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Generation of isospin sum rules in heavy hadron weak decays

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Abstract: Isospin symmetry is the most precise flavor symmetry. In this work, we propose an approach to generate isospin sum rules for heavy hadron decays without the Wigner-Eckhart invariants. The effective Hamiltonian of heavy quark weak decay is fully invariant under a series of isospin lowering operators I_-^n and then the isospin sum rules can be generated through several master formulas. It provides a systematic way to study the isospin symmetry of c- and b-hadron weak decays. The theoretical framework of this approach is presented in detail with the nonleptonic decays of D and B mesons as examples. In addition, the V-/U-spin sum rules are derived in a similar algorithm by replacing I_-^n with V_-^n/U_-^n.

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Contents

1 Introduction

Flavor symmetry, as a powerful tool to analyze heavy meson and baryon weak decays, has been extensively studied in literature [1–67]. It leads to linear relations between amplitudes of some hadronic processes, known as flavor sum rules. Isospin symmetry is the most precise flavor symmetry. Isospin breaking is naively expected as $\delta_I \simeq (m_u - m_d)/\Lambda_{\rm QCD} \sim 1\%$, while V/U-spin breaking is $\delta_{V/U} \simeq m_s/\Lambda_{\rm QCD} \sim 30\%$. Isospin sum rules could provide knowledge on unmeasured channels and be used to extract useful information of hadronic dynamics. For instance, the isospin sum rule of $B \to \pi\pi$ system is critical in the Gronau-London method [68] of determining the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing parameter $\alpha \equiv Arg[V_{td}V_{tb}^*/V_{ud}V_{ub}^*]$.

Flavor sum rules are usually found by observing decay amplitudes expressed by the Wigner-Eckhart invariants [69, 70]. For example, the isospin sum rule of $\overline{B} \to \pi\pi$ system is derived from the isospin decompositions of $B^- \to \pi^0\pi^-$, $\overline{B}^0 \to \pi^+\pi^-$ and $\overline{B}^0 \to \pi^0\pi^0$ modes. A useful method for generating SU(3) sum rules for charm meson decays without the Wigner-Eckhart invariants was proposed in [71]. The effective Hamiltonian of charm quark decay is invariant under operators T_- and S, which allows us to generate SU(3) sum rules through several master formulas. This approach has been extended to singly and doubly charmed baryon decays [72]. However, T_- is a linear combination of isospin and V-spin operators and S is a linear combination of three U-spin operators. Isospin sum rules cannot be generated by T_- and S.

In this work, we propose an approach to generate isospin sum rules by a series of isospin lowering operators I_-^n. Isospin sum rules are derived though several master formulas. Taking the nonleptonic decays of D and B mesons as examples, our method is shown in detail. The V- and U-spin sum rules can also be derived in a similar algorithm by replacing I_-^n with V_-^n and U_-^n. This approach could be easily applied to other decay modes such as heavy baryon decays, multi-body decays, etc. It provides a systematic way to analyze flavor symmetry in c- and b-hadron decays.

The rest of this paper is structured as follows. In section 2, the $D \to PP$ modes are selected as examples to introduce the theoretical framework of generating isospin sum rules. The isospin sum rules of B meson decays are discussed in section 3. Section 4 is a short summary. And the V- and U-spin sum rules are investigated in appendices A and B, respectively.

2 Isospin sum rules in the $D \to PP$ decays

In this section, we present our theoretical framework for generating isospin sum rules, taking the nonleptonic D meson decays as examples. The decays of charm quark are classified as Cabibbo-favored (CF), singly Cabibbo-suppressed (SCS), and doubly Cabibbo-suppressed (DCS) decays. The flavor structures of CF, SCS and DCS decays are $c \to s\bar{d}u$, $c \to d\bar{d}u/s\bar{s}u$, $c \to d\bar{s}u$, respectively. For CF decay, isospin and its third component change as $\Delta I = 1$, $\Delta I_3 = 1$. For SCS decay, isospin and its third component change as $\Delta I = 3/2$ or $1/2$, $\Delta I_3 = 1/2$. And for DCS decay, isospin and its third component change as $\Delta I = 1$ or 0, $\Delta I_3 = 0$. There exists a basis in which isospin sum rules involve only CF, SCS or DCS decays respectively.

The effective Hamiltonian of charm quark decay in the Standard Model (SM) is [73]

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}}\left[\sum_{q=d,s} V_{cq_1}^* V_{uq_2}\left(\sum_{i=1}^{2} C_i(\mu)O_i(\mu)\right) - V_{cb}^* V_{ub}\left(\sum_{i=3}^{6} C_i(\mu)O_i(\mu) + C_{8g}(\mu)O_{8g}(\mu)\right)\right] + h.c., \tag{2.1}$$

where the tree operators are

$$O_1 = (\bar{u}_\alpha q_{2\beta})_{V-A}(\bar{q}_{1\beta}c_\alpha)_{V-A}, \qquad O_2 = (\bar{u}_\alpha q_{2\alpha})_{V-A}(\bar{q}_{1\beta}c_\beta)_{V-A}, \tag{2.2}$$

with α, β being color indices. The QCD penguin operators are

$$\begin{aligned} O_3 &= (\bar{u}_\alpha c_\alpha)_{V-A} \sum_{q'=u,d,s} (\bar{q}'_\beta q'_\beta)_{V-A}, \qquad & O_4 &= (\bar{u}_\alpha c_\beta)_{V-A} \sum_{q'=u,d,s} (\bar{q}'_\beta q'_\alpha)_{V-A}, \\ O_5 &= (\bar{u}_\alpha c_\alpha)_{V-A} \sum_{q'=u,d,s} (\bar{q}'_\beta q'_\beta)_{V+A}, \qquad & O_6 &= (\bar{u}_\alpha c_\beta)_{V-A} \sum_{q'=u,d,s} (\bar{q}'_\beta q'_\alpha)_{V+A}. \end{aligned} \tag{2.3}$$

The chromomagnetic penguin operator is

$$O_{8g} = \frac{g_s}{8\pi^2} m_c \bar{u}_\alpha \sigma_{\mu\nu}(1+\gamma_5)T^a_{\alpha\beta}G^{a\mu\nu}c_\beta, \tag{2.4}$$

which can be included into the penguin operators [74–76]. In the SU(3) picture, the effective Hamiltonian of charm quark decay is written as [62]

$$\mathcal{H}_{\rm eff} = \sum_{i,j,k=1}^{3} H_k^{ij} O_k^{ij} = \sum_{i,j,k=1}^{3} H_k^{ij} (\bar{q}^i q_k)(\bar{q}^j c). \tag{2.5}$$

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The coefficient matrix H is obtained from the map $(\bar{u}q_1)(\bar{q}_2c) \to V^*_{cq_2}V_{uq_1}$ for current-current operators and $(\bar{q}'q')(\bar{u}c) \to -V^*_{cb}V_{ub}$ for penguin operators. Since q_1 and q_2 could be d or s quark and q' could be u, d or s quark according to eq. (2.1), the non-zero H^{ij}_k induced by tree and penguin operators include

$$\begin{aligned}
&\{H^{(0)}\}^{13}_2 = V^*_{cs}V_{ud}, \quad \{H^{(0)}\}^{12}_2 = V^*_{cd}V_{ud}, \quad \{H^{(0)}\}^{13}_3 = V^*_{cs}V_{us}, \quad \{H^{(0)}\}^{12}_3 = V^*_{cd}V_{us},\\
&\{H^{(1)}\}^{11}_1 = -V^*_{cb}V_{ub}, \quad \{H^{(1)}\}^{21}_2 = -V^*_{cb}V_{ub}, \quad \{H^{(1)}\}^{31}_3 = -V^*_{cb}V_{ub},
\end{aligned} \tag{2.6}$$

where superscripts (0) and (1) are used to differentiate the tree and penguin contributions.

The light pseudoscalar meson state is expressed as $|P_\alpha\rangle = (P_\alpha)^i_j|P^i_j\rangle$, in which $|P^i_j\rangle$ is the quark composition $|P^i_j\rangle = |q_i\bar{q}_j\rangle$ and (P_α) is the coefficient matrix. In the SU(3) picture, the pseudoscalar meson octet $|P_8\rangle$ is expressed as

$$|P_8\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}}|\pi^0\rangle + \frac{1}{\sqrt{6}}|\eta_8\rangle, & |\pi^+\rangle, & |K^+\rangle \\ |\pi^-\rangle, & -\frac{1}{\sqrt{2}}|\pi^0\rangle + \frac{1}{\sqrt{6}}|\eta_8\rangle, & |K^0\rangle \\ |K^-\rangle, & |\overline{K}^0\rangle, & -\sqrt{2/3}|\eta_8\rangle \end{pmatrix}. \tag{2.7}$$

The charmed meson state is expressed as $|D_\alpha\rangle = (|D^0\rangle,\ |D^+\rangle,\ |D^+_s\rangle)$. The decay amplitude of $D_\gamma \to P_\alpha P_\beta$ mode can be constructed as

$$\begin{aligned}
\mathcal{A}(D_\gamma \to P_\alpha P_\beta) &= \langle P_\alpha P_\beta|\mathcal{H}_{\rm eff}|D_\gamma\rangle\\
&= \sum_\omega (P_\alpha)^n_m\langle P^n_m|(P_\beta)^s_r\langle P^s_r||H^{jk}_l O^{jk}_l||(D_\gamma)_i|D_i\rangle\\
&= \sum_\omega \langle P^n_m P^s_r|O^{jk}_l|D_i\rangle \times (P_\alpha)^n_m(P_\beta)^s_r H^{jk}_l(D_\gamma)_i\\
&= \sum_\omega X_\omega(C_\omega)_{\alpha\beta\gamma}.
\end{aligned} \tag{2.8}$$

According to the Wigner-Eckhart theorem [69, 70], $X_\omega = \langle P^n_m P^s_r|O^{jk}_l|D_i\rangle$ is the reduced matrix element that is independent of α, β and γ. All information about initial/final states is absorbed into the Clebsch-Gordan (CG) coefficient $(C_\omega)_{\alpha\beta\gamma} = (P_\alpha)^n_m(P_\beta)^s_r H^{jk}_l(D_\gamma)_i$.

