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Matrix quantization of gravitational edge modes

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Abstract: Gravitational subsystems with boundaries carry the action of an infinite-dimensional symmetry algebra, with potentially profound implications for the quantum theory of gravity. We initiate an investigation into the quantization of this corner symmetry algebra for the phase space of gravity localized to a region bounded by a 2-dimensional sphere. Starting with the observation that the algebra $\mathfrak{sdiff}(S^2)$ of area-preserving diffeomorphisms of the 2-sphere admits a deformation to the finite-dimensional algebra $\mathfrak{su}(N)$, we derive novel finite-N deformations for two important subalgebras of the gravitational corner symmetry algebra. Specifically, we find that the area-preserving hydrodynamical algebra $\mathfrak{sdiff}(S^2) \oplus_{\mathcal{L}} \mathbb{R}^{S^2}$ arises as the large-N limit of $\mathfrak{sl}(N,\mathbb{C}) \oplus \mathbb{R}$ and that the full area-preserving corner symmetry algebra $\mathfrak{sdiff}(S^2) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^{S^2}$ is the large-N limit of the pseudo-unitary group $\mathfrak{su}(N,N)$. We find matching conditions for the Casimir elements of the deformed and continuum algebras and show how these determine the value of the deformation parameter N as well as the representation of the deformed algebra associated with a quantization of the local gravitational phase space. Additionally, we present a number of novel results related to the various algebras appearing, including a detailed analysis of the asymptotic expansion of the $\mathfrak{su}(N)$ structure constants, as well as an explicit computation of the full $\mathfrak{diff}(S^2)$ structure constants in the spherical harmonic basis. A consequence of our work is the definition of an area operator which is compatible with the deformation of the area-preserving corner symmetry at finite N.

Keywords: Classical Theories of Gravity, Matrix Models, Models of Quantum Gravity

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1 Introduction

Symmetry has long been a guiding principle in developing and understanding physical theories. This is especially true in quantum gravity, where an absence of experimental constraints forces us to rely on general physical principles such as symmetry to elucidate the conceptual and technical maze.

The defining symmetry of general relativity is the group of diffeomorphisms of spacetime. Being gauge symmetries, these have long been considered to be devoid of physical content and a mere redundancy of description. However, when considering a spacetime with a boundary, the situation changes drastically. This boundary could be a boundary at infinity, with suitable falloff conditions on the fields, or, motivated by considerations of entangling surfaces, could be located at a finite distance. Boundaries force us to consider degrees of freedom—*edge modes*—localized at the boundary which otherwise would be pure gauge. These dynamical variables transform under a symmetry group which, for general relativity in the metric formulation, is given by [1]

$$G_{\mathrm{SL}(2,\mathbb{R})}(S) = \mathrm{Diff}(S) \ltimes \mathrm{SL}(2,\mathbb{R})^S, \tag{1.1}$$

where the *corner* S is the boundary of a spatial or null Cauchy surface for the region under consideration, hence codimension-2 in spacetime.[1] $\mathrm{SL}(2,\mathbb{R})^S$ denotes the group of $\mathrm{SL}(2,\mathbb{R})$-valued functions on S, $\mathrm{Diff}(S)$ the group of diffeomorphisms of S, and $\ltimes$ indicates a semidirect product structure in which the diffeomorphisms act on functions in the usual way via pullbacks. In what follows we will primarily be interested in the Lie algebra of $G_{\mathrm{SL}(2,\mathbb{R})}(S)$:

$$\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S) = \mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S, \tag{1.2}$$

where the subscript $\mathcal{L}$ means the action by Lie derivative. In this work, we make two important restrictions: we consider $3+1$-dimensional spacetimes, so that S is two-dimensional, and restrict S to have the topology of a sphere.

Equipped with a physical system (a region of space) and a physical symmetry group (1.1), we can then follow the spirit of Wigner's approach to quantum mechanics [13]: studying

[1] It has also been shown that this symmetry group of corner-preserving transformations, which we are interested in here, is universal for all diffeomorphism-invariant theories [2]. This symmetry group can be extended to include surface deformations, arising from diffeomorphisms that move the corner S itself. We will not consider this extended group here, but it has been examined in several recent works [2–6]. Recent studies of the corner symmetry group, its extension and its link with the asymptotic symmetry group also include [7–12].

unitary representations of this symmetry group and identifying the carrying space of relevant unitary representations as the Hilbert space of our theory. There are broadly two ways to proceed, which we dub *representation* and *quantization*.[2]

1. **Representation.** In this approach, we construct unitary representations of $G_{\mathrm{SL}(2,\mathbb{R})}(S)$ using existing methods, such as the method of induced representations [16–18] or the method of coadjoint orbits [19–23]. For example, in the case of three-dimensional BMS group, this program of quantization has been carried out by Barnich and Oblak using both induced representation [24] and coadjoint representation [25], and their equivalence has been argued [25, section 4.1]. The first step in studying the representation theory of $G_{\mathrm{SL}(2,\mathbb{R})}(S)$ using coadjoint orbits is obtaining a classification of the orbits, a task that has been completed in our previous work [26]. One then needs to construct the Hilbert space by methods such as geometric quantization [19, 27–30] or brane quantization [31, 32]. It is thus in principle possible to follow this path. However, complications related to the infinite-dimensionality and topological subtleties of the symmetry group (1.1), which are already present for its Diff(S) subgroup, make the implementation of this approach difficult.

2. **Quantization.** Quantum mechanics allows for a more general class of possibilities than the preceding. Namely, it is sufficient to find a group that approaches $G_{\mathrm{SL}(2,\mathbb{R})}(S)$ in a suitable classical limit. A canonical example of this procedure is provided by the quantum mechanics of a single particle in one dimension. The classical phase space of this system, with coordinates x, p (with $\{x, p\} = 1$) carries a representation of the algebra $\mathfrak{sdiff}(\mathbb{R}^2)$ of area-preserving diffeomorphisms of the phase space plane. Quantum-mechanically this symmetry algebra is deformed to an algebra of infinite-dimensional matrices, with the classical symmetry recovered only in the limit $\hbar \to 0$. In this case, quantization preserves the Heisenberg subalgebra generated by x, p, and the constant function 1, but commutators of more general functions of x and p acquire $\mathcal{O}(\hbar)$ corrections due to operator ordering ambiguities.

Here we will follow the second approach, a choice that requires some physical justification. The method based on finding an exact representation of the symmetry algebra carries several drawbacks. The first is that our algebra is infinite-dimensional, and although the representation theory of certain infinite-dimensional algebras such as Virasoro or Kac-Moody is well-developed, very few tools exist for characterizing the unitary representations of the symmetry groups encountered in the present work. Another drawback of such representations is that the operator product of generators at equal points is ill-defined. While this is common in continuum field theory, it is not welcomed for a theory of quantum gravity, as it assumes the continuum structure of spacetime persists even to distances shorter

[2] There is a third possibility, which we might call *polymerization* where the measure on the sphere is taken to be a discrete measure rather than continuous. This possibility is explored in [14] and is close in spirit to the loop gravity approach of quantum gravity [15]. This results in a discrete representation of the Lie group, which is discontinuous i.e. one in which the Lie algebra generators are not differentiable along the sphere. Since we are primarily interested in continuous representations of the Lie algebra, we will not follow this option here.

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than the Planck scale. Instead one would expect a fundamental theory to help resolve such divergences. Composite operators play a central role in the classification of coadjoint orbits of our symmetry group [26] and are therefore essential to its quantization.

The introduction of a deformation of the algebra at the Planck scale is strongly suggested by the finite entropy of black holes and other causal horizons. To see how the symmetry group (1.1) relates to entropy, we first consider the simpler example of Yang-Mills theory. In Yang-Mills theory with gauge group g, the symmetry group analogous to (1.1) is a direct product of the gauge group over points of the surface S [1]:

$$G_{\mathrm{YM}} = g^S := \prod_{x \in S} g(x), \tag{1.3}$$

where each $g(x)$ is an independent copy of the Lie group g at each point x of the surface S. We will use the shorthand notation g^S for this product here and throughout the paper. This infinite-dimensional group can be made precise by including a lattice regulator [33–38]. The states of a region bounded by the surface S transform nontrivially under G, and when decomposed into irreducible representations we obtain a "$\log \dim(R)$" term.

$$S_{\mathrm{YM}} = -\sum_R p_R \log p_R + \sum_R p_R \log(\rho_R) + \sum_R p_R \log \dim(R), \tag{1.4}$$

where p_R is a probability distribution over all irreducible representations of G_{YM}, ρ_R a set of density matrices and $\dim(R)$ the dimension of each irreducible representation. All three terms in (1.4) diverge when the number of points in S is taken to infinity; this is the familiar ultraviolet divergence of the entanglement entropy and comes from the infinite density of ultraviolet degrees of freedom. These ultraviolet divergences are expected in a continuum field theory, and the contribution of the edge modes plays an important role in the relation between the entanglement entropy and the conformal anomaly [36].

In quantum gravity, we don't expect the behavior of continuum quantum field theory to persist into the infinite ultraviolet. At the perturbative level, quantum gravity is much like a gauge theory and there has been much progress in calculating one-loop perturbative quantum-gravitational corrections to the entanglement entropy of gravitons [39–41]. It is however expected that the non-perturbative result for the entropy should be finite and universal [42, 43]. This is in some tension with an infinite-dimensional symmetry group of the form (1.1), which has a continuum of generators, leading to an ultraviolet divergence similar to that of quantum field theory. The continuum limit is already present kinematically in the structure of the commutators of the algebra (1.2) which contain delta functions localized at coincident points of S. A further complication comes from the noncompact $\mathfrak{sl}(2,\mathbb{R})$ factor in (1.2) — unitary representations of noncompact groups are infinite-dimensional and this would lead to infinities in the naïve application of (1.4).

The quantization method we pursue in this work is motivated by an analogy with single-particle quantum mechanics. It was shown in [26] that the coadjoint orbits of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ can be reduced to those of the Wigner little group $\mathfrak{sdiff}(S)$; the latter is the algebra of Hamiltonian transformations of a two-dimensional phase space, which can be quantized in much the same way as the familiar case of the standard (x, p) phase space. This quantization leads to a deformation of the symmetry algebra in which functions on S^2 are replaced with their fuzzy sphere analogs, which are noncommuting hermitian matrices. The resulting deformed algebra is finite-dimensional and isomorphic to $\mathfrak{su}(N)$, where $1/N$ plays the role of a deformation parameter, which has a physical interpretation as a fundamental unit of area. This parameter is analogous to the introduction of Planck's constant h, which effectively discretizes the phase space into a fuzzy space with cells of area h. We note that Planck was motivated by understanding Boltzmann's entropy formula,[3] and it would be remarkable if the same mechanism responsible for the finite Boltzmann entropy could be responsible for a finite Bekenstein-Hawking entropy.

The deformation of $\mathfrak{sdiff}(S)$ we consider is well known: it arises, for example, in string theory in the context of matrix models [45–47]. There is also some similarity to the holographic spacetime model of ref. [48], in which cosmological horizons are replaced with fuzzy spaces. The key distinction here is that rather than taking a "bottom-up" approach and introducing the area-preserving diffeomorphism symmetry by hand, we have derived it "top-down" from the symmetries of general relativity. This allows us to relate the quantum-mechanical generators to geometric quantities in general relativity. Moreover, we will show how to incorporate the boost symmetry of the normal plane, which has not appeared previously in the aforementioned models. This boost symmetry plays an especially important role in the context of horizon thermodynamics, where the global boost generator for Killing horizons is the modular Hamiltonian.

The quantization procedure we consider is accompanied by a deformation of the classical symmetry algebra. These deformations are ubiquitous in physics: two classical examples are the deformation of the Galilean algebra into the semi-simple Lorentz algebra and the deformation of the Poincaré algebra to the semi-simple de Sitter algebra. In both cases, a deformation parameter is needed and the deformation is more stable than the original algebra. In the examples just mentioned these deformation parameters are constants of nature such as the cosmological constant or the inverse speed of light. In the case of deforming $\mathfrak{sdiff}(S)$ to $\mathfrak{su}(N)$, the small parameter is $1/N$. In light of the analogy with quantum mechanics, it is natural to guess that N is related to the area of the surface S in Planck units, which is supported by the matching conditions for Casimir operators described in section 4.2.

The results of section 4.2 indicate that once the deformation has been identified, the quantization procedure is largely constrained by the matching conditions on the Casimirs. However, these conditions fail to fully fix the quantization for two reasons. First, although the matching determines the representation for a folium of the gravitational phase space defined by fixing the values of all Casimirs, the full phase space is in general a sum of several such folia. The quantization is then expected to be a sum of different representations, and it is a nontrivial problem to determine the multiplicity of the representations appearing in the quantization. The second reason that the resulting quantization is not fully determined is that it may not be the case that the deformed algebras explored in this work are unique. In particular, there may be other deformations, or the quantization may proceed by

[3]For a historical account, see [44].

$$
\begin{tikzcd}
\mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S \arrow[r] \arrow[d, shift left, "\text{Large-}N\ \text{Limit}"', leftarrow] \arrow[d, shift right] & \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S \arrow[r] \arrow[d, shift left, leftarrow] \arrow[d, shift right] & \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S \arrow[r] \arrow[d, shift left, leftarrow] \arrow[d, shift right] & \mathfrak{sdiff}(S) \arrow[d, shift left, leftarrow] \arrow[d, shift right, "\text{Matrix Regularization}"] \\
??? & \mathfrak{su}(N,N) \arrow[l] & \mathfrak{sl}(N,\mathbb{C}) \oplus \mathbb{R} \arrow[l] & \mathfrak{su}(N) \arrow[l]
\end{tikzcd}
$$

Figure 1. The corner symmetry algebra $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, its subalgebras and their regularizations. The regularized algebra whose large-N limit is $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ is missing in our analysis.

representing the original, undeformed algebras. The perspective taken in this paper is that the deformations we identify provide nontrivial, finite-dimensional deformed algebras that can be viewed as regulated versions of the continuum algebras. The question of uniqueness of these deformations remains open.

1.1 Main results of the paper

We present here a technical summary of our main results. To make our way toward the quantization of the full group $G_{\mathrm{SL}(2,\mathbb{R})}(S)$, we start by analyzing some of its important subgroups. In the problem of classification of coadjoint orbits of $G_{\mathrm{SL}(2,\mathbb{R})}(S)$, one important subgroup is the group of area-preserving diffeomorphisms of S [26]. Choosing an area form ν on S, this is the subgroup of $\mathrm{Diff}(S)$ that preserves ν

$$
\mathrm{SDiff}(S)_\nu \equiv \{ f \in \mathrm{Diff}(S) \,|\, f^*\nu = \nu \}, \tag{1.5}
$$

whose Lie algebra is denoted as $\mathfrak{sdiff}(S)_\nu$. All area forms on a sphere are diffeomorphic up to an overall scaling, hence we can always restrict attention to the natural volume form on the unit round sphere, and denote the corresponding area-preserving subalgebra simply as $\mathfrak{sdiff}(S)$. It is a celebrated result that $\mathfrak{sdiff}(S)$ can be viewed as a large-N limit of $\mathfrak{su}(N)$,[4]

$$
\mathfrak{sdiff}(S) \quad \underset{\text{Large-}N\ \text{Limit}}{\overset{\text{Matrix Regularization}}{\rightleftarrows}} \quad \mathfrak{su}(N). \tag{1.6}
$$

The large-N limit of $\mathfrak{su}(N)$ has been part of a vast investigation in the past starting with the pioneering work of 't Hooft on QCD [51].

Our goal is to generalize the procedure (1.6) by which the large-N limit of $\mathfrak{su}(N)$ approaches $\mathfrak{sdiff}(S)$ to obtain a new sequence of deformed algebras that limit to the full corner symmetry algebra. This generalization proceeds via the sequence of subalgebras depicted in figure 1. From the full algebra $\mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$ in the upper-left corner, we first fix an area form and consider the subalgebra $\mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$ which preserves this area form. Further fixing a hyperbolic generator of $\mathfrak{sl}(2,\mathbb{R})$ at each point on S reduces the algebra to $\mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S$ consisting of area-preserving diffeomorphisms and pointwise boosts. Finally, fixing the boost generator to zero we are left with the little group $\mathfrak{sdiff}(S)$.

[4]Note that the precise meaning of the large N limit of $\mathfrak{su}(N)$ is ambiguous, and different limiting procedures can result in non-isomorphic infinite-dimensional Lie algebras (see e.g. [47, 49, 50]). The way the limit to $\mathfrak{sdiff}(S)$ should be understood is in terms of quasi-limits, as defined in [50].

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Then starting from the known regularization of $\mathfrak{sdiff}(S)$ by $\mathfrak{su}(N)$ we proceed leftward along the bottom row of the diagram, finding an increasing sequence of finite-dimensional Lie algebras compatible with the $\mathfrak{su}(N)$ regularization of $\mathfrak{sdiff}(S)$.

The regularization procedure is straightforward: we write the mode expansion of the generators on the sphere and look for matrices whose commutators agree with the Poisson brackets up to small corrections. The first subalgebra we consider is:

$$\mathfrak{c}_{\mathbb{R}}(S) = \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S. \tag{1.7}$$

Following [26], we denote the smeared phase space generators of $\mathfrak{sdiff}(S)$ and $\mathbb{R}^S$ as $J[\phi]$ and $N[\lambda]$, respectively. The smearing parameters (ϕ, λ) are both real-valued functions on the sphere. These functions can be expanded in the basis of spherical harmonics $\{Y_\alpha\}$ where α stands for a pair of indices (A, a) with $A \in \{0, 1, \ldots\}$ denoting the total angular momentum quantum number, and $a \in \{-A, \ldots, A\}$ is the magnetic quantum number. The generators in this basis are denoted $J_\alpha \equiv J[Y_\alpha]$, $N_\alpha = N[Y_\alpha]$, and the Poisson brackets of the generators (J_α, N_α) implementing the Hamiltonian action of (1.7) on the gravitational phase space are given by

$$\begin{aligned} \{J_\alpha, J_\beta\} &= C_{\alpha\beta}{}^{\gamma} J_\gamma, \\ \{J_\alpha, N_\beta\} &= C_{\alpha\beta}{}^{\gamma} N_\gamma, \\ \{N_\alpha, N_\beta\} &= 0. \end{aligned} \tag{1.8}$$

The $C_{\alpha\beta}{}^{\gamma}$ are structure constants of the Poisson bracket on S^2,

$$\epsilon^{AB}(\nabla_A Y_\alpha \nabla_B Y_\beta) = C_{\alpha\beta}{}^{\gamma} Y_\gamma, \tag{1.9}$$

where ϵ^{AB} is the inverse of the standard volume form on the unit radius sphere. An explicit expression for $C_{\alpha\beta}{}^{\gamma}$ can be given in terms of Wigner 3j-symbols (see appendix A.1 for details) [52, 53].

As discussed in section 3.3, the matrix regularization of (1.8) is achieved by replacing the generators with matrices of dimension $2N \times 2N$. The generators of the deformed $\mathfrak{sdiff}(S)$ subalgebra correspond to matrices of the form $\widehat{Y}_{\bullet\alpha} = \mathbb{1}_2 \otimes \widehat{Y}_\alpha$, obtained by simply tensoring the fuzzy spherical harmonics with the 2×2 identity matrix. The remaining generators are of the form $\widehat{Y}_{1\alpha} = \rho_1 \otimes \widehat{Y}_\alpha$, where $\rho_1 = \frac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Together, the commutators of $(\widehat{Y}_{\bullet\alpha}, \widehat{Y}_{1\alpha})$ are taken to define the deformed algebra in a $2N$-dimensional representation. Denoting the corresponding generators of the deformed Lie algebra (X_α, Z_α), appropriately rescaled, the Lie brackets take the form

$$\begin{aligned} [X_\alpha, X_\beta] &= \widehat{C}_{\alpha\beta}{}^{\gamma} X_\gamma, \\ [X_\alpha, Z_\beta] &= \widehat{C}_{\alpha\beta}{}^{\gamma} Z_\gamma, \\ [Z_\alpha, Z_\beta] &= -\frac{1}{N^2} \widehat{C}_{\alpha\beta}{}^{\gamma} X_\gamma, \end{aligned} \tag{1.10}$$

where $\widehat{C}_{\alpha\beta}{}^{\gamma}$ denote the structure constants for the fuzzy spherical harmonic commutator, $[\widehat{Y}_\alpha, \widehat{Y}_\beta] = \frac{2i}{N} \widehat{C}_{\alpha\beta}{}^{\gamma} \widehat{Y}_\gamma$. Since $\widehat{C}_{\alpha\beta}{}^{\gamma} \to C_{\alpha\beta}{}^{\gamma}$ as $N \to \infty$, we find that the algebra defined by (X_α, Z_α) approaches the Poisson bracket algebra (1.8) in the large N limit. For finite N, one can show that the algebra is isomorphic to $\mathfrak{sl}(N, \mathbb{C}) \oplus \mathbb{R}$ (after removing the central generator X_{00} which does not generate a diffeomorphism in the continuum algebra). This establishes the following novel large N limit:

$$\mathfrak{c}_{\mathbb{R}}(S) = \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S \quad \underset{\text{Large-}N\text{ Limit}}{\overset{\text{Matrix Regularization}}{\rightleftarrows}} \quad \mathfrak{c}_{\mathbb{R}}(N) \simeq \mathfrak{sl}(N,\mathbb{C}) \oplus \mathbb{R}. \tag{1.11}$$

Going a step further, we then consider an enlargement of the algebra by including the full set of pointwise $\mathfrak{sl}(2,\mathbb{R})$ transformations:

$$\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S) = \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S. \tag{1.12}$$

The Hamiltonian generators of the Poisson bracket algebra in the spherical harmonic basis are now denoted $(J_\alpha, N_{a\alpha})$, with $a = 0,1,2$ an $\mathfrak{sl}(2,\mathbb{R})$ index. The Poisson brackets are given by

$$\begin{aligned} \{J_\alpha, J_\beta\} &= C_{\alpha\beta}{}^{\gamma} J_\gamma, \\ \{J_\alpha, N_{a\beta}\} &= C_{\alpha\beta}{}^{\gamma} N_{a\gamma}, \\ \{N_{a\alpha}, N_{b\beta}\} &= E_{\alpha\beta}{}^{\gamma} \varepsilon_{ab}{}^{c} N_{c\gamma}. \end{aligned} \tag{1.13}$$

where ε_{abc} denotes the Levi-Civita symbol, whose index is raised with the metric $\eta^{cd} = \mathrm{diag}(-1,+1,+1)$ and we have introduced a new set of structure constants $E_{\alpha\beta}{}^{\gamma}$ associated with the commutative product of functions on the sphere, $Y_\alpha Y_\beta = E_{\alpha\beta}{}^{\gamma} Y_\gamma$. Like the $C_{\alpha\beta}{}^{\gamma}$, the $E_{\alpha\beta}{}^{\gamma}$ can be written explicitly in terms of Wigner 3j symbols (see appendix A.1 for the details) [52, 53].

The regularization of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ is obtained in section 3.4 by a similar procedure as the case of $\mathfrak{c}_{\mathbb{R}}(S)$. We construct a $2N$-dimensional representation of the deformed algebra with the matrices $\widehat{Y}_{\bullet\alpha} = \mathbb{1}_2 \otimes \widehat{Y}_\alpha$, and $\widehat{Y}_{a\alpha} = \rho_a \otimes \widehat{Y}_\alpha$, where ρ_a are a basis for $\mathfrak{sl}(2,\mathbb{R})$, defined in equation (3.46). The corresponding Lie algebra generators are denoted $(X_\alpha, Z_{a\alpha})$, rescaled appropriately, and their algebra derived from the $2N$-dimensional representation is given by

$$\begin{aligned} [X_\alpha, X_\beta] &= \widehat{C}_{\alpha\beta}{}^{\gamma} X_\gamma, \\ [X_\alpha, Z_{a\beta}] &= \widehat{C}_{\alpha\beta}{}^{\gamma} Z_{a\gamma}, \\ [Z_{a\alpha}, Z_{b\beta}] &= \widehat{E}_{\alpha\beta}{}^{\gamma} \varepsilon_{ab}{}^{c} Z_{c\gamma} - \frac{1}{N^2} \widehat{C}_{\alpha\beta}{}^{\gamma} \eta_{ab} X_\gamma, \end{aligned} \tag{1.14}$$

where $\widehat{E}_{\alpha\beta}{}^{\gamma}$ are the structure constants for the Jordan product of the fuzzy spherical harmonics, $\widehat{Y}_\alpha \circ \widehat{Y}_\beta = \frac{1}{2}(\widehat{Y}_\alpha \widehat{Y}_\beta + \widehat{Y}_\beta \widehat{Y}_\alpha) = \widehat{E}_{\alpha\beta}{}^{\gamma} \widehat{Y}_\gamma$, which approach $E_{\alpha\beta}{}^{\gamma}$ in the large N limit. It is then readily apparent that the deformed algebra generated by $(X_\alpha, Z_{a\alpha})$ approaches the classical algebra (1.13) as $N \to \infty$. Since the deformed algebra can be shown to be isomorphic to $\mathfrak{su}(N,N)$, this establishes the second novel large N limit in this work,

$$\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S) = \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S \quad \underset{\text{Large-}N\text{ Limit}}{\overset{\text{Matrix Regularization}}{\rightleftarrows}} \quad \mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(N) \simeq \mathfrak{su}(N,N). \tag{1.15}$$

Having established the existence of regularized algebras that approach the three continuum algebras $\mathfrak{sdiff}(S)$, $\mathfrak{c}_{\mathbb{R}}(S)$, and $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ at large N, we turn in section 4 to the analysis of Casimir operators for the deformed and continuum algebras. For each large N limit, we

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demonstrate that the Casimir elements for the deformed algebras approach corresponding Casimir elements of the continuum algebras. We further argue that the matching conditions for the Casimir elements can be used to determine the representation of the deformed algebra that appears in the quantization of the gravitational phase space, and further argue that the matching conditions can also be used to determine the value of the deformation parameter N. We outline how this procedure can be carried out in detail in the case of $\mathfrak{su}(N)$ in section 4.2. We leave a detailed calculation of the matching for the other deformed algebras $\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}$ and $\mathfrak{su}(N,N)$ for future work. Additionally, for the case of $\mathfrak{su}(N,N)$ we identify an operator that can be associated with the dynamical area of the surface, and argue that while it is a Casimir in the continuum algebra, it becomes noncommutative at finite N.

Since the various large N limits considered in this paper rely on properties of the fuzzy spherical harmonics $\widehat{Y}_\alpha$, we collect a number of formulas and conventions related to them in appendix A. In particular, the conventions used for the continuum spherical harmonics are presented in section A.1, and conventions for spin-weighted harmonics, which are used in calculations of structure constants for various differential operators, are given in section A.2. Following that, we review the presentation of the fuzzy spherical harmonics developed in [54] in section A.3. Additionally, we present a novel formula, derived from an identity due to Nomura [55], for the asymptotic limit of the Wigner 6j-symbol appearing in the structure constants for the fuzzy harmonics product, and demonstrate that it immediately provides an expansion of this matrix product order by order in powers of $\frac{1}{N}$. This asymptotic expansion allows us to evaluate subleading corrections to the matrix product beyond the Poisson bracket term. We develop this expansion in section A.4 by determining the $\mathcal{O}(\frac{1}{N^2})$ contribution to the matrix product, showing that it takes the form expected from a valid Moyal product of functions on the sphere. In section A.6 we show that to all orders in $\frac{1}{N}$, the Nomura identity yields the expansion of a specific choice of Moyal product on the sphere.

The majority of this work has focused on the three subalgebras of the full corner symmetry group appearing in the top line of figure 1. Ultimately, however, we are interested in determining the deformation and quantization of the full symmetry algebra $\mathrm{Diff}(S)\ltimes\mathrm{SL}(2,\mathbb{R})^S$. While we do not obtain a deformation of this symmetry algebra due to several conceptual issues related to the form such a deformation should take, we initiate the investigation into such deformations by determining the structure constants of the $\mathfrak{diff}(S)$ algebra, including diffeomorphisms that do not preserve a chosen area form. These structure constants are derived in the spherical harmonic basis in appendix B, and they do not appear to have been presented previously in the literature. These expressions will inform future work into possible deformations of the full symmetry algebra, and also will likely be useful in other contexts in which $\mathfrak{diff}(S)$ algebra appears, such as extended symmetries of asymptotically flat space and celestial holography [56–64]. The remaining appendices include calculational details and proofs of formulas appearing in the main text.

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2 Corner symmetries and their Poisson brackets

In this section, we review some aspects of the corner symmetry algebra $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ and some of its important subalgebras. To prepare the ground for the matrix regularization of these subalgebras, we introduce an explicit basis of generators and give the structure constants of the algebras in this basis.

2.1 Corner symmetry algebra and its subalgebras

As established in [1], in the presence of a finite-distance corner S, general relativity in the metric formulation enjoys a symmetry group, called the *corner symmetry group*, which acts on the dynamical variables, which in the classical analysis correspond to functions on the theory's phase space. The Lie algebra of the corner symmetry group is denoted $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, and consists of two types of transformations: 1) diffeomorphisms that are tangent to the corner S, and 2) generalized boosts that fix S but act on its normal plane. Vector fields on S generate the first type of transformations, forming a $\mathfrak{diff}(S)$ Lie algebra under the vector field Lie bracket, while the second type of transformations are generated by $\mathfrak{sl}(2,\mathbb{R})$-valued functions on S, with the Lie bracket computed pointwise. Denoting the coordinates on S by $\sigma \equiv (\sigma^1, \sigma^2)$, the generators can be packaged together into a pair (ξ, λ), where $\xi(\sigma) = \xi^A(\sigma)\partial_A$ is a vector field on S and $\lambda(\sigma) = \lambda^a(\sigma)\tau_a$ for $a = 0, 1, 2$ belongs to $\mathfrak{sl}(2,\mathbb{R})^S$. Here τ_a are $\mathfrak{sl}(2,\mathbb{R})$ generators, whose Lie brackets are given by $[\tau_a, \tau_b] = \varepsilon_{ab}{}^c\tau_c$ where ε_{abc} is the three-dimensional Levi-Civita symbol $\varepsilon_{012} = 1$ and $\mathfrak{sl}(2,\mathbb{R})$ indices are raised and lowered with the metric $\eta_{ab} = \mathrm{diag}(-1, +1, +1)$. An explicit matrix representation of such generators is

$$\tau_0 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \qquad \tau_1 = \frac{1}{2}\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}. \tag{2.1}$$

The Lie algebra of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ is then given by

$$\Big[(\xi_1, \lambda_1), (\xi_2, \lambda_2)\Big] = \Big([\xi_1, \xi_2]_{\mathrm{Lie}},\ \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 + [\lambda_1, \lambda_2]_{\mathfrak{sl}(2,\mathbb{R})}\Big), \tag{2.2}$$

where $[\cdot,\cdot]_{\mathrm{Lie}}$ denotes the Lie bracket of vector fields on S, and $[\cdot,\cdot]_{\mathfrak{sl}(2,\mathbb{R})}$ is the $\mathfrak{sl}(2,\mathbb{R})$ Lie bracket. Explicitly, we have

$$\begin{gathered} [\xi_1, \xi_2]_{\mathrm{Lie}} := (\xi_1^A\partial_A\xi_2^B - \xi_2^A\partial_A\xi_1^B)\partial_B, \\ \mathcal{L}_{\xi_1}\lambda_2 := \xi_1^A\partial_A\lambda_2, \qquad \mathcal{L}_{\xi_2}\lambda_1 := \xi_2^A\partial_A\lambda_1, \\ [\lambda_1, \lambda_2]_{\mathfrak{sl}(2,\mathbb{R})} := 2\lambda_1^a\lambda_2^b\,\varepsilon_{ab}{}^c\tau_c, \end{gathered} \tag{2.3}$$

As is clear from (2.2), $\mathfrak{diff}(S)$ acts on $\mathfrak{sl}(2,\mathbb{R})^S$ by the Lie derivative and hence the symmetry algebra is

$$\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S) = \mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S, \tag{2.4}$$

where $\oplus_{\mathcal{L}}$ denotes a semidirect sum with an action of the first algebra on the second realized by the Lie derivative.

The subalgebras relevant in this work all involve a restriction of the $\mathfrak{diff}(S)$ algebra to an area-preserving subalgebra, which can be explicitly constructed as follows. Let $\widetilde{n}$ be a positive density on S. Area-preserving diffeomorphisms are generated by divergenceless vector fields ξ^A with respect to $\widetilde{n}$, which satisfy $\partial_A(\widetilde{n}\xi^A) = 0$. Since the Lie bracket of two such vector fields also satisfies this condition, the set of area-preserving diffeomorphisms forms a subalgebra of $\mathfrak{diff}(S)$, which we denote as $\mathfrak{sdiff}_\nu(S)$, where ν is the volume form on the sphere defined as

$$\nu := \frac{1}{2}\nu_{AB}\mathsf{d}\sigma^A \wedge \mathsf{d}\sigma^B, \tag{2.5}$$

where $\nu_{AB} := \widetilde{n}\,\varepsilon_{AB}$, and ε_{AB} is the Levi-Civita symbol with $\varepsilon_{12} = 1$.

An alternative presentation of the area-preserving diffeomorphisms can be given in terms of functions on the sphere, and will serve more convenient when comparing to the regularized algebras in later sections. Since $\nu^{AB} = \varepsilon^{AB}/\widetilde{n}$ (again with $\varepsilon^{12} = 1$) defines a Poisson tensor on the sphere, the corresponding Poisson bracket

$$\{\phi_1, \phi_2\}_\nu := \nu^{AB}\partial_A\phi_1\partial_B\phi_2, \qquad \phi_1, \phi_2 \in C(S), \tag{2.6}$$

where $C(S)$ denotes the space of functions on sphere, acts as a derivation of the function ϕ_1 on ϕ_2. Note that this relation implies that

$$d\phi_1 \wedge d\phi_2 = \nu\,\{\phi_1, \phi_2\}_\nu. \tag{2.7}$$

The vector field ξ^A associated with this derivation is divergenceless, and can be identified with a function ϕ, called the *stream function*, through the relation[5]

$$\xi^B = \nu^{AB}\partial_A\phi. \tag{2.8}$$

For a given vector field ξ^A, this equation determines ϕ up to a constant shift, which can be fixed by requiring that the function ϕ integrate to zero over the sphere. The action of the vector field on functions is then reproduced by taking Poisson brackets with the associated stream function. To make this correspondence clear, we denote a vector field preserving the area form ν corresponding to the stream function ϕ by $\xi^\nu_\phi := \nu^{AB}\partial_A\phi\partial_B$. This vector field is such that

$$[\xi^\nu_\phi, \xi^\nu_\psi] = \xi^\nu_{\{\phi,\psi\}_\nu}, \qquad \xi^\nu_\phi[\psi] = \{\phi, \psi\}_\nu, \tag{2.9}$$

demonstrating that the map from an area-preserving vector field to its stream function is a Lie algebra homomorphism into the Poisson bracket algebra of functions on the sphere. Note that the relation (2.8) implies that the constant function on the sphere is not the stream function of any nonzero vector field, and this function generates the center of the full Poisson algebra. Hence, the Poisson algebra can be viewed as a trivial central extension of the algebra $\mathfrak{sdiff}(S)$ by this constant function.

The area-preserving subalgebra comprises an important component of the main algebra studied in this work, which is the subalgebra of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ that preserves a given volume form $\nu = \widetilde{n}d^2\sigma$. In [26], it was called the *centralizer subalgebra* $\mathfrak{c}_{(\mathfrak{sl}(2,\mathbb{R}),\nu)}(S)$, since it centralizes the SL$(2,\mathbb{R})$ quadratic Casimir operator in the universal enveloping algebra, which defines

[5]This follows from the fact that a divergenceless vector field satisfies $d(\xi\cdot\nu) = 0$, which on the sphere implies that $\xi\cdot\nu = -d\phi$ for some function ϕ.

an area form on the gravitational phase space. All $\mathrm{SL}(2,\mathbb{R})$ transformations preserve the volume form ν, therefore the centralizer subalgebra is simply obtained by restricting the diffeomorphisms appearing in the full algebra $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ to area-preserving ones. This fixes the centralizer subalgebra to be

$$\mathfrak{c}_{(\mathfrak{sl}(2,\mathbb{R}),\nu)}(S) = \mathfrak{sdiff}_\nu(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S. \tag{2.10}$$

Going forward, we will work exclusively with the normalized[6] round sphere volume form, denoted ν_0.[7] For simplicity, we will drop the ν_0 label when working with these algebras, hence we denote

$$\mathfrak{sdiff}(S) := \mathfrak{sdiff}_{\nu_0}(S), \qquad \mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S) := \mathfrak{c}_{(\mathfrak{sl}(2,\mathbb{R}),\nu_0)}(S). \tag{2.11}$$

Moser's theorem [65] implies that any volume form ν of area $A = \int_S \nu$ is, up to a constant multiple, isomorphic to ν_0, meaning there exists a diffeomorphism Φ such that $\Phi^*(\nu) = A\nu_0$. This implies that the different area-preserving diffeomorphism groups are isomorphic to each other

$$\mathfrak{sdiff}_\nu(S) = \Phi(\mathfrak{sdiff}(S)). \tag{2.12}$$

A useful analogy is to compare the $\mathrm{Diff}(S)$ group to the Lorentz group $\mathrm{SO}(3,1)$ and the area preserving $\mathrm{SDiff}_\nu(S)$ group to the rotation subgroup $\mathrm{SO}(3)_p$ preserving the timelike 4-momentum p [26]. The subgroups $\mathrm{SO}(3)_p$ are all isomorphic to the canonical subgroup $\mathrm{SO}(3)$ associated with a reference timelike direction p_0. The isomorphism is such that $\mathrm{SO}(3)_p = g_p \mathrm{SO}(3) g_p^{-1}$ for a boost $g_p p_0 = p$. Within this analogy, the full corner symmetry group $G_{\mathrm{SL}(2,\mathbb{R})}(S)$ is the analog of the Poincaré group, while the centralizer subgroup $C_{\mathrm{SL}(2,\mathbb{R})}$ is analogous to the subgroup $\mathrm{SO}(3) \ltimes \mathbb{R}$ preserving the given direction. Finally, the diffeomorphisms that change the area form are analogous to the boost transformations of the Lorentz group which do not preserve p_0.

An important subalgebra of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ is obtained by considering the one-dimensional subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ generated by a single generator. Taking this generator to be a hyperbolic generator — for example, τ_1 in (2.1) — this subalgebra is isomorphic to $\mathbb{R}$. Considering functions on S valued in this subalgebra rather than the full $\mathfrak{sl}(2,\mathbb{R})$) yields the subalgebra $\mathbb{R}^S \subset \mathfrak{sl}(2,\mathbb{R})^S$, which is just the abelian Lie algebra $C(S)$ of real-valued functions on S. Imposing this restriction on the full algebra (2.4), we end up with the following subalgebra of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}$

$$\mathfrak{g}_{\mathbb{R}}(S) = \mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S. \tag{2.13}$$

This is the hydrodynamical algebra, which is the symmetry algebra of an ideal barotropic fluid [66]. In the present context, this algebra plays an important role in the classification of coadjoint orbits of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}$ [26].

[6] In standard spherical coordinates, $\nu_0 = \frac{1}{4\pi}\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$ and satisfies $\int_S \nu_0 = 1$.

[7] This differs from the "dynamical" measure $\nu = \nu_N$ given by $\nu_N = \sqrt{N_a N^a}\nu_0$ considered in [26], with N_a associated with the $\mathfrak{sl}(2,\mathbb{R})^S$ gravitational Hamiltonian (see section 2.2).

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Imposing the same restriction on the centralizer algebra (2.10), we arrive at another important subalgebra of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}$

$$\mathfrak{c}_{\mathbb{R}}(S) = \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S. \tag{2.14}$$

This algebra appears as the symmetry algebra of a charged particle on a sphere surrounding a magnetic monopole, as recently explored in [67]. Turning off the boost generators in (2.4) and (2.10), gives the subalgebras $\mathfrak{diff}(S)$ and $\mathfrak{sdiff}(S)$ of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}$, respectively. Conversely, turning off the diffeomorphism generators in (2.4) and (2.10), we get the subalgebras $\mathfrak{sl}(2,\mathbb{R})^S$ and $\mathbb{R}^S$ of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}$, respectively. The Lie bracket of each of these subalgebras is obtained by restriction of (2.2) to the corresponding subalgebra. Each of these algebras is the Lie algebra of a subgroup of the full corner symmetry group, which we denote

$$\begin{aligned} G_{\mathrm{SL}(2,\mathbb{R})}(S) &= \mathrm{Diff}(S) \ltimes \mathrm{SL}(2,\mathbb{R})^S, & C_{\mathrm{SL}(2,\mathbb{R})}(S) &= \mathrm{SDiff}(S) \ltimes \mathrm{SL}(2,\mathbb{R})^S, \\ G_{\mathbb{R}}(S) &= \mathrm{Diff}(S) \ltimes \mathbb{R}^S, & C_{\mathbb{R}}(S) &= \mathrm{SDiff}(S) \ltimes \mathbb{R}^S. \end{aligned} \tag{2.15}$$

The algebra inclusions obtained in this section can be summarized in the following diagram:

$$\begin{array}{ccccc} \mathfrak{sdiff}(S) & \subset & \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S & \subset & \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S \\ \cap & & \cap & & \cap \\ \mathfrak{diff}(S) & \subset & \mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S & \subset & \mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S \end{array}$$

This concludes our brief synopsis of the relevant subalgebras of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}$. In later sections, we will focus our attention on the algebras on the top row: $\mathfrak{sdiff}(S)$, $\mathfrak{c}_{\mathbb{R}}(S)$, and $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ and prove that they can be viewed as large-N limits of the finite-dimensional Lie algebras $\mathfrak{su}(N)$, $\mathfrak{sl}(N,\mathbb{C})$, and $\mathfrak{su}(N,N)$, respectively.

