/6_i}. \quad (3.4)$$

We shall see that Q_i and \bar{Q}_i also contribute to the RGEs in the desired way to achieve gauge unification. The full field content of this model also includes extra vector-like fermions Σ and Ξ as shown in table 2. These play a role in the origin of neutrino masses as discussed in section 3.2.

Finally, beyond the minimal set of Higgs doublets introduced in section 2, we shall introduce here three pairs of $\mathbf{5}$, $\bar{\mathbf{5}}$ and $\mathbf{45}$ Higgs representations preserving the cyclic symmetry. The doublets in the $\bar{\mathbf{5}}$ and $\mathbf{45}$ mix, leaving light linear combinations that couple differently to down-quarks and charged leptons in the usual way [31], which we denote as H_i^d .

Therefore, below the GUT scale we effectively have three pairs of Higgs doublets $H_1^{u,d}$, $H_2^{u,d}$ and $H_3^{u,d}$, such that the u - and d -labeled Higgs only couple to up-quarks (and neutrinos) and to down-quarks and charged leptons, respectively, in the spirit of the type II two Higgs doublet model. This choice is motivated to explain the mass hierarchies between the different charged sectors, as originally identified in [1], and could be enforced e.g. by a \mathbb{Z}_2 discrete symmetry. We assume that the third family Higgs $H_3^{u,d}$ are the lightest, they perform electroweak symmetry breaking and provide Yukawa couplings for the third family with $\mathcal{O}(1)$ coefficients if $\tan\beta \approx 20$. In contrast, we assume that the Higgs $H_1^{u,d}$, $H_2^{u,d}$ have masses above the TeV (but much below the GUT scale) and act as messengers of the effective Yukawa couplings for the light families.

Field	SU(5) ₁	SU(5) ₂	SU(5) ₃
F_1	$\bar{5}$	1	1
F_2	1	$\bar{5}$	1
F_3	1	1	$\bar{5}$
T_1	10	1	1
T_2	1	10	1
T_3	1	1	10
χ_1	10	1	1
χ_2	1	10	1
χ_3	1	1	10
Ξ_0	1	1	1
Ξ_{12}	5	$\bar{5}$	1
Ξ_{13}	$\bar{5}$	1	5
Ξ_{23}	1	5	$\bar{5}$
Σ_{atm}	1	10	$\bar{10}$
Σ_{sol}	10	1	$\bar{10}$
Σ_{cyclic}	10	$\bar{10}$	1
Ω_1	24	1	1
Ω_2	1	24	1
Ω_3	1	1	24
H_1^u	5	1	1
H_2^u	1	5	1
H_3^u	1	1	5
$H_1^{\bar{5}}$	$\bar{5}$	1	1
$H_2^{\bar{5}}$	1	$\bar{5}$	1
$H_3^{\bar{5}}$	1	1	$\bar{5}$
H_1^{45}	45	1	1
H_2^{45}	1	45	1
H_3^{45}	1	1	45
Φ_{12}^F	5	$\bar{5}$	1
Φ_{13}^F	$\bar{5}$	1	5
Φ_{23}^F	1	5	$\bar{5}$
Φ_{12}^T	$\bar{10}$	10	1
Φ_{13}^T	10	1	$\bar{10}$
Φ_{23}^T	1	$\bar{10}$	10
Φ_{12}^{45}	1	$\bar{45}$	45
Φ_{13}^{45}	$\bar{45}$	1	45
Φ_{12}^{45}	$\bar{45}$	45	1
Φ^{TFT}	10	5	10
Φ^{FTT}	5	10	10
Φ^{TTF}	10	10	5

Table 2. Fermion and scalar particle content and representations under SU(5)³. F_i and T_i include the chiral fermions of the SM in the usual way, while χ_i , ξ 's and Ξ 's (highlighted in yellow) are vector-like fermions, thus the conjugate partners must be considered. Ω 's, H 's and Φ 's are scalars.

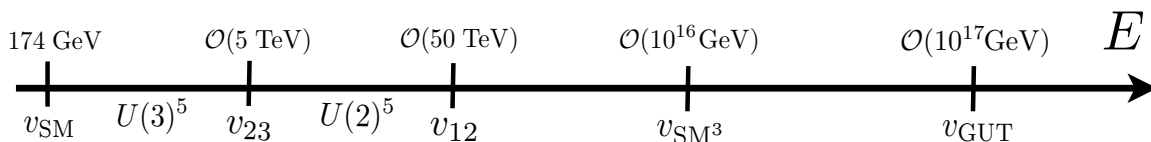


Figure 1. Diagram showing the different scales of spontaneous symmetry breaking in our example model (see also eqs. (3.5)–(3.8)), along with the accidental, approximate flavour symmetries $U(3)^5$ and $U(2)^5$ that arise at low energies.

In detail, we assume that the $SU(5)^3$ group is broken down to the SM through the following symmetry breaking chain

$$SU(5)^3 \xrightarrow{v_{\text{GUT}}} \text{SM}_1 \times \text{SM}_2 \times \text{SM}_3 \quad (3.5)$$

$$\xrightarrow{v_{\text{SM}^3}} SU(3)_{1+2+3} \times SU(2)_{1+2+3} \times U(1)_1 \times U(1)_2 \times U(1)_3 \quad (3.6)$$

$$\xrightarrow{v_{12}} SU(3)_{1+2+3} \times SU(2)_{1+2+3} \times U(1)_{1+2} \times U(1)_3 \quad (3.7)$$

$$\xrightarrow{v_{23}} SU(3)_{1+2+3} \times SU(2)_{1+2+3} \times U(1)_{1+2+3}. \quad (3.8)$$

The $SU(5)^3$ breaking happens at the GUT scale, while the tri-hypercharge breaking may happen as low as the TeV scale, as allowed by current data [1], while the SM^3 breaking step is optional but may be convenient to achieve unification, and may be regarded as free parameter. This second breaking step is performed by the $SU(3)_i$ octets and $SU(2)_i$ triplets contained in $\Omega_i \sim \mathbf{24}_i$. See also figure 1 for an illustrative diagram.

We shall show that within this setup, achieving gauge unification just requires further assuming that the three colour octets that live in $\Omega_i \sim \mathbf{24}_i$ are light, while the remaining degrees of freedom of the adjoint remain very heavy. Before that, we shall study in detail how our model explains the origin of the flavour structure of the SM.

3.1 Charged fermion mass hierarchies and quark mixing

The Higgs doublets in the cyclic $\bar{\mathbf{5}}$ and $\mathbf{45}$ split the couplings of down-quarks and charged leptons in the usual way [31]. We denote as H_i^d the linear combinations that remain light, with their effective couplings to down-quarks and charged leptons given by

$$\tilde{y}_i^d H_i^d T_i F_i \rightarrow y_i^d H_i^d q_i d_i^c + y_i^e H_i^d \ell_i e_i^c, \quad (3.9)$$

where

$$y_i^d = y_i^{\bar{\mathbf{5}}} + y_i^{\overline{\mathbf{45}}}, \quad y_i^e = y_i^{\bar{\mathbf{5}}} - 3y_i^{\overline{\mathbf{45}}}. \quad (3.10)$$

We focus now on the following set of couplings involving the hyperons, the vector-like fermions χ_i and the light linear combinations of Higgs doublets,

$$\begin{aligned} \mathcal{L} \supset & x_{ij} \Phi_{ij}^T T_i \bar{\chi}_j + z_i^u H_i^u \chi_i T_i + z_i^d H_i^d \chi_i F_i \\ & + y_i^u H_i^u T_i T_i + \tilde{y}_i^d H_i^d T_i F_i + f_{ij}^u H_i^u \tilde{H}_j^u \tilde{\Phi}_{ij}^F + f_{ij}^d H_i^d \tilde{H}_j^d \Phi_{ij}^F, \end{aligned} \quad (3.11)$$

where $i, j = 1, 2, 3$, $f_{ij}^{u,d}$ have mass dimension and the rest of the couplings are dimensionless. After integrating out the heavy vector-like fermions χ_i , $\bar{\chi}_i$ and Higgs doublets $H_{1,2}^{u,d}$, we obtain

the following set of effective Yukawa couplings,

$$\mathcal{L} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} c_{11}^u \frac{\phi_{\ell 13}}{M_{H_1^u}} & c_{12}^u \frac{\phi_{\ell 23}}{M_{H_2^u}} \frac{\phi_{q 12}}{M_{Q_2}} & c_{13}^u \frac{\phi_{q 13}}{M_{Q_3}} \\ c_{21}^u \frac{\phi_{\ell 13}}{M_{H_1^u}} \frac{\phi_{q 12}}{M_{Q_1}} & c_{22}^u \frac{\phi_{\ell 23}}{M_{H_2^u}} & c_{23}^u \frac{\phi_{q 23}}{M_{Q_3}} \\ c_{31}^u \frac{\phi_{\ell 13}}{M_{H_1^u}} \frac{\tilde{\phi}_{q 13}}{M_{Q_1}} & c_{32}^u \frac{\phi_{\ell 23}}{M_{H_2^u}} \frac{\tilde{\phi}_{q 23}}{M_{Q_2}} & c_{33}^u \end{pmatrix} \begin{pmatrix} u_1^c \\ u_2^c \\ u_3^c \end{pmatrix} H_3^u \quad (3.12)$$

$$+ \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} c_{11}^d \frac{\tilde{\phi}_{\ell 13}}{M_{H_1^d}} & c_{12}^d \frac{\tilde{\phi}_{\ell 23}}{M_{H_2^d}} \frac{\phi_{q 12}}{M_{Q_2}} & c_{13}^d \frac{\phi_{q 13}}{M_{Q_3}} \\ c_{21}^d \frac{\tilde{\phi}_{\ell 13}}{M_{H_1^d}} \frac{\phi_{q 12}}{M_{Q_1}} & c_{22}^d \frac{\tilde{\phi}_{\ell 23}}{M_{H_2^d}} & c_{23}^d \frac{\phi_{q 23}}{M_{Q_3}} \\ c_{31}^d \frac{\tilde{\phi}_{\ell 13}}{M_{H_1^d}} \frac{\phi_{q 13}}{M_{Q_1}} & c_{32}^d \frac{\tilde{\phi}_{\ell 23}}{M_{H_2^d}} \frac{\phi_{q 23}}{M_{Q_2}} & c_{33}^d \end{pmatrix} \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \end{pmatrix} H_3^d \quad (3.13)$$

$$+ \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix} \begin{pmatrix} c_{11}^e \frac{\tilde{\phi}_{\ell 13}}{M_{H_1^d}} & 0 & 0 \\ 0 & c_{22}^e \frac{\tilde{\phi}_{\ell 23}}{M_{H_2^d}} & 0 \\ 0 & 0 & c_{33}^e \end{pmatrix} \begin{pmatrix} e_1^c \\ e_2^c \\ e_3^c \end{pmatrix} H_3^d + \text{h.c.}, \quad (3.14)$$

where the dimensionless coefficients $c_{ij}^{u,d,e}$ are given by

$$c_{ij}^e = \text{diag} \left(y_1^e \frac{f_{13}^d}{M_{H_1^d}}, y_2^e \frac{f_{23}^d}{M_{H_2^d}}, y_3^e \right). \quad (3.15)$$

$$c_{ij}^{u,d} = \begin{pmatrix} y_1^{u,d} \frac{f_{13}^{u,d}}{M_{H_1^{u,d}}} & x_{12} y_2^{u,d} \frac{f_{23}^{u,d}}{M_{H_2^{u,d}}} & x_{13} z_3^{u,d} \\ x_{21} y_1^{u,d} \frac{f_{13}^{u,d}}{M_{H_1^{u,d}}} & y_2^{u,d} \frac{f_{23}^{u,d}}{M_{H_2^{u,d}}} & x_{23} z_3^{u,d} \\ x_{31} y_1^{u,d} \frac{f_{13}^{u,d}}{M_{H_1^{u,d}}} & x_{32} y_2^{u,d} \frac{f_{23}^{u,d}}{M_{H_2^{u,d}}} & y_3^{u,d} \end{pmatrix}. \quad (3.16)$$

