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Impact of indirect CP violation on $\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}$

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ABSTRACT: The decay $K_S \rightarrow (\mu^+ \mu^-)_{\ell=0}$, with the final muon pair in an angular-momentum zero state, is a sensitive probe of short-distance physics. It has recently been shown how to extract this branching ratio from neutral kaon decay data. We point out that the impact of indirect CP violation on the standard-model prediction of this mode, while nominally of order $|\epsilon_K| \sim 10^{-3}$, is enhanced by a large amplitude ratio and leads to a shift of the branching ratio $\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}$ by a few percent, depending on the size of a relative phase that can be extracted from data. We also update the standard-model prediction of the short-distance contribution.

KEYWORDS: Kaons, CP Violation, Rare Decays

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1 Introduction

CP violation in neutral kaon decays is a very sensitive probe of short-distance dynamics, both within the standard model of particle physics (SM) and beyond. A prime example is the rare decay $K_L \rightarrow \pi^0\nu\bar{\nu}$. It proceeds almost exclusively via CP violation in interference between mixing and decay [1, 2] and is theoretically exceptionally clean [3, 4]. A dedicated experimental program to measure its branching ratio is underway [5].

The amplitude for the decay of neutral kaons into a charged dimuon final state, $K \rightarrow \mu^+\mu^-$, also has a short-distance component; however, it has been believed for a long time that these modes are less sensitive to high-energy dynamics due to very large long-distance (LD) contributions that are hard to control theoretically [6]. However, it has been pointed out in ref. [7] that the interference term between the K_L and K_S components in the time-dependent decay rate of neutral kaons into dileptons receives a large CP-violating contribution, primarily sensitive to short-distance (SD) dynamics.

The dimuon final state resulting from the disintegration of a K_L can have angular-momentum $\ell = 0$ or $\ell = 1$; we denote these final states by $(\mu^+\mu^-)_\ell$. The time-dependent decay rate into either final state of an initial neutral kaon that was tagged as a state K at time $t = 0$ is given by [8, 9]

$$\frac{1}{\mathcal{N}_\ell} \frac{d\Gamma(K(t) \rightarrow (\mu^+\mu^-)_\ell)}{dt} = C_L^\ell e^{-\Gamma_L t} + C_S^\ell e^{-\Gamma_S t} + 2[C_{\text{sin}}^\ell \sin(\Delta M t) + C_{\text{cos}}^\ell \cos(\Delta M t)] e^{-\Gamma t} \tag{1.1}$$

where Γ_S and Γ_L are the K_S and K_L total decay rates, $\Gamma = (\Gamma_S + \Gamma_L)/2$, and $\Delta M = M_L - M_S$, with M_S and M_L the K_S and K_L masses. \mathcal{N}_ℓ is a conventional, final-state dependent normalization factor. The parameters C_L^ℓ , C_S^ℓ , C_{cos}^ℓ , and C_{sin}^ℓ depend on the initial and final states and can be experimentally determined. In practice, the $\ell = 0$ and $\ell = 1$ final states are experimentally indistinguishable. Since there is no interference between these two final states, it follows that only the incoherent sum of the two decay rates can be measured.

More explicitly, only the combinations $C_i \equiv C_i^{\ell=0} + \beta_\mu^2 C_i^{\ell=1}$, where $i = L, S, \cos, \sin$, can be measured (the relative factor β_μ^2 accounts for the phase-space difference of the $\ell = 0$ and $\ell = 1$ final states, see eq. (2.1) below).

Recently, it has been shown in ref. [8] how to use interference data from $K_{L/S} \rightarrow \mu^+ \mu^-$ to extract the angular-momentum $\ell = 0$ part of the branching fraction of K_S into a pair of muons, $\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}$. The main result of ref. [8] is the relation

$$\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0} = \text{Br}(K_L \rightarrow \mu^+ \mu^-) \times \frac{\tau_S}{\tau_L} \times \left(\frac{C_{\text{int}}}{C_L} \right)^2, \quad (1.2)$$

where $\tau_S(\tau_L)$ are the $K_S(K_L)$ lifetimes, and $C_{\text{int}} \equiv (C_{\cos}^2 + C_{\sin}^2)^{1/2}$. Eq. (1.2) is valid for a pure K^0 or \bar{K}^0 beam; see ref. [8] for a discussion of more general cases. *Within the SM*, the branching ratio $\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}$ is fully dominated by SD contributions and can be calculated perturbatively; the remaining theoretical and parametric uncertainties are small. A precise measurement of $\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}$ via eq. (1.2) would, therefore, constitute a sensitive test of the SM.