2.2 Poisson bracket representations

An important property of the above algebras that will be essential in determining their regularizations is that they arise as symmetry algebras of classical phase spaces. Because of this, each algebra deformation considered in section 3 has a natural interpretation in terms of a quantization procedure for the associated phase space. This section describes how the algebras $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ and $\mathfrak{c}_{\mathbb{R}}(S)$ are represented via Poisson brackets on phase spaces and introduces several quantities related to these representations that have direct analogs in the constructions of the deformed algebras.

The algebra $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ was identified in [1] as the symmetry algebra of general relativity restricted to a local subregion bounded by a 2-dimensional surface S. These symmetries arise from diffeomorphisms acting in the vicinity of S, and fail to be pure gauge since the presence of the boundary breaks some of the gauge symmetry of the theory. Instead, these transformations are associated with nonzero Hamiltonians which generate the action of the transformation on the gravitational phase space through Poisson brackets. Hence, given a generator $\xi = \xi^A \partial_A$ or $\lambda = \lambda^a \tau_a$ of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, the corresponding Hamiltonians are given by

$$P[\xi] := \int_S \xi^A \widetilde{P}_A, \qquad N[\lambda] := \int_S \lambda^a \widetilde{N}_a, \tag{2.16}$$

where $\widetilde{P}_A(\sigma)$ and $\widetilde{N}_a(\sigma)$ are quantities related to the geometry of the embedded surface S in spacetime, described in detail in [1, 26]. Since $\widetilde{P}_A$ and $\widetilde{N}_a$ are functions of the dynamical fields in the theory (namely, the metric), the smeared generators $P[\xi]$, $N[\lambda]$ are functions on the phase space. As such, they obey an algebra defined by the Poisson bracket on phase space, which can be shown to satisfy

$$\begin{aligned}\{P[\xi_1], P[\xi_2]\} &= P[[\xi_1, \xi_2]_{\text{Lie}}],\\ \{N[\lambda_1], N[\lambda_2]\} &= N[[\lambda_1, \lambda_2]_{\mathfrak{sl}(2,\mathbb{R})}],\\ \{P[\xi], N[\lambda]\} &= N[\mathcal{L}_\xi[\lambda]],\end{aligned} \tag{2.17}$$

These brackets verify that the Hamiltonians $P[\xi]$, $N[\lambda]$ yield a Poisson bracket representation of the algebra $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$.

When restricting to the area-preserving diffeomorphisms that appear in $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ and $\mathfrak{c}_{\mathbb{R}}(S)$, it is more convenient to parameterize the generators in terms of their stream functions. Since it will be convenient for the vectors to reproduce the Poisson brackets $\{,\}_\epsilon$ associated with the unit radius volume form $\epsilon = 4\pi\nu_0$, we will define the stream function so that $\xi^B_\phi = \epsilon^{AB}\partial_A\phi = \frac{1}{4\pi}\nu_0^{AB}\partial_A\phi$. We can then write the Hamiltonian for an area-preserving diffeomorphism corresponding to the vector field ξ_ϕ as

$$\begin{aligned}P[\xi_\phi] &= \int_S \xi^B_\phi \widetilde{P}_B = \int_S \epsilon^{AB}\partial_A\phi\widetilde{P}_B = \int_S \phi\left(-\frac{1}{4\pi}\varepsilon^{AB}\partial_A P_B\right)\\ &= \int_S \phi\,\widetilde{J} := J[\phi],\end{aligned} \tag{2.18}$$

where[8]

$$P_A := \frac{\widetilde{P}_A}{\widetilde{n}_0}, \qquad J := -\epsilon^{AB}\partial_A P_B = \frac{\widetilde{J}}{\widetilde{n}_0}. \tag{2.19}$$

Using (2.17) and employing the relation (2.9), we have

$$\begin{aligned}\{J[\phi_1], J[\phi_2]\} &= J[\{\phi_1, \phi_2\}_\epsilon],\\ \{J[\phi], N[\lambda]\} &= N[\{\phi, \lambda\}_\epsilon],\\ \{N[\lambda_1], N[\lambda_2]\} &= N[[\lambda_1, \lambda_2]_{\mathfrak{sl}(2,\mathbb{R})}].\end{aligned} \tag{2.20}$$

The Poisson bracket of $\mathfrak{c}_{\mathbb{R}}(S)$ are obtained by simply restricting λ to be proportional to a single $\mathrm{SL}(2,\mathbb{R})$ generator, in which case the last Poisson bracket in (2.20) vanishes.

While the above discussion focused on the specific example of the algebras acting on the gravitational phase space of [1], the various objects that appear in the description have interpretations in term of natural quantities arising for a generic phase space admitting an action of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$. Given any such phase space $\mathcal{P}$, there exists a unique *moment map* $\mu : \mathcal{P} \to \mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)^*$ that sends the phase space to the dual of the Lie algebra, which is a Poisson manifold foliated by the coadjoint orbits [68].[9] One can therefore construct the pullback map μ^* which sends a function on $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)^*$ to a function on $\mathcal{P}$. Since any element

[8]The generator J defined here is $-J$ in [26].

[9]Generically, the image of this map in $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)^*$ will include many different coadjoint orbits; this is implied by the existence of nontrivial Casimir functions on the phase space $\mathcal{P}$.

of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ is naturally associated with a linear function on $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)^*$, the pullback map restricts to an action on $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, and defines a linear map $\mu^* : \mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S) \to C^\infty(\mathcal{P})$; this is just the map that sends a Lie algebra element to its corresponding Hamiltonian on phase space. This map is explicitly described by a quantity $H \in \mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)^* \otimes C^\infty(\mathcal{P})$, i.e. a linear form on the Lie algebra valued in functions on the phase space. The split in $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ and its dual into $\mathfrak{sdiff}(S)$ and $\mathfrak{sl}(2,\mathbb{R})^S$ generators leads to a decomposition of H into two components, $H = (J, N_a)$, with each component coinciding with the functions $J(\sigma)$ and $N_a(\sigma)$ appearing in (2.16) and (2.19).[10] Hence, J and N_a should be viewed as $C^\infty(\mathcal{P})$-valued functions on S.

The above discussion of the moment map can be clarified with a simple example. Let $P = \mathbb{R}^3 \times \mathbb{R}^3$ be the phase space of a non-relativistic particle, with coordinates (q^i, p_j). It is acted upon by the rotation group with generators $X = X_i\sigma^i$ where σ^i is the three dimensional matrix

$$(\sigma_i)_j{}^k = \varepsilon_i{}^k{}_j, \tag{2.21}$$

with indices raised by the standard Euclidean metric. They satisfy the algebra

$$[\sigma_i, \sigma_j] = \varepsilon_{ij}{}^k \sigma_k. \tag{2.22}$$

The matrices σ_i can be taken as a basis for the $\mathfrak{so}(3)$ Lie algebra, and the moment map pullback μ^* sends each matrix to a function on phase space. Defining these functions as L_i, we have that

$$L_i = \mu^* \sigma_i = \varepsilon_{ijk} q^j p^k, \tag{2.23}$$

and they satisfy

$$\{L_i, L_j\} = \varepsilon_{ij}{}^k L_k. \tag{2.24}$$

If we parameterize the dual of the Lie algebra $\mathfrak{so}(3)^*$ with the same matrices σ^i, with a pairing defined by $\langle \sigma^i, \sigma_j \rangle = \delta^i_j$, the moment map μ sending a point (q^i, p_j) in P to a point in $\mathfrak{so}(3)^*$ is therefore given by

$$\mu(q^i, p_j) = \varepsilon_{ijk} \sigma^i q^j p^k. \tag{2.25}$$

The object H in this case is an element of $\mathfrak{so}(3)^* \otimes C^\infty(P)$ given by

$$H = \sigma^i \otimes \varepsilon_{ijk} q^j p^k, \tag{2.26}$$

and we easily verify that it satisfies the defining property

$$\langle H, \sigma_i \rangle = \mu^* \sigma_i = L_i. \tag{2.27}$$

[10] The fact that elements of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)^*$ can be identified with functions on the sphere comes from the existence of a trace provided by the integral over the sphere with respect to the fixed volume form ν_0. This trace gives a canonical identification of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)^*$ with $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, the latter of which is parameterized by functions ϕ and $\mathfrak{sl}(2,\mathbb{R})$-valued functions λ_a.

2.3 Mode expansion of Hamiltonian generators

The determination of the regularized algebras is most easily achieved in an explicit basis for the generators, so in this section we construct such a basis for the Lie algebra $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ and the corresponding phase space generators $J[\phi]$ and $N[\lambda]$ defined in (2.18) and (2.16). The Lie algebra is parametrized by a pair of functions (ϕ, λ^a) on S, where ϕ is real-valued and λ^a is $\mathfrak{sl}(2,\mathbb{R})$-valued; therefore, we need a basis for these spaces of functions. A good choice is the spherical harmonic functions, which we denote as Y_α, where $\alpha = (A, a)$ denotes a pair of integers with $A \in \mathbb{N}$ the total angular momentum and $a \in \{-A, \dots, +A\}$ is the magnetic spherical harmonic number. The conventions employed in this work for the spherical harmonics are detailed in appendix A.1.

The pointwise product and Poisson bracket of spherical harmonic functions yield two types of structure constants,

$$Y_\alpha Y_\beta = E_{\alpha\beta}{}^{\gamma} Y_\gamma, \qquad \{Y_\alpha, Y_\beta\}_\epsilon = C_{\alpha\beta}{}^{\gamma} Y_\gamma. \tag{2.28}$$

The explicit form of $E_{\alpha\beta}{}^{\gamma}$ and $C_{\alpha\beta}{}^{\gamma}$ in terms of Wigner $3j$ symbols are given in equations (A.4) and (A.7). These structure constants are directly used to construct the structure constants of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$. A basis for this algebra is provided by the Y_α and the quantities $Y_{a\alpha} = \tau_a \otimes Y_\alpha$ with τ_a given by (2.1). Applying the relationship between a function and its associated vector field (2.8) as well as the identities (2.9) and (2.28), the Lie brackets are given by

$$\begin{aligned} [Y_\alpha, Y_\beta] &= C_{\alpha\beta}{}^{\gamma} Y_\gamma, \\ [Y_\alpha, Y_{a\beta}] &= C_{\alpha\beta}{}^{\gamma} Y_{a\gamma}, \\ [Y_{a\alpha}, Y_{b\beta}] &= E_{\alpha\beta}{}^{\gamma} \varepsilon_{ab}{}^{c} Y_{c\gamma}. \end{aligned} \tag{2.29}$$

This basis for $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ immediately leads to a basis for the Hamiltonian generators, given by

$$J_\alpha := J[Y_\alpha] = \int_S \nu_0 Y_\alpha J, \qquad N_{a\alpha} := N[Y_\alpha \tau_a] = \int_S \nu_0 Y_\alpha N_a. \tag{2.30}$$

The Poisson bracket relations (2.20) then imply that these basis generators satisfy

$$\begin{aligned} \{J_\alpha, J_\beta\} &= C_{\alpha\beta}{}^{\gamma} J_\gamma, \\ \{J_\alpha, N_{a\beta}\} &= C_{\alpha\beta}{}^{\gamma} N_{a\gamma}, \\ \{N_{a\alpha}, N_{b\beta}\} &= E_{\alpha\beta}{}^{\gamma} \varepsilon_{ab}{}^{c} N_{c\gamma}, \end{aligned} \tag{2.31}$$

which reproduce the structure constants (2.29) of the $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ Lie algebra, as expected. Finally, note that in this basis, the $C^\infty(\mathcal{P})$-valued functions $J(\sigma)$ and $N_a(\sigma)$ discussed in section 2.2 can be written

$$\begin{aligned} J(\sigma) &= \sum_\alpha J_\alpha Y^\alpha(\sigma), \\ N_a(\sigma) &= \sum_\alpha N_{a\alpha} Y^\alpha(\sigma), \end{aligned} \tag{2.32}$$

where $Y^\alpha = \delta^{\alpha\beta} Y_\beta$, with $\delta^{\alpha\beta}$ defined as the inverse of the spherical-harmonic metric defined in equation (A.2).

The generators associated with the subalgebra $\mathfrak{c}_{\mathbb{R}}(S)$ is obtained by restricting the normal generator to be $N_\alpha \equiv N_{1\alpha}$. The expansion in modes simplifies to:

$$\begin{aligned}
\{J_\alpha, J_\beta\} &= C_{\alpha\beta}{}^{\gamma} J_\gamma, \\
\{J_\alpha, N_\beta\} &= C_{\alpha\beta}{}^{\gamma} N_\gamma, \\
\{N_\alpha, N_\beta\} &= 0.
\end{aligned} \tag{2.33}$$

Finally, we mention that although it is not the focus of the present work, one would also like to understand how to lift the algebra deformations identified in section 3 for area-preserving algebras to the full corner symmetry algebra $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$. The main obstacle in doing so lies in the identification of a suitable deformation of the full $\mathfrak{diff}(S)$ algebra compatible with the deformation of its $\mathfrak{sdiff}(S)$ subalgebra. As a first step toward investigating this question, we derive in appendix B the structure constants of $\mathfrak{diff}(S)$, and discuss some ideas and challenges in using these to obtain a suitable deformation of $\mathfrak{diff}(S)$ in section 5.4.

3 Matrix regularizations of classical symmetry algebras

Having reviewed the classical symmetry algebras, we are now interested in exploring their quantization, in the sense described in section 1. We restrict attention to the centralizer algebra $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ and its subalgebras, since these all possess natural candidates for their deformation in terms of finite-dimensional matrix algebras. These matrix algebras arise from promoting the sphere on which the diffeomorphism groups act to a fuzzy sphere and appealing the well-known correspondence between the large N limit of the $\mathfrak{su}(N)$ Lie algebra and $\mathfrak{sdiff}(S)$ [45–47]. Using the mode expansions of the classical algebras obtained in section 2.3, we obtain quantization maps between the classical generators and corresponding sets of matrices and show that the structure constants for the matrix product approach the classical structure constants in the large N limit. The final result is that the respective matrix regularizations of $\mathfrak{sdiff}(S)$, $\mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S$, and $\mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$ are found to be $\mathfrak{su}(N)$, $\mathfrak{sl}(N,\mathbb{C}) \oplus \mathbb{R}$, and $\mathfrak{su}(N,N)$.

3.1 From functions on phase space to linear operators on Hilbert space

Quantization is a procedure that seeks to replace the algebra of functions on a phase space with the algebra of linear operators on a Hilbert space. The quantization map sends each function on phase space to a Hilbert space operator, and the commutators of the quantized observables are required to reproduce the classical Poisson bracket algebra only up to order $\hbar^2$ corrections. These higher-order corrections indicate that the Poisson bracket algebra has been deformed. While it is possible that certain subalgebras remain undeformed by the quantization procedure, generically one expects a deformation to occur whenever one is available. In this case, the classical symmetry algebras discussed in section 2 are modified in the quantum theory.

It is possible to identify at the semiclassical level whether an algebra deformation exists or not. The presence of a quantum deformation implies the existence of a one-parameter family of deformations of the classical Poisson algebra. The deformation of the algebra is encoded in the existence of Poisson 2-cocycles [69], a notion which is intimately related to Lie algebra 2-cocycles [70] and Hochschild 2-cocycles [71] which parameterize deformations of Lie algebras and algebras respectively. As explained in appendix C, a Poisson 2-cocycle for a Poisson manifold M is a map $D : C(M) \times C(M) \to C(M)$, where $C(M)$ denotes the space of smooth functions on M, which is skew-symmetric, is a bi-derivation, and satisfies the Poisson 2-cocycle identity. Explicitly, this means that

$$\begin{aligned} D(f,g) &= -D(g,f), \\ D(f,gh) &= D(f,g)h + gD(f,h), \\ \{f, D(g,h)\} + \{g, D(h,f)\} + \{h, D(f,g)\} &= -[D(f,\{g,h\}) + D(g,\{h,f\}) + D(h,\{f,g\})]. \end{aligned} \tag{3.1}$$

These identities simply imply that the bracket $\{f,g\}_\lambda := \{f,g\} + \lambda D(f,g)$ satisfies the Jacobi identity to first order in λ. As described in appendix C, in each algebra that we study, there exists a Poisson 2-cocycle that controls the quantum deformation. Ultimately we will find that our Lie algebra deformations have a non-perturbative completion at finite $\lambda \sim 1/N^2$ which satisfies the Jacobi identity exactly. After having found such a non-perturbative quantization, the identities (3.1) can be derived by expanding the Jacobi identity in λ.

The quantization procedure also requires that the object H defined in section 2.2 be replaced by its quantum analog, $\widehat{H}$. The classical object H is valued in functions on the phase space, $C^\infty(\mathcal{P})$, which is the space of classical observables. The quantized object $\widehat{H}$ should therefore be valued in the space $\mathcal{L}(\mathcal{H})$ of linear operators on a Hilbert space $\mathcal{H}$, which serves as the space of observables in the quantum theory. Furthermore, since the quantum theory deals with a deformed algebra, $\widehat{H}$ should be a linear map from this deformed algebra into $\mathcal{L}(\mathcal{H})$, as opposed to a map from the classical algebra. Therefore, we see that $\widehat{H} \in \widehat{\mathfrak{g}}^* \otimes \mathcal{L}(\mathcal{H})$, where $\widehat{\mathfrak{g}}$ is the specific deformed algebra under consideration. Generically, the deformed algebra $\widehat{\mathfrak{g}}$ depends on a deformation parameter N, which will be taken to be large in the semiclassical limit. Similar to the classical object, the map defined by $\widehat{H}$ is required to be a homomorphism, up to a constant rescaling, from the deformed Lie algebra $\widehat{\mathfrak{g}}$ into $\mathcal{L}(\mathcal{H})$, which is simply the statement that the image of this map in $\mathcal{L}(\mathcal{H})$ furnishes a linear representation of the deformed algebra. Note that the generators of this representation are taken to be $i\hbar\pi(X)$, with $X \in \widehat{\mathfrak{g}}$ and π denoting a representation. The factor of i ensures that the generators are Hermitian in a unitary representation of the algebra, and the factor of $\hbar$ is included to give the correct proportionality constant between the commutator and the classical Poisson bracket. The value of N and the specific representation π of the deformed algebra that occurs depends on the phase space being quantized: different phase spaces correspond to different deformations and representations. Both N and π can be determined by requiring that the generators reproduce the symmetric product of the classical phase space to leading order in $\hbar$. This is most straightforwardly done by matching the Casimir functions on the classical phase space to the values of corresponding Casimir operators in the representation. This matching procedure is discussed in section 4.

3.2 Matrix regularization of $\mathfrak{sdiff}(S)$

We begin by providing some details on the matrix regularization of $\mathfrak{sdiff}(S)$, the algebra of vector fields preserving a fixed volume form ν_0 discussed in equation (2.11). It is well-known that the regularized algebra is $\mathfrak{su}(N)$ [45–47], and we use this section to illustrate the method for obtaining the large N limit of a matrix algebra that will be subsequently applied to the algebras $\mathfrak{c}_{\mathbb{R}}(S)$ and $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$. The results on the limits of the $\mathfrak{su}(N)$ structure constants obtained in this section will also play a key role in obtaining the matrix regularizations of the other algebras of interest in sections 3.3 and 3.4.

As discussed in section 2.1, a standard presentation of the Lie algebra $\mathfrak{sdiff}(S)$ is in terms of the Poisson brackets of functions on the sphere. In section 2.3, we found that the spherical harmonics Y_α provide a convenient basis for this space of functions. In terms of this basis, a generic function ϕ can be expanded as

$$\phi = \sum_\alpha \phi^\alpha Y_\alpha, \tag{3.2}$$

where ϕ^α are complex constants. Since ϕ is real-valued, the coefficients must satisfy the reality condition

$$(\phi^\alpha)^* = (-1)^a \phi^{\bar{\alpha}}, \tag{3.3}$$

in direct correspondence to the condition (A.3) satisfied by the Y_α, recalling that $\bar{\alpha} = (A, -a)$ for $\alpha = (A, a)$.

In the matrix regularization, the functions Y_α are replaced with fuzzy spherical harmonics $\widehat{Y}_\alpha$, which are $N \times N$ matrices [72]. Our conventions for fuzzy spherical harmonics are spelled out in appendix A.3. These matrices obey a multiplication law with structure constants $\widehat{M}_{\alpha\beta}{}^\gamma$, and satisfy additional normalization and reality conditions:

$$\widehat{Y}_\alpha \widehat{Y}_\beta = \widehat{M}_{\alpha\beta}{}^\gamma \widehat{Y}_\gamma, \qquad \frac{1}{N}\mathrm{Tr}_{\mathbf{N}}(\widehat{Y}_\alpha \widehat{Y}_\beta) = \delta_{\alpha\beta}, \qquad \widehat{Y}_\alpha^\dagger = (-1)^a \widehat{Y}_{\bar{\alpha}}, \tag{3.4}$$

with the metric $\delta_{\alpha\beta}$ defined in (A.2). As discussed in appendix A.3, the multiplication structure constants $\widehat{M}_{\alpha\beta\gamma} \equiv \widehat{M}_{\alpha\beta}{}^\mu \delta_{\gamma\mu}$ can be expressed explicitly in terms of Wigner 3j and 6j symbols according to [54]

$$\widehat{M}_{\alpha\beta\gamma} = \frac{\sqrt{N}}{(-1)^{2J}} \begin{pmatrix} A & B & C \\ a & b & c \end{pmatrix} \begin{Bmatrix} A & B & C \\ J & J & J \end{Bmatrix}, \tag{3.5}$$

with $N = 2J + 1$. The *quantization map* from a function ϕ on the sphere to an $N \times N$ matrix $\widehat{\phi}$ is achieved by expressing $\widehat{\phi}$ in terms of fuzzy spherical harmonics with the same coefficients ϕ^α,

$$\widehat{\phi} = \sum_{\alpha \in I_N} \phi^\alpha \widehat{Y}_\alpha, \tag{3.6}$$

where the sum runs over the index set I_N consisting of all $\alpha = (A, a)$ with $A \le 2J$. Note that this is a finite sum in which all spherical harmonics with $A > 2J$ are truncated. The reality condition in (3.4) for the fuzzy harmonics implies that the quantization map preserves the star structure, $\widehat{\phi^*} = \big(\widehat{\phi}\big)^\dagger$, and since real functions satisfy $\phi^* = \phi$, we see that the quantization map sends them to Hermitian matrices $\widehat{\phi}^\dagger = \widehat{\phi}$.

It is straightforward to see that all possible Hermitian $N \times N$ matrices are obtained as the quantization of some function on the sphere, and hence the full quantized algebra coincides with the algebra of all $N \times N$ Hermitian matrices. The associated Lie algebra obtained by taking commutators is just the standard presentation of the algebra $\mathfrak{u}(N)$. This Lie algebra has a trivial center generated by the matrix $\widehat{Y}_{(0,0)} = \mathbb{I}_N$, which is the quantization of the constant function $Y_{(0,0)}$ on the sphere. Since, as discussed in section 2.1, the constant function does not generate an area-preserving diffeomorphism, we see that the algebra $\mathfrak{sdiff}(S)$ quantizes to the space of matrices with vanishing $\widehat{Y}_{(0,0)}$ component. These are precisely the traceless Hermitian matrices, and hence the quantized Lie algebra is $\mathfrak{su}(N)$.

The classical structure constant relations (2.28) possess corresponding relations for the quantized algebra, coinciding with the symmetric and antisymmetric parts of $\widehat{M}_{\alpha\beta}{}^{\gamma}$,

$$\widehat{E}_{\alpha\beta}{}^{\gamma} = \widehat{M}_{(\alpha\beta)}{}^{\gamma}, \tag{3.7}$$

$$\widehat{C}_{\alpha\beta}{}^{\gamma} = \frac{N}{i}\widehat{M}_{[\alpha\beta]}{}^{\gamma}. \tag{3.8}$$

It is shown in appendix A.3 that in the large-N limit, they approach the classical structure constants

$$\widehat{E}_{\alpha\beta}{}^{\gamma} = E_{\alpha\beta}{}^{\gamma} + \mathcal{O}\left(N^{-2}\right), \tag{3.9}$$

$$\widehat{C}_{\alpha\beta}{}^{\gamma} = C_{\alpha\beta}{}^{\gamma} + \mathcal{O}\left(N^{-2}\right). \tag{3.10}$$

This implies that the quantization map preserves the symmetric product and bracket to order N^{-2} and N^{-3}, respectively

$$\begin{aligned} \widehat{\phi} \circ \widehat{\psi} &= \widehat{\phi\psi} + \mathcal{O}(N^{-2}), \\ [\widehat{\phi}, \widehat{\psi}] &= \frac{2i}{N}\widehat{\{\phi,\psi\}}_{\epsilon} + \mathcal{O}(N^{-3}). \end{aligned} \tag{3.11}$$

where $\widehat{\phi} \circ \widehat{\psi} = \frac{1}{2}(\widehat{\phi}\widehat{\psi} + \widehat{\psi}\widehat{\phi})$ is the symmetrized product.

In this relation, we see that the quantity $2/N$ is playing the role of $\hbar$ in the relation between the commutator and Poisson bracket in (3.11). However, it is not quite correct to equate $\hbar$ with $2/N$, since such a relationship only holds in the special case of the fuzzy sphere, and will not hold for the quantizations of the gravitational phase spaces considered in this work. Instead, recalling that the Poisson bracket $\{\cdot,\cdot\}_{\epsilon}$ is defined for a spherical phase space with area $A = 4\pi$, the correct relation is $\hbar_{\text{fs}} = \frac{A}{2\pi N_{\text{fs}}}$, or equivalently

$$N_{\text{fs}} = \frac{A}{2\pi\hbar_{\text{fs}}}, \qquad \text{(fuzzy sphere)}. \tag{3.12}$$

Here, we have added subscripts "fs" to N and $\hbar$ to emphasize that this relation only holds for the fuzzy sphere, and for other phase spaces (such as the gravitational phase space that is the primary focus of this work), the relation between the two will be different. This relation should be viewed as determining the deformation parameter N_{fs} in terms of the phase space area A and Planck's constant $\hbar_{\rm fs}$.[11] One might worry that this relation is ambiguous since by rescaling the generators $\widehat{\phi}$, $\widehat{\psi}$, one would obtain a similar relation between the commutator and Poisson bracket, but with a rescaled value of $\hbar_{\rm fs}$. Note however that such a rescaling is not possible, as it spoils the first relation in (3.11) for the symmetric product.[12] Hence, the relationship between $N_{\rm fs}$ and $\hbar_{\rm fs}$ is fully determined for a given phase space by requiring that the quantized generators reproduce the symmetric product at leading order in $N_{\rm fs}$, and that the commutator equal the Poisson bracket rescaled by $i\hbar_{\rm fs}$ to leading order in $N_{\rm fs}$.

Finally, we note that the relation $\hbar_{\rm fs} = \frac{2}{N_{\rm fs}}$ for the unit-radius fuzzy sphere allows us to identify the standard normalization for the $\mathfrak{u}(N)$ Lie algebra generators X_α. The fuzzy spherical harmonics $\widehat{Y}_\alpha$ occur in the defining representation $\pi_{\mathbf{N}}$ of $\mathfrak{u}(N)$, i.e. the representation in terms of $N \times N$ Hermitian matrices. Using the relation

$$\pi_{\mathbf{N}}(X_\alpha) = \frac{1}{i\hbar_{\rm fs}}\widehat{Y}_\alpha = \frac{N_{\rm fs}}{2i}\widehat{Y}_\alpha, \tag{3.13}$$

we see that the structure constants for the $\mathfrak{u}(N)$ Lie algebra in the basis X_α are simply $\widehat{C}_{\alpha\beta}{}^{\gamma}$:

$$[X_\alpha, X_\beta] = \widehat{C}_{\alpha\beta}{}^{\gamma} X_\gamma. \tag{3.14}$$

The relation (3.10) then confirms that the large N limit of the $\mathfrak{su}(N)$ Lie algebra in this basis coincides with $\mathfrak{sdiff}(S)$ (recalling that the central generator $X_{(0,0)}$ does not generate a diffeomorphism).

Note that because X_α defines a complex basis for $\mathfrak{u}(N)$, we need to specify a reality condition to identify the real form of the Lie algebra under consideration. This reality condition is an antilinear involution $*$ on the Lie algebra, with the real form determined by the set of generators fixed under the involution. This involution acts on the X_α basis according to

$$X_\alpha^* = (-1)^a X_{\bar{\alpha}}, \tag{3.15}$$

and ensures that in a unitary representation π, the operators $\pi(X_\alpha)$ satisfy $i\pi(X_\alpha^*) = (i\pi(X_\alpha))^\dagger$. One easily verifies that the relation (3.4) for $\widehat{Y}_\alpha^\dagger$ shows that the fuzzy spherical harmonics define a unitary representation of the algebra.

It is also useful to relate the X_α basis to the standard basis of $\mathfrak{u}(N)$ in terms of elementary matrices $E^i{}_j$. The relation is given by

$$X_\alpha = \frac{N}{2i}(\widehat{Y}_\alpha)_i{}^j E^i{}_j, \tag{3.16}$$

[11] Dimensionally, this requires that the phase space area A has the same units as $\hbar_{\rm fs}$. This can be made explicit by defining the symplectic form for the phase space to be $\Omega_{\rm fs} = \gamma\frac{A}{4\pi}\epsilon$, with ϵ the unit-radius spherical volume form, A the area of the sphere in standard units, and γ a parameter with dimensions $[\gamma] = \hbar/(\text{length})^2$. In this case, the relation between $\hbar_{\rm fs}$ and $N_{\rm fs}$ is

$$N_{\rm fs} = \frac{\gamma A}{2\pi\hbar_{\rm fs}}.$$

The relation (3.12) holds in units where $\gamma = 1$.

[12] In more detail, if we work instead with $\widetilde{\phi} \equiv \lambda\widehat{\phi}$, $\widetilde{\psi} \equiv \lambda\widehat{\psi}$, we would instead find $\widetilde{\phi} \circ \widetilde{\psi} = \lambda\widetilde{\phi\psi} + \mathcal{O}(N^{-2}) \neq \widetilde{\phi\psi} + \mathcal{O}(N^{-2})$.

where $(\widehat{Y}_\alpha)_i{}^j$ denotes the ij^{th} component of the matrix $\widehat{Y}_\alpha$, and one can show that the commutation relations (3.14) and reality condition (3.15) imply the standard $\mathfrak{u}(N)$ brackets and involution in the $E^i{}_j$ basis

$$[E^i{}_j, E^k{}_l] = \delta^k_j E^i{}_l - \delta^i_l E^k{}_j, \qquad (E^i{}_j)^* = -\delta_{jk}\delta^{il} E^k{}_l. \tag{3.17}$$

The equivalence of (3.14) and (3.17) follows from the identity (see appendix D.1)

$$(\widehat{Y}_\gamma)_i{}^j \delta^l_k - \delta^j_i (\widehat{Y}_\gamma)_k{}^l = \frac{2i}{N^2} \sum_{\alpha,\beta} \widehat{C}_\gamma{}^{\alpha\beta} (\widehat{Y}_\alpha)_i{}^l (\widehat{Y}_\beta)_k{}^j. \tag{3.18}$$

Following the discussion of section 3.1, the deformed algebra appears when quantizing a classical phase space, and the quantum theory yields a linear representation of the deformed algebra. This representation is characterized by the quantity $\widehat{H} \in \mathfrak{su}(N)^* \otimes \mathcal{L}(\mathcal{H})$, which we instead call $\widehat{J}$ in this section since we are dealing only with the deformation of the $\mathfrak{sdiff}(S)$ algebra, as opposed to the extended algebras $\mathfrak{c}_{\mathbb{R}}(S)$ and $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, which have additional generators. Up to rescaling by $i\hbar$, the fuzzy spherical harmonics $\widehat{Y}_\alpha$ furnish a representation for the (complexification of) the $\mathfrak{su}(N)$ Lie algebra, and hence can be used as an explicit realization of the abstract Lie algebra. These same matrices can be used to parameterize the dual $\mathfrak{su}(N)^*$ by utilizing the trace relation appearing in (3.4). This allows $\widehat{J}$ to instead be viewed as an element of $\text{Mat}(N) \otimes \mathcal{L}(\mathcal{H})$, where $\text{Mat}(N)$ is the space of $N \times N$ matrices, and the generators of the $\mathfrak{su}(N)$ algebra on the quantum Hilbert space are given by

$$\widehat{J}_\alpha = \frac{1}{N} \text{Tr}_{\mathbf{N}} \left(\widehat{J}\widehat{Y}_\alpha \right), \tag{3.19}$$

where the product and trace refer to the $\text{Mat}(N)$ factor of $\widehat{J}$. This relation is the precise analog of the equation (2.30) for the classical generators, yielding the correspondence

$$J_\alpha = \int_S \nu_0 J Y_\alpha, \qquad \underset{\text{Large-}N\text{ Limit}}{\overset{\text{Matrix Regularization}}{\rightleftarrows}} \qquad \widehat{J}_\alpha = \frac{1}{N} \text{Tr}_{\mathbf{N}} \left(\widehat{J}\widehat{Y}_\alpha \right). \tag{3.20}$$

The map $\widehat{J}$ is required to be normalized such that the generators $\widehat{J}_\alpha$ satisfy

$$[\widehat{J}_\alpha, \widehat{J}_\beta] = i\hbar \widehat{C}_{\alpha\beta}{}^\gamma \widehat{J}_\gamma, \tag{3.21}$$

since we recall that $\widehat{J}_\alpha = i\hbar \pi_{\mathcal{P}}(X_\alpha)$, where $\pi_{\mathcal{P}}$ is the representation of the Lie algebra corresponding to the phase space $\mathcal{P}$. Since $\widehat{J}_\alpha$ are a complex basis for the $\mathfrak{u}(N)$ generators, there must be a reality condition imposed to ensure a unitary representation of the Lie algebra. This condition is

$$\widehat{J}^\dagger_\alpha = (-1)^a \widehat{J}_{\bar{\alpha}}, \tag{3.22}$$

in direct analogy with the condition (3.4) satisfied by the fuzzy spherical harmonics. Conversely, the $\mathcal{L}(\mathcal{H})$-valued matrix elements of $\widehat{J}$ can be recovered by summing over the $\widehat{Y}_\alpha$ basis according to

$$\widehat{J}_i{}^j = \sum_{\alpha \in I_N} \widehat{J}_\alpha (\widehat{Y}^\alpha)_i{}^j. \tag{3.23}$$

This is the analog of the classical relation (2.32), with the correspondence being given by

$$J(\sigma) = \sum_\alpha J_\alpha Y^\alpha(\sigma) \qquad \underset{\text{Large-}N\text{ Limit}}{\overset{\text{Matrix Regularization}}{\longleftrightarrow}} \qquad \widehat{J}_i{}^j = \sum_{\alpha\in I_N} \widehat{J}_\alpha (\widehat{Y}^\alpha)_i{}^j. \tag{3.24}$$

With this choice of normalization and the Hermiticity condition (3.22), the operators comprising the matrix elements $\widehat{J}_i{}^j$ satisfy

$$[\widehat{J}_i{}^j, \widehat{J}_k{}^l] = \frac{\hbar N^2}{2}\left(\delta_i^l \widehat{J}_k{}^j - \delta_k^j \widehat{J}_i{}^l\right), \qquad (\widehat{J}_i{}^j)^\dagger = -\delta_{ik}\delta^{jl}\widehat{J}_l{}^k. \tag{3.25}$$

3.3 Matrix regularization of $\mathfrak{c}_{\mathbb{R}}(S)$

Having reviewed the matrix regularization of $\mathfrak{sdiff}(S)$, we turn now to a related deformation of the extended algebra $\mathfrak{c}_{\mathbb{R}}(S) = \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S$. To motivate this deformation, we recall the explicit parameterization of this algebra given in section 2.3. There, the $\mathfrak{sdiff}(S)$ generators were given in terms of spherical harmonics Y_α as before, while the generators of the $\mathbb{R}^S$ algebra were written in terms of $Y_{1\alpha} = \tau_1 \otimes Y_\alpha$, i.e. a tensor product between a function on the sphere and a 2×2 matrix. The idea behind the deformation of the $Y_{1\alpha}$ generators is that we should seek to replace the Y_α appearing in it with its fuzzy version $\widehat{Y}_\alpha$, and simply compute the commutators of the resulting $2N \times 2N$ matrices.[13]

There are two subtleties to implementing this idea in practice. First, the usual quantization of the $\mathfrak{sdiff}(S)$ generators to $N \times N$ matrices clearly will not define a consistent algebra with a set of $2N \times 2N$ matrices. This is easily remedied by simply tensoring with the 2×2 identity matrix $\mathbb{1}_2$, so that the generators of the deformed $\mathfrak{sdiff}(S)$ subalgebra are now $2N \times 2N$ matrices of the form $\widehat{Y}_{\bullet\alpha} = \mathbb{1}_2 \otimes \widehat{Y}_\alpha$.

The second subtlety relates to the relative factor of i in the structure constants for the commutator of the fuzzy harmonics $\widehat{Y}_\alpha$ relative to the Poisson brackets of the classical functions Y_α (see, e.g., (3.11)). This factor of i is simply the "physicist's" convention for parameterizing the $\mathfrak{u}(N)$ Lie algebra in terms of Hermitian matrices, as opposed to the "mathematician's" convention which uses anti-Hermitian matrices, and is necessary because the commutator of two Hermitian matrices is anti-Hermitian. On the other hand, the τ_a basis (2.1) for $\mathfrak{sl}(2,\mathbb{R})$ uses the mathematician's convention in which the structure constants are real. Taking tensor products of a set of matrices in the physicist's convention with a set in the mathematician's convention yields an algebra in the mathematician's convention. To obtain a tensor product algebra in the physicist's convention, both algebras in the tensor product should use this convention. For that reason, we should instead consider a basis ρ_a of $\mathfrak{sl}(2,\mathbb{R})$ in which the structure constants are imaginary. This basis is described in detail in equation (3.46) in the following section, but for the present construction we simply need the form of one of the hyperbolic generators,

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.26}$$

The proposal for the deformation of the $Y_{1\alpha}$ generators is then simply $\widehat{Y}_{1\alpha} = \rho_1 \otimes \widehat{Y}_\alpha$.

[13] A possibly related discussion of the quantization of matrix-valued functions on a fuzzy space is given in [73].

The Lie algebra obtained from the commutators of these matrices can be computed directly by applying an identity for the commutator of a tensor product of matrices,

$$[A \otimes C, B \otimes D] = (A \circ B) \otimes [C, D] + [A, B] \otimes (C \circ D), \tag{3.27}$$

recalling that $A \circ B = \frac{1}{2}(AB + BA)$. Along with the expression (3.11) for the structure constants of the fuzzy spherical harmonics, this immediately yields the algebra

$$[\widehat{Y}_{\bullet\alpha}, \widehat{Y}_{\bullet\beta}] = \frac{2i}{N} \widehat{C}_{\alpha\beta}{}^{\gamma} \widehat{Y}_{\bullet\gamma}, \tag{3.28}$$

$$[\widehat{Y}_{\bullet\alpha}, \widehat{Y}_{1\beta}] = \frac{2i}{N} \widehat{C}_{\alpha\beta}{}^{\gamma} \widehat{Y}_{1\gamma}, \tag{3.29}$$

$$[\widehat{Y}_{1\alpha}, \widehat{Y}_{1\beta}] = -\frac{1}{4} \cdot \frac{2i}{N} \widehat{C}_{\alpha\beta}{}^{\gamma} \widehat{Y}_{\bullet\gamma}. \tag{3.30}$$

Matching this algebra to the classical algebra $\mathfrak{c}_{\mathbb{R}}(S)$ is slightly more subtle than in the case of $\mathfrak{sdiff}(S)$. A reason for the subtlety is the fact that the representation provided by the matrices $(\widehat{Y}_{\bullet\alpha}, \widehat{Y}_{1\alpha})$ is not unitary, since matrices of the form $\phi^{\alpha}\widehat{Y}_{1\alpha}$ with $(\phi^{\alpha})^* = (-1)^a \phi^{\bar{\alpha}}$ are not Hermitian. Hence, this representation does not show up as a quantization of a classical phase space, unlike the example provided by the ordinary fuzzy sphere. Because of this, the matching to the classical phase space generators need not involve a universal rescaling by $i\hbar$; instead, different generators may be scaled by prefactors with different parametric dependence on the deformation parameter N.

We denote the generators of the deformed Lie algebra (X_{α}, Z_{α}), and the $2N$-dimensional representation in which the matrices $(\widehat{Y}_{\bullet\alpha}, \widehat{Y}_{1\alpha})$ live as $\pi_{\mathbf{2N}}$. To obtain the correct large-N limit the matrices $\widehat{Y}_{\bullet\alpha}$ generating the $\mathfrak{su}(N)$ subalgebra should be rescaled as in (3.13),

$$\pi_{\mathbf{2N}}(X_{\alpha}) = \frac{N}{2i} \widehat{Y}_{\bullet\alpha}, \tag{3.31}$$

to obtain the bracket

$$[X_{\alpha}, X_{\beta}] = \widehat{C}_{\alpha\beta}{}^{\gamma} X_{\gamma}, \tag{3.32}$$

as before, which matches the first bracket in the classical algebra (2.33) as $N \to \infty$. The second bracket in (2.33) can be matched for any choice of scaling for the Z_{α} generators. This freedom can be parameterized by a quantity λ defined so that

$$\pi_{\mathbf{2N}}(Z_{\alpha}) = \frac{N\lambda}{i} \widehat{Y}_{1\alpha}. \tag{3.33}$$

The remaining brackets for the Lie algebra are then fully determined to be

$$[X_{\alpha}, Z_{\beta}] = \widehat{C}_{\alpha\beta}{}^{\gamma} Z_{\gamma}, \tag{3.34}$$

$$[Z_{\alpha}, Z_{\beta}] = -\lambda^2 \widehat{C}_{\alpha\beta}{}^{\gamma} X_{\gamma}. \tag{3.35}$$

Here we see that in order to reproduce the final bracket in (2.33), λ must go to zero as $N \to \infty$. While this still leaves some choice in the precise value of λ, the choice $\lambda = \frac{1}{N}$ is most convenient, as it is the value required when determining the deformation of the larger algebra $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ and determining how the deformation of $\mathfrak{c}_{\mathbb{R}}(S)$ embeds into the larger deformed algebra.