It is clear that third family charged fermions get their masses from $\mathcal{O}(1)$ Yukawa couplings to the Higgs doublets $H_3^{u,d}$, where the mass hierarchies $m_{b,\tau}/m_t$ are explained via $\tan \beta \approx \lambda^{-2}$, where $\lambda \simeq 0.224$ is the Wolfenstein parameter. In contrast, quark mixing and the masses of first and second family charged fermions arise from effective Yukawa couplings involving the heavy messengers of the model, once the hyperons develop their VEVs. The heavy Higgs doublets $H_1^{u,d}$ and $H_2^{u,d}$ play a role in the origin of the family mass hierarchies, while the origin of quark mixing involves both the heavy Higgs and the vector-like quarks Q_i and \bar{Q}_i , as shown in figure 2. We fix the various $\langle \phi \rangle / M$ ratios in terms of the Wolfenstein parameter $\lambda \simeq 0.224$

$$\frac{\langle \phi_{q 23} \rangle}{M_{Q_i}} \approx \lambda^2, \quad \frac{\langle \phi_{q 13} \rangle}{M_{Q_i}} \approx \lambda^3, \quad \frac{\langle \phi_{\ell 23} \rangle}{M_{H_2^{u,d}}} \approx \lambda^3, \quad \frac{\langle \phi_{q 12} \rangle}{M_{Q_i}} \approx \lambda, \quad \frac{\langle \phi_{\ell 23} \rangle}{M_{H_1^{u,d}}} \approx \lambda^6. \quad (3.17)$$

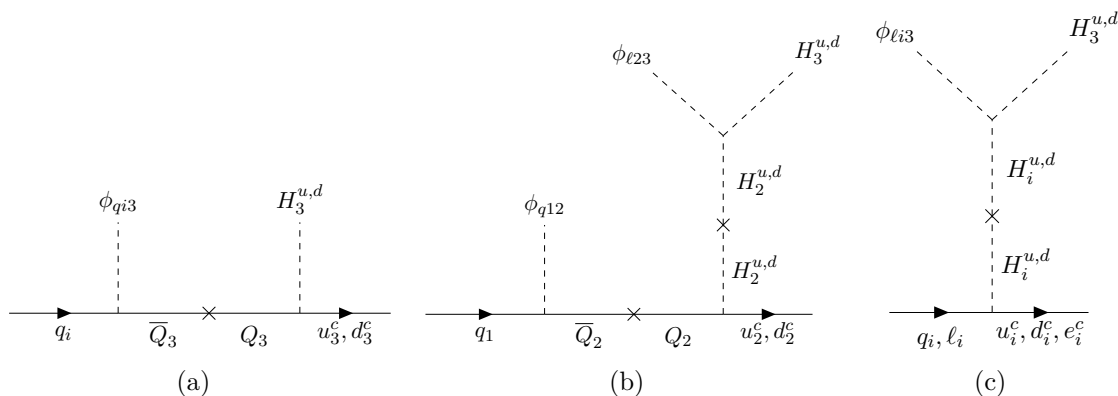


Figure 2. Diagrams in the model which lead to the origin of light charged fermion masses and quark mixing, where $i = 1, 2$.

We notice that the tiny masses of the first family are explained via the hierarchies of Higgs messengers

$$M_{H_3^{u,d}} \ll M_{H_2^{u,d}} \ll M_{H_1^{u,d}}, \quad (3.18)$$

in the spirit of messenger dominance [32]. In other words, the heavy Higgs doublets $H_1^{u,d}$ and $H_2^{u,d}$ can be thought of gaining small effective VEVs from mixing with $H_3^{u,d}$, which are light and perform electroweak symmetry breaking, and these effective VEVs provide naturally small masses for light charged fermions. This is in contrast with the original spirit of tri-hypercharge, where the m_1/m_2 mass hierarchies find their natural origin due to the higher dimension of the effective Yukawa couplings involving the first family [1]. However, we note that in the $SU(5)^3$ framework, the three pairs of Higgs doublets $H_i^{u,d}$ are required by the \mathbb{Z}_3 symmetry, hence it seems natural that they play a role on the origin of fermion masses. Moreover, the introduction of these Higgs provides a very minimal framework to UV-complete the effective Yukawa couplings of tri-hypercharge, which otherwise would require a much larger amount of heavy messengers that are not desired, as they may enhance too much the RGE of the gauge couplings, eventually leading to a non-perturbative gauge coupling at the GUT scale.

The numerical values for the ratios in eq. (3.17) provide the following Yukawa textures (ignoring dimensionless coefficients)

$$\mathcal{L} = (q_1 \ q_2 \ q_3) \begin{pmatrix} \lambda^6 & \lambda^4 & \lambda^3 \\ \lambda^7 & \lambda^3 & \lambda^2 \\ \lambda^9 & \lambda^5 & 1 \end{pmatrix} \begin{pmatrix} u_1^c \\ u_2^c \\ u_3^c \end{pmatrix} v_{\text{SM}} \quad (3.19)$$

$$+ (q_1 \ q_2 \ q_3) \begin{pmatrix} \lambda^6 & \lambda^4 & \lambda^3 \\ \lambda^7 & \lambda^3 & \lambda^2 \\ \lambda^9 & \lambda^5 & 1 \end{pmatrix} \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \end{pmatrix} \lambda^2 v_{\text{SM}} \quad (3.20)$$

$$+ (\ell_1 \ \ell_2 \ \ell_3) \begin{pmatrix} \lambda^6 & 0 & 0 \\ 0 & \lambda^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1^c \\ e_2^c \\ e_3^c \end{pmatrix} \lambda^2 v_{\text{SM}} + \text{h.c.}, \quad (3.21)$$

where v_{SM} is the usual SM electroweak VEV and the fit of the up-quark mass may be improved by assuming a mild difference between $M_{H_1^u}$ and $M_{H_1^d}$. In general, the alignment of the CKM matrix is not predicted but depends on the choice of dimensionless coefficients and on the difference between $M_{H_2^u}$ and $M_{H_2^d}$. Any charged lepton mixing is suppressed by the very heavy masses of the required messengers contained in χ_i and $\bar{\chi}_i$, leading to the off-diagonal zeros in eq. (3.21), in such a way that the PMNS matrix must dominantly arise from the neutrino sector, as we shall see. We notice that a mild hierarchy of dimensionless couplings $y_1^e/y_1^d \approx \lambda^{1.4}$ may be needed to account for the mass hierarchy between the down-quark and the electron.

The larger suppression of the (2, 1), (3, 1) and (3, 2) entries in the quark Yukawa textures ensures a significant suppression of right-handed quark mixing. This is a very desirable feature, given the strong phenomenological constraints on right-handed flavour-changing currents [33, 34]. This way, we expect the model to reproduce the low energy phenomenology of Model 2 in [1], where the VEVs of the 23 and 13 hyperons may be as low as the TeV scale, while the VEVs of the 12 hyperons may be as low as 50 TeV or so. In this manner, we provide the following benchmark values for the mass scales involved in the flavour sector⁵

$$\langle \phi_{q23} \rangle \approx \langle \phi_{q13} \rangle \approx \langle \phi_{\ell23} \rangle \approx \langle \phi_{\ell13} \rangle \sim \mathcal{O}(5 \text{ TeV}), \tag{3.22}$$

$$\langle \phi_{q12} \rangle \sim \mathcal{O}(50 \text{ TeV}), \tag{3.23}$$

$$M_{Q_i} \sim \mathcal{O}(100 \text{ TeV}), \tag{3.24}$$

$$M_{H_2^{u,d}} \sim \mathcal{O}(100 \text{ TeV}), \tag{3.25}$$

$$M_{H_1^{u,d}} \sim \mathcal{O}(10^4 \text{ TeV}). \tag{3.26}$$

3.2 Neutrino masses and mixing

Explaining the observed pattern of neutrino mixing and mass splittings in gauge non-universal theories of flavour is usually difficult, due to the accidental $U(2)^5$ flavour symmetry predicted by these models, which is naively present in the neutrino sector as well. However, exotic variations of the type I seesaw mechanism have been shown to be successful in accommodating neutrino observations within non-universal theories of flavour, see refs. [1, 20]. Here we will incorporate the mechanism of [1], which consists of adding SM singlet neutrinos which carry family hypercharges (although their sum must of course vanish). These neutrinos can be seen as the fermionic counterpart of hyperons, as they will connect the different hypercharge sites, therefore breaking the $U(2)^5$ flavour symmetry in the neutrino sector. In this manner, these neutrinos allow to write effective operators which may provide a successful pattern for neutrino mixing. However, the particular model presented in [1] incorporates SM singlet neutrinos with 1/4 family hypercharge factors, which cannot be obtained from $SU(5)^3$, at least not from representations with dimension smaller than **45**,⁶ according to a search with `GroupMath` [35].

Following the recipe of ref. [1], we start by introducing two right-handed neutrinos: $N_{\text{atm}} \sim (\mathbf{1}, \mathbf{1})_{(0,2/3,-2/3)}$ and $N_{\text{sol}} \sim (\mathbf{1}, \mathbf{1})_{(2/3,0,-2/3)}$, which will be responsible for atmospheric

⁵We note that all VEVs and masses listed here may vary by $\mathcal{O}(1)$ factors, as naturally expected, without affecting our final conclusions.

⁶Since these singlet neutrinos can be seen as the fermionic counterpart of hyperons, the search for $SU(5)^3$ hyperon embeddings shown in appendix B shows that no neutrinos with 1/4 family hypercharge factors are found from $SU(5)^3$ representations with dimension up to **45**.

and solar neutrino mixing, respectively. These neutrinos are embedded in $\Sigma_{23} \sim (\mathbf{1}, \mathbf{10}, \overline{\mathbf{10}})$ and $\Sigma_{13} \sim (\mathbf{10}, \mathbf{1}, \overline{\mathbf{10}})$ representations of $SU(5)^3$, respectively. We also need to add the cyclic permutation N_{cyclic} embedded in $\Sigma_{12} \sim (\mathbf{10}, \overline{\mathbf{10}}, \mathbf{1})$ to preserve the cyclic symmetry of $SU(5)^3$. However, we find that if the ‘‘cyclic’’ neutrino contained in Σ_{12} is much heavier than the other neutrinos, then we can ignore it as it decouples from the seesaw, and we recover the minimal framework of ref. [1]. Finally, in order to cancel gauge anomalies, we choose to make these neutrinos vector-like by introducing the three corresponding conjugate neutrinos.

