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On the high-energy behavior of massive QCD amplitudes

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ABSTRACT: In this note, we propose a factorization formula for gauge-theory scattering amplitudes up to two loops in the high-energy boosted limit. Our formula extends existing results in the literature by incorporating the contributions from massive loops. We derive the new ingredients in our formula using the method of regions with analytic regulators for the rapidity divergences. We verify our results with various form factors and the scattering amplitudes for top-quark pair production. Our results can be used to obtain approximate expressions for complicated two-loop massive amplitudes from simpler massless ones, and can be used to resum the mass logarithms to all orders in the coupling constant.

KEYWORDS: Factorization, Renormalization Group, Higher-Order Perturbative Calculations, Resummation

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1 Introduction

It is well-known that in the high-energy limit, perturbative scattering amplitudes and form factors in gauge theories admit certain factorization properties at the leading power of m/E [1–7], where m denotes the typical mass scale of internal and external particles, and E denotes the typical energy scale of the hard scattering. In quantum chromodynamics (QCD), such a factorization formula up to the two-loop order has been presented in [2]. However, certain contributions from bubble insertions of heavy-quark loops were not incorporated (hereafter we will refer to these contributions as the “ n_h -contributions”). These contributions were considered in quantum electrodynamics (QED) [3], where it was shown that a soft function is needed to account for the soft photon exchanges with a heavy-fermion bubble insertion. Similar contributions to the factorization of differential cross sections have been investigated in $e^+e^- \rightarrow \text{hardons}$ [8, 9]. In the context of heavy-to-light form factors, a similar factorization formula has been presented in [6]. However, as far as we know, there has been no generic treatment of such contributions in non-Abelian gauge theories with an arbitrary number of massive flavors (whose masses can be different). The purpose of this note is to bridge this gap and provide a generic factorization formula that applies to any scattering amplitude up to two loops in the high-energy limit. This can be used to obtain approximate results for complicated two-loop massive amplitudes from the corresponding much simpler massless ones. With suitable renormalization group equations (RGEs), the factorization formula can also be used to resum the large logarithms $\ln^n(m/E)$ to all orders in the coupling constant.

2 The factorization formula

We consider the perturbative scattering amplitude of an arbitrary number of massless or massive partons, with possible colorless particles in addition. In the highly boosted limit

where the energy scale of the hard-scattering process is much larger than the masses of external and internal partons, the scattering amplitude can be factorized as¹

$$\left| \mathcal{M}^{\text{massive}}(\{p\}, \{m\}) \right\rangle = \prod_i \left(\mathcal{Z}_{[i]}^{(m|0)}(\{m\}) \right)^{1/2} \mathcal{S}(\{p\}, \{m\}) \left| \mathcal{M}^{\text{massless}}(\{p\}) \right\rangle, \quad (2.1)$$

where $\{p\}$ denotes the set of all external momenta; $\{m\}$ denotes the set of all parton masses involved in this scattering process; and i runs over all external partons. Note that we have suppressed the dependence of the above functions on the dimensional regulator ϵ and the renormalization scale μ . The \mathcal{Z} -factors can be expanded in α_s as

$$\mathcal{Z}_{[i]}^{(m|0)}(\{m\}) = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^n \mathcal{Z}_{[i]}^{(n)}. \quad (2.2)$$

The coefficients $\mathcal{Z}_{[i]}^{(1)}$ and $\mathcal{Z}_{[i]}^{(2)}$ have been given in [2], up to terms proportional to $n_h^1 n_l^0$, where n_h is the number of heavy flavors, and n_l is the number of massless flavors. The soft function \mathcal{S} also contains $n_h^1 n_l^0$ contributions starting from the second order in α_s . Its presence has been proposed in [3] for QED processes, but has not been extended to more general non-Abelian gauge theories. The purpose of this note is to provide these missing ingredients in the factorization formula up to order α_s^2 .

For illustration, we will take the theory as QCD with all fermions (i.e., quarks) in the fundamental representation, although our results can be easily adapted to any other gauge theories with fermions in any representations. We assume that there are several heavy flavors in the theory, labelled by the index h , with the mass m_h . Up to the second order, the soft function is given by

$$\mathcal{S}(\{p\}, \{m\}) = 1 + \left(\frac{\alpha_s}{4\pi} \right)^2 \sum_{\substack{i,j \\ i \neq j}} (-\mathbf{T}_i \cdot \mathbf{T}_j) \sum_h \mathcal{S}^{(2)}(s_{ij}, m_h^2) + \mathcal{O}(\alpha_s^3), \quad (2.3)$$

where i and j run over all external legs; h runs over all heavy flavors; and

$$\mathcal{S}^{(2)}(s_{ij}, m_h^2) = T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left(-\frac{4}{3\epsilon^2} + \frac{20}{9\epsilon} - \frac{112}{27} - \frac{4\zeta_2}{3} \right) \ln \frac{-s_{ij}}{m_h^2}. \quad (2.4)$$

The boldface \mathbf{T}_i is the color generator for the external parton i which is an operator in the color space [10, 11]. For a final-state quark or an initial-state anti-quark, $(\mathbf{T}_i^a)_{\alpha\beta} = t_{\alpha\beta}^a$; for a final-state anti-quark or an initial-state quark, $(\mathbf{T}_i^a)_{\alpha\beta} = -t_{\beta\alpha}^a$; and for a gluon, $(\mathbf{T}_i^a)_{bc} = -if^{abc}$. The dot product is $\mathbf{T}_i \cdot \mathbf{T}_j \equiv \mathbf{T}_i^a \mathbf{T}_j^a$ with repeated indices summed over.

The \mathcal{Z} -factor for massless quarks starts at α_s^2 , and is given by

$$\mathcal{Z}_{[q]}^{(2)} = \sum_h C_F T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left[\frac{2}{\epsilon^3} + \frac{8}{9\epsilon^2} - \frac{1}{\epsilon} \left(\frac{65}{27} + \frac{2\zeta_2}{3} \right) + \frac{875}{54} + \frac{16\zeta_2}{3} - \frac{20\zeta_3}{3} \right]. \quad (2.5)$$

¹If some of the parton masses are of the same order as the hard scale, the factorization formula needs to be slightly modified. This contribution should be considered as part of the hard-scattering amplitude. See, e.g., [6] in the context of heavy-to-light form factors. We don't consider this case in this work.