As in the case of $\mathfrak{u}(N)$, the generators (X_α, Z_α) yield a complex basis of the deformed Lie algebra, and hence are naturally associated with the complexification of the Lie algebra. The real Lie algebra is obtained by specifying an antilinear involution $*$ and restricting to elements that are fixed under the involution. The resulting reality condition on (X_α, Z_α) is given by

$$X^*_\alpha = (-1)^a X_{\bar{\alpha}}, \qquad Z^*_\alpha = (-1)^a Z_{\bar{\alpha}}. \tag{3.36}$$

The reality condition leads to a criterion for specifying whether a given representation of the Lie algebra is unitary, namely, that $i\pi(X^*_\alpha) = (i\pi(X_\alpha))^\dagger$, and similarly for Z_α. Note that because the generators $\widehat{Y}_{1\alpha}$ in the representation $\pi_{\mathbf{2N}}$ do not satisfy this condition, we see once again that this representation is not unitary.

The Lie algebra defined by the brackets (3.32), (3.34), and (3.35) along with the reality condition (3.36) in fact coincides with $\mathfrak{gl}(N,\mathbb{C})$, viewed as a real Lie algebra, which is the complexification of $\mathfrak{u}(N)$. This can be seen by noting that X_α generate a $\mathfrak{u}(N)$ algebra, and its complexification is obtained by adding generators $\mathtt{i}\otimes X_\alpha$, where $\mathtt{i}$ is an imaginary unit satisfying $\mathtt{i}^2 = -1$.[14] The brackets of the new generators are fixed by assuming $\mathtt{i}$ commutes with the original generators, so $[X_\alpha, \mathtt{i}\otimes X_\beta] = \widehat{C}_{\alpha\beta}{}^\gamma \mathtt{i}\otimes X_\gamma$ and $[\mathtt{i}\otimes X_\alpha, \mathtt{i}\otimes X_\beta] = \widehat{C}_{\alpha\beta}{}^\gamma \mathtt{i}^2 \otimes X_\gamma = -\widehat{C}_{\alpha\beta}{}^\gamma X_\gamma$, which precisely match the brackets (3.34) and (3.35) upon identifying $Z_\alpha = \lambda \mathtt{i}\otimes X_\alpha$. This verifies that (X_α, Z_α) generate the Lie algebra $\mathfrak{gl}(N,\mathbb{C})$.[15] It is worth pointing out that this procedure involving tensoring with the imaginary unit $\mathtt{i}$ is more or less equivalent to the construction of the $\widehat{Y}_{1\alpha}$ generators in the representation $\pi_{\mathbf{2N}}$, with $2\rho_1$ serving as the new imaginary unit $\mathtt{i}$.[16]

The final step in making contact with the continuum algebra $\mathfrak{c}_{\mathbb{R}}(S)$ is to determine which central generators in $\mathfrak{gl}(N,\mathbb{C})$ have classical counterparts on the gravitational phase space. As noted before, the generators X_{00} (i.e. the generator with $\alpha = (A,a) = (0,0)$) do not generate a diffeomorphism of the sphere, and hence should be discarded when matching the continuum algebra. On the other hand, the generator Z_{00} coincides with the global boost in the normal plane to the codimension-2 surface in spacetime, and remains an important part of the continuum algebra. The remaining generators (X_α, Z_α) with $A \geq 1$ produce the simple subalgebra $\mathfrak{sl}(N,\mathbb{C})$. Hence, we can conclude that the deformation of the continuum algebra $\mathfrak{c}_{\mathbb{R}}(S)$ is $\mathfrak{sl}(N,\mathbb{C}) \oplus \mathbb{R}$, with the generator of the central factor $\mathbb{R}$ coinciding with Z_{00}.

[14] The symbol $\mathtt{i}$ is used to distinguish this imaginary unit from the factors of i appearing when using the complex basis X_α for the $\mathfrak{u}(N)$ Lie algebra. The distinction is important, since, for example, the reality condition $(\mathtt{i}\otimes X_\alpha)^* = \mathtt{i}\otimes X^*_\alpha = (-1)^a \mathtt{i}\otimes X_{\bar{\alpha}}$ is essentially equivalent to assuming $\mathtt{i}^* = +\mathtt{i}$.

[15] This can further be verified by constructing generators $E^\pm_\alpha = \frac{1}{2}(X_\alpha \pm \frac{i}{\lambda} Z_\alpha)$, which can be shown to satisfy $[E^\pm_\alpha, E^\pm_\beta] = \widehat{C}_{\alpha\beta}{}^\gamma E^\pm_\gamma$, $[E^\pm_\alpha, E^\mp_\beta] = 0$.

[16] Note that for this algebra, since λ and N may in principle be chosen independently, we could instead take the limit $\lambda \to 0$ before taking $N \to \infty$. This implements an Inönü-Wigner contraction [74] of the algebra $\mathfrak{gl}(N,\mathbb{C})$ to $\mathfrak{u}(N) \ltimes \mathbb{R}^{N^2}$. This contraction is effectively still happening in the large N limit when λ is identified with $\frac{1}{N}$, and explains why the large N limit of the semisimple Lie algebra results in an algebra with instead a semidirect product structure.

While the representation $\pi_{\mathbf{2N}}$ is useful in obtaining the deformed algebra, it has the property that is it not an irreducible representation of $\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}$, as can be seen from the fact that the central generator $\widehat{Y}_{1,00}$ is not proportional to the identity. In section 4.3 when evaluating the Casimir operators for this algebra, it will be useful to instead have an irreducible faithful representation of this algebra. This is given by the standard N-dimensional vector representation $\pi_{\mathbf{N}}$ of $\mathfrak{gl}(N,\mathbb{C})$, in which

$$i\pi_{\mathbf{N}}(X_\alpha)=\frac{N}{2}\widehat{Y}_\alpha,\qquad i\pi_{\mathbf{N}}(Z_\alpha)=iN\lambda\widehat{Y}_\alpha. \tag{3.37}$$

Here it is clear that this representation is just the complexification of the representation of $\mathfrak{u}(N)$ provided by the matrices $\widehat{Y}_\alpha$.

Finally, to relate this algebra to the classical phase space, we should exhibit the map $\widehat{H}\in\left(\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}\right)^*\otimes\mathcal{L}(\mathcal{H})$, with $\mathcal{L}(\mathcal{H})$ chosen to be the space of operators in which the representation of the algebra is valued. Similar to the case of $\mathfrak{u}(N)$, this is equivalent to defining a pair $\widehat{J},\widehat{N}\in\mathrm{Mat}(N)\otimes\mathcal{L}(\mathcal{H})$, which produce a set of generators $(\widehat{J}_\alpha,\widehat{N}_\alpha)$ in $\mathcal{L}(\mathcal{H})$ labeled by the fuzzy spherical harmonics according to

$$\widehat{J}_\alpha=\frac{1}{N}\operatorname{Tr}_{\mathbf{N}}\left(\widehat{J}\widehat{Y}_\alpha\right), \tag{3.38}$$

$$\widehat{N}_\alpha=\frac{1}{N}\operatorname{Tr}_{\mathbf{N}}\left(\widehat{N}\widehat{Y}_\alpha\right). \tag{3.39}$$

The maps $(\widehat{J},\widehat{N})$ must be normalized so that $(\widehat{J}_\alpha,\widehat{N}_\beta)=\Big(i\hbar\,\pi_{\mathcal{P}}(X_\alpha),i\hbar\,\pi_{\mathcal{P}}(Z_\beta)\Big)$, yielding the following algebra satisfied by the generators:

$$[\widehat{J}_\alpha,\widehat{J}_\beta]=i\hbar\,\widehat{C}_{\alpha\beta}{}^{\gamma}\widehat{J}_\gamma, \tag{3.40}$$

$$[\widehat{J}_\alpha,\widehat{N}_\beta]=i\hbar\,\widehat{C}_{\alpha\beta}{}^{\gamma}\widehat{N}_\gamma, \tag{3.41}$$

$$[\widehat{N}_\alpha,\widehat{N}_\beta]=-i\hbar\,\lambda^2\widehat{C}_{\alpha\beta}{}^{\gamma}\widehat{J}_\gamma. \tag{3.42}$$

The correspondence between the classical and quantum generators of the algebra is therefore given by

$$\begin{aligned} J_\alpha&=\int_S\nu_0 JY_\alpha, \\ N_\alpha&=\int_S\nu_0 NY_\alpha, \end{aligned}\qquad \underset{\text{Large-}N\text{ Limit}}{\overset{\text{Matrix Regularization}}{\rightleftarrows}}\qquad \begin{aligned} \widehat{J}_\alpha&=\frac{1}{N}\operatorname{Tr}_{\mathbf{N}}\left(\widehat{J}\widehat{Y}_\alpha\right), \\ \widehat{N}_\alpha&=\frac{1}{N}\operatorname{Tr}_{\mathbf{N}}\left(\widehat{N}\widehat{Y}_\alpha\right). \end{aligned} \tag{3.43}$$

The inverse of this relation expresses the $\mathcal{L}(\mathcal{H})$-valued matrix elements $(\widehat{J}_i{}^j,\widehat{N}_i{}^j)$ as a sum over the generators $(\widehat{J}_\alpha,\widehat{N}_\beta)$,

$$\widehat{J}_i{}^j=\sum_{\alpha\in I_N}\widehat{J}_\alpha\big(\widehat{Y}^\alpha\big)_i{}^j,\qquad \widehat{N}_i{}^j=\sum_{\alpha\in I_N}\widehat{N}_\alpha\big(\widehat{Y}^\alpha\big)_i{}^j. \tag{3.44}$$

As before, the matrix elements $(\widehat{J}_i{}^j,\widehat{N}_i{}^j)$ are the regularized version of the phase space functions $(J(\sigma),N(\sigma))$. Finally, the fact that the representation $\pi_{\mathcal{P}}$ corresponding to the quantization of the phase space $\mathcal{P}$ should be unitary implies that the generators satisfy

$$\widehat{J}_\alpha^\dagger=\widehat{J_\alpha^*}=(-1)^a\widehat{J}_{\bar\alpha},\qquad \widehat{N}_\alpha^\dagger=\widehat{N_\alpha^*}=(-1)^a\widehat{N}_{\bar\alpha}. \tag{3.45}$$

A final comment is in order on the uniqueness of the deformation of $\mathfrak{c}_{\mathbb{R}}(S)$ obtained in this section. Instead of using the generators $\widehat{Y}_{1\alpha}$ to arrive at the deformation, one could instead work with $\widehat{Y}_{0\alpha} = \rho_0 \otimes \widehat{Y}_\alpha$, where $\rho_0 = \frac{1}{2}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ is an elliptic generator of $\mathfrak{sl}(2,\mathbb{R})$ satisfying $\rho_0^2 = +\frac{1}{4}$. The entire discussion goes through as before, with the only change being that the bracket (3.35) now comes with a coefficient $+\lambda^2$. Since $\lambda \to 0$ in the large N limit, this gives the same classical algebra in the limit. The deformed algebra in this case is the compact algebra $\mathfrak{su}(N) \oplus \mathfrak{su}(N) \oplus \mathfrak{u}(1)$, as opposed to $\mathfrak{sl}(N,\mathbb{C}) \oplus \mathbb{R}$. The fact that one can obtain the same algebra as a contraction in different ways is not surprising. A familiar similar example is that of the Euclidean group $\mathfrak{so}(3) \ltimes \mathbb{R}^3$, which can be obtained as an Inönü-Wigner contraction of either the noncompact Lie algebra $\mathfrak{so}(1,3) = \mathfrak{sl}(2,\mathbb{C})$ or the compact Lie algebra $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. While the limiting algebra is the same, the groups are quite different: in particular, the choice of contraction can determine whether the deformed group is compact or noncompact. In the present context, however, the noncompact deformation better matches the nature of the classical algebra in which the additional generators correspond to boosts, as opposed to rotations, of the normal plane of the codimension-2 sphere. Additionally, we will see that the noncompact deformation is the correct choice when embedding into the deformation of the larger algebra $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ considered in the next section.

3.4 Matrix regularization of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$

The determination of the matrix regularization of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ follows a similar procedure to the case of $\mathfrak{c}_{\mathbb{R}}(S)$ considered in section 3.3. Using the explicit parameterization of the algebra given in section 2.3, the $\mathfrak{sdiff}(S)$ generators are again labeled by the spherical harmonics Y_α, and the generators of the $\mathfrak{sl}(2,\mathbb{R})^S$ algebra take the form $Y_{a\alpha} = \tau_a \otimes Y_\alpha$, where τ_a are the basis of $\mathfrak{sl}(2,\mathbb{R})$ given in (2.1). Once again, we determine the deformed algebra by promoting the spherical harmonics appearing in these generators to fuzzy spherical harmonics and computing the matrix commutators. As before, the deformed $\mathfrak{sdiff}(S)$ generators are obtained by tensoring with the 2×2 identity $\widehat{Y}_{\bullet\alpha} = \mathbb{1}_2 \otimes \widehat{Y}_\alpha$. To arrive at the deformed $\mathfrak{sl}(2,\mathbb{R})^S$ generators, we recall the discussion in section 3.3 regarding properties of the tensor product of matrices using physicist's versus mathematician's conventions for the algebra. The conclusion is that in order to obtain a consistent algebra after taking the tensor product, we must use a basis for $\mathfrak{sl}(2,\mathbb{R})$ in which the structure constants are purely imaginary. This basis is given by

$$\rho_0 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad \rho_1 = \frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad \rho_2 = \frac{1}{2}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}, \tag{3.46}$$

whose product satisfies

$$\rho_a\rho_b = -\frac{1}{4}\eta_{ab}\mathbb{1}_2 + \frac{i}{2}\varepsilon_{ab}{}^{c}\rho_c. \tag{3.47}$$

It is useful to recall that this basis arises naturally in the presentation of $\mathfrak{sl}(2,\mathbb{R})$ in terms of the isomorphic algebra $\mathfrak{su}(1,1)$. The latter is the Lie algebra of $\mathrm{SU}(1,1)$, consisting of 2×2 complex matrices g of unit determinant preserving an indefinite Hermitian form h,

$$g^\dagger h g = h, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.48}$$

Expressing g as the exponential of a Lie algebra generator $g = \exp(iT)$, preservation of h translates to the condition

$$T^\dagger h = hT, \tag{3.49}$$

which indeed is satisfied by the matrices ρ_a.

We now take the deformed $\mathfrak{sl}(2,\mathbb{R})^S$ generators to be of the form $\widehat{Y}_{a\alpha} = \rho_a \otimes \widehat{Y}_\alpha$. Again employing the identity (3.27) for the bracket of the tensor product of matrices, we find that the commutators of the matrices $(\widehat{Y}_{\bullet\alpha}, \widehat{Y}_{a\alpha})$ satisfy

$$[\widehat{Y}_{\bullet\alpha}, \widehat{Y}_{\bullet\beta}] = \frac{2i}{N}\widehat{C}_{\alpha\beta}{}^\gamma \widehat{Y}_{\bullet\gamma}, \tag{3.50}$$

$$[\widehat{Y}_{\bullet\alpha}, \widehat{Y}_{a\beta}] = \frac{2i}{N}\widehat{C}_{\alpha\beta}{}^\gamma \widehat{Y}_{a\gamma}, \tag{3.51}$$

$$[\widehat{Y}_{a\alpha}, \widehat{Y}_{b\beta}] = i\varepsilon_{ab}{}^c \widehat{E}_{\alpha\beta}{}^\gamma \widehat{Y}_{c\gamma} - \frac{i}{2N}\eta_{ab}\widehat{C}_{\alpha\beta}{}^\gamma \widehat{Y}_{\bullet\gamma}, \tag{3.52}$$

where the last bracket applies equations (3.7), (3.8), and (3.47) for the structure constants of the symmetric and antisymmetric products of $\widehat{Y}_\alpha$ and of ρ_a.

As in the case of the deformation of $\mathfrak{c}_\mathbb{R}(S)$, these matrices do not provide a unitary representation of a Lie algebra, and hence should not be viewed as a quantization of a phase space. Because of this, when matching to the classical algebra, we are again free to rescale the generators by prefactors with different parametric dependence on N. Unlike the case of $\mathfrak{c}_\mathbb{R}(S)$, in the present context matching to the classical algebra (2.29) fully determines the choice of prefactor. Denoting the basis for the deformed Lie algebra as $(X_\alpha, Z_{a\alpha})$ and $\pi_{\mathbf{2N}}$ the representation in which the matrices $(\widehat{Y}_{\bullet\alpha}, \widehat{Y}_{a\alpha})$ live, the required scaling between the algebra generators and the matrices is given by

$$\pi_{\mathbf{2N}}(X_\alpha) = \frac{N}{2i}\widehat{Y}_{\bullet\alpha}, \tag{3.53}$$

$$\pi_{\mathbf{2N}}(Z_{a\alpha}) = \frac{1}{i}\widehat{Y}_{a\alpha}. \tag{3.54}$$

This implies the following brackets for the deformed Lie algebra generators

$$[X_\alpha, X_\beta] = \widehat{C}_{\alpha\beta}{}^\gamma X_\gamma, \tag{3.55}$$

$$[X_\alpha, Z_{a\beta}] = \widehat{C}_{\alpha\beta}{}^\gamma Z_{a\gamma}, \tag{3.56}$$

$$[Z_{a\alpha}, Z_{b\beta}] = \varepsilon_{ab}{}^c \widehat{E}_{\alpha\beta}{}^\gamma Z_{c\gamma} - \frac{1}{N^2}\eta_{ab}\widehat{C}_{\alpha\beta}{}^\gamma X_\gamma. \tag{3.57}$$

Comparing to (2.29), we see that these brackets match the $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ algebra in the limit $N \to \infty$. Note that the scaling of the generators $Z_{a\alpha}$ in (3.54) corresponds to the preferred choice $\lambda = \frac{1}{N}$ discussed below (3.35) for the similar case of the $\mathfrak{c}_\mathbb{R}(S)$ deformation.

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As before, $(X_\alpha, Z_{a\alpha})$ define a complex basis for the deformed Lie algebra. The reality condition to specify the real Lie algebra again descends from the reality condition for the spherical harmonics, and is given by

$$X_\alpha^* = (-1)^a X_{\bar{\alpha}}, \qquad Z_{a\alpha}^* = (-1)^a Z_{a\bar{\alpha}}. \tag{3.58}$$

This reality condition determines whether a given representation is unitary by the requirement that $i\pi(X_\alpha^*) = (i\pi(X_\alpha))^\dagger$ and similarly for $Z_{a\alpha}$. The fact that the matrices $\widehat{Y}_{1\alpha}$ and $\widehat{Y}_{2\alpha}$ do not satisfy this condition verifies that the representation $\pi_{\mathbf{2N}}$ is not unitary.

In the $\pi_{\mathbf{2N}}$ representation, the combination of generators $(\widehat{Y}_{\bullet\alpha}, \widehat{Y}_{a\alpha})$ that are fixed under the involution $*$ are all of the form $(\mathbb{1}_2 \otimes \widehat{A}, \rho_a \otimes \widehat{B})$, with $\widehat{A}$, $\widehat{B}$ Hermitian $N \times N$ matrices. This characterization of the generators allows us to identify the Lie algebra defined by the brackets (3.55), (3.56), and (3.57). Defining a Hermitian form $\widehat{h}$ of signature (N, N) given by

$$\widehat{h} = h \otimes \mathbb{1}_N = \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix}, \tag{3.59}$$

with h the 2×2 mixed signature Hermitian form from (3.48), we find that the generators preserve $\widehat{h}$ in the sense of satisfying the analogous condition to (3.49):

$$(\mathbb{1}_2 \otimes \widehat{A})^\dagger \widehat{h} = h \otimes \widehat{A}^\dagger = h \otimes \widehat{A} = \widehat{h}(\mathbb{1}_2 \otimes \widehat{A}), \tag{3.60}$$

$$(\rho_a \otimes \widehat{B})^\dagger \widehat{h} = \rho_a^\dagger h \otimes \widehat{B}^\dagger = h\rho_a \otimes \widehat{B} = \widehat{h}(\rho_a \otimes \widehat{B}). \tag{3.61}$$

As preservation of $\widehat{h}$ is the defining property of the Lie algebra $\mathfrak{u}(N, N)$, we immediately conclude that the algebra defined by the generators $(X_\alpha, Z_{a\alpha})$ is $\mathfrak{u}(N, N)$.

Just as in the case of $\mathfrak{u}(N)$, the Lie algebra for $\mathfrak{u}(N, N)$ can be parameterized in a basis of elementary matrices $E^{\mathsf{m}}{}_{\mathsf{n}}$, where $\mathsf{m}, \mathsf{n} = 1, \ldots, 2N$. In terms of these, the $(X_\alpha, Z_{a\alpha})$ generators are given by

$$X_\alpha = \frac{N}{2i} (\widehat{Y}_{\bullet\alpha})_{\mathsf{m}}{}^{\mathsf{n}} E^{\mathsf{m}}{}_{\mathsf{n}}, \tag{3.62}$$

$$Z_{a\alpha} = \frac{1}{i} (\widehat{Y}_{a\alpha})_{\mathsf{m}}{}^{\mathsf{n}} E^{\mathsf{m}}{}_{\mathsf{n}}. \tag{3.63}$$

As usual, the Lie brackets in the $E^{\mathsf{m}}{}_{\mathsf{n}}$ basis are given by (see appendix D.2 for the proof)

$$[E^{\mathsf{m}}{}_{\mathsf{n}}, E^{\mathsf{p}}{}_{\mathsf{q}}] = \delta^{\mathsf{p}}_{\mathsf{n}} E^{\mathsf{m}}{}_{\mathsf{q}} - \delta^{\mathsf{m}}_{\mathsf{q}} E^{\mathsf{p}}{}_{\mathsf{n}}, \tag{3.64}$$

and the involution (3.58) becomes

$$(E^{\mathsf{m}}{}_{\mathsf{n}})^* = -\widehat{h}_{\mathsf{np}} \widehat{h}^{\mathsf{mq}} E^{\mathsf{p}}{}_{\mathsf{q}}, \tag{3.65}$$

where $\widehat{h}_{\mathsf{np}}$ is the Hermitian form defined by (3.59), and $\widehat{h}^{\mathsf{mq}}$ is its inverse. The inverse relation between the two bases is

$$E^{\mathsf{m}}{}_{\mathsf{n}} = \frac{i}{N} \left(\frac{1}{N} X^\alpha (\widehat{Y}_{\bullet\alpha})_{\mathsf{n}}{}^{\mathsf{m}} - 2Z^{a\alpha} (\widehat{Y}_{a\alpha})_{\mathsf{n}}{}^{\mathsf{m}} \right). \tag{3.66}$$

Finally, we recall that the central generator X_{00} is not included when matching to the continuum algebra since the constant function on the sphere does not generate a diffeomorphism. The algebra obtained by excluding this generator from the deformed algebra is then the simple Lie algebra $\mathfrak{su}(N,N)$. We, therefore, arrive at one of our main results, that $\mathfrak{su}(N,N)$ defines a finite-dimensional deformation of the continuum algebra $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, or, equivalently, that the large N limit of $\mathfrak{su}(N,N)$ can be identified with $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$.

Just as $\mathfrak{c}_{\mathbb{R}}(S)$ is the subalgebra of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ obtained by restricting to τ_1 generators of $\mathfrak{sl}(2,\mathbb{R})^S$, the deformation $\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}$ occurs as a subalgebra of $\mathfrak{su}(N,N)$ by including only the $Z_{1\alpha}$ generators. In the $\pi_{\mathbf{2N}}$ representation, this subalgebra can equivalently be characterized as the collection of generators that commute with $2\widehat{Y}_{1,00} = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}$. This is interesting because $(2\widehat{Y}_{1,00})^2 = -\mathbb{1}_{\mathbf{2N}}$, and hence defines a complex structure in this representation. Hence we see that $\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}$ can be viewed as the subalgebra of $\mathfrak{su}(N,N)$ preserving a complex structure in the defining representation.

Having identified the deformed algebra, we can now relate this algebra to the quantization of the classical phase space. This requires specifying the map $\widehat{H}\in\mathfrak{su}(N,N)^*\otimes\mathcal{L}(\mathcal{H})$, with $\mathcal{L}(\mathcal{H})$ the space of operators in which the representation $\pi_{\mathcal{P}}$ defining the quantization is valued. Again, this map can be specified by the quantities $\widehat{J}, \widehat{N}_a \in \mathrm{Mat}(N)\otimes\mathcal{L}(\mathcal{H})$ which yield generators in the representation $\pi_{\mathcal{P}}$ by the relation

$$\widehat{J}_\alpha = \frac{1}{N}\mathrm{Tr}_{\mathbf{N}}\left(\widehat{J}\widehat{Y}_\alpha\right), \tag{3.67}$$

$$\widehat{N}_{a\alpha} = \frac{1}{N}\mathrm{Tr}_{\mathbf{N}}\left(\widehat{N}_a\widehat{Y}_\alpha\right). \tag{3.68}$$

The normalization condition for the maps $(\widehat{J},\widehat{N}_a)$ is again chosen so that $(\widehat{J}_\alpha,\widehat{N}_{a\alpha}) = \left(i\hbar\pi_{\mathcal{P}}(X_\alpha), i\hbar\pi_{\mathcal{P}}(Z_{a\alpha})\right)$, so that the generators satisfy the algebra

$$[\widehat{J}_\alpha,\widehat{J}_\beta] = i\hbar\widehat{C}_{\alpha\beta}{}^{\gamma}\widehat{J}_\gamma, \tag{3.69}$$

$$[\widehat{J}_\alpha,\widehat{N}_{a\beta}] = i\hbar\widehat{C}_{\alpha\beta}{}^{\gamma}\widehat{N}_{a\gamma}, \tag{3.70}$$

$$[\widehat{N}_{a\alpha},\widehat{N}_{b\beta}] = i\hbar\left(\varepsilon_{ab}{}^{c}\widehat{E}_{\alpha\beta}{}^{\gamma}\widehat{N}_{c\gamma} - \frac{1}{N^2}\eta_{ab}\widehat{C}_{\alpha\beta}{}^{\gamma}\widehat{J}_\gamma\right). \tag{3.71}$$

This again produces the correspondence between classical and quantum generators of the algebra,

$$\begin{aligned} J_\alpha &= \int_S \nu_0 J Y_\alpha, \\ N_{a\alpha} &= \int_S \nu_0 N_a Y_\alpha, \end{aligned} \qquad \underset{\text{Large-}N\text{ Limit}}{\overset{\text{Matrix Regularization}}{\rightleftarrows}} \qquad \begin{aligned} \widehat{J}_\alpha &= \frac{1}{N}\mathrm{Tr}_N\left(\widehat{J}\widehat{Y}_\alpha\right), \\ \widehat{N}_{a\alpha} &= \frac{1}{N}\mathrm{Tr}_N\left(\widehat{N}_a\widehat{Y}_\alpha\right). \end{aligned} \tag{3.72}$$

The requirement that the representation be unitary follows from the involution (3.58), and implies that the generators satisfy

$$\widehat{J}_\alpha^\dagger = \widehat{J_\alpha^*} = (-1)^a\widehat{J}_{\bar{\alpha}}, \qquad \widehat{N}_{a\alpha}^\dagger = \widehat{N_{a\alpha}^*} = (-1)^a\widehat{N}_{a\bar{\alpha}}. \tag{3.73}$$

It is also convenient to introduce a set of generators tied to the elementary matrix basis. Using $\widehat{J}$ and $\widehat{N}_a$, we can construct a quantity $\widehat{H} \in \mathrm{Mat}_{2N} \otimes \mathcal{L}(\mathcal{H})$ whose matrix elements are

$$\widehat{H}_{\mathsf{m}}{}^{\mathsf{n}} = \frac{1}{N}\widehat{J}^{\alpha}\big(\widehat{Y}_{\bullet\alpha}\big)_{\mathsf{m}}{}^{\mathsf{n}} - 2\widehat{N}^{a\alpha}\big(\widehat{Y}_{a\alpha}\big)_{\mathsf{m}}{}^{\mathsf{n}}. \tag{3.74}$$

By making the split $\mathsf{m} = (M, i)$, $\mathsf{n} = (N, j)$ where $M, N = 1, 2$ are indices in the 2D representation of $\mathfrak{sl}(2, \mathbb{R})$ and $i, j = 1, \ldots, N$ are $\mathfrak{su}(N)$ indices, this relation can equivalently be expressed in block diagonal form,

$$\widehat{H} = \begin{pmatrix} \frac{1}{N}\widehat{J} + \widehat{N}_0 & -\widehat{N}_1 + i\widehat{N}_2 \\ \widehat{N}_1 + i\widehat{N}_2 & \frac{1}{N}\widehat{J} - \widehat{N}_0 \end{pmatrix}. \tag{3.75}$$

The commutators of the operators $\widehat{H}_{\mathsf{m}}{}^{\mathsf{n}}$ are rescaled relative to the bracket (3.64) according to

$$[\widehat{H}_{\mathsf{m}}{}^{\mathsf{n}}, \widehat{H}_{\mathsf{p}}{}^{\mathsf{q}}] = \hbar N \left(\delta^{\mathsf{q}}_{\mathsf{m}}\widehat{H}_{\mathsf{p}}{}^{\mathsf{n}} - \delta^{\mathsf{n}}_{\mathsf{p}}\widehat{H}_{\mathsf{m}}{}^{\mathsf{q}}\right), \tag{3.76}$$

and the Hermiticity condition they satisfy is

$$\big(\widehat{H}_{\mathsf{m}}{}^{\mathsf{n}}\big)^{\dagger} = -\widehat{h}_{\mathsf{mp}}\widehat{h}^{\mathsf{nq}}\widehat{H}_{\mathsf{q}}{}^{\mathsf{p}}. \tag{3.77}$$

4 The large-N correspondence of Casimirs

The previous section established the existence of deformations of three infinite-dimensional symmetry algebras appearing in gravity into finite-dimensional, semisimple Lie algebras. The quantum theory, however, contains information beyond that in the deformed Lie algebra. In particular, the generators of the deformed symmetry are operators on a Hilbert space, and while the Lie algebra determines the commutators of these operators, the quantum theory depends on the full associative product of the operators, i.e., on anticommutators as well as commutators.[17] The structure of the full operator product depends on the representation of the deformed algebra in which the quantum theory is defined. Hence, in order to understand the quantization of the gravitational phase spaces admitting actions of these algebras, we need a means for determining the appropriate representation of the deformed algebra.

As mentioned in section 3.1, the representation is constrained by matching to the classical algebra of functions on the gravitational phase space. In the limit $\hbar \to 0$, the symmetric product of anticommutators of generators of the algebra is required to reproduce the abelian, associative product of the corresponding functions on the phase space. This matching was already discussed in the simplest example of the fuzzy sphere at the beginning of section 3.2. In that case, the symmetric product of the fuzzy spherical harmonics $\widehat{Y}_\alpha$ was given by $\widehat{Y}_\alpha \circ \widehat{Y}_\beta = \widehat{E}_{\alpha\beta}{}^{\gamma}\widehat{Y}_\gamma$, with the normalization of $\widehat{Y}_\alpha$ chosen so that $\widehat{E}_{\alpha\beta}{}^{\gamma}$ approaches the expression for the classical structure constants $E_{\alpha\beta}{}^{\gamma}$ in the limit $N \to \infty$, as indicated in equation (3.9). This equation in fact determines the representation of $\mathfrak{u}(N)$ associated

[17] Consider, for example, the spin-$\frac{1}{2}$ and spin-1 representations of SO(3). In the former, the anticommutator of two different Pauli matrices is zero, while in the latter the anticommutator of two orthogonal $\mathfrak{so}(3)$ generators is a nonzero symmetric matrix with zeros on the diagonal.

with the quantization of the sphere due to the observation that the classical spherical harmonics Y_α form a complete basis for the algebra of functions on the sphere, and hence their quantization should also share this property, namely, that the operator product of two generators closes on the space of generators. The only representation of $\mathfrak{u}(N)$ possessing this property is the defining representation in terms of $N \times N$ matrices, leading to the conclusion that this is the appropriate representation appearing in the quantization of the sphere. Furthermore, as discussed around equation (3.12), consistently matching the commutators of the generators to the Poisson bracket fixes the deformation parameter N_{fs} to be $\frac{A}{2\pi\hbar_{\text{fs}}}$, where A is the area of the phase space computed from the symplectic form. Hence, in this case, we see that the quantized algebra is fully determined by matching the classical limits of the symmetric and antisymmetric products of operators.

A subtlety arises when applying this reasoning to the gravitational phase space, since the classical generators of the algebra are far from forming a complete basis for functions on the phase space. Generic products of generators become complicated multilocal integrals over the 2-sphere in spacetime, all of which represent independent functions on the phase space. This suggests that the representation yielding the quantization of the phase space will be large, in the sense of containing many operators beyond those corresponding to the Lie algebra generators. These additional operators would then be assigned to the multilocal observables of the classical theory. Determining a representation from properties of these multilocal observables appears daunting; however, the task is drastically simplified by focusing on invariant functionals of the classical symmetry algebra, which are associated with Casimir operators in the quantum theory. These invariant functionals arise from the pullback via the moment map of Casimir functions on the classical coadjoint orbits, and we will find that they reduce, nontrivially, to expressions involving single integrals over the 2-sphere in spacetime. Each such function is shown to coincide uniquely with a Casimir element of the deformed algebra, which are represented as matrices proportional to the identity in an irreducible representation. The c-number proportionality constants largely determine the representation, and hence by matching these c-numbers to the values of the corresponding classical phase space functions, we arrive at a procedure for determining the representation associated with the quantization of the phase space.[18] This matching procedure should also determine the value of the deformation parameter N.

Given their importance for determining the representation of the deformed algebra, in this section, we characterize the Casimir elements of each of the deformed algebras, as well as the Casimir functions on the coadjoint orbits of the classical algebras. Furthermore, we derive the appropriate correspondence between classical and deformed Casimir elements,

[18]This argument assumes that the representation is irreducible, and requires one to consider a subspace of the classical phase space defined by fixing the value of the Casimir functions. More generally, we expect the full phase space to be foliated by several such subspaces, which suggests the full quantum theory will occur in a reducible representation, with each irreducible component coinciding, roughly, with a single leaf of the foliation in the classical phase space. Determining the multiplicity of the representations occurring in this quantization appears to be more challenging. One needs either a natural measure on the space of Casimir functions, possibly arising from the phase space symplectic form itself, or otherwise to find a larger symmetry group that acts transitively on the phase space, whose irreducible representations will occur as reducible representations of the smaller algebras considered here.

which then facilitates the matching procedure needed to determine the representation for the quantization of the gravitational phase space. In the case of $\mathfrak{sdiff}(S)$, we carry out the matching in somewhat more detail to argue that the value of N and the associated representation of $\mathfrak{su}(N)$ are both determined by this procedure.

4.1 $\mathfrak{sdiff}(S)$ and $\mathfrak{su}(N)$

We begin with the application of the above procedure to the algebra $\mathfrak{sdiff}(S)$ and its deformation $\mathfrak{su}(N)$. The key step is to classify the invariants of the two algebras, and to determine the correspondence between the invariants in the large-N limit.

Phase space functions that are invariant under the action of $\mathfrak{sdiff}(S)$ generically arise as pullbacks of Casimir elements of the $\mathfrak{sdiff}(S)$ Lie algebra via the moment map. We recall that the moment map μ for a given phase space admitting an action of $\mathfrak{sdiff}(S)$ sends the phase space $\mathcal{P}$ to the dual of the Lie algebra $\mathfrak{sdiff}(S)^*$, which is itself a phase space admitting a Hamiltonian action of $\mathfrak{sdiff}(S)$ via the coadjoint action [68]. Hence, a classification of the invariant functions for this action on $\mathfrak{sdiff}(S)^*$ leads to a corresponding classification of invariants on the phase space $\mathcal{P}$. Casimir elements of the universal enveloping algebra of $\mathfrak{sdiff}(S)$ define functions on $\mathfrak{sdiff}(S)^*$ via the natural pairing between the Lie algebra and its dual, and the fact that the Casimir elements commute with the Lie algebra translates to the statement that the corresponding functions on $\mathfrak{sdiff}(S)^*$ are invariant under the coadjoint action of $\mathfrak{sdiff}(S)$. This, therefore, gives the link between Casimir elements of the Lie algebra and $\mathfrak{sdiff}(S)$ invariants in the gravitational phase space, thus reducing the problem to determining the Casimir elements of $\mathfrak{sdiff}(S)$.

Before doing so, we first describe the space $\mathfrak{sdiff}(S)^*$ and the coadjoint action in more detail. As discussed in section 2.1, the Lie algebra $\mathfrak{sdiff}(S)$ can be parameterized in terms of stream functions, with each function ϕ on the sphere coinciding with an infinitesimal diffeomorphism. The Lie bracket between two functions ϕ and ψ is defined via the Poisson bracket $\{\phi, \psi\}_\epsilon$ associated with the unit-radius volume form ϵ. It will be convenient in this section to consider the space $C^\infty(S)$ of all smooth functions on the sphere, including the constant function which generates the trivial center of the Poisson algebra of functions, and hence is equivalent to working with the trivially extended algebra $\mathfrak{g} = \mathfrak{sdiff}(S) \oplus \mathbb{R}$. The dual Lie algebra $\mathfrak{g}^*$ can also be parameterized by functions on the sphere due to the natural pairing provided by integration over the sphere. Specifically, for $\phi \in \mathfrak{g}$, $f \in \mathfrak{g}^*$, the pairing is given by

$$\langle f, \phi \rangle = \int_S \nu_0 f \, \phi. \tag{4.1}$$

Note that because $\mathfrak{sdiff}(S)$ is associated with the quotient space of all functions modulo constant shifts, the natural dual $\mathfrak{sdiff}(S)^*$ is given by all functions which integrate to zero, in order to have a consistent pairing by integrating over the sphere.

Since the Poisson bracket of functions defines the Lie algebra on $\mathfrak{g}$, the adjoint action is given in terms of this bracket: $\mathrm{ad}_\phi \psi = \{\phi, \psi\}_\epsilon$. Throughout this section, we will always employ the Poisson bracket $\{\cdot, \cdot\}_\epsilon$, and hence will drop the ϵ subscript. The coadjoint action ad^*_ϕ on $\mathfrak{g}^*$ is defined by

$$\langle \mathrm{ad}^*_\phi f, \psi \rangle = -\langle f, \mathrm{ad}_\phi \psi \rangle. \tag{4.2}$$

Applying the definition (4.1) of the pairing, this implies that

$$\langle \mathrm{ad}^*_\phi f, \psi\rangle = -\int_S \nu_0 f\{\phi,\psi\} = \int_S \nu_0\{\phi, f\}\psi = \langle\{\phi, f\},\psi\rangle. \tag{4.3}$$

We, therefore, conclude that coadjoint action is given by

$$\mathrm{ad}^*_\phi f = \{\phi, f\}, \tag{4.4}$$

and hence agrees with the adjoint action when both spaces $\mathfrak{g}$ and $\mathfrak{g}^*$ are realized as $C^\infty(S)$.

The coadjoint-invariant functions on $\mathfrak{g}^*$ are expressible in terms of the Casimir elements of the $\mathfrak{g}$ universal enveloping algebra. The latter can be constructed as follows. We begin by noting that the identity map Id on $\mathfrak{g}$ is an element of $\mathfrak{g}^*\otimes\mathfrak{g}$, and using the isomorphism between $\mathfrak{g}^*$ and $C^\infty(S)$, we see that it is naturally associated with a Lie-algebra valued function $\mathbb{j}\in C^\infty(S)\otimes\mathfrak{g}$. The fact that $\mathbb{j}$ arises from the identity map implies the relation

$$\langle f, \mathbb{j}\rangle = f, \tag{4.5}$$

where the pairing is taken between the $\mathfrak{g}$ tensor factor of $\mathbb{j}$ and $f\in\mathfrak{g}^*$, and the output is the function on S corresponding to f. Similarly, we have that

$$\int_S \nu_0\,\mathbb{j}\,\phi = \phi, \tag{4.6}$$

where again the output on the right-hand side is the element ϕ of $\mathfrak{g}$ associated with the function ϕ on the sphere. This latter relation implies that, if we instead use the isomorphism between $\mathfrak{g}$ and $C^\infty(S)$, we can view $\mathbb{j}$ as a bilocal function, $\mathbb{j}\in C^\infty(S)\otimes C^\infty(S)$, the relation (4.6) implies that $\mathbb{j}(\sigma,\sigma') = \delta(\sigma-\sigma')$.[19] Additionally, it allows us to obtain a basis $\mathbb{j}_\alpha$ of $\mathfrak{g}$ from the spherical harmonics Y_α,

$$\mathbb{j}_\alpha = \int_S \nu_0\,\mathbb{j} Y_\alpha, \tag{4.7}$$

which, conversely, leads to a mode decomposition of the function $\mathbb{j}(\sigma)$,

$$\mathbb{j}(\sigma) = \sum_\alpha \mathbb{j}_\alpha Y^\alpha(\sigma). \tag{4.8}$$

Equation (4.7) states that the Lie algebra element associated with the spherical harmonic Y_α is given by $\mathbb{j}_\alpha$. It further implies that the Lie bracket with $\phi\in\mathfrak{g}$ should be determined by the Poisson bracket between ϕ and the spherical harmonic Y_α; specifically,

$$[\phi, \mathbb{j}_\alpha] = \int_S \nu_0\,\mathbb{j}\{\phi, Y_\alpha\}. \tag{4.9}$$

This relation can then be used to determine the Lie bracket of ϕ with the $\mathfrak{g}$ factor of $\mathbb{j}$, (see appendix E)

$$[\phi, \mathbb{j}] = -\{\phi, \mathbb{j}\}, \tag{4.10}$$

where the Poisson bracket is evaluated on the $C^\infty(S)$ factor of $\mathbb{j}$.

[19] Since this is a distribution rather than a function, $\mathbb{j}$ actually lies in a larger space than $C^\infty(S)\otimes C^\infty(S)$ that includes distributions, but this technicality does not affect the arguments of this section.

