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Remarks on integrability of $\mathcal{N} = 1$ supersymmetric Ruijsenaars-Schneider three-body models

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ABSTRACT: Integrability of $\mathcal{N} = 1$ supersymmetric Ruijsenaars-Schneider three-body models based upon the potentials $W(x) = \frac{2}{x}$, $W(x) = \frac{2}{\sin x}$, and $W(x) = \frac{2}{\sinh x}$ is proven. The problem of constructing an algebraically resolvable set of Grassmann-odd constants of motion is reduced to finding a triplet of vectors such that all their scalar products can be expressed in terms of the original bosonic first integrals. The supersymmetric generalizations are used to build novel integrable (iso)spin extensions of the respective Ruijsenaars-Schneider three-body systems.

KEYWORDS: Integrable Field Theories, Supersymmetric Effective Theories, Superspaces

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1 Introduction

Supersymmetric extensions of integrable mechanics are usually studied in connection with the superstring theories [1], where they describe dynamics of zero modes, or in the context of microscopic description of near horizon black hole geometries [2]. The question of how a formal supersymmetrization procedure affects integrability has received much less attention. It is generally believed that a superextension of an integrable theory should automatically result in a larger integrable system including fermionic degrees of freedom. If this were the case, supersymmetrization would suggest an efficient way of building new integrable models.

Because the number of fermionic degrees of freedom is in general greater than the number of conserved supercharges at hand, integrability of a supersymmetric extension is not a priori guaranteed. Furthermore, because fermionic integrals of motion are constructed from monomials in Grassmann-odd variables and there does not exist a division by a Grassmann-odd function [3], in order to guarantee integrability in the fermionic sector one has to find constants of motion, which are algebraically resolvable with respect to the fermionic variables. A necessary condition for this is the presence of a linear term in each Grassmann-odd integral of motion.

Aiming at a better understanding of the interrelation between supersymmetry and integrability, in a recent work [4] integrability of an $\mathcal{N} = 1$ supersymmetric extension of the Ruijsenaars-Schneider hyperbolic three-body model [5] was studied in detail. In particular, three functionally independent Grassmann-odd constants of motion were explicitly constructed and their algebraic resolvability was demonstrated. It was also anticipated in [4] that proving integrability of supersymmetric extensions for other variants in [5] should go rather straightforward. As shown below, some of such models present a challenge.

The Ruijsenaars-Schneider systems provide interesting examples of integrable many-body models, equations of motion of which involve particle velocities [5]

$$\ddot{x}_i = \sum_{j \neq i}^n \dot{x}_i \dot{x}_j W(x_{ij}), \tag{1.1}$$

where $x_{ij} = x_i - x_j$, $i, j = 1, \dots, n$, and $W(x)$ is one of the potentials listed below

$$W(x) = \left[\frac{2}{x}, \frac{2}{\sin x}, 2 \cot x, \frac{2}{\sinh x}, 2 \coth x \right]. \tag{1.2}$$

Such systems enjoy symmetries, which form the Poincaré group in 1 + 1 dimensions, and reduce to the celebrated Calogero models [6] in the nonrelativistic limit [5]. By this reason, the former are usually regarded as the relativistic analogues of the latter.

Surprisingly enough, supersymmetric extensions of the relativistic counterparts of the Calogero models remain almost completely unexplored. An $\mathcal{N} = 2$ supersymmetric generalization of the quantum trigonometric Ruijsenaars-Schneider model was constructed in [7] and its eigenfunctions were linked to the Macdonald superpolynomials. Note that the fermionic variables in [7] and their adjoints obey non-standard anticommutation relations which reduce to the conventional ones in the non-relativistic limit only. $\mathcal{N} = 2$ supersymmetric extensions of the rational and hyperbolic three-body models were built in [8] within the Hamiltonian framework. The corresponding supercharges were cubic in the fermionic variables. n -particle $\mathcal{N} = 2$ models were suggested in [9, 10]. The highest power of the fermionic degrees of freedom contributing to the $\mathcal{N} = 2$ supercharges in [9, 10] depends on the number of particles at hand, making the supercharges highly nonlinear. Note that algebraic resolvability of constants of motion in the fermionic sector has not been analyzed in [7–10].

The goal of this work is to extend our recent analysis in [4] of the integrability of an $\mathcal{N} = 1$ supersymmetric Ruijsenaars-Schneider three-body system based upon the potential $W(x) = \frac{2}{\sinh x}$ to other instances listed in (1.2), as well as to the Ruijsenaars-Toda model [11]. Like in [4], we choose to work within the Hamiltonian framework. Our approach includes three steps.

Firstly, subsidiary functions λ_i are built on the phase space parametrized by (x_i, p_i) , $i = 1, 2, 3$, $\{x_i, p_j\} = \delta_{ij}$, which generate the potential $W(x)$ via the Poisson bracket (no summation over repeated indices and $i \neq j$) $\{\lambda_i, \lambda_j\} = \frac{1}{4}W(x_{ij})\lambda_i\lambda_j$. At the same time, they allow one to represent the Hamiltonian in the quadratic form, $H = \lambda_i\lambda_i = I_1$. Two more constants of motion I_2 and I_3 available for a three-body model at hand are expressed in terms of x_i and λ_i as well.

Secondly, a fermionic partner θ_i is considered for each canonical pair (x_i, p_i) , which obeys the Poisson bracket $\{\theta_i, \theta_j\} = -i\delta_{ij}$, and a natural $\mathcal{N} = 1$ supersymmetry charge $Q_1 = \lambda_i\theta_i$ is introduced, which generates the superextended Hamiltonian¹ \mathcal{H} via the Poisson bracket, $\{Q_1, Q_1\} = -i\mathcal{H}$. The latter governs dynamics of the resulting $\mathcal{N} = 1$ supersymmetric Ruijsenaars-Schneider system.

¹Throughout the text, superextensions of the original bosonic quantities are denoted by the same letters written in the calligraphic style.

Thirdly, in order to establish integrability in the fermionic sector, one needs to find two more Grassmann-odd first integrals, the leading terms of which are linear in the Grassmann-odd variables $Q_2 = \mu_i \theta_i + \dots$, $Q_3 = \nu_i \theta_i + \dots$, where dots designate terms cubic in the fermions and $\mu_i(x, \lambda)$, $\nu_i(x, \lambda)$ are specific functions to be fixed below. Because Q_2 and Q_3 are supposed to commute with the superextended Hamiltonian \mathcal{H} , the Poisson brackets between Q_1 , Q_2 , and Q_3 should be conserved over time as well. This follows from the super Jacobi identity. Considering the bosonic limit of expressions contributing to the right hand sides of the respective brackets, one concludes that the scalar products $\lambda_i \mu_i$, $\lambda_i \nu_i$, $\mu_i \mu_i$, $\mu_i \nu_i$, $\nu_i \nu_i$, should all link to the bosonic first integrals (I_1, I_2, I_3) characterizing the original model at hand. It then remains to extract μ_i and ν_i from (I_1, I_2, I_3) . To put it in other words, given a Ruijsenaars-Schneider three-body system with three constants of motion (I_1, I_2, I_3) , our approach to supersymmetrizing it consists in finding a triplet of vectors $(\lambda_i, \mu_i, \nu_i)$, all scalar products of which are expressible in terms of (I_1, I_2, I_3) .

