Constructible reality condition of pseudo entropy via pseudo-Hermiticity

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ABSTRACT: As a generalization of entanglement entropy, pseudo entropy is not always real. The real-valued pseudo entropy has promising applications in holography and quantum phase transition. We apply the notion of pseudo-Hermiticity to formulate the reality condition of pseudo entropy. We find the general form of the transition matrix for which the eigenvalues of the reduced transition matrix possess real or complex pairs of eigenvalues. Further, we find a class of transition matrices for which the pseudo (Rényi) entropies are non-negative. Some known examples which give real pseudo entropy in quantum field theories can be explained in our framework. Our results offer a novel method to generate the transition matrix with real pseudo entropy. Finally, we show the reality condition for pseudo entropy is related to the Tomita-Takesaki modular theory for quantum field theory.

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Contents

1 Introduction

Entanglement entropy (EE), as an entanglement measure, has been investigated in many aspects [1–8]. Specially, in the context of AdS/CFT [9–11], entanglement plays an important role in understanding the emergence of geometry [12, 13], subregion/subregion duality [14, 15] and information paradox of black hole [16, 17].

Recently, a generalization of EE, called *pseudo entropy*, was introduced in [18] via AdS/CFT and post-selection, which may bring us a new understanding of the role of entanglement in quantum field theory (QFT) or gravity. Given a system whose Hilbert space can be divided as $H = H_A \otimes H_{\bar{A}}$, the pseudo entropy of subsystem A is defined as

$$S(\mathcal{T}_A^{\psi|\phi}) = -\text{tr}[\mathcal{T}_A^{\psi|\phi} \log \mathcal{T}_A^{\psi|\phi}], \tag{1.1}$$

where $\mathcal{T}_A^{\psi|\phi} = \text{tr}_{\bar{A}}\left[\mathcal{T}^{\psi|\phi}\right]$, called *reduced transition matrix*, is defined from the transition matrix consisting of two nonorthogonal pure states $|\psi\rangle$ and $|\phi\rangle$

$$\mathcal{T}^{\psi|\phi} = \frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle}. \tag{1.2}$$

Similar to the EE, in practice, especially in QFTs, one uses the replica trick [6] and computes the so-called *pseudo Rényi entropy* instead

$$S^{(n)}(\mathcal{T}_A^{\psi|\phi}) = \frac{1}{1-n} \log \text{tr}[(\mathcal{T}_A^{\psi|\phi})^n]. \tag{1.3}$$

The pseudo entropy (1.1) can be obtained by taking the limit of $n \to 1$ for the above expression.

There are manifold interests driving the study of this quantity. See [19–30] for recent studies. The first comes from holography. In QFTs, the pseudo Rényi entropy can be formulated in the language of the path integral. According to the AdS/CFT, the path integral in CFTs can be translated to the gravitational path integral in AdS. Based on this, it is proposed in [18] that the pseudo entropy of a boundary subsystem is dual to the area of a minimal surface in Euclidean asymptotically AdS spacetime. It is one of the main motivations of [18] to propose this novel quantity. The second interest comes from the experiments of post-selection in quantum measurements [31–33]. It is argued and proved in a two-qubit model that the pseudo entropy characterizes the averaged number of EPR pairs that could have been distilled in the pre-and post-selected systems [18]. Since the post-selection measurement can be incomplete [34, 35], motivated by this, in this article, we would also like to consider a generalization of the transition matrices (1.2) to mixed states. Given two general density matrices ρ_1 and ρ_2 ($\text{tr}[\rho_1\rho_2] \neq 0$) in Hilbert space, we construct the following operator as a mixed-state generalization of the transition matrix.

$$X := \frac{\rho_1\rho_2}{\text{tr}[\rho_1\rho_2]}. \tag{1.4}$$

Eq. (1.4) can be understood as the system being pre-selected and post-selected to mixed states ρ_1 and ρ_2 instead of two pure states, respectively. Note that eq. (1.4) is reduced to eq. (1.2) when $\rho_{1,2}$ are pure. The third interest in studying pseudo entropy comes from quantum many-body systems. The study of pseudo entropy in Ising spin chain [19] indicates that pseudo entropy can be taken as a new order parameter in quantum many-body systems, just like EE [2, 3].

All the above alluring physical interpretations and application of pseudo entropy are implicitly based on such a fact: the pseudo entropy $S(\mathcal{T}_A^{\psi|\phi})$ of a subsystem is real. However,

$S(\mathcal{T}_A^{\psi|\phi})$ should be complex-valued for generic choices of the initial and finite states since the reduced transition matrix, in general, is non-Hermitian. When we construct the initial and final state by a Euclidean path integral with a real-valued action, it seems reasonable that $S(\mathcal{T}_A^{\psi|\phi})$ can be positive. Whereas, for more generic cases, such as the insertion of non-Hermitian operators in Euclidean path integrals, QFTs with Lorentzian signature, and the actual finite-dimensional quantum systems, we have to ask what kind of transition matrices generate positive or non-negative pseudo entropy. It is the central motivation for the present article.

The problem is closely related to non-Hermitian physics, which has been extensively studied recently; see the review [36, 37]. It is found the eigenvalues of the non-Hermitian Hamiltonian can be real. The system satisfying parity-time ($\mathcal{PT}$) symmetry is one of the most important classes [38, 39]. The notion of pseudo-Hermiticity is handy to characterize a class of non-Hermitian matrices with real eigenvalues [40–42] if the matrix has a complete biorthonormal eigenbasis. An operator M is said to be η-pseudo-Hermitian if there exists a Hermitian invertible operator η such that

$$M^\dagger = \eta M \eta^{-1}. \tag{1.5}$$

If η is the identity, the pseudo-Hermitian condition reduces to Hermiticity. The necessary and sufficient conditions for the pseudo-Hermiticity of M are given by the following theorem [40].

Theorem 1. *An operator M with a complete biorthonormal eigenbasis and a discrete spectrum is pseudo-Hermitian if and only if one of the following conditions hold:*

1. The eigenvalues of M are real.

2. The complex eigenvalues come in complex conjugate pairs, and the degeneracy of the eigenvalues are same.

Remark: The existence of a biorthonormal eigenbasis in Hilbert space with respect to M is equivalent to M being diagonalizable. According to the Theorem 1, when the reduced transition matrix $X_A \equiv \mathrm{tr}_{\bar{A}}[X]$ is diagonalizable, the necessary and sufficient condition of $\mathrm{tr}[(X_A)^n]$ being real is that X_A is η_A-pseudo-Hermitian.[1] The pseudo Rényi entropy $S^{(n)}(X_A)$, however, may not be real in this case. Apparently, to guarantee the reality of $S^{(n)}(X_A)$, one should require $\mathrm{tr}[(X_A)^n] > 0$, which gives more constraints on η_A.

Many studies on $\mathcal{PT}$-symmetric or pseudo-Hermitian Hamiltonian mainly focus on finite-dimensional systems. It is expected the main results can also be applicable to QFTs, see, e.g., [43–46] and references therein.

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[1]When X_A is non-diagonalizable, a pseudo-Hermitian X_A can still give a real $\mathrm{tr}[(X_A)^n]$, but the converse is not necessarily true. We show an example in appendix A where $\mathrm{tr}[(X_A)^n]$ is real, but X_A is not pseudo-Hermitian.