In this work, we will critically assess the assumptions made in deriving eq. (1.2). In particular, we will scrutinize the assumption that there is no indirect CP violation in neutral kaon mixing, made in ref. [8]. We will show in section 2 that the relation eq. (1.2) remains true in the presence of indirect CP violation. As a consequence, the $K_S \rightarrow (\mu^+ \mu^-)_{\ell=0}$ branching ratio extracted using eq. (1.2) includes the effect of indirect CP violation, and this effect has to be taken into account in its SM prediction to perform a meaningful comparison. In fact, we will show that the effect, which is naively of order $\epsilon_K \sim 10^{-3}$, can be surprisingly large.

Section 3 contains the main result of this work. We update the numerical value of the SD contribution to the $K_S \rightarrow (\mu^+ \mu^-)_{\ell=0}$ branching ratio, based on the known next-to-next-leading-order QCD and next-to-leading-order electroweak corrections [10–12]. Moreover, we point out that the branching ratio for $K_S \rightarrow (\mu^+ \mu^-)_{\ell=0}$ can receive a sizeable LD correction due to indirect CP violation in the neutral kaon system. While this additional term is naively suppressed by a factor of $|\epsilon_K| = (2.228 \pm 0.011) \times 10^{-3}$ [9], it is enhanced by a large ratio of amplitudes, leading to a correction of up to $\pm 3.7\%$. We show how this LD contribution can, in principle, be fully determined from data and estimate its *maximal* size using the measurement of $\text{Br}(K_L \rightarrow \mu^+ \mu^-)$.

To understand the size of this correction, it may be instructive to recall that an analogous mechanism is at play in the rare decay $K_L \rightarrow \pi^0 \nu \bar{\nu}$. It was shown in ref. [13] that the contribution of indirect CP violation to this CP-violating decay is proportional to $|\epsilon_K|$ times the *real* (CP-conserving) part of the effective Lagrangian, involving charm-quark contributions in addition to the leading top-quark contribution, as well as different CKM input parameters. The net result is an enhancement by roughly a factor of three over the naive estimate, leading to a percent correction of the branching ratio [4, 13]. For the decay $K_S \rightarrow (\mu^+ \mu^-)_{\ell=0}$, a similar mechanism is at play. In this case, however, the real part of the amplitude receives, in addition to a contribution from the real part of the short-distance Lagrangian, a large CP-conserving contribution of a LD two-photon intermediate state. This leads to a much larger correction than naively expected from the smallness of ϵ_K . Our conclusions are contained in section 4.

2 Decay rate and CP structure

In general, a neutral kaon $K_j = K_S, K_L$ decaying into a dimuon final state produces the lepton pair either in the CP-odd angular-momentum $\ell = 0$ state or in the CP-even angular-momentum $\ell = 1$ state. The corresponding total decay rate is given by [14]

$$\Gamma(K_j \rightarrow \mu^+ \mu^-) = \frac{M_{K_j}}{8\pi} \beta_\mu (\beta_\mu^2 |A_1^j|^2 + |A_0^j|^2), \quad (2.1)$$

with $\beta_\mu \equiv (1 - 4m_\mu^2/M_{K_j}^2)^{1/2}$. This defines our normalisation for the angular-momentum amplitudes $A_\ell^j \equiv \langle (\mu^+ \mu^-)_\ell | K_j \rangle$. In other words, the term proportional to $|A_0^j|^2$ corresponds to the lepton pair being produced in the CP-odd $\ell = 0$ state, and the one proportional to $|A_1^j|^2$ to it being produced in the CP-even $\ell = 1$ state.

The flavour and mass eigenstates of the neutral kaons are related by

$$|K_S\rangle = p|K^0\rangle + q|\bar{K}^0\rangle, \quad |K_L\rangle = p|K^0\rangle - q|\bar{K}^0\rangle. \quad (2.2)$$