The next step is adding hyperons that provide effective Yukawa couplings and Majorana masses for the singlet neutrinos. These are summarised in the Dirac and Majorana mass matrices that follow (ignoring the $\mathcal{O}(1)$ dimensionless couplings and the much heavier cyclic neutrinos)

$$m_{D_L} = \frac{1}{M_\xi} \left(\begin{array}{c|cc} & \overline{N}_{\text{sol}} & \overline{N}_{\text{atm}} \\ \hline L_1 & 0 & 0 \\ L_2 & 0 & 0 \\ L_3 & \tilde{\phi}_{u31}^{(-\frac{2}{3}, 0, \frac{2}{3})} & \tilde{\phi}_{u32}^{(0, -\frac{2}{3}, \frac{2}{3})} \end{array} \right) H_u, \quad m_{D_R} = \frac{1}{M_\xi} \left(\begin{array}{c|cc} & N_{\text{sol}} & N_{\text{atm}} \\ \hline L_1 & \phi_{q13}^{(-\frac{1}{6}, 0, \frac{1}{6})} & 0 \\ L_2 & \phi_{u123}^{(-\frac{2}{3}, \frac{1}{2}, \frac{1}{6})} & \phi_{q23}^{(0, -\frac{1}{6}, \frac{1}{6})} \\ L_3 & \phi_{u31}^{(-\frac{2}{3}, 0, \frac{2}{3})} & \phi_{u32}^{(0, -\frac{2}{3}, \frac{2}{3})} \end{array} \right) H_u, \quad (3.27)$$

$$M_L = \left(\begin{array}{c|cc} & \overline{N}_{\text{sol}} & \overline{N}_{\text{atm}} \\ \hline \overline{N}_{\text{sol}} & \tilde{\phi}_{\text{sol}}^{(-\frac{4}{3}, 0, \frac{4}{3})} & 0 \\ \overline{N}_{\text{atm}} & 0 & \tilde{\phi}_{\text{atm}}^{(0, -\frac{4}{3}, \frac{4}{3})} \end{array} \right), \quad M_R = \left(\begin{array}{c|cc} & N_{\text{sol}} & N_{\text{atm}} \\ \hline N_{\text{sol}} & \phi_{\text{sol}}^{(-\frac{4}{3}, 0, \frac{4}{3})} & 0 \\ N_{\text{atm}} & 0 & \phi_{\text{atm}}^{(0, -\frac{4}{3}, \frac{4}{3})} \end{array} \right), \quad (3.28)$$

$$M_{LR} = \left(\begin{array}{c|cc} & N_{\text{sol}} & N_{\text{atm}} \\ \hline \overline{N}_{\text{sol}} & M_{N_{\text{sol}}} & 0 \\ \overline{N}_{\text{atm}} & 0 & M_{N_{\text{atm}}} \end{array} \right), \quad (3.29)$$

where the heavy scale M_ξ is associated to the mass of the heavy vector-like fermions $\xi_0 \sim (\mathbf{1}, \mathbf{1})_{(0,0,0)}$, $\xi_{23} \sim (\mathbf{1}, \mathbf{1})_{(0,1/2,-1/2)}$ (plus cyclic permutations), which are embedded in the representations $\Xi_0 \sim (\mathbf{1}, \mathbf{1}, \mathbf{1})$ and $\Xi_{23} \sim (\mathbf{1}, \mathbf{5}, \overline{\mathbf{5}})$ (plus conjugate, plus cyclic permutations) of $SU(5)^3$. Example diagrams are shown in figure 3. We now construct the full neutrino mass matrix as

$$M_\nu = \left(\begin{array}{c|ccc} & \nu & \overline{N} & N \\ \hline \nu & 0 & m_{D_L} & m_{D_R} \\ \overline{N} & m_{D_L}^T & M_L & M_{LR} \\ N & m_{D_R}^T & M_{LR}^T & M_R \end{array} \right) \equiv \begin{pmatrix} 0 & m_D \\ m_D^T & M_N \end{pmatrix}, \quad (3.30)$$

where we have defined ν as a 3-component vector containing the weak eigenstates of active neutrinos, while N and \overline{N} are 2-component vectors containing the SM singlets N and conjugate neutrinos \overline{N} , respectively. Now we assume that all the hyperons in eqs. (3.28)–(3.29) get VEVs at the scale v_{23} of 23 hypercharge breaking according to eq. (3.7), and we have into

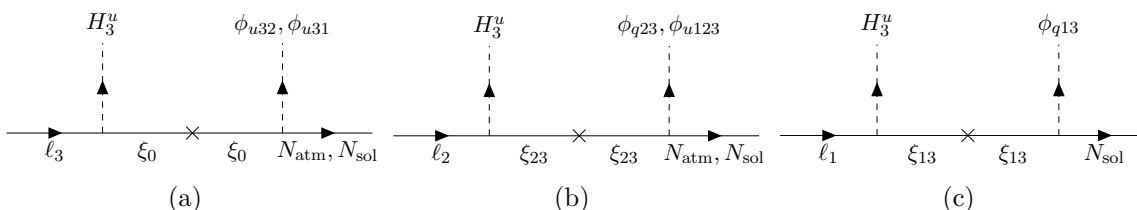


Figure 3. Diagrams leading to effective Yukawa couplings in the neutrino sector.

account that $\langle \phi_{q13} \rangle / \langle \phi_{q23} \rangle \approx \lambda$ as obtained from the discussion of the charged fermion sector in section 3.1. It is also required to assume $M_{N_{\text{atm}}}, M_{N_{\text{sol}}} \lesssim v_{23}$ in order to obtain the observed neutrino mixing with the textures of eqs. (3.28)–(3.29).

Dirac-type masses in $m_{D_{L,R}}$ may be orders of magnitude smaller than the electroweak scale, because they arise from non-renormalisable operators proportional to the SM VEV. In contrast, the eigenvalues of M_N are not smaller than $\mathcal{O}(v_{23})$, which is at least TeV. Therefore, the condition $m_D \ll M_N$ is fulfilled in eq. (3.30) and we can safely apply the seesaw formula to obtain, up to $\mathcal{O}(1)$ factors,

$$m_\nu \simeq m_D M_N^{-1} m_D^T \approx \begin{pmatrix} 1 & 1 & \lambda \\ 1 & 1 & 1 \\ \lambda & 1 & 1 \end{pmatrix} v_{23} \frac{v_{\text{SM}}^2}{M_\xi^2}. \quad (3.31)$$

This is the same texture that was obtained in ref. [1], which is able to accommodate all the observed neutrino mixing angles and mass splittings [36, 37] with $\mathcal{O}(1)$ parameters once the dimensionless coefficients implicit in eq. (3.31) are considered. Remarkably, the singlet neutrinos N_{atm} and N_{sol} get masses around the TeV scale (v_{23}) and contribute to the RGE, while the cyclic neutrino is assumed to get a very heavy vector-like mass and decouples, as mentioned before.

3.3 Gauge coupling unification

In order to ensure that the gauge couplings of our model do indeed unify into a single value at some high energy scale, we must solve their one-loop RGEs, which take the generic form [38]

$$\frac{dg_i}{d \ln \mu} = \frac{g_i^3}{16 \pi^2} b_i. \quad (3.32)$$

The b_i coefficients depend on the specific group G_i , with gauge coupling g_i , and the representations in the model. They are given by

$$b_i = -\frac{11}{3} C_2(G_i) + \frac{4}{3} \kappa S_2(F_i) + \frac{1}{6} \eta S_2(S_i). \quad (3.33)$$

Here μ is the renormalization scale, $C_2(G_i)$ is the quadratic Casimir of the adjoint representation of G_i and $S_2(F_i)$ and $S_2(S_i)$ are the sums of the Dynkin indices of all fermion and scalar non-trivial representations under G_i . Finally, $\kappa = 1$ ($1/2$) for Dirac (Weyl) fermions and $\eta = 2$ (1) for complex (real) scalars.

We computed the b_i coefficients of our model, taking into account not only the gauge group for each energy regime, but also the particle content, since a particle decouples and

Regime	Gauge group	b_i coefficients
1	SM^3	$\left(-\frac{22}{3}, -3, \frac{46}{15}, -\frac{22}{3}, -3, \frac{46}{15}, -\frac{22}{3}, -3, \frac{46}{15}\right)$
2	$SU(3)_c \times SU(2)_L \times U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}$	$\left(0, \frac{11}{3}, \frac{122}{45}, \frac{122}{45}, \frac{46}{15}\right)$
3	$SU(3)_c \times SU(2)_L \times U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}$	$\left(0, \frac{11}{3}, \frac{104}{45}, \frac{104}{45}, \frac{8}{3}\right)$
4	$SU(3)_c \times SU(2)_L \times U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}$	$\left(0, \frac{10}{3}, \frac{19}{9}, \frac{104}{45}, \frac{8}{3}\right)$
5	$SU(3)_c \times SU(2)_L \times U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}$	$\left(0, 3, \frac{19}{9}, \frac{19}{9}, \frac{8}{3}\right)$
6	$SU(3)_c \times SU(2)_L \times U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}$	$\left(-4, -3, \frac{89}{45}, \frac{89}{45}, \frac{38}{15}\right)$
7	$SU(3)_c \times SU(2)_L \times U(1)_{Y_{12}} \times U(1)_{Y_3}$	$\left(-7, -3, \frac{11}{3}, \frac{38}{15}\right)$
8	$SU(3)_c \times SU(2)_L \times U(1)_Y$	$\left(-7, -3, \frac{21}{5}\right)$

Table 3. b_i coefficients of our model. See appendix A for details on the gauge symmetries and particle content at each energy regime.

does not contribute to the running at energies below its mass. The gauge symmetries and particle content at each energy regime are described in detail in appendix A, whereas our results for the b_i coefficients of the model are given in table 3. Finally, we display results for the running of the gauge couplings in figure 4. This figure has been obtained by fixing the intermediate energy scales to

$$\begin{aligned}
 v_{23} &= 5 \text{ TeV}, & v_{12} &= 50 \text{ TeV}, & M_Q &= 100 \text{ TeV}, \\
 M_\Theta &= 100 \text{ TeV}, & M_{H_2^{u,d}} &= 400 \text{ TeV}, & M_{H_1^{u,d}} &= 4 \cdot 10^4 \text{ TeV}, \\
 M_\xi &= 10^{10} \text{ GeV}, & v_{SM^3} &= 6 \cdot 10^{16} \text{ GeV}.
 \end{aligned}
 \tag{3.34}$$

Before discussing gauge coupling unification, we note that we expect radiative corrections to disturb the scales above unless some couplings in the scalar potential are fine-tuned to some extent. In particular, the adjoint scalars Ω_i may couple at tree-level to the light Higgs doublets in order to give corrections to their masses proportional to the GUT scale. This is a consequence of the well-known *hierarchy problem* that afflicts all non-supersymmetric GUTs. To avoid this, perhaps one could build a $SU(5)^3$ tri-unification supersymmetric theory, but this would lead to rapid proton decay via $d = 5$ operators mediated by the coloured Higgs triplet superpartners.⁷ A solution to this would require further model building [6, 9]. Another option would be to include a strongly coupled sector to generate the light scalars as pseudo Goldstone bosons, e.g. one could introduce a strongly coupled $SU(5)$ [5] and generalise the permutation symmetry to \mathbb{Z}_4 in order to enforce a single gauge coupling. This would require further model building beyond the scope of this paper.

⁷In standard supersymmetric $SU(5)$ this contribution also exists but is suppressed by small Yukawa couplings connected to first family fermion masses. In contrast, in $SU(5)^3$ tri-unification all Yukawa couplings are $\mathcal{O}(1)$, including those of the coloured Higgs, hence rapid proton decay happens even if we push the masses of coloured Higgs superpartners to the Planck scale.