2 General form of transition matrix

We seek to construct transition matrix X (1.4) with positive $\text{tr}[(X_A)^n]$ by means of pseudo-Hermiticity. We start with a proposition for transition matrices that strongly suggests a potential connection between pseudo entropy and pseudo-Hermiticity.

Proposition 1. *All transition matrices* (1.4) *in finite-dimensional Hilbert space are pseudo-Hermitian.*

Proof. The statement comes from the fact that the product of any two positive semi-definite matrices can be diagonalized and possesses non-negative eigenvalues. One can construct a complete biorthonormal eigenbasis within the finite-dimensional Hilbert space by utilizing the eigenvectors of a given diagonalizable matrix. Consequently, all finite-dimensional transition matrices of the form (1.4) have a complete biorthonormal eigenbasis and non-negative eigenvalues, satisfying the conditions required for Theorem 1. Then, it follows that all transition matrices are pseudo-Hermitian. $\qquad\square$

Remark: Building to the above proposition, for any transition matrix of the form (1.4) in finite-dimensional Hilbert space, we can construct a η matrix that satisfies eq. (1.5) by using its biorthonormal eigenbasis [40]. As a special case, for a pure state transition matrix (1.2) in d-dimensional Hilbert space, a valid η matrix satisfying eq. (1.5) can be found as[2]

$$\eta = \frac{|\phi\rangle\langle\phi|}{|\langle\phi|\psi\rangle|^2} + \sum_{i=1}^{d-1} |\psi_i\rangle\langle\psi_i|, \tag{2.1}$$

which has an inverse

$$\eta^{-1} = |\psi\rangle\langle\psi| + \left(|\psi_1\rangle - \frac{b^*}{a^*}|\psi\rangle\right)\left(\langle\psi_1| - \frac{b}{a}\langle\psi|\right) + \sum_{i=2}^{d-1} |\psi_i\rangle\langle\psi_i|. \tag{2.2}$$

Although all transition matrices are pseudo-Hermitian, it does not guarantee that the reduced transition matrices share the same property. The two propositions below jointly establish a necessary and sufficient condition for a reduced transition matrix to be pseudo-Hermitian.

Proposition 2. *Any operator $\mathcal{O}$ can be decomposed as*

$$\mathcal{O} = \mathcal{O}_1 + i\mathcal{O}_2, \tag{2.3}$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ are η-pseudo-Hermitian operators, η can be any Hermitian invertible operator.

Proof. For any operator $\mathcal{O}$, we can divide it into two parts

$$\mathcal{O} = \frac{\mathcal{O} + \eta^{-1}\mathcal{O}^\dagger\eta}{2} + i\frac{\mathcal{O} - \eta^{-1}\mathcal{O}^\dagger\eta}{2i}, \tag{2.4}$$

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[2]The $|\psi\rangle$, $|\psi_1\rangle$, $|\psi_2\rangle,\ldots,|\psi_{d-1}\rangle$ in (2.1) form an orthonormal basis in the Hilbert space. Additionally, $|\psi\rangle$ and $|\psi_1\rangle$ satisfy the equation $a|\psi\rangle + b|\psi_1\rangle = |\phi\rangle$.

where η is any Hermitian invertible operator. For η being identity, (2.4) is the well-known result that any operator can be decomposed as linear combinations of two Hermitian operators. Where $\mathcal{O}_1 = \frac{\mathcal{O} + \eta^{-1}\mathcal{O}^\dagger \eta}{2}$ and $\mathcal{O}_2 = \frac{\mathcal{O} - \eta^{-1}\mathcal{O}^\dagger \eta}{2i}$. $\qquad\square$

Proposition 3. $X_{A(\bar{A})}$ *is $\eta_{A(\bar{A})}$-pseudo-Hermitian, if and only if the transition matrix X can be written as*

$$X = X_1 + iX_2, \tag{2.5}$$

where X_1 and X_2 are both η-pseudo-Hermitian with $\eta = \eta_A \otimes \eta_{\bar{A}}$. Further, X_2 satisfies $tr_{\bar{A}(A)}X_2 = 0$.

Proof. Using the result of Proposition 2, let's define the operator

$$\begin{aligned} X_1 &:= \frac{1}{2}(X + \eta^{-1}X^\dagger \eta), \\ X_2 &:= \frac{i}{2}(\eta^{-1}X^\dagger \eta - X). \end{aligned} \tag{2.6}$$

Since $X_{A(\bar{A})}$ is $\eta_{A(\bar{A})}$-pseudo-Hermitian, we have

$$\text{tr}_{\bar{A}}X_2 = \frac{i}{2}[\eta_A^{-1}(\text{tr}_{\bar{A}}X^\dagger)\eta_A - \text{tr}_{\bar{A}}X] = \frac{i}{2}(\eta_A^{-1}X_A^\dagger \eta_A - X_A) = 0. \tag{2.7}$$

Similarly, one could show $\text{tr}_A X_2 = 0$.

If X can be written as (2.5) and $\text{tr}_{\bar{A}}X_2 = 0$, we have $X_A = \text{tr}_{\bar{A}}X_1$ and

$$X_A^\dagger = \text{tr}_{\bar{A}}X^\dagger = \text{tr}_{\bar{A}}(\eta X_1 \eta^{-1}) = \eta_A(\text{tr}_{\bar{A}}X_1)\eta_A^{-1} = \eta_A X_A \eta_A^{-1}. \tag{2.8}$$

Thus X_A is η_A-pseudo-Hermitian. Similarly, we can show $X_{\bar{A}}$ is $\eta_{\bar{A}}$-pseudo-Hermitian. $\qquad\square$

An obvious corollary of Proposition 3 is that a η-pseudo-Hermitian transition matrix with $\eta = \eta_A \otimes \eta_{\bar{A}}$ generates pseudo-Hermitian reduced transition matrices. Let's first focus on the pure state transition matrix $\mathcal{T}^{\psi|\phi}$ and show how to construct a η-pseudo-Hermitian transition matrices. The η-pseudo-Hermiticity gives constraints on the pure states $|\psi\rangle$ and $|\phi\rangle$. We have the following theorem.

Theorem 2. *A transition matrix $\mathcal{T}^{\psi|\phi}$ is η-pseudo-Hermitian , if and only if it can be written as follows.*

$$\mathcal{T}^{\psi|\phi} = \frac{|\psi\rangle\langle\psi|\eta}{\langle\psi|\eta|\psi\rangle}, \tag{2.9}$$

where η is both Hermitian and invertible.