Surprisingly enough, as demonstrated below, while such a procedure works smoothly for the rational potential $W(x) = \frac{2}{x}$, the trigonometric variant $W(x) = \frac{2}{\sin x}$, and its hyperbolic analogue $W(x) = \frac{2}{\sinh x}$, it unexpectedly fails for $W(x) = \cot x$, $W(x) = \coth x$, as well as for the Ruijsenaars-Toda case, meaning that a more sophisticated approach of proving the algebraic resolvability in the fermionic sector of those models is needed. Note that in all the cases in which our construction proves successful, it relies upon specific rational/trigonometric identities (see (2.14), (3.7), and (4.5) below), analogues of which are missing for $W(x) = \cot x$, $W(x) = \coth x$, and the Ruijsenaars-Toda system.

The work is organized as follows. In the next section, an integrable $\mathcal{N} = 1$ supersymmetric extension of the Ruijsenaars-Schneider rational three-body system is constructed. A specific reduction is also discussed, which allows one to build a novel integrable (iso)spin extension of the original bosonic rational model. In section 3.1, a triplet of vectors $(\lambda_i, \mu_i, \nu_i)$ is built, which underlies an integrable $\mathcal{N} = 1$ supersymmetric extension of the Ruijsenaars-Schneider trigonometric three-body model based upon the potential $W(x) = \frac{2}{\sin x}$. A respective integrable (iso)spin extension of the original trigonometric model is proposed as well. Difficulties in obtaining a similar triplet for $W(x) = \cot x$ are summarized in section 3.2. Section 4.1 and 4.2 contain similar analysis of the hyperbolic analogues based upon $W(x) = \frac{2}{\sinh x}$ and $W(x) = \coth x$ producing similar results. In section 5, it is demonstrated that for the Ruijsenaars-Toda system it proves problematic to build a triplet of vectors such that all their scalar products link to first integrals of the original bosonic model. In the concluding section 6, we summarize our results and discuss issues deserving of further study.

Throughout the paper summation over repeated indices is understood unless otherwise stated.

2 $\mathcal{N} = 1$ supersymmetric rational model

The Ruijsenaars-Schneider rational model is described by the differential equations (1.1), in which $W(x_{ij}) = \frac{2}{x_{ij}}$, $x_{ij} = x_i - x_j$, $i, j = 1, \dots, n$, and $x_1 > x_2 > \dots > x_n$. Functionally

independent first integrals, which provide integrability of the system, read

$$\begin{aligned}
 I_1 &= \sum_{i=1}^n \dot{x}_i, \\
 I_2 &= \sum_{i<j}^n \dot{x}_i \dot{x}_j (x_{ij})^2, \\
 I_3 &= \sum_{i<j<k}^n \dot{x}_i \dot{x}_j \dot{x}_k (x_{ij})^2 (x_{ik})^2 (x_{jk})^2, \\
 I_4 &= \sum_{i<j<k<l}^n \dot{x}_i \dot{x}_j \dot{x}_k \dot{x}_l (x_{ij})^2 (x_{ik})^2 (x_{il})^2 (x_{jk})^2 (x_{jl})^2 (x_{kl})^2, \\
 &\dots
 \end{aligned}
 \tag{2.1}$$

where ... designate higher order invariants, which can be constructed likewise.

Our objective in this section is to construct an $\mathcal{N} = 1$ supersymmetric extension of the rational system for the three-body case and to establish its integrability. To this end, it proves convenient to switch to the Hamiltonian formalism [12], within which the model is represented by three mutually commuting constants of motion

$$\begin{aligned}
 I_1 &= \frac{e^{p_1}}{x_{12}x_{13}} + \frac{e^{p_2}}{x_{12}x_{23}} + \frac{e^{p_3}}{x_{13}x_{23}}, & I_2 &= \frac{e^{p_1+p_2}}{x_{13}x_{23}} + \frac{e^{p_1+p_3}}{x_{12}x_{23}} + \frac{e^{p_2+p_3}}{x_{12}x_{13}}, \\
 I_3 &= e^{p_1+p_2+p_3},
 \end{aligned}
 \tag{2.2}$$

the first of which is identified with the Hamiltonian, $H = I_1$. The Poisson bracket is chosen in the conventional form $\{x_i, p_j\} = \delta_{ij}$.

In order to build an $\mathcal{N} = 1$ supersymmetric extension, one first introduces three subsidiary functions

$$\lambda_1 = \frac{e^{\frac{p_1}{2}}}{\sqrt{x_{12}x_{13}}}, \quad \lambda_2 = \frac{e^{\frac{p_2}{2}}}{\sqrt{x_{12}x_{23}}}, \quad \lambda_3 = \frac{e^{\frac{p_3}{2}}}{\sqrt{x_{13}x_{23}}},
 \tag{2.3}$$

which generate the potential $W(x) = \frac{2}{x}$ via the Poisson bracket (no summation over repeated indices and $i \neq j$)

$$\{\lambda_i, \lambda_j\} = \frac{1}{4} W(x_{ij}) \lambda_i \lambda_j.
 \tag{2.4}$$

In terms of λ_i , the Hamiltonian takes on the quadratic form

$$H = \lambda_i \lambda_i,
 \tag{2.5}$$

which is amenable to immediate supersymmetrization.

For most of the calculations to follow, it proves convenient to trade p_i for λ_i , which slightly modifies the canonical bracket (no summation over repeated indices)

$$\{x_i, \lambda_j\} = \frac{1}{2} \delta_{ij} \lambda_j.
 \tag{2.6}$$

The Hamiltonian equations of motion for x_i and λ_i then read

$$\dot{x}_i = \lambda_i^2, \quad \dot{\lambda}_i = \frac{1}{2} \sum_{j \neq i} W(x_{ij}) \lambda_i \lambda_j^2.
 \tag{2.7}$$

These equations prove to maintain their form for other potentials in (1.2). In establishing the supersymmetry algebra below, the following identity (no summation over repeated indices)

$$\{\lambda_i, x_{ij}\lambda_j\} = 0 \tag{2.8}$$

will prove useful.

