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Two-loop integrals of half-BPS six-point functions on a line

Ricardo Rodrigues

Centro de Física do Porto, Departamento de Física e Astronomia,
Faculdade de Ciências da Universidade do Porto,
Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail: up201805119@edu.fc.up.pt

Abstract: We evaluated all two-loop conformal integrals of scalar half-BPS six-point functions in $\mathcal{N} = 4$ SYM restricted to a configuration where all points lie on a line. Moreover, we also computed some of these integrals in the kinematical limit where adjacent points become null-separated. Our results can serve as cross-checks for future works which obtain these integrals for general kinematics or by different methods such as integrability.

Keywords: Conformal and W Symmetry, AdS-CFT Correspondence, Supersymmetric Gauge Theory

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Contents

1 Introduction

The applicability of conformal field theories ranges from the study of critical phenomena to that of quantum gravity. As such, these theories have been intensively studied over the past few decades. Among these, the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory plays a special role not only because it is integrable in the planar limit but also because it is one of the few known examples of an integrable theory in four dimensions. As is the case with most CFT, correlation functions of local operators are one of the most interesting observables to study. For this theory in particular it has been shown to exist certain dualities between some of these correlation functions, scattering amplitudes and Wilson loops [1–4].

The $\mathcal{N} = 4$ SYM theory has also appeared in the first example of the AdS/CFT duality, having been shown to be dual to a type IIB string theory in a $AdS_5 \times S_5$ geometry [5]. Since then, this theory has proved to be of particular usefulness in having a better understanding of strong coupling phenomena, given that AdS/CFT is a strong/weak coupling duality. Within its spectrum a special kind of operators called half-BPS operators are dual to Kaluza-Klein spherical harmonics of the graviton on the sphere. As such, the analysis of correlators of such operators can offer us important information about the internal sector of the dual bulk theory and how its information is encoded in the CFT side of the duality.

The above are some of the reasons why we are concerned with studying correlation functions in $\mathcal{N} = 4$ SYM, in particular of half-BPS operators. These, however, are not always easy to obtain. In fact, although some four-point functions of this theory have been studied both for weak [6–9] and strong [10–15] coupling, where the computations have been taken to a relatively high number of loops, the situation for higher-point functions is not as promising. Five and six-point functions, however, are somewhat of an exception to this as they have been analyzed relatively recently. In particular, the integrands of five-point functions are known up to three loops [16], while the integrals have been explicitly obtained up to two loops when null limits are taken or when we restrict the points to a plane [17]. There have also been some bootstrap approaches to certain five-point functions of this theory at strong coupling in [18, 19]. On the other hand, the integrands of six-point functions have been obtained up to two loops [1, 3, 16] but we still have not been able to obtain the integrated contributions to the corrections.

Here our main concern will be the two loop conformal integrals associated to six-point functions of scalar single-trace half-BPS operators. In particular, since the complexity of these integrals for general kinematics is high enough that we cannot evaluate them explicitly, we will restrict the configuration to a line geometry which simplifies things. In practice this amounts to the study of correlators in a 1d CFT, which is something that has already been explored in recent years. Indeed, in the context of similar one dimensional setups there have been computations of four-point functions of protected operators at weak coupling up to two loops [20] and at strong coupling up to three loops [21] as well as for higher-point correlators, namely six-point functions of half-BPS operators up to one loop [22]. In addition, we will also evaluate some of these integrals when null limits are taken.

The structure of this paper is as follows. In section 2 we introduce the single-trace half-BPS operators, giving special focus to twenty-prime operators, their correlation functions and respective properties. Section 3 describes the procedure to compute the integrals on the line geometry and presents the different types of two loop six-point conformal integrals. Additionally, it tackles some important points about the line geometry and the null limit cases. In section 4 we comment on the results regarding the line configuration and then move on to present the explicit outcome of evaluating one of the integrals in the light-cone limit. In section 5 we make a general assessment of our work and elaborate on possible future directions. Appendix A contains explicit expressions of quantities concerning the integrand of a conformal integral considered in section 3.3. In appendix B we briefly explain how to generalize a result for asymptotic expanded integrals which was necessary to evaluate the integral considered in section 4.

2 Correlation functions

Here, we consider the four dimensional $\mathcal{N} = 4$ SYM theory, with gauge group $\mathrm{SU}(N_c)$, where N_c is the number of colors, which among other fields contains six bosonic scalars denoted by $\Phi^I = (\phi^1, \ldots, \phi^6)$. Moreover, this theory has an associated coupling constant g_{YM} which is related with N_c by the so-called 't Hooft coupling

$$\lambda = g_{\mathrm{YM}}^2 N_c. \tag{2.1}$$

Within the spectrum of this theory there is a class of important operators called the scalar single-trace half-BPS operators, which transform in the symmetric and traceless representation of the $\mathrm{SO}(6)_R \simeq \mathrm{SU}(4)_R$ global symmetry group. They are generally written as

$$\mathcal{O}_k(x,Y) = Y^{i_1} \dots Y^{i_k} \,\mathrm{tr}\left(\phi^{i_1}(x) \dots \phi^{i_k}(x)\right), \tag{2.2}$$

where $k \geq 2$ is an integer, Y^i are six-dimensional null polarization vectors ($Y^i \cdot Y^i = 0$) contracting with the $\mathrm{SO}(6)_R$ indices, and the trace is taken over the indices of $\mathrm{SU}(N_c)$, which are omitted for simplicity. Such operators $\mathcal{O}_k$ have conformal dimension $\Delta_{\mathcal{O}_k} = k$.

An interesting property of these operators is that their two- and three-point functions are protected from quantum corrections by supersymmetry [23], which means they remain unchanged independently of the theory's coupling. On the other hand, while their higher-point functions no longer enjoy this property they are of extreme importance to us as they encode useful information about the theory itself and its holographic dual.

In general, any planar n-point function of scalar single-trace half-BPS operators is given by

$$\langle \mathcal{O}_{k_1}\, \mathcal{O}_{k_2} \dots \mathcal{O}_{k_n} \rangle = \sum_{\ell=0}^{\infty} \lambda^{2\ell}\, G_n^{(\ell)}, \tag{2.3}$$

while the non-planar contributions are subleading in the large N_c limit. In the above expression $G_n^{(0)}$ concerns the free correlator, whereas the terms with $\ell \geq 1$ account for loop corrections, which are given by specific spacetime integrals. In practice, the integrands of these corrections can be computed through the Lagrangian insertion method [1, 3, 7, 23, 24], in which they are given by $(n+\ell)$-point functions with ℓ insertions of the Lagrangian density operator. However, as we start considering more points and/or higher loops computing the integrands becomes harder as it requires the evaluation of higher-point correlators. Additionally, once we have the integrands we still have to compute the integrals which can be quite challenging.

Inside this special class of operators one of the most important and studied ones is the lowest length half-BPS operator ($k=2$), which is usually denoted by $20'$ operator

$$O_2(x,Y) \equiv O_{20'}(x,Y) = Y^{i_1}\, Y^{i_2}\, \mathrm{tr}\left(\phi^{i_1}(x)\, \phi^{i_2}(x)\right). \tag{2.4}$$

It belongs to the stress-tensor supermultiplet along with the stress-tensor itself, the Lagrangian density operator and others. Given the above definition, we expect the correlation functions of these operators to be given in terms of polynomials of polarization vectors that have weight two in each Y_i.

The two- and three-point functions of the twenty-prime operators are completely fixed by conformal symmetry and supersymmetry to be given by

$$\langle \mathcal{O}_{20'}(x_1,Y_1)\, \mathcal{O}_{20'}(x_2,Y_2)\rangle = \left(\frac{Y_{12}}{x_{12}^2}\right)^2, \tag{2.5}$$

$$\langle \mathcal{O}_{20'}(x_1,Y_1)\, \mathcal{O}_{20'}(x_2,Y_2)\, \mathcal{O}_{20'}(x_3,Y_3)\rangle = \frac{C_{20'20'20'}\, Y_{12}Y_{13}Y_{23}}{(x_{12}^2 x_{13}^2 x_{23}^2)}, \tag{2.6}$$

where we have used the notation $x_{ij}^2 \equiv (x_i - x_j)^2$, $Y_{ij} \equiv Y_i \cdot Y_j$, and where $C_{20'20'20'}$ is the structure constant of its three-point function, which is independent of the theory's coupling. On the other hand, the higher-point functions, which are not protected by supersymmetry, are completely non-trivial.

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Figure 1. Diagrams representing the Y_{ij} structures of the twenty-prime four-point function.