Proof. Assume the transition matrix $\mathcal{T}^{\psi|\phi} = \frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle}$ is η-pseudo-Hermitian, we have

$$\frac{\eta|\psi\rangle\langle\phi|\eta^{-1}}{\langle\phi|\psi\rangle} = \frac{|\phi\rangle\langle\psi|}{\langle\psi|\phi\rangle}. \tag{2.10}$$

It leads to

$$\eta|\psi\rangle = \frac{\langle\psi|\eta|\psi\rangle}{\langle\psi|\phi\rangle}|\phi\rangle. \tag{2.11}$$

Taking the above formula into (1.2), one can see the transition matrix $\mathcal{T}^{\psi|\phi}$ is of the form (2.9). On the other hand, it is easy to show that if a transition matrix $\mathcal{T}$ takes the form of eq. (2.9), where η is an invertible Hermitian matrix, $\mathcal{T}$ is η-pseudo-Hermitian. $\qquad\square$

Remark: In the Theorem 2, the pure states $|\psi\rangle$ and $|\phi\rangle$ of the transition matrix $\mathcal{T}^{\psi|\phi}$ are assumed to be non-orthogonal. For the case of $\langle\phi|\psi\rangle = 0$, one may consider the matrix $\mathcal{T}'^{\psi|\phi} = |\psi\rangle\langle\phi|$. If it is η-pseudo-Hermitian, we have

$$\eta|\psi\rangle\langle\phi|\eta^{-1} = |\phi\rangle\langle\psi|. \tag{2.12}$$

To satisfy the above condition, it is necessary that $\langle\psi|\eta|\phi\rangle = \langle\phi|\eta|\psi\rangle \neq 0$, otherwise we would have $\eta|\psi\rangle = 0$, which is impossible for $|\psi\rangle \neq 0$. Further, we have the relation

$$|\phi\rangle = \frac{\eta|\psi\rangle}{\langle\psi|\eta|\phi\rangle}. \tag{2.13}$$

Therefore, the transition matrix $|\psi\rangle\langle\phi|$ is given by

$$\mathcal{T}'^{\psi|\phi} = \frac{|\psi\rangle\langle\psi|\eta}{\langle\psi|\eta|\phi\rangle}, \tag{2.14}$$

for the case $\langle\psi|\phi\rangle = 0$.

A η-pseudo-Hermitian transition matrix $\mathcal{T}^{\psi|\phi}$ with $\eta = \eta_A \otimes \eta_{\bar{A}}$ generates pseudo-Hermitian reduced transition matrices $\mathcal{T}_A^{\psi|\phi}$ and $\mathcal{T}_{\bar{A}}^{\psi|\phi}$. Whereas, the pseudo-Hermiticity of $\mathcal{T}_{A(\bar{A})}^{\psi|\phi}$ only guarantees $\text{tr}[(\mathcal{T}_{A(\bar{A})}^{\psi|\phi})^n]$ is real, not necessarily positive. One can expect to give more constraints on $\eta_{A(\bar{A})}$ to make $\text{tr}[(\mathcal{T}_{A(\bar{A})}^{\psi|\phi})^n] > 0$. The following theorem provides a construction in which the eigenvalues of the reduced transition matrix are all non-negative and not all zeros so that $\text{tr}[(\mathcal{T}_{A(\bar{A})}^{\psi|\phi})^n] > 0$.

Theorem 3. $\mathcal{T}^{\psi|\phi}$ is η-pseudo-Hermitian with $\eta = \eta_A \otimes \eta_{\bar{A}}$.

1. *If $\eta_{A(\bar{A})}$ is positive or negative definite operator, the eigenvalues of $\mathcal{T}_{A(\bar{A})}^{\psi|\phi}$ are real.*

2. *If η_A is positive or negative definite and $\eta_{\bar{A}}$ is positive or negative definite too, then the eigenvalues of $\mathcal{T}_{A(\bar{A})}^{\psi|\phi}$ are non-negative and not all zeros.*

Proof. In general, $\mathcal{T}_{A(\bar{A})}^{\psi|\phi}$ are expected to have complex eigenvalues for arbitrary Hermitian $\eta_{A(\bar{A})}$. Let us define

$$\tilde{\mathcal{T}}_A := \frac{\text{tr}_{\bar{A}}\left[|\psi\rangle\langle\psi|\eta_{\bar{A}}\right]}{\langle\psi|\eta|\psi\rangle}. \tag{2.15}$$

It is obvious that $\tilde{\mathcal{T}}_A$ is a Hermitian operator. By using (2.9) we have

$$\mathcal{T}_A^{\psi|\phi} = \text{tr}_{\bar{A}}\mathcal{T}^{\psi|\phi} = \tilde{\mathcal{T}}_A\eta_A. \tag{2.16}$$

Assume that η_A is a positive (negative) definite operator. There exists a Hermitian (skew-Hermitian) invertible operator $\eta_A^{1/2}$ such that $(\eta_A^{1/2})^2 = \eta_A$. Thus we have

$$\eta_A^{1/2}\mathcal{T}_A^{\psi|\phi}\eta_A^{-1/2} = \eta_A^{1/2}\tilde{\mathcal{T}}_A\eta_A^{1/2}. \tag{2.17}$$

$\mathcal{T}_A^{\psi|\phi}$ is similar to the operator on the right-hand side of the above equation, which is Hermitian. Thus, the eigenvalues of $\mathcal{T}_A^{\psi|\phi}$ are real.

If further assuming $\eta_{\bar{A}}$ is positive (negative) definite, we have

$$\tilde{\mathcal{T}}_A = \frac{\mathrm{tr}_{\bar{A}}\left[\eta_{\bar{A}}^{1/2}|\psi\rangle\langle\psi|\eta_{\bar{A}}^{1/2}\right]}{\langle\psi|\eta|\psi\rangle}, \tag{2.18}$$

where we define the Hermitian (skew-Hermitian) invertible operator $\eta_{\bar{A}}^{1/2}$ and use the cyclic property of partial trace. It is not hard to show $\eta_A^{1/2}\tilde{\mathcal{T}}_A\eta_A^{1/2}$ is always positive semi-definite and not null, the eigenvalues of which are non-negative and not all zeros.[3] Therefore, using (2.17), we have proved the eigenvalues of $\mathcal{T}_A^{\psi|\phi}$ are non-negative and not all zeros. By a similar process, one could show the eigenvalues of $\mathcal{T}_{\bar{A}}^{\psi|\phi}$ are all non-negative and not all zeros. $\qquad\square$

Since the pure state transition matrix (1.2) has unit trace, for a transition matrix $\mathcal{T}^{\psi|\phi}$ satisfying condition 2 in Theorem 3, we have $0 < \mathrm{tr}[(\mathcal{T}_A^{\psi|\phi})^n] \leq 1$ ($n \geq 2$). Hence, we construct a class of transition matrices for which the pseudo (Rényi) entropies (1.3) are non-negative.

Remark: The above theorem can be generalized to the case where $|\psi\rangle$ and $|\phi\rangle$ are orthogonal to each other. Building on (2.14), for the second case (η_A is positive or negative definite and $\eta_{\bar{A}}$ is positive or negative definite too), one has an expression similar to eq. (2.17),

$$\eta_A^{1/2}\mathcal{T}_A'^{\psi|\phi}\eta_A^{-1/2} = \eta_A^{1/2}\tilde{T}_A'\eta_A^{1/2}, \tag{2.19}$$

where $\tilde{T}_A' = \mathrm{tr}_{\bar{A}}\left(\eta_{\bar{A}}^{1/2}|\psi\rangle\langle\psi|\eta_{\bar{A}}^{1/2}\right)/\langle\psi|\eta|\phi\rangle$ is a positive or negative semi-definite matrix depending on $\eta_{\bar{A}}$ and the value of $\langle\psi|\eta|\phi\rangle$. Thus the right-hand side of (2.19) is positive or negative semi-definite, depending on $\eta_{A(\bar{A})}$ and $\langle\psi|\eta|\phi\rangle$. In what follows, we assume that it is positive semi-definite, which can always be achieved by redefining $|\phi\rangle$ as $-|\phi\rangle$.

So far, we have studied the construction of the pure state transition matrix using the pseudo-Hermiticity to obtain non-negative pseudo (Rényi) entropy. The following theorem shows that, for generic cases (i.e., the transition matrix takes the form (1.4)), the construction we find is closely related to the results of the pure state transition matrix.