The one-loop \mathcal{Z} -factor for gluons has been given in [2], and reads

$$\mathcal{Z}_{[g]}^{(1)} = \sum_h T_F \left(\frac{\mu^2}{m_h^2} \right)^\epsilon \left(-\frac{4}{3\epsilon} - \frac{2\zeta_2}{3}\epsilon + \frac{4\zeta_3}{9}\epsilon^2 \right). \quad (2.6)$$

The two-loop coefficient can be split into three parts:

$$\mathcal{Z}_{[g]}^{(2)} = \left(\mathcal{Z}_{[g]}^{(1)} \right)^2 + \frac{4}{3\epsilon} (n_l + n_h) T_F \mathcal{Z}_{[g]}^{(1)} + \sum_h \mathcal{Z}_{[g]}^{(2),h}, \quad (2.7)$$

where

$$\begin{aligned} \mathcal{Z}_{[g]}^{(2),h} = & C_A T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left[\frac{2}{\epsilon^3} + \frac{34}{9\epsilon^2} - \frac{2}{\epsilon} \left(\frac{22}{9} \ln \frac{\mu^2}{m_h^2} + \frac{64}{27} - \zeta_2 \right) \right. \\ & \left. + \frac{22}{9} \ln^2 \frac{\mu^2}{m_h^2} + \frac{358}{27} + \frac{4\zeta_2}{3} - 4\zeta_3 \right] \\ & - C_F T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left(\frac{2}{\epsilon} + 15 \right). \end{aligned} \quad (2.8)$$

Finally, we discuss the \mathcal{Z} -factor for a heavy flavor Q . The one-loop coefficient has again been given in [2]. The two-loop coefficient can be split into three parts:

$$\mathcal{Z}_{[Q]}^{(2)} = \mathcal{Z}_{[Q]}^{(2),l} + \mathcal{Z}_{[Q]}^{(2),Q} + \sum_{h \neq Q} \mathcal{Z}_{[Q]}^{(2),h}, \quad (2.9)$$

where $\mathcal{Z}_{[Q]}^{(2),l}$ contains contributions from gluon loops and light-quark loops; $\mathcal{Z}_{[Q]}^{(2),Q}$ denotes the contribution from a loop insertion of Q itself; and $\mathcal{Z}_{[Q]}^{(2),h}$ denotes the contribution from a loop insertion of heavy flavors other than Q . We do not give the explicit expressions for $\mathcal{Z}_{[Q]}^{(1)}$ and $\mathcal{Z}_{[Q]}^{(2),l}$ here since they are n_h -independent, and refer the readers to [2]. The new ingredients $\mathcal{Z}_{[Q]}^{(2),Q}$ and $\mathcal{Z}_{[Q]}^{(2),h}$ are given by

$$\begin{aligned} \mathcal{Z}_{[Q]}^{(2),Q} = & C_F T_F \left[\frac{2}{\epsilon^3} + \frac{1}{\epsilon^2} \left(\frac{4}{3} \ln \frac{\mu^2}{m_Q^2} + \frac{8}{9} \right) + \frac{1}{\epsilon} \left(\frac{4}{9} \ln \frac{\mu^2}{m_Q^2} - \frac{65}{27} - 2\zeta_2 \right) \right. \\ & \left. - \frac{4}{9} \ln^3 \frac{\mu^2}{m_Q^2} - \frac{2}{9} \ln^2 \frac{\mu^2}{m_Q^2} - \left(\frac{274}{27} + \frac{16\zeta_2}{3} \right) \ln \frac{\mu^2}{m_Q^2} + \frac{5107}{162} - \frac{70\zeta_2}{9} - \frac{4\zeta_3}{9} \right], \quad (2.10) \\ \mathcal{Z}_{[Q]}^{(2),h} = & \mathcal{Z}_{[Q]}^{(2),Q} + C_F T_F \left\{ -\frac{8}{3\epsilon^2} H(0, x) + \frac{1}{\epsilon} \left[\frac{64}{3} H(0, 0, x) - \left(\frac{8}{9} + \frac{16}{3} \ln \frac{\mu^2}{m_Q^2} \right) H(0, x) \right] \right. \\ & + \frac{332x^2}{9} - \frac{332}{9} - 128H(0, 0, 0, x) + \left(\frac{128}{3} \ln \frac{\mu^2}{m_Q^2} + 48x^4 - \frac{128}{9} \right) H(0, 0, x) \\ & + \left(-\frac{16}{3} \ln^2 \frac{\mu^2}{m_Q^2} - \frac{16}{9} \ln \frac{\mu^2}{m_Q^2} + \frac{224x^2}{9} - \frac{8\zeta_2}{3} + \frac{16}{27} \right) H(0, x) \\ & \left. + \left(24x^4 - \frac{220x^3}{3} + 36x + \frac{8}{9} \right) \zeta_2 - \frac{16\zeta_3}{3} + \frac{32}{3} H(0, -1, 0, x) - \frac{32}{3} H(0, 1, 0, x) \right\} \end{aligned}$$

$$\begin{aligned}
 & + \left(24x^4 - \frac{220x^3}{9} + 12x - \frac{104}{9} \right) H(1, 0, x) \\
 & - \left(24x^4 + \frac{220x^3}{9} - 12x - \frac{104}{9} \right) H(-1, 0, x),
 \end{aligned} \tag{2.11}$$

where $x = m_h/m_Q$ and H denotes the harmonic polylogarithms (HPLs) [12]. It is easy to check that $\mathcal{Z}_{[Q]}^{(2),h} = \mathcal{Z}_{[Q]}^{(2),Q}$ when $x = 1$, i.e., $m_h = m_Q$.

3 Derivation and validation

In this section, we show how to derive the n_h -contributions to the \mathcal{Z} -factors and the soft function, and demonstrate the validity of our factorization formula in the context of various form factors and the scattering amplitudes for $t\bar{t}$ production.

3.1 The derivation

In the highly-boosted limit, the scattering amplitudes can be described in the soft-collinear effective theory [13–15]. For each external leg i , we introduce a light-like vector n_i^μ with space-components along the same direction of p_i , and a light-like vector \bar{n}_i^μ opposite to p_i . They satisfy

$$n_i^2 = \bar{n}_i^2 = 0, \quad n_i \cdot \bar{n}_i = 2. \tag{3.1}$$

We define a small expansion parameter $\lambda \sim m/\sqrt{|s|}$, where m is a representative mass scale, and s is a representative of s_{ij} . The relevant momentum modes are

$$\begin{aligned}
 \text{hard} : k^\mu & \sim \sqrt{|s|}, \\
 n_i\text{-collinear} : (n_i \cdot k, \bar{n}_i \cdot k, k_\perp) & \sim \sqrt{|s|} (\lambda^2, 1, \lambda), \\
 \text{soft} : k^\mu & \sim \sqrt{|s|} \lambda.
 \end{aligned}$$

The contributions from the hard region correspond to the massless amplitudes $\mathcal{M}^{\text{massless}}$, while the collinear and soft regions produce the \mathcal{Z} -factors and the soft function \mathcal{S} in eq. (2.1).

We are concerned with the two-loop contributions to \mathcal{Z} and \mathcal{S} with a massive bubble insertion, which involve the double-collinear integrals and the double-soft integrals, depicted in figure 1. Note that without the massive bubble, the relevant integrals are well-defined, which give rise to the non- n_h terms in the \mathcal{Z} -factors (the soft integrals without the massive bubble are scaleless and vanish). However, with the massive bubble insertion, these integrals become rapidity-divergent (similar as the Sukadov problem with a massive vector boson [16, 17]), and one needs to introduce an additional regulator. To this end we employ the analytic regulator of [16, 18–20]. With this regulator, the double-soft contributions vanish, and we only need to consider the double-collinear integrals.