Let us then start by considering the four-point function, which is the lowest of the higher-point correlators. From a simple analysis it is possible to infer that the polynomial in the polarization vectors can contain up to two different structures. In particular, all the structures we can find are

$$Y_{12}^2\, Y_{34}^2, \qquad Y_{12}\, Y_{23}\, Y_{34}\, Y_{41}, \tag{2.7}$$

which correspond to the diagrams in figure 1, plus the ones obtained from permuting the positions. As such, we can write the four-point function as

$$\langle \mathcal{O}_{20'}(x_1, Y_1)\, \mathcal{O}_{20'}(x_2, Y_2)\, \mathcal{O}_{20'}(x_3, Y_3)\, \mathcal{O}_{20'}(x_4, Y_4) \rangle = \left(\frac{Y_{12}\, Y_{34}}{x_{12}^2\, x_{34}^2} \right)^2 \mathcal{G}_4(u, v; \sigma, \tau), \tag{2.8}$$

where the conformal and the R-symmetry[1] cross-ratios are respectively defined as

$$\begin{aligned} u &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}), \\ \sigma &= \frac{Y_{12} Y_{34}}{Y_{13} Y_{24}} = \alpha\bar{\alpha}, \quad \tau = \frac{Y_{14} Y_{23}}{Y_{13} Y_{24}} = (1-\alpha)(1-\bar{\alpha}). \end{aligned} \tag{2.9}$$

More can be said about the cross-ratios function $\mathcal{G}$. Indeed, superconformal Ward identities impose extra non-trivial conditions on it [13, 25, 26] which in due turn enforce it to have a particular structure, namely

$$\mathcal{G}_4(z, \bar{z}; \alpha, \bar{\alpha}) = \mathcal{G}_4^{(0)}(z, \bar{z}; \alpha, \bar{\alpha}) + R\, \mathcal{F}_4(z, \bar{z}), \tag{2.10}$$

where $\mathcal{G}_4^{(0)}(z, \bar{z}; \alpha, \bar{\alpha})$ is the free part of the correlator $G_4^{(0)}$ with the position-dependent prefactor extracted, R is given by

$$R = \frac{(1 - z\,\alpha)\,(1 - \bar{z}\,\alpha)\,(1 - z\,\bar{\alpha})\,(1 - \bar{z}\,\bar{\alpha})}{(1 - z)\,(1 - \bar{z})\,\alpha^2\,\bar{\alpha}^2}, \tag{2.11}$$

and $\mathcal{F}_4(z, \bar{z})$ contains all the loop-corrections mentioned in (2.3) also without the prefactor. This latter function is important in the sense that it is contains all the dynamical information.

When it comes to the four-point function of twenty-prime operators, this comprises one of the most well studied four-point functions in all conformal field theories. Indeed, the integrands of the corrections for this object are known in the weak coupling perturbative

[1] R-symmetry is a symmetry that is present in supersymmetric theories and which is responsible by transforming the different supercharges into each other. In this case, it will be equivalent to the SO(6) symmetry associated with rotations among the six scalar fields Φ^I.

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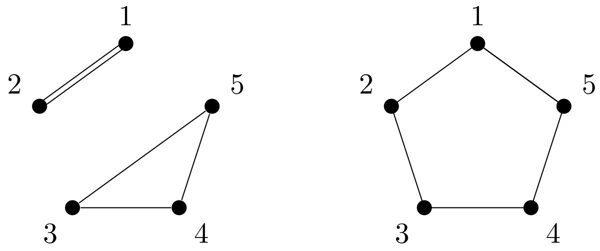

Figure 2. Diagrams representing the Y_{ij} structures of the twenty-prime five-point function.

regime up to ten loops [27, 28]. In spite of this, the conformal integrals have only been evaluated up to three loops for fixed cross-ratios [29] and up to five loops in the Euclidean OPE limit [9]. As for the strong coupling regime, the first few orders of the $1/N$ corrections have been obtained by means of conformal bootstrap methods [14, 30–32].

The five-point function of these same operators is also interesting to look at. Performing a similar analysis as for the four-point case we can see that the allowed structures in the null polarization vectors are

$$Y_{12}^2 Y_{34} Y_{45} Y_{53}, \quad Y_{12} Y_{23} Y_{34} Y_{45} Y_{51}, \tag{2.12}$$

corresponding to the diagrams of figure 2, together with the ones obtained from indices permutations. Consequently, we can write this correlation function as

$$\langle \mathcal{O}_{20'}(x_1, Y_1) \dots \mathcal{O}_{20'}(x_5, Y_5) \rangle = \frac{x_{13}^2 Y_{12}^2 Y_{34}^2 Y_{15} Y_{35}}{x_{12}^4 x_{34}^4 x_{15}^2 x_{35}^2 Y_{13}} \mathcal{G}_5(u_i; \sigma_i), \tag{2.13}$$

which depends on five conformal cross-ratios $u_1, \dots, u_5$ defined as

$$u_1 = \frac{x_{12}^2 x_{35}^2}{x_{13}^2 x_{25}^2}, \qquad u_{i+1} = u_i|_{x_j \to x_{j+1}}, \tag{2.14}$$

where the index j is taken modulo 5, and on the R-symmetry cross-ratios $\sigma_1, \dots, \sigma_5$ given as

$$\sigma_1 = \frac{Y_{12} Y_{35}}{Y_{13} Y_{25}}, \qquad \sigma_{i+1} = \sigma_i|_{Y_j \to Y_{j+1}}, \tag{2.15}$$

with j also taken modulo 5. Similarly, the cross-ratios function $\mathcal{G}_5$ also needs to satisfy superconformal Ward identities like in the four-point case. However, as far as we know no explicit solution was found for five points and we do not know if it should have a similar structure to (2.10) or not.

Regarding the status of the five-point function corrections, the current situation is as referred previously. In particular, while the integrands have been obtained up to three loops [16], the integrals have only been computed up to 2 loops for special cases, such as when null limits are taken or when all the points are set to the plane [17].

Next, we have the twenty-prime six-point function, which is of particular interest to us since we will concern ourselves with the two-loop six-point conformal integrals. For this case, the polynomial in Y_i which describes this function should contain up to four different kinds of structures

$$Y_{12}^2 Y_{34}^2 Y_{56}^2, \quad Y_{12}^2 Y_{34} Y_{45} Y_{56} Y_{63}, \quad Y_{12} Y_{23} Y_{31} Y_{45} Y_{56} Y_{64}, \quad Y_{12} Y_{23} Y_{34} Y_{45} Y_{56} Y_{61}, \tag{2.16}$$

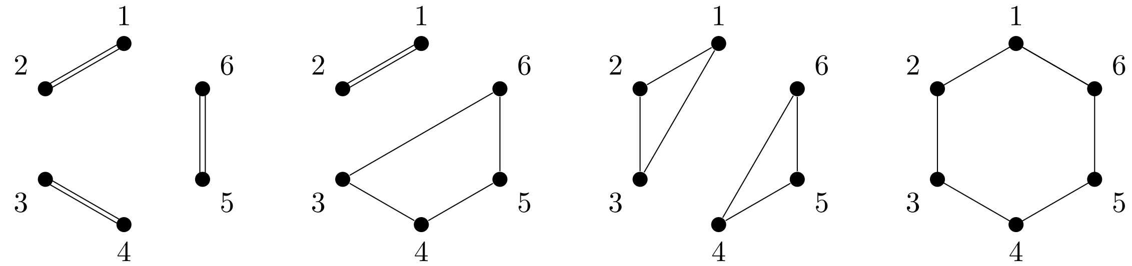

Figure 3. Diagrams representing the Y_{ij} structures of the twenty-prime six-point function.

which can be seen from the diagrams of figure 3, together with all the possible permutations. Therefore, a reasonable choice to represent this six-point function is

$$\langle \mathcal{O}_{20'}(x_1, Y_1) \ldots \mathcal{O}_{20'}(x_6, Y_6) \rangle = \left(\frac{Y_{12}\, Y_{34}\, Y_{56}}{x_{12}^2\, x_{34}^2\, x_{56}^2} \right)^2 \mathcal{G}_6\left(u_i, U_i; \sigma_i, \tau_i\right), \tag{2.17}$$

which now depends on nine cross-ratios $u_1, \ldots, u_6$ and $U_1, \ldots, U_3$ defined as

$$u_1 = \frac{x_{12}^2 x_{35}^2}{x_{13}^2 x_{25}^2}, \quad u_{i+1} = u_i|_{x_j \to x_{j+1}}, \quad U_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad U_{i+1} = U_i|_{x_j \to x_{j+1}}, \tag{2.18}$$

where the superscript in x_j is taken modulo 6, and where the respective R-symmetry cross-ratios $\sigma_1, \ldots, \sigma_6$ and $\tau_1, \ldots, \tau_3$ are given by the above expressions with the squared distances x_{ij}^2 replaced by the products of the null polarization vectors Y_{ij}.