As the second step, each canonical pair (x_i, p_i) is accompanied by a real Grassmann-odd partner θ_i , obeying the Poisson bracket²

$$\{\theta_i, \theta_j\} = -i\delta_{ij}, \tag{2.9}$$

and a natural $\mathcal{N} = 1$ supersymmetry charge is introduced

$$Q_1 = \lambda_i\theta_i, \tag{2.10}$$

which via the Poisson bracket generates the superextended Hamiltonian

$$\{Q_1, Q_1\} = -i\mathcal{H} = -i\mathcal{I}_1, \quad \mathcal{H} = \lambda_i\lambda_i + \frac{i}{4}W(x_{ij})\lambda_i\lambda_j\theta_i\theta_j. \tag{2.11}$$

As was explained in the Introduction, in order to establish integrability in the fermionic sector, one needs to find two more Grassmann-odd first integrals, the leading terms of which are linear in the Grassmann-odd variables

$$Q_2 = \mu_i\theta_i + \dots, \quad Q_3 = \nu_i\theta_i + \dots,$$

where \dots designate terms cubic in the fermions and $\mu_i(x, \lambda)$, $\nu_i(x, \lambda)$ are specific functions to be fixed below. Because Q_2 and Q_3 are supposed to commute with the superextended Hamiltonian \mathcal{H} , the following Poisson brackets

$$\begin{aligned} \{Q_1, Q_2\} &= -i\lambda_i\mu_i + \dots, & \{Q_1, Q_3\} &= -i\lambda_i\nu_i + \dots, & \{Q_2, Q_2\} &= -i\mu_i\mu_i + \dots, \\ \{Q_2, Q_3\} &= -i\mu_i\nu_i + \dots, & \{Q_3, Q_3\} &= -i\nu_i\nu_i + \dots, \end{aligned}$$

where \dots stand for terms quadratic in the Grassmann-odd variables, should be conserved over time as well. This follows from the super Jacobi identities. Considering the bosonic limit of the expressions contributing to the right hand sides, one concludes that the scalar products

$$\lambda_i\mu_i, \quad \lambda_i\nu_i, \quad \mu_i\mu_i, \quad \mu_i\nu_i, \quad \nu_i\nu_i,$$

should all link to the bosonic first integrals (2.2) characterizing the model at hand.

Rewriting (2.2) in terms of the subsidiary functions (2.3)

$$I_1 = \lambda_i\lambda_i, \quad I_2 = \frac{1}{2}\lambda_i^2\lambda_j^2x_{ij}^2, \quad I_3 = \left(\frac{1}{3!}\epsilon_{ijk}\lambda_i\lambda_j\lambda_kx_{ij}x_{ik}x_{jk}\right)^2, \tag{2.12}$$

²The conventional fermionic kinetic term $\frac{1}{2}\int dt\theta_i\dot{\theta}_i$ gives rise to the second class constraints $p_{\theta_i} - \frac{1}{2}\theta_i = 0$, where $p_{\theta_i} = \frac{\partial\mathcal{L}}{\partial\dot{\theta}_i}$ is the momentum canonically conjugate to θ_i , $\mathcal{L} = \frac{1}{2}\theta_i\dot{\theta}_i$ is the Lagrangian density, and the right derivative with respect to the Grassmann-odd variables is used. Introducing the conventional Dirac bracket and eliminating p_{θ_i} from the consideration by resolving the second class constraints, one arrives at (2.9).

where ϵ_{ijk} is the Levi-Civita totally antisymmetric symbol with $\epsilon_{123} = 1$, one obtains a natural candidate for the vector μ_i , which underpins Q_2

$$I_2 = \mu_i \mu_i, \quad \mu_i = \frac{1}{2} \epsilon_{ijk} \lambda_j \lambda_k x_{jk}, \quad \lambda_i \mu_i = 0, \quad (2.13)$$

where the last equality holds due to the identity

$$x_{12} - x_{13} + x_{23} = 0. \quad (2.14)$$

In obtaining (2.13), the properties of the Levi-Civita symbol

$$\epsilon_{ijk} \epsilon_{lpk} = \delta_{il} \delta_{jp} - \delta_{ip} \delta_{jl}, \quad \epsilon_{ijk} \epsilon_{ljk} = 2\delta_{il} \quad (2.15)$$

proved useful. Then it is straightforward to verify that

$$Q_2 = \mu_i \theta_i = \frac{1}{2} \epsilon_{ijk} \lambda_j \lambda_k x_{jk} \theta_i, \quad (2.16)$$

Poisson commutes with Q_1 and, hence, it is conserved over time as a consequence of the super Jacobi identity involving the triplet (Q_1, Q_1, Q_2) . In verifying the relation $\{Q_1, Q_2\} = 0$, the identity (2.14) was used.

Computing the Poisson bracket of Q_2 with itself, one obtains the superextension of the original bosonic first integral I_2

$$\{Q_2, Q_2\} = -i\mathcal{I}_2, \quad \mathcal{I}_2 = \frac{1}{2} \lambda_i^2 \lambda_j^2 x_{ij}^2 + \frac{i}{8} (\epsilon_{ijk} \theta_i \theta_j \lambda_i \lambda_j) (\epsilon_{plk} W(x_{pl}) x_{pk} x_{lk} \lambda_k^2), \quad (2.17)$$

which rightly commutes with $\mathcal{H} = \mathcal{I}_1$ in (2.11). The third constant of motion I_3 in (2.12) does not acquire fermionic contributions and maintains its form after the superextension, $\mathcal{I}_3 = I_3$, which is a manifestation of the invariance of the resulting system under the translation $x'_i = x_i + a$.

The construction of Q_3 is less straightforward, however. It appears problematic to represent I_3 as a scalar product of a vector ν_i with itself. Another option, which will ultimately prove correct, is to take ν_i entering Q_3 as the vector product of λ_i , upon which Q_1 is constructed, and μ_i , which underlies Q_2 . A contribution to Q_3 , which is cubic in the Grassmann-odd variables, is then found directly from the conservation equation $\{Q_3, \mathcal{H}\} = 0$.

Yet another possibility to build Q_3 is to make recourse to higher order fermionic invariants available for the case at hand. Taking into account the equations of motion in the fermionic sector

$$\dot{\theta}_i = \frac{1}{2} \sum_{j \neq i}^3 W(x_{ij}) \lambda_i \lambda_j \theta_j, \quad W(x_{ij}) = \frac{2}{x_{ij}}, \quad (2.18)$$

one readily obtains a cubic integral of motion

$$\Omega = \frac{i}{3!} \epsilon_{ijk} \theta_i \theta_j \theta_k = i\theta_1 \theta_2 \theta_3, \quad (2.19)$$

which is conserved over time as a consequence of the Grassmann-valued nature of the variable θ_i : $\theta_1^2 = \theta_2^2 = \theta_3^2 = 0$. The Poisson brackets of Ω and Q_1, Q_2 can then be used to build lower order fermionic invariants

$$\begin{aligned} \{Q_1, \Omega\} &= -i\Lambda, & \Lambda &= \frac{i}{2}\epsilon_{ijk}\lambda_i\theta_j\theta_k, \\ \{Q_2, \Lambda\} &= Q_3, & Q_3 &= x_{ij}\lambda_j^2\lambda_i\theta_i + \frac{1}{4}\epsilon_{ijk}x_{ij}W(x_{jk})\lambda_i\lambda_j\lambda_k\Omega, \end{aligned} \quad (2.20)$$

the last of which is the desired third fermionic constant of motion. It is straightforward to verify that the leading term in $Q_3 = \nu_i\theta_i + \dots$ is indeed constructed as the vector product of λ_i and μ_j

$$\nu_i = x_{ij}\lambda_i\lambda_j^2 = \epsilon_{ijk}\lambda_j\mu_k, \quad \nu_i\nu_i = I_1I_2, \quad \nu_i\lambda_i = 0, \quad \nu_i\mu_i = 0, \quad (2.21)$$

with μ_k defined in (2.13). In the leftmost equation in (2.21) and in the text below no summation over repeated indices carrying a hat symbol is understood.