We will need to perform the calculations for external massless-quarks, external massive-quarks and external gluons. For convenience, we work with the two-point quark-anti-quark vector form factors and the gluon-gluon scalar form factor with outgoing external momenta p_1 and p_2 . In the center-of-mass frame, the relevant light-like directions are $n^\mu = n_1^\mu = (1, 0, 0, 1)$

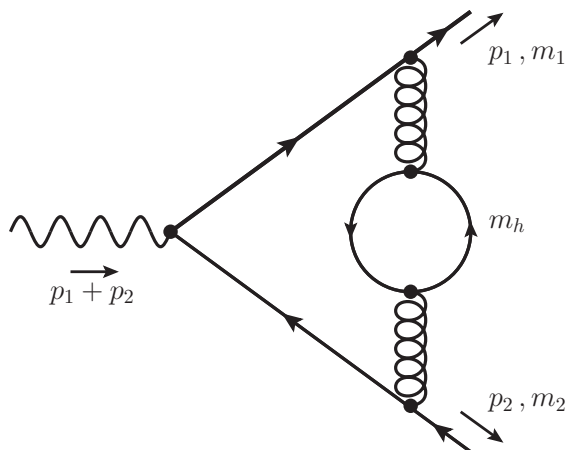


Figure 1. Two-loop Feynman diagram with a massive bubble for the $q\bar{q}$ -vector vertex.

and $\bar{n}^\mu = n^\mu = (1, 0, 0, -1)$. In this frame, p_1 and p_2 have no perpendicular components, and can be written as

$$p_1^\mu = p_1^+ \frac{\bar{n}^\mu}{2} + p_1^- \frac{n^\mu}{2}, \quad p_2^\mu = p_2^+ \frac{\bar{n}^\mu}{2} + p_2^- \frac{n^\mu}{2}, \quad (3.2)$$

where $p_i^+ \equiv n \cdot p_i$ and $p_i^- \equiv \bar{n} \cdot p_i$. In the counting of (3.2), we have $p_1^+ \sim p_2^- \sim \lambda^2 \sqrt{|s|}$ and $p_1^- \sim p_2^+ \sim \sqrt{|s|}$. By convention, we refer to the n -collinear modes as “collinear” and the \bar{n} -collinear modes as “anti-collinear”, respectively. We set $p_1^2 = m_1^2$, $p_2^2 = m_2^2$ and $(p_1 + p_2)^2 = s$. In the $q\bar{q}$ case, m_1 and m_2 can be zero or non-zero, while they are always zero in the gg case. The mass of the internal quark is taken to be m_h .

We first discuss the $q\bar{q}$ case with $m_1 = m_2 = m_Q$. Here, the $m_Q = m_h$ case has been considered in ref. [21], and we will consider the more general case where m_h and m_Q can be different. The $q\bar{q}$ -vector vertex can be decomposed as

$$\Gamma^\mu(p_1, p_2) = F_1(s, m_Q^2, m_h^2) \gamma^\mu + \frac{1}{2m_Q} F_2(s, m_Q^2, m_h^2) i \sigma^{\mu\nu} (p_1 + p_2)_\nu, \quad (3.3)$$

where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. The form factors F_i can be extracted by applying a set of projectors given in [21], and be expressed as linear combinations of scalar Feynman integrals. These Feynman integrals can be defined in the following generic form

$$I_{\{a_i\}} \equiv \mu^{4\epsilon} \int \frac{dk_1}{(2\pi)^d} \frac{dk_2}{(2\pi)^d} \frac{1}{[k_1^2 - m_h^2]^{a_1}} \frac{1}{[k_2^2 - m_h^2]^{a_2}} \frac{1}{[(k_1 + k_2)^2]^{a_3}} \frac{1}{[(k_1 + k_2 - p_1)^2 - m_1^2]^{a_4}} \\ \times \frac{(-\tilde{\mu}^2)^\nu}{[(k_1 + k_2 + p_2)^2 - m_2^2]^{a_5 + \nu}} \frac{1}{[(k_1 - p_1)^2]^{a_6}} \frac{1}{[(k_1 + p_2)^2]^{a_7}}, \quad (3.4)$$

where ν is the rapidity regulator and $\tilde{\mu}$ is the corresponding scale introduced to balance the mass dimension. In our calculation, the last two propagators are actually irreducible scalar products (ISPs) with $a_6, a_7 \leq 0$. We need to calculate the above integrals in the double-collinear (cc) and double-anti-collinear ($\bar{c}\bar{c}$) regions (recall that the soft region integrals vanish). Note that $F_2 = 0$ at the leading power (LP) in λ , so we only need to calculate the

form factor F_1 . The contributions from the cc and $\bar{c}\bar{c}$ regions are given by

$$\begin{aligned}
 F_{1,cc}^{(2),\text{bare}}(s, m_Q^2, m_h^2) &= C_F T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left(\frac{1}{\nu} + \ln \frac{\tilde{\mu}^2}{-s} \right) \left(\frac{4}{3\epsilon^2} - \frac{20}{9\epsilon} + \frac{112}{27} + \frac{4\zeta_2}{3} \right) \\
 &+ C_F T_F \left(\frac{\mu^2}{m_Q^2} \right)^{2\epsilon} \left\{ -\frac{4}{3\epsilon^3} - \frac{1}{\epsilon^2} \left[\frac{4}{3} - \frac{8}{3} H(0, x) \right] - \frac{1}{\epsilon} \left[\frac{8\zeta_2}{3} + \frac{56}{9} + \frac{16}{3} H(0, 0, x) \right] \right. \\
 &- \frac{4\zeta(3)}{9} - \left(\frac{20x^3}{3} - 36x + \frac{80}{9} \right) \zeta_2 + \frac{40x^2}{9} - \frac{2504}{81} + \left(\frac{40x^2}{9} + \frac{8\zeta_2}{3} - \frac{224}{27} \right) H(0, x) \\
 &- \left(\frac{20x^3}{9} - 12x - \frac{88}{9} \right) H(-1, 0, x) - \left(\frac{20x^3}{9} - 12x + \frac{88}{9} \right) H(1, 0, x) - \frac{80}{9} H(0, 0, x) \\
 &\left. + \frac{16}{3} H(0, -1, 0, x) - \frac{16}{3} H(0, 1, 0, x) \right\}, \tag{3.5}
 \end{aligned}$$

$$\begin{aligned}
 F_{1,\bar{c}\bar{c}}^{(2),\text{bare}}(s, m_Q^2, m_h^2) &= C_F T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left(\frac{1}{\nu} + \ln \frac{\tilde{\mu}^2}{m_h^2} \right) \left(-\frac{4}{3\epsilon^2} + \frac{20}{9\epsilon} - \frac{112}{27} - \frac{4\zeta_2}{3} \right) \\
 &+ C_F T_F \left(\frac{\mu^2}{m_Q^2} \right)^{2\epsilon} \left\{ \frac{2}{3\epsilon^3} - \frac{1}{\epsilon^2} \left[\frac{28}{9} + \frac{16}{3} H(0, x) \right] - \frac{1}{\epsilon} \left[\frac{2\zeta_2}{3} + \frac{212}{27} - \frac{64}{9} H(0, x) \right. \right. \\
 &- \left. \frac{80}{3} H(0, 0, x) \right] - \frac{40\zeta_3}{9} - \left(\frac{20x^3}{3} - 36x + \frac{32}{3} \right) \zeta_2 + \frac{40x^2}{9} - \frac{1652}{81} \\
 &+ \left(\frac{40x^2}{9} - \frac{16\zeta_2}{3} - \frac{16}{9} \right) H(0, x) - \frac{112}{3} H(0, 0, x) - \left(\frac{20x^3}{9} - 12x - \frac{88}{9} \right) H(-1, 0, x) \\
 &- \left(\frac{20x^3}{9} - 12x + \frac{88}{9} \right) H(1, 0, x) + \frac{16}{3} H(0, -1, 0, x) - 128 H(0, 0, 0, x) \\
 &\left. - \frac{16}{3} H(0, 1, 0, x) \right\}. \tag{3.6}
 \end{aligned}$$

Note that the exchange symmetry between the cc and $\bar{c}\bar{c}$ regions are broken since we have only introduced the analytic regulator on the propagator associated with m_2 . It is possible to obtain a more symmetric result by introducing two regulators ν_1 and ν_2 for the two propagators associated with m_1 and m_2 . This makes the calculations and the intermediate results more complicated, with the same final results for the \mathcal{Z} -factors and the soft function. Hence, we choose the simpler regularization scheme here and in the following.