Concerning the corrections of this object, the integrands are only known up to 2 loops [1, 3, 16], but the respective integrals have not yet been explicitly computed up to this level. The simplest kind of these integrals, the double-box integrals, have been expressed in terms of elliptic functions and had its symbol obtained for when null limits are taken in [28, 33]. Moreover, they have also been studied for more general kinematics in [34], where the symbol was determined. However, the explicit result of these integrals for general kinematics was not yet obtained. Fortunately, it has been observed for higher point functions that under certain kinematical limits the associated conformal integrals simplify. Therefore, although we might not be able to evaluate the integrals in full generality we can compute them in several simpler cases and try to learn something from it. Here we shall consider two distinct circumstances: a configuration in which the operators lie in a line, and another where adjacent points will be taken to become null separated from each other $x_{i,i+1}^2 \to 0$, also known as the light-cone limit.

It is also important to mention that the higher-point functions have interesting non-trivial properties, namely the Drukker-Plefka twist [35] and the chiral algebra twist [36], which impose extra constraints on the correlators. In particular, since both these properties are verified already at the level of the integrands, they automatically have to be satisfied by the integrals as well.

Furthermore, the results we will obtain can not only be used to provide useful information, such as CFT data, but they can also serve later as cross-checks. Indeed, if these integrals happen to be computed for general kinematics or by means of other methods, one can always restrict the results to the same kinematics as here and see if both results agree. This has already been the case for four- and five-point correlation functions, where the results

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from the evaluation of conformal integrals matched those obtained by means of integrability methods [37–40]. In particular, integrability can be used to obtain the correlators as a power series over the cross-ratios, where generally one also considers the points to lie on a plane or some other simpler kinematics. Because the results these methods provide us are already full integrated functions, simply knowing the integrands of the ℓ loop corrections is not enough. As such, evaluating the conformal integrals for the six-point function, even if for rather particular kinematics, would already be useful to compare with results from integrability. Furthermore, given that from the integrability viewpoint it is hard to make an identification between a certain integrability contribution and a conformal integral, our results could also help us make predictions regarding this issue.

3 Integrals

Our goal is to compute two-loop six-point conformal integrals of the form

$$I = \int \frac{d^d x_7\, d^d x_8}{x_{78}^2 \prod_{i=1}^{6} \prod_{j=7}^{8} (x_{ij}^2)^{a_{ij}}}, \tag{3.1}$$

where a_{ij} are integers that satisfy $\sum_i a_{ij} = d-1$. Below we explain a possible way of evaluating these integrals which was the one used here to compute the integrals for the line configuration.

3.1 Methodology

The first step of the computation is to use conformal symmetry to simplify the integrand. In particular, we send one of the external points to infinity. Although the choice of the point is completely arbitrary it is good practice to choose one of the points that appears more times in the denominator or one that appears in the numerator. Additionally, it is also useful to send one point to lie at the origin and another to a fixed position, usually chosen to be $(1, 0, \dots)$.

In case the remaining integrand has no position-dependent numerator, the next step is to perform the Schwinger parametrization (see [41] for example), which makes use of the following identity

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\alpha\, \alpha^{n-1} e^{-\alpha A}. \tag{3.2}$$

Upon using this equality, the spacetime integrals transform into Gaussian integrals, which can easily be done, and we obtain integrals over the Schwinger parameters [42, 43]

$$I(a_{ij}) = \Gamma(\omega) \left(\prod_i \int_0^\infty \frac{d\alpha_i\, \alpha_i^{a_i - 1}}{\Gamma(a_i)} \right) \frac{\Psi^{\omega - \frac{d}{2}}}{\Phi^{\frac{d}{2}}}\, \delta(1 - \alpha_j), \tag{3.3}$$

where Ψ and Φ are graph polynomials resulting from the spacetime integrations, the index i runs over the propagators appearing in the integral (3.1), with a_i being their powers, and where the superficial degree of divergence ω for any number of loops ℓ is given by

$$\omega = \sum_i a_i - \frac{d}{2} \cdot \ell. \tag{3.4}$$

The Dirac delta at the end fixes some arbitrary Schwinger parameter α_j to one.

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Finally, the integrals over the parameters α_i can be evaluated by means of the computer package `HyperInt` [43], provided that we consider some simple geometry or kinematical limits which simplify the integrands. At the end, we use the function `fibrationBasis`, belonging to the same package, that expresses the result in a way that is suitable for expansion around 0 with respect to the provided arguments and the specified order.

When it is the case that even after using conformal symmetry we still have a position-dependent numerator in the integrand the procedure is slightly different. This has to do with the fact that the identity (3.2) has problems for negative n, in which case the Gamma function has poles. Nevertheless, the procedure does not differ significantly from what was just explained, requiring us to differentiate with respect to some additional Schwinger parameters and evaluate them at zero. We give a more detailed explanation of this in an example below but we refer the reader to [44] for a complete description.

3.2 Two-loop half-BPS six-point integrals

The two-loop correction of single-trace half-BPS six-point functions involves conformal integrals of three different types

$$\mathcal{B}_{123,456} \equiv \int \frac{d^4x_7\, d^4x_8}{(x_{17}^2 x_{27}^2 x_{37}^2)\, x_{78}^2\, (x_{48}^2 x_{58}^2 x_{68}^2)}, \tag{3.5}$$

$$\mathcal{P}_{1,23,456} \equiv \int \frac{x_{28}^2\, d^4x_7\, d^4x_8}{(x_{17}^2 x_{27}^2 x_{37}^2)\, x_{78}^2\, (x_{18}^2 x_{48}^2 x_{58}^2 x_{68}^2)}, \tag{3.6}$$

$$\mathcal{T}_{14,23,56} \equiv \int \frac{x_{28}^2\, x_{57}^2\, d^4x_7\, d^4x_8}{(x_{17}^2 x_{27}^2 x_{37}^2 x_{47}^2)\, x_{78}^2\, (x_{18}^2 x_{48}^2 x_{58}^2 x_{68}^2)}. \tag{3.7}$$

Following the same convention as [16][2] we will refer to the $\mathcal{B}$ integrals as double-box integrals, $\mathcal{P}$ as penta-box integrals and $\mathcal{T}$ as double-penta integrals. Due to the symmetries of the integrands, each of these conformal integrals are invariant under certain permutations of positions. For example, switching $1 \leftrightarrow 2$ in the $\mathcal{B}$ integral above amounts to the same integral. Consequently, the number of different integrals for each type is: 10 $\mathcal{B}$'s, 120 $\mathcal{P}$'s and 180 $\mathcal{T}$'s. Luckily, we do not have to compute all of these integrals, which would be quite cumbersome, but only those with which we can obtain all others from cross-ratios transformations.

Ideally, we would like to be able to compute these integrals in the most general configuration of the six-point function. Unfortunately, we have not yet been able to evaluate integrals of the sort expressed in (3.1) under general kinematics. Nonetheless, we can still study them in specific cases for which the integrands simplify. In principle, this allows us to compute the integrals and obtain closed form results in terms of well defined functions.

3.3 Line geometry

One of the possibilities is to consider a configuration in which all points lie in the same line. We thus start by using conformal symmetry to send three points to 0, 1 and ∞, that we respectively choose to be x_1, x_2 and x_6, and then impose the remaining points to lie on a

[2] The conformal integrals presented here are the same as the ones from [16] with the difference being that we are using a different notation and we have ignored the prefactors.