Using this decomposition, we can express the experimental parameters of the time-dependent decay rate in eq. (1.1) in terms of the K_L/K_S amplitudes.¹ In the PDG conventions [9], this implies for $K(t=0) = K^0$ (“ K^0 beam”)

$$\begin{aligned} C_L^\ell &= \frac{1}{2|p|^2} |A_\ell^L|^2, & C_{\sin}^\ell &= \frac{1}{2|p|^2} \text{Im}\{(A_\ell^S)^* A_\ell^L\}, \\ C_S^\ell &= \frac{1}{2|p|^2} |A_\ell^S|^2, & C_{\cos}^\ell &= \frac{1}{2|p|^2} \text{Re}\{(A_\ell^S)^* A_\ell^L\}, \end{aligned} \quad (2.3)$$

and analogously for $K(t=0) = \bar{K}^0$ (“ \bar{K}^0 beam”)

$$\begin{aligned} C_L^\ell &= \frac{1}{2|q|^2} |A_\ell^L|^2, & C_{\sin}^\ell &= -\frac{1}{2|q|^2} \text{Im}\{(A_\ell^S)^* A_\ell^L\}, \\ C_S^\ell &= \frac{1}{2|q|^2} |A_\ell^S|^2, & C_{\cos}^\ell &= -\frac{1}{2|q|^2} \text{Re}\{(A_\ell^S)^* A_\ell^L\}. \end{aligned} \quad (2.4)$$

Eqs. (2.3) and (2.4) agree with the corresponding equations in ref. [8], which are written in terms of the flavour-basis amplitudes. Note that, according to eq. (2.1), the normalisation factors for the $\ell = 0$ and $\ell = 1$ final states are related by $\mathcal{N}_1 = \beta_\mu^2 \mathcal{N}_0 = 0.82 \mathcal{N}_0$. Note that the global $1/|p|^2$ and $1/|q|^2$ factors can be absorbed into the normalisation factor \mathcal{N}_ℓ . We only keep them to conform with the PDG conventions [9].

With these definitions, it is now straightforward to derive eq. (1.2) *under the sole assumption that $A_1^L = 0$* . We have

$$\frac{\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}}{\text{Br}(K_L \rightarrow \mu^+ \mu^-)} = \frac{|A_0^S|^2}{|A_0^L|^2} \times \frac{\tau_L}{\tau_S} = \frac{|A_0^S|^2 |A_0^L|^2}{|A_0^L|^4} \times \frac{\tau_L}{\tau_S} = \left(\frac{C_{\text{int}}}{C_L} \right)^2 \times \frac{\tau_L}{\tau_S}, \quad (2.5)$$

where we used that $C_L = C_L^0$ and $C_{\text{int}} = C_{\text{int}}^0$ if $A_1^L = 0$. It is clear from this derivation that eq. (1.2) is true also in the presence of indirect CP violation, since we use the amplitudes

¹For kaon mixing, we find it more transparent to express the C 's in terms of the amplitudes of the mass eigenstates, rather than the flavor eigenstates as in refs. [8, 9].

for the exact weak mass eigenstates. The presence of indirect CP violation is not something that can be switched off by choice — it is experimental fact, and it will necessarily affect the experimental determination of $\text{Br}(K_S \rightarrow \mu^+\mu^-)_{\ell=0}$ based on eq. (1.2). It follows that indirect CP violation should be taken into account in the SM prediction of this branching ratio. This is the main purpose of this work.

As a preparation for the remainder of this work we now briefly discuss the CP structure of the decays $K_j \rightarrow \mu^+\mu^-$. First, we consider the case that indirect CP violation is absent in $K \rightarrow \mu^+\mu^-$ (i.e. $\epsilon_K = 0$). For clarity, we will then explicitly use the even and odd CP eigenstates

$$|K_1\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle), \quad |K_2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle), \quad (2.6)$$

where we adopted phase conventions such that $\text{CP}|K^0\rangle = -|\bar{K}^0\rangle$, $\text{CP}|\bar{K}^0\rangle = -|K^0\rangle$. We have $|K_S\rangle \rightarrow |K_1\rangle$ and $|K_L\rangle \rightarrow |K_2\rangle$ in the limit $\epsilon_K \rightarrow 0$. In the SM, the genuine SD contributions are induced by the effective Lagrangian²

$$\mathcal{L}_{\text{eff, SD}}^{|\Delta S|=1} = \frac{2G_F^2 M_W^2}{\pi^2} (V_{ts}^* V_{td} Y_t + V_{cs}^* V_{cd} Y_{\text{NL}}) Q_\mu + \text{h.c.} \quad (2.7)$$

Here, Y_{NL} and Y_t are functions of $x_t \equiv m_t^2/M_W^2$ and $x_c \equiv m_c^2/M_W^2$, respectively, with m_t and m_c the top- and charm-quark masses, M_W the W -boson mass, and the local operator is defined as

$$Q_\mu = (\bar{s}_L \gamma^\nu d_L)(\bar{\mu}_L \gamma_\nu \mu_L). \quad (2.8)$$