The Casimir elements of $\mathfrak{g}$ can then be obtained straightforwardly by taking products of $\mathbb{j}$ with itself. The expression $\mathbb{j}^n$ is interpreted as a $\mathfrak{g}^{\otimes n}$-valued function on the sphere, with the product taken within the $C^\infty(S)$ factor of each $\mathbb{j}$.[20] To arrive at the Casimir elements, we integrate this object over the sphere,

$$c_n = \int_S \nu_0 \, \mathbb{j}^n. \tag{4.11}$$

Verifying that c_n commutes with every element of $\mathfrak{g}$ comes from a straightforward application of (4.10):

$$\begin{aligned}
[\phi, c_n] &= \int_S \nu_0 \sum_{k=1}^{n} \mathbb{j}^{k-1}[\phi, \mathbb{j}]\mathbb{j}^{n-k} \\
&= -\int_S \nu_0 \sum_{k=1}^{n} \mathbb{j}^{k-1}\{\phi, \mathbb{j}\}\mathbb{j}^{n-k} \\
&= -\int_S \nu_0 \{\phi, \mathbb{j}^n\} = 0,
\end{aligned} \tag{4.12}$$

since any Poisson bracket integrates to zero over the sphere.

When matching to the Casimirs of the deformed algebra $\mathfrak{u}(N)$, it is useful to have an expression of c_n in a specific basis. This can be obtained immediately from the mode decomposition (4.8) of $\mathbb{j}$,

$$c_n = \mathbb{j}_{\alpha_1}\mathbb{j}_{\alpha_2} \dots \mathbb{j}_{\alpha_n} \int_S \nu_0 Y^{\alpha_1} \dots Y^{\alpha_n} \equiv \mathbb{j}_{\alpha_1} \dots \mathbb{j}_{\alpha_n} d^{\alpha_1 \dots \alpha_n}, \tag{4.13}$$

where the second equation defines the totally symmetric tensor $d^{\alpha_1 \dots \alpha_n}$. We will later see that this tensor matches a corresponding tensor $\widehat{d}^{\alpha_1 \dots \alpha_n}$ defined for $\mathfrak{u}(N)$ as $N \to \infty$.

The functions on $\mathfrak{g}^*$ associated with the Casimir elements c_n are obtained by the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. For $f \in \mathfrak{g}^*$, we have that

$$c_n[f] = \int_S \nu_0 \langle \mathbb{j}, f \rangle^n = \int_S \nu_0 \, f^n, \tag{4.14}$$

where we have applied the relation (4.5). In the current context in which our Lie algebra involves $\mathfrak{sdiff}(S)$, these Casimir functions are called enstrophies, due to a close analogy between the $\mathfrak{sdiff}(S)$ coadjoint orbits and 2D fluid dynamics on the sphere [26, 75, 76]. These Casimir functions can be pulled back to the gravitational phase space via the moment map μ. This pullback is readily obtained from the relation $\mu^* \mathbb{j} = J$, where J is the function on S defined in section 2.2. The result of the pullback of the Casimir functions is a set of $\mathfrak{sdiff}(S)$ invariants on the gravitational phase space, coinciding with the gravitational enstrophies discussed in [26]. These invariants are given explicitly by

$$C_n = \int_S \nu_0 J^n. \tag{4.15}$$

[20] If we instead identify each factor of $\mathfrak{g}$ with a function on the sphere, the expression $\mathbb{j}^n$ would be interpreted as an $(n+1)$-local function on the sphere consisting of products of delta functions, i.e. $\mathbb{j}^n(\sigma_1, \dots \sigma_n; \sigma) = \prod_{i=1}^{n} \delta(\sigma_i - \sigma)$. However, in matching to the Casimirs of the deformed algebra, it is more convenient to use the abstract Lie algebra as opposed to the representation in terms of functions on the sphere.

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Note that the invariants can equivalently be expressed by pulling back the mode decomposition (4.13) to the gravitational phase space. Since $\mathbb{j}_\alpha$ pulls back to the generator J_α on the phase space, we see that C_n is equivalently expressed as

$$C_n = J_{\alpha_1} \dots J_{\alpha_n} d^{\alpha_1 \dots \alpha_n}. \tag{4.16}$$

Since each J_α is given by an integral over S, the expression (4.16) naively appears to be a complicated object involving multiple integrals over the sphere. The fact that it localizes to a single integral as in (4.15) comes from special properties of the tensor $d^{\alpha_1 \dots \alpha_n}$, which produces delta functions when contracted into the J_α in the expression for C_n, resulting in a single integral expression.[21]

The Casimir elements for the deformed algebra $\widehat{\mathfrak{g}} = \mathfrak{u}(N)$ can be obtained in an analogous manner. We now define an object $\widehat{\mathbb{j}} \in \mathrm{Mat}_{N\times N} \otimes \widehat{\mathfrak{g}}$ normalized so that

$$\frac{1}{N} \mathrm{Tr}_{\mathbf{N}} \left(\widehat{\mathbb{j}} \cdot \widehat{Y}_\alpha \right) = X_\alpha, \tag{4.17}$$

where X_α are the basis elements for the $\mathfrak{u}(N)$ Lie algebra introduced in section 3.2. The mode decompositions of the $\widehat{\mathbb{j}}$ matrix elements are therefore given by

$$\widehat{\mathbb{j}}_i{}^j = \sum_\alpha X_\alpha (\widehat{Y}^\alpha)_i{}^j. \tag{4.18}$$

Similar to the classical relation (4.10), the Lie bracket between $\widehat{\mathbb{j}}$ and a Lie algebra element X_α can be expressed as (see appendix E)

$$[X_\alpha, \widehat{\mathbb{j}}]_{\widehat{\mathfrak{g}}} = -\frac{N}{2i} [\widehat{Y}_\alpha, \widehat{\mathbb{j}}], \tag{4.19}$$

where the bracket on the right hand side is the matrix commutator evaluated on the $\mathrm{Mat}_{N\times N}$ factor of $\widehat{\mathbb{j}}$.

Invariant elements of the $\widehat{\mathfrak{g}}$ tensor algebra arise from products of $\widehat{\mathbb{j}}$ with itself, $\widehat{\mathbb{j}}^n$, where the product is taken within the $\mathrm{Mat}_{N\times N}$ factor of $\widehat{\mathbb{j}}$ and the resulting matrix is valued in $\widehat{\mathfrak{g}}^{\otimes n}$, which therefore defines an element of the universal enveloping algebra. Taking a trace over the matrix factor yields the Casimir element,

$$\widehat{c}_n = \frac{1}{N} \mathrm{Tr}_{\mathbf{N}} \widehat{\mathbb{j}}^n, \tag{4.20}$$

[21] As an example, since $J_\alpha = \int_S d\sigma Y_\alpha(\sigma) J(\sigma)$ and $d^{\alpha\beta} = \int_S d\sigma Y^\alpha(\sigma) Y^\beta(\sigma)$, evaluating (4.16) for C_2, we get

$$\begin{aligned} C_2 &= \int d\sigma_1 \int d\sigma_2 \int d\sigma_3 J(\sigma_1) J(\sigma_2) Y_\alpha(\sigma_1) Y_\beta(\sigma_2) Y^\alpha(\sigma_3) Y^\beta(\sigma_3) \\ &= \int d\sigma_1 \int d\sigma_2 \int d\sigma_3 J(\sigma_1) J(\sigma_2) \delta(\sigma_1 - \sigma_3) \delta(\sigma_2 - \sigma_3) \\ &= \int d\sigma_1 J^2(\sigma_1). \end{aligned}$$

Computations for the higher Casimirs C_n show that delta functions appear in a similar manner, always leading to a single integral expression.

which can be shown to commute with $\widehat{\mathfrak{g}}$ using (4.19):

$$\begin{aligned}[X_\alpha, \widehat{c}_n]_{\widehat{\mathfrak{g}}} &= \frac{1}{N}\sum_{k=1}^{n} \mathrm{Tr}\left(\widehat{\mathbb{j}}^{k-1}[X_\alpha, \widehat{\mathbb{j}}\,]_{\widehat{\mathfrak{g}}}\,\widehat{\mathbb{j}}^{n-k}\right) \\ &= -\frac{N}{2i}\frac{1}{N}\sum_{k=1}^{n} \mathrm{Tr}\left(\widehat{\mathbb{j}}^{k-1}[\widehat{Y}_\alpha \widehat{\mathbb{j}}\,]\,\widehat{\mathbb{j}}^{n-k}\right) \\ &= -\frac{1}{2i}\,\mathrm{Tr}\left([\widehat{Y}_\alpha, \widehat{\mathbb{j}}^n]\right) = 0.\end{aligned} \tag{4.21}$$

This can also be expressed in the X_α basis for $\widehat{\mathfrak{g}}$ by applying the mode decomposition (4.18)

$$\widehat{c}_n = X_{\alpha_1}\dots X_{\alpha_n}\frac{1}{N}\,\mathrm{Tr}\left(\widehat{Y}^{\alpha_1}\dots\widehat{Y}^{\alpha_n}\right) \equiv X_{\alpha_1}\dots X_{\alpha_n}\widehat{d}^{\alpha_1\dots\alpha_n}, \tag{4.22}$$

where the second equality defines the coefficients $\widehat{d}^{\alpha_1\dots\alpha_n}$.

As demonstrated in appendix E, in the large-N limit the deformed Casimir coefficients $\widehat{d}^{\alpha_1\dots\alpha_n}$ approach the classical Casimir coefficients $d^{\alpha_1\dots\alpha_n}$ for the Lie algebra $\mathfrak{sdiff}(S)\oplus\mathbb{R}$,

$$\widehat{d}^{\alpha_1\dots\alpha_n} = d^{\alpha_1\dots\alpha_n} + \mathcal{O}(N^{-1}). \tag{4.23}$$

In this sense, the Casimir elements of the deformed algebra approach those of the classical algebra in the large-N limit.[22] In particular, it implies that in the representation $\pi_{\mathcal{P}}$ associated with the gravitational phase space, the quantization of the deformed Casimir elements

$$\widehat{C}_n = \widehat{J}_{\alpha_1}\dots\widehat{J}_{\alpha_n}\widehat{d}^{\alpha_1\dots\alpha_n} = (i\hbar)^n\pi_{\mathcal{P}}(\widehat{c}_n), \tag{4.24}$$

must match the value of the classical invariants C_n, given by (4.15), up to $\mathcal{O}(\hbar^2)$ and $\mathcal{O}(N^{-2})$ corrections. The prefactor of $(i\hbar)^n$ appears due to the normalization condition $\widehat{J}_\alpha = i\hbar\pi_{\mathcal{P}}(X_\alpha)$.

4.2 Matching Casimirs

Having determined the correspondence between the Casimir operators $\widehat{C}_n$ and the classical gravitational invariants C_n, we next show that this correspondence can be used to determine the deformation parameter N and the appropriate representation of $\mathfrak{u}(N)$ associated with the quantization of the gravitational phase space. This matching makes use of the explicit characterization of large-N representations of $\mathfrak{u}(N)$ and the associated Casimir operators that has been developed in previous investigations on matrix models (see, e.g. [77]).

In order to take advantage of these results, we first need to express the Casimir elements $\widehat{c}_n$ given in (4.22) in terms of the standard expressions for the Casimirs in the elementary matrix basis ${E^i}_j$ for $\mathfrak{u}(N)$, described in (3.16) and (3.17). Using the identity satisfied by the fuzzy spherical harmonics (derived in appendix D.1)

$$\delta^{\alpha\beta}{(\widehat{Y}_\alpha)_i}^j{(\widehat{Y}_\beta)_k}^l = N\delta_i^l\delta_k^j, \tag{4.25}$$

[22] This agreement between the Casimir elements c_n and $\widehat{c}_n$ requires that n is held fixed as $N\to\infty$.

we find the expression for $\widehat{c}_k$ in the $E^i{}_j$ basis,

$$\begin{aligned}\widehat{c}_k &= \left(\frac{N}{2i}\right)^k \frac{1}{N} (\widehat{Y}^{\alpha_1})_{i_1}{}^{i_2} (\widehat{Y}_{\alpha_1})_{m_1}{}^{n_1} E^{m_1}{}_{n_1} \dots (\widehat{Y}^{\alpha_k})_{i_k}{}^{i_1} (\widehat{Y}_{\alpha_k})_{m_k}{}^{n_k} E^{m_k}{}_{n_k} \\ &= \frac{N^{2k-1}}{(2i)^k} E^{i_2}{}_{i_1} E^{i_3}{}_{i_2} \dots E^{i_1}{}_{i_k}.\end{aligned} \tag{4.26}$$

Up to a permutation of the order of the $E^i{}_j$ generators,[23] this shows that the Casimir elements $\widehat{c}_k$ are rescaled by a factor of $\frac{N^{2k-1}}{(2i)^k}$ relative to the standard $\mathfrak{u}(N)$ Casimirs

$$\widetilde{c}_k = E^{i_1}{}_{i_2} E^{i_2}{}_{i_3} \dots E^{i_k}{}_{i_1}. \tag{4.27}$$

In a given irreducible representation R of $\mathfrak{u}(N)$, the Casimir element $\widetilde{c}_k$ is given by a number $\widetilde{c}(R)$ times the identity. This number is matched to the corresponding invariant functional on the gravitational phase space in order to determine the representation R and deformation parameter N. Noting the rescaling by $(i\hbar)^k$ implied by equation (4.24) and the additional prefactor in (4.26) relating $\widehat{c}_k$ and $\widetilde{c}_k$, the matching between $\widehat{C}_k$ and C_k implies that

$$C_k = \left(\frac{\hbar N}{2}\right)^k N^{k-1} \widetilde{c}_k(R), \tag{4.28}$$

showing that the gravitational enstrophies C_k directly determine the values of the Casimirs $\widetilde{c}(R)$ in the representation associated with the quantization of the phase space.

While a detailed determination of the representation from these matching relations depends on the precise values of the enstrophies C_k, we can use generic properties of the Casimirs at large N to determine a scaling relation for the deformation parameter N. The relation is that [77]

$$\widetilde{c}_k(R) \sim N^{k-1} n, \tag{4.29}$$

where n denotes the number of boxes in the Young diagram for the representation R, with the precise coefficient and subleading corrections depending on the shape of the Young diagram.

To arrive at the desired relation for N, we would like to determine how the gravitational enstrophies C_k scale with the area of the surface S. This requires relating the normalization conventions for the J_α generators given in section 2.3 to the convention employed in reference [26]. Consistently relating the normalization conventions (see appendix E) leads to the relation

$$C_k = \left(\frac{A}{16\pi G}\right)^k \int_S \nu_0 \left(\frac{-AW}{4\pi}\right)^k, \tag{4.30}$$

where A is the area of the surface, and W is the outer curvature scalar associated with curvature of the normal bundle of S [26, 78]. Since the quantity AW is a dimensionless, order 1 function on the sphere, we see that the integral in (4.30) only contributes an order 1 coefficient to each C_k. Hence, the scaling relation for C_k with area is

$$C_k \sim \left(\frac{A}{16\pi G}\right)^k. \tag{4.31}$$

[23]This reordering will affect a detailed matching for the Casimirs including subleading corrections in $\frac{1}{N}$, but should not affect the large-N scaling derived in this section.

Together with the matching equation (4.28) and the large N scaling of the Casimirs (4.29), this implies that

$$\left(\frac{A}{16\pi G}\right)^k \sim \left(\frac{\hbar N^3}{2}\right)^k \frac{n}{N^2}. \tag{4.32}$$

In order to satisfy this scaling for all values of k, it must be that the number of boxes n in the Young diagram of the representation scales like N^2, and further that N scales as

$$N \sim \left(\frac{A}{8\pi G\hbar}\right)^{\frac{1}{3}}. \tag{4.33}$$

Since $\frac{A}{8\pi G\hbar}$ is associated with the entropy of the codimension-2 surface S, we find that this relation says that $S \sim N^3$. This stands in contrast with standard holographic examples, where typically the entropy of a black hole scales with N^2.[24] However, we note that this computation should be taken with a grain of salt, since we are only analyzing the $\mathfrak{sdiff}(S)$ symmetry algebra, which is a subalgebra of the full gravitational symmetry algebra. In particular, the $\mathfrak{sdiff}(S)$ subalgebra does not include boosts, whose Noether charge in gravity is typically associated with the entropy of black holes. Hence, although the calculations of this section give a proof of principle for how the Casimir matching should work, we should not immediately draw any conclusions from these computations in relation to entropy in gravitational applications. Instead, we should look to complete the matching conditions in the extended algebras $\mathfrak{c}_{\mathbb{R}}(S)$ or $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, or even the full gravitational algebra $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, which may yield a more sensible relation between the entropy and deformation parameter N.

The results of this section has demonstrated how the Casimir matching can be done in principle to determine the representation; however, it would be interesting to carry out this matching in more detail. Doing so would yield a precise relation between the entropy and deformation parameter N. Furthermore, we should expect to be able to relate the function W on the sphere to the shape of the Young diagram of the representation in the large N limit. We leave this more detailed matching as an interesting direction for future work.

4.3 $\mathfrak{c}_{\mathbb{R}}(S)$ and $\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}$

We now turn to the first extended algebra appearing in the gravitational phase space, $\mathfrak{c}_{\mathbb{R}}(S)$, which was shown in section 3.3 to arise as a large N limit of the finite-dimensional algebra $\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}$. Following the same procedure as in section 4.1, we begin by describing the coadjoint orbits of the classical algebra $\mathfrak{c}_{\mathbb{R}}(S)$ and use them to determine the Casimir elements of the algebra. We then identify the Casimir elements of the deformed algebra, and determine the appropriate matching condition between these Casimirs and their classical analogs.

As in section 4.1, it is convenient to work with the algebra $\mathfrak{g} = \mathfrak{c}_{\mathbb{R}}(S)\oplus\mathbb{R}$, where the additional central generator corresponds to a constant function on the sphere. The Lie

[24] However, this scaling is far from universal. A counterexample is provided by ABJM theory, which is dual to quantum gravity in AdS_4 [79] and for which the entropy scales as $S = N^{\frac{3}{2}}$. There are also examples of brane configurations in string theory with triple intersections in which the number of states can scale as N^3 [80], reminiscent of the scaling found here.

algebra $\mathfrak{g}$ is then parameterized by a pair of functions (ϕ, α) on the sphere, and the dual of the Lie algebra is similarly parameterized by a pair of functions (f, a). The pairing is given by the integral over the sphere,

$$\langle (f, a), (\phi, \alpha) \rangle = \int_S \nu_0 \, (f\phi + a\alpha) \, . \tag{4.34}$$

The adjoint action of the Lie algebra on itself is given by $\mathrm{ad}_{(\phi,\alpha)}(\psi, \beta) = (\{\phi, \psi\}, \{\phi, \beta\} - \{\psi, \alpha\})$, which, along with the pairing (4.34) determines the coadjoint action to be (see appendix E)

$$\mathrm{ad}^*_{(\phi,\alpha)}(f, a) = (\{\phi, f\} + \{\alpha, a\}, \{\phi, a\}). \tag{4.35}$$

This equation indicates that both f and a transform as scalars under $\mathfrak{sdiff}(S)$ transformations, but f has a nontrivial transformation law under the $\mathbb{R}^S$ subalgebra, which leads to some subtleties in obtaining a full set of Casimir invariants.

To construct the Casimir elements of $\mathfrak{g}$, it is convenient to introduce a pair of $\mathfrak{g}$-valued functions on the sphere $(\mathbb{j}, \mathbb{n})$ in analogy with the construction of section 4.1, satisfying

$$\langle (f, a), \mathbb{j} \rangle = f, \tag{4.36}$$

$$\langle (f, a), \mathbb{n} \rangle = a, \tag{4.37}$$

where the left-hand side evaluates the pairing between (f, a) and the $\mathfrak{g}$ factors of $\mathbb{j}$ and $\mathbb{n}$, and the right hand side returns the functions on the sphere associated with f and a. These pairing relations can then be used to determine the Lie bracket between an element (ϕ, α) of $\mathfrak{g}$ and the $\mathfrak{g}$ factors of $\mathbb{j}$ and $\mathbb{n}$ (see appendix E). The result is

$$[(\phi, \alpha), \mathbb{j}] = -\{\phi, \mathbb{j}\} - \{\alpha, \mathbb{n}\}, \tag{4.38}$$

$$[(\phi, \alpha), \mathbb{n}] = -\{\phi, \mathbb{n}\}. \tag{4.39}$$

The Casimir elements are now obtained by examining products of the form $\mathbb{j}^m \mathbb{n}^n$, interpreted as a $\mathfrak{g}^{\otimes(m+n)}$-valued function on the sphere, again with the product taken within the $C^\infty(S)$ factor of each $\mathbb{j}, \mathbb{n}$. Taking integrals of these over the sphere gives a set of candidate Casimir elements,

$$c_{mn} = \int_S \nu_0 \mathbb{j}^m \mathbb{n}^n. \tag{4.40}$$

From the relations (4.38) and (4.39), one can verify that c_{mn} commute with all $\mathfrak{sdiff}(S)$ generators in $\mathfrak{g}$:

$$[(\phi, 0), c_{mn}] = -\int_S \nu_0 \{\phi, \mathbb{j}^m \mathbb{n}^n\} = 0. \tag{4.41}$$

However, there is an additional constraint coming from demanding invariance with respect to the $\mathbb{R}^S$ generators $(0, \alpha)$:[25]

$$\begin{aligned} [(0, \alpha), c_{mn}] &= -m \int_S \nu_0 \, \mathbb{j}^{m-1} \{\alpha, \mathbb{n}\} \mathbb{n}^n \\ &= m(m-1) \int_S \nu_0 \, \mathbb{j}^{m-2} \mathbb{n}^n \{\mathbb{j}, \mathbb{n}\} \alpha. \end{aligned} \tag{4.42}$$

[25]These steps require that we employ the identities $\mathbb{j}\mathbb{n} = \mathbb{n}\mathbb{j}$, $\{\mathbb{n}, \mathbb{n}\} = 0$, and $\{\mathbb{n}, \mathbb{j}\} \propto \mathbb{n}$, none of which are immediately obvious due to $\mathbb{j}$ and $\mathbb{n}$ being Lie algebra valued. These identities are derived in appendix E.

Since $\{\mathbb{j}, \mathbb{n}\} \neq 0$, this quantity will vanish for all choices of the function α only if $m = 0$ or $m = 1$. Hence, these define two sets of Casimir elements for $\mathfrak{g}$,

$$c_{0n} = \int_S \nu_0 \, \mathbb{n}^n, \qquad c_{1n} = \int_S \nu_0 \, \mathbb{j} \mathbb{n}^n, \qquad n = 1, 2, \ldots. \tag{4.43}$$

The associated functions on $\mathfrak{g}^*$ that are invariant under the coadjoint action are given by

$$c_{0n}[(f, a)] = \int_S \nu_0 \, a^n, \qquad c_{1n}[(f, a)] = \int_S \nu_0 \, f a^n. \tag{4.44}$$

These pull back to invariant functions on the gravitational phase space is given by

$$C_{0n} = \int_S \nu_0 \, N^n, \qquad C_{1n} = \int_S \nu_0 \, J N^n. \tag{4.45}$$

We now would like to relate the classical Casimirs to the Casimir elements of the deformed algebra, which we take to be $\widehat{\mathfrak{g}} = \mathfrak{gl}(N, \mathbb{C})$, which is the appropriate algebra to limit to the classical algebra $\mathfrak{c}_{\mathbb{R}}(S) \oplus \mathbb{R}$. To identify the Casimirs of $\widehat{\mathfrak{g}}$, we can proceed analogously to section 4.1 and define $\widehat{\mathbb{j}}, \widehat{\mathbb{n}} \in \mathrm{Mat}_{N\times N} \otimes \widehat{\mathfrak{g}}$, normalized such that

$$\frac{1}{N} \mathrm{Tr}_{\mathbf{N}} \left(\widehat{\mathbb{j}} \cdot \widehat{Y}_\alpha \right) = X_\alpha, \tag{4.46}$$

$$\frac{1}{N} \mathrm{Tr}_{\mathbf{N}} \left(\widehat{\mathbb{n}} \cdot \widehat{Y}_\alpha \right) = Z_\alpha, \tag{4.47}$$

where X_α, Z_α are the generators of the $\mathfrak{gl}(N, \mathbb{C})$ algebra defined in section 3.3. This implies the following mode decomposition of the matrix elements of $\widehat{\mathbb{j}}, \widehat{\mathbb{n}}$,

$$\widehat{\mathbb{j}}_i{}^j = \sum_\alpha X_\alpha (\widehat{Y}^\alpha)_i{}^j, \tag{4.48}$$

$$\widehat{\mathbb{n}}_i{}^j = \sum_\alpha Z_\alpha (\widehat{Y}^\alpha)_i{}^j. \tag{4.49}$$

The Casimir elements are most easily identified by forming complex combinations of $\widehat{\mathbb{j}}$ and $\widehat{\mathbb{n}}$. These arise naturally by noting that the complexified generators $E^\pm_\alpha = \frac{1}{2}(X_\alpha \pm i N Z_\alpha)$ satisfy

$$[E^\pm_\alpha, E^\pm_\beta] = \widehat{C}_{\alpha\beta}{}^\gamma E^\pm_\gamma, \tag{4.50}$$

$$[E^+_\alpha, E^-_\beta] = 0. \tag{4.51}$$

The associated Lie-algebra-valued matrices $\widehat{\mathbb{e}}^\pm$ defined by the condition

$$\frac{1}{N} \mathrm{Tr}_{\mathbf{N}} \left(\widehat{\mathbb{e}}^\pm \cdot \widehat{Y}_\alpha \right) = E^\pm_\alpha, \tag{4.52}$$

are then related to $\widehat{\mathbb{j}}, \widehat{\mathbb{n}}$ by

$$\widehat{\mathbb{e}}^\pm = \frac{1}{2} \left(\widehat{\mathbb{j}} \pm i N \widehat{\mathbb{n}} \right). \tag{4.53}$$

The Lie brackets between $\widehat{\mathbb{e}}^\pm$ and the Lie algebra elements $E^\pm_\alpha$ can be shown to satisfy

$$[E^\pm_\alpha, \widehat{\mathbb{e}}^\pm]_{\widehat{\mathfrak{g}}} = -\frac{N}{2i} [\widehat{Y}_\alpha, \widehat{\mathbb{e}}^\pm], \tag{4.54}$$

$$[E^\mp_\alpha, \widehat{\mathbb{e}}^\pm]_{\widehat{\mathfrak{g}}} = 0. \tag{4.55}$$

From this, it follows that two sets of Casimir elements can be formed according to

$$\widehat{c}_n^{\pm} = \frac{1}{N}\,\mathrm{Tr}_{\mathbf{N}}\left[\left(\frac{2\widehat{\mathbb{e}}^{\pm}}{N}\right)^n\right]. \tag{4.56}$$

Demonstrating that $\widehat{c}_n^{\pm}$ commute with $\widehat{\mathfrak{g}}$ proceeds analogously to the computation leading to (4.21).

The scaling with N chosen for the normalization of $\widehat{c}_n^{\pm}$ is needed in order to obtain a good large N limit. Expanding out the expression for $\widehat{c}_n^{\pm}$ in terms of $\widehat{\mathbb{j}}, \widehat{\mathbb{n}}$, we find that

$$\widehat{c}_n^{\pm} = \frac{(\pm i)^n}{N}\,\mathrm{Tr}_{\mathbf{N}}(\widehat{\mathbb{n}}^n) + \frac{(\pm i)^{n-1}n}{N}\,\mathrm{Tr}_{\mathbf{N}}(\widehat{\mathbb{j}}\widehat{\mathbb{n}}^{n-1}) + \mathcal{O}(N^{-2}). \tag{4.57}$$

The appropriate objects to match to the classical Casimir elements (4.43) are the linear combinations

$$\widehat{c}_{0n} = \frac{1}{2i^n}\left(\widehat{c}_n^{+} + (-1)^n\widehat{c}_n^{-}\right) = \frac{1}{N}\,\mathrm{Tr}_{\mathbf{N}}(\widehat{\mathbb{n}}^n) + \mathcal{O}(N^{-2}), \tag{4.58}$$

$$\widehat{c}_{1n} = \frac{N}{2ni^{n-1}}\left(\widehat{c}_n^{+} - (-1)^n\widehat{c}_n^{-}\right) = \frac{1}{N}\,\mathrm{Tr}_{\mathbf{N}}(\widehat{\mathbb{j}}\widehat{\mathbb{n}}^{n-1}) + \mathcal{O}(N^{-2}). \tag{4.59}$$

Just as in section 4.1, one can show that the deformed Casimirs $\widehat{c}_{0n}$, $\widehat{c}_{1n}$ approach their classical counterparts c_{0n}, c_{1n}, in the sense that their coefficients when expressed in the (X_α, Z_α) basis approach the classical coefficients. Once again, this is a consequence of the relation (4.23). Furthermore, the corresponding quantum operators $\widehat{C}_{0n}, \widehat{C}_{1n}$ obtained in the representation $\pi_{\mathcal{P}}$ corresponding to the quantization of the phase space are given by

$$\widehat{C}_{0n} = (i\hbar)^n\pi_{\mathcal{P}}(\widehat{c}_{0n}), \qquad \widehat{C}_{1n} = (i\hbar)^n\pi_{\mathcal{P}}(\widehat{c}_{1n}), \tag{4.60}$$

and these should be matching to the classical invariants C_{0n}, C_{1n} defined on the gravitational phase space. Since $\widehat{C}_{0n,1n}$ are proportional to the identity in an irreducible representation of $\widehat{\mathfrak{g}}$, they can be matched as c-numbers according to

$$C_{0n,1n} = \widehat{C}_{0n,1n} + \mathcal{O}(\hbar^2) + \mathcal{O}(N^{-2}). \tag{4.61}$$

Just as in section 4.2, this matching relation should determine the representation of $\widehat{\mathfrak{g}}$ as well as the value of the deformation parameter N. Carrying out the matching in detail would require an in-depth enumeration of the unitary irreducible representations of $\mathfrak{gl}(N, \mathbb{C})$, which is beyond the scope of the present work, but would nevertheless be a fruitful direction for future investigations. In carrying out this matching, the results of [81] are likely relevant.

4.4 $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ and $\mathfrak{su}(N, N)$

Finally, we consider the largest extended algebra $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, which was shown in section 3.4 to appear in the large N limit of the semisimple, finite-dimensional algebra $\mathfrak{su}(N, N)$. As in previous sections, we begin the analysis by describing the coadjoint orbits of the classical algebra $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, and use these to identify the Casimir elements. We then show that these Casimirs naturally match onto corresponding Casimirs of the deformed algebra, and this matching condition can once again be used to determine the representation appearing in the quantization of the classical phase space.

As before, we work with the trivially extended algebra $\mathfrak{g} = \mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S) \oplus \mathbb{R}$ for convenience, which is naturally associated with the large N limit of $\mathfrak{u}(N,N) = \mathfrak{su}(N,N) \oplus \mathbb{R}$. The Lie algebra is parameterized by a pair of functions on the sphere (ϕ, α^a), with ϕ scalar valued and α^a valued in $\mathfrak{sl}(2,\mathbb{R})$, with the index $a = 0,1,2$ denoting the components of the function in a basis. We will utilize the τ_a basis for $\mathfrak{sl}(2,\mathbb{R})$ given in equation (2.1) in which the structure constants are real. The dual lie algebra $\mathfrak{g}^*$ is similarly parameterized by a pair of functions (f, a_a), again with f scalar-valued and a_a $\mathfrak{sl}(2,\mathbb{R})$-valued, and the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is given by

$$\langle (f, a_b), (\phi, \alpha^a) \rangle = \int_S \nu_0 (f\phi + a_a \alpha^a). \tag{4.62}$$

Given the expression for the adjoint action of the Lie algebra on itself, $\mathrm{ad}_{(\phi,\alpha^a)}(\psi, \beta^b) = (\{\phi,\psi\}, \{\phi, \beta^b\} - \{\psi, \alpha^b\} + [\alpha,\beta]^b_{\mathfrak{sl}(2,\mathbb{R})})$, where $[\alpha,\beta]^c_{\mathfrak{sl}(2,\mathbb{R})} = \alpha^a \beta^b \varepsilon_{ab}{}^c$, the coadjoint action is given by

$$\mathrm{ad}^*_{(\phi,\alpha^a)}(f, a_b) = \left(\{\phi, f\} + \{\alpha^b, a_b\}, \{\phi, a_b\} + [\alpha, a]_b^{\mathfrak{sl}(2,\mathbb{R})} \right). \tag{4.63}$$

Note that this coadjoint action for $\mathfrak{g}$ is closely related to the action for the larger symmetry group $\mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$ examined in [26], upon replacing Lie derivatives with Poisson brackets. The action (4.63) indicates that f and a_b transform as scalars under diffeomorphisms of the sphere, and a_b transforms in the adjoint representation under $\mathfrak{sl}(2,\mathbb{R})$ transformations. However, f transforms inhomogeneously under $\mathfrak{sl}(2,\mathbb{R})$ transformations, and this is the main challenge to deal with when looking for invariant functions under the coadjoint action.

Rather than working with $\mathfrak{g}$-valued functions $(\mathbb{j}, \mathbb{n}_a)$ to construct Casimir elements as in previous sections, here it will be more convenient to look directly for invariant functions on the orbits, after which expressions for the Casimir elements can be determined. A first set of invariants is readily obtained by noting that the $\mathfrak{sl}(2,\mathbb{R})$ quadratic Casimir $a^2 = a_b a^b$ transforms as a scalar function on the sphere, and hence the moments of this function will be fully invariant under $\mathfrak{sdiff}(S)$ and $\mathfrak{sl}(2,\mathbb{R})$ transformations. This leads to the first set of Casimir functions

$$c_{2n}[(f, a_b)] = \int_S \nu_0 \left(a^2\right)^n. \tag{4.64}$$

These invariants are the analogs of the c_{0n} Casimirs of the algebra $\mathfrak{c}_{\mathbb{R}}(S)$ defined in (4.43), since both are independent of f. Note that these Casimirs have no analog in the larger algebra $\mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$ examined in [26], since in that case, there is no fixed volume form ν_0, and hence the only natural volume form on the sphere comes from the $\mathfrak{sl}(2,\mathbb{R})$ quadratic Casimir itself. Because of this, there is no meaningful way to construct moments of the quadratic Casimir when working with the larger algebra, since in that case it transforms as a density as opposed to a scalar.

On the other hand, we should expect a second set of Casimirs that are the analogs of the c_{1n} Casimirs of $\mathfrak{c}_{\mathbb{R}}(S)$ in equation (4.43). Additionally, the construction of Casimir functions for the larger symmetry algebra $\mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$ in [26] lead to a set of generalized enstrophies constructed from moments of a scalar vorticity w which contains a cubic term in the $\mathfrak{sl}(2,\mathbb{R})$ generators. Since the vorticity arose naturally as an object constructed from f that is invariant under $\mathfrak{sl}(2,\mathbb{R})$ transformations, the expectation is that a similar object should arise in the classification of invariants of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$. By examining to what extent such a vorticity can be defined from the $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ orbit data, we will obtain a prescription for constructing the second set of Casimirs for this algebra.

In the larger algebra, the vorticity is constructed from the orbit data, which consists of a densitized 1-form $\widetilde{p}_A$ and an $\mathfrak{sl}(2,\mathbb{R})$-valued density $\widetilde{a}_b$. The $\mathfrak{sl}(2,\mathbb{R})$ quadratic Casimir constructed from $\widetilde{a}_a$ determines a dynamical volume form ν, which is related to the fixed volume form ν_0 by the relation

$$\nu = \nu_0\sqrt{a^2}, \tag{4.65}$$

where a_b is the $\mathfrak{sl}(2,\mathbb{R})$-valued scalar appearing in the $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ orbit data. We assume throughout this section that $a^2 > 0$, which defines the positive area orbits, as are relevant for gravitational applications. This volume form then allows us to construct a de-densitized one-form p_A satisfying $|\nu|p_A = \widetilde{p}_A$. Note that when specializing to $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ orbits, $\widetilde{p}_A$ is related to the associated scalar stream function f by a similar relation as in equation (2.19), which, taking into account the relation (4.65) between the fixed and dynamical volume forms, is given by

$$f = -\epsilon^{BA}\partial_B(|a|p_A), \tag{4.66}$$

where $|a| \equiv \sqrt{a^2}$. Defining $p^0_A = |a|p_A$, this relation equivalently can be expressed as $*_\epsilon f = -dp^0$. Similarly, we can define a de-densitized $\mathfrak{sl}(2,\mathbb{R})$ function by the equation $\widetilde{a}_b = \widehat{a}_b|\nu|$, which is related to the $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ orbit data by $a_b = |a|\widehat{a}_b$.

With all this in hand, we can examine the expression for the vorticity in terms of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ orbit data. Using the results of section 4.2 of [26], the vorticity 2-form $\bar{w}$ is given by

$$\begin{aligned}\bar{w} &= dp - \frac{1}{2}\varepsilon_{abc}\widehat{a}^a d\widehat{a}^b \wedge d\widehat{a}^c \\ &= -\frac{*_\epsilon f}{|a|} - \frac{1}{2}\varepsilon_{abc}\frac{a^a}{|a|}d\left(\frac{a^b}{|a|}\right)\wedge d\left(\frac{a^c}{|a|}\right) - \frac{d|a|}{|a|^2}\wedge p^0.\end{aligned} \tag{4.67}$$

While the first two terms in this expression are well-defined functions of the $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ orbit data (f, a_b), the third term is not, since it depends explicitly on the one-form p^0. Although p^0 is related to f by the equation $*_\epsilon f = -dp^0$, this expression only determines p^0 up to shifts by exact forms, $p^0 \to p^0 + dA$. This means that under an $\mathfrak{sdiff}(S)$ transformation generated by $\xi^A = \epsilon^{BA}\nabla_B\phi$, p^0 will transform anomalously as $\delta_\xi p^0 = \mathcal{L}_\xi p^0 + dA_\xi$, where A_ξ is a scalar function depending on the precise procedure employed to construct a unique p^0 from a given f.[26] This similarly implies an anomalous transformation of $\bar{w}$ under $\mathfrak{sdiff}(S)$ transformations:

$$\delta_\xi\bar{w} = \mathcal{L}_\xi\bar{w} - \frac{d|a|}{|a|^2}\wedge dA_\xi. \tag{4.68}$$

[26] An example of such a procedure is to select a fixed metric on the sphere, and impose that $d\star p^0 = 0$, where $\star$ is the natural dualization associated with this metric. Such a condition fixes the shift ambiguity in p^0, but introduces dependence on the fixed background metric.

Nevertheless, $\bar{w}$ retains the important property of being invariant under $\mathfrak{sl}(2,\mathbb{R})^S$ transformations

Because of the anomalous transformation property (4.68), arbitrary moments of the vorticity scalar $*_\epsilon \bar{w}$ will not yield invariant functions on the $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ coadjoint orbits. However, a set of invariants analogous to the c_{1n} Casimirs for $\mathfrak{c}_{\mathbb{R}}(S)$ described in (4.43) can be obtained when integrating a single factor of $\bar{w}$ against a function of $|a|$. Under $\mathfrak{sdiff}(S)$ transformations, we have that

$$\begin{aligned}\delta_\xi\Big(\bar{w}|a|^2F'(|a|)\Big) &= \mathcal{L}_\xi\Big(\bar{w}|a|^2F'(|a|)\Big) + F'(|a|)d|a| \wedge dA_\xi \\ &= \mathcal{L}_\xi\Big(\bar{w}|a|^2F'(|a|)\Big) + d\Big(F(|a|)dA_\xi\Big).\end{aligned} \tag{4.69}$$

Since this is an exact form, integrating $\bar{w}|a|^2F'(|a|)$ over the sphere will yield an invariant for the orbit:

$$c_F[(f,a)] = \int_S \bar{w}|a|^2F'(|a|). \tag{4.70}$$

Applying the definition (4.67) and using that $*_\epsilon f = 4\pi\nu_0 f$, $da^b \wedge da^c = 4\pi\nu_0\{a^b,a^c\}_\epsilon$, and $dp^0 = - *_\epsilon f$, this can be reexpressed as

$$c_F[(f,a_b)] = -4\pi\int_S \nu_0\left(f\left[|a|F'(|a|) + F(|a|)\right] + \frac{F'(|a|)}{|a|}\varepsilon_{abc}a^a\{a^b,a^c\}\right), \tag{4.71}$$

which is now manifestly a function of the orbit data (f,a_b). These can be expressed as a set of polynomial invariants by choosing $F(|a|) = \frac{-1}{4\pi}|a|^{2n}$, producing

$$c_{2n+1}[(f,a_b)] = \int_S \nu_0\Big((2n+1)fa^2 + n\varepsilon_{abc}a^a\{a^b,a^c\}\Big)(a^2)^{n-1}, \tag{4.72}$$

which are the desired analogs of the c_{1n} invariants from (4.43) for the $\mathfrak{c}_{\mathbb{R}}(S)$ algebra. It is possible to check directly that this expression is invariant under the coadjoint action. In the appendix E it is shown that, if we call the integrand of (4.72) w_{2n+1}, we obtained that the coadjoint action gives

$$ad^*_{(\phi,\alpha^a)}w_{2n+1} = \{\phi, x_{2n+1}\} + \{\alpha^a, a_a a^{2n}\}, \tag{4.73}$$

which implies the invariance of its sphere integral (4.72).

A somewhat strange feature is that only the even n values of $c_{0,n}$ from the $\mathfrak{c}_{\mathbb{R}}(S)$ algebra match onto the Casimirs c_{2n} for $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$, and similarly only the odd n values of the $c_{1,n}$ Casimirs match onto the c_{2n+1} Casimirs of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$. This discrepancy occurs due to the requirement that only integer powers of a^2 appear in (4.64) and (4.72), as is necessary to obtain Casimirs that are polynomial in the generators. An additional set of non-polynomial Casimirs for $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ can be obtained by allowing odd powers of $|a| = \sqrt{a_a a^a}$ to appear in these expressions, and they would give analogs of the remaining $\mathfrak{c}_{\mathbb{R}}(S)$ Casimirs $c_{0,n}$ with n odd and $c_{1,n}$ with n even. Such square roots are relevant in the discussion of the area operator for the deformed algebra at the end of this section.