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line by fixing $x_{ij}^2 = (z_i - z_j)(\bar{z}_i - \bar{z}_j)$ with $z_i = \bar{z}_i$. In this configuration, the cross-ratios u_i, U_i of the six-point function (2.18) are related to the z_i cross-ratios by

$$\begin{aligned} u_1 &= \frac{z_2^2\,(1-z_5)^2}{(z_2-z_5)^2}, \quad u_2 = \frac{(z_2-1)^2}{(z_2-z_4)^2}, \quad u_3 = \frac{(1-z_4)^2\,z_5^2}{z_4^2\,(1-z_5)^2}, \quad u_4 = \frac{(z_4-z_5)^2}{(z_2-z_5)^2}, \quad u_5 = \frac{1}{z_5^2}, \\ U_1 &= \frac{1}{z_4^2}, \qquad U_2 = \frac{(z_2-z_4)^2\,z_5^2}{z_4^2\,(z_2-z_5)^2}, \quad U_3 = \frac{(1-z_5)^2}{(z_2-z_5)^2}. \end{aligned} \tag{3.8}$$

Relatedly, the choice of cross-ratios with which we work can be crucial for the evaluation of the integral. To see this let us consider the one-loop correction to the four point function of scalars in four dimensions

$$\int \frac{d^4x_0}{x_{01}^2\,x_{02}^2\,x_{03}^2\,x_{04}^2}. \tag{3.9}$$

Following the procedure mentioned above we start by sending $x_4 \to \infty$, we then introduce the Schwinger parameters and integrate over x_0. After doing this we obtain the integral

$$\int \frac{d\alpha_1 d\alpha_2}{(1+\alpha_1+\alpha_2)(\alpha_1 + v\,\alpha_2 + u\,\alpha_1\,\alpha_2)^2}, \tag{3.10}$$

where the third Schwinger parameter has been fixed to 1 by the Dirac delta. It is not hard to do one of the two above integrals. If we choose to do the one in α_1 we find that this double integral gets reduced to

$$\int d\alpha_2\, \frac{1 + \{1 + u - v\,[1 - \log(v\alpha_2) + \log\left((1+\alpha_2)(1+u\,\alpha_2)\right)]\}\,\alpha_2 + u\,\alpha_2^2}{v\,\alpha_2\,\left[1 + \alpha_2(1 - v + u(1+\alpha_2))\right]^2}. \tag{3.11}$$

The issue with the above expression is that some of the factors in the denominator are not linear in the integration variable. We want to avoid these kinds of terms as the method used by `HyperInt` requires the integrands to factor linearly. Generally, we can overcome this problem simply by using different variables. In this particular case if we use the cross-ratios z and $\bar{z}$ instead of u and v the integral over Schwinger parameters is given by

$$\int \frac{d\alpha_1 d\alpha_2}{(1+\alpha_1+\alpha_2)((1-z)(1-\bar{z})\alpha_2 + \alpha_1(1+z\bar{z}\alpha_2))^2}, \tag{3.12}$$

and the one we obtain after performing one of the integrations is

$$\int d\alpha_2\, \frac{1+\alpha_2\left\{z+\bar{z}+(1-z)\,(1-\bar{z})\left[\log[(1-z)\,(1-\bar{z})\,\alpha_2] - \log[(1+\alpha_2)\,(1+z\bar{z}\alpha_2)]\right] + z\,\bar{z}\,\alpha_2\right\}}{(1-z)\,(1-\bar{z})\,\alpha_2\,(1+z\alpha_2)^2\,(1+\bar{z}\alpha_2)^2}. \tag{3.13}$$

Now, we no longer have terms in α_2 as the one found above, thus making the second integral more tractable.

For higher-point functions the choice of variables such as not to have these non-linear terms is in general not straightforward. For the six-point integrals considered here (3.5), (3.6), (3.7) there might exist some possible choice that allows us to only have linearized terms in the integrands, which in turn makes the integrals easier to evaluate. As far as we know such a choice was not yet found. Therefore, we opted to work with the $z, \bar{z}$ cross-ratios and further restrict to the line configuration $z = \bar{z}$ in order to avoid the non-linear terms and be able to compute the integrals.

As mentioned previously, some of the integrals will still have numerator terms even after using conformal symmetry to fix some of the points. We would now like to illustrate what the procedure for these cases is when evaluating them on a line. For this, we will consider one of the double-penta integrals, namely

$$\mathcal{T}_{16,23,45} = \int \frac{x_{28}^2\, x_{47}^2\, d^4x_7\, d^4x_8}{x_{17}^2 x_{27}^2 x_{37}^2 x_{78}^2 x_{18}^2 x_{48}^2 x_{58}^2}, \tag{3.14}$$

where we have already used conformal symmetry to send $x_6 \to \infty$. Following the recipe we gave before, we then use Schwinger parametrization for every term in the integrand and integrate over the Gaussian space-time integrals. Next, we restrict the points to the plane, send $x_1 \to 0$ and $x_3 \to 1$ and further restrict to the line kinematics $z_i = \bar{z}_i$. The outcome is an integral over Schwinger parameters of the form (3.3), with the graph polynomials Ψ and Φ as given in appendix A. Their expressions depend on nine Schwinger parameters, given that we have nine squared distances terms, but should only depend on seven since it is the number of terms on the denominator. This has to do with the aforementioned fact that the Schwinger parametrization as in (3.2) is problematic for the numerator terms. In these cases what we must do according to [44] is to use the more general form of the Schwinger trick (see [41] for example). This will involve taking two derivatives for each of the two extra Schwinger parameters α_8 and α_9 evaluated at zero. Since one of the two derivatives for each Schwinger parameter cancels the respective integration, the integral will then be given by

$$\mathcal{T}_{16,23,45} \propto \prod_{i=1}^{7} \int_0^\infty d\alpha_i\, \delta(1-\alpha_j)\, \frac{\partial}{\partial \alpha_8} \left[\frac{\partial}{\partial \alpha_9} \left[\frac{\Psi}{\Phi^2} \right]_{\alpha_9=0} \right]_{\alpha_8=0}, \tag{3.15}$$

where we already used that $d = 4$ and $\omega = 1$ for this case. A small comment is in order here. In particular, although we can take the two derivatives and then evaluate the integrals, we found that in some cases the resulting integrands after the derivatives are complex enough that the integrals take a very long time to be evaluated with `HyperInt`. What one must do is to take one derivative, evaluate a few of the integrals over Schwinger parameters, say α_1, α_3 and α_4 for example,[3] take the other derivative and only then compute the remaining integrals. This is really a trial and error game that has to be played for each integral to see what works best. At the end, we use `fibrationBasis` to write the result in a way that is suitable for being expanded in the limit of small arguments.

Finally, we would just like to note that because on the line configuration we will be working with general cross-ratios z_i, it is sufficient to evaluate only one conformal integral of each type. All others follow from appropriately transforming the cross-ratios according to the respective positions permutations.

[3]The order of integration on the α's is also important to prevent the evaluation of the integrals from taking too long. To find the best order one uses the functions `irreducibles`, `cgReduction` and `suggestIntegrationOrder` described in [43].

3.4 Light-cone limit and asymptotic expansions

Another choice that can be taken in order to simplify the conformal integrals is to consider physically relevant kinematical limits. In particular, limits in which the leading order terms of these integrals are relatively easier to compute. One of these cases is the light-cone limit, where we take $x^2_{i,i+1} \to 0$. When in this kinematics the conformal integrals can be separated in distinct regions according to whether the integrations variables are small or not and use certain identities that make the integrals easier. This procedure is called asymptotic expansion method (see [45–47] for example).

Let us then consider we are taking the null limit $x^2_{12} \to 0$. The idea behind the asymptotic expansions is that we may start by separating each integration into two regions: one where the integration variable is small and thus comparable to the small parameter x^2_{12}, and another where this variable is big such that dropping the squared distance is a good approximation. Given that we are dealing with two-loop integrals, we will have four regions:

$$\text{I. } x_7 \ll 1, x_8 \ll 1; \quad \text{II. } x_7 \ll 1, x_8 \gg 1; \quad \text{III. } x_7 \gg 1, x_8 \ll 1; \quad \text{IV. } x_7 \gg 1, x_8 \gg 1. \tag{3.16}$$

In general, when we take the light-cone limit between any two points, an ℓ-loop integral will give rise to 2^ℓ regions. After separating the integral in these four regions, it is possible to simplify the integrand by making use of the identity

$$\frac{1}{(x^2_{ij})^b} = \frac{1}{(x^2_{1j})^b} \sum_{n=0}^{\infty} \binom{-b}{n} \frac{(x^2_{1i} - 2x_{1i} \cdot x_{1j})^n}{(x^2_{1j})^n}, \tag{3.17}$$

where x_i is assumed to be small and x_j to be large. Naturally these expansions have to be done in a suitable manner for each region.