Thus, $(\lambda_i, \mu_i, \nu_i)$ do form a triplet of vectors, all scalar products of which can be expressed in terms of the bosonic first integrals (2.12). The latter fact will prove important in subsequent sections, where other variants of the Ruijsenaars-Schneider model will be studied.

At this point, algebraic resolvability of the fermionic constants of motion (Q_1, Q_2, Q_3) with respect to the variables $(\theta_1, \theta_2, \theta_3)$ can be easily established. Because the cubic term Ω is itself conserved over time, the expressions for (Q_1, Q_2, Q_3) can be put into the linear algebraic form $A_{ij}\theta_j = B_i$, where B_i is a specific vector function, which can be read off from (Q_1, Q_2, Q_3) , and A_{ij} is the matrix involving three rows $A_{1i} = \lambda_i, A_{2i} = \mu_i, A_{3i} = \nu_i = \epsilon_{ijk}\lambda_j\mu_k$. Because the determinant of A_{ij} is equal to the square of the area of a parallelogram formed by the vectors λ_i and μ_i , the matrix A_{ij} is invertible and, hence, the system of equations for θ_i is algebraically resolvable: $\theta_i = (A^{-1})_{ij}B_j$. Taking the resulting expressions and computing the product $\theta_1\theta_2\theta_3$, one can then link the higher order invariant Ω to (Q_1, Q_2, Q_3) and $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$.

As was mentioned in the Introduction, a formal supersymmetrization procedure is expected to provide an efficient way of generating integrable extensions of known integrable systems. Concluding this section, we discuss how the $\mathcal{N} = 1$ supersymmetric model above can be used to build a novel integrable (iso)spin extension of the Ruijsenaars-Schneider rational three-body model.

For the case at hand, the fermionic sector is described by three Grassmann-odd variables $\theta_i, i = 1, 2, 3$, which obey the first order differential equations (2.18). The corresponding general solution involves three Grassmann-odd constants of integration. Denoting them by α, β , and γ and taking into account $\alpha^2 = \beta^2 = \gamma^2 = 0$, one gets the natural decompositions

$$\theta_i = \alpha\varphi_{i1} + \beta\varphi_{i2} + \gamma\varphi_{i3} + i\alpha\beta\gamma\varphi_{i4}, \quad x_i = x_{i0} + i\alpha\beta x_{i1} + i\alpha\gamma x_{i2} + i\beta\gamma x_{i3}, \quad (2.22)$$

where components accompanying α, β , and γ are real *bosonic* functions of the temporal variable t . Substituting (2.22) into the Hamiltonian equations of motion of the superextended system and analyzing monomials in α, β, γ on both sides, one turns them into a system of ordinary differential equations for usual real-valued functions. The latter provides an integrable extension of the original Ruijsenaars-Schneider rational model.

The resulting system is rather bulky and hard to interpret. A simple and tractable extension arises if one focuses on a particular solution for which $\beta = \gamma = 0$

$$\theta_i = \alpha\varphi_i, \quad \alpha^2 = 0, \tag{2.23}$$

φ_i being a real-valued bosonic function to be interpreted below as describing (iso)spin degrees of freedom. In this case, all terms quadratic or cubic in the fermionic variables vanish owing to $\alpha^2 = 0$ and the $\mathcal{N} = 1$ superextension above reduces to the original Ruijsenaars-Schneider equations (1.1), which are accompanied by the linear differential equations for φ_i

$$\dot{\varphi}_i = \frac{1}{2} \sum_{j \neq i}^3 W(x_{ij}) \sqrt{\dot{x}_i \dot{x}_j} \varphi_j, \quad W(x_{ij}) = \frac{2}{x_{ij}}. \tag{2.24}$$

The latter system inherits from its $\mathcal{N} = 1$ supersymmetric progenitor three first integrals

$$I_4 = \sqrt{\dot{x}_i} \varphi_i, \quad I_5 = \frac{1}{2} \epsilon_{ijk} \sqrt{\dot{x}_i \dot{x}_j} x_{ij} \varphi_k, \quad I_6 = -x_{ij} \dot{x}_i \sqrt{\dot{x}_j} \varphi_j, \tag{2.25}$$

which descend from (Q_1, Q_2, Q_3) , and admits one extra constant of motion

$$I_7 = \varphi_i \varphi_i, \tag{2.26}$$

which implies that φ_i can be interpreted as internal degrees of freedom parametrizing a two-sphere. To the best of our knowledge, such integrable (iso)spin extension of the Ruijsenaars-Schneider rational three-body model is new.

3 $\mathcal{N} = 1$ supersymmetric trigonometric models

3.1 The case of $W(x) = \frac{2}{\sin x}$

The potentials listed in (1.2) contain two trigonometric variants, the first of which is described by the equations of motion (1.1) involving $W(x_{ij}) = \frac{2}{\sin x_{ij}}$, with $x_{ij} = x_i - x_j$, $i, j = 1, \dots, n$, $x_1 > x_2 > \dots > x_n$. The system is characterized by the first integrals

$$\begin{aligned} I_1 &= \sum_{i=1}^n \dot{x}_i, \\ I_2 &= \sum_{i < j}^n \dot{x}_i \dot{x}_j \tan^2 \left(\frac{x_{ij}}{2} \right), \\ I_3 &= \sum_{i < j < k}^n \dot{x}_i \dot{x}_j \dot{x}_k \tan^2 \left(\frac{x_{ij}}{2} \right) \tan^2 \left(\frac{x_{ik}}{2} \right) \tan^2 \left(\frac{x_{jk}}{2} \right), \\ I_4 &= \sum_{i < j < k < l}^n \dot{x}_i \dot{x}_j \dot{x}_k \dot{x}_l \tan^2 \left(\frac{x_{ij}}{2} \right) \tan^2 \left(\frac{x_{ik}}{2} \right) \tan^2 \left(\frac{x_{il}}{2} \right) \tan^2 \left(\frac{x_{jk}}{2} \right) \tan^2 \left(\frac{x_{jl}}{2} \right) \tan^2 \left(\frac{x_{kl}}{2} \right), \\ &\dots \end{aligned} \tag{3.1}$$

where ... stand for higher order invariants, which are constructed in a similar fashion.

Like before, the Hamiltonian formulation for such a three-body model is constructed in terms the subsidiary functions

$$\begin{aligned}\lambda_1 &= e^{\frac{p_1}{2}} \sqrt{\cot\left(\frac{x_{12}}{2}\right) \cot\left(\frac{x_{13}}{2}\right)}, & \lambda_2 &= e^{\frac{p_2}{2}} \sqrt{\cot\left(\frac{x_{12}}{2}\right) \cot\left(\frac{x_{23}}{2}\right)}, \\ \lambda_3 &= e^{\frac{p_3}{2}} \sqrt{\cot\left(\frac{x_{13}}{2}\right) \cot\left(\frac{x_{23}}{2}\right)},\end{aligned}\tag{3.2}$$

where p_i are momenta canonically conjugate to the coordinates x_i , $\{x_i, p_j\} = \delta_{ij}$, which generate the potential $W(x) = \frac{2}{\sin x}$ via the Poisson bracket (no summation over repeated indices and $i \neq j$)

$$\{\lambda_i, \lambda_j\} = \frac{1}{4} W(x_{ij}) \lambda_i \lambda_j, \quad W(x_{ij}) = \frac{2}{\sin x_{ij}}.\tag{3.3}$$

In terms of λ_i , three functionally independent integrals of motion in involution take on the form

$$\begin{aligned}I_1 &= \lambda_i \lambda_i, & I_2 &= \frac{1}{2} \lambda_i^2 \lambda_j^2 \tan^2\left(\frac{x_{ij}}{2}\right), \\ I_3 &= \left(\frac{1}{3!} \epsilon_{ijk} \lambda_i \lambda_j \lambda_k \tan\left(\frac{x_{ij}}{2}\right) \tan\left(\frac{x_{ik}}{2}\right) \tan\left(\frac{x_{jk}}{2}\right)\right)^2,\end{aligned}\tag{3.4}$$

the first of which is identified with the Hamiltonian $I_1 = H$. The Hamiltonian equations of motion read as in (2.7), but this time they involve $W(x_{ij}) = \frac{2}{\sin x_{ij}}$.