The Casimir functions $c_{2n}[(f,a_b)], c_{2n+1}[(f,a_b)]$ on the coadjoint orbits arise from Casimir elements of the algebra $\mathfrak{g}$. These elements can be obtained from the functional expressions by constructing the $\mathfrak{g}$-valued functions on S $(\mathbb{j}, \mathbb{n}_b)$, normalized such that

$$\langle (f,a_b), \mathbb{j}\rangle = f, \tag{4.74}$$

$$\langle (f,a_b), \mathbb{n}_a\rangle = a_a, \tag{4.75}$$

where, as before, the left-hand side evaluates the pairing between (f,a_b) and the $\mathfrak{g}$ factors of $\mathbb{j}$ and $\mathbb{n}_a$, and the right-hand side returns the functions on the sphere associated with f and a_a. The Casimir elements of $\mathfrak{g}$ are then obtained by replacing f with $\mathbb{j}$ and a_b with $\mathbb{n}_b$ in the expressions (4.64) and (4.72), giving[27]

$$c_{2n} = \int_S \nu_0\, (\mathbb{n}_a\mathbb{n}^a)^n, \tag{4.76}$$

$$c_{2n+1} = \int_S \nu_0 \left((2n+1)\mathbb{j}\mathbb{n}_a\mathbb{n}^a + n\varepsilon_{abc}\mathbb{n}^a\{\mathbb{n}^b,\mathbb{n}^c\}\right)\left(\mathbb{n}_d\mathbb{n}^d\right)^{n-1}. \tag{4.77}$$

With the expressions for the classical Casimirs in hand, we can now turn to matching these to the large N limit of the Casimirs of the deformed algebra $\widehat{\mathfrak{g}} = \mathfrak{u}(N,N)$. To obtain convenient expressions for the deformed Casimirs, we begin by constructing the objects $(\widehat{\mathbb{j}}, \widehat{\mathbb{n}}_a)$ which are elements of $\mathrm{Mat}_{N\times N}\otimes\widehat{\mathfrak{g}}$, normalized according to

$$\frac{1}{N}\mathrm{Tr}_{\mathbf{N}}\left(\widehat{\mathbb{j}}\cdot\widehat{Y}_\alpha\right) = X_\alpha, \tag{4.78}$$

$$\frac{1}{N}\mathrm{Tr}_{\mathbf{N}}\left(\widehat{\mathbb{n}}_a\cdot\widehat{Y}_\alpha\right) = Z_{a\alpha}, \tag{4.79}$$

where $(X_\alpha, Z_{a\alpha})$ are the basis of $\widehat{\mathfrak{g}}$ introduced in section 3.4. These objects can be assembled into a single $2N\times 2N$ matrix valued in $\widehat{\mathfrak{g}}$ by tensoring with the 2×2 matrices $(\mathbb{1}_2, \rho_a)$. The resulting object $\widehat{\mathbb{h}}$ given by

$$\widehat{\mathbb{h}} = \left(\frac{1}{N}\mathbb{1}_2\otimes\widehat{\mathbb{j}} - 2\rho^a\otimes\widehat{\mathbb{n}}_a\right) = \begin{pmatrix} \frac{1}{N}\widehat{\mathbb{j}} + \widehat{\mathbb{n}}_0 & -\widehat{\mathbb{n}}_1 + i\widehat{\mathbb{n}}_2 \\ \widehat{\mathbb{n}}_1 + i\widehat{\mathbb{n}}_2 & \frac{1}{N}\widehat{\mathbb{j}} - \widehat{\mathbb{n}}_0 \end{pmatrix}, \tag{4.80}$$

can then be shown to satisfy the key relations

$$\left[X_\alpha, \widehat{\mathbb{h}}\right]_{\widehat{\mathfrak{g}}} = -\frac{N}{2i}\left[\mathbb{1}_2\otimes\widehat{Y}_\alpha, \widehat{\mathbb{h}}\right], \tag{4.81}$$

$$\left[Z_{a\alpha}, \widehat{\mathbb{h}}\right]_{\widehat{\mathfrak{g}}} = i\left[\rho_a\otimes\widehat{Y}_\alpha, \widehat{\mathbb{h}}\right], \tag{4.82}$$

[27]There are subtleties related to the ordering of the Lie algebra elements in these expressions, due to the fact, derived in appendix E, that $\mathbb{n}_a(\sigma)\mathbb{n}_b(\sigma') = \mathbb{n}_b(\sigma')\mathbb{n}_a(\sigma) + \delta(\sigma-\sigma')\varepsilon_{ab}{}^c\mathbb{n}_c(\sigma')$, and hence, for example, the quantities $\mathbb{n}_a\mathbb{n}^a\mathbb{n}_b\mathbb{n}^b$ and $\mathbb{n}_a\mathbb{n}_b\mathbb{n}^a\mathbb{n}^b$ differ by divergent coefficients. However, any choice of ordering for the Lie algebra elements define the same Casimir function on the coadjoint orbits, and furthermore any choice of ordering for c_{2n} and c_{2n+1} yields objects in the center of the universal enveloping algebra. We will not worry too much about this ordering for the remainder of this section since the Casimir functions on the orbits are the important quantities to work with to determine the representation for the quantization of the phase space. However, as we will see, it is interesting that the large N limit of the $\mathfrak{u}(N,N)$ Casimirs picks out a preferred ordering, and it would be interesting to understand how this preferred ordering could be obtained directly from the classical algebra.

where the brackets on the right-hand side denote a matrix commutator with the $\mathrm{Mat}_{2N\times 2N}$ factor of $\widehat{\mathbb{h}}$. Note that $\widehat{\mathbb{h}}$ is closely related to the quantity $\widehat{H}$ defined in equation (3.74), since the latter is an operator-valued matrix acting in a representation $\pi_{\mathcal{P}}$ of $\mathfrak{u}(N,N)$ coinciding with a quantization of the phase space. The exact relation between the two quantities is $\widehat{H} = i\hbar\pi_{\mathcal{P}}(\widehat{\mathbb{h}})$. The relative coefficients of the $\widehat{\mathbb{j}}$ and $\widehat{\mathbb{n}}$ terms in (4.80) are chosen to ensure the relation (4.82) holds.

Since the action of any Lie algebra element on $\widehat{\mathbb{h}}$ can be expressed as a matrix commutator acting on $\widehat{\mathbb{h}}$, the Casimir elements can be formed as in previous sections by taking powers $\widehat{\mathbb{h}}^n$, where the product refers to the matrix product on the $\mathrm{Mat}_{2N\times 2N}$ factor, thus producing an element of $\mathrm{Mat}_{2N\times 2N}\otimes\widehat{\mathfrak{g}}^{\otimes n}$. It is then straightforward to verify that

$$\widehat{c}_n = \frac{1}{2N}\,\mathrm{Tr}_{\mathbf{2N}}\,\widehat{\mathbb{h}}^n, \tag{4.83}$$

commute with the action of $\widehat{\mathfrak{g}}$, and hence define the Casimir elements of the deformed algebra.

In order to match these Casimirs to those of the classical algebra, we make use of the following relations at large N (see appendix E):

$$[\widehat{\mathbb{j}},\widehat{\mathbb{n}}_a] \equiv \widehat{\mathbb{j}}\widehat{\mathbb{n}}_a - \widehat{\mathbb{n}}_a\widehat{\mathbb{j}} = \mathcal{O}(N^{-1}), \tag{4.84}$$

$$[\widehat{\mathbb{n}}_a,\widehat{\mathbb{n}}_b] \equiv \widehat{\mathbb{n}}_a\widehat{\mathbb{n}}_b - \widehat{\mathbb{n}}_b\widehat{\mathbb{n}}_a = \frac{2i}{N}\widehat{\{\mathbb{n}_a,\mathbb{n}_b\}} + [\widehat{\mathbb{n}}_a,\widehat{\mathbb{n}}_b]_{\widehat{\mathfrak{g}}} + \mathcal{O}(N^{-3}). \tag{4.85}$$

We can then expand (4.83) in terms of $\widehat{\mathbb{j}}$ and $\widehat{\mathbb{n}}_a$ using (4.80). Beginning with the even case, all terms involving $\widehat{\mathbb{j}}$ are suppressed by $\frac{1}{N}$, hence to leading order we have that

$$\begin{aligned}\widehat{c}_{2n} &= \frac{1}{2N}\,\mathrm{Tr}_{\mathbf{2N}}\left[(-2\rho^{a_1}\otimes\widehat{\mathbb{n}}_{a_1})(-2\rho^{b_1}\otimes\widehat{\mathbb{n}}_{b_1})(-2\rho^{a_2}\otimes\widehat{\mathbb{n}}_{a_2})\dots(-2\rho^{b_n}\otimes\widehat{\mathbb{n}}_{b_n})\right]\\ &= \frac{1}{2N}(-2)^{2n}\,\mathrm{Tr}_{\mathbf{2}}\left(\rho^{a_1}\rho^{b_1}\dots\rho^{a_n}\rho^{b_n}\right)\mathrm{Tr}_{\mathbf{N}}\left(\widehat{\mathbb{n}}_{a_1}\widehat{\mathbb{n}}_{b_1}\dots\widehat{\mathbb{n}}_{a_n}\widehat{\mathbb{n}}_{b_n}\right).\end{aligned} \tag{4.86}$$

In the trace over the ρ^a matrices, we can expand the products pairwise using (3.47) to get $\rho^{a_i}\rho^{b_i} = -\frac{1}{4}\eta^{a_ib_i}\mathbb{1}_2 + \frac{i}{2}\varepsilon^{a_ib_ic_i}\rho_{c_i}$. Each term involving $\varepsilon^{a_ib_ic_i}$ produces a commutator in $\widehat{\mathbb{n}}_{a_i}\widehat{\mathbb{n}}_{b_i}$ appearing in the second trace in (4.86). We can then apply (4.85) to find that this commutator can be replaced with the $\widehat{\mathfrak{g}}$ Lie bracket, up to subleading terms in $\frac{1}{N}$. While these Lie brackets survive in the large N limit, they simply produce elements of the universal enveloping algebra with different orderings of the Lie algebra elements, all of which map to the same classical function on the coadjoint orbits. Hence, we can drop these terms when matching to the classical Casimir (4.76). Thus, keeping only the term proportional to the identity in each pairwise product of the ρ^a matrices, (4.86) evaluates to

$$\widehat{c}_{2n} = (-1)^n\eta^{a_1b_1}\dots\eta^{a_nb_n}\frac{1}{N}\,\mathrm{Tr}_{\mathbf{N}}\left(\widehat{\mathbb{n}}_{a_1}\dots\widehat{\mathbb{n}}_{a_n}\right) + \mathcal{O}(N^{-1}). \tag{4.87}$$

Demonstrating that this Casimir matches the classical expression (4.76) follows immediately from the same argument as in section 4.1, by expressing each Casimir in the Lie algebra basis, and applying the large N relation (4.23). This results in

$$\widehat{c}_{2n} \to (-1)^n c_{2n} + \mathcal{O}(N^{-1}). \tag{4.88}$$

For the odd Casimirs, the leading piece in the contribution involving only $\widehat{\mathbb{n}}_a$ terms will be suppressed in the large N limit, and hence we need to keep the first-order terms in the $\frac{1}{N}$ expansion. The leading behavior at large N is then given by

$$\begin{aligned}\widehat{c}_{2n+1} = \frac{1}{2N}\Big[&(-2)^{2n+1}\,\mathrm{Tr}_{\mathbf{2}}\left(\rho^c\rho^{a_1}\rho^{b_1}\dots\rho^{a_n}\rho^{b_n}\right)\mathrm{Tr}_{\mathbf{N}}\left(\widehat{\mathbb{n}}_c\widehat{\mathbb{n}}_{a_1}\widehat{\mathbb{n}}_{b_1}\dots\widehat{\mathbb{n}}_{a_n}\widehat{\mathbb{n}}_{b_n}\right)\\ &(-2)^{2n}\,\mathrm{Tr}_{\mathbf{2}}\left(\rho^{a_1}\rho^{b_1}\dots\rho^{a_n}\rho^{b_n}\right)\mathrm{Tr}_{\mathbf{N}}\left(\frac{(2n+1)}{N}\,\widehat{\mathbb{j}}\,\widehat{\mathbb{n}}_{a_1}\widehat{\mathbb{n}}_{b_1}\dots\widehat{\mathbb{n}}_{a_n}\widehat{\mathbb{n}}_{b_n}\right)\Big],\end{aligned} \tag{4.89}$$

where we have applied (4.84) in the second term to move $\widehat{\mathbb{j}}$ to the left in each term in which it appears. In the first line of (4.89), we again expand out the products of ρ^a matrices in pairs. The terms involving only identity matrices in this product multiply with a term proportional to $\mathrm{Tr}_{\mathbf{2}}\,\rho^c$, which vanishes. Hence we need to keep all terms with one factor of $\varepsilon^{a_ib_ic_i}$, since, as before, each such term will produce a factor of $\frac{1}{N}$ due to commutations of the $\widehat{\mathbb{n}}_a$ in the second trace. Applying equation (4.85) and dropping terms involving $\widehat{\mathfrak{g}}$ Lie brackets, we find for the first line

$$(-1)^n\frac{n}{N}\frac{1}{N}\,\mathrm{Tr}_{\mathbf{N}}\left(\varepsilon^{cab}\,\widehat{\mathbb{n}}_c\widehat{\{\mathbb{n}_a,\mathbb{n}_b\}}\widehat{\mathbb{n}}_{a_2}\widehat{\mathbb{n}}^{a_2}\dots\widehat{\mathbb{n}}_{a_n}\widehat{\mathbb{n}}^{a_n}\right). \tag{4.90}$$

For the second line of (4.89), we can simply keep all terms proportional to the identity in each pairwise ρ^a product. This term then evaluates to

$$(-1)^n\frac{(2n+1)}{N}\frac{1}{N}\,\mathrm{Tr}_{\mathbf{N}}\left(\widehat{\mathbb{j}}\,\widehat{\mathbb{n}}_{a_1}\widehat{\mathbb{n}}^{a_1}\dots\widehat{\mathbb{n}}_{a_n}\widehat{\mathbb{n}}^{a_n}\right). \tag{4.91}$$

Combining these terms and again applying the large N relationship (4.23), we find that $\widehat{c}_{2n+1}$ approaches the classical Casimir after rescaling by N,

$$\widehat{c}_{2n+1} \to \frac{(-1)^n}{N}c_{2n+1} + \mathcal{O}(N^{-2}). \tag{4.92}$$

We can translate these correspondences to matching conditions for the Casimir operators in the representation $\pi_{\mathcal{P}}$ by recalling that each Lie algebra element is rescaled by $i\hbar$ in the representation. Hence we can define the operators

$$\begin{aligned}\widehat{C}_{2n} &= (-1)^n(i\hbar)^{2n}\pi_{\mathcal{P}}(\widehat{c}_{2n}),\\ \widehat{C}_{2n+1} &= N(-1)^n(i\hbar)^{2n+1}\pi_{\mathcal{P}}(\widehat{c}_{2n+1}),\end{aligned} \tag{4.93}$$

which can be matched to the classical invariant functions on the gravitational phase space, obtained by pulling back the orbit invariants (4.64) and (4.72) via the moment map. This matching should again determine the deformation parameter N as well as the representation of $\mathfrak{su}(N,N)$ corresponding to a quantization of the phase space. Carrying out this matching in detail requires a thorough investigation into the unitary representations of $\mathfrak{su}(N,N)$, which we leave for future work.

It is interesting to examine in more detail the quadratic Casimir, whose full expression is

$$\widehat{c}_2 = -\frac{1}{N}\,\mathrm{Tr}_{\mathbf{N}}(\widehat{\mathbb{n}}_a\widehat{\mathbb{n}}^a) + \frac{1}{N^2}\frac{1}{N}\,\mathrm{Tr}_{\mathbf{N}}(\widehat{\mathbb{j}}^2). \tag{4.94}$$

The first term is the piece that survives in the large N limit, and is related to the area of the surface embedded in spacetime. The area operator can be defined by

$$\widehat{A} = \frac{1}{N}\operatorname{Tr}_{\mathbf{N}}\sqrt{\widehat{\mathfrak{n}}_a\widehat{\mathfrak{n}}^a}, \tag{4.95}$$

where the square root should be interpreted an object that yields the square root of the operator $\pi_{\mathcal{P}}(\widehat{\mathfrak{n}}_a\widehat{\mathfrak{n}}^a)$ in a representation of the algebra. Here we find that at infinite N, the area operator is a Casimir of the continuum group, as first demonstrated in [1]. However, at finite N, this operator is not a Casimir, and instead is a hyperbolic element of $\mathfrak{su}(N,N)$, up to higher order corrections in the universal enveloping algebra. This suggests that the area operator becomes noncommutative after including finite N corrections to the algebra. This result is reminiscent of the recent investigations into large N algebras in holography [82–85], where, in particular, the failure of the area operator to be central upon including $\frac{1}{N}$ corrections leads to a deformation of the associated von Neumann algebras from type III to type II. It would be interesting to further explore the connection between the noncommutativity of the area operator in the present context and the appearance of deformed von Neumann algebras in holography.

5 Conclusion and future work

In this work, we have undertaken the first steps of studying, at a quantum level, the symmetries of a finite region of space identified in ref. [1]. Inspired by the fact, shown in ref. [26] that the Lie algebra of the Wigner little group is

$$\mathfrak{sdiff}(S) = \lim_{N\to\infty}\mathfrak{su}(N), \tag{5.1}$$

we have looked for a deformation of the corner symmetry algebra which would generalize (5.1). In extending the symmetry to include boost transformations of the normal plane, we arrived at two generalizations of the matrix regularization (5.1) to noncompact groups:

$$\mathfrak{sdiff}(S)\oplus_{\mathcal{L}}\mathbb{R}^S = \lim_{N\to\infty}\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}, \tag{5.2}$$

$$\mathfrak{sdiff}(S)\oplus_{\mathcal{L}}\mathfrak{sl}(2,\mathbb{R})^S = \lim_{N\to\infty}\mathfrak{su}(N,N). \tag{5.3}$$

These deformations nontrivially combine the diffeomorphisms of the sphere with normal boosts such that in the large-N limit the semidirect sum structure is recovered. While we have established identities (5.2), (5.3) at the level of the structure constants, in section 4 we extended this analysis to Casimir invariants of the groups, showing that the large-N limits of the well-known $\mathfrak{sl}(N,\mathbb{C})$ and $\mathfrak{su}(N,N)$ Casimirs yield the complete set of invariants of identified for the corresponding infinite-dimensional Lie algebras. The Casimirs allow us to determine the representation of the symmetry group in terms of physical properties of the surface S, and in particular, allow us to argue for a particular scaling of the deformation parameter N. The Casimirs also give a set of commuting operators at the quantum level. Interestingly, the area operator, which was shown in ref. [26] to play a special role in the classification of orbits, is not among the Casimirs but becomes noncentral at finite N. This fact remains puzzling but may have implications for black hole entropy for which the area plays a crucial role.

Our work opens up many potential avenues for future works, so we spend the majority of this section identifying the most interesting future directions.

5.1 Detailed Casimir matching

In section 4.2, we outlined the Casimir matching procedure for the case of $\mathfrak{su}(N)$. There we found that the matching conditions in an irreducible representation determine how the deformation parameter N scales with $\frac{A}{4G\hbar}$, as displayed in equation (4.33). It would be quite interesting to carry out this matching in more detail to not only determine the value of N but to also identify the representation that should be employed in the quantization of the phase space. In the large N limit, we should expect to find a relation between the shape of the Young diagram for the representation and the function W on the sphere, or the associated measured Reeb graph derived from W, which, as explained in [26, 76], is an additional invariant of the continuum algebra $\mathfrak{sdiff}(S)$. Given the large amount of literature related to the large N limits of representations of $\mathfrak{su}(N)$ (see, e.g. [77]), it seems likely that this more detailed matching would be achievable.

The scaling $\frac{A}{4G\hbar} \sim N^3$ identified in equation (4.33) is somewhat odd from the perspective of AdS holography or matrix models, in which it is more common for the entropy to scale like N^2_{AdS}. This suggests that there might be an issue with trying to only quantize the $\mathfrak{sdiff}(S)$ subalgebra in the process of attempting to obtain an understanding of the entropy of the surface S. Instead, it seems likely that one would need to work with one of the enlarged algebras $\mathfrak{c}_{\mathbb{R}}(S)$ or $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ to obtain a sensible relation for the entropy from the Casimir matching procedure.

This motivates further investigating the large N representation theory of the deformed algebras $\mathfrak{sl}(N,\mathbb{C}) \oplus \mathbb{R}$ and $\mathfrak{su}(N,N)$. Unfortunately, the literature on the unitary representations of these groups is somewhat sparse. The unitary representations of $\mathfrak{gl}(N,\mathbb{C})$ were classified in [81], and some results on $\mathfrak{su}(N,N)$ are given in [86]. A standard reference on the general theory of representations of semisimple groups is [87]. It would be interesting to investigate this representation theory in more detail, and to identify which representations occur in the large N limit when matching to the continuum algebras $\mathfrak{c}_{\mathbb{R}}(S)$ and $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$. Ultimately, one would hope to be able to identify the analog of equation (4.33) for these groups, which may yield the expected relation between N^2 and the entropy $\frac{A}{4G\hbar}$.

Alternatively, it may be that the N^2 scaling is not appropriate for the localized gravitational subregions considered here, and the N appearing in our algebra deformation is a priori a different entity that the N_{AdS} appearing in holography. In our context, N appears as a deformation parameter for the corner symmetry algebra and corresponds to a measure of the corner surface area in Planck units. On the other hand N_{AdS} appearing in holography as a label for the boundary gauge group is related to the ratio of the cosmological scale and the Planck scale through the relation $N^2_{\text{AdS}} \sim (\hbar G\Lambda)^{-1}$ [88, 89] in four spacetime dimensions (although, see footnote 24 for situations with different parametric dependence of the entropy on N). It is natural to expect some functional relation between the corner N and the holographic N_{AdS}. The exact nature of this relationship is not established at this stage.

5.2 Computation of characters

We have introduced large-N limits of the groups $\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}$ and $\mathfrak{su}(N,N)$ and shown the continuum limit of the structure constants as well as the Casimir invariants. These invariants allow us to establish a correspondence not only between the finite and infinite-dimensional Lie groups, but also their representations. It would be interesting to see how much of the finite-dimensional representation theory can be carried over to the large-N limit. In particular, it would be interesting to compare characters of the finite-dimensional Lie group representations to those of their continuum counterparts.

Group characters are especially important in physics because they are essentially quantum-mechanical partition functions, encoding the number of states in each irreducible representation as a function of the physical values of the generators. In the gravitational context, the most important character is that of the global boost, which, in the deformed algebras, coincides with the $\mathbb{R}$ factor of $\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}$, or a generic hyperbolic generator in $\mathfrak{su}(N,N)$. This operator plays an important role in both black hole thermodynamics, where it defines the time-translation symmetry associated with Killing horizons, and in quantum field theory where it defines the modular Hamiltonian of a quantum field theory restricted to a half-space or conformal field theory restricted to a sphere. The boost character is therefore essential in relating the entropy of horizons — which is controlled by the density of states — to the value of the charges, which are determined by the horizon geometry.

An important first step would be to understand the relation between character formulas for $\mathfrak{su}(N)$ and $\mathfrak{sdiff}(S)$ in the limit of large N. On the $\mathfrak{su}(N)$ side, the large-N limit of the characters can be obtained from the Itzykson-Zuber integral formula [90]. This large-N limit was studied in [91] which expressed the leading asymptotics of the character in terms of the complex inviscid Burgers equation (or the Hopf equation) [92, 93] whose solutions have been studied in ref. [94]. The characters have the leading-order behavior $\exp(N^2F_0+F_1+\dots)$ where F_0 is an on-shell action and F_1 the first subleading correction in the $1/N^2$ expansion. Independently, certain characters of $\mathfrak{sdiff}(S)$ have been calculated using the Atiyah-Bott localization formula, and take the form of divergent sums [95, equation (3.19)]. It would therefore be interesting to understand whether these $\mathfrak{sdiff}(S)$ characters can be obtained as appropriate limits of the $\mathfrak{su}(N)$ characters. Since the leading term of the $\mathfrak{su}(N)$ character diverges at large N, it cannot be calculated within $\mathfrak{sdiff}(S)$ — rather, we expect it to appear as a divergence that must be renormalized away. Having subtracted this leading divergence one expects to find agreement between the renormalized $\mathfrak{sdiff}(S)$ characters and $1/N$ corrections to the $\mathfrak{su}(N)$ characters: the latter would appear as corrections to the leading-order result of ref. [91].

An important next question is whether the large-N calculation of characters can be extended to large-N limits of $\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}$ and $\mathfrak{su}(N,N)$ and related to character formulas for $\mathfrak{c}_{\mathbb{R}}(S)$ and $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ respectively. Such characters can in principle be computed from the analog of Kirillov's character formula for reductive groups [96], and we expect similar divergent behavior of characters seen for $\mathfrak{su}(N)$ to hold for $\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}$ and $\mathfrak{su}(N,N)$.

In the case of the noncompact groups $\mathfrak{sl}(N,\mathbb{C})\oplus\mathbb{R}$ and $\mathfrak{su}(N,N)$ the calculation of characters plays a further important role. Since unitary representations of noncompact groups are infinite-dimensional, the direct analog of the formula (1.4) for Yang-Mills theory cannot apply. Instead, we expect the global boost K to have a nonzero expectation value which leads to an insertion of $\exp(-2\pi K)$ in the partition function. This suggests it is characters of the global boost K (or suitable analytic continuations thereof), and not dimensions, which are the relevant quantities for counting states in representations of $\mathfrak{sl}(N,\mathbb{C}) \oplus \mathbb{R}$ and $\mathfrak{su}(N,N)$.

The chief physical application of such characters is in understanding the entropy of a region of space bounded by the corner S. The characters give a way to organize the computation of entropy, see [40] for a concrete example. They would in principle give a way of calculating the entanglement spectrum in terms of geometric properties of the surface S, which would be an intriguing application of the formalism developed in ref. [1] and further explored in ref. [26] and this work.

5.3 Topological aspects of large-N limit

In section 4.1, we obtained a correspondence between the Casimirs of $\mathfrak{su}(N)$ and an associated set of Casimirs for the continuum algebra $\mathfrak{sdiff}(S)$, which coincide with generalized enstrophies of incompressible hydrodynamics. However, a complete classification of the invariants of $\mathfrak{sdiff}(S)$ involves additional topological information contained in the measured Reeb graph of the function W on the sphere — see ref. [76] for a proof and ref. [26] for discussion in the context of the corner symmetry algebra. Each coadjoint orbit of $\mathfrak{sdiff}(S)$ is labeled by a function W on S^2, and the Reeb graph encodes the topology of the level sets of W. This raises the question of how this topological data arises from the large N limit of $\mathfrak{su}(N)$. Since the invariants $\widehat{c}_k$, $k = 2, \ldots, N$ comprise a complete set of Casimirs for $\mathfrak{su}(N)$, there appears to be no topological data present at finite N. Instead, the topology is contained in the way the limit $N \to \infty$ is taken. In this limit, the topology of the surface S restricts the allowed representations of $\mathfrak{su}(N)$ that have good infinite N limits, and different topologies should single out different representations. In order to make this connection more precise, one would like to obtain the Reeb graph from some property of the large N limit, such as the shape of the Young diagrams for the allowed representations. At finite N, the object corresponding to the function W is a hermitian matrix $\widehat{W}$, and a natural way to approach the large-N limit is to study the trace of the resolvent, $\mathrm{Tr}[(\lambda I - \widehat{W})^{-1}]$ as $N \to \infty$. In this limit the trace of the resolvent develops a branch cut, and the discontinuity across the cut encodes the spectral density of $\widehat{W}$ and hence all of the Casimirs. A natural conjecture is that the topology of this branch cut is related to the topology of the Reeb graph. An intriguing possibility arises from the observation that trivalent vertices in the Reeb graph are associated with logarithmic singularities in the eigenvalue density of $\widehat{W}$. It is then tempting to conjecture that the Reeb graph data is encoded in the branching structure of the resolvent as $N \to \infty$.

A related topological consideration comes from the interpretation of the finite N algebra as a sum over all possible topologies of the surface S [49, 97]. This is related to the fact that $\mathfrak{su}(N)$ can reproduce the group of area-preserving diffeomorphisms of any Riemann surface as $N \to \infty$, depending on how the limit is taken. For example, we could instead work with torus harmonics Y_{mn} as opposed to spherical harmonics, and these admit a finite N deformation to fuzzy torus harmonics $\widehat{Y}_{mn}$ which satisfy an $\mathfrak{su}(N)$ algebra [98–100]. Therefore, at finite N the fuzzy torus harmonics must be expressible in terms of fuzzy spherical harmonics by a change of basis,

$$\widehat{Y}_{mn} = \sum_{\alpha} B^{\alpha}_{mn} \widehat{Y}_{\alpha}. \tag{5.4}$$

This change of basis becomes singular in the large N limit, reflecting the fact that this limit requires one to choose a basis appropriate to the set of smooth functions in the limiting topology. It would be quite interesting to explore ideas related to the finite N algebra and sums over the topologies of the surface in more detail.

A different topological aspect arising from the larger groups $\mathfrak{su}(N,N)$ and $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ is related to nontrivial $\mathfrak{sl}(2,\mathbb{R})$-bundles over S. These were argued to be closely associated with nonzero NUT charges for the surface [26]. The natural question is whether the information of these nontrivial bundles could somehow be encoded in the $\mathfrak{su}(N,N)$ regularization of section 3.4. One possibility is that the information of these nontrivial bundles could be only emergent as we take $N \to \infty$, similar to the emergence of the topology of S discussed above. Note that the continuum algebra is different from $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}(S)$ when working with nontrivial bundles: rather than taking the form of a semidirect product, the symmetry algebra is instead a nontrivial extension of $\mathfrak{sdiff}(S)$ by $\mathfrak{sl}(2,\mathbb{R})^S$. Presumably these algebras could be obtained by considering a different large N limit involving twisted generators $\widetilde{X}_\alpha = X_\alpha + \frac{N}{2} A_\alpha{}^{\mu a} Z_{a\mu}$, with the tensor $A_\alpha{}^{\mu a}$ subject to some consistency conditions needed to ensure a good large N limit. Note that $\widetilde{X}_\alpha$ are divergent in the original large N limit, implying that these generators lead to a different continuum algebra which conjecturally coincides with the symmetry algebra associated with nontrivial $\mathfrak{sl}(2,\mathbb{R})$ bundles. The tensor $A_\alpha{}^{\mu a}$ would then be related to the curvature of a connection on the resulting $\mathfrak{sl}(2,\mathbb{R})$ bundle, which characterizes the Lie algebra 2-cocycle defining the extension, as discussed in [26, appendix A]. It is thus conceivable that the data of different topologies of S along with different $\mathfrak{sl}(2,\mathbb{R})$ bundles are contained in the finite N algebra $\mathfrak{su}(N,N)$.

Finally, throughout this work, we have eliminated the central generator X_{00} since it arises from the constant function on the sphere, which does not generate a diffeomorphism in the continuum algebras. However, a question remains as to whether the charges associated with this central generator should be nonzero in the quantum theory. It would be interesting to investigate this, and determine whether these central charges bear any relation to the NUT charges discussed above.

5.4 Deformation of the full diffeomorphism algebra

This work has focused on three subalgebras of the full corner symmetry algebra, all of which involve area-preserving diffeomorphisms as opposed to the full diffeomorphism algebra of S^2. Nevertheless, this raises the question whether the deformations considered here could eventually be lifted to the full corner symmetry algebra $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}(S) = \mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$, as suggested by figure 1. The main challenge here would be to determine the deformation for the full $\mathfrak{diff}(S)$ algebra, after which one may be able to extend it to the corner symmetry algebra following similar techniques as employed in this paper. Our initial investigations on this topic involve an explicit computation of the structure constants for $\mathfrak{diff}(S)$, which are derived in detail in appendix B. However, there are several indications that any deformation of this algebra will involve a more complicated procedure than the analogous problem for $\mathfrak{sdiff}(S)$. Recently, a no-go theorem for the existence of such a linear deformation of $\mathfrak{diff}(S)$ was proven in [101]. This suggests that the full corner symmetry algebra does not admit such a deformation, although the possibility remains that $\mathfrak{diff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$ is deformable even though $\mathfrak{diff}(S)$ itself is not.

A more likely possibility is that the deformation would involve a nonlinear algebra, such as those appearing in the theory of quantum groups. Relatedly, one might consider looking for a deformation of a larger algebra containing $\mathfrak{diff}(S)$, such as the higher spin Schouten algebra of all symmetric multivector fields on the sphere. This higher spin picture is naturally associated with the higher spin-weighted spherical harmonics, whose deformation was suggested in appendix A.3 to be a set of rectangular matrices. These matrices are associated with changes in the value of N, and hence one might conjecture that the natural deformation of this higher spin algebra involves a sum over all possible values of the deformation parameter N. It is possible that a deformation of this higher spin algebra can be consistently defined, and only in the classical limit do the $\mathfrak{diff}(S)$ generators close to form a subalgebra. We leave further investigation into these ideas to future work.

Furthermore, we have restricted our attention to the part of the corner symmetry algebra that preserves the corner S and exclude the so-called corner deformations, which move S itself. By including normal translations of the corner, we would instead end up with the symmetry group [2–6, 10, 12]

$$(\mathrm{Diff}(S) \ltimes \mathrm{SL}(2,\mathbb{R})^S) \ltimes (\mathbb{R}^2)^S. \tag{5.5}$$

It has been shown in ref. [3] that this is the maximal subalgebra of the diffeomorphism group of the bulk spacetime that is associated to an isolated corner S. Therefore, the full regularization of corner symmetry should include this generalization, and it would be interesting to explore deformations of this algebra as well.

5.5 Other algebra deformations

As indicated in figure 1, this work identified a natural nested sequence of deformed algebras $\mathfrak{su}(N) \subset \mathfrak{sl}(N,\mathbb{C}) \oplus \mathbb{R} \subset \mathfrak{su}(N,N)$ coinciding with the continuum algebra inclusions $\mathfrak{sdiff}(S) \subset \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S \subset \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$. In particular, the intermediate algebra $\mathfrak{sl}(N,\mathbb{C}) \oplus \mathbb{R}$ arises as the subalgebra of $\mathfrak{su}(N,N)$ preserving a complex structure, which in the $\pi_{\mathbf{2N}}$ representation is just the matrix $2\widehat{Y}_{1,00}$. It is noteworthy that a number of other interesting algebras appear as intermediate steps between $\mathfrak{su}(N,N)$ and $\mathfrak{su}(N)$. In particular, if one instead looks for the algebra preserving the paracomplex structure $2\widehat{Y}_{0,00} = \left(\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{smallmatrix}\right)$, the result is the maximal compact subalgebra $\mathfrak{su}(N) \oplus \mathfrak{su}(N) \oplus \mathfrak{u}(1)$. This algebra may be relevant as a corner symmetry algebra in Euclidean signature, where one is interested in rotations instead of boosts in the normal plane. We can also form the algebra $\mathfrak{sp}(2N,\mathbb{R}) \oplus \mathbb{R}$ as the set of generators $\widehat{A}$ preserving a real structure, meaning that $\widehat{A}^* J_r = J_r \widehat{A}$, with J_r a matrix satisfying $J_r^* J_r = \mathbb{1}$ and $*$ denotes complex conjugation. This matrix can be taken to be $J_r = 2\widehat{Y}_{2,00} = \left(\begin{smallmatrix} 0 & -i\mathbb{1} \\ -i\mathbb{1} & 0 \end{smallmatrix}\right)$. Finally, one can obtain the quaternionic orthogonal algebra $\mathfrak{so}^*(2N) \oplus \mathbb{R}$ by restricting to generators $\widehat{B}$ that preserve a pseudoreal structure, meaning $\widehat{B}^* J_p = J_p \widehat{B}$, with J_p satisfying $J_p^* J_p = -\mathbb{1}$. Such a pseudoreal structure is given by $2\widehat{Y}_{1,00} = \left(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$. It is an interesting question whether these other intermediate algebras have large N limits in terms of diffeomorphism algebras of S.

In a different vein, we note that the limit of $\mathfrak{su}(N,N)$ to the continuum algebra required a specific scaling of the generators according to (3.54). There exists a different scaling of generators that also yields a finite limit as $N \to \infty$: we can rescale the $\mathfrak{sdiff}(S)$ generators according to

$$\widetilde{X}_\alpha = \frac{1}{N^2} X_\alpha. \tag{5.6}$$

In terms of these, the algebra becomes

$$[\widetilde{X}_\alpha, \widetilde{X}_\beta] = \frac{1}{N^2} \widehat{C}_{\alpha\beta}{}^\gamma \widetilde{X}_\gamma, \tag{5.7}$$

$$[\widetilde{X}_\alpha, Z_{a\beta}] = \frac{1}{N^2} \widehat{C}_{\alpha\beta}{}^\gamma Z_{a\gamma}, \tag{5.8}$$

$$[Z_{a\alpha}, Z_{b\beta}] = \varepsilon_{ab}{}^c \widehat{E}_{\alpha\beta}{}^\gamma Z_{c\gamma} - \eta_{ab} \widehat{C}_{\alpha\beta}{}^\gamma \widetilde{X}_\gamma. \tag{5.9}$$

The $N \to \infty$ limit now implements a different contraction of the algebra in which the generators $\widetilde{X}_\alpha$ become central, and (5.9) indicates that the resulting algebra is a nontrivial central extension of the sphere algebra $\mathfrak{sl}(2,\mathbb{R})^S$ [102]. These centrally extended sphere algebras have been explored, for example, in [53, 103], and it is interesting to see that they arise from a nontrivial limit of the matrix algebra $\mathfrak{su}(N,N)$. Whether this limit has any bearing on the quantization of the corner symmetry algebra remains to be seen.

5.6 Connections to holography

Although the algebras considered in this work arose as deformations of classical algebras arising from a bulk gravitational theory, there are several connections between these deformations and features of holographic models of quantum gravity. Many examples of holography arise as matrix models, which naturally are associated with $\mathfrak{su}(N)$ symmetry [79, 104–106]. Indeed, the supermembranes arising in string theory and M-theory were the original context in which the identification of $\mathfrak{sdiff}(S)$ as the large N limit of $\mathfrak{su}(N)$ arose [45–47]. Related ideas appear in the holographic spacetime model of reference [48]. Such examples give a motivation for considering the deformed algebras described in this paper, and describe models where the exact diffeomorphism symmetry is an emergent symmetry in the low-energy, classical theory. While we have approached the question from the perspective of gravitational theory, it would be interesting to obtain deformations of the corner symmetry algebra from a more fundamental UV theory. There has been some progress in understanding the closely related concept of "entangling branes" in string field theory [107] and in topological string theory [108–112] but their precise relation to symmetries in the emergent gravitational theory remains unclear.

Finally, the large N limits considered in the present work have interesting connections to recent work on von Neumann algebras arising in the large N limit of holographic conformal field theories [82–85, 113]. Particularly intriguing is the fact that the area operator defined in equation (4.95) is central at infinite N, but becomes noncentral upon including perturbative $\frac{1}{N}$ corrections. This bears some resemblance to aspects of the crossed product construction considered in [84], where, in particular, it was important to realize that the area operator is singular in the quantum theory, and only becomes a well-defined operator after adding the bulk modular hamiltonian to it, which accounts for the noncommutativity at subleading order in Newton's constant. A fruitful future direction for the present work is to try to make this connection more precise, and look to understand the corner symmetries and their deformations in terms of von Neumann algebras.

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A Spherical harmonics and fuzzy spherical harmonics

In this appendix, we define the conventions used for continuum spherical harmonics which are used as an explicit basis of functions on the unit sphere. In section A.1, we describe the structure constants for multiplication and the Poisson bracket with respect to this basis. The conventions for spin-weighted spherical harmonics, which are used when evaluating structure constants for differential operators on the sphere, are subsequently presented in section A.2. We then describe the basis of fuzzy spherical harmonics in section A.3 as finite-dimensional Hermitian matrices, and review the standard result showing that the structure constants for the commutator of these matrices approaches the structure constants of $\mathfrak{sdiff}(S)$. The matrix product is given by a simple formula in terms of the Wigner $6j$ symbol, and we present an expression for it that immediately yields the large N expansion of the product to any desired order. We demonstrate the utility of this formula in section A.4 by determining the $\mathcal{O}(\frac{1}{N^2})$ correction to the matrix product, and verifying that it takes the form of a Fedosov $\star$-product for the sphere, viewed as a symplectic manifold.

A.1 Spherical harmonics

We use spherical harmonics $Y_{Aa}(\theta,\varphi)$ where $A \in \mathbb{N}$ and $a \in \{-A, \dots, +A\}$. We work with the Racah normalization convention and the Condon-Shortley phase, which imply

$$\int_S \nu_0\, Y_{Aa}(\theta,\varphi) Y_{Bb}(\theta,\varphi) = (-1)^a \,\frac{\delta_{A,B}\delta_{a,-b}}{(2A+1)}, \tag{A.1}$$

where $\nu_0 = \frac{1}{4\pi} \cdot \frac{1}{2}\epsilon_{AB} \mathsf{d}\sigma^A \wedge \mathsf{d}\sigma^B = \frac{1}{4\pi}\sin\theta \, \mathsf{d}\theta \wedge \mathsf{d}\varphi$ is the unit-normalized volume form on the sphere. It will be convenient to adopt a condensed index notation $\alpha = (A, a)$ in which Y_α is shorthand for the spherical harmonic functions $Y_{Aa}(\theta, \varphi)$. Then (A.1) defines a real metric

$$\delta_{\alpha\beta} = \frac{(-1)^a}{(2A+1)}\delta_{A,B}\delta_{a,-b}, \tag{A.2}$$

on the vector space of functions on the sphere. The α indices will be raised and lowered with this real metric. Complex conjugation and orientation reversal act as

$$(Y_{A,a}(\theta,\varphi))^* = (-1)^a Y_{A,-a}(\theta,\varphi), \qquad (Y_{A,a}(\pi-\theta,\varphi+\pi))^* = (-1)^A Y_{A,-a}(\theta,\varphi). \tag{A.3}$$

The multiplication structure constants $E_{\alpha\beta}{}^\gamma$ are defined via $Y_\alpha Y_\beta = E_{\alpha\beta}{}^\gamma Y_\gamma$. Their explicit values are given in terms of Wigner $3j$ symbols [114] according to

$$E_{\alpha\beta}{}^\gamma = (-1)^c\,(2C+1)\begin{pmatrix} A & B & C \\ a & b & -c \end{pmatrix}\begin{pmatrix} A & B & C \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.4}$$

Lowering one index with the metric (A.2) gives the totally symmetric tensor $E_{\alpha\beta\gamma}$:

$$E_{\alpha\beta\gamma} = \begin{pmatrix} A & B & C \\ a & b & c \end{pmatrix}\begin{pmatrix} A & B & C \\ 0 & 0 & 0 \end{pmatrix} = \int_S \nu_0 \, Y_\alpha Y_\beta Y_\gamma. \tag{A.5}$$

Note that this is nonzero only when $A + B + C$ is even.