For the sake of exemplification let us then consider one of the simpler six-point conformal integrals, namely (3.5). Following the aforementioned prescription we can write

$$\begin{aligned}
\mathcal{B}_{123,456} &= \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4, \\
\mathcal{B}_1 &= \sum_{n,m,k} \int \frac{d^dx_7 d^dx_8\, (2x_{13}\cdot x_{17} - x^2_{17})^n (2x_{14}\cdot x_{18} - x^2_{18})^m (2x_{15}\cdot x_{18} - x^2_{18})^k}{(x^2_{13})^{1+n}\, (x^2_{14})^{1+m}\, (x^2_{15})^{1+k}\, x^2_{17}\, x^2_{27}\, x^2_{78}}, \\
\mathcal{B}_2 &= \sum_{n,m} \int \frac{d^dx_7 d^dx_8\, (2x_{13}\cdot x_{17} - x^2_{17})^n (2x_{17}\cdot x_{18} - x^2_{17})^m}{(x^2_{13})^{1+n}\, (x^2_{18})^{1+m}\, x^2_{17}\, x^2_{27}\, x^2_{48}\, x^2_{58}}, \\
\mathcal{B}_4 &= \sum_{n} \int \frac{d^dx_7 d^dx_8\, (2x_{12}\cdot x_{17} - x^2_{12})^n}{(x^2_{17})^{2+n}\, x^2_{37}\, x^2_{48}\, x^2_{58}\, x^2_{78}},
\end{aligned} \tag{3.18}$$

where for simplicity we have used conformal symmetry to send $x_6 \to \infty$. The reason why we have not given the explicit expression for $\mathcal{B}_3$ is that it is scaleless [45, 48] and thus evaluates to zero. Moreover, we point out to the dimensional regularization $d = 4 - \epsilon$ performed in the integrals measures. This is a key step of the asymptotic expansion method which allows to regulate the possible divergences of each region integral. Importantly, the final result should not depend on ϵ since the conformal integrals had no dependence on it to start with.

Hopefully the above expressions convey the central idea of this method, which is that we obtain several integrals that are easier to evaluate than the original one. If needed, however, we could further take more null limits, namely $x_{34}^2 \to 0$ and $x_{56}^2 \to 0$, and follow the same path to obtain several integrals which in practice are simpler to compute.

To compute the integrals in (3.18) we can start by dropping the terms x_{12}^2, x_{17}^2 and x_{18}^2 inside the parenthesis terms in the numerators since they are subleading. Afterwards, we can obtain the final expressions for these integrals by making use of the following result [49]

$$\int \frac{d^d x_7\, d^d x_8\, (x_{15}\cdot x_{57})^{k_1}\, (x_{25}\cdot x_{58})^{k_2}}{x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2} = c_{k_1,k_2}\, \frac{(x_{15}\cdot x_{56})^{k_1} (x_{25}\cdot x_{56})^{k_2}}{(x_{56}^2)^{5-d}}, \tag{3.19}$$

with

$$c_{k_1,k_2} = \binom{k_1+k_2}{k_1} \sum_{j=0}^{k_2} (-1)^j \binom{k_2}{j} \frac{1}{1+j+k_1} \left[6\zeta_3 + S_{1,2}(k_1+j) - S_{2,1}(k_1+j)\right], \tag{3.20}$$

where $S_{i,j}$ are harmonic sums defined recursively as

$$S_{a,\mathbf{b}}(N) = \sum_{n=1}^{N} \frac{(\mathrm{sign}(a))^n}{n^{|a|}}\, S_{\mathbf{b}}(n), \quad \text{with } S_{\emptyset}(N) = 1, \quad a, b_i \in \mathbb{Z} \setminus \{0\}. \tag{3.21}$$

Even though our integrals might have more terms in the numerator than what appears in the equality (3.19), this does not prevent us from using it, as explained in appendix B.

In general, when we take kinematical limits it is no longer true that we can obtain all of the conformal integrals simply by computing one integral of each kind and appropriately transforming the cross-ratios. This stems from the fact that in these cases some of these integrals become independent of each other. Therefore, what must be done is that we must compute those integrals with which we can generate all others in these limits. In our case, it is possible to infer that the conformal integrals we need to assess are

$$\mathcal{B}_{123,456}, \qquad \mathcal{B}_{135,246}, \tag{3.22}$$

for the double-box integrals,

$$\mathcal{P}_{2,13,456}, \quad \mathcal{P}_{3,12,456}, \quad \mathcal{P}_{3,14,256}, \quad \mathcal{P}_{3,15,246}, \tag{3.23}$$

for the penta-box integrals, and

$$\mathcal{T}_{46,15,23}, \quad \mathcal{T}_{46,15,32}, \quad \mathcal{T}_{56,14,23}, \quad \mathcal{T}_{56,14,32}, \quad \mathcal{T}_{56,12,34}, \quad \mathcal{T}_{26,15,34}, \quad \mathcal{T}_{24,16,35}, \tag{3.24}$$

for the double-penta integrals.[4] As it so happens, some of these integrals are easier to evaluate than others, in particular $\mathcal{T}_{56,14,23}$, $\mathcal{T}_{56,14,32}$ and $\mathcal{T}_{56,12,34}$. Here we decided to compute only one of these three integrals, namely $\mathcal{T}_{56,14,23}$, when the light-cone limit $x_{56}^2 \to 0$ is taken, although the procedure for the other two would be exactly the same. The reason why these integrals are easier to evaluate than the remaining ones is because when we take a null limit and separate into asymptotic regions we only have to compute one region integral as the others either evaluate to zero or are subdominant, as will be explained below. On the other hand, the more

[4] Naturally, we are free to permute the indices and obtain another set of independent integrals, although we gain nothing new.

complicated integrals are still cumbersome to compute even if we take additional null limits. Indeed, in these cases not only would we have to consider more regions as we would also have to compute the majority of the integrals in each region in contrast to what happened in the simplest cases. Nonetheless, we still evaluated all of these integrals (3.22), (3.23), (3.24) for a line configuration whose results can be find in an supplementary `Mathematica` file.

Now that we have given some insights about the kinematics and respective procedures to deal with the integrals at hand, we will proceed to present the obtained results and respective comments.

4 Results and analysis

4.1 Line configuration

We will start by analyzing the results obtained from evaluating the integrals in the line configuration. We have chosen to compute all ten double-box integrals, given that the number is fairly reasonable. However, when it comes to the penta-box integrals and the double-penta integrals, we have only evaluated the integrals $\mathcal{P}_{1,23,456}$ and the ones in (3.23) and $\mathcal{T}_{16,23,45}$ and the integrals in (3.24), respectively. Indeed, not only would it be quite time-consuming to compute all others but it would also be inefficient since one can obtain the remaining ones by finding how the cross-ratios transform under the indices permutations, as mentioned in section 3.2. For instance, under the exchange of points $2 \leftrightarrow 6$, the cross-ratios would transform as

$$z_2 \to 1 - z_2, \quad z_4 \to \frac{z_4\,(z_2 - 1)}{z_2 - z_4}, \quad z_5 \to \frac{z_5\,(z_2 - 1)}{z_2 - z_5}. \tag{4.1}$$

Afterwards, we must expand the final expression accordingly to the kinematical limit we want to analyze.

All the integrals were computed in four different cases, such that the following kinematical regimes could be taken afterwards if one wanted, namely: $z_2, z_4, z_5 \to 0$; $z_2, z_5 \to 0$, $z_4 \to 1$; $z_2, z_4 \to 0$ and $z_5 \to \infty$; $z_2 \to 0$, $z_4 \to 1$ and $z_5 \to \infty$ (with the remaining points fixed by conformal symmetry as mentioned before). The results can be found in the aforementioned auxiliary `Mathematica` file. We point out that throughout this file we have introduced two new variables defined as

$$y = 1 - z_4, \qquad w = \frac{1}{z_5}, \tag{4.2}$$

which when taken to zero allow for the kinematical limits mentioned above.

Importantly, we found that all of the results were given in terms of rational functions of the cross-ratios z_i, zeta functions

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{4.3}$$

with $s \in \mathbb{C}$, and hyperlogarithm functions defined as

$$\mathrm{Hlog}(z, [\sigma_1, \ldots, \sigma_r]) = \int_0^z \frac{dz'}{z' - \sigma}\,\mathrm{Hlog}(z', [\sigma_2, \ldots, \sigma_r]), \quad \text{with } \mathrm{Hlog}(z) = 1, \tag{4.4}$$

where σ_i,$z \in \mathbb{C}$.

Interestingly enough, we observed that in all of the results the simplest type of integrals $\mathcal{B}$ factorized as a product of a common cross-ratios prefactor and a linear combination of hyperlogarithms and zeta functions. For instance, in the case in which the limits $z_i \to 0$ $(i = 2, 4, 5)$ could be taken later, we found one of these integrals to be given by

$$\mathcal{B}_{123,456} = \frac{2}{z_2(z_2-1)(z_4-z_5)}\Big(\mathrm{Hlog}(z_2,[z_4,0,z_5]) - \mathrm{Hlog}(z_2,[z_4,1,z_5]) + \dots\Big). \tag{4.5}$$

However, in the case of the $\mathcal{P}$ and $\mathcal{T}$ integrals this property is no longer observed due to the non-existence of a common cross-ratios prefactor.