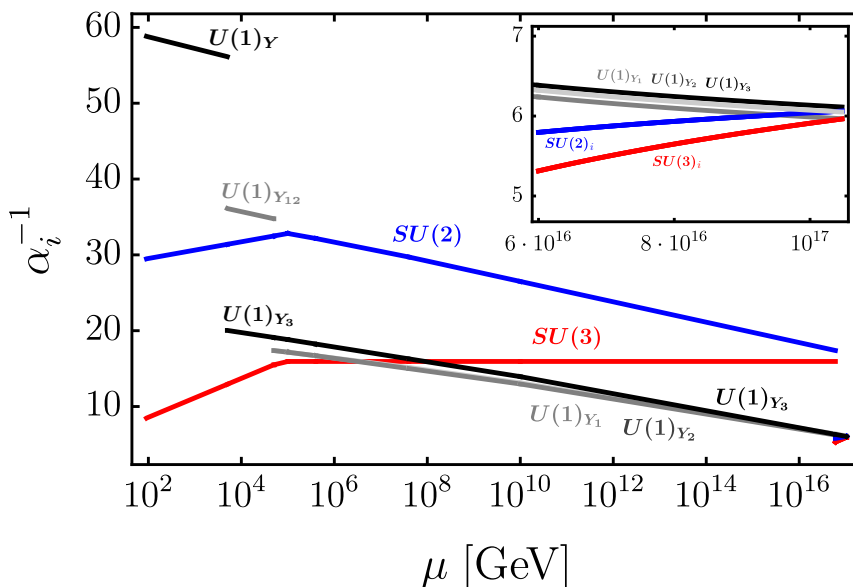


Figure 4. Running of the gauge couplings. The red lines correspond to the SU(3) gauge couplings, the blue ones to the SU(2) gauge couplings and the black/grey ones to the U(1) gauge couplings. A zoom-in with the high-energy region close to the unification scale is also shown. These results have been obtained with $v_{23} = 5$ TeV, $v_{12} = 50$ TeV, $M_Q = 100$ TeV, $M_{H_2^{u,d}} = 400$ TeV, $M_{H_1^{u,d}} = 4 \cdot 10^4$ TeV, $M_\xi = 10^{10}$ GeV and $v_{SM^3} = 6 \cdot 10^{16}$ GeV. The discontinuities in the plot are due to the gauge coupling matching conditions that apply at each symmetry breaking step, see main text and appendix A.

Nevertheless, without the need of imposing supersymmetry nor extra dynamics, the nine gauge couplings of the SM³ group unify at a very high unification scale $M_{GUT} \approx 10^{17}$ GeV, slightly above the SM³ breaking scale, with a unified gauge coupling $g_{GUT} \approx 1.44$. We note the important role played by the three Θ_i colour octets embedded into $\Omega_i \sim \mathbf{24}_i$, and by the Q_i vector-like quarks which also act as heavy messengers of the flavour theory, which are crucial to modify the running of the SU(3) and SU(2) gauge couplings in order to achieve unification. We also remark that the discontinuities in figures 4 and 5 are due to the gauge coupling matching conditions that apply at the steps in which the U(1)_Y group is decomposed into two (first discontinuity) and three hypercharges (second discontinuity) and in which the SU(3) and SU(2) groups are decomposed into one for each family (third discontinuity), see more details in appendix A.

Even though the \mathbb{Z}_3 symmetry gets broken at the SM³ breaking scale, it stays approximately conserved at low energies, down to the tri-hypercharge breaking scale, and only the running of U(1)_{Y₃} is slightly different from that of the other two hypercharge groups. In fact, the gauge couplings of the U(1)_{Y₁} and U(1)_{Y₂} groups almost overlap and cannot be distinguished in figure 4. This can be easily understood by inspecting the b_i coefficients on table 3. Then, the matching conditions at $v_{12} = 50$ TeV split the low energy $g_{Y_{12}}$ and g_{Y_3} couplings, which become clearly different: $g_{Y_{12}}(v_{12}) \approx 0.59$ and $g_{Y_3}(v_{12}) \approx 0.79$. Finally, at $v_{23} = 5$ TeV one recovers the standard SU(3)_c × SU(2)_L × U(1)_Y gauge group, which remains unbroken down to the electroweak scale.

In order to study how unification changes with the scale of SM³ breaking, v_{SM^3} , we consider the values $v_{SM^3} = 5 \cdot 10^{15}$ GeV and $v_{SM^3} = 5 \cdot 10^{17}$ GeV and fix the rest of the

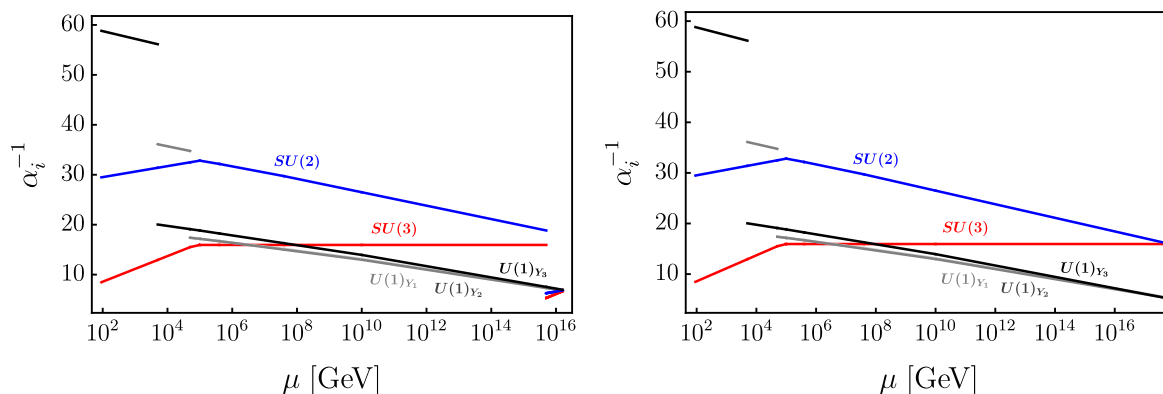


Figure 5. Running of the gauge couplings. Colour code as in figure 4. These results have been obtained with $v_{\text{SM}^3} = 5 \cdot 10^{15}$ GeV (left) and $v_{\text{SM}^3} = 5 \cdot 10^{17}$ GeV (right). The rest of the intermediate scales have been chosen as in eq. (3.34). The discontinuities in the plots are due to the gauge coupling matching conditions that apply at each symmetry breaking step, see main text and appendix A.

intermediate scales as in eq. (3.34). Results for the running of the gauge couplings in these two scenarios are shown in figure 5. On the left-hand side we show the case $v_{\text{SM}^3} = 5 \cdot 10^{15}$ GeV whereas on the right-hand side we display our results for $v_{\text{SM}^3} = 5 \cdot 10^{17}$ GeV. In the first case, our choice of SM^3 breaking scale leads to unification of the gauge couplings at a relatively low scale, $M_{\text{GUT}} \approx 1.8 \cdot 10^{16}$ GeV. This is potentially troublesome, as it may lead to too fast proton decay, as explained below. In contrast, when the SM^3 breaking scale is chosen to be very high, as in the second scenario, unification also gets delayed to much higher energies. In fact, we note that with our choice $v_{\text{SM}^3} = 5 \cdot 10^{17}$ GeV, gauge coupling unification already takes place at the SM^3 breaking scale, $M_{\text{GUT}} \approx v_{\text{SM}^3}$. In this case, $\text{SU}(5)^3$ breaks *directly* to the tri-hypercharge group $\text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_{Y_1} \times \text{U}(1)_{Y_2} \times \text{U}(1)_{Y_3}$ and there is no intermediate SM^3 scale. Finally, the impact of v_{SM^3} is further illustrated in figure 6. Here we show the relation between M_{GUT} , v_{SM^3} and $\alpha_{\text{GUT}}^{-1} = 4\pi/g_{\text{GUT}}^2$. These two plots have been made by varying v_{SM^3} and all the other intermediate scales fixed as in eq. (3.34). The left-hand side of this figure confirms that larger v_{SM^3} values lead to higher unification scales and smaller gaps between these two energy scales. The right-hand side of the figure shows the relation between the unified gauge coupling and the GUT scale. Again, the larger M_{GUT} (or, equivalently, larger v_{SM^3}) is, the larger g_{GUT} (and smaller α_{GUT}^{-1}) becomes. In particular, in this plot g_{GUT} ranges from ~ 1.30 to ~ 1.53 .

3.4 Proton decay

As in any GUT, proton decay is a major prediction in our setup. In standard $\text{SU}(5)$ the most relevant proton decay mode is usually $p \rightarrow e^+ \pi^0$. This process is induced by the tree-level exchange of the X gauge bosons contained in the $\mathbf{24}$ (adjoint) representation, such as the $(\mathbf{3}, \mathbf{2})_{\frac{5}{6}}$ vector leptoquark. Integrating out these heavy vector leptoquarks leads to effective dimension-6 operators⁸ that violate both baryon and lepton numbers, for instance $qqq\ell$. The

⁸We note that a coloured Higgs triplet in our model also contributes to dimension-6 proton decay operator, with a similar size as the gauge leptoquark contribution, hence just changing the Wilson coefficients by $\mathcal{O}(1)$ factors.

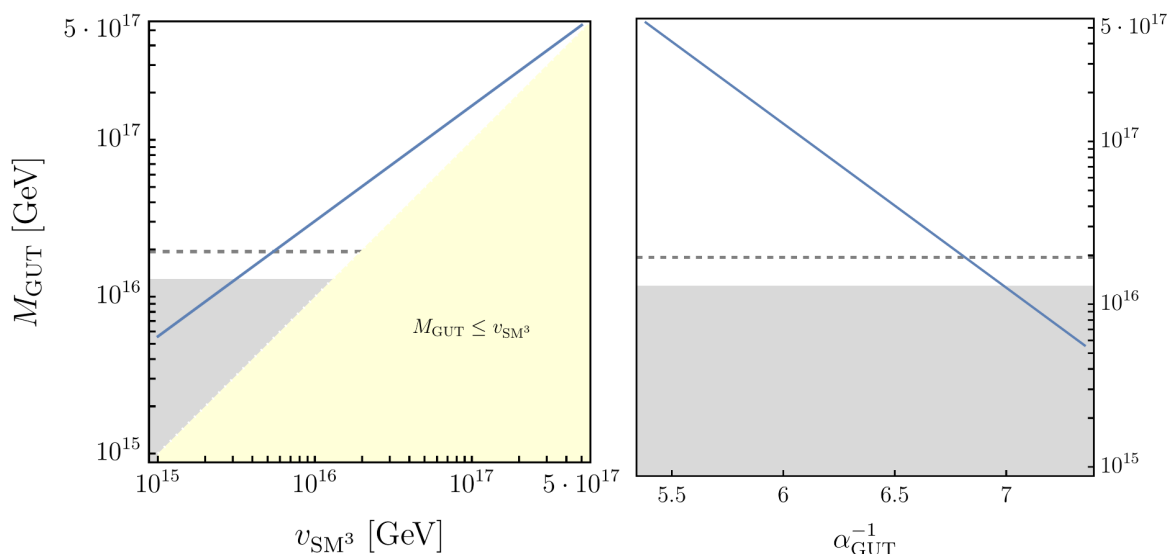


Figure 6. M_{GUT} as a function of v_{SM^3} (left) and α_{GUT}^{-1} (right). The SM^3 breaking scale v_{SM^3} varies in these plots, while the rest of the intermediate scales have been fixed to the values in eq. (3.34). The shaded grey region is excluded by the existing Super-Kamiokande 90% C.L. limit on the $p \rightarrow e^+\pi^0$ lifetime, $\tau(p \rightarrow e^+\pi^0) > 2.4 \cdot 10^{34}$ years [39], whereas the horizontal dashed line corresponds to the projected Hyper-Kamiokande sensitivity at 90% C.L. after 20 years of runtime, $\tau(p \rightarrow e^+\pi^0) > 1.2 \cdot 10^{35}$ years, obtained in [40]. See section 3.4 for details on the proton decay calculation. Finally, the shaded yellow region on the left-hand plot is excluded due to $M_{\text{GUT}} \leq v_{\text{SM}^3}$.

resulting proton life time can be roughly estimated as

$$\tau_p \approx \tau_p^{\text{app}} = \frac{m_X^4}{\alpha_{\text{GUT}}^2 m_p^5}, \quad (3.35)$$

where m_X is the mass of the heavy leptoquark, $m_p \approx 0.938 \text{ GeV}$ is the proton mass and $\alpha_{\text{GUT}} = g_{\text{GUT}}^2/(4\pi)$ is the value of the fine structure constant at the unification scale. For a comprehensive review on proton decay we refer to [41].