In general, the flavor sum rules are derived by writing decay amplitudes and combining several modes to form a polygon in the complex plane. However, this method is laborious and unsystematic. The authors of ref. [71] proposed an approach to generate flavor sum rules for charmed meson decays without the Wigner-Eckhart invariants. The idea is that if there is an operator T under which $TH = 0$, it follows that

$$\langle P_\alpha P_\beta|T\mathcal{H}_{\rm eff}|D_\gamma\rangle = \sum_\omega \langle P^n_m P^s_r|O^{jk}_l|D_i\rangle \times (P_\alpha)^n_m(P_\beta)^s_r(TH)^{jk}_l(D_\gamma)_i = 0. \tag{2.9}$$

Operator T can be applied to the initial/final states rather than the effective Hamiltonian. Then the l.h.s. of eq. (2.9) is turned into a sum of several decay amplitudes and eq. (2.9) becomes a flavor sum rule.

It is found in ref. [71] that the effective Hamiltonian is invariant under T_- and S, i.e., $T_-H = 0$, $SH = 0$. T_- and S are expressed as [71]

$$T_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -\lambda^2 & \lambda \end{pmatrix}, \tag{2.10}$$

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where λ is a Wolfenstein parameter given by $\lambda \approx 0.225$ [77]. S is a linear combination of U-spin operators and T_- is a linear combination of isospin and V-spin operators,

$$S = -\lambda U_3 - \lambda^2 U_- + U_+, \qquad T_- = I_- + \lambda V_-. \tag{2.11}$$

The SU(3) sum rules generated by S and T_- are U-spin sum rules and combinations of isospin and V-spin sum rules, respectively. Besides, the premise of $T_- H = 0$ and $SH = 0$ is that the CKM matrix elements in charm decay are approximated to be $V_{ud} \approx 1$, $V_{us} \approx \lambda$, $V_{cd} \approx -\lambda$, $V_{cs} \approx 1$. If the next order correction is included in the Wolfenstein parametrization, we have $T_- H \neq 0$ and $SH \neq 0$. So the SU(3) sum rules generated though S and T_- dependent on the Wolfenstein approximation of the CKM matrix elements in charm sector. S and T_- cannot be used to construct SU(3) sum rules of b-hadron decay.

The three operators associated with isospin are I_3, I_+ and I_-. In this work, we try to establish the master formulas of isospin sum rules through the isospin lowering operator I_-, where I_- is expressed as

$$I_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.12}$$

To achieve this goal, we decompose the four-quark operator O^{ij}_k to SU(3) irreducible representations, $3 \otimes \overline{3} \otimes 3 = 3_p \oplus 3_t \oplus \overline{6} \oplus 15$. The explicit decomposition is [71]

$$O^{ij}_k = \frac{1}{8} O(15)^{ij}_k + \frac{1}{4}\epsilon^{ijl} O(\overline{6})_{lk} + \delta^j_k \left(\frac{3}{8} O(3_t)^i - \frac{1}{8} O(3_p)^i\right) + \delta^i_k \left(\frac{3}{8} O(3_p)^j - \frac{1}{8} O(3_t)^j\right). \tag{2.13}$$

According to the map rule below eq. (2.5), the non-zero coefficients corresponding to the tree operators include

$$\begin{aligned}
&\{H^{(0)}(\overline{6})\}_{22} = -2V^*_{cs}V_{ud}, \quad \{H^{(0)}(\overline{6})\}_{23} = (V^*_{cd}V_{ud} - V^*_{cs}V_{us}), \quad \{H^{(0)}(\overline{6})\}_{33} = 2V^*_{cd}V_{us},\\
&\{H^{(0)}(15)\}^{11}_1 = -2(V^*_{cd}V_{ud} + V^*_{cs}V_{us}), \quad \{H^{(0)}(15)\}^{13}_2 = 4V^*_{cs}V_{ud}, \quad \{H^{(0)}(15)\}^{12}_3 = 4V^*_{cd}V_{us},\\
&\{H^{(0)}(15)\}^{12}_2 = 3V^*_{cd}V_{ud} - V^*_{cs}V_{us}, \quad \{H^{(0)}(15)\}^{13}_3 = 3V^*_{cs}V_{us} - V^*_{cd}V_{ud},\\
&\{H^{(0)}(3_t)\}^1 = V^*_{cd}V_{ud} + V^*_{cs}V_{us}.
\end{aligned} \tag{2.14}$$

The non-zero coefficients corresponding to the penguin operators include

$$\{H^{(1)}(3_t)\}^1 = -V^*_{cb}V_{ub}, \qquad \{H^{(1)}(3_p)\}^1 = -3V^*_{cb}V_{ub}. \tag{2.15}$$

The $\overline{6}$ representation can be written in matrix form as $[H^{(0)}(\overline{6})]^i_j = [H^{(0)}(\overline{6})]_{ij}$ with

$$[H^{(0)}(\overline{6})] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\,V^*_{cs}V_{ud} & (V^*_{cd}V_{ud} - V^*_{cs}V_{us}) \\ 0 & (V^*_{cd}V_{ud} - V^*_{cs}V_{us}) & 2\,V^*_{cd}V_{us} \end{pmatrix}. \tag{2.16}$$

Under the isospin lowering operator I_-, $[H^{(0)}(\overline{6})]$ is transformed as

$$I_-[H^{(0)}(\overline{6})] = I_- \cdot [H^{(0)}(\overline{6})] + I_- \cdot [H^{(0)}(\overline{6})]^T = 0, \tag{2.17}$$

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where symbol "·" represents the dot product of two matrices and superscript T represents the transposition of matrix. The three 3-dimensional presentations are written in matrix form as

$$[H^{(0)}(3_t)] = (\, V_{cd}^* V_{ud} + V_{cs}^* V_{us},\, 0,\, 0\,), \tag{2.18}$$

$$[H^{(1)}(3_t)] = (\, -V_{cb}^* V_{ub},\, 0,\, 0\,), \tag{2.19}$$

$$[H^{(1)}(3_p)] = (\, -3V_{cb}^* V_{ub},\, 0,\, 0\,). \tag{2.20}$$

Under the isospin lowering operator I_-, $[H^{(0,1)}(3_{t,p})]$ are transformed as

$$I_-[H^{(0,1)}(3_{t,p})] = [H^{(0,1)}(3_{t,p})] \cdot I_- = 0. \tag{2.21}$$

One can find $[H^{(0)}(\overline{6})]$ and $[H^{(0,1)}(3_{t,p})]$ are zero under I_-. The $\underline{15}$ representation is written in matrix form as $\{[H^{(0)}(15)]_i\}^k_j = [H^{(0)}(15)]^{ij}_k$ with

$$[H^{(0)}(15)]_1 = \begin{pmatrix} -2\,(V_{cd}^* V_{ud} + V_{cs}^* V_{us}) & 0 & 0 \\ 0 & (3\,V_{cd}^* V_{ud} - V_{cs}^* V_{us}) & 4\,V_{cs}^* V_{ud} \\ 0 & 4\,V_{cd}^* V_{us} & (3\,V_{cs}^* V_{us} - V_{cd}^* V_{ud}) \end{pmatrix}, \tag{2.22}$$

$$[H^{(0)}(15)]_2 = \begin{pmatrix} 0 & 0 & 0 \\ (3\,V_{cd}^* V_{ud} - V_{cs}^* V_{us}) & 0 & 0 \\ 4\,V_{cd}^* V_{us} & 0 & 0 \end{pmatrix}, \tag{2.23}$$

$$[H^{(0)}(15)]_3 = \begin{pmatrix} 0 & 0 & 0 \\ 4\,V_{cs}^* V_{ud} & 0 & 0 \\ (3\,V_{cs}^* V_{us} - V_{cd}^* V_{ud}) & 0 & 0 \end{pmatrix}. \tag{2.24}$$

The tensor transformation law of $[H^{(0)}(15)]$ under I_- is

$$\{I_-[H^{(0)}(15)]_i\}^k_j = 2\,\{[H^{(0)}(15)]_{(i} \cdot I_-\}^k_{j)} - \{I_- \cdot [H^{(0)}(15)]_i\}^k_j. \tag{2.25}$$

Under the isospin lowering operator I_-, $[H^{(0)}(15)]_{2,3}$ are zero but $[H^{(0)}(15)]_1$ is non-zero,

$$I_-[H^{(0)}(15)]_{2,3} = 0, \qquad I_-[H^{(0)}(15)]_1 = \begin{pmatrix} 0 & 0 & 0 \\ 8\,V_{cd}^* V_{ud} & 0 & 0 \\ 8\,V_{cd}^* V_{us} & 0 & 0 \end{pmatrix} \neq 0. \tag{2.26}$$

Thereby, the isospin lowering operator I_- cannot be used to construct flavor sum rules directly like T_- and S.

Notice the matrix $I_-[H^{(0)}(15)]_1$ does not include $V_{cs}^* V_{ud}$. By combining with eqs. (2.17), (2.21) and (2.26), it is found that the Hamiltonian of CF decay is zero under I_-, i.e., $I_- H_{\rm CF} = 0$. Observe that the matrix $I_-[H^{(0)}(15)]_1$ is invariant under I_-,

$$I_-^2[H^{(0)}(15)]_1 = I_-\{I_-[H^{(0)}(15)]_1\} = 0. \tag{2.27}$$

So the Hamiltonian of SCS and DCS decays is zero under I_-^2, i.e., $I_-^2\, H_{\rm SCS,DCS} = 0$. In fact, we can define a series of operators I_-^n. The Hamiltonian of charm quark decay is zero under I_-^n if $n \geq 2$ since $I_- 0 = 0$. If the operator T in eq. (2.9) is replaced by I_-^n, eq. (2.9) will be an abstract isospin sum rule,

$$\langle P_\alpha P_\beta | I_-^n \mathcal{H}_{\rm eff} | D_\gamma \rangle = \sum_\omega \langle P_m^n P_r^s | O_l^{jk} | D_i \rangle \times (P_\alpha)_m^n (P_\beta)_r^s (I_-^n H)_l^{jk} (D_\gamma)_i = 0. \tag{2.28}$$

The derivation of eqs. (2.17), (2.21), (2.26) and (2.27) does not involve the values of CKM matrix elements. So the isospin sum rules generated from eq. (2.28) do not rely on any approximation of the CKM matrix.