This effective Lagrangian originates from electroweak box and penguin diagrams [15] and generates only the $A_{\ell=0}^j$ amplitudes. Hence, in the limit of vanishing indirect CP violation its contribution is CP-conserving for $K_2 \rightarrow \mu^+\mu^-$ and CP-violating for $K_1 \rightarrow \mu^+\mu^-$. However, within the SM, the muon pair can also be produced via a two-photon intermediate state originating from operators of the full $|\Delta S| = 1$ Lagrangian other than Q_μ . Contrary to the Q_μ contribution, the two-photon intermediate state generates both the $\ell = 0$ and the $\ell = 1$ amplitudes. The two-photon contribution is completely LD dominated and nearly CP conserving [6]; hence, the $\ell = 1$ amplitude can be neglected for $K_2 \rightarrow \mu^+\mu^-$, but not for $K_1 \rightarrow \mu^+\mu^-$ [6].

An important comment is in order. As we have shown, the only assumption in deriving eq. (1.2) is the vanishing of the amplitude A_L^1 . Since A_S^1 is non-zero, indirect CP violation is expected to induce a small but non-zero contribution to the amplitude A_L^1 , of order $\epsilon_K \times A_S^1$. (This contribution has to be added to any amplitude $K_2 \rightarrow (\mu^+\mu^-)_{\ell=1}$ that may be present due to direct CP violation.) We therefore expect A_L^1 to be non-zero, with an absolute size that is small but hard to quantify. With A_L^1 nonzero, we can no longer identify the experimentally determined parameters C_L and C_{int} with C_L^0 and C_{int}^0 , respectively; hence, eq. (1.2) is no longer exactly valid. Since the numerical value of A_L^1 is not well known, it is hard to make quantitative statements, but based solely on the contribution to A_L^1 arising from

²We use a normalization of the effective Lagrangian conforming with the convention in ref. [12] to transparently include perturbative corrections to Y_t (see section 3.1).

indirect CP violation, the relation in eq. (1.2) will receive a correction at the percent level.³ Further study of this issue seems worthwhile; this is, however, beyond the scope of this work.

3 Br($K_S \rightarrow \mu^+ \mu^-$) $_{\ell=0}$ in the standard model

In this section, we will show that the amplitude A_0^S receives two contributions: the first involves only the imaginary part of the effective Lagrangian, and the second, proportional to ϵ_K , involves the real part of the effective Lagrangian. Throughout, we neglect all effects that are of higher powers in ϵ_K .

To obtain this decomposition we recall that, in the presence of indirect CP violation, the neutral kaon mass eigenstates are related to the flavor eigenstates via eq. (2.2). For the following argument, it will be convenient to switch to the “traditional” notation⁴ $p = (1 + \bar{\epsilon})/\sqrt{2}$, $q = -(1 - \bar{\epsilon})/\sqrt{2}$, with the approximation $\bar{\epsilon} \approx \epsilon_K$ [13]. This gives

$$|K_S\rangle = \frac{1 + \epsilon_K}{\sqrt{2}}|K^0\rangle - \frac{1 - \epsilon_K}{\sqrt{2}}|\bar{K}^0\rangle, \quad |K_L\rangle = \frac{1 + \epsilon_K}{\sqrt{2}}|K^0\rangle + \frac{1 - \epsilon_K}{\sqrt{2}}|\bar{K}^0\rangle, \quad (3.1)$$

and we can write (note that $\mathcal{L}_{\text{eff}}^{\Delta S=-1} = (\mathcal{L}_{\text{eff}}^{\Delta S=1})^\dagger$)

$$A_0^S \equiv \langle (\mu^+ \mu^-)_{\ell=0} | \mathcal{L}_{\text{eff}}^{|\Delta S|=1} | K_S \rangle = + \frac{1 + \epsilon_K}{\sqrt{2}} \langle (\mu^+ \mu^-)_{\ell=0} | \mathcal{L}_{\text{eff}}^{\Delta S=1} | K^0 \rangle - \frac{1 - \epsilon_K}{\sqrt{2}} \langle (\mu^+ \mu^-)_{\ell=0} | \mathcal{L}_{\text{eff}}^{\Delta S=-1} | \bar{K}^0 \rangle. \quad (3.2)$$