The structure of invariants (3.4) suggests introducing two vectors

$$\mu_i = \frac{1}{2} \epsilon_{ijk} \lambda_j \lambda_k \tan\left(\frac{x_{jk}}{2}\right), \quad \nu_i = \lambda_i^2 \tan\left(\frac{x_{ij}}{2}\right) = \epsilon_{ijk} \lambda_j \mu_k,\tag{3.5}$$

which accompany λ_i in (3.2). A remarkable property of the triplet $(\lambda_i, \mu_i, \nu_i)$ is that all their scalar products can be expressed in terms of the first integrals (3.4)

$$\begin{aligned}\lambda_i \lambda_i &= I_1, & \mu_i \mu_i &= I_2, & \nu_i \nu_i &= I_1 I_2 - I_3, \\ \lambda_i \mu_i &= -\sqrt{I_3}, & \lambda_i \nu_i &= 0, & \mu_i \nu_i &= 0.\end{aligned}\tag{3.6}$$

When computing $\lambda_i \mu_i$ the trigonometric identity

$$\tan\left(\frac{x_{12}}{2}\right) - \tan\left(\frac{x_{13}}{2}\right) + \tan\left(\frac{x_{23}}{2}\right) = -\tan\left(\frac{x_{12}}{2}\right) \tan\left(\frac{x_{13}}{2}\right) \tan\left(\frac{x_{23}}{2}\right)\tag{3.7}$$

was used. The latter is an analogue of (2.14), which underpins the rational case.

An integrable $\mathcal{N} = 1$ supersymmetric extension of the trigonometric model at hand is built upon the triplet $(\lambda_i, \mu_i, \nu_i)$ in a remarkably succinct way³

$$Q_1 = \lambda_i \theta_i, \quad Q_2 = \mu_i \theta_i, \quad Q_3 = \nu_i \theta_i - \{\lambda_i, \mu_i\} \Omega,\tag{3.8}$$

where θ_i are the fermionic degrees of freedom obeying $\{\theta_i, \theta_j\} = -i\delta_{ij}$ and $\Omega = \frac{i}{3!} \epsilon_{ijk} \theta_i \theta_j \theta_k$ is the cubic invariant similar to that used in the previous section. Superextensions of the original bosonic first integrals (3.4) are found by computing the Poisson brackets

$$\{Q_1, Q_1\} = -i\mathcal{I}_1, \quad \{Q_2, Q_2\} = -i\mathcal{I}_2, \quad \{Q_1, Q_2\} = i\sqrt{I_3},\tag{3.9}$$

³In the unfolded form, the Poisson bracket entering (3.8) reads $\{\lambda_i, \mu_i\} = -\frac{1}{4} \epsilon_{pjk} \tan\left(\frac{x_{pj}}{2}\right) W(x_{jk}) \lambda_p \lambda_j \lambda_k$.

which yield

$$\mathcal{I}_1 = \mathcal{H} = \lambda_i \lambda_i + \frac{i}{4} W(x_{ij}) \lambda_i \lambda_j \theta_i \theta_j, \quad \mathcal{I}_2 = \mu_i \mu_i - \frac{i}{4} W(x_{ij}) \mu_i \mu_j \theta_i \theta_j, \quad \mathcal{I}_3 = I_3. \quad (3.10)$$

Like before, I_3 in (3.4) does not acquire fermionic corrections in the process of supersymmetrization, which reflects the invariance of the system under the translation $x'_i = x_i + a$. In obtaining (3.10) the following relations (no summation over repeated indices and $i \neq j$)

$$\{\lambda_i, \tan\left(\frac{x_{ij}}{2}\right) \lambda_j\} = 0, \quad \{\mu_i, \mu_j\} = -\frac{1}{4} W(x_{ij}) \mu_i \mu_j, \quad \{\lambda_i, \mu_j\} = \frac{1}{4} \delta_{ij} \lambda_j \mu_j \sum_{k \neq i} W(x_{ik})$$

proved useful. The algebraic resolvability of the first integrals (3.8) with respect to the variables θ_i is established by repeating the argument in the preceding section.

Concluding this section, we display an integrable (iso)spin extension of the trigonometric three-body model based upon the potential $W(x) = \frac{2}{\sin x}$. It is built following the recipe in the preceding section. The (iso)spin degrees of freedom obey the differential equations

$$\dot{\varphi}_i = \frac{1}{2} \sum_{j \neq i}^3 W(x_{ij}) \sqrt{\dot{x}_i \dot{x}_j} \varphi_j, \quad W(x_{ij}) = \frac{2}{\sin x_{ij}}, \quad (3.11)$$

which are characterized by three constants of motion

$$I_4 = \sqrt{\dot{x}_i} \varphi_i, \quad I_5 = \frac{1}{2} \epsilon_{ijk} \sqrt{\dot{x}_i \dot{x}_j} \tan\left(\frac{x_{ij}}{2}\right) \varphi_k, \quad I_6 = -\tan\left(\frac{x_{ij}}{2}\right) \dot{x}_i \sqrt{\dot{x}_j} \varphi_j, \quad (3.12)$$

originating from the supercharges (Q_1, Q_2, Q_3) in (3.8). The sector also admits an extra integral of motion $I_7 = \varphi_i \varphi_i$ describing the geometry of the subspace parametrized by the internal degrees of freedom. To the best of our knowledge, such an integrable extension of the Ruijsenaars-Schneider trigonometric three-body model is new.

3.2 The case of $W(x) = 2 \cot x$

The second trigonometric model builds upon the potential $W(x) = 2 \cot x$ and the set of functionally independent first integrals

$$\begin{aligned} I_1 &= \sum_{i=1}^n \dot{x}_i, \\ I_2 &= \sum_{i < j}^n \dot{x}_i \dot{x}_j \sin^2 x_{ij}, \\ I_3 &= \sum_{i < j < k}^n \dot{x}_i \dot{x}_j \dot{x}_k \sin^2 x_{ij} \sin^2 x_{ik} \sin^2 x_{jk}, \\ I_4 &= \sum_{i < j < k < s}^n \dot{x}_i \dot{x}_j \dot{x}_k \dot{x}_s \sin^2 x_{ij} \sin^2 x_{ik} \sin^2 x_{is} \sin^2 x_{jk} \sin^2 x_{js} \sin^2 x_{ks}, \\ &\dots \end{aligned} \quad (3.13)$$

where ... denote higher order invariants of a similar structure.