Furthermore, we noted that in all the obtained results the integrals only contained hyperlogarithms with up to three parameters in the arguments list (see (4.4)), in which case it is known [50, 51] that these can be expressed solely in terms of polylogarithms,[5] defined as

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \qquad \text{with } \ \mathrm{Li}_1(z) = -\log(1-z), \tag{4.6}$$

for $s \in \mathbb{N}$ and $z \in \mathbb{C}$. This contrasts to what would happen if the list of parameters had four or more parameters, in which case the hyperlogarithms can be expressed in terms of the so-called multiple polylogarithms

$$\mathrm{Li}_{s_1,\dots,s_r}(z_1,\dots,z_r) = \sum_{0<k_1<\dots<k_r}^{\infty} \frac{z_1^{k_1}\dots z_r^{k_r}}{k_1^{k_1}\dots k_r^{s_r}}, \qquad s_1,\dots,s_r \in \mathbb{N}. \tag{4.7}$$

Another important aspect that is worth pointing out is the divergences that come up in these integrals. If the above expression (4.5) is expanded according to the limits $z_2 \to 0$, $z_4 \to 0$, $z_5 \to 0$ in this respective order, one finds

$$\begin{aligned}\mathcal{B}_{123,456} \underset{z_i\to 0}{\sim} \frac{1}{z_5^2}\Big[&2\log(z_5)^2 - 4\log(z_2)\log(z_5) - 2\log(z_4)^2 + 4\log(z_2)\log(z_4) \\ &+ 6\log(z_5) - 2\log(z_4) - 4\log(z_2) + 4\Big] + \frac{2}{z_5} + \dots,\end{aligned} \tag{4.8}$$

where the dots denote subleading terms. From this expression we can clearly see both power-law and logarithmic divergences. To gain intuition about the former we must think about the physical significance of the limits we are taking. Indeed, we are sending three operators positioned at z_2, z_4 and z_5 to lie infinitesimally close to the operator at the origin, which in practice corresponds to taking the OPE between these four operators by the same order we are taking the limits. Therefore, let us recall the expression of the OPE between two primary scalars for simplicity

$$\mathcal{O}_1(0)\,\mathcal{O}_2(z) = \sum_k C_{12k}\,|z|^{\Delta_k-\Delta_1-\Delta_2}\Big[\mathcal{O}_k(z) + \dots\Big] \tag{4.9}$$

where the $\dots$ denote the contributions of the descendants and where for simplicity we are only representing the exchange of a scalar operator. From the above expression it is not

[5]We did not present the results in terms of polylogarithms since the Hlog basis gives shorter expressions. Moreover, this transformation introduces spurious branch cuts which should be carefully dealt with.

surprising that power-law divergences might show up due to the factor $|z|^{\Delta_k-\Delta_1-\Delta_2}$, provided that the conformal dimensions of the external operators and of the exchanged operator are such that the exponent is negative. Additionally, equation (4.8) tells us that the order with which we take the limits is relevant for the divergences as in this case we only see power-law divergences in the z_5, corresponding to the last operator whose OPE is taken. With this interpretation, the factor $1/z_5^2$ for example, corresponds to the contribution in the successive OPEs $z_2 \to 0$, $z_4 \to 0$, $z_5 \to 0$, respectively, of a particular choice of exchanged operators with conformal dimensions such that the exponent of z_5 is -2. On the other hand, the logarithmic divergences come about due to the existence of anomalous dimensions γ_n, generally defined as

$$\Delta = \Delta_0 + \epsilon\gamma_1 + \epsilon^2\gamma_2 + \ldots, \tag{4.10}$$

where Δ_0 is the dimension in the free theory and ϵ is some infinitesimally small parameter. When expanding the same factors $|z|^{-\Delta}$ appearing in the OPE, in the considered limits, we obtain something of the schematic form

$$|z|^{-\Delta} \underset{z\to 0}{\sim} |z|^{\Delta_0}\left(1 - \epsilon\gamma_1 \log(|z|) - \ldots\right), \tag{4.11}$$

which include divergent logarithmic terms in these limits. Naturally, a similar interpretation is also valid for the other limits mentioned above.

4.2 Light-cone limit

We now present the results obtained from evaluating one of the simplest conformal integrals from (3.22), (3.23), (3.24) in the light-cone limit $x_{56}^2 \to 0$, namely the double-penta integral $\mathcal{T}_{56,14,23}$, given by

$$\mathcal{T}_{56,14,23} = \int \frac{x_{18}^2\, x_{27}^2\, d^dx_7\, d^dx_8}{x_{17}^2 x_{28}^2 x_{38}^2 x_{47}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2}\,. \tag{4.12}$$

In order to evaluate it in the light-cone limit we follow the integration by regions that we explained above. However, for this integral we will only need to concern with the first region of (3.16), which will be denoted by $\mathcal{T}_I$, as we will only be interested in the leading order results. Indeed, from a simple analysis of the leading order behavior in x_{56}^2, it is possible to infer that $\mathcal{T}_I$ diverges as $(x_{56}^2)^{-1-2\epsilon}$ in the light-cone limit, while others are subleading.

After using the identity (3.17) accordingly to the assumptions of the first region, we end up with

$$\begin{aligned}\mathcal{T}_I = &\sum_{n_1,n_2,n_3,n_4,n_5} \frac{1}{(x_{15}^2)^{n_1+n_4}\,(x_{25}^2)^{n_2+n_5}\,(x_{35}^2)^{n_3}} \times \\ &\int \frac{d^dx_7\, d^dx_8\,(-2x_{15}\cdot x_{57})^{n_1}(-2x_{25}\cdot x_{58})^{n_2}(-2x_{35}\cdot x_{58})^{n_3}(2x_{15}\cdot x_{58})^{n_4}(2x_{25}\cdot x_{57})^{n_5}}{x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2}\end{aligned} \tag{4.13}$$

where the sums in n_4 and n_5 run only over $\{0,1\}$[6] and where for simplicity we have sent $x_4 \to \infty$. Moreover, we have dropped the terms x_{57}^2 and x_{58}^2 inside the parentheses in the numerator terms as they are subleading relatively to the ones we have kept.

[6]The reason why these sums run only over $\{0,1\}$ is because they come from the expansion of the numerators. Therefore, the associated binomial coefficients will be $\binom{1}{n}$, which only make sense for $0 \leq n \leq 1$.

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To evaluate this integral we use (3.19) while following the reasoning of appendix B and get to the final result

$$\mathcal{T}_I = \sum_{n_1,n_2,n_3,n_4,n_5} \frac{(-1)^{n_1+n_2+n_3}\, c_{n_1+n_5,n_2+n_3+n_4} (2x_{15}\cdot x_{56})^{n_1+n_4} (2x_{25}\cdot x_{56})^{n_2+n_5} (2x_{35}\cdot x_{56})^{n_3}}{(x_{15}^2)^{n_1+n_4}\,(x_{25}^2)^{n_2+n_5}\,(x_{35}^2)^{n_3}\,(x_{56}^2)^{5-d}}\,. \tag{4.14}$$

We may take the limit $\epsilon \to 0$ such that $(5-d) \to 1$ and explicitly see that this integral indeed diverges as $(x_{56}^2)^{-1}$ in the light-cone limit. It could also be beneficial to express this result in terms of the six-point cross-ratios, which could render its analysis simpler. This will not be done here since it would make the expression longer and thus not possible to fit it in a single line anymore. Nevertheless, the procedure would consist on using the identity

$$x_{i5}\cdot x_{56} = -\frac{1}{2}\left(x_{i5}^2 + x_{56}^2 - x_{i6}^2\right), \tag{4.15}$$

where we can drop x_{56}^2 in the light-cone limit, and finally use the definition of the cross-ratios (2.18).

5 Discussion

Correlation functions of $\mathcal{N}=4$ SYM, especially of half-BPS operators, have been a subject of interest for some decades as they provide us information about the CFT itself and also about its dual theory in AdS. The simpler non-trivial of these are the four-point functions, which have been studied both in the weak [6–9] and strong [10–15] regimes of the coupling and whose loop corrections are known up to a relatively high order. When it comes to correlators with even more points, however, not so much is known. This stems from the contrasting difficulty between knowing the integrands and actually being able to evaluate the conformal integrals which give the ℓ-loop corrections. In the case of five-point functions, the first few orders of the corrections for some of these functions have already been obtained under certain kinematical limits [17]. On the other hand, the integrands of the six-point functions are known up to 2-loops [16] and the respective integrals have been analyzed in [34], even though they were not able to explicitly evaluate them.