Here, we explicitly display the flavor-changing weak interactions, induced by higher dimension operators, as these are the only terms that potentially involve a weak phase; i.e., $\mathcal{L}_{\text{eff}}^{|\Delta S|=1}$ denotes the full $|\Delta S| = 1$ effective Lagrangian, not just the Q_μ operator. All effects of the strong and electromagnetic interactions are understood to be taken into account implicitly in the matrix elements. Using the CP invariance of QCD and QED, together with the well-known transformation properties of currents and states under CP (see, e.g., ref. [16]), it is straightforward to show that

$$\langle (\mu^+ \mu^-)_{\ell=0} | \mathcal{L}_{\text{eff}}^{\Delta S=-1} | \bar{K}^0 \rangle = \langle (\mu^+ \mu^-)_{\ell=0} | (\mathcal{L}_{\text{eff}}^{\Delta S=1})^* | K^0 \rangle. \quad (3.3)$$

³If we loosen the assumption $A_1^L = 0$, the relation in eq. (1.2) receives corrections because in this case $C_L \neq C_L^0$ and $C_{\text{int}} \neq C_{\text{int}}^0$ (see the derivation in eq. (2.5)). In the limit of $|A_1^L| \ll |A_0^L|$, the leading corrections multiplying the right side of eq. (1.2) are

$$r = 1 - 2\beta_\mu^2 \frac{|A_1^L|}{|A_0^L|} \frac{|A_1^S|}{|A_0^S|} \cos(\phi_0 - \phi_1) + \beta_\mu^4 \frac{|A_1^L|^2}{|A_0^L|^2} \frac{|A_1^S|^2}{|A_0^S|^2} (4 \cos^2(\phi_0 - \phi_1) - 1) + \mathcal{O}(|A_1^L|^3/|A_0^L|^3),$$

with the strong phases $\phi_\ell \equiv \arg \{ (A_\ell^S)^* A_\ell^L \}$. To estimate the size of the corrections, we use $|A_0^S| = 2.64 \times 10^{-13}$ and $|A_0^L| = 2.22 \times 10^{-12}$ which we derive below, and the estimate in ref. [7] for the long-distance contribution to the branching ratio for $K_L \rightarrow \mu^+ \mu^-$ to obtain $|A_1^S| = 1.58 \times 10^{-12}$, with an unspecified uncertainty. For the correction to be below a percent we must have $|A_1^L| \lesssim 2 \times 10^{-15}$ for $\cos(\phi_0 - \phi_1) \sim \mathcal{O}(1)$ and $|A_1^L| \lesssim 5 \times 10^{-14}$ if $\cos(\phi_0 - \phi_1) \sim 0$. This can be compared to a naive estimate for A_1^L , namely, $A_1^L = A_1^L|_{\epsilon_K=0} + \epsilon_K A_1^S$. Using $A_1^L|_{\epsilon_K=0} = 0$, this gives $|A_1^L| \sim 3.52 \times 10^{-15}$. We see that corrections to eq. (1.2) of the order of one percent are expected.

⁴We neglected the tiny correction factor $1/\sqrt{1 + |\bar{\epsilon}|^2}$ in the coefficients.

Note that the complex conjugation acts only on the Wilson coefficients in the Lagrangian. Combining eqs. (3.2) and (3.3), we obtain

$$A_0^S \equiv \langle (\mu^+ \mu^-)_{\ell=0} | \mathcal{L}_{\text{eff}}^{|\Delta S|=1} | K_S \rangle = \overbrace{\sqrt{2}i \langle (\mu^+ \mu^-)_{\ell=0} | \text{Im}(\mathcal{L}_{\text{eff}}^{\Delta S=1}) | K^0 \rangle}^{=A_0^S|_{\epsilon_K=0} \text{ (see section 3.1)}} + \underbrace{\sqrt{2}\epsilon_K \langle (\mu^+ \mu^-)_{\ell=0} | \text{Re}(\mathcal{L}_{\text{eff}}^{\Delta S=1}) | K^0 \rangle}_{\text{(see section 3.2)}}. \quad (3.4)$$

We see that, in the first term, only the imaginary part of the Wilson coefficients in the effective Lagrangian contributes. To the extent that we assume that the LD contributions are CP conserving, this term arises only from the operator Q_μ and can be calculated using perturbation theory, see section 3.1. The second term, proportional to ϵ_K , is dominated by LD contributions that are hard to calculate. We will show in section 3.2 that they can be estimated from data.