An $\mathcal{N} = 1$ supersymmetric extension of the three-body model at hand is constructed following the general pattern above. It suffices to consider three subsidiary functions

$$\lambda_1 = \frac{e^{\frac{p_1}{2}}}{\sqrt{\sin x_{12} \sin x_{13}}}, \quad \lambda_2 = \frac{e^{\frac{p_2}{2}}}{\sqrt{\sin x_{12} \sin x_{23}}}, \quad \lambda_3 = \frac{e^{\frac{p_3}{2}}}{\sqrt{\sin x_{13} \sin x_{23}}}, \quad (3.14)$$

which generate the potential $W(x) = 2 \cot x$ via the Poisson bracket (no summation over repeated indices and $i \neq j$)

$$\{\lambda_i, \lambda_j\} = \frac{1}{4} W(x_{ij}) \lambda_i \lambda_j, \quad W(x_{ij}) = 2 \cot x_{ij}. \quad (3.15)$$

Then one introduces a superpartner θ_i for each canonical pair (x_i, p_i) , and finally builds the linear supercharge $Q_1 = \lambda_i \theta_i$. The latter generates the superextended Hamiltonian via the Poisson bracket, $\{Q_1, Q_1\} = -i\mathcal{H}$.

In order to obtain two more supercharges, one rewrites three available first integrals in terms of λ_i

$$I_1 = \lambda_i \lambda_i, \quad I_2 = \frac{1}{2} \lambda_i^2 \lambda_j^2 \sin^2 x_{ij}, \quad I_3 = \left(\frac{1}{3!} \epsilon_{ijk} \lambda_i \lambda_j \lambda_k \sin x_{ij} \sin x_{ik} \sin x_{jk} \right)^2, \quad (3.16)$$

and then tries to extract from them two more vectors μ_i and ν_i suitable for building Q_2 and Q_3 , respectively. At this point, one reveals a problem, however. Using I_2 in order to construct μ_i

$$\mu_i = \frac{1}{2} \epsilon_{ijk} \lambda_j \lambda_k \sin x_{jk}, \quad \mu_i \mu_i = I_2, \quad (3.17)$$

just like we did in our previous examples, one immediately finds that the scalar product of μ_i and λ_i is not conserved over time⁴

$$\lambda_i \mu_i = \frac{\sqrt{I_3}}{2 \cos\left(\frac{x_{12}}{2}\right) \cos\left(\frac{x_{13}}{2}\right) \cos\left(\frac{x_{23}}{2}\right)}, \quad (\lambda_i \mu_i)' \neq 0. \quad (3.18)$$

Note that in two previous cases the equalities $\lambda_i \mu_i = 0$ and $\lambda_i \mu_i = -\sqrt{I_3}$ held due to the identities (2.14) and (3.7), respectively, which link to the specific form of the potential $W(x)$ for each respective case. For the model under consideration

$$\sin x_{12} - \sin x_{13} + \sin x_{23} \neq \sin x_{12} \sin x_{13} \sin x_{23} \quad (3.19)$$

and, hence, $\lambda_i \mu_i$ fails to be proportional to $\sqrt{I_3}$.

One could try to treat $\sqrt{I_3}$ as a scalar product of λ_i and $\nu_i = -\frac{1}{2} \epsilon_{ijk} \cos x_{ij} \cos x_{ik} \sin x_{jk} \lambda_j \lambda_k$, which would rely upon the trigonometric identity

$$-\sin x_{12} \cos x_{13} \cos x_{23} + \sin x_{13} \cos x_{12} \cos x_{23} - \sin x_{23} \cos x_{12} \cos x_{13} = \sin x_{12} \sin x_{13} \sin x_{23}.$$

Yet, at the next step one would immediately find that $\nu_i \nu_i$ is not conserved over time. In a similar fashion, one could try to regard I_2 in (3.16) as the scalar product of λ_i and $\nu_i = \frac{1}{2} \lambda_i \lambda_j^2 \sin^2 x_{ij}$, which would again result in the nonconservation of $\nu_i \nu_i$ over time.

Thus, despite our anticipation in [4] that proving integrability in the fermionic sector should go rather straightforward for each $\mathcal{N} = 1$ supersymmetric variant of the Ruijsenaars-Schneider three-body system, the trigonometric model above presents a challenge. In the next sections, we shall see more examples of such a kind.

⁴Of course, one can reshuffle the components of μ_i without changing $I_2 = \mu_i \mu_i$. Unfortunately, this arbitrariness does not help to improve $(\lambda_i \mu_i)' \neq 0$.

4 $\mathcal{N} = 1$ supersymmetric hyperbolic models

4.1 The case of $W(x) = \frac{2}{\sinh x}$

The trigonometric models above have two hyperbolic analogues, which we discuss in this section.⁵ The first variant is based upon $W(x) = \frac{2}{\sinh x}$ and it was studied in our recent work [4]. Referring the reader to [4] for more details, we proceed directly to the subsidiary vector λ_i

$$\begin{aligned}\lambda_1 &= e^{\frac{p_1}{2}} \sqrt{\coth\left(\frac{x_{12}}{2}\right) \coth\left(\frac{x_{13}}{2}\right)}, & \lambda_2 &= e^{\frac{p_2}{2}} \sqrt{\coth\left(\frac{x_{12}}{2}\right) \coth\left(\frac{x_{23}}{2}\right)}, \\ \lambda_3 &= e^{\frac{p_3}{2}} \sqrt{\coth\left(\frac{x_{13}}{2}\right) \coth\left(\frac{x_{23}}{2}\right)},\end{aligned}\tag{4.1}$$

and its two companions

$$\mu_i = \frac{1}{2} \epsilon_{ijk} \lambda_j \lambda_k \tanh\left(\frac{x_{jk}}{2}\right), \quad \nu_i = \lambda_i^2 \tanh\left(\frac{x_{ij}}{2}\right) = \epsilon_{ijk} \lambda_j \mu_k.\tag{4.2}$$

In accord with our analysis above, in order to establish integrability in the fermionic sector of the corresponding $\mathcal{N} = 1$ supersymmetric extension, it suffices to verify that all scalar products between $(\lambda_i, \mu_i, \nu_i)$ can be expressed in terms of the first integrals characterizing the case

$$\begin{aligned}I_1 &= \lambda_i \lambda_i, & I_2 &= \frac{1}{2} \lambda_i^2 \lambda_j^2 \tanh^2\left(\frac{x_{ij}}{2}\right), \\ I_3 &= \left(\frac{1}{3!} \epsilon_{ijk} \lambda_i \lambda_j \lambda_k \tanh\left(\frac{x_{ij}}{2}\right) \tanh\left(\frac{x_{ik}}{2}\right) \tanh\left(\frac{x_{jk}}{2}\right)\right)^2.\end{aligned}\tag{4.3}$$

An easy calculation yields

$$\begin{aligned}\lambda_i \lambda_i &= I_1, & \mu_i \mu_i &= I_2, & \nu_i \nu_i &= I_1 I_2 - I_3, \\ \lambda_i \mu_i &= \sqrt{I_3}, & \lambda_i \nu_i &= 0, & \mu_i \nu_i &= 0,\end{aligned}\tag{4.4}$$

meaning that $(\lambda_i, \mu_i, \nu_i)$ do pass the test. Note that, like in our integrable examples above, the equality $\lambda_i \mu_i = \sqrt{I_3}$ appeals to the specific identity

$$\tanh\left(\frac{x_{12}}{2}\right) - \tanh\left(\frac{x_{13}}{2}\right) + \tanh\left(\frac{x_{23}}{2}\right) = \tanh\left(\frac{x_{12}}{2}\right) \tanh\left(\frac{x_{13}}{2}\right) \tanh\left(\frac{x_{23}}{2}\right),\tag{4.5}$$

which holds for the hyperbolic functions at hand. The construction of an integrable $\mathcal{N} = 1$ supersymmetric extension and the respective (iso)spin reduction is then straightforward [4].