The abstract isospin sum rule (2.28) becomes explicit isospin sum rules by applying I_-^n to initial/final states and computing the coefficients expanded by initial/final states as bases [72]. Under the isospin lowering operator I_-, we have

$$I_- | D_\gamma \rangle = \sum_\alpha | D_\alpha \rangle \langle D_\alpha | I_- | D_\gamma \rangle = \sum_\alpha (D^\alpha)^j [I_-]_j^i (D_\gamma)_i | D_\alpha \rangle = \sum_\alpha \{[I_-]_D\}_\gamma^\alpha | D_\alpha \rangle. \tag{2.29}$$

$[I_-]_D$ is the coefficient matrix of $I_- | D_\gamma \rangle$ expanded by $| D_\alpha \rangle$, which is derived to be

$$[I_-]_D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.30}$$

The isospin lowering operator I_- acting on a pseudoscalar meson octet is a commutator,

$$\begin{aligned} I_- \langle [P_8]_\alpha | &= [I_-, \langle [P_8]_\alpha |] = I_- \cdot \langle [P_8]_\alpha | - \langle [P_8]_\alpha | \cdot I_- \\ &= \sum_\beta \mathrm{Tr}\{[\, I_-, [P_8]_\alpha \,] \cdot [P_8]_\beta^T\} \langle [P_8]_\beta | = \sum_\beta \{[I_-]_{P_8}\}_\alpha^\beta \langle [P_8]_\beta |, \end{aligned} \tag{2.31}$$

where $[I_-]_{P_8}$ is coefficient matrix of commutator $[\, I_-, \langle [P_8]_\alpha | \,]$ expanded by $\langle [P_8]_\beta |$. If we define pseudoscalar meson octet as

$$\langle [P_8]_\beta | = (\langle \pi^+ |,\ \langle \pi^0 |,\ \langle \pi^- |,\ \langle K^+ |,\ \langle K^0 |,\ \langle \overline{K}^0 |,\ \langle K^- |,\ \langle \eta_8 |), \tag{2.32}$$

we get

$$[I_-]_{P_8} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.33}$$

With the matrices $[I_-]_D$ and $[I_-]_{P_8}$, the sum of decay amplitudes generated by I_- is written as

$$SumI_-[\gamma, \alpha, \beta] = \sum_\mu \Big[\{[I_-]_{P_8}\}_\alpha^\mu \mathcal{A}_{\gamma \to \mu\beta} + \{[I_-]_{P_8}\}_\beta^\mu \mathcal{A}_{\gamma \to \alpha\mu} + \{[I_-]_D\}_\gamma^\mu \mathcal{A}_{\mu \to \alpha\beta} \Big]. \tag{2.34}$$

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Since the effective Hamiltonian of CF decay is invariant under I_-, $SumI_-[\gamma,\alpha,\beta]$ is zero if it is a sum of amplitudes of several CF decay channels. An isospin sum rule is generated via eq. (2.34) if appropriate α, β and γ are selected. I_- is the isospin lowering operator. I_- acting on the final/initial state lowers/arises I_3 by one. Then α, β can be chosen as the states with the maximal I_3, and γ can be chosen as the state with the minimal I_3. In the $D \to PP$ decays, the choice of $\{\gamma,\alpha,\beta\} = \{D^0,\pi^+,\overline{K}^0\}$ generates an isospin sum rule as

$$SumI_-[D^0,\pi^+,\overline{K}^0] = -\sqrt{2}\,\mathcal{A}(D^0 \to \pi^0\overline{K}^0) - \mathcal{A}(D^0 \to \pi^+K^-) + \mathcal{A}(D^+ \to \pi^+\overline{K}^0) = 0. \tag{2.35}$$

For the SCS and DCS decays, equation $I_-^2 H_{\rm SCS,DCS} = 0$ indicates that the isospin sum rules are obtained by acting I_- on the final and initial states twice. Specifically, the isospin sum rule of singly Cabibbo-suppressed $D \to \pi\pi$ system is generated by I_-^2 with $\{\gamma,\alpha,\beta\} = \{D^0,\pi^+,\pi^+\}$,

$$\begin{aligned}SumI_-^2[D^0,\pi^+,\pi^+] &= -\sqrt{2}SumI_-[D^0,\pi^+,\pi^0] - \sqrt{2}SumI_-[D^0,\pi^0,\pi^+] + SumI_-[D^+,\pi^+,\pi^+]\\ &= 4\big[\mathcal{A}(D^0\to\pi^0\pi^0) - \mathcal{A}(D^0\to\pi^+\pi^-) - \sqrt{2}\mathcal{A}(D^+\to\pi^+\pi^0)\big] = 0.\end{aligned} \tag{2.36}$$

The isospin rum rule of doubly Cabibbo-suppressed $D \to K\pi$ decays is generated by I_-^2 with $\{\gamma,\alpha,\beta\} = \{D^0,\pi^+,K^+\}$,

$$\begin{aligned}SumI_-^2[D^0,\pi^+,K^+] &= SumI_-[D^0,\pi^+,K^0] - \sqrt{2}SumI_-[D^0,\pi^0,K^+] + SumI_-[D^+,\pi^+,K^+]\\ &= -2\big[\sqrt{2}\mathcal{A}(D^0\to\pi^0K^0) + \mathcal{A}(D^0\to\pi^-K^+)\\ &\quad + \sqrt{2}\mathcal{A}(D^+\to\pi^0K^+) - \mathcal{A}(D^+\to\pi^+K^0)\big] = 0.\end{aligned} \tag{2.37}$$

The three isospin sum rules derived from I_- and I_-^2 are consistent with the results given by ref. [71].

From above analysis, it is found that the isospin sum rules are obtained by applying I_-^n to the initial/final states if the effective Hamiltonian is invariant under I_-^n. Isospin sum rules can also be generated by isospin raising operators I_+^n. The results are the same as the ones derived from I_-^n. One should note that not arbitrary choices of $\{\gamma,\alpha,\beta\}$ and I_-^n generate isospin sum rules. There are two requirements for $\{\gamma,\alpha,\beta\}$ and I_-^n. Firstly, the choices of $\{\gamma,\alpha,\beta\}$ and I_-^n should be associated with physical amplitudes. For example, the choice of $\{\gamma,\alpha,\beta\} = \{D^+,\pi^+,\overline{K}^0\}$ and I_- cannot generate an isospin sum rule. It because that I_- is a QED charge lowering operator and then I_- acting on $\{D^+,\pi^+,\overline{K}^0\}$ cannot derive charge preserving decay amplitudes. The choice of $\{\gamma,\alpha,\beta\} = \{D^0,K^+,K^+\}$ and I_-^2 cannot generate an isospin sum rule since I_- does not change strangeness and $\Delta S = -2$ amplitudes are forbidden in charm decay. Secondly, the isospin sum rule is generated by I_-^n only if $n \geq 1$ for CF decay and $n \geq 2$ for SCS and DCS decays. For example, the sum of amplitudes derived by the choice of $\{\gamma,\alpha,\beta\} = \{D^0,K^+,\overline{K}^0\}$ and I_- is not zero because it is a sum of SCS amplitudes and $I_-H_{\rm SCS} \neq 0$,

$$SumI_-[D^0,K^+,\overline{K}^0] = \mathcal{A}(D^0 \to K^0\overline{K}^0) - \mathcal{A}(D^0 \to K^+K^-) + \mathcal{A}(D^+ \to K^+\overline{K}^0) \neq 0. \tag{2.38}$$

The change of strangeness in CF, SCS, DCS transitions are $\Delta S = -1$, 0 and 1, respectively. I_-^n cannot change strangeness, then we can distinguish the three decay modes though ΔS in $\gamma \to \alpha\beta$. Considering that I_- is a QED charge lowering operator, we conclude the selection rule of $\{\gamma, \alpha, \beta\}$. The choice of $\{\gamma, \alpha, \beta\}$ corresponding to a $\Delta Q = 1$ and $\Delta S = -1$ amplitude produces an isospin sum rule of CF mode. The choice of $\{\gamma, \alpha, \beta\}$ corresponding to a $\Delta Q = 2$ and $\Delta S = 0$ amplitude produces an isospin sum rule of SCS mode. And the choice of $\{\gamma, \alpha, \beta\}$ corresponding to a $\Delta Q = 2$ and $\Delta S = 1$ amplitude produces an isospin sum rule of DCS mode. For other choices, no sum rule is generated. In the $D \to PP$ decays, there are only four choices of $\{\gamma, \alpha, \beta\}$ satisfying above selection rule, $\{\gamma, \alpha, \beta\} = \{D^0, \pi^+, \overline{K}^0\}$, $\{D^0, \pi^+, \pi^+\}$, $\{D^0, \pi^+, K^+\}$ and $\{D_s^+, \pi^+, \pi^+\}$. The first three choices generate the isospin sum rules (2.35)$\sim$(2.37) respectively. And the choice of $\{\gamma, \alpha, \beta\} = \{D_s^+, \pi^+, \pi^+\}$ generates an isospin sum rule as

$$SumI_-\,[D_s^+, \pi^+, \pi^+] = -2\sqrt{2}\,\mathcal{A}(D_s^+ \to \pi^+\pi^0) = 0. \tag{2.39}$$

The approach for generating isospin sum rules can be extended to other decay modes such as B meson decays, heavy baryon decays, multi-body decays, etc. It provides a programmatic way to derive isospin sum rules for heavy hadron decays. In the next section, the applications of our method in the $\overline{B} \to DP$ and $\overline{B} \to PP$ decays are discussed. For the isospin sum rules of other heavy hadron decays and the phenomenological discussions, we will leave them in the future work. In addition, the V- and U-spin sum rules are derived by V_-^n and U_-^n operators with $D \to PP$, $\overline{B} \to DP$ and $\overline{B} \to PP$ decays as examples in appendices A and B.