The Poisson bracket of two functions on the sphere is defined as

$$\{f, g\} = \epsilon^{AB}\nabla_A f \nabla_B g, \tag{A.6}$$

where ϵ^{AB} is the negative inverse of the standard area form ϵ_{AB} on the unit sphere.[28]

The structure constants for the Poisson bracket $C_{\alpha\beta}{}^\gamma$ are defined by $\{Y_\alpha, Y_\beta\} = C_{\alpha\beta}{}^\gamma Y_\gamma$. The expression for these structure constants is [52, 53]

$$C_{\alpha\beta}{}^\gamma = -i(-1)^c(2C+1)\delta^1_{[A+B+C]}\,[A]_1[B]_1\begin{pmatrix} A & B & C \\ a & b & -c \end{pmatrix}\begin{pmatrix} A & B & C \\ 1 & -1 & 0 \end{pmatrix}, \tag{A.7}$$

where we have defined

$$[A]_m = \sqrt{\frac{(A+m)!}{(A-m)!}}, \tag{A.8}$$

and $\delta^n_{[A]}$ is equal to 1 if $A = n \bmod(2)$ and equal to zero otherwise. Note that these are nonvanishing only when $A + B + C$ is odd. These structure constants can be derived using identities for spin-weighted spherical harmonics, discussed in section A.2.

It is also convenient to introduce a symmetric bracket constructed from the round sphere metric,

$$\langle f, g\rangle = q^{AB}\nabla_A f \nabla_B g. \tag{A.9}$$

[28]It is an antisymmetric tensor normalised by the condition $\epsilon^{12} = 1/\sqrt{q}$ where q is the metric determinant in the coordinate chosen. Note that this Poisson bracket differs from the Poisson bracket $\{,\}_{\nu_0}$ defined relative to the unit area volume form ν_0 by a factor of $\frac{1}{4\pi}$.

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Its structure constants ${G_{\alpha\beta}}^{\gamma}$ defined by $\langle Y_\alpha, Y_\beta\rangle = {G_{\alpha\beta}}^{\gamma} Y_\gamma$ are given by a similar expression

$$ {G_{\alpha\beta}}^{\gamma} = -(-1)^c (2C+1)\delta^0_{[A+B+C]}\,[A]_1[B]_1 \begin{pmatrix} A & B & C \\ a & b & -c \end{pmatrix} \begin{pmatrix} A & B & C \\ 1 & -1 & 0 \end{pmatrix}, \tag{A.10} $$

which are nonvanishing only when $A+B+C$ is even. These structure constants have a simple relation to the product structure constants ${E_{\alpha\beta}}^{\gamma}$ arising from the identity,

$$ \int_S \nu_0 \nabla^A Y_\alpha \nabla_A Y_\beta Y_\gamma = -\frac{1}{2}\int_S \nu_0 \left(\nabla^2 Y_\alpha Y_\beta Y_\gamma + Y_\alpha \nabla^2 Y_\beta Y_\gamma - Y_\alpha Y_\beta \nabla^2 Y_\gamma\right), \tag{A.11} $$

which then implies

$$ {G_{\alpha\beta}}^{\gamma} = \frac{1}{2}\Big(A^{(1)} + B^{(1)} - C^{(1)}\Big){E_{\alpha\beta}}^{\gamma}, \tag{A.12} $$

where

$$ A^{(1)} = ([A]_1)^2 = A(A+1), \tag{A.13} $$

is minus Laplacian eigenvalue on the sphere, i.e. $\nabla^2 Y_\alpha = -A^{(1)} Y_\alpha$.

A.2 Spin-weighted spherical harmonics

Just as the ordinary spherical harmonics provide a basis with respect to which functions on the sphere can be decomposed, the spin-weighted spherical harmonics Y^s_α [115] yield a convenient basis for decomposing tensorial objects and differential operators on the sphere. They are most easily described by introducing the holomorphic coordinate on the sphere,

$$ z = e^{i\varphi}\cot\frac{\theta}{2}, \tag{A.14} $$

so that the metric is given by

$$ ds^2 = \frac{1}{P^2} dz d\bar{z}, \tag{A.15} $$

$$ P = \frac{1}{2}(1+z\bar{z}). \tag{A.16} $$

A complex null basis for the tangent space is provided by

$$ m^A = \sqrt{2} P \partial^A_z, \quad \bar{m}^A = \sqrt{2} P \partial^A_{\bar{z}}, \tag{A.17} $$

which satisfy

$$ m\cdot m = \bar{m}\cdot\bar{m} = 0, \quad m\cdot\bar{m} = 1. \tag{A.18} $$

The metric and volume form on the sphere are expressed in terms of the holomorphic basis by

$$ q_{AB} = \bar{m}_A m_B + m_A \bar{m}_B, \tag{A.19} $$

$$ \epsilon_{AB} = i(\bar{m}_A m_B - m_A \bar{m}_B). \tag{A.20} $$

A quantity is defined to have spin weight s if under the phase rotation $m^a \to e^{i\psi} m^a$, it transforms with a factor of $e^{is\psi}$. A general traceless symmetric tensor $T_{A_1\ldots A_n}$ on the sphere has a decomposition in terms of objects $(T, \bar{T})$ of spin weights $(n, -n)$ via

$$T_{A_1\ldots A_n} = T\bar{m}_{A_1}\ldots\bar{m}_{A_n} + \bar{T}m_{A_1}\ldots m_{A_n}. \tag{A.21}$$

Any function of spin weight s can be decomposed in terms of the spin-weighted harmonics Y^s_α,[29] which form a basis for functions of the given spin weight. Note that the spin-0 harmonics are simply the usual spherical harmonics discussed in section A.1. Goldberg et. al. [115] give explicit expressions for Y^s_α and a detailed discussion of their properties; here, we will simply quote the relevant properties needed in this work. Complex conjugation acts via

$$(Y^s_{lm})^* = (-1)^{l+s}Y^{-s}_{l,-m}, \tag{A.22}$$

and continuing to use the Racah normalization, the integral over the sphere of a product is given by

$$\int_S \nu_0 Y^s_{Aa}Y^{-s}_{Bb} = (-1)^{s+a}\frac{\delta_{A,B}\delta_{a,-b}}{(2A+1)}. \tag{A.23}$$

The differential operators $\eth$ and $\bar{\eth}$ defined in [115] act as spin-weight raising and lowering operators on Y^s_α, whose action is given explicitly by

$$\eth Y^s_\alpha = \frac{[A]_{s+1}}{[A]_s}\,Y^{s+1}_\alpha, \tag{A.24}$$

$$\bar{\eth}Y^s_\alpha = -\frac{[A]_s}{[A]_{s-1}}\,Y^{s-1}_\alpha. \tag{A.25}$$

As a consequence, we have that these operators satisfy the relations

$$[\bar{\eth},\eth]Y^s_\alpha = 2sY^s_\alpha, \qquad (\bar{\eth}\eth + \eth\bar{\eth})Y^s_\alpha = -2[A(A+1) - s^2]Y^s_\alpha. \tag{A.26}$$

The derivative operators $m^A\nabla_A$ and $\bar{m}^A\nabla_A$ are closely related to $\eth$, $\bar{\eth}$ when acting on totally symmetric traceless tensors, as is seen by the following relations:

$$m^A\nabla_A(Y^s_\alpha\bar{m}^{B_1}\ldots\bar{m}^{B_s}) = \frac{1}{\sqrt{2}}\eth Y^s_\alpha\bar{m}^{B_1}\ldots\bar{m}^{B_s}, \tag{A.27}$$

$$m^A\nabla_A(Y^{-s}_\alpha m^{B_1}\ldots m^{B_s}) = \frac{1}{\sqrt{2}}\eth Y^{-s}_\alpha m^{B_1}\ldots m^{B_s}, \tag{A.28}$$

$$\bar{m}^A\nabla_A(Y^s_\alpha\bar{m}^{B_1}\ldots\bar{m}^{B_s}) = \frac{1}{\sqrt{2}}\bar{\eth}Y^s_\alpha\bar{m}^{B_1}\ldots\bar{m}^{B_s}, \tag{A.29}$$

$$\bar{m}^A\nabla_A(Y^{-s}_\alpha m^{B_1}\ldots m^{B_s}) = \frac{1}{\sqrt{2}}\bar{\eth}Y^{-s}_\alpha m^{B_1}\ldots m^{B_s}. \tag{A.30}$$

Using these, we can write the gradient and curl of Y_{lm} in terms of spin-weighted harmonics by

$$q^{CB}\nabla_C Y_\alpha = \frac{[A]_1}{\sqrt{2}}\left(Y^1_\alpha\bar{m}^B - Y^{-1}_\alpha m^B\right), \tag{A.31}$$

$$\epsilon^{CB}\nabla_C Y_\alpha = \frac{-i[A]_1}{\sqrt{2}}\left(Y^1_\alpha\bar{m}^B + Y^{-1}_\alpha m^B\right). \tag{A.32}$$

[29] We use the notation Y^s_{lm} instead of the more standard ${}_sY_{lm}$ for ease of readability.

The vectors (A.31) and (A.32) respectively coincide with the pure-spin electric and magnetic vector harmonics, defined in e.g. [116], after normalizing by a factor of $\frac{1}{[A]_1}$. This terminology refers to the transformation properties of these vectors under parity. More general higher order differential operators acting on Y_α can be evaluated similarly. We define the following operator

$$\Delta_{A_1 \ldots A_s} = \nabla_{(A_1} \ldots \nabla_{A_s)} - \text{traces}, \tag{A.33}$$

which is symmetric and traceless by definition. For example,

$$\Delta_{AB} = \nabla_{(A}\nabla_{B)} - \frac{1}{2} q_{AB} \nabla^2. \tag{A.34}$$

Then the following relation can be shown by inductively applying the above identities

$$\Delta_{B_1 \ldots B_s} Y_\alpha = \frac{[A]_s}{2^{s/2}} \Big(Y_\alpha^s \bar{m}_{B_1} \ldots \bar{m}_{B_s} + (-1)^s Y_\alpha^{\text{-}s} m_{B_1} \ldots m_{B_s} \Big). \tag{A.35}$$

The final relation that is useful in obtaining structure constants for differential operators is the triple integral identity, which generalizes (A.5),

$$\int_S \nu_0 Y_\alpha^i Y_\beta^j Y_\gamma^k = \begin{pmatrix} A & B & C \\ a & b & c \end{pmatrix} \begin{pmatrix} A & B & C \\ \text{-}i & \text{-}j & \text{-}k \end{pmatrix}. \tag{A.36}$$

For example, this equation, along with the gradient and curl expressions (A.31), (A.32), provides a straightforward means of evaluating the integrals $\int_S \nu_0 \{Y_\alpha, Y_\beta\} Y_\gamma$ and $\int_S \nu_0 \langle Y_\alpha, Y_\beta \rangle Y_\gamma$ involving the Poisson bracket (A.6) and symmetric bracket (A.9), and this leads directly to the expressions (A.7) and (A.10) for their structure constants.

A.3 Fuzzy spherical harmonics

The fuzzy sphere replaces the algebra of functions on the sphere by a noncommutative matrix algebra, corresponding to the fundamental representation of the SU(N) Lie algebra. As with the continuum algebra, these matrices decompose into representations of SU(2), and hence can be labeled by fuzzy spherical harmonics $\widehat{Y}_\alpha$, with $\alpha = (A, a)$ again denoting the SU(2) representation indices. As shown in [54], the matrix elements of the fuzzy harmonics can be given explicitly in terms of a $3j$-symbol according to

$$(\widehat{Y}_\alpha)_i{}^j = \sqrt{N}\,(-1)^{J-j} \begin{pmatrix} A & J & J \\ a & i & \text{-}j \end{pmatrix}, \tag{A.37}$$

where $N = 2J+1$. The range of the A index is $0 \le A \le 2J$, since for $A > 2J$ the expression (A.37) vanishes, and J can be an integer or half integer. The fuzzy haronics satisfy the reality condition $\widehat{Y}_{Aa}^\dagger = (-1)^a \widehat{Y}_{A,-a}$ in direct analogy with the continuum harmonics, and are normalized to satisfy

$$\frac{1}{N} \operatorname{Tr}\left(\widehat{Y}_\alpha \widehat{Y}_\beta\right) = \frac{(-1)^a}{2A+1} \delta_{AB} \delta_{a,-b} = \delta_{\alpha\beta}, \tag{A.38}$$

where the real metric $\delta_{\alpha\beta}$ agrees with the expression for the continuum harmonics. The matrices $\widehat{Y}_\alpha$ form a basis for traceless $N \times N$ matrices, which in general are not Hermitian.

However, for each value of α, one can form the Hermitian combinations $\widehat{Y}_\alpha + \widehat{Y}^\dagger_\alpha$ and $i(\widehat{Y}_\alpha - \widehat{Y}^\dagger_\alpha)$, just as one would form real combinations of the complex continuum harmonics Y_α. These Hermitian combinations thus provide a matrix version of real-valued functions, and since Hermitian matrices generate the Lie algebra of $\mathrm{SU}(N)$, we see that the matrix regularization of the algebra of real functions on the sphere coincides with $\mathfrak{su}(N)$.

The product of two fuzzy harmonics can be defined via structure constants, $\widehat{Y}_\alpha \widehat{Y}_\beta = \widehat{M}_{\alpha\beta}{}^\gamma \widehat{Y}_\gamma$, explicitly given in terms of the Wigner $6j$ symbol [114] by [54, 117]

$$\widehat{M}_{\alpha\beta}{}^\gamma = \sqrt{N}(2C+1)(-1)^{2J+c} \begin{pmatrix} A & B & C \\ a & b & -c \end{pmatrix} \begin{Bmatrix} A & B & C \\ J & J & J \end{Bmatrix}, \tag{A.39}$$

or more symmetrically with the γ index lowered using the metric (A.38) as

$$\widehat{M}_{\alpha\beta\gamma} = \frac{1}{N} \operatorname{Tr}\left(\widehat{Y}_\alpha \widehat{Y}_\beta \widehat{Y}_\gamma\right) = \frac{\sqrt{N}}{(-1)^{2J}} \begin{pmatrix} A & B & C \\ a & b & c \end{pmatrix} \begin{Bmatrix} A & B & C \\ J & J & J \end{Bmatrix}. \tag{A.40}$$

It is convenient to define a deformed $3j$ symbol

$$\begin{bmatrix} A & B & C \\ 0 & 0 & 0 \end{bmatrix}_N := \frac{\sqrt{N}}{(-1)^{2J}} \begin{Bmatrix} A & B & C \\ J & J & J \end{Bmatrix}, \tag{A.41}$$

so that the structure constants take the form

$$\widehat{M}_{\alpha\beta\gamma} = \begin{pmatrix} A & B & C \\ a & b & c \end{pmatrix} \begin{bmatrix} A & B & C \\ 0 & 0 & 0 \end{bmatrix}_N, \tag{A.42}$$

directly analogous to the continuum equation (A.5). We can further decompose these structure constants into their symmetric $\widehat{E}_{\alpha\beta\gamma}$ and antisymmetric $\widehat{C}_{\alpha\beta\gamma}$ pieces on α and β,

$$\widehat{M}_{\alpha\beta\gamma} = \widehat{E}_{\alpha\beta\gamma} + \frac{i}{N} \widehat{C}_{\alpha\beta\gamma}, \tag{A.43}$$

and we will see below that as $N \to \infty$, $\widehat{E}_{\alpha\beta\gamma}$ and $\widehat{C}_{\alpha\beta\gamma}$ approach their classical counterparts, $E_{\alpha\beta\gamma}$ and $C_{\alpha\beta\gamma}$, defined in section A.1.

The large-N expansion of the structure constants $\widehat{M}_{\alpha\beta\gamma}$ can be obtained by employing a remarkable identity by Nomura [55, eq. (2.22)] that expresses the $6j$ symbol as a single sum in which each term involves a single $3j$ symbol.[30] Applied to the deformed $3j$ symbol (A.41), this identity yields

$$\begin{bmatrix} A & B & C \\ 0 & 0 & 0 \end{bmatrix}_N = \left[\frac{\theta_N(A)\theta_N(B)}{\theta_N(C)}\right]^{\frac{1}{2}} \times \sum_{m=0}^{\min(A,B)} \frac{N!}{(N+m)!m!} [A]_m [B]_m \begin{pmatrix} A & B & C \\ m & -m & 0 \end{pmatrix}, \tag{A.44}$$

where we have made the definitions

$$\theta_N(A) := \frac{(N+A)!(N-A-1)!}{N!(N-1)!} = \frac{(N+A)\dots(N+1)}{(N-A)\dots(N-1)}, \tag{A.45}$$

[30] Note that Nomura [55] uses a nonstandard normalization for the $6j$ symbol, and with the standard normalization [114], the factor of $(2e+1)^{-\frac{1}{2}}$ that appears in Nomura's equation (2.22) should be left out.

and $[A]_m$ is defined in (A.8). The identity (A.44) is valid assuming B is an integer, and holds for J either integer or half integer. Each term in the sum (A.44) is suppressed by an additional factor of $\frac{1}{N}$, and hence this sum manifestly yields the large N expansion of the matrix product of fuzzy spherical harmonics. The prefactor to the sum has the following expansion at large N,

$$1+\frac{1}{2N}\left(A^{(1)}+B^{(1)}-C^{(1)}\right)+\frac{1}{8N^2}\left(A^{(1)}+B^{(1)}-C^{(1)}\right)^2+\ldots. \tag{A.46}$$

The leading order term in the deformed $3j$ symbol expansion is then seen to be

$$\begin{bmatrix} A & B & C \\ 0 & 0 & 0 \end{bmatrix}_N = \begin{pmatrix} A & B & C \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{O}\left(1/N\right). \tag{A.47}$$

Substituting this expression into the structure constants (A.40), we see that the leading order piece $\widehat{M}^{(0)}_{\alpha\beta\gamma}$ coincides exactly with the continuum commutative product structure constants $E_{\alpha\beta\gamma}$ (A.5).

Equation (A.41) straightforwardly yields the first subleading correction to the deformed symbol $3j$ symbol,

$$\frac{1}{N}\left[\frac{1}{2}\left(A^{(1)}+B^{(1)}-C^{(1)}\right)\begin{pmatrix} A & B & C \\ 0 & 0 & 0 \end{pmatrix}+[A]_1[B]_1\begin{pmatrix} A & B & C \\ 1 & -1 & 0 \end{pmatrix}\right]. \tag{A.48}$$

Using the relations (A.11) and (A.10) between the $E_{\alpha\beta\gamma}$ and $G_{\alpha\beta\gamma}$ structure constants, we see that the terms with $A+B+C$ even cancel in (A.48), leaving only the odd piece,

$$\frac{1}{N}\delta^1_{[A+B+C]}\,[A]_1[B]_1\begin{pmatrix} A & B & C \\ 1 & -1 & 0 \end{pmatrix}. \tag{A.49}$$

Comparing to (A.7), this determines the first-order correction to the structure constants (A.40) in terms of the continuum Poisson bracket structure constants $C_{\alpha\beta\gamma}$

$$\widehat{M}^{(1)}_{\alpha\beta\gamma}=\frac{i}{N}C_{\alpha\beta\gamma}. \tag{A.50}$$

This verifies that the matrix product of the fuzzy harmonics $\widehat{Y}_\alpha$ takes the desired form of a valid $\star$-product in the sense of deformation quantization of the algebra of functions on the sphere (see e.g. [118–120]); namely, it has the expansion

$$\widehat{Y}_\alpha\cdot\widehat{Y}_\beta=\widehat{Y_\alpha Y_\beta}+\frac{i\hbar}{2}\widehat{\{Y_\alpha,Y_\beta\}}+\mathcal{O}(\hbar^2), \tag{A.51}$$

with $\hbar=\frac{2}{N}$. In fact, we can fix the value of $\hbar$ more precisely by recalling that Poisson brackets of the $l=1$ continuum harmonics generate an $\mathfrak{su}(2)$ algebra, and by requiring that the matrix commutator exactly reproduce this algebra in the sense

$$[\widehat{Y}_{1,m},\widehat{Y}_{1,m'}]=i\hbar\,\widehat{\{Y_{1,m},Y_{1,m'}\}}, \tag{A.52}$$

the value of $\hbar$ is determined to be

$$\hbar=\frac{1}{[J]_1}=\frac{2}{\sqrt{N^2-1}}=\frac{2}{N}+\mathcal{O}(N^{-3}), \tag{A.53}$$

which can be derived by evaluating the exact structure constants for the matrix product (A.40) in terms of the $6j$ symbol

$$\begin{Bmatrix} 1 & 1 & 1 \\ J & J & J \end{Bmatrix} = \frac{(-1)^{2J}}{\sqrt{N}} \frac{\sqrt{6}}{3\sqrt{N^2-1}}. \tag{A.54}$$

Since the Poisson bracket $\{,\}$ is defined with respect to a unit radius sphere with area $A = 4\pi$, the result $\hbar = \frac{2}{N}$ is consistent with the standard relation $N = \frac{A}{2\pi\hbar}$ between the dimension N of the quantum Hilbert space and the volume A of the classical phase space.

As an aside, we note that the definition of the deformed $3j$ symbol (A.41) can be extended to nonzero magnetic quantum numbers by the equation

$$\begin{bmatrix} A & B & C \\ i & j & k \end{bmatrix}_N = \frac{\sqrt{N}}{(-1)^{2J}} \begin{Bmatrix} A & B & C \\ J-k & J & J+i \end{Bmatrix}, \tag{A.55}$$

where $i+j+k=0$. The Nomura identity [55] in this case yields the expression

$$\begin{aligned}\begin{bmatrix} A & B & C \\ i & j & k \end{bmatrix}_N =& \left[\frac{N(N+i-A-1)!(N+A+i)!(N-k+i-B-1)!(N-k+i+B)!}{((N+2i)!)^2(N-k-C-1)!(N-k+C)!}\right]^{\frac{1}{2}} \\ &\times \sum_{m=0}^{\min(A-i,\,B+j)} \frac{(N+2i)!}{(N+2i+m)!m!} \frac{[A]_{i+m}\,[B]_j}{[B]_{j-m}\,[A]_i} \begin{pmatrix} A & B & C \\ i+m & j-m & k \end{pmatrix}. \end{aligned} \tag{A.56}$$

The prefactor in this expression approaches 1 as $N\to\infty$, and hence at leading order the deformed $3j$ symbol approaches the usual $3j$ symbol,

$$\begin{bmatrix} A & B & C \\ i & j & k \end{bmatrix}_N = \begin{pmatrix} A & B & C \\ i & j & k \end{pmatrix} + \mathcal{O}(N^{-1}), \tag{A.57}$$

which is equivalent to a known asymptotic formula for the $6j$ symbol in terms of a $3j$ symbol [121], although equation (A.56) additionally produces all subleading corrections to this asymptotic formula.

The motivation for the definition (A.55) lies in a product relation for a fuzzy version of spin-weighted spherical harmonics $\widehat{Y}^s_\alpha$, which can be defined as rectangular matrices whose row and column dimension differ by the spin weight s,

$$[\widehat{Y}^s_\alpha]_i{}^j = \sqrt{N}(-1)^{J-j} \begin{pmatrix} A & J+s & J \\ a & i & \text{-}j \end{pmatrix}. \tag{A.58}$$

These matrices can be multiplied by appropriately adjusting the value of J to ensure that the number of columns of the first matrix matches the number of rows of the second. The deformed $3j$ symbol then appears in the structure constants for this matrix multiplication, which, similar to equation (A.40), can be characterized by a trace of a triple product,

$$\widehat{M}^{ijk}_{\alpha\beta\gamma} = \frac{1}{N}\,\mathrm{Tr}\left(\widehat{Y}^i_\alpha \widehat{Y}^j_\beta \widehat{Y}^k_\gamma\right) = \begin{pmatrix} A & B & C \\ a & b & c \end{pmatrix} \begin{bmatrix} A & B & C \\ \text{-}i & \text{-}j & \text{-}k \end{bmatrix}_N. \tag{A.59}$$

This equation is the fuzzy analog of the continuum triple integral expression (A.36). The above proposal for a fuzzy version of the spin-weighted harmonics has not been considered previously, and may provide some hints at determining a deformation of the full diffeomorphism algebra of the sphere $\mathfrak{diff}(S)$. We leave investigation into this idea to future work.

A.4 Expansion of the matrix product

As mentioned above, the identity (A.44) provides a means of expanding the matrix product of the fuzzy harmonics to higher order in $\frac{1}{N}$. This can be used to show that the matrix product takes the form of a valid $\star$-product. Such a product is a deformation of the commutative product of functions of the sphere that admits a formal expansion in powers of $\hbar$ of the form [118–120]

$$f \star g = fg + \sum_{n=1}^{\infty} \left(\frac{i\hbar}{2}\right)^n \frac{1}{n!} C^{(n)}(f,g), \tag{A.60}$$

with

$$C^{(1)}(f,g) = \{f,g\}, \tag{A.61}$$

and with each higher-order term $C^{(n)}(f,g)$ given by a bidifferential operator of order at most n, whose highest order piece takes the form expected from a Moyal product,

$$C^{(n)}(f,g) = \epsilon^{A_1B_1} \ldots \epsilon^{A_nB_n} (\nabla_{A_1} \ldots \nabla_{A_n} f)(\nabla_{B_1} \ldots \nabla_{B_n} g) + B^{(n)}(f,g), \tag{A.62}$$

with the differential order of $B^{(n)}(f,g)$ strictly less than n. As an application of the utility of the formula (A.44), we demonstrate here that the $\mathcal{O}(N^{-2})$ term in the structure constants for the matrix product (A.40) is precisely of this form. From (A.44) and (A.46), the $\mathcal{O}(N^{-2})$ term in the deformed $3j$ symbol is

$$\frac{1}{N^2}\left[\frac{([A]_1^2+[B]_1^2-[C]_1^2)^2}{8}\begin{pmatrix} A & B & C \\ 0 & 0 & 0 \end{pmatrix} + \frac{[A]_1^2+[B]_1^2-[C]_1^2-2}{2}[A]_1[B]_1\begin{pmatrix} A & B & C \\ 1 & -1 & 0 \end{pmatrix}\right.$$
$$\left. + \frac{1}{2}[A]_2[B]_2\begin{pmatrix} A & B & C \\ 2 & -2 & 0 \end{pmatrix}\right]. \tag{A.63}$$

This expression simplifies using a recursion identity for the $3j$ symbols [122, 123],

$$[A]_2[B]_2\begin{pmatrix} A & B & C \\ 2 & -2 & 0 \end{pmatrix} = -([A]_1^2+[B]_1^2-[C]_1^2-2)[A]_1[B]_1\begin{pmatrix} A & B & C \\ 1 & -1 & 0 \end{pmatrix}$$
$$- [A]_1^2[B]_1^2\begin{pmatrix} A & B & C \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.64}$$

to give

$$\frac{1}{N^2}\left[\frac{[A]_1^4+[B]_1^4+[C]_1^4-2[A]_1^2[B]_1^2-2[A]_1^2[C]_1^2-2[B]_1^2[C]_1^2}{8}\right]\begin{pmatrix} A & B & C \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.65}$$

which is notably totally symmetric in A, B, C, and only nonzero for $A+B+C$ even.

We now look for the second order bidifferential operator $C^{(2)}(f,g)$ that yields the expression (A.65) when acting on the continuum harmonics. The structure constants for $C^{(2)}(\cdot,\cdot)$ are defined by

$$C^{(2)}_{\alpha\beta\gamma} = \int_S \nu_0\, C^{(2)}(Y_\alpha, Y_\beta) Y_\gamma. \tag{A.66}$$

Since we expect the highest order term in this operator to take the Moyal product form as in equation (A.62), we begin by evaluating the structure constants

$$\Pi^{(2)}_{\alpha\beta\gamma} = \int_S \nu_0\, \epsilon^{AB}\epsilon^{CD}(\nabla_A\nabla_C Y_\alpha)(\nabla_B\nabla_D Y_\beta)Y_\gamma. \tag{A.67}$$

First using $\epsilon^{AB}\epsilon^{CD} = q^{AC}q^{BD} - q^{AD}q^{BC}$ and $\nabla_{[A}\nabla_{B]}Y_\beta = 0$, we have that

$$\begin{aligned}
\epsilon^{AB}\epsilon^{CD}(\nabla_A\nabla_C Y_\alpha)(\nabla_B\nabla_D Y_\beta) &= \nabla^2 Y_\alpha \nabla^2 Y_\beta - (\nabla^A\nabla^B Y_\alpha)(\nabla_{(A}\nabla_{B)}Y_\beta)\\
&= \nabla^2 Y_\alpha \nabla^2 Y_\beta - \left(\Delta^{AB} + \frac{1}{2}q^{AB}\nabla^2\right)Y_\alpha \left(\Delta_{AB} + \frac{1}{2}q_{AB}\nabla^2\right)Y_\beta\\
&= \frac{1}{2}\nabla^2 Y_\alpha \nabla^2 Y_\beta - (\Delta^{AB}Y_\alpha)(\Delta_{AB}Y_\beta).
\end{aligned} \tag{A.68}$$

We can expand the second term in spin-weighted harmonics using (A.35) to obtain

$$\begin{aligned}
-(\Delta^{AB}Y_\alpha)(\Delta_{AB}Y_\beta) &= -\frac{[A]_2[B]_2}{4}\left(Y^2_\alpha \bar{m}^A\bar{m}^B + Y^{-2}_\alpha m^A m^B\right)\left(Y^2_\beta \bar{m}_A\bar{m}_B + Y^{-2}_\beta m_A m_B\right)\\
&= -\frac{[A]_2[B]_2}{4}(Y^2_\alpha Y^{-2}_\beta + Y^{-2}_\alpha Y^2_\beta).
\end{aligned} \tag{A.69}$$

The contribution of this term to the structure constants (A.67) then follows directly from the triple integral identity (A.36),

$$\begin{aligned}
-\frac{[A]_2[B]_2}{4}\int_S \nu_0\left(Y^2_\alpha Y^{-2}_\beta Y_\gamma + Y^{-2}_\alpha Y^2_\beta Y_\gamma\right) &= -\frac{[A]_2[B]_2}{4}\begin{pmatrix} A & B & C\\ a & b & c\end{pmatrix}\left[\begin{pmatrix} A & B & C\\ -2 & 2 & 0\end{pmatrix} + \begin{pmatrix} A & B & C\\ 2 & -2 & 0\end{pmatrix}\right]\\
&= -\frac{[A]_2[B]_2}{2}\delta^0_{[A+B+C]}\begin{pmatrix} A & B & C\\ a & b & c\end{pmatrix}\begin{pmatrix} A & B & C\\ 2 & -2 & 0\end{pmatrix},
\end{aligned} \tag{A.70}$$

where we recall that $\delta^0_{[A+B+C]}$ is 1 if $A+B+C$ is even, and 0 otherwise. The other contribution to (A.67) is

$$\frac{1}{2}\int_S \nu_0\, \nabla^2 Y_\alpha \nabla^2 Y_\beta Y_\gamma = \frac{[A]_1^2[B]_1^2}{2}\begin{pmatrix} A & B & C\\ a & b & c\end{pmatrix}\begin{pmatrix} A & B & C\\ 0 & 0 & 0\end{pmatrix}. \tag{A.71}$$

So we find the structure constants

$$\Pi^{(2)}_{\alpha\beta\gamma} = \frac{\delta^0_{[A+B+C]}}{2}\begin{pmatrix} A & B & C\\ a & b & c\end{pmatrix}\left[[A]_1^2[B]_1^2\begin{pmatrix} A & B & C\\ 0 & 0 & 0\end{pmatrix} - [A]_2[B]_2\begin{pmatrix} A & B & C\\ 2 & -2 & 0\end{pmatrix}\right]. \tag{A.72}$$

This can be simplified using the $3j$ symbol recursion identities (A.64) and [122, 123]

$$[A]_1[B]_1\begin{pmatrix} A & B & C\\ 1 & -1 & 0\end{pmatrix} = -\frac{1}{2}([A]_1^2 + [B]_1^2 - [C]_1^2)\begin{pmatrix} A & B & C\\ 0 & 0 & 0\end{pmatrix}, \tag{A.73}$$

valid for even $A+B+C$, which reduces the bracketed term in (A.72) to

$$-\frac{\left([A]_1^4 + [B]_1^4 + [C]_1^4 - 2[A]_1^2[B]_1^2 - 2[A]_1^2[C]_1^2 - 2[B]_1^2[C]_1^2\right)}{2}\begin{pmatrix} A & B & C\\ 0 & 0 & 0\end{pmatrix} - 2[A]_1[B]_1\begin{pmatrix} A & B & C\\ 1 & -1 & 0\end{pmatrix}. \tag{A.74}$$

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The first term matches the expression (A.65) appearing at second order in the large N expansion of the matrix product structure constants. The remaining term is a correction that appears in the structure constant $G_{\alpha\beta\gamma}$ for the symmetric bracket (A.9). This then shows that

$$C^{(2)}_{\alpha\beta\gamma} = \Pi^{(2)}_{\alpha\beta\gamma} - G_{\alpha\beta\gamma}, \tag{A.75}$$

or equivalently, that the bidifferential operator $C^{(2)}(f,g)$ is given by

$$C^{(2)}(f,g) = \epsilon^{A_1B_1}\epsilon^{A_2B_2}(\nabla_{A_1}\nabla_{A_2}f)(\nabla_{B_1}\nabla_{B_2}g) - q^{AB}\nabla_A f\nabla_B g, \tag{A.76}$$

where the second term in this expression corresponds to the term $B^{(2)}(f,g)$ in the general expression (A.62) the expansion of the $\star$-product. One noticeable property of $C^{(2)}$ is that it is a symmetric bidifferential operator. As we show in the next section we also have that $C^{(3)}$ is a skew-symmetric bidifferential operator. This means that the first correction to the commutator is of order $\hbar^2$: $[f,g] = i\hbar\left(C^{(1)}(f,g) - \frac{1}{4}\hbar^2 C^{(3)}(f,g) + \mathcal{O}(\hbar^3)\right)$

The appearance of this correction $B^{(2)}(f,g)$ to the naive Moyal product at $\mathcal{O}(\hbar^2)$ deserves some attention. When applying the procedure of Fedosov quantization to construct an associative $\star$-product on a symplectic manifold, one generically finds nontrivial $B^{(n)}(f,g)$ terms that account for effects coming from the curvature of a chosen symplectic connection ∇_A [124]. In the simplest application of the Fedosov construction, however, such curvature corrections only occur at $\mathcal{O}(\hbar^3)$ or higher, whereas the fuzzy matrix product generates such a correction at $\mathcal{O}(\hbar^2)$. Nevertheless, there is no inconsistency in finding such terms at $\mathcal{O}(\hbar^2)$, since the Fedosov procedure contains certain gauge ambiguities that affect the precise expression for the $\star$-product, and these ambiguities can affect the $\mathcal{O}(\hbar^2)$ terms [125]. The correction appearing in (A.76) can arise in two different ways. The first is as an ambiguity in how one constructs a flat connection on the Weyl bundle of the symplectic manifold, which is not uniquely determined even after specifying a symplectic connection. The second way it can appear simply comes from the standard ambiguity in the quantization map sending a classical function Y_α to its quantum operator $\widehat{Y_\alpha}$. In general, one is free to correct this map at higher order in $\hbar$, and a shift of the form $\widehat{Y_\alpha} \to \widehat{Y_\alpha} + \lambda\hbar^2\left(\widehat{\nabla^2 Y_\alpha}\right) + \ldots$ can generate corrections at $\mathcal{O}(\hbar^2)$ in the $\star$-product as were found above.

In the present context, the appearance of a nontrivial $B^{(2)}(\cdot,\cdot)$ at $\mathcal{O}(\hbar^2)$ in the $\star$-product ensures the desirable property that the second-order structure constants $C^{(2)}_{\alpha\beta\gamma}$ are totally symmetric in the indices α, β, γ. This symmetry follows from the permutation symmetry of the columns of the $6j$ symbol that appears in the fully non-perturbative structure constants $\widehat{M}_{\alpha\beta\gamma}$ for the matrix product (A.40). It would be interesting to investigate in future work whether there is some deeper meaning to this correction that appears in the $\star$-product.

A.5 Parity of the matrix product

An interesting feature exhibited by the matrix product structure constants $\widehat{M}_{\alpha\beta\gamma}$ is that the lowest order term in the large N expansion is nonzero only when $A+B+C$ is even, the $\mathcal{O}(N^{-1})$ term is nonvanishing only for $A+B+C$ odd, and, as calculated in section A.4, the $\mathcal{O}(N^{-2})$ term is again nonzero only for $A+B+C$ even. Given the expression (A.40) for the structure constants, this translates to the statement that the $\mathcal{O}(N^0)$ and $\mathcal{O}(N^{-2})$ terms in

$\widehat{M}_{\alpha\beta\gamma}$ are totally symmetric tensors, while the $\mathcal{O}(N^{-1})$ term is totally antisymmetric. This conclusion follows from the fact that the $6j$-symbol is totally symmetric under permutations of its columns, while the $3j$-symbol satisfies $\left(\begin{smallmatrix} B & A & C \\ b & a & c \end{smallmatrix}\right) = (-1)^{A+B+C} \left(\begin{smallmatrix} A & B & C \\ a & b & c \end{smallmatrix}\right)$. Here we will show that this pattern persists to all orders in the $\frac{1}{N}$ expansion, namely, that only even powers of N^{-1} appear in $\widehat{M}_{\alpha\beta\gamma}$ when $A+B+C$ is even, and only odd powers of N^{-1} appear when $A+B+C$ is odd. Given the decomposition (A.43) of $\widehat{M}_{\alpha\beta\gamma}$ into its symmetric and antisymmetric parts, this statement then implies that $\widehat{E}_{\alpha\beta\gamma}$ and $\widehat{C}_{\alpha\beta\gamma}$ both admit large N expansions involving only even powers of N^{-1}.

We will say that the rescaled $6j$ symbol $\begin{bmatrix} A & B & C \\ 0 & 0 & 0 \end{bmatrix}_N = \frac{\sqrt{N}}{(-1)^{2J}} \begin{Bmatrix} A & B & C \\ J & J & J \end{Bmatrix}$ with A, B, C integers satisfies N-parity if its expansion in N^{-1} involves only powers with the same parity as $A+B+C$. To prove the claim that $\begin{bmatrix} A & B & C \\ 0 & 0 & 0 \end{bmatrix}_N$ satisfies N-parity, we begin by noting that as a base case, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}_N$ has an expansion involving only odd powers of N^{-1}, as is apparent from its exact expression obtained from (A.54). Similarly, from the exact expressions

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_N = 1, \qquad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}_N = \frac{-1}{\sqrt{3}}, \tag{A.77}$$

we see that for these lowest values for which $A+B+C$ is even, only N^0-terms appear, and hence N-parity is satisfied. Since $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_N = 0$, this base case for $A+B+C$ odd also trivially satisfies N-parity.

To proceed with an inductive proof to higher values of A, B, C, we apply the following recursion relation for $6j$-symbols [126] (see [127] for a geometrical interpretation of this identity in quantum gravity)

$$A\, E_N(A+1) \begin{Bmatrix} A+1 & B & C \\ J & J & J \end{Bmatrix} = -F(A) \begin{Bmatrix} A & B & C \\ J & J & J \end{Bmatrix} - (A+1)E_N(A) \begin{Bmatrix} A-1 & B & C \\ J & J & J \end{Bmatrix}, \tag{A.78}$$

where

$$E_N(A) = N\sqrt{1 - \frac{A^2}{N^2}}\, A\sqrt{[A^2 - (B-C)^2][(B+C+1)^2 - A^2]}, \tag{A.79}$$

admits an expansion in odd powers of N^{-1}, and

$$F(A) = (2A+1)A^{(1)}(B^{(1)} + C^{(1)} - A^{(1)}), \tag{A.80}$$

is independent of N. Since J (and hence N) is fixed, the recursion relation (A.78) also applies to the rescaled $6j$-symbols $\begin{bmatrix} A & B & C \\ 0 & 0 & 0 \end{bmatrix}_N$.