The above bare cc and $\bar{c}\bar{c}$ contributions contain rapidity divergences in the form $1/\nu$, as well as ultraviolet (UV) and infrared (IR) divergences as poles in ϵ . When adding together, the $1/\nu$ singularities cancel, along with logarithms of $\tilde{\mu}$. However, there remains a term proportional to $\ln(-s/m_h^2)$. This give rise to the two-loop soft function $\mathcal{S}^{(2)}(s, m_h^2)$ defined in eq. (2.4). Note that a similar soft function has been given in ref. [3] in QED, with a slight difference: the overall factor $\ln(-s/m_h^2)$ in (2.4) is replaced by $\ln(-s/m_Q^2)$ in eq. (19) of ref. [3] (converted to our notation). We will see that our convention (2.4) is more convenient as it remains the same for massless external legs.

After removing the rapidity divergent parts, we can perform the UV renormalization and extract $\mathcal{Z}_{[Q]}^{(2),h}$. The UV-renormalized form factor is given by

$$F_1(s, m_Q^2, m_h^2) = Z_Q \left[F_1^{\text{bare}}(s, m_Q^2, m_h^2) \Big|_{\alpha_s^{\text{bare}} \rightarrow Z_{\alpha_s} \alpha_s} \right], \tag{3.7}$$

where Z_{α_s} is the $\overline{\text{MS}}$ renormalization constant for the strong coupling α_s , and Z_Q is the on-shell field renormalization constant for the massive quark. Neither the one-loop $Z_Q^{(1)}$ nor the one-loop $F_1^{(1),\text{bare}}$ depends on massive quark loops. Hence, for the n_h -contributions, we only need the n_h -dependent parts of $Z_Q^{(2)}$ and $Z_{\alpha_s}^{(1)}$. We collect these renormalization constants in appendix A. Finally, we have

$$\begin{aligned} \mathcal{Z}_{[Q]}^{(2),h} &= F_{1,cc}^{(2),\text{bare}}(s, m_Q^2, m_h^2) + F_{1,\bar{c}\bar{c}}^{(2),\text{bare}}(s, m_Q^2, m_h^2) \\ &\quad + Z_{\alpha_s}^{(1),h} \left[F_{1,c}^{(1),\text{bare}}(s, m_Q^2) + F_{1,\bar{c}}^{(1),\text{bare}}(s, m_Q^2) \right] + Z_Q^{(2)} - C_F \mathcal{S}^{(2)}(s, m_h^2), \end{aligned} \quad (3.8)$$

where the one-loop collinear and anti-collinear contributions $F_{1,c/\bar{c}}^{(1),\text{bare}}$ can be found in ref. [21]. Putting everything together, we arrive at the expression in eq. (2.11).

We now move to the case where both external legs are massless, i.e., $m_1 = m_2 = 0$. The contributions from cc and $\bar{c}\bar{c}$ regions are given by

$$\begin{aligned} F_{1,cc}^{(2),\text{bare}}(s, 0, m_h^2) &= C_F T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left(\frac{1}{\nu} + \ln \frac{\tilde{\mu}^2}{-s} \right) \left(\frac{4}{3\epsilon^2} - \frac{20}{9\epsilon} + \frac{112}{27} + \frac{4\zeta_2}{3} \right) \\ &\quad + C_F T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left[\frac{4}{3\epsilon^2} - \frac{1}{\epsilon} \left(\frac{8}{9} + \frac{4\zeta_2}{3} \right) + \frac{88}{27} + \frac{32\zeta_2}{9} - \frac{4\zeta_3}{3} \right], \end{aligned} \quad (3.9)$$

$$\begin{aligned} F_{1,\bar{c}\bar{c}}^{(2),\text{bare}}(s, 0, m_h^2) &= C_F T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left(\frac{1}{\nu} + \ln \frac{\tilde{\mu}^2}{m_h^2} \right) \left(-\frac{4}{3\epsilon^2} + \frac{20}{9\epsilon} - \frac{112}{27} - \frac{4\zeta_2}{3} \right) \\ &\quad + C_F T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left[\frac{2}{\epsilon^3} - \frac{4}{9\epsilon^2} - \frac{1}{\epsilon} \left(\frac{68}{27} - \frac{2\zeta_2}{3} \right) + \frac{124}{9} + \frac{16\zeta_2}{9} - \frac{16\zeta_3}{3} \right]. \end{aligned} \quad (3.10)$$

Note that the rapidity divergent parts of eqs. (3.9) and (3.10) are the same as those of eqs. (3.5) and (3.6), respectively. Hence, we may extract the same two-loop soft function $\mathcal{S}^{(2)}(s, m_h^2)$ here. After that, we can obtain the \mathcal{Z} -factor for massless quarks as

$$\mathcal{Z}_{[q]}^{(2),h} = F_{1,cc}^{(2),\text{bare}}(s, 0, m_h^2) + F_{1,\bar{c}\bar{c}}^{(2),\text{bare}}(s, 0, m_h^2) + Z_q^{(2)} - C_F \mathcal{S}^{(2)}(s, m_h^2), \quad (3.11)$$

where $Z_q^{(2)}$ is the two-loop on-shell wave function renormalization constant for the massless quark q . Here we have used the fact that the one-loop collinear and anti-collinear contributions $F_{1,c/\bar{c}}^{(1),\text{bare}}(s, 0, m_h^2)$ are scaleless and vanish. The above expression leads to eq. (2.5).

The cases where $m_1 \neq m_2$ can be similarly dealt with. In fact, it is easy to see that the double-collinear integrals are independent of m_2 . In this region, the 5th and 7th propagators in eq. (3.4) need to be expanded as

$$-(k_1 + k_2 + p_2)^2 + m_2^2 \rightarrow -\bar{n} \cdot (k_1 + k_2) n \cdot p_2, \quad (k_1 + p_2)^2 \rightarrow \bar{n} \cdot k_1 n \cdot p_2. \quad (3.12)$$

The integrals therefore do not involve $p_2^2 = m_2^2$. Therefore, the double-collinear contributions simply leads to $F_{1,cc}^{(2),\text{bare}}(s, m_1^2, m_h^2)$. Its expression can be taken from eq. (3.5) or (3.9), depending on whether m_1 is zero or not. Similarly, the double-anti-collinear integrals are independent of m_1^2 and lead to $F_{1,\bar{c}\bar{c}}^{(2),\text{bare}}(s, m_2^2, m_h^2)$. It is then clear that the same soft function $\mathcal{S}^{(2)}(s, m_h^2)$ is produced after cancelling the rapidity divergences. The remaining terms coincide with the sum $(\mathcal{Z}_{[1]}^{(2),h} + \mathcal{Z}_{[2]}^{(2),h})/2$.