3 Isospin sum rules in the B meson decays

3.1 Isospin sum rules in the $\overline{B} \to DP$ decays

The effective Hamiltonian of $b \to c\bar{u}q$ transition is given by [73]

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}} \sum_{q=d,s} V_{cb}V_{uq}^*\,[C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)] + h.c., \tag{3.1}$$

where the tree operators are

$$O_1 = (\bar{q}_\alpha u_\beta)_{V-A}(\bar{c}_\beta b_\alpha)_{V-A}, \qquad O_2 = (\bar{q}_\alpha u_\alpha)_{V-A}(\bar{c}_\beta b_\beta)_{V-A}. \tag{3.2}$$

In the SU(3) picture, O_j^i is decomposed into irreducible representations as $3 \otimes \overline{3} = 8 \oplus 1$. The non-zero CKM components include

$$\{H^{(0)}(8)\}_1^2 = V_{cb}V_{ud}^*, \qquad \{H^{(0)}(8)\}_1^3 = V_{cb}V_{us}^*. \tag{3.3}$$

$H^{(0)}(8)$ is written in matrix form as

$$[H^{(0)}(8)] = \begin{pmatrix} 0 & V_{cb}V_{ud}^* & V_{cb}V_{us}^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

Under I_-^n, $[H^{(0)}(8)]$ is transformed as

$$I_-[H^{(0)}(8)] = I_- \cdot [H^{(0)}(8)] - [H^{(0)}(8)] \cdot I_- = \begin{pmatrix} -V_{cb}V_{ud}^* & 0 & 0 \\ 0 & V_{cb}V_{ud}^* & V_{cb}V_{us}^* \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

$$I_-^2[H^{(0)}(8)] = I_-\{I_-[H^{(0)}(8)]\} = \begin{pmatrix} 0 & 0 & 0 \\ -2V_{cb}V_{ud}^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.6}$$

$$I_-^3[H^{(0)}(8)] = I_-\{I_-\{I_-[H^{(0)}(8)]\}\} = 0. \tag{3.7}$$

The effective Hamiltonian of $b \to c\bar{u}d$ ($b \to c\bar{u}s$) transition is zero under I_-^n with $n \geq 3$ ($n \geq 2$). So the isospin sum rules of $b \to c\bar{u}d$ ($b \to c\bar{u}s$) transition can be generated by I_-^n if $n \geq 3$ ($n \geq 2$).

Under the isospin lowering operator I_-, $[I_-]_{\overline{B}} = [I_-]_D$ if the $\overline{B}$ meson anti-triplet is defined as $|\overline{B}_\alpha\rangle = (|B^-\rangle, |\overline{B}^0\rangle, |\overline{B}^0_s\rangle)$. The sum of decay amplitudes generated from $\overline{B}_\gamma \to D_\alpha P_\beta$ under I_- is

$$SumI_-[\gamma,\alpha,\beta] = \sum_\mu \Big[\{[I_-]_D^T\}^\mu_\alpha \mathcal{A}_{\gamma\to\mu\beta} + \{[I_-]_{P_8}\}^\mu_\beta \mathcal{A}_{\gamma\to\alpha\mu} + \{[I_-]_{\overline{B}}\}^\mu_\gamma \mathcal{A}_{\mu\to\alpha\beta}\Big]. \tag{3.8}$$

The transposition of matrix $[I_-]_D$ is arisen from the initial-final transformation of D meson anti-triplet. With eq. (3.8), isospin sum rules in the $\overline{B} \to DP$ modes are derived to be

$$\begin{aligned} SumI_-^3[B^-,D^+,\pi^+] &= -SumI_-^2[B^-,D^0,\pi^+] - \sqrt{2}SumI_-^2[B^-,D^+,\pi^0] + SumI_-^2[\overline{B}^0,D^+,\pi^+] \\ &= 2\sqrt{2}SumI_-[B^-,D^0,\pi^0] - 2SumI_-[B^-,D^+,\pi^-] \\ &\quad -2SumI_-[\overline{B}^0,D^0,\pi^+] - 2\sqrt{2}SumI_-[\overline{B}^0,D^+,\pi^0] \\ &= 6\big[\mathcal{A}(B^-\to D^0\pi^-) + \sqrt{2}\mathcal{A}(\overline{B}^0\to D^0\pi^0) - \mathcal{A}(\overline{B}^0\to D^+\pi^-)\big] = 0, \end{aligned} \tag{3.9}$$

$$\begin{aligned} SumI_-^2[B^-,D^+,\overline{K}^0] &= -SumI_-[B^-,D^0,\overline{K}^0] - SumI_-[B^-,D^+,K^-] + SumI_-[\overline{B}^0,D^+,\overline{K}^0] \\ &= 2\big[\mathcal{A}(B^-\to D^0K^-) - \mathcal{A}(\overline{B}^0\to D^0\overline{K}^0) - \mathcal{A}(\overline{B}^0\to D^+K^-)\big] = 0, \end{aligned} \tag{3.10}$$

$$\begin{aligned} SumI_-^2[\overline{B}^0_s,D^+,\pi^+] &= -SumI_-[\overline{B}^0_s,D^0,\pi^+] - \sqrt{2}SumI_-[\overline{B}^0_s,D^+,\pi^0] \\ &= 2\big[\sqrt{2}\mathcal{A}(\overline{B}^0_s\to D^0\pi^0) - \mathcal{A}(\overline{B}^0_s\to D^+\pi^-)\big] = 0. \end{aligned} \tag{3.11}$$

3.2 Isospin sum rules in the $\overline{B} \to PP$ decays

The effective Hamiltonian of $b \to u\bar{u}q$ transition is given by [73]

$$\begin{aligned} \mathcal{H}_{\rm eff} &= \frac{G_F}{\sqrt{2}} \sum_{q=d,s} \left[V_{ub}^*V_{uq}\left(\sum_{i=1}^2 C_i^u(\mu)O_i^u(\mu)\right) + V_{cb}^*V_{cq}\left(\sum_{i=1}^2 C_i^c(\mu)O_i^c(\mu)\right)\right] \\ &\quad - \frac{G_F}{\sqrt{2}} \sum_{q=d,s} \left[V_{tb}V_{tq}^*\left(\sum_{i=3}^{10} C_i(\mu)O_i(\mu) + C_{7\gamma}(\mu)O_{7\gamma}(\mu) + C_{8g}(\mu)O_{8g}(\mu)\right)\right] + h.c.. \end{aligned} \tag{3.12}$$

The tree operators are

$$\begin{aligned} O_1^u &= (\bar{q}_\alpha u_\beta)_{V-A}(\bar{u}_\beta b_\alpha)_{V-A}, & O_2^u &= (\bar{q}_\alpha u_\alpha)_{V-A}(\bar{u}_\beta b_\beta)_{V-A}, \\ O_1^c &= (\bar{q}_\alpha c_\beta)_{V-A}(\bar{c}_\beta b_\alpha)_{V-A}, & O_2^c &= (\bar{q}_\alpha c_\alpha)_{V-A}(\bar{c}_\beta b_\beta)_{V-A}. \end{aligned} \tag{3.13}$$

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The QCD penguin operators are

$$
\begin{aligned}
O_3 &= (\bar{q}_\alpha b_\alpha)_{V-A} \sum_{q'=u,d,s} (\bar{q}'_\beta q'_\beta)_{V-A}, & O_4 &= (\bar{q}_\alpha b_\beta)_{V-A} \sum_{q'=u,d,s} (\bar{q}'_\beta q'_\alpha)_{V-A}, \\
O_5 &= (\bar{q}_\alpha b_\alpha)_{V-A} \sum_{q'=u,d,s} (\bar{q}'_\beta q'_\beta)_{V+A}, & O_6 &= (\bar{q}_\alpha b_\beta)_{V-A} \sum_{q'=u,d,s} (\bar{q}'_\beta q'_\alpha)_{V+A}.
\end{aligned} \tag{3.14}
$$

The QED penguin operators are

$$
\begin{aligned}
O_7 &= \frac{3}{2}(\bar{q}_\alpha b_\alpha)_{V-A} \sum_{q'=u,d,s} e_{q'} (\bar{q}'_\beta q'_\beta)_{V+A}, & O_8 &= \frac{3}{2}(\bar{q}_\alpha b_\beta)_{V-A} \sum_{q'=u,d,s} e_{q'} (\bar{q}'_\beta q'_\alpha)_{V+A}, \\
O_9 &= \frac{3}{2}(\bar{q}_\alpha b_\alpha)_{V-A} \sum_{q'=u,d,s} e_{q'} (\bar{q}'_\beta q'_\beta)_{V-A}, & O_{10} &= \frac{3}{2}(\bar{q}_\alpha b_\beta)_{V-A} \sum_{q'=u,d,s} e_{q'} (\bar{q}'_\beta q'_\alpha)_{V-A}.
\end{aligned} \tag{3.15}
$$

The electromagnetic penguin and chromomagnetic penguin operators are

$$
\begin{aligned}
O_{7\gamma} &= \frac{e}{8\pi^2} m_b \bar{q}_\alpha \sigma_{\mu\nu}(1+\gamma_5) F^{\mu\nu} b_\alpha, \\
O_{8g} &= \frac{g_s}{8\pi^2} m_b \bar{q}_\alpha \sigma_{\mu\nu}(1+\gamma_5) T^a_{\alpha\beta} G^{a\mu\nu} b_\beta.
\end{aligned} \tag{3.16}
$$