Now, assuming we have shown that $\begin{bmatrix} A & B & C \\ 0 & 0 & 0 \end{bmatrix}_N$ satisfies N-parity for all $A \leq K$, with B, C fixed, the recursion identity (A.78) implies that

$$\begin{bmatrix} K+1 & B & C \\ 0 & 0 & 0 \end{bmatrix}_N = -\frac{F(K)}{K E_N(K+1)} \begin{bmatrix} K & B & C \\ 0 & 0 & 0 \end{bmatrix}_N - \frac{(K+1)E_N(K)}{K E_N(K+1)} \begin{bmatrix} K-1 & B & C \\ 0 & 0 & 0 \end{bmatrix}_N. \tag{A.81}$$

Since $\frac{F(K)}{KE_N(K+1)}$ involves only odd powers of N^{-1} in its expansion, the first term on the right-hand side above will have an N^{-1} expansion with powers of the opposite parity of the expansion of $\begin{bmatrix} K & B & C \\ 0 & 0 & 0 \end{bmatrix}_N$. Similarly, the expansion of $\frac{(K+1)E_N(K)}{KE_N(K+1)}$ involves only even powers of N^{-1}, and so the second term on the right-hand side will have an N^{-1} expansion with powers of N with the same parity as the expansion of $\begin{bmatrix} K-1 & B & C \\ 0 & 0 & 0 \end{bmatrix}_N$. Hence, the N^{-1} expansion of both terms on the right-hand side in the above relation only involves powers of N^{-1} with the same parity as $K+B+C-1$, which is the same as $K+B+C+1$. We therefore see that $\begin{bmatrix} K+1 & B & C \\ 0 & 0 & 0 \end{bmatrix}_N$ satisfies N-parity, proving the inductive step. Due to the fact that $\begin{bmatrix} A & B & C \\ 0 & 0 & 0 \end{bmatrix}_N$ is totally symmetric under permutations of its columns, the same inductive argument applies to $\begin{bmatrix} A & K+1 & C \\ 0 & 0 & 0 \end{bmatrix}_N$ and $\begin{bmatrix} A & B & K+1 \\ 0 & 0 & 0 \end{bmatrix}_N$, and we can conclude that $\begin{bmatrix} A & B & C \\ 0 & 0 & 0 \end{bmatrix}_N$ satisfies N-parity for all nonnegative integers A, B, C.

A.6 Star product and Nomura identity

In this section, we demonstrate that the product arising from the $6j$-symbol (A.40) can be viewed as a valid star product to all orders in $\frac{1}{N}$, and further show that the Nomura identity [55] arises precisely from the $\hbar$ expansion of this star product. The star product on the sphere can be induced from a rotationally-invariant star product on $\mathbb{R}^3$ via the natural embedding of the sphere in this space. In Cartesian coordinates x^a, $a = 1, 2, 3$, this star product is given by [128–131]

$$f \star g = fg + \sum_{n=1}^{\infty} C_n\left(\frac{\hbar}{r}\right) J^{a_1 b_1} \dots J^{a_n b_n}\, \partial_{a_1} \dots \partial_{a_n} f \partial_{b_1} \dots \partial_{b_n} g, \tag{A.82}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the sphere radius and

$$\begin{aligned} J^{ab}(x) &\equiv r^2 \delta^{ab} - x^a x^b + i\, r \varepsilon^{ab}{}_c x^c, \\ C_n\left(\frac{\hbar}{r}\right) &\equiv \frac{\left(\frac{\hbar}{r}\right)^n}{n!\left(1-\frac{\hbar}{r}\right)\dots\left(1-(n-1)\frac{\hbar}{r}\right)}. \end{aligned} \tag{A.83}$$

J^{ab} is covariant under rotations and therefore the $\star$-product is rotationally-invariant. Moreover, since $J^{ab}x_b = 0$ we have that $f \star r = r \star f = rf$, and hence it can be restricted to the sphere. However, we would like to know the expression in terms of intrinsic coordinates on the sphere and not the above embedding Euclidean coordinates. Denoting the restriction $J|_S$ by J obtained by the embedding $\iota : S \hookrightarrow \mathbb{R}^3$, we have

$$J^{AB} = \frac{\partial \sigma^A}{\partial x^c} \frac{\partial \sigma^B}{\partial x^d} J^{cd}, \tag{A.84}$$

where $\sigma^A = \sigma^A(x^1, x^2, x^3)$ are a set of coordinates on sphere. Since we are sitting on a sphere, r is a constant, which we could set to one. We however keep the radius as r to make the formulas general.

We would like to write the star product in the holomorphic polarization. Celestial coordinates provide an appropriate means for doing so. We use the relation between celestial coordinates $(z, \bar{z})$ and Euclidean coordinates (x^1, x^2, x^3) in the north-pole patch $S - \{(0, 0, r)\}$ given by

$$x^1 = \frac{(z+\bar{z})}{1+\bar{z}z}r, \qquad x^2 = \frac{-i(z-\bar{z})}{1+\bar{z}z}r, \qquad x^3 = \frac{-1+\bar{z}z}{1+\bar{z}z}r,$$
$$z = \frac{x^1+ix^2}{r-x^3}, \qquad \bar{z} = \frac{x^1-ix^2}{r-x^3}. \tag{A.85}$$

Using (A.84), we find

$$\begin{aligned} J^{zz} &= 0, & J^{z\bar{z}} &= 0, \\ J^{\bar{z}z} &= (1+\bar{z}z)^2, & J^{\bar{z}\bar{z}} &= 0. \end{aligned} \tag{A.86}$$

Similarly, the relations between celestial coordinates $(z, \bar{z})$ and Euclidean coordinates (x^1, x^2, x^3) in the south-pole patch $S - \{(0, 0, -r)\}$ are

$$z = \frac{x^1+ix^2}{r+x^3}, \qquad \bar{z} = \frac{x^1+ix^2}{r+x^3}. \tag{A.87}$$

which by using (A.84) gives

$$\begin{aligned} J^{zz} &= 0, & J^{z\bar{z}} &= (1+\bar{z}z)^2, \\ J^{\bar{z}z} &= 0, & J^{\bar{z}\bar{z}} &= 0. \end{aligned} \tag{A.88}$$

Hence, the sphere star product in the holomorphic polarization is parameterized by a weight $\lambda = -\frac{r}{\hbar}$ where r is the sphere radius and $\hbar$ is the formal deformation parameter is known to all orders in perturbation theory. It is explicitly given by

$$f \star_H g = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} C_H^{(n)}(f, g), \tag{A.89}$$

Here the label H stands for Holomorphic. The holomorphic deformation cocycles are

$$C_n^H(f, g) := \frac{1}{\prod_{p=0}^{n-1}(r - p\hbar)} J^{A_1B_1} \dots J^{A_nB_n} (\nabla_{A_1} \dots \nabla_{A_n} f)(\nabla_{B_1} \dots \nabla_{B_n} g). \tag{A.90}$$

Here $J^{AB} = q^{AB} - i\epsilon^{AB}$ is the standard Hermitian form on the sphere. Expanding the expression (A.90) in powers of $\hbar$ gives a relation between the $\{C_H^{(n)}\}$ and the $\{C^{(m)}\}$ introduced in (A.60).

It can be expressed in terms of the null complex frame field m^A introduced earlier.

$$J^{AB} = 2\bar{m}^A m^B. \tag{A.91}$$

This and the definition of the spin-raising differential operator $\eth$ given in (A.27) means that we can write the holomorphic star product parametrized by the weight $\lambda := -\frac{r}{\hbar}$ more concisely as

$$f \star_H g = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\bar{\eth}^n f)\,(\eth^n g)}{\prod_{p=0}^{n-1}(\lambda+p)}. \tag{A.92}$$

Using the commutation relations (A.26) we see that the operators $X = \eth$, $Y = -\bar{\eth}$, $H = 2\widehat{s}$, where $\widehat{s}$ is the operator that measures the spin of the observable, form an $\mathfrak{su}(2)$ algebra

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{A.93}$$

It is usually convenient to formalize the construction of the star product as resulting from the composition of the multiplication operator of functions $m : C(S) \times C(S) \to C(S)$ with the deformation operator $\mathcal{F} : C(S) \times C(S) \to C(S) \times C(S)$. The deformation operator encodes the non-triviality of the star product. The star product can therefore be written in an algebraic form as $F \star_H G = m[\mathcal{F}_H(F \otimes G)]$ where

$$\begin{aligned} \mathcal{F}_H &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{Y^n \otimes X^n}{\prod_{p=0}^{n-1}(\lambda+p)} \\ &= 1 + \lambda^{-1} Y \otimes X + \frac{\lambda^{-2}}{2} Y^2 \otimes X^2 + \frac{\lambda^{-3}}{6}(Y^3 \otimes X^3 - 3Y^2 \otimes X^2) + \mathcal{O}(\lambda^{-4}). \end{aligned} \tag{A.94}$$

In this representation, the weight λ is the eigenvalue of $-H$ on the SL(2) module representing the sphere sections [130, equation (6)].

Other equivalent star-product which are covariant under the rotation group can be obtained after a reparameterization $F \to \theta[F]$ where $\theta = 1 + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \theta_n$ and θ_n is a differential operator of order n which is invariant under rotation[31] and invertible. The new star product defined by θ from the holomorphic product is $\theta[F \star_\theta G] = \theta[F] \star \theta[G]$ which in terms of the operation $\mathcal{F}$ means

$$f \star_\theta g = m[\mathcal{F}_\theta(f \otimes g)], \qquad \mathcal{F}_\theta = (\Delta\theta)^{-1} \mathcal{F}_H(\theta \otimes \theta), \tag{A.95}$$

where Δ is the coproduct[32] of differential operators $\Delta(\nabla_A) = \nabla_A \otimes \mathbb{1} + \mathbb{1} \otimes \nabla_A$ and m is the multiplication of functions. An interesting subclass of $\star$-products are the parity symmetric ones which are such that $C_n(f, g) = (-1)^n C_n(g, f)$. The parity symmetric star product are such that the star commutator $[f, g] = f \star g - g \star f$ and the star-symmetrized product $f \circ g = \frac{1}{2}(f \star g + g \star f)$ only involves even powers of $\hbar$. We have seen in section A.5 that the $6j$ star product arising from the fuzzy sphere is parity symmetric. We can now evaluate the holomorphic star product on a basis. Using that

$$\bar{\eth}^n Y_\alpha = (-1)^n [A]_n Y_\beta^{-n}, \qquad \eth^n Y_\beta = [B]_n Y_\beta^n. \tag{A.96}$$

[31]This means that θ is a function of the Laplacian operator

$$\nabla^2 := q^{AB} \nabla_A \nabla_B = \frac{1}{2}(\nabla\bar{\nabla} + \bar{\nabla}\nabla),$$

on the sphere.

[32]The coproduct is a morphism of differential operators $\Delta(D_1 D_2) = \Delta(D_1)\Delta(D_2)$ such that $Dm(F \otimes G) = m(\Delta(D) F \otimes G)$ for and differential operator D.

We find that the star product

$$\int_S (Y_\alpha \star_\theta Y_\beta) Y_\gamma = \frac{\theta(A)\theta(B)}{\theta(C)} \sum_{n=0}^{\infty} \frac{[A]_n [B]_n}{n! \prod_{p=0}^{n-1}(\lambda+p)} \int_S (Y_\alpha^{-n} Y_\beta^{n} Y_\gamma),$$
$$= \begin{pmatrix} A & B & C \\ a & b & c \end{pmatrix} \frac{\theta(A)\theta(B)}{\theta(C)} \sum_{n=0}^{\infty} \frac{\Gamma(\lambda)}{n!\Gamma(\lambda+n)} [A]_n [B]_n \begin{pmatrix} A & B & C \\ n & -n & 0 \end{pmatrix}, \quad \text{(A.97)}$$

where we use equation (A.36) in the last equality. We see that, quite remarkably, one recovers the Nomura expression (A.44) provided we chose

$$\lambda = N+1, \qquad \theta(A) = \sqrt{\frac{\Gamma(\lambda+A)}{\Gamma(\lambda-A)}}. \quad \text{(A.98)}$$

Note that the infinite sum truncates since $[A]_n = 0$ when $A > n$.

It is curious that the specific choice (A.98) for $\theta(A)$ is needed to reproduce the $\star$-product derived from the $6j$ symbol described in section A.3. We have seen in (A.76) that the $6j$ star product is parity symmetric to all orders in $\hbar$. It would be interesting to have a different star product derivation of this property. More generally, it would be really interesting to have an independent argument for choosing the form (A.98) for θ and relating it to the $6j$-symbol. Such an argument would give us an independent derivation of the Nomura identity [55]. Note that the star product algebra described here appears in the physics literature as a higher spin symmetry algebra called $hs[\lambda]$ [101, 132–134].

B Structure constants for $\mathfrak{diff}(S^2)$

In this appendix we show how to parameterize the $\mathfrak{diff}(S^2)$ Lie algebra and structure constants in terms of two functions on the sphere, and also derive explicit expressions for the structure constants in a spherical harmonic basis. Working in this basis ensures that all generators correspond to smooth vector fields on the sphere. The results on the explicit form of these structure constants given in equations (B.26), (B.27), and (B.28) are novel (although see [101] for some partial results). In particular, they differ from treatments such as [135] based on commuting holomorphic and antiholomorphic subalgebras, most of whose generators possess singularities on the sphere.

We begin by fixing a round metric q_{AB} on the sphere, which also determines a preferred volume form ϵ_{AB}, as in equations (A.19) and (A.20). We can then decompose an arbitrary vector ξ^A on the sphere into a curl and a gradient according to the Hodge decomposition,

$$\xi^A = \epsilon^{BA}\nabla_B \phi_\xi + q^{BA}\nabla_B \psi_\xi. \quad \text{(B.1)}$$

Given an arbitrary vector ξ^A, its constituent functions (ϕ_ξ, ψ_ξ) can be determined according to the equations

$$\psi_\xi = \frac{1}{\nabla^2}\nabla_A \xi^A, \quad \text{(B.2)}$$
$$\phi_\xi = \frac{1}{\nabla^2}\nabla_A \left(\epsilon^A{}_B \xi^B\right). \quad \text{(B.3)}$$

Here, $\frac{1}{\nabla^2}$ is the operator that inverts the Laplacian $\nabla^2 = q^{AB}\nabla_A\nabla_B$ associated with the metric q_{AB} on the sphere. Since the constant functions lie in the kernel of ∇^2, the inverse $\frac{1}{\nabla^2}$ is defined to produce a function with no constant piece, which we take to mean a function that integrates to zero with respect to the volume form ϵ_{AB}. In the spherical harmonic basis Y_{lm}, this space of functions is spanned by all harmonics with $l \geq 1$.

We can therefore decompose the space of all vectors on the sphere into subspaces of pure curl and pure gradient vectors:

$$B^A_\phi = \epsilon^{BA}\nabla_B\phi, \tag{B.4}$$

$$E^A_\phi = q^{BA}\nabla_B\phi. \tag{B.5}$$

We will call the B^A_ϕ vectors "magnetic" and the E^A_ϕ vectors "electric", in line with their properties under parity transformations [116]. Note that the tensor $\epsilon^A{}_B$ defines an integrable almost complex structure that maps magnetic and electric vectors into each other according to

$$\epsilon \cdot B_\phi = E_\phi, \qquad \epsilon \cdot E_\phi = -B_\phi. \tag{B.6}$$

The effect of multiplying the generators B^A_ϕ and E^A_ϕ by a scalar function can be expressed in terms of the antisymmetric and symmetric brackets defined in (A.6), (A.9):

Lemma B.1. *Multiplication by a scalar function λ acts on the vectors B_ϕ, E_ϕ according to*

$$\lambda B_\phi = B_{\frac{1}{\nabla^2}\left(\langle\lambda,\phi\rangle + \lambda\nabla^2\phi\right)} - E_{\frac{1}{\nabla^2}\{\lambda,\phi\}}, \tag{B.7}$$

$$\lambda E_\phi = B_{\frac{1}{\nabla^2}\{\lambda,\phi\}} + E_{\frac{1}{\nabla^2}\left(\langle\lambda,\phi\rangle + \lambda\nabla^2\phi\right)}. \tag{B.8}$$

Proof. These identities come from applying (B.2) and (B.3) to extract the electric and magnetic potentials of the resulting vector:

$$\nabla_A(\lambda\epsilon^A{}_B B^B_\phi) = \nabla_A(\lambda\nabla^A\phi) = \langle\lambda,\phi\rangle + \lambda\nabla^2\phi, \tag{B.9}$$

$$\nabla_A(\lambda B^A_\phi) = \epsilon^{BA}\nabla_B\phi\nabla_A\lambda = -\{\lambda,\phi\}, \tag{B.10}$$

which then leads to (B.7). An analogous computation leads to (B.8), which can also be obtained by acting with the complex structure ϵ on (B.7) and using (B.6). □

Before computing the Lie brackets of these vector fields, we will need some identities satisfied by the brackets $\langle\cdot,\cdot\rangle$ and $\{\cdot,\cdot\}$:

Lemma B.2. *The brackets $\langle\cdot,\cdot\rangle$ and $\{\cdot,\cdot\}$ satisfy*

$$\{\phi,\{\psi,\lambda\}\} - \{\psi,\{\phi,\lambda\}\} = \{\{\phi,\psi\},\lambda\}, \tag{B.11}$$

$$\{\phi,\langle\psi,\lambda\rangle\} - \langle\psi,\{\phi,\lambda\}\rangle = -\{\langle\phi,\psi\rangle,\lambda\} + \{\phi,\lambda\}\nabla^2\psi, \tag{B.12}$$

$$\langle\phi,\langle\psi,\lambda\rangle\rangle - \langle\psi,\langle\phi,\lambda\rangle\rangle = \{\{\phi,\psi\},\lambda\} + \langle\phi,\lambda\rangle\nabla^2\psi - \langle\psi,\lambda\rangle\nabla^2\phi. \tag{B.13}$$

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Proof. We handle each case separately:

- Proof of (B.11): This identity is simply the statement of the Jacobi identity for the Poisson bracket $\{\cdot,\cdot\}$. We can explicitly check it as follows:

$$\begin{aligned}\{\{\phi,\psi\},\lambda\} &= \epsilon^{AB}\epsilon^{CD}\nabla_C(\nabla_A\phi\nabla_B\psi)\nabla_D\lambda \\ &= \epsilon^{AB}\epsilon^{CD}(\nabla_C\nabla_A\phi\nabla_B\psi\nabla_D\lambda + \nabla_A\phi\nabla_C\nabla_B\psi\nabla_D\lambda) \\ &= \epsilon^{AB}\epsilon^{CD}(\nabla_A\phi\nabla_B(\nabla_C\psi\nabla_D\lambda) - \nabla_A\phi\nabla_C\psi\nabla_B\nabla_D\lambda \\ &\qquad + \nabla_B\psi\nabla_A(\nabla_C\phi\nabla_D\lambda) - \nabla_B\psi\nabla_C\phi\nabla_A\nabla_D\lambda) \\ &= \{\phi,\{\psi,\lambda\}\} - \{\psi,\{\phi,\lambda\}\}.\end{aligned} \tag{B.14}$$

- Proof of (B.12): This can be derived straightforwardly by first evaluating the Lie derivative of ϵ^{AB} with respect to the vector E_ψ. Since the resulting tensor remains antisymmetric in its indices, we must have $\mathcal{L}_{E_\psi}\epsilon^{AB} = \alpha\epsilon^{AB}$ with $\alpha = \frac{1}{2}\epsilon_{AB}\mathcal{L}_{E_\psi}\epsilon^{AB} = -\frac{1}{2}\epsilon^{AB}\mathcal{L}_{E_\psi}\epsilon_{AB}$. Then since $\mathcal{L}_{E_\psi}\epsilon_{AB} = (\mathrm{div}K_\psi)\epsilon_{AB} = \nabla^2\psi\,\epsilon_{AB}$, we conclude that $\mathcal{L}_{E_\psi}\epsilon^{AB} = -\nabla^2\psi\,\epsilon^{AB}$. We then evaluate the nested brackets

$$\begin{aligned}\langle\psi,\{\phi,\lambda\}\rangle &= \mathcal{L}_{E_\psi}(\epsilon^{AB}\nabla_A\phi\nabla_B\lambda) \\ &= (\mathcal{L}_{E_\psi}\epsilon^{AB})\nabla_A\phi\nabla_B\lambda + \epsilon^{AB}(\nabla_A(\mathcal{L}_{E_\psi}\phi)\nabla_B\lambda + \nabla_A\phi\nabla_B(\mathcal{L}_{E_\psi}\lambda)) \\ &= -\nabla^2\psi\,\{\phi,\lambda\} + \{\langle\psi,\phi\rangle,\lambda\} + \{\phi,\langle\psi,\lambda\rangle\}.\end{aligned} \tag{B.15}$$

- Proof of (B.13): To derive this, we first work out an expression for $\nabla^A\nabla^B\phi\nabla_A\psi\nabla_B\lambda$ in terms of the symmetric brackets:

$$\begin{aligned}\nabla^A\nabla^B\phi\nabla_A\psi\nabla_B\lambda &= \nabla^A(\nabla^B\phi\nabla_B\lambda)\nabla_A\psi - \nabla^B\phi\nabla_A\psi\nabla^A\nabla_B\lambda \\ &= \langle\langle\phi,\lambda\rangle,\psi\rangle - \langle\phi,\langle\psi,\lambda\rangle\rangle + \nabla^B\phi\nabla_B\nabla_A\psi\nabla^A\lambda \\ &= \langle\langle\phi,\lambda\rangle,\psi\rangle - \langle\phi,\langle\psi,\lambda\rangle\rangle + \langle\langle\phi,\psi\rangle,\lambda\rangle - \nabla^A\nabla^B\phi\nabla_B\psi\nabla_A\lambda,\end{aligned} \tag{B.16}$$

  and hence

$$\nabla^A\nabla^B\phi\nabla_A\psi\nabla_B\lambda = \frac{1}{2}\Big(\langle\langle\phi,\lambda\rangle,\psi\rangle + \langle\langle\phi,\psi\rangle,\lambda\rangle - \langle\phi,\langle\psi,\lambda\rangle\rangle\Big). \tag{B.17}$$

  We can then evaluate the nested bracket $\{\{\phi,\psi\},\lambda\}$ by first exploiting the relation $\epsilon^{AB}\epsilon^{CD} = q^{AC}q^{BD} - q^{AD}q^{BC}$, and then applying (B.17) to obtain

$$\begin{aligned}\{\{\phi,\psi\},\lambda\} &= \epsilon^{AB}\epsilon^{CD}\nabla_A(\nabla_C\phi\nabla_D\psi)\nabla_B\lambda \\ &= (q^{AC}q^{BD} - q^{AD}q^{BC})\nabla_A(\nabla_C\phi\nabla_D\psi)\nabla_B\lambda \\ &= \nabla^2\phi\langle\psi,\lambda\rangle + \nabla^A\phi\nabla_A\nabla^B\psi\nabla_B\lambda - \nabla_A\nabla^B\phi\nabla^A\psi\nabla_B\lambda - \langle\phi,\lambda\rangle\nabla^2\psi \\ &= \langle\phi,\langle\psi,\lambda\rangle\rangle - \langle\psi,\langle\phi,\lambda\rangle\rangle + \langle\psi,\lambda\rangle\nabla^2\phi - \langle\phi,\lambda\rangle\nabla^2\psi.\end{aligned} \tag{B.18}$$

□

The Lie brackets of B_ϕ and E_ψ can now be computed by examining how these vectors act on scalar functions. Any vector field acts as a derivation on the space of functions, and these derivations can be expressed in terms of the brackets,

$$B_\phi(\lambda) = B_\phi^A\nabla_A\lambda = \{\phi,\lambda\}, \tag{B.19}$$

$$E_\phi(\lambda) = E_\phi^A\nabla_A\lambda = \langle\phi,\lambda\rangle. \tag{B.20}$$

The Lie bracket of two vector fields is then given by the commutator of the two associated derivations acting on a function. The left hand side of the bracket identities in Lemma B.2 expresses the three options for these commutators, and the expressions on the right hand side give the equivalent derivation acting on the function λ. Hence, these identities immediately allow us to write down expressions for the Lie brackets of the vector fields:

$$[B_\phi, B_\psi] = B_{\{\phi,\psi\}}, \tag{B.21}$$
$$[B_\phi, E_\psi] = -B_{\langle\phi,\psi\rangle} + \nabla^2\psi\, B_\phi, \tag{B.22}$$
$$[E_\phi, E_\psi] = B_{\{\phi,\psi\}} + \nabla^2\psi\, E_\phi - \nabla^2\phi\, E_\psi. \tag{B.23}$$

Then using the identities in Lemma B.1 for multiplication of a vector by a scalar function, we can reduce equations (B.22) and (B.23) to

$$[B_\phi, E_\psi] = E_{\frac{1}{\nabla^2}\{\phi,\nabla^2\psi\}} - B_{\langle\phi,\psi\rangle-\frac{1}{\nabla^2}\left(\langle\phi,\nabla^2\psi\rangle+\nabla^2\phi\nabla^2\psi\right)}, \tag{B.24}$$
$$[E_\phi, E_\psi] = B_{\{\phi,\psi\}-\frac{1}{\nabla^2}\left(\{\nabla^2\phi,\psi\}+\{\phi,\nabla^2\psi\}\right)} + E_{\frac{1}{\nabla^2}\left(\langle\phi,\nabla^2\psi\rangle-\langle\nabla^2\phi,\psi\rangle\right)}. \tag{B.25}$$

We can now explicitly parameterize the generators and structure constants by decomposing the potentials (ϕ,ψ) in a spherical harmonic basis. Following the conventions and notation of appendix A.1, we let Y_α denote a spherical harmonic with $\alpha = (A,a)$ denoting its total angular momentum A and magnetic quantum number a, with $-A \le a \le A$. We then employ the shorthand $B_\alpha = B_{Y_\alpha}$, $E_\alpha = E_{Y_\alpha}$ to denote the generators in the spherical harmonic basis. Since the Lie brackets (B.21), (B.24), and (B.25) are expressed in terms of the Poisson bracket, symmetric bracket, and product of functions, we can express the right hand sides of these equations using the structure constants for these operations given in equations (A.7), (A.10), and (A.4). Recalling also that $\nabla^2 Y_\alpha = -A^{(1)}Y_\alpha$, $\frac{1}{\nabla^2}Y_\alpha = \frac{-1}{A^{(1)}}Y_\alpha$ with $A^{(1)} = A(A+1)$ defined in (A.13), the structure constants are immediately found to be

$$[B_\alpha, B_\beta] = C_{\alpha\beta}{}^{\gamma} B_\gamma, \tag{B.26}$$
$$[B_\alpha, E_\beta] = \frac{B^{(1)}}{C^{(1)}} C_{\alpha\beta}{}^{\gamma} E_\gamma + \frac{1}{C^{(1)}}\left((B^{(1)} - C^{(1)})G_{\alpha\beta}{}^{\gamma} - A^{(1)}B^{(1)}E_{\alpha\beta}{}^{\gamma}\right) B_\gamma, \tag{B.27}$$
$$[E_\alpha, E_\beta] = \frac{C^{(1)} - A^{(1)} - B^{(1)}}{C^{(1)}} C_{\alpha\beta}{}^{\gamma} B_\gamma + \frac{B^{(1)} - A^{(1)}}{C^{(1)}} G_{\alpha\beta}{}^{\gamma} E_\gamma. \tag{B.28}$$

Restricting these relations to $A = B = 1$, we find that the algebra of the six generators (B_{1a}, E_{1a}) closes, and reduces to

$$[B_{1a}, B_{1b}] = C_{(1a)(1b)}{}^{(1c)} B_{1c}, \tag{B.29}$$
$$[B_{1a}, E_{1b}] = C_{(1a)(1b)}{}^{(1c)} E_{1c}, \tag{B.30}$$
$$[E_{1a}, E_{1b}] = -C_{(1a)(1b)}{}^{(1c)} B_{1c}. \tag{B.31}$$

This algebra is readily recognized as the $\mathfrak{sl}(2,\mathbb{C})$ subalgebra of $\mathfrak{diff}(S^2)$ consisting of the six globally defined conformal Killing vectors of S^2.

Finally, it is interesting to note that we can form a new holomorphic basis for this algebra by forming combinations of B_α and E_α that are eigenvectors for the complex structure $\epsilon^A{}_B$.

These are given by $F^+_\alpha = E_\alpha + iB_\alpha$ and $F^-_\alpha = -E_\alpha + iB_\alpha$, which satisfy $\epsilon \cdot F^\pm_\alpha = \pm i F^\pm_\alpha$. The F^+_α form a subalgebra within (the complexification of) the full diffeomorphism algebra, as do F^-_α, which follows from the fact that the complex structure $\epsilon^A{}_B$ is integrable.[33]

The algebra in the (F^+_α, F^-_α) basis can be computed following a similar method as in the (B_α, E_α) basis. Given an arbitrary scalar function ϕ, we can construct holomorphic and antiholomorphic vector fields $F^\pm_\phi = \pm E_\phi + iB_\phi$. These vector fields act on functions as derivations, and this action can be equivalently expressed in terms of two new brackets for scalar functions $(\cdot,\cdot)_\pm$, defined by

$$(\phi, \psi)_\pm = \pm\langle \phi, \psi\rangle + i\{\phi, \psi\}. \tag{B.32}$$

Note that these are neither symmetric nor antisymmetric, but instead satisfy

$$(\phi, \psi)_+ = -(\psi, \phi)_-. \tag{B.33}$$

The action of the vector fields on a function λ can then be expressed as

$$F^\pm_\phi(\lambda) = (\phi, \lambda)_\pm. \tag{B.34}$$

Next we note that the multiplication of $F^\pm_\phi$ by a scalar function can be derived from the relations (B.7), (B.8), and leads to

$$\lambda F^\pm_\phi = F^\pm_{\frac{1}{\nabla^2}\left(\mp(\lambda,\phi)_\mp + \lambda\nabla^2\phi\right)}. \tag{B.35}$$

Interestingly, unlike the vectors B_ϕ, E_ϕ, scalar multiplication maps the set of vectors F^+_ϕ into themselves, and similarly for F^-_ϕ.

The Lie bracket of these vector fields is most straightforwardly obtained by computing the nested relations of the $(\cdot,\cdot)_\pm$ brackets. These follow from the relations in Lemma B.2, and lead to

$$\Big(\phi, (\psi,\lambda)_+\Big)_+ - \Big(\psi, (\phi,\lambda)_+\Big)_+ = \nabla^2\psi(\phi,\lambda)_+ - \nabla^2\phi(\psi,\lambda)_+, \tag{B.36}$$

$$\Big(\phi, (\psi,\lambda)_-\Big)_- - \Big(\psi, (\phi,\lambda)_-\Big)_- = -\nabla^2\psi(\phi,\lambda)_- + \nabla^2\phi(\psi,\lambda)_-, \tag{B.37}$$

$$\Big(\phi, (\psi,\lambda)_-\Big)_+ - \Big(\psi, (\phi,\lambda)_+\Big)_- = \Big((\phi,\psi)_+,\lambda\Big)_+ - \nabla^2\psi(\phi,\lambda)_+ + \Big((\phi,\psi)_+,\lambda\Big)_- - \nabla^2\phi(\psi,\lambda)_-. \tag{B.38}$$

[33] Recall that an integrable complex structure is one in which the Nijenhuis tensor $N^A{}_{BC}$ vanishes. This tensor is defined by the relation

$$N(X,Y) = [X,Y] + \epsilon\cdot([\epsilon\cdot X, Y] + [X, \epsilon\cdot Y]) - [\epsilon\cdot X, \epsilon\cdot Y],$$

where X, Y are vectors, and the brackets are vector field Lie brackets. The vanishing of this tensor implies that eigenvectors $X^\pm$ of the complex structure ϵ form a subalgebra, i.e.

$$\epsilon[X^\pm, Y^\pm] = \pm i[X^\pm, Y^\pm].$$

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These then imply relations for the brackets of the $F_\phi^\pm$ vector fields, which can now be expressed as

$$[F_\phi^+, F_\psi^+] = \nabla^2\psi\, F_\phi^+ - \nabla^2\phi\, F_\psi^+, \tag{B.39}$$
$$[F_\phi^-, F_\psi^-] = -\nabla^2\psi\, F_\phi^- + \nabla^2\phi\, F_\psi^-, \tag{B.40}$$
$$[F_\phi^+, F_\psi^-] = F^+_{(\phi,\psi)_+} - \nabla^2\psi\, F_\phi^+ + F^-_{(\phi,\psi)_+} - \nabla^2\phi\, F_\psi^-. \tag{B.41}$$

We then can apply the formula (B.35) for scalar multiplication acting on the vectors to derive

$$[F_\phi^+, F_\psi^+] = F^+_{\frac{1}{\nabla^2}\left((\phi,\nabla^2\psi)_+ + (\nabla^2\phi,\psi)_-\right)}, \tag{B.42}$$
$$[F_\phi^-, F_\psi^-] = F^-_{\frac{1}{\nabla^2}\left((\phi,\nabla^2\psi)_- + (\nabla^2\phi,\psi)_+\right)}, \tag{B.43}$$
$$[F_\phi^+, F_\psi^-] = F^+_{(\phi,\psi)_+ - \frac{1}{\nabla^2}\left((\phi,\nabla^2\psi)_+ + \nabla^2\phi\nabla^2\psi\right)} + F^-_{(\phi,\psi)_+ - \frac{1}{\nabla^2}\left((\nabla^2\phi,\psi)_+ + \nabla^2\phi\nabla^2\psi\right)}. \tag{B.44}$$

Note that these relations explicitly verify that the F_ϕ^+ vectors form a subalgebra, as do the F_ϕ^- vectors, but these two subalgebras do not commute.

From these relations, the structure constants in the spherical harmonic basis follow straightforwardly. Defining the structure constants for the $(\cdot,\cdot)_\pm$ brackets according to $(Y_\alpha, Y_\beta)_\pm = {}^\pm H_{\alpha\beta}{}^\gamma Y_\gamma$, we see that they are related to $C_{\alpha\beta}{}^\gamma$ and $G_{\alpha\beta}{}^\gamma$ via

$${}^\pm H_{\alpha\beta}{}^\gamma = \pm G_{\alpha\beta}{}^\gamma + iC_{\alpha\beta}{}^\gamma. \tag{B.45}$$

Using the expressions (A.7) and (A.10) for the G and C structure constants and lowering an index with the metric $\delta_{\alpha\beta}$ defined by (A.2), we can express ${}^\pm H_{\alpha\beta\gamma}$ explicitly in terms of $3j$-symbols according to

$${}^\pm H_{\alpha\beta\gamma} = \mp[A]_1[B]_1 \begin{pmatrix} A & B & C \\ a & b & c \end{pmatrix} \begin{pmatrix} A & B & C \\ \mp 1 & \pm 1 & 0 \end{pmatrix}. \tag{B.46}$$

In terms of these, we can immediately translate the expressions (B.42), (B.43), and (B.44) into formulas for the structure constants in the spherical harmonic basis:

$$[F_\alpha^+, F_\beta^+] = \frac{1}{C^{(1)}}\left(B^{(1)}\, {}^+H_{\alpha\beta}{}^\gamma + A^{(1)}\, {}^-H_{\alpha\beta}{}^\gamma\right) F_\gamma^+, \tag{B.47}$$
$$[F_\alpha^-, F_\beta^-] = \frac{1}{C^{(1)}}\left(B^{(1)}\, {}^-H_{\alpha\beta}{}^\gamma + A^{(1)}\, {}^+H_{\alpha\beta}{}^\gamma\right) F_\gamma^-, \tag{B.48}$$
$$\begin{aligned}[F_\alpha^+, F_\beta^-] = &\frac{1}{C^{(1)}}\left(\left(C^{(1)} - B^{(1)}\right) {}^+H_{\alpha\beta}{}^\gamma + A^{(1)}B^{(1)}E_{\alpha\beta}{}^\gamma\right) F_\gamma^+ \\ &+ \frac{1}{C^{(1)}}\left(\left(C^{(1)} - A^{(1)}\right) {}^+H_{\alpha\beta}{}^\gamma + A^{(1)}B^{(1)}E_{\alpha\beta}{}^\gamma\right) F_\gamma^-.\end{aligned} \tag{B.49}$$

C Algebra deformation

The deformation of symmetry algebras has a long and fruitful history in physics: deformation of the abelian phase-space algebra into a centrally extended algebra with physical parameter $\hbar$ is at the core of the discovery of quantum mechanics. Another type of physical deformation involves deforming a semi-direct product algebra into a semi-simple algebra which is much more regular; a standard mathematical reference on the theory of deformation of Poisson algebras, Lie algebras, and algebras is [69]. Two key examples are

1. the deformation of the Poincaré group into the de Sitter group. The former can be obtained from the latter by contraction [74]. The deformation parameter is the cosmological constant Λ.

2. the deformation of the Galilean group into the Poincaré algebra. Again, the former can be obtained from the latter by contraction [74]. The deformation parameter is the inverse of the speed of light c.

The corner symmetry algebra $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}$ is a semi-direct product. It is therefore natural to look for a deformation of $\mathfrak{g}_{\mathfrak{sl}(2,\mathbb{R})}$ which is semi-simple. In this paper, we focused on the deformation of $\mathfrak{sdiff}(S)$ and of the centralizer algebras $\mathfrak{c}_{\mathbb{R}} = \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathbb{R}^S$ and $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})} = \mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$.

What we propose here is a deformation of $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}$, denoted $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}[\lambda]$, which will prove to be invaluable at the quantum level. The deformation parameter is a real parameter λ. It is a new constant that still needs to be interpreted in a physical term as a constant of nature and which, we hope, could be promoted to the same status as $\hbar$, Λ and c have reached. One proposal for such a dimensionless deformation parameter is that it is given as a measure of the ratio of the Planck scale over the cosmological scale, which is the only universal dimensionless number we naturally encounter in quantum gravity. In our analysis and for irreducible representations we have seen through the Casimir matching procedure described in section 4 that $\lambda \sim \frac{1}{N^2}$ is related to the quantum of the area associated with the corner sphere.

Deformations of Poisson algebras arise as follows. Let M be a Poisson manifold, meaning it is equipped with a bilinear map on functions $\{\cdot,\cdot\} : C(M) \times C(M) \to C(M)$,[34] which satisfies antisymmetry, Leibniz properties, and the Jacobi identity. The deformation of a Poisson algebra is characterized by a Poisson two-cocycle, that is, a map $D : C(M) \times C(M) \to C(M)$ which is a skew-symmetric bi-derivation and satisfies the Poisson 2-cocycle identity. Explicitly this means that

$$\begin{aligned} D(f,g) &= -D(g,f), \\ D(f,gh) &= D(f,g)h + gD(f,h), \\ \{f,D(g,h)\} + \{g,D(h,f)\} + \{h,D(f,g)\} &= -[D(f,\{g,h\}) + D(g,\{h,f\}) + D(h,\{f,g\})]. \end{aligned} \tag{C.1}$$

These identities simply imply that the deformed bracket $\{f,g\}_\lambda := \{f,g\} + \lambda D(f,g)$ satisfies the Jacobi identity to first order in λ. A Poisson 2-cocycle is trivial if it can be written in terms of a 1-cocycle $D_1 : C(M) \to C(M)$ as

$$D(f,g) = D_1(\{f,g\}) - \{D_1(f),g\} - \{f,D_1(g)\}, \tag{C.2}$$

where $D_1(f)$ is a differential operator. In such a case, the Poisson deformation is trivial and simply amounts to a redefinition of the variables $f \to f - \lambda D_1(f)$. Note that the last

[34] Here, $C(M)$ is the space of smooth functions on M.

identity in (C.1) and the 1-cocycle deformation (C.2) can be written as $\delta D(f,g,h)=0$ and $D=\delta D_1$, respectively, where δ is the Chevalley coboundary operator [136].

The existence of deformation for $\mathfrak{sdiff}(S)$ follows straightforwardly from the construction of the star product operation done in sections A.4 and A.6: From the analysis done there, the fact that $C^{(2)}$ is symmetric that $C^{(3)}$ is skew-symmetric and the proof of associativity of the star product, we know that $C^{(3)}$ is a Poisson cocycle for the sphere Poisson bracket $\{Y_\alpha,Y_\beta\}_\epsilon = C_{\alpha\beta}{}^\gamma Y_\gamma$. In particular if we define $C^{(3)}(Y_\alpha,Y_\beta):=D_{\alpha\beta}{}^\gamma$ this means that

$$C_{\alpha\delta}{}^\sigma D_{\beta\gamma}{}^\delta + D_{\alpha\delta}{}^\sigma C_{\beta\gamma}{}^\delta + \mathrm{cycl}[\alpha,\beta,\gamma]=0, \tag{C.3}$$

where $\mathrm{cycl}[\alpha,\beta,\gamma]$ means that we perform a cyclic permutation of the indices.