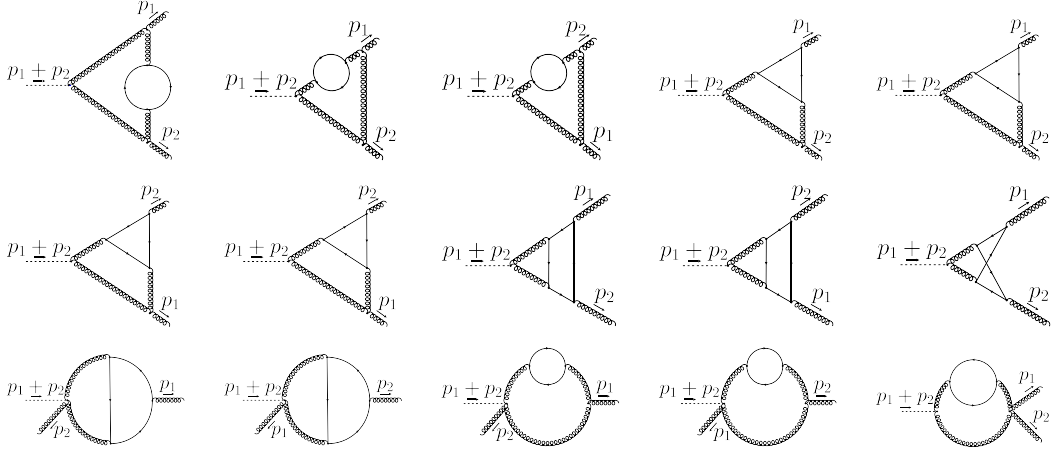


Figure 2. Two-loop Feynman diagrams involving a massive quark loop for the gluon form factor.

Finally, we apply the same methodology to the gluon case and derive $\mathcal{Z}_{[g]}^{(2),h}$. We start from the gluon scalar form factor generated from the effective Lagrangian

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4} H G_a^{\mu\nu} G_{a,\mu\nu}. \quad (3.13)$$

From the effective H - $g(p_1)$ - $g(p_2)$ vertex $\Gamma_{gg}^{\mu\nu}$, we project out the scalar form factor as

$$F_{gg} = \frac{p_1 \cdot p_2 g_{\mu\nu} - p_{1,\mu} p_{2,\nu} - p_{1,\nu} p_{2,\mu}}{2(1-\epsilon)} \Gamma_{gg}^{\mu\nu}. \quad (3.14)$$

We use QGRAF [22] to generate the relevant two-loop Feynman diagrams involving a massive quark loop, shown in figure 2. We then use FORM [23] to manipulate the resulting expressions. In the gluon cases, there are quark-box and quark-triangle insertions besides the bubble insertions. Using the method of regions, we find that, in the high-energy limit at LP, the diagrams with a quark-box insertion do not contribute in the cc and $\bar{c}\bar{c}$ regions. To determine $\mathcal{Z}_{[g]}^{(2),h}$, we therefore only need to consider the contributions from the quark-triangle and bubble insertions. The scalar integrals appearing in these diagrams can be expressed in the following generic form

$$I_{\{a_i\}}^A \equiv \mu^{4\epsilon} \int \frac{dk_1}{(2\pi)^d} \frac{dk_2}{(2\pi)^d} \frac{1}{[k_1^2 - m_h^2]^{a_1}} \frac{1}{[k_2^2 - m_h^2]^{a_2}} \frac{1}{[(k_1 + k_2)^2]^{a_3}} \frac{1}{[(k_1 + k_2 - p_1)^2]^{a_4}} \\ \times \frac{(-\tilde{\mu}^2)^\nu}{[(k_1 + k_2 + p_2)^2]^{a_5 + \nu}} \frac{1}{[(k_1 - p_1)^2 - m_h^2]^{a_6}} \frac{1}{[(k_1 + p_2)^2 - m_h^2]^{a_7}}. \quad (3.15)$$

Note that only the first diagram in figure 2 contains rapidity divergences. Hence, we may set $\nu \rightarrow 0$ from the beginning for the other diagrams. Only the cc and $\bar{c}\bar{c}$ regions are relevant for $\mathcal{Z}_{[g]}^{(2),h}$, and their contributions are given by

$$F_{gg,cc}^{(2),\text{bare}}(s, m_h^2) = C_A T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left(\frac{1}{\nu} + \ln \frac{\tilde{\mu}^2}{-s} \right) \left(\frac{4}{3\epsilon^2} - \frac{20}{9\epsilon} + \frac{112}{27} + \frac{4\zeta_2}{3} \right) \\ + C_A T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left[\frac{5}{6\epsilon^2} - \frac{11}{36\epsilon} + \frac{179}{216} + \frac{5\zeta_2}{6} \right], \quad (3.16)$$

$$\begin{aligned}
 F_{gg,\bar{c}\bar{c}}^{(2),\text{bare}}(s, m_h^2) &= C_A T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left(\frac{1}{\nu} + \ln \frac{\tilde{\mu}^2}{m_h^2} \right) \left(-\frac{4}{3\epsilon^2} + \frac{20}{9\epsilon} - \frac{112}{27} - \frac{4\zeta_2}{3} \right) \\
 &+ C_A T_F \left(\frac{\mu^2}{m_h^2} \right)^{2\epsilon} \left[\frac{2}{\epsilon^3} - \frac{17}{18\epsilon^2} - \frac{1}{\epsilon} \left(\frac{209}{108} - 2\zeta_2 \right) + \frac{817}{72} - \frac{17\zeta_2}{18} - 4\zeta_3 \right]. \quad (3.17)
 \end{aligned}$$

Note that the rapidity divergent parts of eqs. (3.16) and (3.17) are the same as those of eqs. (3.5) and (3.6), respectively, with the color factor $C_F \rightarrow C_A$. We may then extract the same two-loop soft function $\mathcal{S}^{(2)}(s, m_h^2)$ here. After that, we extract the \mathcal{Z} -factor for gluon according to

$$\mathcal{Z}_{[g]}^{(2),h} = F_{gg,cc}^{(2),\text{bare}}(s, m_h^2) + F_{gg,\bar{c}\bar{c}}^{(2),\text{bare}}(s, m_h^2) + Z_g^{(2)} - C_A \mathcal{S}^{(2)}(s, m_h^2), \quad (3.18)$$

where $Z_g^{(2)}$ is the two-loop on-shell renormalization constant for gluon fields. The above expression leads to eq. (2.8).

3.2 The quark form factors

We now turn to the validation of our factorization formula (2.1) in the context of quark form factors. We will only be concerned with the n_h -dependent part, since it is the new result of this work. On the right-hand side of (2.1), we need the two-loop massless quark form factor with a massless quark-bubble insertion. This can be extracted from [24, 25], and is given by

$$\begin{aligned}
 F_{1,q\bar{q}}^{(2),l}(s) &= C_F T_F \left[-\frac{2}{\epsilon^3} + \frac{1}{\epsilon^2} \left(\frac{4L_s}{3} - \frac{8}{9} \right) + \frac{1}{\epsilon} \left(\frac{65}{27} - \frac{20L_s}{9} + 2\zeta_2 \right) \right. \\
 &\quad \left. - \frac{4L_s^3}{9} + \frac{38L_s^2}{9} - \frac{418L_s}{27} - \frac{8\zeta_2 L_s}{3} + \frac{4085}{162} + \frac{46\zeta_2}{9} + \frac{4\zeta_3}{9} \right], \quad (3.19)
 \end{aligned}$$

where $L_s = \ln(-s/\mu^2)$.