Here, we obtained a closed form expression for some of the two loop six-point integrals on a line configuration, which amounts to knowing all of them since they are related by cross-ratios transformations. For this we have adopted the Schwinger parameterization and used the `HyperInt` package [42, 43], which allows to compute Feynman integrals in the Schwinger parameters representation. We have found in all the obtained results that the final expressions were given by rational functions of the cross-ratios z_i, hyperlogarithm functions and the zeta functions. Moreover, we took the kinematical limits mentioned in section 4 for some results and observed that they possess both power-law and logarithmic divergences. We have given physical arguments to explain where they come from, in particular by thinking of these limits from an OPE perspective and bearing in mind anomalous dimensions. As stated in section 2, the integrals we are computing must satisfy the Drukker-Plefka twist and the chiral algebra twist since this is already true at the integrand level. Using the obtained results we verified that both properties indeed hold at the level of the integrals as well, as was to be expected.

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In addition, we also evaluated one of the double-penta integrals in the kinematics where two adjacent points become null separated. Although we only considered one of the three integrals which we claimed were easier to compute in this kinematical limit the other two can be obtained by following the exact same path. In order to evaluate this integral we implemented the asymptotic expansion method. In spite of the higher level of complexity, it would be nice to compute the remaining independent integrals (3.22), (3.23), (3.24) in the same kinematical limit in the future, as they are crucial for obtaining the full two-loop corrections and extract physical information. Taking additional null limits, despite simplifying, still requires us to evaluate a significant number of integrals in the several asymptotic regions, thus making it more complicated than the case we computed here.

As was mentioned several times throughout this paper, we chose to consider the line configuration and the light-cone limit in order to simplify the integrands and be able to evaluate the integrals. This stems from the high complexity of the same integrals for general configurations, which so far has prevented us from computing them. It would be nice if we could somehow work around this problem and obtain the results for a general disposition of the six-point function. Naturally, this ought to be quite challenging so we could start off by taking one point out of the line and do the same analysis. We would then continue to take more points out of the line and gradually move on to more general but harder configurations and assess these integrals.

Finally, we could try to tackle these issues from an integrability-based approach, as was done for four- and five-point functions [37–40], and compare the outcomes with our results.

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A Graph Polynomials Ψ and Φ

Here we give the explicit expressions for the graph polynomials Φ and Ψ that are obtained after writing the double-penta integral $\mathcal{T}_{16,23,45}$ in the Schwinger parameters representation (3.3). In particular, these are given by

$$\begin{aligned}\Psi = &\alpha_1\alpha_2 + \alpha_1\alpha_5 + \alpha_1\alpha_6 + \alpha_1\alpha_7 + \alpha_1\alpha_8 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_2\alpha_7 + \alpha_2\alpha_9 + \alpha_3\alpha_5\\ &+ \alpha_3\alpha_6 + \alpha_3\alpha_7 + \alpha_3\alpha_8 + \alpha_4\alpha_5 + \alpha_4\alpha_6 + \alpha_4\alpha_7 + \alpha_4\alpha_8 + \alpha_5\alpha_7 + \alpha_5\alpha_9 + \alpha_6\alpha_7\\ &+ \alpha_6\alpha_9 + \alpha_7\alpha_8 + \alpha_7\alpha_9 + \alpha_8\alpha_9,\end{aligned} \tag{A.1}$$

$$
\begin{aligned}
\Phi =\ & \alpha_1\alpha_2\alpha_3 z_2^2 + \alpha_2\alpha_3\alpha_4 z_2^2 + \alpha_1\alpha_3\alpha_5 z_2^2 + \alpha_3\alpha_4\alpha_5 z_2^2 + \alpha_1\alpha_3\alpha_6 z_2^2 + \alpha_3\alpha_4\alpha_6 z_2^2 \\
& + \alpha_1\alpha_3\alpha_7 z_2^2 + \alpha_2\alpha_3\alpha_7 z_2^2 + \alpha_3\alpha_4\alpha_7 z_2^2 + \alpha_3\alpha_5\alpha_7 z_2^2 + \alpha_3\alpha_6\alpha_7 z_2^2 + \alpha_1\alpha_2\alpha_8 z_2^2 \\
& + \alpha_1\alpha_3\alpha_8 z_2^2 + \alpha_2\alpha_3\alpha_8 z_2^2 + \alpha_2\alpha_4\alpha_8 z_2^2 + \alpha_3\alpha_4\alpha_8 z_2^2 + \alpha_1\alpha_5\alpha_8 z_2^2 + \alpha_3\alpha_5\alpha_8 z_2^2 \\
& + \alpha_4\alpha_5\alpha_8 z_2^2 + \alpha_1\alpha_6\alpha_8 z_2^2 + \alpha_3\alpha_6\alpha_8 z_2^2 + \alpha_4\alpha_6\alpha_8 z_2^2 + \alpha_1\alpha_7\alpha_8 z_2^2 + \alpha_2\alpha_7\alpha_8 z_2^2 \\
& + \alpha_4\alpha_7\alpha_8 z_2^2 + \alpha_5\alpha_7\alpha_8 z_2^2 + \alpha_6\alpha_7\alpha_8 z_2^2 + \alpha_2\alpha_3\alpha_9 z_2^2 + \alpha_3\alpha_5\alpha_9 z_2^2 + \alpha_3\alpha_6\alpha_9 z_2^2 \\
& + \alpha_3\alpha_7\alpha_9 z_2^2 + \alpha_2\alpha_8\alpha_9 z_2^2 + \alpha_3\alpha_8\alpha_9 z_2^2 + \alpha_5\alpha_8\alpha_9 z_2^2 + \alpha_6\alpha_8\alpha_9 z_2^2 + \alpha_7\alpha_8\alpha_9 z_2^2 \\
& - 2\alpha_2\alpha_3\alpha_4 z_2 - 2\alpha_3\alpha_4\alpha_5 z_2 - 2\alpha_3\alpha_4\alpha_6 z_2 - 2\alpha_3\alpha_4\alpha_7 z_2 - 2z_4\alpha_3\alpha_5\alpha_7 z_2 \\
& - 2z_5\alpha_3\alpha_6\alpha_7 z_2 - 2\alpha_3\alpha_4\alpha_8 z_2 - 2z_4\alpha_1\alpha_5\alpha_8 z_2 - 2z_4\alpha_3\alpha_5\alpha_8 z_2 - 2z_4\alpha_4\alpha_5\alpha_8 z_2 \\
& - 2z_5\alpha_1\alpha_6\alpha_8 z_2 - 2z_5\alpha_3\alpha_6\alpha_8 z_2 - 2z_5\alpha_4\alpha_6\alpha_8 z_2 - 2\alpha_4\alpha_7\alpha_8 z_2 - 2z_4\alpha_5\alpha_7\alpha_8 z_2 \\
& - 2z_5\alpha_6\alpha_7\alpha_8 z_2 - 2z_4\alpha_2\alpha_3\alpha_9 z_2 - 2z_4\alpha_3\alpha_5\alpha_9 z_2 - 2z_4\alpha_3\alpha_6\alpha_9 z_2 - 2z_4\alpha_3\alpha_7\alpha_9 z_2 \\
& - 2z_4\alpha_3\alpha_8\alpha_9 z_2 - 2z_4\alpha_5\alpha_8\alpha_9 z_2 - 2z_5\alpha_6\alpha_8\alpha_9 z_2 - 2z_4\alpha_7\alpha_8\alpha_9 z_2 + \alpha_1\alpha_2\alpha_4 \\
& + \alpha_2\alpha_3\alpha_4 + z_4^2\alpha_1\alpha_2\alpha_5 + z_4^2\alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_4\alpha_5 + z_4^2\alpha_2\alpha_4\alpha_5 + \alpha_3\alpha_4\alpha_5 + z_5^2\alpha_1\alpha_2\alpha_6 \\
& + z_5^2\alpha_2\alpha_3\alpha_6 + \alpha_1\alpha_4\alpha_6 + z_5^2\alpha_2\alpha_4\alpha_6 + \alpha_3\alpha_4\alpha_6 + z_4^2\alpha_1\alpha_5\alpha_6 + z_5^2\alpha_1\alpha_5\alpha_6 \\
& - 2z_4z_5\alpha_1\alpha_5\alpha_6 + z_4^2\alpha_3\alpha_5\alpha_6 + z_5^2\alpha_3\alpha_5\alpha_6 - 2z_4z_5\alpha_3\alpha_5\alpha_6 + z_4^2\alpha_4\alpha_5\alpha_6 + z_5^2\alpha_4\alpha_5\alpha_6 \\
& - 2z_4z_5\alpha_4\alpha_5\alpha_6 + \alpha_1\alpha_4\alpha_7 + \alpha_2\alpha_4\alpha_7 + \alpha_3\alpha_4\alpha_7 + z_4^2\alpha_1\alpha_5\alpha_7 + z_4^2\alpha_2\alpha_5\alpha_7 \\
& + z_4^2\alpha_3\alpha_5\alpha_7 + z_4^2\alpha_4\alpha_5\alpha_7 - 2z_4\alpha_4\alpha_5\alpha_7 + \alpha_4\alpha_5\alpha_7 + z_5^2\alpha_1\alpha_6\alpha_7 + z_5^2\alpha_2\alpha_6\alpha_7 \\
& + z_5^2\alpha_3\alpha_6\alpha_7 + z_5^2\alpha_4\alpha_6\alpha_7 - 2z_5\alpha_4\alpha_6\alpha_7 + \alpha_4\alpha_6\alpha_7 + z_4^2\alpha_5\alpha_6\alpha_7 + z_5^2\alpha_5\alpha_6\alpha_7 \\
& - 2z_4z_5\alpha_5\alpha_6\alpha_7 + \alpha_1\alpha_4\alpha_8 + \alpha_3\alpha_4\alpha_8 + z_4^2\alpha_1\alpha_5\alpha_8 + z_4^2\alpha_3\alpha_5\alpha_8 + z_4^2\alpha_4\alpha_5\alpha_8 \\
& + z_5^2\alpha_1\alpha_6\alpha_8 + z_5^2\alpha_3\alpha_6\alpha_8 + z_5^2\alpha_4\alpha_6\alpha_8 + \alpha_4\alpha_7\alpha_8 + z_4^2\alpha_5\alpha_7\alpha_8 + z_5^2\alpha_6\alpha_7\alpha_8 \\
& + z_4^2\alpha_1\alpha_2\alpha_9 + z_4^2\alpha_2\alpha_3\alpha_9 + z_4^2\alpha_2\alpha_4\alpha_9 - 2z_4\alpha_2\alpha_4\alpha_9 + \alpha_2\alpha_4\alpha_9 + z_4^2\alpha_1\alpha_5\alpha_9 \\
& + z_4^2\alpha_2\alpha_5\alpha_9 + z_4^2\alpha_3\alpha_5\alpha_9 + z_4^2\alpha_4\alpha_5\alpha_9 - 2z_4\alpha_4\alpha_5\alpha_9 + \alpha_4\alpha_5\alpha_9 + z_4^2\alpha_1\alpha_6\alpha_9 \\
& + z_5^2\alpha_2\alpha_6\alpha_9 + z_4^2\alpha_3\alpha_6\alpha_9 + z_4^2\alpha_4\alpha_6\alpha_9 - 2z_4\alpha_4\alpha_6\alpha_9 + \alpha_4\alpha_6\alpha_9 + z_4^2\alpha_5\alpha_6\alpha_9 \\
& + z_5^2\alpha_5\alpha_6\alpha_9 - 2z_4z_5\alpha_5\alpha_6\alpha_9 + z_4^2\alpha_1\alpha_7\alpha_9 + z_4^2\alpha_2\alpha_7\alpha_9 + z_4^2\alpha_3\alpha_7\alpha_9 + z_4^2\alpha_4\alpha_7\alpha_9 \\
& - 2z_4\alpha_4\alpha_7\alpha_9 + \alpha_4\alpha_7\alpha_9 + z_4^2\alpha_6\alpha_7\alpha_9 + z_5^2\alpha_6\alpha_7\alpha_9 - 2z_4z_5\alpha_6\alpha_7\alpha_9 + z_4^2\alpha_1\alpha_8\alpha_9 \\
& + z_4^2\alpha_3\alpha_8\alpha_9 + z_4^2\alpha_4\alpha_8\alpha_9 - 2z_4\alpha_4\alpha_8\alpha_9 + \alpha_4\alpha_8\alpha_9 + z_4^2\alpha_5\alpha_8\alpha_9 + z_5^2\alpha_6\alpha_8\alpha_9 + z_4^2\alpha_7\alpha_8\alpha_9 .
\end{aligned} \tag{A.2}
$$