⁵Note that the hyperbolic versions follow from the trigonometric models by the formal substitution $x_i \rightarrow ix_i$. For completeness of the presentation, we briefly discuss them in this section.

4.2 The case of $W(x) = 2 \coth x$

The second hyperbolic model builds upon the potential $W(x) = 2 \coth x$ and the set of functionally independent integrals of motion

$$\begin{aligned}
 I_1 &= \sum_{i=1}^n \dot{x}_i, \\
 I_2 &= \sum_{i<j}^n \dot{x}_i \dot{x}_j \sinh^2(x_{ij}), \\
 I_3 &= \sum_{i<j<k}^n \dot{x}_i \dot{x}_j \dot{x}_k \sinh^2(x_{ij}) \sinh^2(x_{ik}) \sinh^2(x_{jk}), \\
 I_4 &= \sum_{i<j<k<s}^n \dot{x}_i \dot{x}_j \dot{x}_k \dot{x}_s \sinh^2(x_{ij}) \sinh^2(x_{ik}) \sinh^2(x_{is}) \sinh^2(x_{jk}) \sinh^2(x_{js}) \sinh^2(x_{ks}), \\
 &\dots
 \end{aligned}
 \tag{4.6}$$

where \dots denote higher order invariants, which are constructed likewise.

Focusing on the three-body case, introducing momenta p_i canonically conjugate to the configuration space variables x_i , the conventional Poisson bracket $\{x_i, p_j\} = \delta_{ij}$, and the Hamiltonian function

$$H = \frac{e^{p_1}}{\sinh(x_{12}) \sinh(x_{13})} + \frac{e^{p_2}}{\sinh(x_{12}) \sinh(x_{23})} + \frac{e^{p_3}}{\sinh(x_{13}) \sinh(x_{23})} = I_1,
 \tag{4.7}$$

one can represent the system in the Hamiltonian form. Two extra integrals of motion read

$$I_2 = \frac{e^{p_1+p_2}}{\sinh(x_{13}) \sinh(x_{23})} + \frac{e^{p_1+p_3}}{\sinh(x_{12}) \sinh(x_{23})} + \frac{e^{p_2+p_3}}{\sinh(x_{12}) \sinh(x_{13})}, \quad I_3 = e^{p_1+p_2+p_3}.
 \tag{4.8}$$

It is straightforward to verify that (I_1, I_2, I_3) are functionally independent and mutually commuting, which guarantees the Liouville integrability.

Like in all our examples above, in order to construct an $\mathcal{N} = 1$ supersymmetric extension, it suffices to build three subsidiary functions

$$\lambda_1 = \frac{e^{\frac{p_1}{2}}}{\sqrt{\sinh(x_{12}) \sinh(x_{13})}}, \quad \lambda_2 = \frac{e^{\frac{p_2}{2}}}{\sqrt{\sinh(x_{12}) \sinh(x_{23})}}, \quad \lambda_3 = \frac{e^{\frac{p_3}{2}}}{\sqrt{\sinh(x_{13}) \sinh(x_{23})}},$$

which obey the Poisson bracket (no summation over repeated indices and $i \neq j$)

$$\{\lambda_i, \lambda_j\} = \frac{1}{4} W(x_{ij}) \lambda_i \lambda_j, \quad W(x_{ij}) = 2 \coth(x_{ij}),
 \tag{4.9}$$

then one introduces the superpartners θ_i of (x_i, p_i) and builds the supersymmetry generator $Q_1 = \lambda_i \theta_i$. The latter generates the superextended Hamiltonian via the Poisson bracket, $\{Q_1, Q_1\} = -i\mathcal{H}$.

In the search for two more Grassmann-odd constants of motion $Q_2 = \mu_i \theta_i + \dots$ and $Q_3 = \nu_i \theta_i + \dots$, one rewrites (4.7), (4.8) in terms of λ_i

$$I_1 = \lambda_i \lambda_i, \quad I_2 = \frac{1}{2} \lambda_i^2 \lambda_j^2 \sinh^2 x_{ij}, \quad I_3 = \left(\frac{1}{3!} \epsilon_{ijk} \lambda_i \lambda_j \lambda_k \sinh x_{ij} \sinh x_{ik} \sinh x_{jk} \right)^2,$$

and then considers a feasible candidate for μ_i

$$\mu_i = \frac{1}{2} \epsilon_{ijk} \lambda_j \lambda_k \sinh x_{jk}, \quad \mu_i \mu_i = I_2. \quad (4.10)$$

Yet, although $\mu_i \mu_i$ is conserved over time, $\lambda_i \mu_i$ is not

$$\lambda_i \mu_i = -\frac{\sqrt{I_3}}{2 \cosh\left(\frac{x_{12}}{2}\right) \cosh\left(\frac{x_{13}}{2}\right) \cosh\left(\frac{x_{23}}{2}\right)}, \quad (\lambda_i \mu_i)' \neq 0. \quad (4.11)$$

The latter fact links to the inequality

$$\sinh x_{12} - \sinh x_{13} + \sinh x_{23} \neq \sinh x_{12} \sinh x_{13} \sinh x_{23}, \quad (4.12)$$

which prevents $\lambda_i \mu_i$ from being proportional to $\sqrt{I_3}$.⁶

Thus, similarly to its trigonometric partner discussed in section 3.2, one faces a problem in establishing integrability in the fermionic sector of the $\mathcal{N} = 1$ supersymmetric hyperbolic system at hand, which calls for a more sophisticated analysis.

5 $\mathcal{N} = 1$ supersymmetric Ruijsenaars-Toda model

Our next example is the Ruijsenaars-Toda periodic lattice, which is described by the equations of motion [11]

$$\ddot{x}_i = \dot{x}_{i+1} \dot{x}_i W(x_{i+1} - x_i) - \dot{x}_i \dot{x}_{i-1} W(x_i - x_{i-1}), \quad W(x - y) = \frac{g^2 e^{x-y}}{1 + g^2 e^{x-y}}, \quad (5.1)$$

where $i = 1, \dots, N$, g is a coupling constant. The boundary conditions

$$x_0 = x_N, \quad x_{N+1} = x_1 \quad (5.2)$$

are assumed to hold.