Theorem 4. *Any diagonalizable η-pseudo-Hermitian matrix M can be expressed as*

$$M = \sum_i m_i \mathcal{T}^{\psi_i|\phi_i}, \tag{2.20}$$

where m_i are real, $\mathcal{T}^{\psi_i|\phi_i}$ are η-pseudo-Hermitian transition matrices between two pure states $|\psi_i\rangle$ and $|\phi_i\rangle$, which take the form (2.9) or (2.14).

The proof can be found in appendix C. By combining Proposition 1 with the above theorem, we can conclude that any mixed state transition matrix X of the form (1.4) in finite-dimensional Hilbert space can be decomposed into the sum of a series of pure state transition matrices of the form (2.9) or (2.14), that is,

$$X = \frac{\rho_1\rho_2}{\mathrm{tr}[\rho_1\rho_2]} = \sum_i m_i \mathcal{T}^{\psi_i|\phi_i}, \tag{2.21}$$

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[3]Please refer to appendix B for a proof.

where m_i are real and both X and $\mathcal{T}^{\psi_i|\phi_i}$ are pseudo-Hermitian under some η matrix. We may further recast X into a more general form than (2.21) by combining Theorem 4 with Proposition 2.

Corollary 1. *For any transition matrix X of form* (1.4)*, assuming that there is a Hermitian invertible matrix η such that $X \pm \eta^{-1}X^\dagger\eta$ are diagonalizable, then X can be expressed as*

$$X = \sum_i x_i^1 \mathcal{T}_1^{\psi_i|\phi_i} + i\sum_j x_j^2 \mathcal{T}_2^{\psi_j|\phi_j}, \tag{2.22}$$

where $\mathcal{T}_\alpha^{\psi_i|\phi_i}$ are η-pseudo-Hermitian matrices which take the form (2.9) *or* (2.14) *and x_i^α are real ($\alpha = 1, 2$).*

Combining Proposition 1, Theorem 3, and Theorem 4, we get the second corollary as a summary of our results for the mixed state transition matrix.

Corollary 2. *The transition matrix X taking the form* (1.4) *is η-pseudo-Hermitian with $\eta = \eta_A \otimes \eta_{\bar{A}}$. Assume both η_A and $\eta_{\bar{A}}$ are positive or negative definite operators. If m_i in eq.* (2.21) > 0*, the eigenvalues of X_A and $X_{\bar{A}}$ are non-negative and not all zeros.*

Remark: One can show Corollary 2 by using the fact that the linear combinations of positive semi-definite operators with positive coefficients are still positive semi-definite. It can be slightly generalized. Once some coefficients, namely the elements in a subset $\{m_a\}$, are negative, and the transition matrix $\mathcal{T}^{\psi_a|\phi_a}$ satisfies $\text{tr}_{A(\bar{A})} \sum_a m_a \mathcal{T}^{\psi_a|\phi_a} = 0$, one can show the eigenvalues of $X_{A(\bar{A})}$ are non-negative and not all zeros.

Since X has unit trace, one can show $0 < \text{tr}[(X_{A(\bar{A})})^n] \leq 1$. Then Corollary 2 provides an approach to generate transition matrix X, such that the pseudo Rényi entropies of $X_{A(\bar{A})}$ are non-negative. The above condition is sufficient to have non-negative pseudo Rényi entropy. In appendix F.2, we show an example of a finite dimension. $\eta_{A(\bar{A})}$ are neither positive nor negative, the pseudo Rényi entropy is positive for integers $n \geq 2$.

3 Construction of the transition matrix

Next, we shall explore the concrete implementation of pseudo-Hermitian transition matrices satisfying Theorem 3 in finite-dimensional quantum systems and QFTs, respectively.[4] Before that, we would like to review the algebraic view of QFTs, which will be useful in the following discussion.

Let's consider a bipartite system whose Hilbert space $\mathcal{H}$ is divided into two subspaces $\mathcal{H}_A$ and $\mathcal{H}_{\bar{A}}$ of the same dimension. Let's denote $\mathcal{R}_{A(\bar{A})}$ to be the algebra of operators working on $\mathcal{H}_{A(\bar{A})}$. The algebra of operators for the total system is given by $\mathcal{R} = \mathcal{R}_A \otimes \mathcal{R}_{\bar{A}}$. Choosing a reference state $|\Psi\rangle$, one could generate the states in $\mathcal{H}$ by acting the operators in $\mathcal{R}_{A(\bar{A})}$ on $|\Psi\rangle$. A state $|\Psi\rangle$ in $\mathcal{H}$ is said to be cyclic for an algebra such as $\mathcal{R}_A$ if the set $\{a|\Psi\rangle\}$, $a \in \mathcal{R}_A$ is dense in $\mathcal{H}$. For the finite-dimensional case, say d-dimension, one could choose the state

$$|\Psi\rangle := \sum_{k=1}^{d} c_k |k\rangle_A \otimes |k\rangle_{\bar{A}}, \tag{3.1}$$

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[4]Considering a lattice-regularized QFT may ensure that Theorem 3 can be applied legally.

with $c_k \neq 0$, $|k\rangle_{A(\bar{A})}$ are basis of $\mathcal{H}_{A(\bar{A})}$. It can be proved that $|\Psi\rangle$ is cyclic for the algebra $\mathcal{R}_A$ and $\mathcal{R}_{\bar{A}}$. For the infinite-dimensional case, in the framework of algebraic QFT, one could also construct the local algebra $\mathcal{R}(A)$ consisting of the local operators supported in the open region A. The Reeh-Schlieder theorem [47] states that the vacuum state $|0\rangle$ is cyclic for the algebra $\mathcal{A}$ associated with any bounded open region A. Therefore, one could construct any pure state $|\psi\rangle$ in $\mathcal{H}$ by only using the operators in $\mathcal{R}(A)$ or $\mathcal{R}(\bar{A})$. In appendix D, we give a brief review of the associated aspects of algebraic QFT, see also [48, 49].

Assume the state $|\Psi\rangle$ is cyclic for the algebra $\mathcal{R}(A)$, there exist a and $\tilde{a}$ such that

$$a|\Psi\rangle = |\psi\rangle, \quad \tilde{a}|\Psi\rangle = |\phi\rangle. \tag{3.2}$$

Substituting the above expression into eq. (2.9), the η-pseudo-Hermitian transition matrices can be expressed as

$$\mathcal{T}^{\psi|\phi} = \frac{a|\Psi\rangle\langle\Psi|a^\dagger\eta}{\langle\Psi|a^\dagger\eta a|\Psi\rangle}, \tag{3.3}$$

where η, according to Proposition 3, is the direct product of two invertible Hermitian matrices η_A and $\eta_{\bar{A}}$.