The massive-massive quark form factor with a massive quark-bubble insertion in the special case $m_1 = m_2 = m_h$ can be extracted from [26–28], where m_1 and m_2 denote the masses of two external quarks, and m_h is the mass of the internal bubble. The result reads

$$\begin{aligned}
 F_{1,Q\bar{Q}}^{(2),h}(s, m_h^2, m_h^2, m_h^2) &= C_F T_F \left[\frac{8}{3\epsilon} \left(L_m - L_m^2 - L_m L_s \right) - \frac{28L_m^3}{9} - \frac{4L_s^3}{9} - \frac{8L_m^2 L_s}{3} \right. \\
 &\quad \left. + \frac{38L_m^2}{9} + \frac{38L_s^2}{9} + \frac{40L_m L_s}{9} - \frac{20\zeta_2 L_m}{3} - 4\zeta_2 L_s \right. \\
 &\quad \left. - \frac{386L_m}{27} - \frac{530L_s}{27} + \frac{1532}{27} - \frac{8\zeta_2}{3} \right], \quad (3.20)
 \end{aligned}$$

where $L_m = \ln(\mu^2/m_h^2)$. It is straightforward to check that it satisfies the factorization formula:

$$F_{1,Q\bar{Q}}^{(2),h}(s, m_h^2, m_h^2, m_h^2) = F_{1,q\bar{q}}^{(2),l}(s) + C_F \mathcal{S}^{(2)}(s, m_h^2) + \mathcal{Z}_{[Q]}^{(2),h}(m_h^2, m_h^2). \quad (3.21)$$

We now turn to the massive-massless quark form factor with a massive quark-bubble insertion, in the special case $m_1 = m_h$. This can be extracted from [29–32], and the

expression is

$$\begin{aligned}
F_{Q\bar{q}}^{(2),h}(s, m_h^2, m_h^2) = C_F T_F & \left[\frac{4L_m}{3\epsilon^2} + \frac{1}{\epsilon} \left(\frac{10L_m}{3} - \frac{2L_m^2}{3} - \frac{8L_m L_s}{3} + \frac{2\zeta_2}{3} \right) \right. \\
& - \frac{14L_m^3}{9} - \frac{4L_s^3}{9} - \frac{8L_m^2 L_s}{3} + \frac{47L_m^2}{9} + \frac{38L_s^2}{9} + \frac{40L_m L_s}{9} \\
& \left. - \frac{14\zeta_2 L_m}{3} - 4\zeta_2 L_s - \frac{314L_m}{27} - \frac{530L_s}{27} + \frac{7951}{162} + \frac{35\zeta_2}{9} - \frac{28\zeta_3}{9} \right]. \quad (3.22)
\end{aligned}$$

The corresponding factorization formula can be verified to hold:

$$F_{Q\bar{q}}^{(2),h}(s, m_h^2, m_h^2) = F_{q\bar{q}}^{(2),l}(s) + C_F \mathcal{S}^{(2)}(s, m_h^2) + \frac{1}{2} \mathcal{Z}_{[Q]}^{(2),h}(m_h^2, m_h^2) + \frac{1}{2} \mathcal{Z}_{[q]}^{(2),h}(m_h^2). \quad (3.23)$$

Finally, we consider the two-loop massless-massless quark form factor with a massive-quark bubble insertion. This is not available in the literature, so we have to calculate that ourselves. There is only one diagram and it is straightforward to write down the loop integrands. For the loop integrals, we refer to the next sub-section, since they will also appear in the gluon-gluon form factors. The unrenormalized two-loop form factor is then given by

$$\begin{aligned}
F_{q\bar{q}}^{(2),h,\text{bare}} = C_F T_F & \left[\frac{8}{3\epsilon^3} + \frac{1}{\epsilon^2} \left(\frac{8L_m}{3} - \frac{8L_s}{3} + 4 \right) \right. \\
& + \frac{1}{\epsilon} \left(-\frac{8L_m L_s}{3} + \frac{4L_m^2}{3} + 4L_m + \frac{4L_s^2}{3} - 4L_s + \frac{29}{3} \right) \\
& - \frac{8}{3} L_m^2 L_s + \frac{40L_m L_s}{9} + \frac{56L_m^2}{9} - \frac{8\zeta_2 L_m}{3} - \frac{296L_m}{27} - \frac{8L_s^3}{9} \\
& \left. + \frac{56L_s^2}{9} - \frac{8\zeta_2 L_s}{3} - \frac{818L_s}{27} - \frac{112\zeta_3}{9} + \frac{76\zeta_2}{9} + \frac{10301}{162} \right]. \quad (3.24)
\end{aligned}$$

After renormalization, we get

$$\begin{aligned}
F_{q\bar{q}}^{(2),h}(s, m_h^2) = C_F T_F & \left[\frac{8L_m}{3\epsilon^2} + \frac{1}{\epsilon} \left(-\frac{8L_m L_s}{3} + \frac{4L_m^2}{3} + 4L_m + \frac{4\zeta_2}{3} \right) \right. \\
& - \frac{56\zeta_3}{9} - \frac{8}{3} L_m^2 L_s + \frac{40L_m L_s}{9} + \frac{56L_m^2}{9} - \frac{8\zeta_2 L_m}{3} \\
& \left. - \frac{242L_m}{27} - \frac{4L_s^3}{9} + \frac{38L_s^2}{9} - 4\zeta_2 L_s - \frac{530L_s}{27} + \frac{94\zeta_2}{9} + \frac{3355}{81} \right]. \quad (3.25)
\end{aligned}$$

Again, we find that it satisfies the factorization formula:

$$F_{q\bar{q}}^{(2),h}(s, m_h^2) = F_{q\bar{q}}^{(2),l}(s) + C_F \mathcal{S}^{(2)}(s, m_h^2) + \mathcal{Z}_{[q]}^{(2),h}(m_h^2). \quad (3.26)$$

3.3 The gluon form factor

For the gluon form factor, we again consider the diagrams in figure 2. We now need to calculate the integrals without region expansion. The integrals can be categorized into two different families. The first family (family A) covers all planar diagrams, and is defined in eq. (3.15). These integrals can be expressed in terms of multiple polylogarithms (MPLs),

and can be found in [33–36]. The second family (family B) corresponds to the non-planar diagram (the last one in the second row of figure 2), and is defined as

$$I_{\{a_i\}}^B \equiv \mu^{4\epsilon} \int \frac{dk_1}{(2\pi)^d} \frac{dk_2}{(2\pi)^d} \frac{1}{[k_1^2]^{a_1}} \frac{1}{[(k_1 - p_1 - p_2)^2]^{a_2}} \frac{1}{[k_2^2 - m_h^2]^{a_3}} \frac{1}{[(k_2 + p_1)^2 - m_h^2]^{a_4}} \\ \times \frac{1}{[(k_1 + k_2)^2 - m_h^2]^{a_5}} \frac{1}{[(k_1 + k_2 - p_2)^2 - m_h^2]^{a_6}} \frac{1}{[(k_2 + p_2)^2]^{a_7}}. \quad (3.27)$$