In the SU(3) picture, the coefficient matrices induced by $O^u_{1,2}$ are

$$
[H^{(0,u)}(\overline{6})] = \begin{pmatrix} 0 & -V_{ub}V^*_{us} & V_{ub}V^*_{ud} \\ -V_{ub}V^*_{us} & 0 & 0 \\ V_{ub}V^*_{ud} & 0 & 0 \end{pmatrix}, \tag{3.17}
$$

$$
[H^{(0,u)}(15)]_1 = \begin{pmatrix} 0 & 3V_{ub}V^*_{ud} & 3V_{ub}V^*_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.18}
$$

$$
[H^{(0,u)}(15)]_2 = \begin{pmatrix} 3V_{ub}V^*_{ud} & 0 & 0 \\ 0 & -2V_{ub}V^*_{ud} & -V_{ub}V^*_{us} \\ 0 & 0 & -V_{ub}V^*_{ud} \end{pmatrix}, \tag{3.19}
$$

$$
[H^{(0,u)}(15)]_3 = \begin{pmatrix} 3V_{ub}V^*_{us} & 0 & 0 \\ 0 & -V_{ub}V^*_{us} & 0 \\ 0 & -V_{ub}V^*_{ud} & -2V_{ub}V^*_{us} \end{pmatrix}, \tag{3.20}
$$

$$
[H^{(0,u)}(3_t)] = (\,0,\ V_{ub}V^*_{ud},\ V_{ub}V^*_{us}\,). \tag{3.21}
$$

The coefficient matrix induced by $O^c_{1,2}$ is

$$
[H^{(0,c)}(3_t)] = (\,0,\ V_{cb}V^*_{cd},\ V_{cb}V^*_{cs}\,). \tag{3.22}
$$

The coefficient matrices induced by penguin operators are

$$
[H^{(1)}(3_t)] = (\,0,\ -V_{tb}V^*_{td},\ -V_{tb}V^*_{ts}\,), \qquad [H^{(1)}(3_p)] = (\,0,\ -3V_{tb}V^*_{td},\ -3V_{tb}V^*_{ts}\,). \tag{3.23}
$$

One can find all the 3-dimensional presentations have the structure of $[H(3)] = (0,\, a,\, b)$.

Under the operators I_-^n, $[H^{(0,u)}(\overline{6})]$, $[H^{(0,u)}(15)]_i$, $[H(3)]$ are transformed as

$$I_-^2[H^{(0,u)}(\overline{6})] = I_-\{I_-[H^{(0,u)}(\overline{6})]\} = I_- \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2V_{ub}V_{us}^* & 2V_{ub}V_{ud}^* \\ 0 & 0 & 0 \end{pmatrix} = 0, \tag{3.24}$$

$$\begin{aligned} I_-^3[H^{(0,u)}(15)]_1 &= I_-\{I_-\{I_-[H^{(0,u)}(15)]_1\}\} = I_-\left\{ I_- \begin{pmatrix} 6V_{ub}V_{ud}^* & 0 & 0 \\ 0 & -5V_{ub}V_{ud}^* & -4V_{ub}V_{us}^* \\ 0 & 0 & -V_{ub}V_{ud}^* \end{pmatrix} \right\} \\ &= I_- \begin{pmatrix} 0 & 0 & 0 \\ -16V_{ub}V_{ud}^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0, \end{aligned} \tag{3.25}$$

$$I_-^2[H^{(0,u)}(15)]_2 = I_-\{I_-[H^{(0,u)}(15)]_2\} = I_- \begin{pmatrix} 0 & 0 & 0 \\ -5V_{ub}V_{ud}^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0, \tag{3.26}$$

$$I_-^2[H^{(0,u)}(15)]_3 = I_-\{I_-[H^{(0,u)}(15)]_3\} = I_- \begin{pmatrix} 0 & 0 & 0 \\ -4V_{ub}V_{us}^* & 0 & 0 \\ -V_{ub}V_{ud}^* & 0 & 0 \end{pmatrix} = 0, \tag{3.27}$$

$$I_-^2[H(3)] = I_-\{I_-[H(3)]\} = I_- \begin{pmatrix} a & 0 & 0 \end{pmatrix} = 0. \tag{3.28}$$

The isospin sum rules of $b \to u\overline{u}d$ ($b \to u\overline{u}s$) transition can be generated by I_-^n in the case of $n \geq 3$ ($n \geq 2$). The sum of decay amplitudes generated from $\overline{B}_\gamma \to P_\alpha P_\beta$ under I_- is

$$SumI_-[\gamma,\alpha,\beta] = \sum_\mu \left[\{[I_-]_{P_8}\}^\mu_\alpha \mathcal{A}_{\gamma\to\mu\beta} + \{[I_-]_{P_8}\}^\mu_\beta \mathcal{A}_{\gamma\to\alpha\mu} + \{[I_-]_{\overline{B}}\}^\mu_\gamma \mathcal{A}_{\mu\to\alpha\beta} \right]. \tag{3.29}$$

With eq. (3.29), the isospin sum rules of $\overline{B} \to PP$ modes are derived to be

$$\begin{aligned} SumI_-^3[B^-,\pi^+,\pi^+] =& -\sqrt{2}SumI_-^2[B^-,\pi^0,\pi^+] - \sqrt{2}SumI_-^2[B^-,\pi^+,\pi^0] + SumI_-^2[\overline{B}^0,\pi^+,\pi^+] \\ =& -2SumI_-[B^-,\pi^+,\pi^-] + 4SumI_-[B^-,\pi^0,\pi^0] - 2SumI_-[B^-,\pi^-,\pi^+] \\ & -2\sqrt{2}SumI_-[\overline{B}^0,\pi^+,\pi^0] - 2\sqrt{2}SumI_-[\overline{B}^0,\pi^0,\pi^+] \\ =& 12\big[\sqrt{2}\mathcal{A}(B^-\to\pi^0\pi^-) + \mathcal{A}(\overline{B}^0\to\pi^0\pi^0) - \mathcal{A}(\overline{B}^0\to\pi^+\pi^-)\big] = 0, \end{aligned} \tag{3.30}$$

$$\begin{aligned} SumI_-^2[B^-,\pi^+,\overline{K}^0] =& -\sqrt{2}SumI_-[B^-,\pi^0,\overline{K}^0] - SumI_-[B^-,\pi^+,K^-] + SumI_-[\overline{B}^0,\pi^+,\overline{K}^0] \\ =& 2\big[\sqrt{2}\mathcal{A}(B^-\to\pi^0K^-) - \mathcal{A}(B^-\to\pi^-\overline{K}^0) \\ & -\sqrt{2}\mathcal{A}(\overline{B}^0\to\pi^0\overline{K}^0) - \mathcal{A}(\overline{B}^0\to\pi^+K^-)\big] = 0, \end{aligned} \tag{3.31}$$

$$\begin{aligned} SumI_-^2[\overline{B}^0_s,\pi^+,\pi^+] =& -\sqrt{2}SumI_-[\overline{B}^0_s,\pi^0,\pi^+] - \sqrt{2}SumI_-[\overline{B}^0_s,\pi^+,\pi^0] \\ =& 4\big[\mathcal{A}(\overline{B}^0_s\to\pi^0\pi^0) - \mathcal{A}(\overline{B}^0_s\to\pi^+\pi^-)\big] = 0. \end{aligned} \tag{3.32}$$

According to eqs. (2.39), (3.11) and (3.32), the branching fractions of $D_s^+ \to \pi^+\pi^0$, $\overline{B}^0_s \to D^0\pi^0$, $\overline{B}^0_s \to D^+\pi^-$, $\overline{B}^0_s \to \pi^+\pi^-$ and $\overline{B}^0_s \to \pi^0\pi^0$ satisfy following equations under isospin symmetry,

$$\mathcal{B}r(D_s^+ \to \pi^+\pi^0) = 0, \tag{3.33}$$

$$\mathcal{B}r(\overline{B}_s^0 \to D^+\pi^-) = 2\,\mathcal{B}r(\overline{B}_s^0 \to D^0\pi^0), \tag{3.34}$$

$$\mathcal{B}r(\overline{B}_s^0 \to \pi^+\pi^-) = 2\,\mathcal{B}r(\overline{B}_s^0 \to \pi^0\pi^0), \tag{3.35}$$

where the identical factor in the $\overline{B}_s^0 \to \pi^0\pi^0$ channel is considered. So we suggest to measure the branching fractions of $D_s^+ \to \pi^+\pi^0$, $\overline{B}_s^0 \to D^0\pi^0$, $\overline{B}_s^0 \to D^+\pi^-$, $\overline{B}_s^0 \to \pi^+\pi^-$ and $\overline{B}_s^0 \to \pi^0\pi^0$ modes to test the isospin symmetry. The branching fraction of $\overline{B}_s^0 \to \pi^+\pi^-$ mode has been measured by many experiments and averaged to be $(7.0 \pm 1.0) \times 10^{-7}$ [77]. And the upper limits of $\mathcal{B}r(D_s^+ \to \pi^+\pi^0)$ and $\mathcal{B}r(\overline{B}_s^0 \to \pi^0\pi^0)$ are given by 1.2×10^{-4} and 2.1×10^{-4}, respectively [77]. It is significant to perform a more precise measurement for above five channels in the future.

4 Summary

Flavor symmetry is a model-independent tool to analyze heavy meson and baryon decays. The flavor invariants are independent of the detailed dynamics and determined by fitting experimental data. In this work, we propose a simple algorithm to generate the isospin, V-spin and U-spin sum rules of heavy hadron decays. We found that the effective Hamiltonian of heavy quark decay is fully invariant under a series of lowering operators I_-^n, V_-^n and U_-^n. The isospin, V-spin and U-spin sum rules can be generated from several master formulas without the Wigner-Eckhart invariants. Taking the two-body decays of D and B mesons as examples, our approach is presented in detail. In addition, we suggest to measure the branching fractions of $D_s^+ \to \pi^+\pi^0$, $\overline{B}_s^0 \to D^0\pi^0$, $\overline{B}_s^0 \to D^+\pi^-$, $\overline{B}_s^0 \to \pi^+\pi^-$ and $\overline{B}_s^0 \to \pi^0\pi^0$ modes to test the isospin symmetry.