3.1 Finite-dimensional examples

For simplicity, in the following, we choose c_k in (3.1) to be positive. One can assume the operators a, η_A, and $\eta_{\bar{A}}$ in eq. (3.3) are of the form $a = \sum_{ij} a_{ij}|i\rangle_A {}_A\langle j|$, $\eta_A = \eta_{mn}|m\rangle_A {}_A\langle n|$ and $\eta_{\bar{A}} = \bar{\eta}_{mn}|m\rangle_{\bar{A}} {}_{\bar{A}}\langle n|$, where the matrices η_{mn} and $\bar{\eta}_{mn}$ are invertible and Hermitian. Then we construct the transition matrix $\mathcal{T}^a$ by using the formula (3.3) as follows

$$\mathcal{T}_A^a = \mathcal{N} \sum_{\substack{j,k \\ i',j',k'}} a_{jk'} c_{k'} a_{j'i'}^* c_{i'} \eta_{j'k} \bar{\eta}_{i'k'} |j\rangle_A {}_A\langle k|, \tag{3.4}$$

where $\mathcal{N}$ is the normalization. We present some numerical η-pseudo-Hermitian transition matrices in appendix F.1.

3.2 Examples in QFTs

3.2.1 2-dimensional rational CFTs

The first example in QFTs is the real-time evolution of pseudo Rényi entropy in 2-dimensional CFTs considered in [27]. The subsystem A is taken to be $[-L, L]$ ($L > 0$).[5] Consider the transition matrix

$$\mathcal{T}^{\mathcal{O}} := \frac{\mathcal{O}(x_1, t_1)|0\rangle\langle 0|\mathcal{O}(x_2, t_2)}{\langle 0|\mathcal{O}(x_2, t_2)\mathcal{O}(x_1. t_1)|0\rangle}, \tag{3.5}$$

where $\mathcal{O}$ is assumed to be the Hermitian primary operator. Consider the case with $t_1 = t_2 = -t$ and $x_1 = -x_2$. We find for some rational CFT models $\mathrm{tr}[(\mathcal{T}_A^{\mathcal{O}})^2]$ is always real [27].

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[5]Note that A is chosen to be $[0, L]$ in [27], which, according to the spatial translation symmetry, does not affect the final conclusion.

Assume the Hamiltonian H commutes with the parity P. The transition matrix can be written as

$$\mathcal{T}^{\mathcal{O}} = \frac{\mathcal{O}(x_1, -t)|0\rangle\langle 0|\mathcal{O}^\dagger(x_1, -t)\mathrm{P}}{\langle 0|\mathcal{O}^\dagger(x_1, -t)\mathrm{P}\mathcal{O}(x_1, -t)|0\rangle}. \tag{3.6}$$

It shows that $\mathcal{T}^{\mathcal{O}}$ is P-pseudo-Hermitian according to eq. (2.9). Since the subsystem A and $\bar{A}$ are invariant under parity P, we could decompose $\mathrm{P} = \mathrm{P}_A \otimes \mathrm{P}_{\bar{A}}$, where $\mathrm{P}_{A(\bar{A})}$ acts on $A(\bar{A})$. We can schematically write $|\phi(x)\rangle$ $(x \in A)$ and $|\phi(\bar{x})\rangle$ $(\bar{x} \in \bar{A})$ as the basis for subsystem A and $\bar{A}$. The action of $\mathrm{P}_{A(\bar{A})}$ is regarded as $\mathrm{P}_A|\phi(x)\rangle = |\phi(-x)\rangle$. By definition $-x \in A$ if $x \in A$. P_A maps the basis of A into itself. The parity operators are invertible. Therefore, $\mathcal{T}^{\mathcal{O}}_{A(\bar{A})}$ is $\mathrm{P}_{A(\bar{A})}$-pseudo-Hermitian. This leads to $\mathrm{tr}[(\mathcal{T}^{\mathcal{O}}_{A(\bar{A})})^2]$ is real, which is consistent with the results in [27]. Again, our results in this paper predict $\mathrm{tr}[(\mathcal{T}^{\mathcal{O}}_{A(\bar{A})})^n]$ should be real for any positive integer $n \geq 2$.

3.2.2 General QFTs in Minkowski space

Our next example aims to demonstrate the construction of a transition matrix generating non-negative pseudo Rényi entropies within the context of general QFT in Minkowski space. We will see that it is closely related to Tomita-Takesaki theory.[6] Consider a QFT that resides in a d-dimensional Minkowski space $\mathcal{R}_{1,d-1}$. The metric on $\mathcal{R}_{1,d-1}$ is given by $ds^2 = -dt^2 + dx^2 + d\vec{y}^2$, where $\vec{y}$ are coordinates of $(d-2)$-dimensional Euclidean space. We choose the subsystem A to be the half-space $\{(x, \vec{y})|x < 0\}$ and $\bar{A}$ its complement. The local algebra $\mathcal{R}_A$ is given by operators located at the left Rindler wedge $\mathcal{W}_A := \{(t, x, \vec{y})|x < -|t|\}$, which is the causal domain of A. The operators can be constructed by the smeared field $\int d^d x f(x^\mu)\phi(x^\mu)$ with the functions f supported in $\mathcal{W}_A$. Similarly, the algebra $\mathcal{R}_{\bar{A}}$ is associated with the right wedge $\mathcal{W}_{\bar{A}} := \{(t, x, \vec{y})|x > |t|\}$.

There exists an antiunitary operator J_Ω, called modular conjugation, that exchanges the algebras $\mathcal{W}_A$ and $\mathcal{W}_{\bar{A}}$ according to the Tomita-Takesaki theory. For a given Hermitian operator $\phi(t, x, \vec{y})$, J_Ω acts as

$$J_\Omega \phi(t, x, \vec{y}) J_\Omega = \phi(-t, -x, \vec{y}). \tag{3.7}$$

It has been proved that $J_\Omega = \mathrm{CRT}$, where C and T are charge and time reversal operators, and R is the reflection $x \to -x$ while keeping other coordinates invariant [48, 50]. According to eq. (3.7), for any operator $\mathcal{O}_A \in \mathcal{R}_A$ we can define the operator $\mathcal{O}_{\bar{A}} := J_\Omega \mathcal{O}_A J_\Omega \in \mathcal{R}_{\bar{A}}$. For any pure state $|\psi\rangle$ there exists $\mathcal{O}_A \in \mathcal{R}_A$ such that $|\psi\rangle$ can be approximated by $\mathcal{O}_A|0\rangle$ by the cyclic property of the vacuum state $|0\rangle$ for the algebra $\mathcal{R}_A$. Define the transition matrix

$$\mathcal{T}^{\mathcal{O}_A} = \frac{\mathcal{O}_A|0\rangle\langle 0|\mathcal{O}_{\bar{A}}}{\langle 0|\mathcal{O}_A\mathcal{O}_{\bar{A}}|0\rangle}, \tag{3.8}$$

where $\mathcal{O}_{\bar{A}} = J_\Omega \mathcal{O}_A J_\Omega$. It can be shown the spectra of $\mathcal{T}^{\mathcal{O}_A}_A$ are non-negative. Thus the pseudo Rényi entropy is non-negative. To show this result, we need to use the modular theory of QFTs. By using the modular theory

$$\mathcal{T}^{\mathcal{O}_A} = \frac{\mathcal{O}_A|0\rangle\langle 0|\mathcal{O}_A^\dagger \Delta_\Omega^{1/2}}{\langle 0|\mathcal{O}_A^\dagger \Delta_\Omega^{1/2} \mathcal{O}_A|0\rangle}, \tag{3.9}$$