This family involves elliptic integrals. In the high-energy limit $|s| \gg m_h^2$, the asymptotic expressions for integrals in family B can be found in [37]. For the UV renormalization, we need Z_g for the gluon field, Z_{α_s} for the strong coupling, as well as Z_λ for the effective operator. These renormalization constants are collected in appendix A. After renormalization, the two-loop gluon form factor with a massive quark loop is given by

$$F_{gg}^{(2),h}(s, m_h^2) = \frac{1}{N_c} T_F \left(-8\zeta_3 + 2L_m - 2L_s + \frac{56}{3} \right) \\ + N_c T_F \left[\frac{16L_m}{3\epsilon^2} + \frac{1}{\epsilon} \left(-\frac{16L_m L_s}{3} + \frac{8L_m^2}{3} + \frac{88L_m}{9} + \frac{8\zeta_2}{3} \right) \right. \\ \left. - \frac{64\zeta_3}{9} - 4L_m^2 L_s + \frac{4}{3} L_m L_s^2 + \frac{40L_m L_s}{9} + \frac{4L_m^3}{9} + \frac{64L_m^2}{9} \right. \\ \left. + \frac{8\zeta_2 L_m}{3} - \frac{422L_m}{27} - \frac{4L_s^3}{9} + \frac{20L_s^2}{9} + \frac{46L_s}{27} + \frac{4\zeta_2}{9} - \frac{2270}{81} \right], \quad (3.28)$$

where N_c is the number of colors. Here, it is worth mentioning that the massive quark-loop contributions to $F_{gg}^{(2)}$ consist of three parts: the contribution from a single massive quark loop as shown above, the contribution from two massive quark loops (which we denote as the n_h^2 contribution), and the contribution from a massive quark loop and a massless quark loop (which we denote as the $n_h n_l$ contribution). The last two parts arise from renormalization, and are not included in the above expression. In the factorization formula, the n_h^2 and $n_h n_l$ contributions are taken care of by the $(\mathcal{Z}_{[g]}^{(1)})^2$ term and the $(n_l + n_h)\mathcal{Z}_{[g]}^{(1)}$ term in eq. (2.7), which were already known in [2].

For the purpose of this work, we only need to check the factorization property of $F_{gg}^{(2),h}$. For that we need the one-loop gluon form factor without quark loops, and the two-loop gluon form factor with a massless quark loop. They can be extracted from [25, 38] and are given by

$$F_{gg}^{(1),\text{no-quark}} = N_c T_F \left[-\frac{4}{\epsilon^2} + \frac{1}{\epsilon} \left(4L_s - \frac{22}{3} \right) + 2\zeta_2 - 2L_s^2 + \epsilon \left(\frac{28\zeta_3}{3} + \frac{2L_s^3}{3} - 2\zeta_2 L_s - 4 \right) \right. \\ \left. + \epsilon^2 \left(-\frac{28\zeta_3 L_s}{3} - \frac{L_s^4}{6} + \zeta_2 L_s^2 + 4L_s + \frac{47\zeta_4}{4} - 12 \right) \right], \quad (3.29)$$

$$F_{gg}^{(2),l} = \frac{1}{N_c} T_F \left(-\frac{1}{\epsilon} - 8\zeta_3 - 2L_s + \frac{67}{6} \right) \\ + N_c \left[-\frac{14}{3\epsilon^3} + \frac{1}{\epsilon^2} \left(4L_s - \frac{26}{3} \right) + \frac{1}{\epsilon} \left(-\frac{4L_s^2}{3} - \frac{20L_s}{9} + \frac{2\zeta_2}{3} + \frac{155}{27} \right) \right. \\ \left. + 4\zeta_3 + \frac{20L_s^2}{9} + \frac{4\zeta_2 L_s}{3} + \frac{158L_s}{27} - \frac{10\zeta_2}{3} - \frac{5905}{162} \right]. \quad (3.30)$$

It is then straightforward to check the factorization formula:

$$F_{gg}^{(2),h}(s, m_h^2) = F_{gg}^{(2),l}(s) + \mathcal{Z}_{[g]}^{(1)}(m_h^2) F_{gg}^{(1),\text{no-quark}}(s) + \mathcal{Z}_{[g]}^{(2),h}(m_h^2) + N_c \mathcal{S}^{(2)}(s, m_h^2). \quad (3.31)$$

3.4 The scattering amplitudes for $t\bar{t}$ production

As the last and the most complicated validation, we consider the $q\bar{q} \rightarrow t\bar{t}$ and $gg \rightarrow t\bar{t}$ partonic processes. The two-loop amplitudes in the high-energy limit have been calculated in [39, 40]. The purely massless amplitudes for $q\bar{q} \rightarrow q'\bar{q}'$ and $gg \rightarrow q'\bar{q}'$ are given in [41, 42]. In contrast to the form factor cases where the color structure is simple, the soft function in $t\bar{t}$ production is a non-trivial operator in color space. To deal with this, we follow the same strategy of [43]. For the $q_l + \bar{q}_k \rightarrow t_i + \bar{t}_j$ channel we choose the color basis as

$$|c_1\rangle = \delta_{ij}\delta_{kl}, \quad |c_2\rangle = (t^a)_{ij} (t^a)_{kl}, \quad (3.32)$$

and for the $g_a + g_b \rightarrow t_i + \bar{t}_j$ channel we use

$$|c_1\rangle = \delta^{ab}\delta_{ij}, \quad |c_2\rangle = if^{abc} (t^c)_{ij}, \quad |c_3\rangle = d^{abc} (t^c)_{ij}, \quad (3.33)$$

where i, j, k, l, a, b are color indices. The scattering amplitudes are then projected into the above bases:

$$|\mathcal{M}\rangle = \sum_I \mathcal{M}_I |c_I\rangle. \quad (3.34)$$

Using this decomposition, we may rewrite eq. (2.1) as

$$\mathcal{M}_I^{\text{massive}} = \sum_J \prod_i \left(\mathcal{Z}_{[i]}^{(m|0)} \right)^{1/2} \mathcal{S}_{IJ} \mathcal{M}_J^{\text{massless}}, \quad (3.35)$$

where the soft matrix elements are defined by

$$\mathcal{S}_{IJ} = \frac{\langle c_I | \mathcal{S} | c_J \rangle}{\langle c_I | c_I \rangle}. \quad (3.36)$$

For the $q\bar{q}$ channel, the matrix elements of $\mathbf{T}_i \cdot \mathbf{T}_j$ in \mathcal{S} (see eq. (2.3)) are given by

$$\begin{aligned} \mathbf{T}_1 \cdot \mathbf{T}_2 &= \mathbf{T}_3 \cdot \mathbf{T}_4 = \begin{pmatrix} -C_F & 0 \\ 0 & \frac{1}{2N_c} \end{pmatrix}, \\ \mathbf{T}_1 \cdot \mathbf{T}_3 &= \mathbf{T}_2 \cdot \mathbf{T}_4 = \begin{pmatrix} 0 & -\frac{C_F}{2N_c} \\ -1 & -\frac{N_c^2-2}{2N_c} \end{pmatrix}, \\ \mathbf{T}_2 \cdot \mathbf{T}_3 &= \mathbf{T}_1 \cdot \mathbf{T}_4 = \begin{pmatrix} 0 & \frac{C_F}{2N_c} \\ 1 & -\frac{1}{N_c} \end{pmatrix}. \end{aligned} \quad (3.37)$$