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A V-spin sum rules

In this appendix, we derive the V-spin sum rules in the $D \to PP$, $\overline{B} \to DP$ and $\overline{B} \to PP$ modes. The V-spin lowering operator V_- is

$$V_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{A.1}$$

In the charm quark decay, $[H^{(0)}(\overline{6})]$, $[H^{(0)}(15)]_i$, $[H^{(0,1)}(3_{t,p})]$ are transformed under V_-^n as

$$V_-[H^{(0)}(\overline{6})] = 0, \qquad V_-[H^{(0)}(15)]_{2,3} = 0, \qquad V_-[H^{(0,1)}(3_{t,p})] = 0, \tag{A.2}$$

$$V_-^2[H^{(0)}(15)]_1 = V_- \begin{pmatrix} 0 & 0 & 0 \\ 8V_{cs}^*V_{ud} & 0 & 0 \\ 8V_{cs}^*V_{us} & 0 & 0 \end{pmatrix} = 0. \tag{A.3}$$

So the V-spin sum rules of DCS transition can be generated by V_-^n if $n \geq 1$, and the V-spin sum rules of CF and SCS transitions can be generated by V_-^n if $n \geq 2$. The coefficient matrix $[V_-]_D$ is derived to be

$$[V_-]_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{A.4}$$

The coefficient matrix $[V_-]_{P_8}$ is derived to be

$$[V_-]_{P_8} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2} \\ 0 & 0 & 0 & -\frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.5}$$

The sum of decay amplitudes generated from $D_\gamma \to P_\alpha P_\beta$ under V_- is

$$SumV_-[\gamma, \alpha, \beta] = \sum_\mu \left[\{[V_-]_{P_8}\}^\mu_\alpha \mathcal{A}_{\gamma\to\mu\beta} + \{[V_-]_{P_8}\}^\mu_\beta \mathcal{A}_{\gamma\to\alpha\mu} + \{[V_-]_D\}^\mu_\gamma \mathcal{A}_{\mu\to\alpha\beta}\right]. \tag{A.6}$$

The V-spin sum rules in the $D \to PP$ modes are derived to be

$$\begin{aligned} SumV_-^2[D^0,\pi^+,K^+] =& -\frac{SumV_-[D^0,\pi^+,\pi^0]}{\sqrt{2}} - \sqrt{\frac{3}{2}} SumV_-[D^0,\pi^+,\eta_8] \\ & + SumV_-[D^0,\overline{K}^0,K^+] + SumV_-[D_s^+,\pi^+,K^+] \\ =& 2\mathcal{A}(D_s^+\to K^+\overline{K}^0) - \sqrt{6}\mathcal{A}(D_s^+\to\pi^+\eta_8) - \sqrt{2}\mathcal{A}(D_s^+\to\pi^+\pi^0) \\ & -\sqrt{6}\mathcal{A}(D^0\to\overline{K}^0\eta_8) - \sqrt{2}\mathcal{A}(D^0\to\pi^0\overline{K}^0) \\ & -2\mathcal{A}(D^0\to\pi^+K^-) = 0, \end{aligned} \tag{A.7}$$

$$\begin{aligned} SumV_-^2[D^0,K^+,K^+] =& -\frac{SumV_-[D^0,\pi^0,K^+]}{\sqrt{2}} - \frac{SumV_-[D^0,K^+,\pi^0]}{\sqrt{2}} - \sqrt{\frac{3}{2}} SumV_-[D^0,K^+,\eta_8] \\ & -\sqrt{\frac{3}{2}} SumV_-[D^0,\eta_8,K^+] + SumV_-[D_s^+,K^+,K^+] \\ =& -2\sqrt{6}\mathcal{A}(D_s^+\to K^+\eta_8) - 2\sqrt{2}\mathcal{A}(D_s^+\to\pi^0K^+) + 3\mathcal{A}(D^0\to\eta_8\eta_8) \\ & -4\mathcal{A}(D^0\to K^+K^-) + 2\sqrt{3}\mathcal{A}(D^0\to\pi^0\eta_8) \\ & +\mathcal{A}(D^0\to\pi^0\pi^0) = 0, \end{aligned} \tag{A.8}$$

$$\begin{aligned} SumV_-[D^0,K^+,K^0] =& \mathcal{A}(D_s^+\to K^+K^0) - \sqrt{\frac{3}{2}}\mathcal{A}(D^0\to K^0\eta_8) \\ & -\frac{\mathcal{A}(D^0\to\pi^0K^0)}{\sqrt{2}} - \mathcal{A}(D^0\to\pi^-K^+) = 0. \end{aligned} \tag{A.9}$$

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In the $b \to c\bar{u}q$ transition, $[H^{(0)}(8)]$ is transformed under V_-^n as

$$V_-^3[H^{(0)}(8)] = V_- \left\{ V_- \begin{pmatrix} -V_{cb}V_{us}^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & V_{cb}V_{ud}^* & V_{cb}V_{us}^* \end{pmatrix} \right\} = V_- \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2V_{cb}V_{us}^* & 0 & 0 \end{pmatrix} = 0. \tag{A.10}$$

So the V-spin sum rules of $b \to c\bar{u}d$ transition can be generated by V_-^n if $n \geq 2$, and the V-spin sum rules of $b \to c\bar{u}s$ transition can be generated by V_-^n if $n \geq 3$. Under the V-spin lowering operator V_-, we have $[V_-]_{\overline{B}} = [V_-]_D$. The sum of decay amplitudes generated from $\overline{B}_\gamma \to D_\alpha P_\beta$ under V_- is

$$SumV_-[\gamma,\alpha,\beta] = \sum_\mu \Big[\{[V_-]_D^T\}_\alpha^\mu \mathcal{A}_{\gamma\to\mu\beta} + \{[V_-]_{P_8}\}_\beta^\mu \mathcal{A}_{\gamma\to\alpha\mu} + \{[V_-]_{\overline{B}}\}_\gamma^\mu \mathcal{A}_{\mu\to\alpha\beta} \Big]. \tag{A.11}$$

The V-spin sum rules in the $\overline{B} \to DP$ modes are derived to be

$$\begin{aligned} SumV_-^2[B^-,D_s^+,K^0] &= -SumV_-[B^-,D^0,K^0] - SumV_-[B^-,D_s^+,\pi^-] + SumV_-[\overline{B}_s^0,D_s^+,K^0] \\ &= -2\big[\mathcal{A}(\overline{B}_s^0\to D_s^+\pi^-) + \mathcal{A}(\overline{B}_s^0\to D^0K^0) - \mathcal{A}(B^-\to D^0\pi^-)\big] = 0, \end{aligned} \tag{A.12}$$

$$\begin{aligned} SumV_-^2[\overline{B}^0,D_s^+,K^+] &= -SumV_-[\overline{B}^0,D^0,K^+] - \frac{SumV_-[\overline{B}^0,D_s^+,\pi^0]}{\sqrt{2}} \\ &\quad - \sqrt{\frac{3}{2}} SumV_-[\overline{B}^0,D_s^+,\eta_8] \\ &= -2\mathcal{A}(\overline{B}^0\to D_s^+K^-) + \sqrt{6}\mathcal{A}(\overline{B}^0\to D^0\eta_8) \\ &\quad + \sqrt{2}\mathcal{A}(\overline{B}^0\to D^0\pi^0) = 0, \end{aligned} \tag{A.13}$$

$$\begin{aligned} SumV_-^3[B^-,D_s^+,K^+] &= -SumV_-^2[B^-,D^0,K^+] - \frac{SumV_-^2[B^-,D_s^+,\pi^0]}{\sqrt{2}} \\ &\quad - \sqrt{\frac{3}{2}} SumV_-^2[B^-,D_s^+,\eta_8] + SumV_-^2[\overline{B}_s^0,D_s^+,K^+] \\ &= \sqrt{2}SumV_-[B^-,D^0,\pi^0] + \sqrt{6}SumV_-[B^-,D^0,\eta_8] \\ &\quad - 2SumV_-[B^-,D_s^+,K^-] - 2SumV_-[\overline{B}_s^0,D^0,K^+] \\ &\quad - \sqrt{2}SumV_-[\overline{B}_s^0,D_s^+,\pi^0] - \sqrt{6}SumV_-[\overline{B}_s^0,D_s^+,\eta_8] \\ &= 3\big[-2\mathcal{A}(\overline{B}_s^0\to D_s^+K^-) + \sqrt{6}\mathcal{A}(\overline{B}_s^0\to D^0\eta_8) \\ &\quad + \sqrt{2}\mathcal{A}(\overline{B}_s^0\to D^0\pi^0) + 2\mathcal{A}(B^-\to D^0K^-)\big] = 0. \end{aligned} \tag{A.14}$$

In the $b \to u\bar{u}q$ transition, $[H^{(0,u)}(\overline{6})]$, $[H^{(0,u)}(15)]_i$, $[H(3)]$ are transformed under V_-^n as

$$V_-^2[H^{(0,u)}(\overline{6})] = V_- \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2V_{ub}V_{us}^* & 2V_{ub}V_{ud}^* \end{pmatrix} = 0, \tag{A.15}$$

$$V_-^3[H^{(0,u)}(15)]_1 = V_-\left\{V_-\begin{pmatrix} 6V_{ub}V_{us}^* & 0 & 0 \\ 0 & -V_{ub}V_{us}^* & 0 \\ 0 & -4V_{ub}V_{ud}^* & -5V_{ub}V_{us}^* \end{pmatrix}\right\}$$

$$= V_-\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -16V_{ub}V_{us}^* & 0 & 0 \end{pmatrix} = 0, \tag{A.16}$$

$$V_-^2[H^{(0,u)}(15)]_2 = V_-\{V_-[H^{(0,u)}(15)]_2\} = V_-\begin{pmatrix} 0 & 0 & 0 \\ -V_{ub}V_{us}^* & 0 & 0 \\ -4V_{ub}V_{ud}^* & 0 & 0 \end{pmatrix} = 0, \tag{A.17}$$

$$V_-^2[H^{(0,u)}(15)]_3 = V_-\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -5V_{ub}V_{us}^* & 0 & 0 \end{pmatrix} = 0, \tag{A.18}$$

$$V_-^2[H(3)] = V_-(\, b\ 0\ 0\,) = 0. \tag{A.19}$$

So the V-spin sum rules of $b \to u\overline{u}d$ ($b \to u\overline{u}s$) transition can be generated by V_-^n if $n \geq 2$ ($n \geq 3$). The sum of decay amplitudes generated from $\overline{B}_\gamma \to P_\alpha P_\beta$ under V_- is

$$SumV_-[\gamma,\alpha,\beta] = \sum_\mu \left[\{[V_-]_{P_8}\}^\mu_\alpha \mathcal{A}_{\gamma\to\mu\beta} + \{[V_-]_{P_8}\}^\mu_\beta \mathcal{A}_{\gamma\to\alpha\mu} + \{[V_-]_{\overline{B}}\}^\mu_\gamma \mathcal{A}_{\mu\to\alpha\beta}\right]. \tag{A.20}$$