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[6]We briefly review the Tomita-Takesaki theory and its application in algebraic QFTs in appendix D.
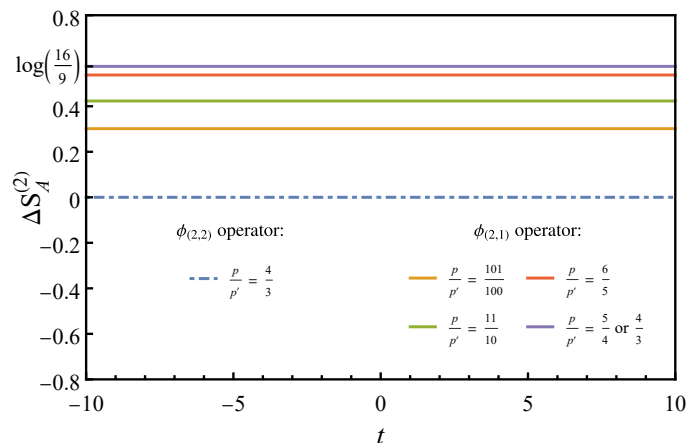
Figure 1. The excess of the 2nd pseudo Rényi entropy $\Delta S_A^{(2)}$ ($\Delta S_A^{(2)} \equiv S_A^{(2)} - S_{A;vac}^{(2)}$, where $S_{A;vac}^{(2)}$ denotes the 2nd Rényi entropy of A when the total system is in the vacuum) of the transition matrix $\mathcal{T}_A \equiv \text{tr}_{\bar{A}} \frac{\mathcal{O}(x,t)|0\rangle\langle 0|\mathcal{O}(-x,-t)}{\langle 0|\mathcal{O}(-x,-t)\mathcal{O}(x,t)|0\rangle}$ in the minimal models $\mathcal{M}(p,p')$. We study the case of $\mathcal{O} = \phi_{(2,2)}$ (dot-dashed line) and $\mathcal{O} = \phi_{(2,1)}$ (solid line), respectively. One novel feature is that the 2nd pseudo entropy is real and time-independent.

where Δ_Ω is the modular operator, a positive Hermitian operator. $\mathcal{T}^{\mathcal{O}_A}$ takes the same form as (3.3). Thus it is $\Delta_\Omega^{1/2}$-pseudo-Hermitian. Further, $\Delta_\Omega^{1/2} = e^{-\pi K_A} \otimes e^{\pi K_{\bar{A}}}$. By using Theorem 3 and the fact that $e^{-\pi K_A} = e^{-\pi K_A/2}e^{-\pi K_A/2}$ and $e^{\pi K_{\bar{A}}} = e^{\pi K_{\bar{A}}/2}e^{\pi K_{\bar{A}}/2}$ we conclude the eigenvalues of $\mathcal{T}_A^{\mathcal{O}_A}$ are all positive. In appendix E we show more general examples with positive pseudo Rényi entropy.

One could check the above result by evaluating the pseudo Rényi entropy by QFT methods. To move on, let's focus on 2D CFTs.[7] Consider the transition matrix (3.5). If $x_1 = -x_2$ and $t_1 = -t_2$, $J_\Omega \mathcal{O}(x_1, t_1)J_\Omega = \mathcal{O}(x_2, t_2)$, thus the pseudo Rényi entropy is expected to be real in this case. The results are shown in figure 1 for models in rational CFTs.

3.3 Comment on an exceptional case in the 2-dimensional free scalar theory

In [18], the authors studied the pseudo-Rényi entropy of locally excited states in a massless scalar field theory that resides in two-dimensional Euclidean space.[8] The states of interest $|\psi\rangle$ and $|\phi\rangle$ are chosen as

$$|\psi\rangle = e^{-aH}\tilde{\mathcal{O}}(x=0)|0\rangle, \quad |\phi\rangle = e^{-a'H}\mathcal{O}(x=0)|0\rangle, \tag{3.10}$$

where H is the Hamiltonian of CFT, a and a' are cutoff to avoid UV divergence. The inserted operators $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are defined as

$$\mathcal{O} = e^{\frac{i}{2}\phi} + e^{-\frac{i}{2}\phi}, \quad \tilde{\mathcal{O}} := e^{\frac{i}{2}\phi} + e^{i\theta}e^{-\frac{i}{2}\phi}, \quad \left(\theta \in [-\pi, \pi]\right). \tag{3.11}$$

---

[7]In appendix H, we give an overview of the replica method to compute the nth pseudo Rényi entropy in 2D CFTs.

[8]In Euclidean space the coordinates are τ, x or $w = x - i\tau$, $\bar{w} = x + i\tau$, and the action is given by $S = \int dw d\bar{w} \partial_w \phi \partial_{\bar{w}} \phi$.

– 11 –

In [18], the authors showed that the 2nd and 3rd pseudo Rényi entropies of $\mathcal{T}_A^{\psi|\phi}$ are real for subsystem A with finite length, utilizing the replica method and conformal mapping.

To clarify the reality of the pseudo Rényi entropy within our framework, it is necessary to construct a transition matrix that satisfies condition 2 in Theorem 3. However, it should be noted that in this particular case, proving the pseudo-Hermiticity of the transition matrix $\mathcal{T}^{\psi|\phi}$ is a challenging task, as we cannot directly apply Proposition 1 from finite-dimensional quantum mechanics to QFT.

Let us first consider the case of $\theta \neq 0$ to show that $\mathcal{T}^{\psi|\phi}$ is pseudo-Hermitian. One first defines the momentum operator [51]

$$\pi_0 := \frac{i}{4\pi} \int dx \partial_\tau \phi, \tag{3.12}$$

which satisfies the following commutation relation

$$[\pi_0, \mathcal{V}_\alpha] = \alpha \mathcal{V}_\alpha, \quad (\mathcal{V}_\alpha \equiv : e^{i\alpha\phi} :). \tag{3.13}$$

One can also show π_0 is a Hermitian operator and commutes with Hamiltonian H. By using the commutator (3.13) and the Baker-Campbell-Hausdorff formula, we have

$$e^{\lambda\pi_0} \mathcal{V}_\alpha e^{-\lambda\pi_0} = e^{\alpha\lambda} \mathcal{V}_\alpha. \tag{3.14}$$

The transition matrix $\mathcal{T}^{\psi|\phi}$ then can be written as

$$\mathcal{T}^{\psi|\phi} = \frac{|\Phi\rangle\langle\Phi|\eta_\Phi}{\langle\Phi|\eta_\Phi|\Phi\rangle}, \tag{3.15}$$

with

$$\eta_\Phi := e^{-i\theta\pi_0} e^{-(a'-a)H}, \quad |\Phi\rangle := e^{-aH} (e^{\frac{i}{2}\phi(0)} e^{-\frac{i}{2}\theta} + e^{-\frac{i}{2}\phi(0)} e^{\frac{i}{2}\theta}) |0\rangle. \tag{3.16}$$

We further introduce an operator $\tilde{\mathcal{T}}^{\psi|\phi}$

$$\tilde{\mathcal{T}}^{\psi|\phi} := \frac{(e^{\frac{i}{2}\phi(0)} e^{-\frac{i}{2}\theta} + e^{-\frac{i}{2}\phi(0)} e^{\frac{i}{2}\theta}) |0\rangle\langle 0| (e^{\frac{i}{2}\phi(0)} e^{-\frac{i}{2}\theta} + e^{-\frac{i}{2}\phi(0)} e^{\frac{i}{2}\theta})}{2\cos\frac{\theta}{2} \langle e^{\frac{i}{2}\phi(0)} e^{-(a+a')H} e^{-\frac{i}{2}\phi(0)} \rangle} e^{-i\theta\pi_0},$$

$$\mathcal{T}^{\psi|\phi} = e^{-aH} \tilde{\mathcal{T}}^{\psi|\phi} e^{-a'H}. \tag{3.17}$$