B Asymptotic expanded integrals

When evaluating the integral (4.13) in the light-cone limit $x_{56}^2 \to 0$ we claimed that we could make use of the equality (3.19) even though our integral has more terms than what appeared in this identity. Here we will briefly explain the basic idea of why such result can be used for our case, as it is merely a generalization. For this we will consider a simple example.

Let us then suppose that we have an integral very similar to the one appearing in the left-hand side of (3.19) but with an extra term of the sort $(v \cdot x_{57})^k$, namely

$$
\int \frac{d^d x_7\, d^d x_8\, (v_1 \cdot x_{57})^{k_1}\, (v_2 \cdot x_{57})^{k_2}\, (w \cdot x_{58})^a}{x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2}, \tag{B.1}
$$

where v_i and w are some vectors, which in practice are going to be distances x_{ij} as in (3.19), whereas k_i and a are integers.

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In order to use the result (3.19) we need to combine the first two terms in the numerator into a single one. This can be achieved by making use of the following identity

$$\sum_{k_1,k_2=0}^{\infty} \binom{k_1+k_2}{k_2} (v_1 \cdot x_{57})^{k_1} (v_2 \cdot x_{57})^{k_2} q^{k_2} = \sum_{n=0}^{\infty} \left((v_1 + q\, v_2) \cdot x_{57}\right)^n, \tag{B.2}$$

where q is an auxiliary parameter which helps us keeping track of the coefficients associated to the different powers of $(v_1 \cdot x_{57})$ and $(v_2 \cdot x_{57})$ and that must be evaluated at $q = 1$ in the end. Hence, if we sum the previous integral with the binomial coefficients

$$\sum_{k_1,k_2=0}^{\infty} \binom{k_1+k_2}{k_2} \int d^d x_7\, d^d x_8\, \frac{(v_1 \cdot x_{57})^{k_1}\, (v_2 \cdot x_{57})^{k_2}\, q^{k_2}\, (w \cdot x_{58})^a}{x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2}, \tag{B.3}$$

and use the previous equality, we get

$$\sum_{n=0}^{\infty} \int \frac{d^d x_7\, d^d x_8\, \left((v_1 + q\, v_2) \cdot x_{57}\right)^n\, (w \cdot x_{58})^a}{x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2} = \sum_{n=0}^{\infty} c_{n,a} \frac{\left((v_1 + q\, v_2) \cdot x_{56}\right)^n (w \cdot x_{56})^a}{(x_{56}^2)^{5-d}}, \tag{B.4}$$

where the right-hand side results from direct application of (3.19).

We now expand the binomial in the numerator

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,a} \binom{n}{m} \frac{(v_1 \cdot x_{56})^{n-m}\, (v_2 \cdot x_{56})^m\, q^m\, (w \cdot x_{56})^a}{(x_{56}^2)^{5-d}}, \tag{B.5}$$

and then reorganize the sum by making the transformations $n \to k_1 + k_2$ and $m \to k_2$, such that we can compare with (B.3). Doing so we obtain

$$\sum_{k_1,k_2=0}^{\infty} \binom{k_1+k_2}{k_2} c_{k_1+k_2,a} \frac{(v_1 \cdot x_{56})^{k_1}\, (v_2 \cdot x_{56})^{k_2}\, q^{k_2}\, (w \cdot x_{56})^a}{(x_{56}^2)^{5-d}}, \tag{B.6}$$

which when compared with the aforementioned equation and matching the orders in the auxiliary parameter q yields the following equality

$$\int \frac{d^d x_7\, d^d x_8\, (v_1 \cdot x_{57})^{k_1}\, (v_2 \cdot x_{57})^{k_2}\, (w \cdot x_{58})^a}{x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2} = c_{k_1+k_2,a} \frac{(v_1 \cdot x_{56})^{k_1}\, (v_2 \cdot x_{56})^{k_2}\, (w \cdot x_{56})^a}{(x_{56}^2)^{5-d}}. \tag{B.7}$$

Therefore, we observe that the generalization of (3.19) is very close to the result itself but involves considering the exponents of the additional terms in the labels of the coefficients $c_{n,m}$. We would like to point out that these arguments remain valid whenever we have any number of these terms of the sort $(v \cdot x_{57})^k$ as we can employ this procedure several times to put them into a single binomial and then expand them again after integration. Naturally, this is also true under $7 \leftrightarrow 8$ since x_7 and x_8 are integration variables.

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