With eq. (A.20), the V-spin sum rules in the $\overline{B} \to PP$ modes are derived to be

$$\begin{aligned}
SumV_-^2[B^-,K^+,K^0] =& -\frac{SumV_-[B^-,\pi^0,K^0]}{\sqrt{2}} - SumV_-[B^-,K^+,\pi^-] \\
&-\sqrt{\frac{3}{2}}SumV_-[B^-,\eta_8,K^0] + SumV_-[\overline{B}^0_s,K^+,K^0] \\
=& -\sqrt{6}\mathcal{A}(\overline{B}^0_s\to K^0\eta_8) - \sqrt{2}\mathcal{A}(\overline{B}^0_s\to\pi^0K^0) \\
&-2\mathcal{A}(\overline{B}^0_s\to\pi^-K^+) - 2\mathcal{A}(B^-\to K^0K^-) \\
&+\sqrt{6}\mathcal{A}(B^-\to\pi^-\eta_8) + \sqrt{2}\mathcal{A}(B^-\to\pi^0\pi^-) = 0,
\end{aligned} \tag{A.21}$$

$$\begin{aligned}
SumV_-^2[\overline{B}^0,K^+,K^+] =& -\frac{SumV_-[\overline{B}^0,\pi^0,K^+]}{\sqrt{2}} - \frac{SumV_-[\overline{B}^0,K^+,\pi^0]}{\sqrt{2}} \\
&-\sqrt{\frac{3}{2}}SumV_-[\overline{B}^0,\eta_8,K^+] - \sqrt{\frac{3}{2}}SumV_-[\overline{B}^0,K^+,\eta_8] \\
=& 3\mathcal{A}(\overline{B}^0\to\eta_8\eta_8) - 4\mathcal{A}(\overline{B}^0\to K^+K^-) \\
&+2\sqrt{3}\mathcal{A}(\overline{B}^0\to\pi^0\eta_8) + \mathcal{A}(\overline{B}^0\to\pi^0\pi^0) = 0,
\end{aligned} \tag{A.22}$$

$$\begin{aligned}
SumV_-^3[B^-,K^+,K^+] =& SumV_-^2[\overline{B}^0_s,K^+,K^+] - \frac{SumV_-^2[B^-,\pi^0,K^+]}{\sqrt{2}} - \frac{SumV_-^2[B^-,K^+,\pi^0]}{\sqrt{2}} \\
&-\sqrt{\frac{3}{2}}SumV_-^2[B^-,K^+,\eta_8] - \sqrt{\frac{3}{2}}SumV_-^2[B^-,\eta_8,K^+] \\
=& SumV_-[B^-,\pi^0,\pi^0] + \sqrt{3}SumV_-[B^-,\pi^0,\eta_8] - 2SumV_-[B^-,K^+,K^-] \\
&-2SumV_-[B^-,K^-,K^+] + \sqrt{3}SumV_-[B^-,\eta_8,\pi^0] \\
&+3SumV_-[B^-,\eta_8,\eta_8] - \sqrt{2}SumV_-[\overline{B}^0_s,\pi^0,K^+]
\end{aligned}$$

$$
\begin{aligned}
&-\sqrt{2}SumV_-[\overline{B}^0_s,K^+,\pi^0]-\sqrt{6}SumV_-[\overline{B}^0_s,K^+,\eta_8]\\
&-\sqrt{6}SumV_-[\overline{B}^0_s,\eta_8,K^+]\\
=&3\big[3\mathcal{A}(\overline{B}^0_s\to\eta_8\eta_8)-4\mathcal{A}(\overline{B}^0_s\to K^+K^-)+2\sqrt{3}\mathcal{A}(\overline{B}^0_s\to\pi^0\eta_8)\\
&+\mathcal{A}(\overline{B}^0_s\to\pi^0\pi^0)+2\sqrt{6}\mathcal{A}(B^-\to K^-\eta_8)\\
&+2\sqrt{2}\mathcal{A}(B^-\to K^-\pi^0)\big]=0. \qquad \text{(A.23)}
\end{aligned}
$$

B U-spin sum rules

In this appendix, we derive the U-spin sum rules in the $D\to PP$, $\overline{B}\to DP$ and $\overline{B}\to PP$ modes. The U-spin lowering operator U_- is

$$
U_-=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}. \qquad \text{(B.1)}
$$

In charm quark decay, $[H^{(0)}(\overline{6})]$, $[H^{(0)}(15)]_i$, $[H^{(0,1)}(3_{t,p})]$ are transformed under U_-^n as

$$
U_-^2[H^{(0)}(\overline{6})]=U_-\begin{pmatrix}0&0&0\\0&0&0\\0&-4V^*_{cs}V_{ud}&2(V^*_{cd}V_{ud}-V^*_{cs}V_{us})\end{pmatrix}=0, \qquad \text{(B.2)}
$$

$$
U_-[H^{(0,1)}(3_{t,p})]=0, \qquad \text{(B.3)}
$$

$$
\begin{aligned}
U_-^3[H^{(0)}(15)]_1=&U_-\left\{U_-\begin{pmatrix}0&0&0\\0&4V^*_{cs}V_{ud}&0\\0&4(V^*_{cs}V_{us}-V^*_{cd}V_{ud})&-4V^*_{cs}V_{ud}\end{pmatrix}\right\}\\
=&U_-\begin{pmatrix}0&0&0\\0&0&0\\-4V^*_{cs}V_{ud}&-4V^*_{cs}V_{ud}&0\end{pmatrix}=0, \qquad \text{(B.4)}
\end{aligned}
$$

$$
U_-^3[H^{(0)}(15)]_2=U_-\left\{U_-\begin{pmatrix}0&0&0\\4V^*_{cs}V_{ud}&0&0\\4(V^*_{cs}V_{us}-V^*_{cd}V_{ud})&0&0\end{pmatrix}\right\}=U_-\begin{pmatrix}0&0&0\\0&0&0\\-8V^*_{cs}V_{ud}&0&0\end{pmatrix}=0, \qquad \text{(B.5)}
$$

$$
U_-^2[H^{(0)}(15)]_3=U_-\begin{pmatrix}0&0&0\\0&0&0\\-4V^*_{cs}V_{ud}&0&0\end{pmatrix}=0, \qquad \text{(B.6)}
$$

The U-spin sum rules of DCS, SCS and CF transitions are generated by U_-^n if $n\geq 1$, $n\geq 2$ and $n\geq 3$, respectively. The coefficient matrix $[U_-]_D$ is derived to be

$$
[U_-]_D=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}. \qquad \text{(B.7)}
$$

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The coefficient matrix $[U_-]_{P_8}$ is derived to be

$$[U_-]_{P_8} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{6}}{2} & 0 & 0 & 0 \end{pmatrix}. \tag{B.8}$$

The sum of decay amplitudes generated from $D_\gamma \to P_\alpha P_\beta$ under U_- is

$$SumU_-[\gamma,\alpha,\beta] = \sum_\mu \Big[\{[U_-]_{P_8}\}^\mu_\alpha \mathcal{A}_{\gamma\to\mu\beta} + \{[U_-]_{P_8}\}^\mu_\beta \mathcal{A}_{\gamma\to\alpha\mu} + \{[U_-]_D\}^\mu_\gamma \mathcal{A}_{\mu\to\alpha\beta}\Big]. \tag{B.9}$$

The U-spin sum rules in the $D \to PP$ modes are derived to be

$$\begin{aligned}
SumU^3_-[D^+,K^+,K^0] &= SumU^2_-[D^+_s \to K^+K^0] - \sqrt{\frac{3}{2}}\,SumU^2_-[D^+ \to K^+\eta_8] \\
&\quad + \frac{SumU^2_-[D^+ \to \pi^0 K^+]}{\sqrt{2}} - SumU^2_-[D^+ \to \pi^+K^0] \\
&= -\sqrt{6}\,SumU_-[D^+_s \to K^+\eta_8] + \sqrt{2}\,SumU_-[D^+_s \to \pi^0K^+] \\
&\quad - 2\,SumU_-[D^+_s \to \pi^+K^0] - 2\,SumU_-[D^+ \to K^+\overline{K}^0] \\
&\quad + \sqrt{6}\,SumU_-[D^+ \to \pi^+\eta_8] - \sqrt{2}\,SumU_-[D^+ \to \pi^+\pi^0] \\
&= -6\,\mathcal{A}(D^+_s \to K^+\overline{K}^0) + 3\sqrt{6}\,\mathcal{A}(D^+_s \to \pi^+\eta_8) \\
&\quad - 3\sqrt{2}\,\mathcal{A}(D^+_s \to \pi^+\pi^0) + 6\,\mathcal{A}(D^+ \to \pi^+\overline{K}^0) = 0, &\text{(B.10)} \\
SumU^3_-[D^0,K^0,K^0] &= -\sqrt{6}\,SumU^2_-[D^0 \to K^0\eta_8] + \sqrt{2}\,SumU^2_-[D^0 \to \pi^0K^0] \\
&= 3\,SumU_-[D^0 \to \eta_8\eta_8] - 4\,SumU_-[D^0 \to K^0\overline{K}^0] \\
&\quad - 2\sqrt{3}\,SumU_-[D^0 \to \pi^0\eta_8] + SumU_-[D^0 \to \pi^0\pi^0] = \\
&= 6\sqrt{6}\,\mathcal{A}(D^0 \to \overline{K}^0\eta_8) - 6\sqrt{2}\,\mathcal{A}(D^0 \to \pi^0\overline{K}^0) = 0, &\text{(B.11)} \\
SumU^2_-[D^+,K^+,K^0] &= SumU_-[D^+_s \to K^+K^0] - \sqrt{\frac{3}{2}}\,SumU_-[D^+ \to K^+\eta_8] \\
&\quad + \frac{SumU_-[D^+ \to \pi^0K^+]}{\sqrt{2}} - SumU_-[D^+ \to \pi^+K^0] \\
&= -\sqrt{6}\,\mathcal{A}(D^+_s \to K^+\eta_8) + \sqrt{2}\,\mathcal{A}(D^+_s \to \pi^0K^+) - 2\,\mathcal{A}(D^+_s \to \pi^+K^0) \\
&\quad - 2\,\mathcal{A}(D^+ \to K^+\overline{K}^0) + \sqrt{6}\,\mathcal{A}(D^+ \to \pi^+\eta_8) \\
&\quad - \sqrt{2}\,\mathcal{A}(D^+ \to \pi^+\pi^0) = 0, &\text{(B.12)} \\
SumU^2_-[D^0,K^0,K^0] &= -\sqrt{6}\,SumU_-[D^0 \to K^0\eta_8] + \sqrt{2}\,SumU_-[D^0 \to \pi^0K^0] \\
&= 3\,\mathcal{A}(D^0 \to \eta_8\eta_8) - 4\,\mathcal{A}(D^0 \to K^0\overline{K}^0) \\
&\quad - 2\sqrt{3}\,\mathcal{A}(D^0 \to \pi^0\eta_8) + \mathcal{A}(D^0 \to \pi^0\pi^0) = 0, &\text{(B.13)}
\end{aligned}$$