3.1 Perturbative CP-violating contribution from $\text{Im}(\mathcal{L}_{\text{eff}}^{|\Delta S|=1})$

The contribution to $\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}$ proportional to $|A_0^S|_{\epsilon_K=0}|^2$ from the first term in eq. (3.4) involves the imaginary part of the $|\Delta S| = 1$ Wilson coefficients. It is thus proportional to the short-distance top-quark contribution of the effective Lagrangian in eq. (2.7) proportional to $\text{Im}(V_{ts}^* V_{td}) \times Y_t = A^2 \lambda^5 \bar{\eta} \times Y_t$. In what follows, we denote this contribution to the branching ratio by $\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}^{\text{pert.}}$.

The Y_t function receives perturbative QCD and EW corrections. With respect to QCD, there is no renormalization-group (RG) evolution for the Q_μ operator, thus the QCD corrections for Y_t are the same as for the $B_q \rightarrow \mu^+ \mu^-$ decays. They have been computed with next-to-next-to-leading accuracy in ref. [11]; the residual scale uncertainty is below 0.1% on Y_t . Here, we simply update the numerical value for the top quark. The next-to-leading-order EW corrections have been computed for the $B_q \rightarrow \mu^+ \mu^-$ decays in ref. [10]. In this case, there is a mixed QCD \times QED RG evolution that involves operator mixing. To consistently include the EW corrections we extend the evolution down to 2 GeV (see ref. [17] and appendix B in ref. [10]); the residual scheme and scale uncertainty is 0.5% on Y_t . Including both next-to-next-to-leading-order QCD and next-to-leading-order EW corrections we find for $\mu_{\text{low}} = 2$ GeV:

$$Y_t = 0.931 \pm 0.001|_{\text{QCD}} \pm 0.005|_{\text{EW}}, \quad (3.5)$$

where the uncertainties corresponding to residual scale and scheme uncertainties. All input is taken from ref. [9], in particular we use for the top-quark mass $M_t^{\text{pole}} = 172.5(7)$ GeV from the cross-section measurements. The W -boson mass is not a primary input, we calculate it as a function of the Z -boson mass, the Higgs-boson mass, and the strong and the electromagnetic coupling constants α_s and α , respectively (see ref. [18]).

The CP-violating contribution to the decay $K_S \rightarrow (\mu^+ \mu^-)_{\ell=0}$ from the imaginary part of the effective Lagrangian in eq. (2.7) then reads [6, 8]

$$\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}^{\text{pert.}} = \frac{\beta_\mu \tau_S}{16\pi M_K} A^4 \lambda^{10} \bar{\eta}^2 \left| \frac{2G_F^2 M_W^2}{\pi^2} f_K M_K m_\mu Y_t \right|^2. \quad (3.6)$$

Using eq. (3.5), $f_K = 155.7(3)$ MeV [19], and all remaining input parameters from ref. [9] (apart from the W -boson mass that is not an independent input parameter — see above) we find

$$\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}^{\text{pert.}} = 1.70(02)_{\text{QCD/EW}}(01)_{f_K}(19)_{\text{param.}} \times 10^{-13}. \quad (3.7)$$

See the discussion at the end of section 3.2 for the error budget of the parametric uncertainties. Using the normalisation as in eq. (2.1), this branching ratio corresponds to $|A_0^S|_{\epsilon_K=0}| = 2.64 \times 10^{-13}$.

3.2 Contribution proportional to ϵ_K and $\text{Re}(\mathcal{L}_{\text{eff}}^{|\Delta S|=1})$

After having calculated the perturbative contribution to the $K_S \rightarrow (\mu^+ \mu^-)_{\ell=0}$ decay rate, it remains to estimate the correction proportional to ϵ_K . We recall that the $K_L \rightarrow \mu^+ \mu^-$ decay rate is fully dominated by the amplitude A_0^L . To relate this amplitude to the term in eq. (3.4) proportional to ϵ_K , we use eq. (3.3) to rewrite

$$\begin{aligned} 2\langle(\mu^+ \mu^-)_{\ell=0}|\text{Re}(\mathcal{L}_{\text{eff}}^{\Delta S=1})|K^0\rangle &= \langle(\mu^+ \mu^-)_{\ell=0}|\mathcal{L}_{\text{eff}}^{\Delta S=1}|K^0\rangle + \langle(\mu^+ \mu^-)_{\ell=0}|(\mathcal{L}_{\text{eff}}^{\Delta S=1})^*|K^0\rangle \\ &= \langle(\mu^+ \mu^-)_{\ell=0}|\mathcal{L}_{\text{eff}}^{\Delta S=1}|K^0\rangle + \langle(\mu^+ \mu^-)_{\ell=0}|\mathcal{L}_{\text{eff}}^{\Delta S=-1}|\bar{K}^0\rangle \\ &= \sqrt{2}\underbrace{\langle(\mu^+ \mu^-)_{\ell=0}|\mathcal{L}_{\text{eff}}^{|\Delta S|=1}|K_L\rangle}_{\equiv A_0^L} + \mathcal{O}(\epsilon_K). \end{aligned} \quad (3.8)$$