Since the momentum operator π_0 is Hermitian, $e^{-i\theta\pi_0}$ is a unitary operator. Then, η_Φ is not a Hermitian operator. The transition matrix (3.15) seems to be different from the general form of the pseudo-Hermitian operator (2.9). It doesn't mean the transition matrix cannot be pseudo-Hermitian. The transition matrix $\tilde{\mathcal{T}}^{\psi|\phi} \propto |\Psi\rangle\langle\Psi|U$ with the unitary operator $U := e^{-i\theta\pi_0}$ and the state $|\Psi\rangle := (e^{\frac{i}{2}\phi(0)} e^{-\frac{i}{2}\theta} + e^{-\frac{i}{2}\phi(0)} e^{\frac{i}{2}\theta}) |0\rangle$. $\tilde{\mathcal{T}}^{\psi|\phi}$ can be η-pseudo-Hermitian if the unitary operator U and the state $|\Psi\rangle$ satisfy certain constraints. It is necessary that η, U, and $|\Psi\rangle$ should satisfy that $U\eta|\Psi\rangle = |\Psi\rangle$.[9] Equivalently, the operator $\eta' := U\eta$ should satisfy the condition (G.3) (G.4) or a weaker condition (G.5). One could show the transition matrix $\mathcal{T}^{\psi|\phi}$ is $\eta e^{-(a'-a)H}$-pseudo-Hermitian if $\tilde{\mathcal{T}}^{\psi|\phi}$ is η-pseudo-Hermitian and η commutes with H. Assume $\tilde{\mathcal{T}}^{\psi|\phi}$ is η-pseudo-Hermitian, we have

$$\eta\tilde{\mathcal{T}}^{\psi|\phi}\eta^{-1} = \tilde{\mathcal{T}}^{\psi|\phi\dagger}. \tag{3.18}$$

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[9]Please refer to appendix G for more details.

By using the above result and the definition (3.17) one could show

$$(\eta e^{-(a'-a)H})\mathcal{T}^{\psi|\phi}(\eta e^{-(a'-a)H})^{-1} = \eta e^{-a'H}\tilde{\mathcal{T}}^{\psi|\phi}e^{-aH}\eta^{-1} = e^{-a'H}(\tilde{\mathcal{T}}^{\psi|\phi})^{\dagger}e^{-aH} = \mathcal{T}^{\psi|\phi\dagger},$$
(3.19)

where in the second step we use η commutes with H and (3.18). It is expected that there exists an operator η' satisfying the condition (G.5), or equally $\eta'\pi_0\eta' = -\pi_0$. A candidate for η' is the time reflection operator Θ [52, 53], which gives the transformation $\Theta\phi(\tau, x)\Theta = \phi(-\tau, x)$ and $\Theta\partial_\tau\phi(\tau)\Theta = -\partial_\tau\phi(\tau)$. Thus we have $\Theta\pi_0\Theta = -\pi_0$. Further, one should note that Θ commutes with H.

We next consider the simpler case $\theta = 0$. One can readily find that the transition matrix $\mathcal{T}^{\psi|\phi}$ in this case is $e^{-(a'-a)H}$-pseudo-Hermitian.

It seems that our analysis can only come to a standstill here. If we want to continue applying Theorem 3, we are faced with the problem of decomposing $e^{-(a'-a)H}$ into the tensor product of operators in A and $\bar{A}$, regardless of whether θ is zero or not. A naive speculation could be $e^{-(a'-a)H} = e^{-(a'-a)H_A} \otimes e^{-(a'-a)H_{\bar{A}}}$, where $H_{A(\bar{A})} = \int_{A(\bar{A})} dx T_{00}$. However, if one takes the QFT as a lattice model, this decomposition would be ambiguous. It seems there exists some interaction term between A and $\bar{A}$.[10] We could consider a lattice-regularized massless free scalar to see this clearly. The lattice-regularized Hamiltonian of it is given by

$$H = \sum_{n=-\infty}^{+\infty} \frac{1}{2}a\left\{\dot{\phi}_n^2 + \frac{1}{a^2}(\phi_{n+1} - \phi_n)^2\right\}, \quad \left(\dot{\phi}_n \equiv \frac{d}{dt}\phi_n\right),$$
(3.20)

where a is the lattice size which is so-called regulator. From eq. (3.20), it is evident that the term $(\phi_{n+1} - \phi_n)^2/a^2$ contributes to the interaction term at the boundary between A and $\bar{A}$. For instance, in the case where $A = [0, \infty)$, the interaction term takes the form $H_{\text{int}} = \frac{1}{2a}(\phi_0 - \phi_{-1})^2$, where $\phi_0 \equiv \phi(x = 0) \in A$, $\phi_{-1} \equiv \phi(x = -a) \in \bar{A}$. Thus, it cannot be decomposed H into $H_A \otimes H_{\bar{A}}$ naively in the lattice model. In the continuous limit $a \to 0$, it seems the interaction term becomes a boundary term. But it is hard to estimate whether this boundary term would be important for our purpose. For the lattice model with interactions, the problem would be more subtle.

On the other hand, we would like to emphasize that the η matrix that ensures the transition matrix satisfies the pseudo-Hermiticity condition (1.5) is not unique. Our results only provide one possibility, and the question of whether a decomposable η matrix exists is an intriguing one that deserves further investigation.

4 Discussion

Real-valued pseudo entropies have robust potential applications in holography, quantum information, and quantum many-body physics. In this article, we explore the reality condition of pseudo entropy utilizing the notion of pseudo-Hermiticity. We have extended the concept of transition matrices from pure states to more general mixed states (1.4), and have demonstrated that all finite-dimensional transition matrices are pseudo-Hermitian

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[10]We thank the anonymous referee for pointing this out to us.

(see Proposition 1). We derive the equivalent condition for the reduced transition matrix to be pseudo-Hermitian (see Proposition 3). On this basis, we construct a class of transition matrices that are η-pseudo-Hermitian and have non-negative eigenvalues (not all zeros) by selecting special η matrices (see Theorem 3 about pure state transition matrices and Corollary 2 about mixed state transition matrices). Thus, we construct a set of transition matrices that can generate non-negative pseudo (Rényi) entropy of arbitrary order of subsystems.

Based on our constructions, we perform a series of numerical tests in finite dimensional quantum systems. We subsequently work out several non-trivial examples in quantum field theories, including Euclidean QFT with non-Hermitian operator insertion, and QFTs with Lorentzian signature. The core idea is to find an appropriate η matrix so that the transition matrix is η-pseudo-Hermitian and satisfy the conditions of Theorem 3. In terms of these examples, we learn that the η-pseudo-Hermiticity is associated with the various aspects, e.g., algebraic structures of QFT, modular Hamiltonian, parity, etc.

The notion of pseudo-Hermiticity has been extensively studied for the non-Hermitian systems. It originates from non-Hermitian matrices, which are diagonalizable and own a complete biorthonormal eigenbasis. In the current work, we mainly focus on the pseudo-Hermiticity of the reduced transition matrix to construct the real-valued pseudo entropy. If the reduced transition matrix is non-pseudo-Hermitian which means it is not diagonalizable, one possible way to find the reality condition of pseudo entropy is writing the matrix into Jordan form for the non-diagonalizable cases [36, 54]. It is an interesting direction to achieve the real-valued pseudo entropy by going beyond the pseudo-Hermiticity.