In our model there are three $\text{SU}(5)$ groups. This implies a larger number (three times as many) of vector leptoquarks, potentially affecting the proton lifetime. However, due the special flavour structure of our setup, only one of the leptoquark *generations* couples directly to the first fermion generation. The other two couple to the first SM fermion generation only via mixing. However, given that in our setup the three generation leptoquarks get the same mass, in practice the gauge leptoquark phenomenology is that of conventional (flavour universal) $\text{SU}(5)$. One can easily estimate that for $m_X = 10^{17} \text{ GeV}$ and $g_{\text{GUT}} \sim 1.5$, the proton life time is $\tau_p \sim 10^{38}$ years, well above the current experimental limit, $\tau(p \rightarrow e^+\pi^0) > 2.4 \cdot 10^{34}$ years at 90% C.L. [39]. Therefore, a large unification scale suffices to guarantee that our model respects the current limits on the proton lifetime. In fact, such a long life time is beyond the reach of near future experiments, which will increase the current limit by about one order of magnitude [40].

Gauge group	γ_{iL} coefficients	γ_{iR} coefficients
SM ³	$(2, \frac{9}{4}, \frac{23}{20}, 0, 0, 0, 0, 0, 0)$	$(2, \frac{9}{4}, \frac{11}{20}, 0, 0, 0, 0, 0, 0)$
SU(3) _c × SU(2) _L × U(1) _{Y₁} × U(1) _{Y₂} × U(1) _{Y₃}	$(2, \frac{9}{4}, \frac{23}{20}, 0, 0)$	$(2, \frac{9}{4}, \frac{11}{20}, 0, 0)$
SU(3) _c × SU(2) _L × U(1) _{Y₁₂} × U(1) _{Y₃}	$(2, \frac{9}{4}, \frac{23}{20}, 0)$	$(2, \frac{9}{4}, \frac{11}{20}, 0)$
SU(3) _c × SU(2) _L × U(1) _Y	$(2, \frac{9}{4}, \frac{23}{20})$	$(2, \frac{9}{4}, \frac{11}{20})$

Table 4. Anomalous dimension coefficients $\gamma_{iL,R}$ for proton decay operators in our model.

A more precise determination of the $p \rightarrow e^+ \pi^0$ decay width is [41, 42]

$$\Gamma(p \rightarrow e^+ \pi^0) = \frac{m_p}{8} \pi \left(1 - \frac{m_{\pi^0}^2}{m_p^2}\right)^2 A_L^2 \frac{\alpha_{\text{GUT}}^2}{M_{\text{GUT}}^4} \times \left[A_{SL}^2 |C_L|^2 \left| \langle \pi^0 | (ud)_{LU} | p \rangle \right|^2 + A_{SR}^2 |C_R|^2 \left| \langle \pi^0 | (ud)_{RU} | p \rangle \right|^2 \right] \quad (3.36)$$

where $A_L \approx 1.247$ accounts for the QCD RGE from the M_Z scale to m_p [41]. In contrast, $A_{SL(R)}$ accounts for the short-distance RGE from the GUT scale to M_Z , given by

$$A_{SL(R)} = \prod_A^{M_Z \leq M_A \leq M_{\text{GUT}}} \prod_i \left[\frac{\alpha_i(M_{A+1})}{\alpha_i(M_A)} \right]^{\frac{\gamma_{iL(R)}}{b_i}}, \quad (3.37)$$

where b_i and γ_i denote the β coefficients and the anomalous dimensions computed at one-loop in tables 3 and 4, respectively, for the various intermediate scales M_A . The matrix elements appearing in eq. (3.36) are given by [43]

$$\langle \pi^0 | (ud)_{LU} | p \rangle = 0.134(5)(16) \text{ GeV}^2, \quad (3.38)$$

$$\langle \pi^0 | (ud)_{RU} | p \rangle = -0.131(4)(13) \text{ GeV}^2, \quad (3.39)$$

where the errors (shown in the parenthesis) denote statistical and systematic uncertainties, respectively. Given that in our model we have three SU(5) groups, we actually have three generations of the usual SU(5) leptoquarks, coupling only to their corresponding family of chiral fermions. However, since the three SU(5)_i groups are all broken down to their SM_i subgroups at the same scale, in practice the model reproduces the phenomenology of a flavour universal leptoquark as in conventional SU(5), albeit with the specific fermion mixing predicted by our model as shown in section 3.1. The effect of fermion mixing is encoded via the coefficients C_L and C_R [41]

$$C_L = (V_{uc}^\dagger V_u)^{11} (V_{ec}^\dagger V_d)^{11} + (V_{uc}^\dagger V_u V_{\text{CKM}})^{11} (V_{ec}^\dagger V_d V_{\text{CKM}}^\dagger)^{11}, \quad (3.40)$$

$$C_R = (V_{uc}^\dagger V_u)^{11} (V_{dc}^\dagger V_e)^{11}, \quad (3.41)$$

where $V_{\text{CKM}} = V_u^\dagger V_d$. Notice that even though our flavour model predicts non-generic fermion mixing, the alignment of the CKM matrix is not univocally predicted but relies on the choice of dimensionless coefficients. Assuming the CKM mixing to originate mostly from the down sector we find $C_L \simeq 1.946$ and $C_R \simeq 0.999$, while if the CKM mixing originates mostly from the up sector we find very similar coefficients as $C_L \simeq 1.946$ and $C_R \simeq 0.974$.

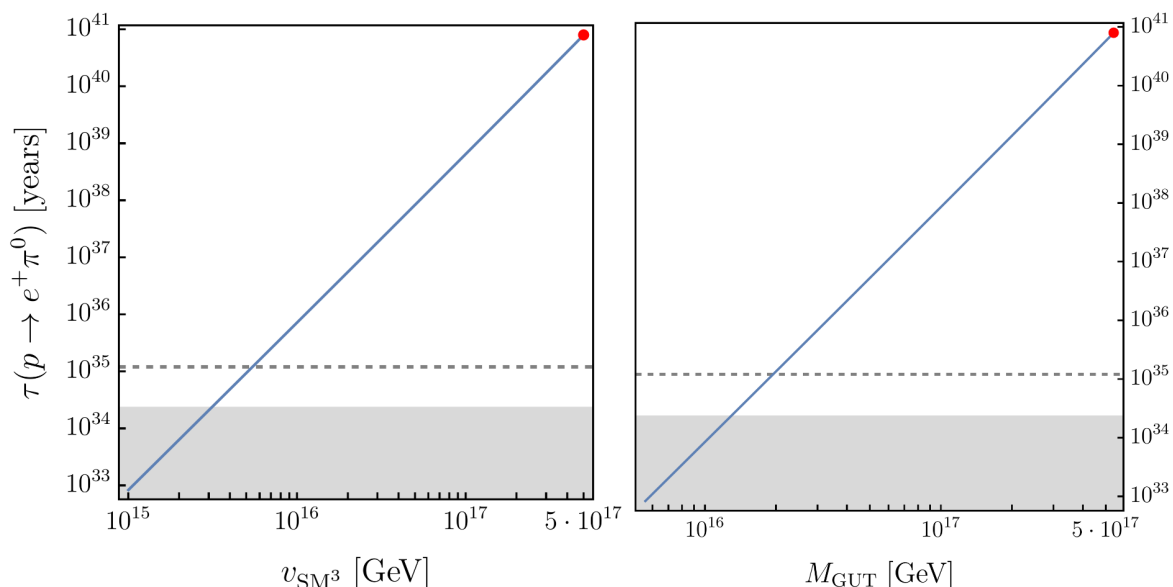


Figure 7. $\tau(p \rightarrow e^+\pi^0)$ as a function of v_{SM^3} (left) and M_{GUT} (right). The SM^3 breaking scale v_{SM^3} varies in these plots, while the rest of the intermediate scales have been fixed to the values in eq. (3.34). The choice $v_{\text{SM}^3} = 5 \cdot 10^{17}$ GeV, which leads to $M_{\text{GUT}} \approx v_{\text{SM}^3}$, is highlighted with a red point in both plots. The shaded grey region is excluded by the existing Super-Kamiokande 90% C.L. limit on the $p \rightarrow e^+\pi^0$ lifetime, $\tau(p \rightarrow e^+\pi^0) > 2.4 \cdot 10^{34}$ years [39], whereas the horizontal dashed line corresponds to the projected Hyper-Kamiokande sensitivity at 90% C.L. after 20 years of runtime, $\tau(p \rightarrow e^+\pi^0) > 1.2 \cdot 10^{35}$ years, obtained in [40].

Therefore, the prediction for proton decay is robust and independent of the alignment of the CKM to excellent accuracy.

We show our numerical results for the $p \rightarrow e^+\pi^0$ lifetime in figure 7. Again, v_{SM^3} varies in the left panel of this figure, while the rest of intermediate scales have been chosen as in eq. (3.34). The right panel shows an equivalent plot with the $p \rightarrow e^+\pi^0$ lifetime as a function of M_{GUT} . This figure provides complementary information to that already shown in figure 6. In both cases we have used the precise determination of the lifetime in eq. (3.36), but we note that the estimate in eq. (3.35) actually provides a very good approximation, with $\tau_p/\tau_p^{\text{app}} \in [0.5, 1.2]$ in the parameter region covered in figure 7. The current Super-Kamiokande 90% C.L. limit on the $p \rightarrow e^+\pi^0$ lifetime, $\tau(p \rightarrow e^+\pi^0) > 2.4 \cdot 10^{34}$ years [39], excludes values of the GUT scale below $M_{\text{GUT}} \sim 1.3 \cdot 10^{16}$ GeV, while the projected Hyper-Kamiokande sensitivity at 90% C.L. after 20 years of runtime, $\tau(p \rightarrow e^+\pi^0) > 1.2 \cdot 10^{35}$ years [40], would push this limit on the unification scale in our model to $M_{\text{GUT}} \sim 2 \cdot 10^{16}$ GeV. Therefore, our model will be probed in the next round of proton decay searches, although large regions of the parameter space predict a long proton lifetime, well beyond any foreseen experiment.

4 Conclusions

In this paper we have discussed $\text{SU}(5)^3$ with cyclic symmetry as a possible GUT. The basic idea of such a tri-unification is that there is a separate $\text{SU}(5)$ for each fermion family, with the light Higgs doublet(s) arising from the third family $\text{SU}(5)$, providing a basis for charged

fermion mass hierarchies. We have set out a general framework in which a class of such models which have been proposed in the literature, including $U(1)_Y^3$, $SU(2)_L^3$ and other related models, may have an ultraviolet completion in terms of $SU(5)^3$ tri-unification.