To obtain a nontrivial Poisson deformation of $\mathfrak{sdiff}(S)$, one first uses the fact that D is a bi-derivation. This implies that the knowledge of D on arbitrary functions is entirely determined by the knowledge of $D(J_\alpha,J_\beta)$, where, following the notation (2.30), $J_\alpha=J[Y_\alpha]$ is a basis for $C(M)$, since $D(f,g)=\sum_{\alpha,\beta}\frac{\partial f}{\partial J_\alpha}\frac{\partial g}{\partial J_\beta}D(J_\alpha,J_\beta)$. The identity (C.3) then implies that there exists a Poisson deformation of $\mathfrak{sdiff}(S)$ Poisson algebra simply given by

$$D(J_\alpha,J_\beta)=J[C^{(3)}(Y_\alpha,Y_\beta)]=D_{\alpha\beta}{}^\gamma J_\gamma. \tag{C.4}$$

The deformation of $\mathfrak{c}_\mathbb{R}(S)$ goes along the same line. From the differentiability property, one learns that it is enough to give the prescription on the generators (J_α,N_α) with bracket given in (2.33). One chooses

$$\begin{aligned}
D(J_\alpha,J_\beta)&=D_{\alpha\beta}{}^\gamma J_\gamma,\\
D(J_\alpha,N_\beta)&=D_{\alpha\beta}{}^\gamma N_\gamma,\\
D(N_\alpha,N_\beta)&=-C_{\alpha\beta}{}^\gamma J_\gamma.
\end{aligned} \tag{C.5}$$

To verify the cocycle property $\delta D(P_0,P_1,P_2)=0$ with P_i denoting the arbitrary Lie-algebra generators and the coboundary δ acting as

$$\begin{aligned}
\delta D(P_0,P_1,P_2):=&\{P_0,D(P_1,P_2)\}-\{P_1,D(P_0,P_2)\}+\{P_2,D(P_0,P_1)\}\\
&-D(\{P_0,P_1\},P_2)+D(\{P_0,P_2\},P_1)-D(\{P_1,P_2\},P_0),
\end{aligned} \tag{C.6}$$

we need to investigate 4 different cases depending on whether the argument is: $(J_\alpha,J_\beta,J_\gamma)$, $(J_\alpha,J_\beta,N_\gamma)$,$(J_\alpha,N_\beta,N_\gamma)$ or $(N_\alpha,N_\beta,N_\gamma)$. The proof goes by inspection of each case separately. The cocycle identity (C.3) proves the first two cases, and the Jacobi identity of the sphere bracket proves the following two cases:

$$\begin{aligned}
\delta D(J_\alpha,J_\beta,J_\gamma)&=\left(C_{\alpha\delta}{}^\sigma D_{\beta\gamma}{}^\delta + D_{\alpha\delta}{}^\sigma C_{\beta\gamma}{}^\delta + \mathrm{cycl}[\alpha,\beta,\gamma]\right)J_\sigma=0,\\
\delta D(J_\alpha,J_\beta,N_\gamma)&=\left(C_{\alpha\delta}{}^\sigma D_{\beta\gamma}{}^\delta + D_{\alpha\delta}{}^\sigma C_{\beta\gamma}{}^\delta + \mathrm{cycl}[\alpha,\beta,\gamma]\right)N_\sigma=0,\\
\delta D(J_\alpha,N_\beta,N_\gamma)&=\left(C_{\alpha\delta}{}^\sigma C_{\beta\gamma}{}^\delta + \mathrm{cycl}[\alpha,\beta,\gamma]\right)J_\sigma=0,\\
\delta D(N_\alpha,N_\beta,N_\gamma)&=\left(C_{\alpha\delta}{}^\sigma C_{\beta\gamma}{}^\delta + \mathrm{cycl}[\alpha,\beta,\gamma]\right)N_\sigma=0.
\end{aligned} \tag{C.7}$$

We can finally describe the deformation for the algebra $\mathfrak{c}_{\mathfrak{sl}(2,\mathbb{R})}$ given by (2.31). This deformation involves the coefficient $C^{(2)}$ and we denote $C^{(2)}(Y_\alpha,Y_\beta):=F_{\alpha\beta}{}^\gamma Y_\gamma$ which is symmetric under the exchange (α,β). ${F_{\alpha\beta}}^{\gamma}$ is essentially the N^{-2} term in the expansion of ${\widehat{E}_{\alpha\beta}}^{\gamma}$. From the associativity of the star product, we obtain that

$$Y_\alpha \circ (Y_\gamma \circ Y_\beta) - (Y_\alpha \circ Y_\gamma) \circ Y_\beta = [Y_\gamma, [Y_\alpha, Y_\beta]], \tag{C.8}$$

where $[f,g] = f\star g - g\star f$ and $f\circ g = \frac{1}{2}(f\star g + g\star f)$. Expanding this identity at second order we obtain that

$$({E_{\beta\gamma}}^{\delta}{F_{\alpha\delta}}^{\sigma} + {F_{\beta\gamma}}^{\delta}{E_{\alpha\delta}}^{\sigma}) - ({E_{\gamma\alpha}}^{\delta}{F_{\beta\delta}}^{\sigma} + {F_{\gamma\alpha}}^{\delta}{E_{\beta\delta}}^{\sigma}) = {C_{\alpha\beta}}^{\delta}{C_{\gamma\delta}}^{\sigma}. \tag{C.9}$$

The deformation cocycle is now taken to be

$$\begin{aligned} D(J_\alpha, J_\beta) &= {D_{\alpha\beta}}^{\gamma} J_\gamma, \\ D(J_\alpha, N_{a\beta}) &= {D_{\alpha\beta}}^{\gamma} N_{a\gamma}, \\ D(N_{a\alpha}, N_{b\beta}) &= {\varepsilon_{ab}}^{c}{F_{\alpha\beta}}^{\gamma} N_{c\gamma} - \eta_{ab}{C_{\alpha\beta}}^{\gamma} J_\gamma. \end{aligned} \tag{C.10}$$

One sees that this deformation restricts to the previous one if one chooses $N_\alpha = N_{1\alpha}$.

The proof for the cocycle identities follows similarly. We need to look at them case by case. The proof or the combinations $(J_\alpha, J_\beta, J_\gamma)$, $(J_\alpha, J_\beta, N_{c\gamma})$ is the same as before. For the combination $(N_{a\alpha}, N_{b\beta}, N_{c\gamma})$ we use that

$$\begin{aligned} D(N_{a\alpha}, \{N_{b\beta}, N_{c\gamma}\}) &= {\varepsilon_{bc}}^{d}{\varepsilon_{ad}}^{s}{E_{\beta\gamma}}^{\delta}{F_{\alpha\delta}}^{\sigma} N_{s\sigma} - \varepsilon_{bca}{E_{\beta\gamma}}^{\delta}{C_{\alpha\delta}}^{\sigma} J_\sigma, \\ \{N_{a\alpha}, D(N_{b\beta}, N_{c\gamma})\} &= {\varepsilon_{bc}}^{d}{\varepsilon_{ad}}^{s}{F_{\beta\gamma}}^{\delta}{E_{\alpha\delta}}^{\sigma} N_{s\sigma} - \eta_{bc}{C_{\beta\gamma}}^{\delta}{C_{\alpha\delta}}^{\sigma} N_{a\sigma}. \end{aligned} \tag{C.11}$$

To evaluate the sum over the cyclic permutation, we first use that ${E_{\beta\gamma}}^{\delta}{C_{\alpha\delta}}^{\sigma}+\mathrm{cycl}[\alpha,\beta,\gamma]=0$ which follows from the fact that the Poisson bracket is a bi-derivation. Then, we collect the terms proportional to $\eta_{ab}N_{c\sigma}$ which are proven to be proportional[35] to the identity (C.9).

Finally for the combination $(J_\alpha, N_{b\beta}, N_{c\gamma})$ we use that

$$\begin{aligned} D(J_\alpha, \{N_{b\beta}, N_{c\gamma}\}) &= {\varepsilon_{bc}}^{s}{E_{\beta\gamma}}^{\delta}{D_{\alpha\delta}}^{\sigma} N_{s\sigma}, \\ \{J_\alpha, D(N_{b\beta}, N_{c\gamma})\} &= {\varepsilon_{bc}}^{s}{F_{\beta\gamma}}^{\delta}{C_{\alpha\delta}}^{\sigma} N_{s\sigma} - \eta_{bc}{C_{\beta\gamma}}^{\delta}{C_{\alpha\delta}}^{\sigma} J_\sigma, \\ D(N_{b\beta}, \{N_{c\gamma}, J_\alpha\}) &= {\varepsilon_{bc}}^{s}{C_{\gamma\alpha}}^{\delta}{F_{\beta\delta}}^{\sigma} N_{s\sigma} - \eta_{bc}{C_{\gamma\alpha}}^{\delta}{C_{\beta\delta}}^{\sigma} J_\sigma, \\ \{N_{b\beta}, D(N_{c\gamma}, J_\alpha)\} &= {\varepsilon_{bc}}^{s}{D_{\gamma\alpha}}^{\delta}{E_{\beta\delta}}^{\sigma} N_{s\sigma} - \eta_{bc}{C_{\gamma\alpha}}^{\delta}{C_{\beta\delta}}^{\sigma} J_\sigma. \end{aligned} \tag{C.12}$$

The cocycle identity then follows from the expansion of the differential identity for the star product

$$[Y_\alpha, Y_\beta \circ Y_\gamma] = [Y_\alpha, Y_\beta] \circ Y_\gamma + Y_\beta \circ [Y_\alpha, Y_\gamma]. \tag{C.13}$$

In components this means that

$${E_{\beta\gamma}}^{\delta}{D_{\alpha\delta}}^{\sigma} + {F_{\beta\gamma}}^{\delta}{C_{\alpha\delta}}^{\sigma} = ({C_{\gamma\alpha}}^{\delta}{F_{\beta\delta}}^{\sigma} - {D_{\gamma\alpha}}^{\delta}{E_{\beta\delta}}^{\sigma}) + ({C_{\beta\alpha}}^{\delta}{F_{\gamma\delta}}^{\sigma} - {D_{\beta\alpha}}^{\delta}{E_{\gamma\delta}}^{\sigma}), \tag{C.14}$$

which completes the proof.

[35] One simply needs to use that ${\varepsilon_{bc}}^{d}{\varepsilon_{ad}}^{s} = \delta^s_c\eta_{ab} - \delta^s_b\eta_{ac}$.

D $\mathfrak{su}(N)$ and $\mathfrak{su}(N,N)$ relations

In this appendix, we establish some key $\mathfrak{su}(N)$ and $\mathfrak{su}(N,N)$ identities.

D.1 $\mathfrak{su}(N)$

We start with the facts necessary for the proof of equivalence between (3.14) and (3.17). In particular, we prove (3.18). We have seen that in the fundamental and the adjoint representations of $\mathfrak{su}(N)$, which we denote by $\pi_{\mathbf{N}}$ and $\pi_{\mathbf{ad}}$, respectively, we have

$$[\pi_{\mathbf{N}}(X_\alpha)]_i{}^j = \frac{N}{2i}[\widehat{Y}_\alpha]_i{}^j, \qquad [\pi_{\mathbf{ad}}(X_\alpha)]_\beta{}^\gamma = \widehat{C}_{\alpha\beta}{}^\gamma. \tag{D.1}$$

In the following we denote $i,j \in \{1,\dots,N\}$ the vectorial indices and $\alpha,\beta \in \{1,\dots,N^2-1\}$ are the adjoint indices. We denote by $U = \exp(u^\alpha X_\alpha)$ to be an abstract group element. It is well-known that the adjoint action is simply given by

$$\rho_{\mathbf{Ad}}(U)\widehat{Y}^\beta := \pi_{\mathbf{N}}(U)\widehat{Y}^\beta\pi_{\mathbf{N}}(U^{-1}). \tag{D.2}$$

This relation can be written in components in terms of the components $\pi_{\mathbf{N}}U$ and $\pi_{\mathbf{ad}}U$ as which in components means

$$\sum_{\alpha\in I_N}(\widehat{Y}^\alpha)_i{}^l[\pi_{\mathbf{Ad}}(U)]_\alpha{}^\beta = [\pi_{\mathbf{N}}(U)]_i{}^j[\pi_{\mathbf{N}}(U^{-1})]_k{}^l(\widehat{Y}^\beta)_j{}^k. \tag{D.3}$$

Next, we establish the relationship

$$\sum_\beta(\widehat{Y}^\beta)_j{}^k(\widehat{Y}_\beta)_{k'}{}^{j'} = N\delta_j^{j'}\delta_{k'}^k. \tag{D.4}$$

To see this just contract the l.h.s. with $(\widehat{Y}^\alpha)_{j'}{}^{k'}$. We get that this is equal to $\sum_\beta(\widehat{Y}^\beta)_j{}^k\mathrm{Tr}_{\mathbf{N}}(\widehat{Y}_\beta\widehat{Y}^\alpha) = N(\widehat{Y}^\beta)_j{}^k$, where we used that $\mathrm{Tr}_{\mathbf{N}}(\widehat{Y}_\beta\widehat{Y}^\alpha) = N\delta_\beta^\alpha$. Since $\widehat{Y}^\alpha$ with $\alpha\in I_N$ is a complete basis of matrices we get the desired equality. Using this identity and contracting (D.3) with $(\widehat{Y}_\beta)_{k'}{}^{j'}$ and summing over β gives the relation

$$\sum_{\alpha,\beta}(\widehat{Y}^\alpha)_i{}^l[\pi_{\mathbf{Ad}}(U)]_\alpha{}^\beta(\widehat{Y}_\beta)_{k'}{}^{j'} = N\,[\pi_{\mathbf{N}}(U)]_i{}^{j'}[\pi_{\mathbf{N}}(U^{-1})]_{k'}{}^l. \tag{D.5}$$

If we expand $U = \exp(u^\gamma X_\gamma)$ to first order in u^γ, we get from (D.1) that

$$(\widehat{Y}_\gamma)_i{}^j\delta_k^l - \delta_i^j(\widehat{Y}_\gamma)_k{}^l = \frac{2i}{N^2}\sum_{\alpha,\beta}\widehat{C}_\gamma{}^{\alpha\beta}(\widehat{Y}_\alpha)_i{}^l(\widehat{Y}_\beta)_k{}^j. \tag{D.6}$$

Next consider the matrix elements of $\widehat{J}$, which in two different bases are

$$X_\alpha = \frac{N}{2i}\sum_{i,j}E^j{}_i(\widehat{Y}_\alpha)_j{}^i, \qquad E^j{}_i = \frac{2i}{N^2}\sum_{\alpha\in I_N}X^\alpha(\widehat{Y}_\alpha)_i{}^j, \tag{D.7}$$

where $(\widehat{Y}_\alpha)_i{}^j$ denotes the ij^{th} component of the matrix $\widehat{Y}_\alpha$ (see appendix A.3 for details). The proof of equivalence between (3.14) and (3.17) and goes as follows. We first assume (3.14) and prove (3.17) as follows

$$\begin{aligned}
[E^j{}_i, E^l{}_k] &= \left(\frac{2i}{N^2}\right)^2 \sum_{\alpha,\beta\in I_N} [X^\alpha, X^\beta](\widehat{Y}_\alpha)_i{}^j(\widehat{Y}_\beta)_k{}^l \\
&= \left(\frac{2i}{N^2}\right)^2 \sum_{\alpha,\beta\in I_N} \widehat{C}^{\alpha\beta}{}_\gamma X^\gamma (\widehat{Y}_\alpha)_i{}^j(\widehat{Y}_\beta)_k{}^l \\
&= \frac{2i}{N^2} \sum_{\gamma\in I_N} X^\gamma (\delta_i{}^l(\widehat{Y}_\gamma)_k{}^j - \delta^j{}_k(\widehat{Y}_\gamma)_i{}^l) \\
&= \delta_i{}^l E^j{}_k - \delta_k{}^j E^l{}_i,
\end{aligned} \tag{D.8}$$

which is the $\mathfrak{su}(N)$ defining relation (3.17) and we have use the identity (D.6). Using this, one can show the following

$$[\pi_{\mathbf{N}}(E^j{}_i)]_a{}^b = \delta^j_a \delta^b_i, \tag{D.9}$$

then the commutator of any two matrices $A := A_j{}^i E_i{}^j$ and $B := B_j{}^i E_i{}^j$ is given by

$$\begin{aligned}
[A, B] &= A_j{}^i B_l{}^k [E^j{}_i, E^l{}_k] = A_j{}^i B_l{}^k (\delta_i{}^l E^j{}_k - \delta_k{}^j E^l{}_i) \\
&= (AB)_j{}^k E^j{}_k - (BA)_l{}^i E^l{}_i \\
&= [A, B]_j{}^i E^j{}_i.
\end{aligned} \tag{D.10}$$

Conversely, we can derive (3.14) from (3.17) easily as follows

$$\begin{aligned}
[X_\alpha, X_\beta] &= \frac{N^2}{(2i)^2} [E^j{}_i, E^l{}_k](\widehat{Y}_\alpha)_j{}^i(\widehat{Y}_\beta)_l{}^k \\
&= \frac{N^2}{(2i)^2} (\delta_i{}^l E^j{}_k - \delta_k{}^j E^l{}_i)(\widehat{Y}_\alpha)_j{}^i(\widehat{Y}_\beta)_l{}^k \\
&= \frac{N^2}{(2i)^2} E^j{}_i [\widehat{Y}_\alpha, \widehat{Y}_\beta]_j{}^i \\
&= \frac{N}{2i} \widehat{C}_{\alpha\beta}{}^\gamma E^j{}_i (\widehat{Y}_\gamma)_j{}^i \\
&= \widehat{C}_{\alpha\beta}{}^\gamma X_\gamma.
\end{aligned} \tag{D.11}$$

This completes the proof of equivalence of (3.14) and (3.17). The equivalence of (3.21) and (3.25) then follows.

D.2 $\mathfrak{su}(N, N)$

We can provide a similar identity for $\mathfrak{su}(N, N)$. The Lie algebra generators in the vector representations are

$$\widehat{Y}_{\bullet\alpha} = \mathbb{1}_2 \otimes \widehat{Y}_\alpha, \qquad \widehat{Y}_{a\alpha} = \rho_a \otimes \widehat{Y}_\alpha. \tag{D.12}$$

We see that

$$\frac{1}{2}(\mathbb{1}_2 \otimes \mathbb{1}_2) - 2(\rho_a \otimes \rho^a) = \mathbb{1}_{\mathrm{Mat}(2)}. \tag{D.13}$$

Therefore, using (D.4) which states that $\sum_{\alpha\in I_N}(\widehat{Y}_\alpha\otimes\widehat{Y}^\alpha) = N\mathbb{1}_{\mathrm{Mat}(\mathbf{N})}$ we have the identity decomposition

$$\sum_{\alpha\in I_N}\left(\frac{1}{2}\widehat{Y}_{\bullet\alpha}\otimes\widehat{Y}^{\bullet\alpha} - 2\sum_{a=0,1,2}\widehat{Y}_{a\alpha}\otimes\widehat{Y}^{a\alpha}\right) = N\mathbb{1}_{\mathrm{Mat}(2N)}. \tag{D.14}$$

From this, we can show that

$$\begin{aligned}
(\widehat{Y}_{\bullet\gamma})_{\mathsf{n}}{}^{\mathsf{q}}\delta_{\mathsf{p}}{}^{\mathsf{m}} - \delta_{\mathsf{n}}{}^{\mathsf{q}}(\widehat{Y}_{\bullet\gamma})_{\mathsf{p}}{}^{\mathsf{n}} &= \frac{2i}{N^2}\sum_{\alpha,\beta\in I_N}\widehat{C}_\gamma{}^{\alpha\beta}\big(\tfrac{1}{2}(\widehat{Y}_{\bullet\alpha})_{\mathsf{n}}{}^{\mathsf{m}}\left(\widehat{(}Y_{\bullet\beta})_{\mathsf{n}}{}^{\mathsf{m}} - 2\sum_{a,b}\eta^{ab}(\widehat{Y}_{a\alpha})_{\mathsf{n}}{}^{\mathsf{m}}(\widehat{Y}_{b\beta})_{\mathsf{p}}{}^{\mathsf{q}}\right), \\
(\widehat{Y}_{c\gamma})_{\mathsf{n}}{}^{\mathsf{q}}\delta_{\mathsf{p}}{}^{\mathsf{m}} - \delta_{\mathsf{n}}{}^{\mathsf{q}}(\widehat{Y}_{c\gamma})_{\mathsf{p}}{}^{\mathsf{n}} &= \frac{2i}{N^2}\sum_{\alpha,\beta\in I_N}\widehat{C}_\gamma{}^{\alpha\beta}\left((\widehat{Y}_{c\alpha})_{\mathsf{n}}{}^{\mathsf{m}}(\widehat{(Y}_{\bullet\beta})_{\mathsf{n}}{}^{\mathsf{m}} - (\widehat{Y}_{\bullet\alpha})_{\mathsf{n}}{}^{\mathsf{m}}((\widehat{Y}_{c\beta})_{\mathsf{n}}{}^{\mathsf{m}}\right) \\
&\quad - \frac{2i}{N^4}\sum_{\alpha,\beta\in I_N}\widehat{C}_\gamma{}^{\alpha\beta}\sum_{a,b}\varepsilon_c{}^{ab}(\widehat{Y}_{a\alpha})_{\mathsf{n}}{}^{\mathsf{m}}(\widehat{Y}_{b\beta})_{\mathsf{p}}{}^{\mathsf{q}},
\end{aligned}$$

where $\mathsf{m} = (A, i)$ with $A = 1, 2$ and $i = 1, \ldots, N$. These identities are exactly what is needed to establish that

$$[E^{\mathsf{m}}{}_{\mathsf{n}}, E^{\mathsf{p}}{}_{\mathsf{q}}] = \delta^{\mathsf{p}}_{\mathsf{n}}E^{\mathsf{m}}{}_{\mathsf{q}} - \delta^{\mathsf{m}}_{\mathsf{q}}E^{\mathsf{p}}{}_{\mathsf{n}}, \tag{D.15}$$

where we have defined the $\mathfrak{su}(N,N)$ generator

$$E^{\mathsf{m}}{}_{\mathsf{n}} := \frac{i}{N}\left(\frac{1}{N}X^\alpha\big(\widehat{Y}_{\bullet\alpha}\big)_{\mathsf{n}}{}^{\mathsf{m}} - 2Z^{a\alpha}\big(\widehat{Y}_{a\alpha}\big)_{\mathsf{n}}{}^{\mathsf{m}}\right). \tag{D.16}$$

The equality (D.14) can also be written in terms of the $\mathfrak{su}(N,N)$ element E as

$$(\pi_{\mathbf{2N}}\otimes\mathbb{1}_{\mathbf{2N}})E = \mathbb{1}_{\mathrm{Mat}(\mathbf{2N})}, \tag{D.17}$$

where $\pi_{\mathbf{2N}}$ is the representation (3.54).

E Identities for Casimir computations

In this appendix, we collect a number of computations relevant for the discussion of Casimirs for the continuum and deformed algebras described in section 4.

Proof of (4.10). This equation can be derived using the mode decomposition (4.8) of $\widehat{\mathbb{j}}(\sigma)$. This gives

$$\begin{aligned}
[\phi, \mathbb{j}(\sigma)]_{\mathfrak{g}} &= [\phi, \mathbb{j}_\alpha]_{\mathfrak{g}}\, Y^\alpha(\sigma) \\
&= \int_S d\sigma'\{\phi, Y_\alpha\}(\sigma')\, \mathbb{j}(\sigma')Y^\alpha(\sigma) \\
&= -\int_S d\sigma'\{\phi, \mathbb{j}\}(\sigma')\, Y_\alpha(\sigma')Y^\alpha(\sigma) \\
&= -\int_S d\sigma'\{\phi, \mathbb{j}\}(\sigma')\, \delta(\sigma - \sigma') \\
&= -\{\phi, \mathbb{j}\}(\sigma).
\end{aligned} \tag{E.1}$$

Proof of (4.19). Here we use the mode decomposition (4.18) of $\widehat{\mathbb{j}}$:

$$\begin{aligned}
[X_\alpha, \widehat{\mathbb{j}}]_{\widehat{\mathfrak{g}}} &= [X_\alpha, X_\beta]_{\widehat{\mathfrak{g}}} \widehat{Y}^\beta \\
&= \widehat{C}_{\alpha\beta}{}^\gamma X_\gamma \widehat{Y}^\beta \\
&= -\frac{N}{2i}[\widehat{Y}_\alpha, \widehat{Y}^\gamma] X_\gamma \\
&= -\frac{N}{2i}[\widehat{Y}_\alpha, \widehat{\mathbb{j}}].
\end{aligned} \tag{E.2}$$

Proof of (4.23). We begin with the expression for $d_{\alpha_1\ldots\alpha_n}$, which, employing (2.28) and (A.2), is given by

$$\begin{aligned}
d_{\alpha_1\ldots\alpha_n} &= \int_S \nu_0 \, Y_{\alpha_1} \ldots Y_{\alpha_n} \\
&= E_{\alpha_1\alpha_2}{}^{\beta_1} E_{\beta_1\alpha_3}{}^{\beta_2} \ldots E_{\beta_{n-3}\alpha_{n-1}}{}^{\beta_{n-2}} \int_S \nu_0 Y_{\beta_{n-2}} Y_{\alpha_n} \\
&= E_{\alpha_1\alpha_2}{}^{\beta_1} \ldots E_{\beta_{n-3}\alpha_{n-1}\alpha_n}.
\end{aligned} \tag{E.3}$$

Note that for $n=2$, the expression instead reads $d_{\alpha\beta} = \int_S \nu_0 \, Y_\alpha Y_\beta = \delta_{\alpha\beta}$. On the other hand, employing (3.4), we have

$$\begin{aligned}
\widehat{d}_{\alpha_1\ldots\alpha_n} &= \frac{1}{N} \mathrm{Tr}_{\mathbf{N}}(\widehat{Y}_{\alpha_1} \ldots \widehat{Y}_{\alpha_n}) \\
&= \widehat{M}_{\alpha_1\alpha_2}{}^{\beta_1} \widehat{M}_{\beta_1\alpha_3}{}^{\beta_2} \ldots \widehat{M}_{\beta_{n-3}\alpha_{n-1}}{}^{\beta_{n-2}} \frac{1}{N} \mathrm{Tr}_{\mathbf{N}}(\widehat{Y}_{\beta_{n-2}} \widehat{Y}_{\alpha_n}) \\
&= \widehat{M}_{\alpha_1\alpha_2}{}^{\beta_1} \ldots \widehat{M}_{\beta_{n-2}\alpha_{n-1}\alpha_n}.
\end{aligned} \tag{E.4}$$

Then using that $\widehat{M}_{\alpha\beta}{}^\gamma = \widehat{E}_{\alpha\beta}{}^\gamma + \frac{i}{N}\widehat{C}_{\alpha\beta}{}^\gamma$, we find that

$$\widehat{d}_{\alpha_1\ldots\alpha_n} = \widehat{E}_{\alpha_1\alpha_2}{}^{\beta_1} \ldots \widehat{E}_{\beta_{n-2}\alpha_{n-1}\alpha_n} + \mathcal{O}(N^{-1}). \tag{E.5}$$

Since $\widehat{E}_{\alpha\beta}{}^\gamma = E_{\alpha\beta}{}^\gamma + \mathcal{O}(N^{-2})$ (see sections A.3 and A.4), this shows that

$$\widehat{d}_{\alpha_1\ldots\alpha_n} = d_{\alpha_1\ldots\alpha_n} + \mathcal{O}(N^{-1}). \tag{E.6}$$

Proof of (4.30). The Casimir matching considered in section 4.2 requires us to determine the relation between the J_α Hamiltonians defined by equation (2.30), and the gravitational charges constructed in [26]. The latter were defined in terms of the vector fields ξ^A on S, while the former are written in terms of the stream functions Y_α. According to the conventions of section 2.2, these are related by $\xi^A_{Y_\alpha} = \epsilon^{BA}\partial_B Y_\alpha = \frac{1}{4\pi}\nu_0^{BA}\partial_B Y_\alpha$. The Hamiltonians in [26] were written in terms of a 1-form density $\widetilde{P}_A$ which has a geometrical interpretation in spacetime as a component of a connection on the normal bundle of S. The relation for the charges in terms of this is then given by

$$
\begin{aligned}
J_\alpha &= \frac{1}{16\pi G}\int_S \widetilde{P}_A\, \xi^A_{Y_\alpha} \\
&= \frac{A}{16\pi G}\int_S \nu_0 P_A\, \xi^A_{Y_\alpha} \\
&= \frac{A}{16\pi G}\int_S \nu_0 \frac{1}{4\pi}\nu_0^{AB} P_A \partial_B Y_\alpha \\
&= \frac{A}{16\pi G}\int_S \frac{1}{4\pi} dY_\alpha \wedge P \\
&= \frac{-A}{16\pi G}\int_S Y_\alpha \frac{dP}{4\pi} \\
&= \frac{A}{16\pi G}\int_S \nu_0 Y_\alpha \left(\frac{-AW}{4\pi}\right),
\end{aligned}
\tag{E.7}
$$

where, following the conventions of [26], we have used that $\widetilde{P}_A$ is related to P_A via the physical volume form $\nu = A\nu_0$, so that $\widetilde{P}_A = \nu P_A = A\nu_0 P_A$, and the last equality uses that dP is related to the outer curvature scalar W according to $dP = W\nu = AW\nu_0$. As a curvature scalar, W has dimensions $[\text{length}]^{-2}$, and A has dimensions $[\text{length}]^2$ in 4 spacetime dimensions, so the function AW is dimensionless. The final integral in (E.7) is therefore dimensionless, and hence J_α has dimensions of angular momentum, as expected since the $\mathfrak{diff}(S)$ charges are generalizations of angular momentum. Comparing to equation (2.30), we see that the function $J(\sigma)$ is related to the geometrical data according to

$$
J(\sigma) = \frac{A}{16\pi G}\left(\frac{-AW(\sigma)}{4\pi}\right). \tag{E.8}
$$

Plugging this relation into the expression (4.15) for the gravitational Casimirs then immediately reproduces equation (4.30).

Proof of (4.35). Using the definition of the coadjoint action and the pairing (4.34) for $\mathfrak{c}_{\mathbb{R}}(S)$, we compute

$$
\begin{aligned}
\langle \mathrm{ad}^*_{(\phi,\alpha)}(f,a),(\psi,\beta)\rangle &= -\langle (f,a),(\{\phi,\psi\},\{\phi,\beta\}-\{\psi,\alpha\})\rangle \\
&= -\int_S \nu_0\Big(f\{\phi,\psi\} + a(\{\phi,\beta\}+\{\alpha,\psi\})\Big) \\
&= \int_S \nu_0\Big(\{\phi,f\}\psi + \{\phi,a\}\beta + \{\alpha,a\}\psi\Big) \\
&= \langle (\{\phi,f\}+\{\alpha,a\},\{\phi,a\}),(\psi,\beta)\rangle,
\end{aligned}
\tag{E.9}
$$

which determines the coadjoint action to be (4.35).

Proof of (4.38) and (4.39). These relations are derived as follows:

$$
\begin{aligned}
\langle (f,a),[(\phi,\alpha),\mathbb{j}]_{\mathfrak{g}}\rangle &= -\langle \mathrm{ad}^*_{(\phi,\alpha)}(f,a),\mathbb{j}\rangle \\
&= -\langle (\{\phi,f\}+\{\alpha,a\},\{\phi,a\}),\mathbb{j}\rangle \\
&= -\{\phi,f\}-\{\alpha,a\} \\
&= \langle (f,a),-\{\phi,\mathbb{j}\}-\{\alpha,\mathbb{n}\}\rangle,
\end{aligned}
\tag{E.10}
$$

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verifying (4.38). Similarly,

$$\begin{aligned}\langle (f,a),[(\phi,\alpha),\mathbb{n}]_{\mathfrak{g}}\rangle &= -\langle \mathrm{ad}^*_{(\phi,\alpha)}(f,a),\mathbb{n}\rangle \\ &= -\langle (\{\phi,f\}+\{\alpha,a\},\{\phi,a\}),\mathbb{n}\rangle \\ &= -\{\phi,a\} \\ &= \langle (f,a),-\{\phi,\mathbb{n}\}\rangle,\end{aligned} \tag{E.11}$$

verifying (4.39).

Identities for deriving (4.42) The Lie-algebra-valued functions $\mathbb{j}$ and $\mathbb{n}$ can be shown to satisfy

$$\begin{aligned}\mathbb{j}\mathbb{n} &= \mathbb{j}_\alpha \mathbb{n}_\beta Y^\alpha Y^\beta \\ &= \mathbb{n}_\beta \mathbb{j}_\alpha Y^\alpha Y^\beta + [\mathbb{j}_\alpha,\mathbb{n}_\beta]_{\mathfrak{g}} E^{\alpha\beta\gamma} Y_\gamma \\ &= \mathbb{n}\mathbb{j} + C_{\alpha\beta}{}^{\gamma}\mathbb{n}_\gamma E^{\alpha\beta\gamma}Y_\gamma \\ &= \mathbb{n}\mathbb{j}.\end{aligned} \tag{E.12}$$

$$\begin{aligned}\{\mathbb{n},\mathbb{n}\} &= \mathbb{n}_\alpha\mathbb{n}_\beta\{Y^\alpha,Y^\beta\} \\ &= \mathbb{n}_\alpha\mathbb{n}_\beta C^{\alpha\beta\gamma}Y_\gamma \\ &= \frac{1}{2}[\mathbb{n}_\alpha,\mathbb{n}_\beta]_{\mathfrak{g}} C^{\alpha\beta\gamma}Y_\gamma = 0.\end{aligned} \tag{E.13}$$

$$\begin{aligned}\{\mathbb{j},\mathbb{n}\} &= \mathbb{j}_\alpha\mathbb{n}_\beta C^{\alpha\beta\gamma}Y_\gamma \\ &= \frac{1}{2}[\mathbb{j}_\alpha,\mathbb{n}_\beta]_{\mathfrak{g}} C^{\alpha\beta\gamma}Y_\gamma \\ &= \frac{1}{2}C_{\alpha\beta}{}^{\mu}C^{\alpha\beta\gamma}\mathbb{n}_\mu Y_\gamma.\end{aligned} \tag{E.14}$$

Although the sum over α and β in this final expression is divergent, we can take $C_{\alpha\beta}{}^{\mu}C^{\alpha\beta\gamma}$ to be proportional to $\delta^{\mu\gamma}$ times a divergent coefficient. This then demonstrates that $\{\mathbb{j},\mathbb{n}\} \propto \mathbb{n}$.

Identity satisfied by $\mathbb{n}_a$. The quantities $\mathbb{n}_a(\sigma)$ and $\mathbb{n}_b(\sigma')$ do not commute due to being valued in a Lie algebra, but instead have a $\delta-$function contribution coming from coincident points. This can be derived by

$$\begin{aligned}\mathbb{n}_a(\sigma)\mathbb{n}_b(\sigma') &= \mathbb{n}_{a\alpha}\mathbb{n}_{b\beta}Y^\alpha(\sigma)Y^\beta(\sigma') \\ &= \mathbb{n}_{b\beta}\mathbb{n}_{a\alpha}Y^\alpha(\sigma)Y^\beta(\sigma') + \varepsilon_{ab}{}^{c}E_{\alpha\beta}{}^{\gamma}\mathbb{n}_{c\gamma}Y^\alpha(\sigma)Y^\beta(\sigma') \\ &= \mathbb{n}_b(\sigma')\mathbb{n}_a(\sigma) + \varepsilon_{ab}{}^{c}\mathbb{n}_{c\gamma}\int_S d\sigma'' Y_\alpha(\sigma'')Y_\beta(\sigma'')Y^\gamma(\sigma'')Y^\alpha(\sigma)Y^\beta(\sigma') \\ &= \mathbb{n}_b(\sigma')\mathbb{n}_a(\sigma) + \varepsilon_{ab}{}^{c}\mathbb{n}_{c\gamma}\int d\sigma''\delta(\sigma''-\sigma)\delta(\sigma''-\sigma')Y^\gamma(\sigma'') \\ &= \mathbb{n}_b(\sigma')\mathbb{n}_a(\sigma) + \varepsilon_{ab}{}^{c}\delta(\sigma-\sigma')\mathbb{n}_c(\sigma).\end{aligned} \tag{E.15}$$

Proof of (4.84) and (4.85). These relations are once again derived using the mode decompositions of $\widehat{\mathbb{j}}$ and $\widehat{\mathbb{n}}_a$:

$$
\begin{aligned}
\widehat{\mathbb{j}}\widehat{\mathbb{n}}_a &= X_\alpha Z_{a\beta}\widehat{Y}^\alpha\widehat{Y}^\beta \\
&= Z_{a\beta}X_\alpha\widehat{Y}^\beta\widehat{Y}^\alpha + Z_{a\beta}X_\alpha[\widehat{Y}^\alpha,\widehat{Y}^\beta] + [X_\alpha, Z_{a\beta}]_{\widehat{\mathfrak{g}}}\widehat{Y}^\alpha\widehat{Y}^\beta \\
&= \widehat{\mathbb{n}}_a\widehat{\mathbb{j}} + Z_{a\beta}X_\alpha[\widehat{Y}^\alpha,\widehat{Y}^\beta] + \widehat{C}_{\alpha\beta}{}^\mu Z_{\mu a}\widehat{Y}^\alpha\widehat{Y}^\beta \\
&= \widehat{\mathbb{n}}_a\widehat{\mathbb{j}} + Z_{a\beta}X_\alpha[\widehat{Y}^\alpha,\widehat{Y}^\beta] + \frac{1}{2}[X_\alpha, Z_{a\beta}]_{\widehat{\mathfrak{g}}}[\widehat{Y}^\alpha,\widehat{Y}^\beta] \\
&= \widehat{\mathbb{n}}_a\widehat{\mathbb{j}} + \frac{i}{N}(Z_{a\beta}X_\alpha + X_\alpha Z_{a\beta})\widehat{C}^{\alpha\beta\mu}\widehat{Y}_\mu \\
&= \widehat{\mathbb{n}}_a\widehat{\mathbb{j}} + \mathcal{O}(N^{-1}).
\end{aligned}
\tag{E.16}
$$

$$
\begin{aligned}
\widehat{\mathbb{n}}_a\widehat{\mathbb{n}}_b &= Z_{a\alpha}Z_{b\beta}\widehat{Y}^\alpha\widehat{Y}^\beta \\
&= Z_{b\beta}Z_{a\alpha}\widehat{Y}^\beta\widehat{Y}^\alpha + [Z_{a\alpha}, Z_{b\beta}]_{\widehat{\mathfrak{g}}}\widehat{Y}^\alpha\widehat{Y}^\beta + Z_{a\alpha}Z_{b\beta}[\widehat{Y}^\alpha,\widehat{Y}^\beta] \\
&= \widehat{\mathbb{n}}_b\widehat{\mathbb{n}}_a + [\widehat{\mathbb{n}}_a,\widehat{\mathbb{n}}_b]_{\widehat{\mathfrak{g}}} + \frac{2i}{N}Z_{a\alpha}Z_{b\beta}\widehat{\{Y_\alpha, Y_\beta\}} + \mathcal{O}(N^{-3}) \\
&= \widehat{\mathbb{n}}_b\widehat{\mathbb{n}}_a + [\widehat{\mathbb{n}}_a,\widehat{\mathbb{n}}_b]_{\widehat{\mathfrak{g}}} + \frac{2i}{N}\widehat{\{\mathbb{n}_a, \mathbb{n}_b\}} + \mathcal{O}(N^{-3}).
\end{aligned}
\tag{E.17}
$$

Proof of (4.73). We now put a derivation of (4.73) which gives an alternate derivation that (4.72) is a Casimir.

$$
\begin{aligned}
\mathrm{ad}^*_{(\phi,\alpha)}(fa^{2n}) &= (\mathrm{ad}^*_{(\phi,\alpha)}f)a^{2n} + f\mathrm{ad}^*_{(\phi,\alpha)}a^{2n} \\
&= \{\phi, f\}_{\nu 0}a^{2n} + \{\alpha_a, a^a\}_{\nu 0}a^{2n} + f\{\phi, a^{2n}\}_{\nu 0} \\
&= \{\phi, fa^{2n}\}_{\nu 0} + \{\alpha^a, a_a\}_{\nu 0}a^{2n},
\end{aligned}
\tag{E.18}
$$

and also

$$
\begin{aligned}
\mathrm{ad}^*_{(\phi,\alpha)}(\varepsilon_{abc}\{a^a, a^b\}a^c\, a^{2(n-1)}) &= \{\phi, \varepsilon_{abc}\{a^a, a^b\}a^c a^{2(n-1)}\} \\
&\quad + \varepsilon_{abc}(\{[\alpha, a]^a, a^b\}a^c + \{a^a, [\alpha, a]^b\}a^c + \{a^a, a^b\}[\alpha, a]^c)a^{2(n-1)} \\
&= \{\phi, \varepsilon_{abc}\{a^a, a^b\}a^c\, a^{2(k-1)}\} + 2\varepsilon_{abc}\varepsilon^{ade}(\{\alpha_d, a^b\}a^c a_e)a^{2(n-1)} \\
&= \{\phi, \varepsilon_{abc}\{a^a, a^b\}a^c\, a^{2(n-1)}\} - 2(\{\alpha_b, a^b\}a^c a_c - \{\alpha_c, a^b\}a^c a_b)a^{2(n-1)} \\
&= \{\phi, \varepsilon_{abc}\{a^a, a^b\}a^c\, a^{2(n-1)}\} - 2\{\alpha_b, a^b\}a^{2n} + \{\alpha_c, a^2\}a^c a^{2(n-1)} \\
&= \{\phi, \varepsilon_{abc}\{a^a, a^b\}a^c\, a^{2(n-1)}\} - \frac{2n+1}{n}\{\alpha_b, a^b\}a^{2n} + \frac{1}{n}\{\alpha_b, a^{2n}a^b\}.
\end{aligned}
\tag{E.19}
$$

We can thus define

$$
w_n := \left[(2n+1)fa^2 + n\varepsilon_{abc}\{a^a, a^b\}a^c\right]a^{2(n-1)},
\tag{E.20}
$$

which transforms as

$$
\mathrm{ad}^*_{(\phi,\alpha)}w_n = \{\phi, w_n\} + \{\alpha_a, a^{2n}a^a\}.
\tag{E.21}
$$

The proof that (4.72) is a Casimir then follows from the fact that the integral of (E.21) on S vanishes, i.e. we have

$$
\int_S \{f, g\}_\nu = 0, \qquad \forall f, g \in C(S).
\tag{E.22}
$$

This can be proven as follows. Let (M, ω) be an $2n$-dimensional symplectic manifold with symplectic form ω. The volume form on M is given by ω^n. Then, for any two functions $f, g \in C(M)$, we have

$$\begin{aligned} \{f, g\}\omega^n &= \omega(X_f, X_g)\omega^n \\ &= X_g(f)\omega^n = \mathcal{L}_{X_g} f\omega^n \\ &= \mathcal{L}_{X_g}(f\omega^n) = (\mathsf{d}\iota_{X_g} + \iota_{X_g}\mathsf{d})f\omega^n \\ &= \mathsf{d}\iota_{X_g}(f\omega^n) \\ &= \mathsf{d}(f\iota_{X_g}\omega^n), \end{aligned} \tag{E.23}$$

where X_g is the Hamiltonian vector field associated with g defined by $\iota_{X_g}\omega := \mathsf{d}g$, and in the fourth equality we used the fact that the Lie derivative of the symplectic form along a Hamiltonian vector field vanishes; this can be seen as follows

$$\mathcal{L}_{X_g}\omega = (\mathsf{d}\iota_{X_g} + \iota_{X_g}\mathsf{d})\omega = \mathsf{d}(\mathsf{d}g) + 0 = 0,$$

where we have used the closeness of the symplectic form $\mathsf{d}\omega = 0$. We thus have

$$\int_M \omega^n \{f, g\} = \int_M \mathsf{d}(f\iota_{X_g}\omega^n) = \int_{\partial M} f\iota_{X_g}\omega^n. \tag{E.24}$$

In the case of sphere $M = S$, $\omega = \sqrt{q}\epsilon_{AB}\mathsf{d}\sigma^A \wedge \mathsf{d}\sigma^B$, $\partial M = \emptyset$, and $\omega^n\{f, g\} \to \{f, g\}_\nu$. Therefore, (E.24) vanishes and we thus end up with the desired result.

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