Combining this with eq. (3.4) we have

$$A_0^S = A_0^S|_{\epsilon_K=0} + \epsilon_K A_0^L + \mathcal{O}(\epsilon_K^2). \quad (3.9)$$

For the decay rate, we need the absolute value squared of the amplitude. Keeping terms up to first order in ϵ_K , we find

$$|A_0^S|^2 = |A_0^S|_{\epsilon_K=0}|^2 + 2\text{Re}\left\{\left(A_0^S|_{\epsilon_K=0}\right)^* A_0^L \epsilon_K\right\} + \mathcal{O}(\epsilon_K^2). \quad (3.10)$$

To linear order in ϵ_K , we can approximate $A_0^S|_{\epsilon_K=0}$ by just A_0^S in the term proportional to ϵ_K . Defining the relative phase $\phi_0 = \arg\{(A_0^S)^* A_0^L\}$ as in ref. [8], we have

$$\text{Re}\left\{\left(A_0^S\right)^* A_0^L \epsilon_K\right\} = \frac{|\epsilon_K|}{\sqrt{2}} |A_0^S| |A_0^L| (\cos \phi_0 - \sin \phi_0), \quad (3.11)$$

where we used [9]

$$\epsilon_K \approx \frac{1+i}{\sqrt{2}} |\epsilon_K|. \quad (3.12)$$

Therefore, the branching ratio including the effects of indirect CP violation is obtained via

$$\begin{aligned} \text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0} &= \text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}^{\text{pert.}} \\ &\times \left(1 + \sqrt{2} |\epsilon_K| \frac{|A_0^L|}{|A_0^S|} (\cos \phi_0 - \sin \phi_0)\right), \end{aligned} \quad (3.13)$$

up to tiny corrections of order ϵ_K^2 . Even without further knowledge about the size of the phase ϕ_0 we can estimate the maximal size of the correction to the branching ratio, by

observing that $|\cos \phi_0 - \sin \phi_0| \leq \sqrt{2}$ and extracting $|A_0^L|$ from data ($|A_0^S| \simeq 2.64 \times 10^{-13}$ has been calculated in the previous section). Using the fact that the $K_L \rightarrow \mu^+ \mu^-$ decay is completely dominated by the $\ell = 0$ amplitude, the value of $|A_0^L|$ can be extracted from the measured branching ratio $\text{Br}(K_L \rightarrow \mu^+ \mu^-) = (6.84 \pm 0.11) \times 10^{-9}$ [9]. Using eq. (2.1) and the K_L lifetime $\tau_L = 5.116(21) \times 10^{-8} \text{ s}$ [9], we find $|A_0^L| \simeq 2.22(2) \times 10^{-12}$. Together with $|\epsilon_K| = (2.228 \pm 0.011) \times 10^{-3}$ [9], this leads to a maximal correction factor of (1 ± 0.037) such that

$$\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0} = \text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}^{\text{pert.}} \times r_{\text{ICPV}}, \quad (3.14)$$

where

$$r_{\text{ICPV}} = 1 + 0.037 \frac{\cos \phi_0 - \sin \phi_0}{\sqrt{2}}. \quad (3.15)$$

Accordingly, without any knowledge of the phase, the SM prediction of $\text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0}$ is afflicted with an uncertainty of up to $\pm 3.7\%$ that was previously not accounted for.

Interestingly, it has been shown recently how to extract precise information on the relative phase ϕ_0 from $K_L \rightarrow \mu^+ \mu^-$ and $K_L \rightarrow \gamma\gamma$ data [20]. It was found that $\cos^2 \phi_0 = 0.96 \pm 0.03$, where the error includes both experimental and theoretical uncertainties, see ref. [20] for details. This leads to four allowed values for the relative phase ϕ_0 , with two corresponding to a negative sign of $\cos \phi_0$, and another two corresponding to a positive sign. The negative sign is preferred from theoretical arguments, as discussed in detail in refs. [6, 20]. The resulting branching ratios are shown in table 1. For the CKM input we have used the current SM fit from ref. [9] giving $\lambda = 0.22500(67)$, $A = 0.826_{-0.015}^{+0.018}$, $\bar{\rho} = 0.159(10)$, $\bar{\eta} = 0.348(10)$. We see that the non-parametric uncertainty, arising from unknown higher-order perturbative corrections and the kaon decay constant, is of the order of 1%, similar as for the $K_L \rightarrow \pi^0 \nu \bar{\nu}$ mode. The dominant (relative) parametric uncertainties arise from the CKM input parameters and are 8.7% from A , 5.7% from $\bar{\eta}$, 3.0% from λ , as well as 1.3% from the top quark mass; the remaining parametric uncertainties (including those related to the relative phase ϕ_0) are at the permil level and thus negligible.