The main analysis in the paper was concerned with a particular embedding of the tri-hypercharge model $U(1)_Y^3$ into $SU(5)^3$ with cyclic symmetry. We showed that a rather minimal example can account for all the quark and lepton (including neutrino) masses and mixing parameters. This same example can also satisfy the constraints of gauge coupling unification into the cyclic $SU(5)^3$ gauge group, by assuming minimal multiplet splitting, together with a set of relatively light colour octet scalars. The approximate conservation of the cyclic symmetry at low energies is also crucial to achieve gauge unification. The heavy messengers required to generate the flavour structure also modify the RGE in the desired way, highlighting the minimality of the framework.

Finally, we have also studied proton decay in this example, and presented the predictions of the proton lifetime in the dominant $e^+\pi^0$ channel. The results depend on the scale at which the three SM gauge groups break down into their diagonal non-Abelian subgroup together with tri-hypercharge, which is a free parameter in this model, enabling the proton lifetime to escape the existing Super-Kamiokande bound, but be possibly observable at Hyper-Kamiokande. In this manner, the signals on proton decay may allow to test the model at high scales, while low energy signals associated with tri-hypercharge enable the model to be tested by collider and flavour experiments. We conclude that $SU(5)^3$ tri-unification reconciles the idea of gauge non-universality with the idea of gauge coupling unification, opening up the possibility to build consistent non-universal descriptions of Nature that are valid all the way up to the scale of grand unification.

Acknowledgments

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A Energy regimes, symmetries and particle content

We describe the symmetries and particle content of our model at each energy regime between the GUT and electroweak scales.

Regime 1: $SU(5)^3$ breaking scale \rightarrow $(SU(3) \times SU(2) \times U(1))^3$ breaking scale

As a result of $SU(5)^3$ breaking, each of the fermion representations F_i and T_i becomes charged under an $SU(3) \times SU(2) \times U(1)$ factor. Regarding the rest of the fields, most get

masses at the $M_{\text{GUT}} \sim v_{\text{GUT}}$ unification scale and decouple. We will assume that only those explicitly required at low energies remain light. For instance, out of all the components of the Ω_i scalars, only the Θ_i and Δ_i states, belonging to the adjoint representations of $\text{SU}(3)_i$ and $\text{SU}(2)_i$, respectively, remain in the particle spectrum. Similarly, only some SM singlets in the Φ_i scalar fields are assumed to be present at this energy scale. For instance, this is the case of $\Phi_{\ell 23}$, contained in $\Phi_{23}^{(5)}$, a $(\mathbf{1}, \mathbf{5}, \bar{\mathbf{5}})$ representation of $\text{SU}(5)^3$, as shown in table 8. These representations eventually become the tri-hypercharge hyperons at lower energies. Similarly, the Q_i vector-like quarks in the χ_i and $\bar{\chi}_i$ multiplets are also assumed to be present at this energy scale. The full fermion and scalar particle content of the model in this energy regime is shown in table 5.

Regime 2: $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))^3$ breaking scale $\rightarrow \xi$ scale

The $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))^3$ gauge symmetry gets broken by the non-zero VEVs of the Θ_i and Δ_i scalars. The Θ_i octets break $\text{SU}(3)_1 \times \text{SU}(3)_2 \times \text{SU}(3)_3 \rightarrow \text{SU}(3)_{1+2+3} \equiv \text{SU}(3)_c$, while the Δ_i triplets play an analogous role for the $\text{SU}(2)$ factors. We assume these two breakings to take place simultaneously at $v_{\text{SM}^3} = \langle \Theta_i \rangle = \langle \Delta_i \rangle$, slightly below the GUT scale. As a result of this, the remnant symmetry is the tri-hypercharge group [1], $\text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_{Y_1} \times \text{U}(1)_{Y_2} \times \text{U}(1)_{Y_3}$:

$$(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))^3 \xrightarrow{\langle \Theta_i \rangle, \langle \Delta_i \rangle} \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_{Y_1} \times \text{U}(1)_{Y_2} \times \text{U}(1)_{Y_3} \quad (\text{A.1})$$

The gauge couplings above (g_{s_i} and g_{L_i} , with $i = 1, 2, 3$) and below (g_s and g_L) the breaking scale verify the matching relations

$$\frac{g_{s_1} g_{s_2} g_{s_3}}{\sqrt{g_{s_1}^2 g_{s_2}^2 + g_{s_1}^2 g_{s_3}^2 + g_{s_2}^2 g_{s_3}^2}} = g_s, \quad (\text{A.2})$$

$$\frac{g_{L_1} g_{L_2} g_{L_3}}{\sqrt{g_{L_1}^2 g_{L_2}^2 + g_{L_1}^2 g_{L_3}^2 + g_{L_2}^2 g_{L_3}^2}} = g_L, \quad (\text{A.3})$$

which are equivalent to

$$\alpha_{s_1}^{-1} + \alpha_{s_2}^{-1} + \alpha_{s_3}^{-1} = \alpha_s^{-1}, \quad (\text{A.4})$$

$$\alpha_{L_1}^{-1} + \alpha_{L_2}^{-1} + \alpha_{L_3}^{-1} = \alpha_L^{-1}, \quad (\text{A.5})$$

with $\alpha_i^{-1} = 4\pi/g_i^2$.

The main difference with respect to the original tri-hypercharge model [1] is that a complete ultraviolet completion for the generation of the flavour structure is provided in our setup. As already explained, we achieve this with the hyperons and vector-like fermions present in the particle spectrum, which originate from $\text{SU}(5)^3$ representations. We assume N_{cyclic} as well as the conjugate representation \bar{N}_{cyclic} to be decoupled at this energy scale. Similarly, the Δ_i triplets are also assumed to get masses of the order of the SM^3 breaking scale and decouple. The resulting fermion and scalar particle content of the model is shown in table 6.

Field	SU(3) ₁	SU(2) ₁	U(1) ₁	SU(3) ₂	SU(2) ₂	U(1) ₂	SU(3) ₃	SU(2) ₃	U(1) ₃
q_1	3	2	$\frac{1}{6}$	1	1	0	1	1	0
u_1^c	$\bar{\mathbf{3}}$	1	$-\frac{2}{3}$	1	1	0	1	1	0
d_1^c	$\bar{\mathbf{3}}$	1	$\frac{1}{3}$	1	1	0	1	1	0
ℓ_1	1	2	$-\frac{1}{2}$	1	1	0	1	1	0
e_1^c	1	1	1	1	1	0	1	1	0
q_2	1	1	0	3	2	$\frac{1}{6}$	1	1	0
u_2^c	1	1	0	$\bar{\mathbf{3}}$	1	$-\frac{2}{3}$	1	1	0
d_2^c	1	1	0	$\bar{\mathbf{3}}$	1	$\frac{1}{3}$	1	1	0
ℓ_2	1	1	0	1	2	$-\frac{1}{2}$	1	1	0
e_2^c	1	1	0	1	1	1	1	1	0
q_3	1	1	0	1	1	0	3	2	$\frac{1}{6}$
u_3^c	1	1	0	1	1	0	$\bar{\mathbf{3}}$	1	$-\frac{2}{3}$
d_3^c	1	1	0	1	1	0	$\bar{\mathbf{3}}$	1	$\frac{1}{3}$
ℓ_3	1	1	0	1	1	0	1	2	$-\frac{1}{2}$
e_3^c	1	1	0	1	1	0	1	1	1
ξ_0	1	1	0	1	1	0	1	1	0
ξ_{12}	1	1	$\frac{1}{2}$	1	1	$-\frac{1}{2}$	1	1	0
ξ_{13}	1	1	$\frac{1}{2}$	1	1	0	1	1	$-\frac{1}{2}$
ξ_{23}	1	1	0	1	1	$\frac{1}{2}$	1	1	$-\frac{1}{2}$
Q_1	3	2	$\frac{1}{6}$	1	1	0	1	1	0
Q_2	1	1	0	3	2	$\frac{1}{6}$	1	1	0
Q_3	1	1	0	1	1	0	3	2	$\frac{1}{6}$
N_{atm}	1	1	0	1	1	$\frac{2}{3}$	1	1	$-\frac{2}{3}$
N_{sol}	1	1	$\frac{2}{3}$	1	1	0	1	1	$-\frac{2}{3}$
N_{cyclic}	1	1	$\frac{2}{3}$	1	1	$-\frac{2}{3}$	1	1	0
Θ_1	8	1	0	1	1	0	1	1	0
Θ_2	1	1	0	8	1	0	1	1	0
Θ_3	1	1	0	1	1	0	8	1	0
Δ_1	1	3	0	1	1	0	1	1	0
Δ_2	1	1	0	1	3	0	1	1	0
Δ_3	1	1	0	1	1	0	1	3	0
H_1^u	1	2	$\frac{1}{2}$	1	1	0	1	1	0
H_1^d	1	2	$-\frac{1}{2}$	1	1	0	1	1	0
H_2^u	1	1	0	1	2	$\frac{1}{2}$	1	1	0
H_2^d	1	1	0	1	2	$-\frac{1}{2}$	1	1	0
H_3^u	1	1	0	1	1	0	1	2	$\frac{1}{2}$
H_3^d	1	1	0	1	1	0	1	2	$-\frac{1}{2}$
$\Phi_{\ell 12}$	1	1	$\frac{1}{2}$	1	1	$-\frac{1}{2}$	1	1	0
$\Phi_{\ell 13}$	1	1	$\frac{1}{2}$	1	1	0	1	1	$-\frac{1}{2}$
$\Phi_{\ell 23}$	1	1	0	1	1	$\frac{1}{2}$	1	1	$-\frac{1}{2}$
$\Phi_{q 12}$	1	1	$-\frac{1}{6}$	1	1	$\frac{1}{6}$	1	1	0
$\Phi_{q 13}$	1	1	$-\frac{1}{6}$	1	1	0	1	1	$\frac{1}{6}$
$\Phi_{q 23}$	1	1	0	1	1	$-\frac{1}{6}$	1	1	$\frac{1}{6}$
$\Phi_{u 12}$	1	1	$-\frac{2}{3}$	1	1	$\frac{2}{3}$	1	1	0
$\Phi_{u 13}$	1	1	$-\frac{2}{3}$	1	1	0	1	1	$\frac{2}{3}$
$\Phi_{u 23}$	1	1	0	1	1	$-\frac{2}{3}$	1	1	$\frac{2}{3}$

Table 5. Fermion and scalar representations under $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))^3$ in energy regime 1. Fermions highlighted in yellow belong to a vector-like pair and thus have a conjugate representation not shown in this table.