Introducing momenta p_i canonically conjugate to the configuration space variables x_i and the conventional Poisson bracket, $\{x_i, p_j\} = \delta_{i,j}$, one finds that the boundary conditions (5.2) imply

$$\{x_{i+1}, p_j\} = \delta_{i+1,j} + \delta_{i,N} \delta_{j,1}, \quad \{x_{i-1}, p_j\} = \delta_{i-1,j} + \delta_{i,1} \delta_{j,N}. \quad (5.3)$$

The latter relations can be used to verify that the positive definite Hamiltonian (no sum with respect to i in the second relation)

$$H = e^{p_i} \left(1 + g^2 e^{x_{i+1} - x_i}\right) = \lambda_i \lambda_i, \quad \lambda_i = e^{\frac{p_i}{2}} \sqrt{1 + g^2 e^{x_{i+1} - x_i}} \quad (5.4)$$

does put (5.1) into the Hamiltonian form. The subsidiary functions λ_i obey the structure relations [13]

$$\{\lambda_i, \lambda_j\} = \frac{1}{4} \lambda_i \lambda_j \left(W(x_{i+1} - x_i) [\delta_{i+1,j} + \delta_{i,N} \delta_{j,1}] - W(x_{j+1} - x_j) [\delta_{i,j+1} + \delta_{i,1} \delta_{j,N}] \right). \quad (5.5)$$

⁶One could try to regard I_2 or $\sqrt{I_3}$ as the scalar product of λ_i with $\nu_i = \frac{1}{2} \lambda_i \lambda_j^2 \sinh^2 x_{ij}$ or $\nu_i = \frac{1}{3!} \epsilon_{ijk} \lambda_j \lambda_k \sinh x_{ij} \sinh x_{ik} \sinh x_{jk}$, respectively. Yet, $\nu_i \nu_i$ would not be conserved over time.

Focusing on the three-body case and introducing a fermionic partner θ_i for each bosonic canonical pair (x_i, p_i) , one immediately obtains an $\mathcal{N} = 1$ supersymmetric extension of the model at hand, which is governed by the supersymmetry charge $Q_1 = \lambda_i \theta_i$. The latter generates the superextended Hamiltonian via the Poisson bracket, $\{Q_1, Q_1\} = -i\mathcal{H}$.

Representing three mutually commuting first integrals in terms of λ_i

$$\begin{aligned}
 I_1 = H &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, & I_2 &= \frac{\lambda_1^2 \lambda_2^2}{1 + g^2 e^{x_2 - x_1}} + \frac{\lambda_1^2 \lambda_3^2}{1 + g^2 e^{x_1 - x_3}} + \frac{\lambda_2^2 \lambda_3^2}{1 + g^2 e^{x_3 - x_2}}, \\
 I_3 &= \frac{\lambda_1^2 \lambda_2^2 \lambda_3^2}{(1 + g^2 e^{x_2 - x_1})(1 + g^2 e^{x_3 - x_2})(1 + g^2 e^{x_1 - x_3})}, & & (5.6)
 \end{aligned}$$

one can then verify that it proves problematic to construct a vector μ_i , $I_2 = \mu_i \mu_i$, such that $\lambda_i \mu_i$ is conserved over time. Thus, similarly to the examples in section 3.2 and 4.2, the system does not pass our simple integrability test and a more sophisticated analysis is needed.

6 Conclusion

To summarize, in this work integrability of $\mathcal{N} = 1$ supersymmetric Ruijsenaars-Schneider three-body models based upon the potentials $W(x) = \frac{2}{x}$, $W(x) = \frac{2}{\sin x}$, and $W(x) = \frac{2}{\sinh x}$ was proven. The problem of constructing an algebraically resolvable set of Grassmann-odd constants of motion was reduced to building a triplet of vectors such that all their scalar products are expressible in terms of the original bosonic first integrals. The supersymmetric generalizations were then used to build novel integrable (iso)spin extensions of the respective Ruijsenaars-Schneider three-body systems.

In cases where our method succeeded, it relied upon specific rational/trigonometric identities ((2.14), (3.7), and (4.5)). The absence of similar identities presented an obstacle for establishing integrability of the $\mathcal{N} = 1$ supersymmetric three-body systems relying upon $W(x) = 2 \cot x$, $W(x) = 2 \coth x$, and the Ruijsenaars-Toda potential. It is important to understand whether this is a purely technical problem or something more fundamental lies behind it.

Another question deserving of further study is the construction of a Lax pair in the fermionic sector of the $\mathcal{N} = 1$ supersymmetric systems constructed in this work. Within the Lax formalism, constants of motion link to $\text{Tr} L^n$, $n = 1, 2, \dots$, where L is the Lax matrix. Given three Grassmann-odd integrals of motion (Q_1, Q_2, Q_3) , the leading terms of which are linear in the fermionic variables, the first trace is usually related to the supersymmetry charge, $\text{Tr} L = Q_1$. It is interesting to study whether the higher traces $\text{Tr} L^n$, with $n > 1$, factorize as the products of (Q_1, Q_2, Q_3) , or an alternative Lax pair can be built for each member of the triplet (Q_1, Q_2, Q_3) . A related issue is how the Lax pairs acting in the bosonic and fermionic sectors transform under the $\mathcal{N} = 1$ supersymmetry transformations.

An extension of the present analysis to the case of more than three interacting (super)particles is worth studying as well. It is intriguing to see whether the construction of n supercharges can be reduced to purely algebraic problem of building n vectors, all scalar products of which link to n first integrals characterizing the original bosonic model. Note, however, that examples are known in the literature, when integrability essentially depends on

the number of particles. The classic instance is the system of n point vortices on a plane, which is integrable for $n = 1, 2, 3$ only [14].

From the Lie-theoretic standpoint, the Ruijsenaars-Schneider n -body systems are built upon root vectors of the simple Lie algebra \mathcal{A}_{n-1} . Integrable generalizations, which link to root vectors of other classical Lie algebras, were proposed in [15]. The construction of supersymmetric extensions of the models in [15] and the study of their integrability is an interesting avenue to explore.

Apart from root vectors underlying the classical Lie algebras, one can also consider deformed root systems (see reviews [16, 17] and references therein). For example, a deformation of the \mathcal{A}_2 root system amounts to keeping x_{12} intact and changing x_{13} and x_{23} as follows

$$x_{13} \rightarrow x_1 - \sqrt{m}x_3, \quad x_{23} \rightarrow x_2 - \sqrt{m}x_3,$$

where m is a real deformation parameter. Interestingly enough, integrability of the deformed models of the Calogero type relies upon specific rational/trigonometric identities which look akin to those revealed in this paper.⁷ Notably, the relations (2.14), (3.7), and (4.5) continue to hold true after the deformation was implemented. To the best of our knowledge, integrability of the Ruijsenaars-Schneider-type models involving the deformed \mathcal{A}_{n-1} root system has not yet been established. A detailed analysis of this issue as well as the study of possible supersymmetric extensions represent interesting open problems to tackle.

A generalization of the present research to encompass various supersymmetric extensions of the Calogero model is worth studying as well. In the latter regard, the similarity transformation in [18] might prove helpful.

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⁷The author thanks an anonymous JHEP reviewer for drawing his attention to this fact as well as for revealing the review [16].

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