$$SumU_-[D^+,K^+,K^0]=\mathcal{A}(D_s^+\to K^+K^0)-\sqrt{\frac{3}{2}}\mathcal{A}(D^+\to K^+\eta_8)$$
$$+\frac{\mathcal{A}(D^+\to\pi^0K^+)}{\sqrt{2}}-\mathcal{A}(D^+\to\pi^+K^0)=0, \tag{B.14}$$
$$SumU_-[D^0,K^0,K^0]=-\sqrt{6}\mathcal{A}(D^0\to K^0\eta_8)+\sqrt{2}\mathcal{A}(D^0\to\pi^0K^0)=0. \tag{B.15}$$

One should note the U-spin sum rules derived by U_-^n do not dependent on the Wolfenstein approximation of the CKM matrix.

In the $b\to c\bar{u}q$ transition, $[H^{(0)}(8)]$ is transformed under U_-^n as

$$U_-^2[H^{(0)}(8)]=U_-\begin{pmatrix}0 & -V_{cb}V_{us}^* & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=0. \tag{B.16}$$

So the U-spin sum rules of $b\to c\bar{u}d$ transition are generated by U_-^n if $n\geq 1$, and the U-spin sum rules of $b\to c\bar{u}s$ transition are generated by U_-^n if $n\geq 2$. Under the U-spin lowering operator U_-, $[U_-]_{\overline{B}}=[U_-]_D$. The sum of decay amplitudes generated from $\overline{B}_\gamma\to D_\alpha P_\beta$ under U_- is

$$SumU_-[\gamma,\alpha,\beta]=\sum_\mu\Big[\{[U_-]_D^T\}^\mu_\alpha\mathcal{A}_{\gamma\to\mu\beta}+\{[U_-]_{P_8}\}^\mu_\beta\mathcal{A}_{\gamma\to\alpha\mu}+\{[U_-]_{\overline{B}}\}^\mu_\gamma\mathcal{A}_{\mu\to\alpha\beta}\Big]. \tag{B.17}$$

The U-spin sum rules in the $\overline{B}\to DP$ modes are

$$SumU_-[B^0,D^0,K^0]=\mathcal{A}(\overline{B}{}^0_s\to D^0K^0)-\sqrt{\frac{3}{2}}\mathcal{A}(\overline{B}{}^0\to D^0\eta_8)+\frac{\mathcal{A}(\overline{B}{}^0\to D^0\pi^0)}{\sqrt{2}}=0, \tag{B.18}$$
$$SumU_-[B^0,D_s^+,\pi^-]=\mathcal{A}(\overline{B}{}^0_s\to D_s^+\pi^-)+\mathcal{A}(\overline{B}{}^0\to D_s^+K^-)-\mathcal{A}(\overline{B}{}^0\to D^+\pi^-)=0, \tag{B.19}$$
$$\begin{aligned}SumU_-^2[B^0,D^0,K^0]&=SumU_-[\overline{B}{}^0_s\to D^0K^0]-\sqrt{\frac{3}{2}}SumU_-[\overline{B}{}^0\to D^0\eta_8]\\&\quad+\frac{SumU_-[\overline{B}{}^0\to D^0\pi^0]}{\sqrt{2}}\\&=\sqrt{2}\mathcal{A}(\overline{B}{}^0_s\to D^0\pi^0)-\sqrt{6}\mathcal{A}(\overline{B}{}^0_s\to D^0\eta_8)-2\mathcal{A}(\overline{B}{}^0\to D^0\overline{K}{}^0)=0,\end{aligned} \tag{B.20}$$
$$\begin{aligned}SumU_-^2[B^0,D_s^+,\pi^-]&=SumU_-[\overline{B}{}^0_s\to D_s^+\pi^-]+SumU_-[\overline{B}{}^0\to D_s^+K^-]\\&\quad-SumU_-[\overline{B}{}^0\to D^+\pi^-]\\&=2\Big[\mathcal{A}(\overline{B}{}^0_s\to D_s^+K^-)-\mathcal{A}(\overline{B}{}^0_s\to D^+\pi^-)-\mathcal{A}(\overline{B}{}^0\to D^+K^-)\Big]=0.\end{aligned} \tag{B.21}$$

In the $b\to u\bar{u}q$ transition, $[H^{(0,u)}(\overline{6})]$, $[H^{(0,u)}(15)]_i$, $[H(3)]$ are transformed under U_-^n as

$$U_-^2[H^{(0,u)}(\overline{6})]=U_-\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -2V_{ub}V_{us}^* & 0 & 0\end{pmatrix}=0, \tag{B.22}$$
$$U_-^2[H^{(0,u)}(15)]_1=U_-\begin{pmatrix}0 & 3V_{ub}V_{us}^* & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=0, \tag{B.23}$$

$$U_-^2[H^{(0,u)}(15)]_2 = U_- \begin{pmatrix} 3V_{ub}V_{us}^* & 0 & 0 \\ 0 & -2V_{ub}V_{us}^* & 0 \\ 0 & 0 & -V_{ub}V_{us}^* \end{pmatrix} = 0, \tag{B.24}$$

$$U_-^2[H^{(0,u)}(15)]_3 = U_- \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -V_{ub}V_{us}^* & 0 \end{pmatrix} = 0, \tag{B.25}$$

$$U_-^2[H(3)] = U_-(\,0\;\, b\;\, 0\,) = 0. \tag{B.26}$$

So the U-spin sum rules of $b \to u\bar{u}d$ ($b \to u\bar{u}s$) transition can be generated by U_-^n if $n \geq 1$ ($n \geq 2$). The sum of decay amplitudes generated from $\overline{B}_\gamma \to P_\alpha P_\beta$ under U_- is

$$SumU_-[\gamma, \alpha, \beta] = \sum_\mu \Big[\{[U_-]_{P_8}\}^\mu_\alpha \mathcal{A}_{\gamma\to\mu\beta} + \{[U_-]_{P_8}\}^\mu_\beta \mathcal{A}_{\gamma\to\alpha\mu} + \{[U_-]_{\overline{B}}\}^\mu_\gamma \mathcal{A}_{\mu\to\alpha\beta}\Big]. \tag{B.27}$$

With eq. (B.27), the U-spin sum rules of $\overline{B} \to PP$ modes are derived to be

$$SumU_-[B^-,\pi^-,K^0] = \mathcal{A}(B^-\to K^0K^-) - \sqrt{\frac{3}{2}}\mathcal{A}(B^-\to\pi^-\eta_8) + \frac{\mathcal{A}(B^-\to\pi^0\pi^-)}{\sqrt{2}} = 0, \tag{B.28}$$

$$SumU_-[B^0,K^+,\pi^-] = \mathcal{A}(\overline{B}{}^0_s\to\pi^-K^+) + \mathcal{A}(\overline{B}{}^0\to K^+K^-) - \mathcal{A}(\overline{B}{}^0\to\pi^+\pi^-) = 0, \tag{B.29}$$

$$\begin{aligned} SumU_-^2[B^-,\pi^-,K^0] &= SumU_-[B^-\to K^0K^-] - \sqrt{\frac{3}{2}}SumU_-[B^-\to\pi^-\eta_8] \\ &\quad + \frac{SumU_-[B^-\to\pi^0\pi^-]}{\sqrt{2}} \\ &= \sqrt{2}\mathcal{A}(B^-\to\pi^0K^-) - \sqrt{6}\mathcal{A}(B^-\to K^-\eta_8) \\ &\quad - 2\mathcal{A}(B^-\to\pi^-\overline{K}{}^0) = 0, \end{aligned} \tag{B.30}$$

$$\begin{aligned} SumU_-^2[B^0,K^+,\pi^-] &= SumU_-[\overline{B}{}^0_s\to\pi^-K^+] + SumU_-[\overline{B}{}^0\to K^+K^-] \\ &\quad - SumU_-[\overline{B}{}^0\to\pi^+\pi^-] \\ &= 2\Big[\mathcal{A}(\overline{B}{}^0_s\to K^+K^-) - \mathcal{A}(\overline{B}{}^0_s\to\pi^+\pi^-) - \mathcal{A}(\overline{B}{}^0\to\pi^+K^-)\Big] = 0. \end{aligned} \tag{B.31}$$

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