Field	SU(3) _c	SU(2) _L	U(1) _{Y₁}	U(1) _{Y₂}	U(1) _{Y₃}
q_1	3	2	$\frac{1}{6}$	0	0
u_1^c	$\bar{\mathbf{3}}$	1	$-\frac{2}{3}$	0	0
d_1^c	$\bar{\mathbf{3}}$	1	$\frac{1}{3}$	0	0
ℓ_1	1	2	$-\frac{1}{2}$	0	0
e_1^c	1	1	1	0	0
q_2	3	2	0	$\frac{1}{6}$	0
u_2^c	$\bar{\mathbf{3}}$	1	0	$-\frac{2}{3}$	0
d_2^c	$\bar{\mathbf{3}}$	1	0	$\frac{1}{3}$	0
ℓ_2	1	2	0	$-\frac{1}{2}$	0
e_2^c	1	1	0	1	0
q_3	3	2	0	0	$\frac{1}{6}$
u_3^c	$\bar{\mathbf{3}}$	1	0	0	$-\frac{2}{3}$
d_3^c	$\bar{\mathbf{3}}$	1	0	0	$\frac{1}{3}$
ℓ_3	1	2	0	0	$-\frac{1}{2}$
e_3^c	1	1	0	0	1
ξ_0	1	1	0	0	0
ξ_{12}	1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0
ξ_{13}	1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$
ξ_{23}	1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$
Q_1	3	2	$\frac{1}{6}$	0	0
Q_2	3	2	0	$\frac{1}{6}$	0
Q_3	3	2	0	0	$\frac{1}{6}$
N_{atm}	1	1	0	$\frac{2}{3}$	$-\frac{2}{3}$
N_{sol}	1	1	$\frac{2}{3}$	0	$-\frac{2}{3}$
Θ_1	8	1	0	0	0
Θ_2	8	1	0	0	0
Θ_3	8	1	0	0	0
H_1^u	1	2	$\frac{1}{2}$	0	0
H_1^d	1	2	$-\frac{1}{2}$	0	0
H_2^u	1	2	0	$\frac{1}{2}$	0
H_2^d	1	2	0	$-\frac{1}{2}$	0
H_3^u	1	2	0	0	$\frac{1}{2}$
H_3^d	1	2	0	0	$-\frac{1}{2}$
$\phi_{\ell 12}$	1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0
$\phi_{\ell 13}$	1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$
$\phi_{\ell 23}$	1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$
$\phi_{q 12}$	1	1	$-\frac{1}{6}$	$\frac{1}{6}$	0
$\phi_{q 13}$	1	1	$-\frac{1}{6}$	0	$\frac{1}{6}$
$\phi_{q 23}$	1	1	0	$-\frac{1}{6}$	$\frac{1}{6}$
$\phi_{u 12}$	1	1	$-\frac{2}{3}$	$\frac{2}{3}$	0
$\phi_{u 13}$	1	1	$-\frac{2}{3}$	0	$\frac{2}{3}$
$\phi_{u 23}$	1	1	0	$-\frac{2}{3}$	$\frac{2}{3}$

Table 6. Fermion and scalar representations under $SU(3)_c \times SU(2)_L \times U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}$ in energy regimes 2, 3, 4, 5 and 6. Some states in this table get decoupled at intermediate scales and are not present at all energy regimes, see text for details. Fermions highlighted in yellow belong to a vector-like pair and thus have a conjugate representation not shown in this table.

Regime 3: ξ scale $\rightarrow H_1$ scale

The next energy threshold is given by the ξ singlets, responsible for the flavour structure of the neutrino sector, with masses $M_\xi \sim 10^{10}$ GeV. At this scale, the ξ_0 as well as the ξ_{12} , ξ_{13} , ξ_{23} and their conjugate representations are integrated out and no longer contribute to the running of the gauge couplings. The gauge symmetry does not change and stays the same as in the previous energy regime. The resulting particle spectrum is that of table 6 removing the ξ singlet fermions.

Regime 4: H_1 scale $\rightarrow H_2$ scale

At energies of the order of $M_{H_1^{u,d}} \sim 10^4$ TeV, the $H_1^{u,d}$ scalar doublets decouple from the particle spectrum of the model. Again, the gauge symmetry does not change. The particle spectrum at this stage is that shown on table 6 removing the ξ singlet fermions and the $H_1^{u,d}$ scalar doublets.

Regime 5: H_2 scale $\rightarrow Q, \Theta$ scale

At energies of the order of $M_{H_2^{u,d}} \sim 100$ TeV, the $H_2^{u,d}$ scalar doublets decouple from the particle spectrum of the model. As in the previous two energy thresholds, the gauge symmetry remains the same. The particle spectrum at this stage is that shown on table 6 removing the ξ singlet fermions and the $H_{1,2}^{u,d}$ scalar doublets.

Regime 6: Q, Θ scale $\rightarrow \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_{Y_1} \times \text{U}(1)_{Y_2} \times \text{U}(1)_{Y_3}$ breaking scale

At $M_Q \lesssim M_{H_2^{u,d}}$, the Q_i vector-like quarks and the Θ_i colour octets decouple from the particle spectrum of the model. As in the previous two energy thresholds, the gauge symmetry is not altered. The particle spectrum at this stage is that shown on table 6 removing the ξ singlet fermions, the $H_{1,2}^{u,d}$ scalar doublets, the Q_i vector-like quarks and the Θ_i colour octets.

Hyperons are responsible for the breaking of the tri-hypercharge symmetry. In a first hypercharge breaking step, $\text{U}(1)_{Y_1} \times \text{U}(1)_{Y_2} \times \text{U}(1)_{Y_3}$ gets broken to $\text{U}(1)_{Y_{12}} \times \text{U}(1)_{Y_3}$, where $Y_{12} = Y_1 + Y_2$, by the non-zero VEV of the ϕ_{q12} hyperon, $v_{12} = \langle \phi_{q12} \rangle \sim 50$ TeV:

$$\text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_{Y_1} \times \text{U}(1)_{Y_2} \times \text{U}(1)_{Y_3} \xrightarrow{\langle \phi_{q12} \rangle} \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_{Y_{12}} \times \text{U}(1)_{Y_3}. \tag{A.6}$$

The gauge couplings above (g_{Y_1} and g_{Y_2}) and below ($g_{Y_{12}}$) the breaking scale verify the matching relation

$$\frac{g_{Y_1} g_{Y_2}}{\sqrt{g_{Y_1}^2 + g_{Y_2}^2}} = g_{Y_{12}}, \tag{A.7}$$

which is equivalent to

$$\alpha_{Y_1}^{-1} + \alpha_{Y_2}^{-1} = \alpha_{Y_{12}}^{-1}. \tag{A.8}$$

The ‘‘12 hyperons’’ $\phi_{\ell 12}$, ϕ_{q12} and ϕ_{u12} get masses of the order of $\langle \phi_{q12} \rangle$ and decouple at this stage. We also assume the Θ_i colour octets to be integrated out at the tri-hypercharge breaking scale. The resulting fermion and scalar particle content is shown in table 7.

Field	$SU(3)_c$	$SU(2)_L$	$U(1)_{Y_{12}}$	$U(1)_{Y_3}$
q_1	3	2	$\frac{1}{6}$	0
u_1^c	$\bar{\mathbf{3}}$	1	$-\frac{2}{3}$	0
d_1^c	$\bar{\mathbf{3}}$	1	$\frac{1}{3}$	0
ℓ_1	1	2	$-\frac{1}{2}$	0
e_1^c	1	1	1	0
q_2	3	2	$\frac{1}{6}$	0
u_2^c	$\bar{\mathbf{3}}$	1	$-\frac{2}{3}$	0
d_2^c	$\bar{\mathbf{3}}$	1	$\frac{1}{3}$	0
ℓ_2	1	2	$-\frac{1}{2}$	0
e_2^c	1	1	1	0
q_3	3	2	0	$\frac{1}{6}$
u_3^c	$\bar{\mathbf{3}}$	1	0	$-\frac{2}{3}$
d_3^c	$\bar{\mathbf{3}}$	1	0	$\frac{1}{3}$
ℓ_3	1	2	0	$-\frac{1}{2}$
e_3^c	1	1	0	1
N_{atm}	1	1	$\frac{2}{3}$	$-\frac{2}{3}$
N_{sol}	1	1	$\frac{2}{3}$	$-\frac{2}{3}$
H_3^u	1	2	0	$\frac{1}{2}$
H_3^d	1	2	0	$-\frac{1}{2}$
$\phi_{\ell 13}$	1	1	$\frac{1}{2}$	$-\frac{1}{2}$
$\phi_{\ell 23}$	1	1	$\frac{1}{2}$	$-\frac{1}{2}$
$\phi_{q 13}$	1	1	$-\frac{1}{6}$	$\frac{1}{6}$
$\phi_{q 23}$	1	1	$-\frac{1}{6}$	$\frac{1}{6}$
$\phi_{u 13}$	1	1	$-\frac{2}{3}$	$\frac{2}{3}$
$\phi_{u 23}$	1	1	$-\frac{2}{3}$	$\frac{2}{3}$

Table 7. Fermion and scalar representations under $SU(3)_c \times SU(2)_L \times U(1)_{Y_{12}} \times U(1)_{Y_3}$ in energy regime 7. Fermions highlighted in yellow belong to a vector-like pair and thus have a conjugate representation not shown in this table.

Regime 7: $SU(3)_c \times SU(2)_L \times U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}$ breaking scale $\rightarrow SU(3)_c \times SU(2)_L \times U(1)_{Y_{12}} \times U(1)_{Y_3}$ breaking scale

The $SU(3)_c \times SU(2)_L \times U(1)_{Y_{12}} \times U(1)_{Y_3}$ gauge symmetry also gets broken by hyperon VEVs, leaving as a remnant the conventional SM gauge symmetry with $Y = Y_{12} + Y_3 = Y_1 + Y_2 + Y_3$. In this case, the hyperons responsible for the breaking are $\phi_{\ell 13}$, $\phi_{\ell 23}$, $\phi_{q 13}$ and $\phi_{q 23}$, which get VEVs of the order of $v_{23} \sim 5 \text{ TeV}$:

$$SU(3)_c \times SU(2)_L \times U(1)_{Y_{12}} \times U(1)_{Y_3} \xrightarrow{\langle \phi_{\ell 13, 23} \rangle, \langle \phi_{q 13, 23} \rangle} SU(3)_c \times SU(2)_L \times U(1)_Y. \tag{A.9}$$

The gauge couplings above ($g_{Y_{12}}$ and g_{Y_3}) and below (g_Y) the breaking scale verify the matching relation

$$\frac{g_{Y_{12}} g_{Y_3}}{\sqrt{g_{Y_{12}}^2 + g_{Y_3}^2}} = g_Y, \tag{A.10}$$

which is equivalent to

$$\alpha_{Y_{12}}^{-1} + \alpha_{Y_3}^{-1} = \alpha_Y^{-1}. \tag{A.11}$$

All the remaining hyperons as well as the neutrino mass messengers N_{atm} and N_{sol} (as well as their conjugate representations) decouple at this stage. The resulting particle spectrum is that of a two Higgs doublet model, with universal charges for all fermions.

Regime 8: $SU(3)_c \times SU(2)_L \times U(1)_{Y_{12}} \times U(1)_{Y_3}$ breaking scale $\rightarrow SU(3)_c \times SU(2)_L \times U(1)_Y$ breaking scale

Finally, at the scale v_{SM} , the electroweak symmetry gets broken in the usual way, by the VEVs of the $H_3^{u,d}$ scalar doublets:

$$SU(3)_c \times SU(2)_L \times U(1)_Y \xrightarrow{\langle H_3^{u,d} \rangle} SU(3)_c \times U(1)_{\text{em}} \tag{A.12}$$

B Hyperons from $SU(5)^3$

Tables 8 and 9 list all possible hyperon embeddings in $SU(5)^3$ representations with dimension up to 45. These tables have been obtained with the help of GroupMath [35].