We note in this context that in ref. [21] the ratio $R_{\text{SL}} = \text{Br}(K_S \rightarrow \mu^+ \mu^-)_{\ell=0} / \text{Br}(K_L \rightarrow \pi^0 \nu \bar{\nu})$ has been suggested as a null-test of the SM that is free of CKM input parameters.⁵ The effect of indirect CP violation calculated here amounts to a small correction to R_{SL} of the order of a few percent, in which, however, the CKM parameters do not cancel.

4 Discussion and conclusions

The decay $K_S \rightarrow (\mu^+ \mu^-)_{\ell=0}$ is almost purely CP violating and dominated, in the SM and many of its extensions, by short-distance contributions that can be calculated perturbatively with high precision. It has been demonstrated in refs. [7, 8] that this branching ratio can, in principle, be extracted from neutral kaon interference data, thus adding a new precision test of the SM in the kaon sector.

⁵The dependence of this ratio on λ^2 , as it appears in ref. [21], is spurious in the sense that it assumes the specific parameterization of the hadronic matrix elements for the $K_{\ell 3}$ decays employed in the fit in ref. [3].

$\cos(\phi_0)$	$\sin(\phi_0)$	r_{ICPV}	$\text{Br}(K_S \rightarrow \mu^+\mu^-)_{\ell=0}$
0.98	0.20	1.021	$1.74(2)(19) \times 10^{-13}$
0.98	-0.20	1.031	$1.75(2)(19) \times 10^{-13}$
-0.98	0.20	0.969	$1.65(2)(18) \times 10^{-13}$
-0.98	-0.20	0.979	$1.67(2)(18) \times 10^{-13}$

Table 1. Predictions for the $K_S \rightarrow (\mu^+\mu^-)_{\ell=0}$ branching ratio in dependence on the different values for the strong phase ϕ_0 . The first uncertainty comprises the missing higher-order perturbative corrections as well as the uncertainty in f_K , while the second uncertainty is parametric (dominated by the uncertainty in the CKM input parameters).

In this work, we revisited the SM prediction of $\text{Br}(K_S \rightarrow \mu^+\mu^-)_{\ell=0}$. To the extent that indirect CP violation in the neutral kaon sector is neglected, the only relevant hadronic input parameter is the kaon decay constant, which is known with permil accuracy. The SD contribution is comprised by the loop function Y_t [15]; results for the NNLO QCD [11] and NLO electroweak [10] corrections are available in the literature and have been combined here into a state-of-the-art SM prediction.

Upon close inspection, it turns out that eq. (1.2), relating the searched-for branching ratio to measurable quantities, comprises the effects of indirect CP violation. It is, therefore, mandatory to include these effects also in the SM prediction. Maybe somewhat surprisingly, the impact of indirect CP violation on the branching ratio is enhanced over the naive estimate $\epsilon_K \sim 10^{-3}$ by about an order of magnitude, due to non-perturbative effects of the strong interactions that are hard to calculate. Estimating the size of the correction with the help of available data, we found that this can shift the SM prediction by up to $\pm 3.7\%$; an effect that was previously unaccounted for. Using a recent estimate of the relative phase ϕ_0 that was determined up to a four-fold ambiguity, we find that the correction to the branching ratio is either $\pm 2\%$ or $\pm 3\%$ (see table 1), with the negative values preferred from theoretical arguments.

Finally, we briefly summarize the assumptions that enter at various stages of the analysis (see ref. [8] for details). We start with an assumption that we *do not* need to make: as shown in this work, CP violation in neutral kaon mixing can consistently be taken into account. We made, however, the following assumptions: (i) the only source of CP violation is in the $\ell = 0$ amplitude; and (ii) the long-distance physics is CP conserving. Both assumptions are satisfied to very good approximation in the SM. Assumption (i) is also valid in all SM extensions in which the leading operator is vectorial.

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