For the gg channel, they are

$$\mathbf{T}_1 \cdot \mathbf{T}_2 = \begin{pmatrix} -N_c & 0 & 0 \\ 0 & -\frac{N_c}{2} & 0 \\ 0 & 0 & -\frac{N_c}{2} \end{pmatrix},$$

$$\begin{aligned}
 \mathbf{T}_3 \cdot \mathbf{T}_4 &= \begin{pmatrix} -C_F & 0 & 0 \\ 0 & \frac{1}{2N_c} & 0 \\ 0 & 0 & \frac{1}{2N_c} \end{pmatrix}, \\
 \mathbf{T}_1 \cdot \mathbf{T}_3 = \mathbf{T}_2 \cdot \mathbf{T}_4 &= \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -1 & -\frac{N_c}{4} & -\frac{N_c^2-4}{4N_c} \\ 0 & -\frac{N_c}{4} & -\frac{N_c}{4} \end{pmatrix}, \\
 \mathbf{T}_2 \cdot \mathbf{T}_3 = \mathbf{T}_1 \cdot \mathbf{T}_4 &= \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1 & -\frac{N_c}{4} & \frac{N_c^2-4}{4N_c} \\ 0 & \frac{N_c}{4} & -\frac{N_c}{4} \end{pmatrix}.
 \end{aligned} \tag{3.38}$$

Plugging the above into the factorization formula, we are able to reproduce the results of [39, 40] from the purely massless amplitudes of [41, 42], including all the n_h -related terms. This provides a strong check on our factorization formula and the expressions of the \mathcal{Z} -factors and the soft function.

4 Summary and outlook

In summary, in this work we have proposed a generic factorization formula for scattering amplitudes up to two loops in the high-energy boosted limit. Our formula completes that of [2] by incorporating the contributions from massive loops (the n_h -contributions). The formula involves a soft function as an operator in color space, as well as modified \mathcal{Z} -factors with n_h -dependence. Using the method of regions with analytic regulators for the rapidity divergences, we derive these new ingredients through explicit calculations. We verify our results using various form factors. In particular, we have computed two form factors that were not available in the literature: the massive loop contributions to the quark-quark vector form factor and to the gluon-gluon scalar form factor. Furthermore, the scattering amplitudes for $t\bar{t}$ production provide a strong validation of our results, with non-trivial color structures.

Our factorization formula can be used to obtain approximate expressions for complicated two-loop massive amplitudes from the corresponding much simpler massless ones. While such an approximation is only valid in the high-energy limit, it can be combined with low-energy approximations (e.g., in the threshold limit or the soft limit) to get reasonable results in the whole phase space. This may find applications in cutting-edge problems such as the NNLO QCD corrections to $t\bar{t}Z$, $t\bar{t}H$ $t\bar{t}j$ and $t\bar{t}t\bar{t}$ production processes. Combined with suitable RGEs, our formula can also be used to resum the large mass logarithms to all orders in the coupling constant, either in the high-energy boosted limit alone, or in the double boosted/threshold limit considered in [44–47].

Finally, it will be interesting to think about the extension of the factorization formula to three loops and beyond. In particular, there have been concerns about extra momentum regions (other than hard, soft and collinear ones) in scalar integrals [48]. These extra regions cancel out at the two-loop level in QED [3, 21], and clearly also in QCD since our region analysis correctly reproduces the full results. It remains to be investigated whether this kind of cancellation also happens at higher-loop orders. It is also worth asking whether

the three-loop soft function involves correlations among three external legs, similar to the two-loop anomalous dimension governing the IR divergences of massive amplitudes [43], and how to determine this kind of contributions through explicit calculations. This will be left for future investigations.

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A Renormalization constants

In this appendix we collect the renormalization constants relevant to our calculations. All renormalization constants are expanded in terms of the renormalized strong coupling α_s . The renormalization constants for the strong coupling and the Hgg effective coupling λ are given by

$$\begin{aligned} Z_{\alpha_s} &= 1 - \left(\frac{\alpha_s}{4\pi}\right) \frac{\beta_0}{\epsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right), \\ Z_\lambda &= 1 - \left(\frac{\alpha_s}{4\pi}\right) \frac{\beta_0}{\epsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{\epsilon}\right), \end{aligned} \tag{A.1}$$

where the β -function coefficients are

$$\begin{aligned} \beta_0 &= \frac{11}{3}C_A + \frac{4}{3}T_F n_f, \\ \beta_1 &= \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F n_f - 4C_F T_F n_f. \end{aligned} \tag{A.2}$$

The renormalization constant for the light-quark field is

$$Z_q = 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 C_F T_F \sum_h \left(\frac{\mu^2}{m_h^2}\right)^{2\epsilon} \left(\frac{1}{\epsilon} - \frac{5}{6}\right).$$

The two-loop renormalization constant for the massive quark field with mass m_Q arising from a massive quark loop with mass m_h is given by

$$\begin{aligned} Z_{Q,n_h}^{(2)} &= C_F T_F \left\{ \left[\frac{1}{\epsilon} \left(1 + 4 \ln \frac{\mu^2}{m_Q^2} \right) + 6 \ln^2 \frac{\mu^2}{m_Q^2} + \frac{22}{3} \ln \frac{\mu^2}{m_Q^2} + \frac{947}{18} - 30\zeta_2 \right] \right. \\ &\quad + \sum_{h \neq Q} \left[\frac{1}{\epsilon} \left(1 + 4 \ln \frac{\mu^2}{m_Q^2} - 8H(0, x) \right) + 6 \ln^2 \frac{\mu^2}{m_Q^2} + \frac{22}{3} \ln \frac{\mu^2}{m_Q^2} + \frac{443}{18} \right. \\ &\quad \left. \left. + \left(10 - 36x - 60x^3 + 24x^4 \right) \zeta_2 + 28x^2 + \left(\frac{32}{3} - 16 \ln \frac{\mu^2}{m_Q^2} + 16x^2 \right) H(0, x) \right] \right\} \end{aligned}$$

$$\begin{aligned}
 & - \left(8 + 12x + 20x^3 + 24x^4 \right) H(-1, 0, x) + \left(8 - 12x - 20x^3 + 24x^4 \right) H(1, 0, x) \\
 & + \left(32 + 48x^4 \right) H(0, 0, x) \Bigg\}, \tag{A.3}
 \end{aligned}$$

where $x = m_h/m_Q$. For the gluon field renormalization, we only give the terms with a single massive quark loop (i.e., we drop the n_h^2 and the $n_h n_l$ terms):

$$\begin{aligned}
 Z_g^{(1),h} &= T_F \left[-\frac{4}{3\epsilon} - \frac{4L_m}{3} - \epsilon \left(\frac{2L_m^2}{3} + \frac{2\zeta_2}{3} \right) + \epsilon^2 \left(\frac{4\zeta_3}{9} - \frac{2L_m^3}{9} - \frac{2\zeta_2 L_m}{3} \right) \right], \\
 Z_g^{(2),h} &= T_F \left\{ C_A \left[\frac{35}{9\epsilon^2} + \frac{1}{\epsilon} \left(\frac{26L_m}{9} - \frac{5}{2} \right) + \frac{4L_m^2}{9} - 5L_m + \frac{13\zeta_2}{9} + \frac{13}{12} \right] \right. \\
 & \quad \left. + C_F \left(-\frac{2}{\epsilon} - 4L_m - 15 \right) \right\}, \tag{A.4}
 \end{aligned}$$

where $L_m = \ln(\mu^2/m_h^2)$.

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