Hyperon			SU(5) ³ representations
0	$-\frac{1}{3}$	$\frac{1}{3}$	$(1, 5, \bar{5}), (1, 5, \overline{45}), (1, 45, \bar{5}), (1, 45, \overline{45}), (24, 5, \bar{5}), (24, 5, \overline{45}), (24, 45, \bar{5}), (24, 45, \overline{45})$
0	$\frac{1}{2}$	$-\frac{1}{2}$	$(1, 5, \bar{5}), (1, 5, \overline{45}), (1, 45, \bar{5}), (1, 45, \overline{45}), (24, 5, \bar{5}), (24, 5, \overline{45}), (24, 45, \bar{5}), (24, 45, \overline{45})$
0	$-\frac{2}{3}$	$\frac{2}{3}$	$(1, 10, \bar{10}), (1, 10, \overline{40}), (1, 15, \bar{15}), (1, 35, \bar{35}), (1, 35, \overline{40}), (1, 40, \bar{10}), (1, 40, \bar{35}), (1, 40, \overline{40}),$ $(24, 10, \bar{10}), (24, 10, \bar{15}), (24, 10, \bar{35}), (24, 10, \overline{40}), (24, 15, \bar{10}), (24, 15, \bar{15}), (24, 15, \overline{40}),$ $(24, 35, \bar{10}), (24, 35, \bar{35}), (24, 35, \overline{40}), (24, 40, \bar{10}), (24, 40, \bar{15}), (24, 40, \bar{35}), (24, 40, \overline{40})$
0	$\frac{1}{6}$	$-\frac{1}{6}$	$(1, 10, \bar{10}), (1, 10, \bar{15}), (1, 10, \overline{40}), (1, 15, \bar{10}), (1, 15, \bar{15}), (1, 15, \overline{40}), (1, 35, \bar{35}),$ $(1, 35, \overline{40}), (1, 40, \bar{10}), (1, 40, \bar{15}), (1, 40, \bar{35}), (1, 40, \overline{40}), (24, 10, \bar{10}), (24, 10, \bar{15}),$ $(24, 10, \bar{35}), (24, 10, \overline{40}), (24, 15, \bar{10}), (24, 15, \bar{15}), (24, 15, \bar{35}), (24, 15, \overline{40}), (24, 35, \bar{10}),$ $(24, 35, \bar{15}), (24, 35, \bar{35}), (24, 35, \overline{40}), (24, 40, \bar{10}), (24, 40, \bar{15}), (24, 40, \bar{35}), (24, 40, \overline{40})$
0	1	-1	$(1, 10, \bar{10}), (1, 15, \bar{15}), (1, 35, \bar{35}), (1, 40, \overline{40}), (24, 10, \bar{10}), (24, 10, \bar{15}), (24, 10, \overline{40}),$ $(24, 15, \bar{10}), (24, 15, \bar{15}), (24, 35, \bar{35}), (24, 35, \overline{40}), (24, 40, \bar{10}), (24, 40, \bar{35}), (24, 40, \overline{40})$
0	$\frac{5}{6}$	$-\frac{5}{6}$	$(1, 24, 24), (24, 24, 24)$
0	$-\frac{3}{2}$	$\frac{3}{2}$	$(1, 35, \bar{35}), (1, 40, \overline{40}), (24, 35, \bar{35}), (24, 35, \overline{40}), (24, 40, \bar{35}), (24, 40, \overline{40})$
0	$\frac{4}{3}$	$-\frac{4}{3}$	$(1, 45, \overline{45}), (24, 45, \overline{45})$
0	$-\frac{7}{6}$	$\frac{7}{6}$	$(1, 45, \overline{45}), (24, 45, \overline{45})$

Table 8. Hyperons charged under two individual hypercharge groups and their SU(5)³ origin. All SU(5)³ representations that involve up to $\mathbf{45}$ and $\overline{\mathbf{45}}$ of SU(5) are included. Other hyperons can be obtained by reordering the hypercharge values or by conjugating the SU(5)³ representations.

Hyperon	$SU(5)^3$ representations
$-\frac{1}{3} \quad -\frac{1}{3} \quad \frac{2}{3}$	$(5, 5, \overline{10}), (5, 5, \overline{15}), (5, 5, \overline{40}), (5, 45, \overline{10}), (5, 45, \overline{15}), (5, 45, \overline{35}), (5, 45, \overline{40}), (45, 5, \overline{10}), (45, 5, \overline{15}), (45, 5, \overline{35}), (45, 5, \overline{40}), (45, 45, \overline{10}), (45, 45, \overline{15}), (45, 45, \overline{35}), (45, 45, \overline{40})$
$-\frac{1}{3} \quad \frac{1}{2} \quad -\frac{1}{6}$	$(5, 5, \overline{10}), (5, 5, \overline{15}), (5, 5, \overline{40}), (5, 45, \overline{10}), (5, 45, \overline{15}), (5, 45, \overline{35}), (5, 45, \overline{40}), (45, 5, \overline{10}), (45, 5, \overline{15}), (45, 5, \overline{35}), (45, 5, \overline{40}), (45, 45, \overline{10}), (45, 45, \overline{15}), (45, 45, \overline{35}), (45, 45, \overline{40})$
$\frac{1}{2} \quad \frac{1}{2} \quad -1$	$(5, 5, \overline{10}), (5, 5, \overline{15}), (5, 45, \overline{10}), (5, 45, \overline{15}), (5, 45, \overline{40}), (45, 5, \overline{10}), (45, 5, \overline{15}), (45, 5, \overline{40}), (45, 45, \overline{10}), (45, 45, \overline{15}), (45, 45, \overline{35}), (45, 45, \overline{40})$
$-\frac{1}{3} \quad -\frac{1}{2} \quad \frac{5}{6}$	$(5, \overline{5}, 24), (5, \overline{45}, 24), (45, \overline{5}, 24), (45, \overline{45}, 24)$
$-\frac{1}{3} \quad -\frac{2}{3} \quad 1$	$(5, 10, 10), (5, 10, 40), (5, 15, 35), (5, 15, 40), (5, 35, 15), (5, 40, 10), (5, 40, 15), (5, 40, 40), (45, 10, 10), (45, 10, 15), (45, 10, 40), (45, 15, 10), (45, 15, 35), (45, 15, 40), (45, 35, 10), (45, 35, 15), (45, 35, 40), (45, 40, 10), (45, 40, 15), (45, 40, 40)$
$-\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{6}$	$(5, 10, 10), (5, 10, 15), (5, 10, 35), (5, 10, 40), (5, 15, 10), (5, 15, 15), (5, 15, 35), (5, 15, 40), (5, 35, 10), (5, 35, 15), (5, 35, 40), (5, 40, 10), (5, 40, 15), (5, 40, 35), (5, 40, 40), (45, 10, 10), (45, 10, 15), (45, 10, 35), (45, 10, 40), (45, 15, 10), (45, 15, 15), (45, 15, 35), (45, 15, 40), (45, 35, 10), (45, 35, 15), (45, 35, 35), (45, 35, 40), (45, 40, 10), (45, 40, 15), (45, 40, 35), (45, 40, 40)$
$\frac{1}{2} \quad -\frac{2}{3} \quad \frac{1}{6}$	$(5, 10, 10), (5, 10, 15), (5, 10, 40), (5, 15, 35), (5, 15, 40), (5, 35, 10), (5, 35, 15), (5, 35, 40), (5, 40, 10), (5, 40, 15), (5, 40, 40), (45, 10, 10), (45, 10, 15), (45, 10, 35), (45, 10, 40), (45, 15, 10), (45, 15, 15), (45, 15, 35), (45, 15, 40), (45, 35, 10), (45, 35, 15), (45, 35, 35), (45, 35, 40), (45, 40, 10), (45, 40, 15), (45, 40, 35), (45, 40, 40)$
$\frac{1}{2} \quad 1 \quad -\frac{3}{2}$	$(5, 10, 40), (5, 15, 35), (5, 15, 40), (45, 10, 40), (45, 15, 35), (45, 15, 40), (45, 40, 40)$
$-\frac{1}{3} \quad -1 \quad \frac{4}{3}$	$(5, \overline{10}, 45), (5, \overline{40}, 45), (45, \overline{10}, 45), (45, \overline{15}, 45), (45, \overline{35}, 45), (45, \overline{40}, 45)$
$\frac{1}{2} \quad \frac{2}{3} \quad -\frac{7}{6}$	$(5, \overline{10}, 45), (5, \overline{35}, 45), (5, \overline{40}, 45), (45, \overline{10}, 45), (45, \overline{15}, 45), (45, \overline{35}, 45), (45, \overline{40}, 45)$
$-\frac{1}{3} \quad -\frac{5}{6} \quad \frac{7}{6}$	$(5, 24, \overline{45}), (45, 24, \overline{45})$
$\frac{1}{2} \quad \frac{5}{6} \quad -\frac{4}{3}$	$(5, 24, \overline{45}), (45, 24, \overline{45})$
$-\frac{1}{3} \quad \frac{3}{2} \quad -\frac{7}{6}$	$(5, \overline{40}, 45), (45, \overline{35}, 45), (45, \overline{40}, 45)$
$-\frac{2}{3} \quad -\frac{2}{3} \quad \frac{4}{3}$	$(10, 10, 45), (10, 15, 45), (10, 40, 45), (15, 10, 45), (15, 40, 45), (35, 35, 45), (35, 40, 45), (40, 10, 45), (40, 15, 45), (40, 35, 45), (40, 40, 45)$
$\frac{1}{6} \quad 1 \quad -\frac{7}{6}$	$(10, 10, 45), (10, 15, 45), (10, 40, 45), (15, 10, 45), (15, 15, 45), (15, 40, 45), (35, 40, 45), (40, 10, 45), (40, 15, 45), (40, 40, 45)$
$-\frac{2}{3} \quad -\frac{1}{6} \quad \frac{5}{6}$	$(10, \overline{10}, 24), (10, \overline{15}, 24), (10, \overline{35}, 24), (10, \overline{40}, 24), (15, \overline{10}, 24), (15, \overline{15}, 24), (15, \overline{40}, 24), (35, \overline{10}, 24), (35, \overline{15}, 24), (35, \overline{35}, 24), (35, \overline{40}, 24), (40, \overline{10}, 24), (40, \overline{15}, 24), (40, \overline{35}, 24), (40, \overline{40}, 24)$
$\frac{1}{6} \quad -1 \quad \frac{5}{6}$	$(10, \overline{10}, 24), (10, \overline{15}, 24), (10, \overline{40}, 24), (15, \overline{10}, 24), (15, \overline{15}, 24), (15, \overline{40}, 24), (35, \overline{35}, 24), (35, \overline{40}, 24), (40, \overline{10}, 24), (40, \overline{15}, 24), (40, \overline{35}, 24), (40, \overline{40}, 24)$
$-\frac{2}{3} \quad -\frac{5}{6} \quad \frac{3}{2}$	$(10, 24, \overline{40}), (35, 24, \overline{35}), (35, 24, \overline{40}), (40, 24, \overline{35}), (40, 24, \overline{40})$
$\frac{1}{6} \quad -\frac{3}{2} \quad \frac{4}{3}$	$(10, 40, 45), (15, 40, 45), (40, 40, 45)$
$\frac{1}{6} \quad -\frac{4}{3} \quad \frac{7}{6}$	$(10, \overline{45}, \overline{45}), (15, \overline{45}, \overline{45}), (35, \overline{45}, \overline{45}), (40, \overline{45}, \overline{45})$

Table 9. Hyperons charged under the three individual hypercharge groups and their $SU(5)^3$ origin. All $SU(5)^3$ representations that involve up to $\mathbf{45}$ and $\overline{\mathbf{45}}$ of $SU(5)$ are included. Other hyperons can be obtained by reordering the hypercharge values or by conjugating the $SU(5)^3$ representations.

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