A Non-pseudo-Hermitian X_A with real $\text{tr}[(X_A)^n]$

Let us consider a 4 qubits system $\mathcal{S}$ (2 qubits each for A and $\bar{A}$) and a transition matrix $X = \frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle}$ acting on its Hilbert space $\mathcal{H}_{\mathcal{S}} \equiv H_A \otimes H_{\bar{A}}$, where $|\psi\rangle$ and $|\phi\rangle$ are two non-orthogonal quantum states living in $H_{\mathcal{S}}$,

$$|\psi\rangle = \frac{1}{2}|00\rangle_A|00\rangle_{\bar{A}} + \frac{1}{2}|01\rangle_A|01\rangle_{\bar{A}} + \frac{1}{2}|10\rangle_A|10\rangle_{\bar{A}} + \frac{1}{2}|11\rangle_A|11\rangle_{\bar{A}},$$

$$|\psi_\perp\rangle = \frac{i}{4}|00\rangle_A|00\rangle_{\bar{A}} + \frac{i}{4}|01\rangle_A|01\rangle_{\bar{A}} - \frac{i}{4}|10\rangle_A|10\rangle_{\bar{A}} - \frac{i}{4}|11\rangle_A|11\rangle_{\bar{A}} + \frac{\sqrt{3}}{2}|11\rangle_A|10\rangle_{\bar{A}},$$

$$|\phi\rangle \equiv \frac{\sqrt{2}}{2}|\psi\rangle + \frac{\sqrt{2}}{2}|\psi_\perp\rangle, \qquad (\langle\phi|\phi\rangle = \langle\psi|\psi\rangle = \langle\psi_\perp|\psi_\perp\rangle = 1, \ \langle\psi|\psi_\perp\rangle = 0). \tag{A.1}$$

The reduced transition matrix of the subsystem A, obtained by tracing out the d.o.f. of $\bar{A}$, is given by

$$X_A \equiv \operatorname{tr}_{\bar{A}} X \tag{A.2}$$

$$= \left(\frac{1}{4} - \frac{i}{8}\right)\left(|00\rangle_{AA}\langle 00| + |01\rangle_{AA}\langle 01|\right) + \left(\frac{1}{4} + \frac{i}{8}\right)\left(|10\rangle_{AA}\langle 10| + |11\rangle_{AA}\langle 11|\right) + \frac{\sqrt{3}}{4}|01\rangle\langle 11|.$$

Building on (A.2), it's more useful to write down the matrix formulation of X_A,

$$X_A = \begin{pmatrix} \frac{1}{4} - \frac{i}{8} & & & \\ & \frac{1}{4} - \frac{i}{8} & & \\ & & \frac{1}{4} + \frac{i}{8} & \frac{\sqrt{3}}{4} \\ & & & \frac{1}{4} + \frac{i}{8} \end{pmatrix}, \tag{A.3}$$

which is an upper triangular 4×4 matrix and cannot be diagonalized. It can be found from (A.3) that the eigenvalues of X_A consist of two complex conjugate pairs, which renders $\operatorname{tr}[(X_A)^n]$ real. On the other hand, we have

$$X_A^{\dagger} = \begin{pmatrix} \frac{1}{4} + \frac{i}{8} & & & \\ & \frac{1}{4} + \frac{i}{8} & & \\ & & \frac{1}{4} - \frac{i}{8} & \\ & & \frac{\sqrt{3}}{4} & \frac{1}{4} - \frac{i}{8} \end{pmatrix}. \tag{A.4}$$

Although $X_A^{\dagger}$ has the same eigenvalues as X_A, they are not similar. This is because they have different Jordan standard forms, which is read from the fact that the Jordan blocks of the same eigenvalue of two matrices are different. Therefore, we know that X_A is non-pseudo-Hermitian.

B A proof of Theorem 3

In this appendix, we show that $\eta_A^{1/2}\tilde{\mathcal{T}}_A\eta_A^{1/2}$ is positive semi-definite and not null in the following four cases: $\mathfrak{a}$) η_A and $\eta_{\bar{A}}$ are both positive definite; $\mathfrak{b}$) η_A and $\eta_{\bar{A}}$ are both negative definite; $\mathfrak{c}$) η_A is positive definite and $\eta_{\bar{A}}$ is negative definite; $\mathfrak{d}$) η_A is negative definite and $\eta_{\bar{A}}$ is positive definite.

Case $\mathfrak{a}$. Since η_A and $\eta_{\bar{A}}$ are both positive definite, we first have $\langle\psi|\eta|\psi\rangle = \langle\psi|\eta_A \otimes \eta_{\bar{A}}|\psi\rangle > 0$ for any $|\psi\rangle \neq 0$.[11] We also know that $\eta_A^{1/2}$ and $\eta_{\bar{A}}^{1/2}$ are both positive definite. Therefore, we find that $\tilde{\mathcal{T}}_A$ (2.18) is positive semi-definite and not null.[12] It follows that $\eta_A^{1/2}\tilde{\mathcal{T}}_A\eta_A^{1/2}$ is positive semi-definite and not null.

Case $\mathfrak{b}$. We first have $\langle\psi|\eta|\psi\rangle > 0$ for any $|\psi\rangle \neq 0$. Since $(-\eta_{\bar{A}})^{1/2}$ is positive definite in this case, $\tilde{\mathcal{T}}_A$ can be rewritten as

$$\tilde{\mathcal{T}}_A = \frac{\operatorname{tr}_{\bar{A}}\left[|\psi\rangle\langle\psi|\eta_{\bar{A}}\right]}{\langle\psi|\eta|\psi\rangle} = -\frac{\operatorname{tr}_{\bar{A}}\left[(-\eta_{\bar{A}})^{1/2}|\psi\rangle\langle\psi|(-\eta_{\bar{A}})^{1/2}\right]}{\langle\psi|\eta|\psi\rangle}. \tag{B.1}$$

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[11]Note that the eigenvalues of η consist of the product of the respective eigenvalues of η_A and $\eta_{\bar{A}}$.

[12]Note that $\tilde{\mathcal{T}}_A$ can have zero eigenvalues, depending on the choice of $|\psi\rangle$.

From above we conclude that $\tilde{\mathcal{T}}_A$ is negative semi-definite and not null. On the other hand, the square root of η_A can be rewritten as $\eta_A^{1/2} = i(-\eta_A)^{1/2}$, where $(-\eta_A)^{1/2}$ is positive definite. It follows that $\eta_A^{1/2}\tilde{\mathcal{T}}_A\eta_A^{1/2} = (-\eta_A)^{1/2}(-\tilde{\mathcal{T}}_A)(-\eta_A)^{1/2}$ is positive semi-definite and not null.

Case 𝔠. In the third case, we first know that $\langle\psi|\eta|\psi\rangle < 0$ for any $|\psi\rangle \neq 0$ and $(-\eta_{\bar{A}})^{1/2}$ is positive definite, which leads to $\tilde{\mathcal{T}}_A$ (B.1) being positive semi-definite and not null. Since $\eta_{\bar{A}}^{1/2}$ is positive definite, $\eta_A^{1/2}\tilde{\mathcal{T}}_A\eta_A^{1/2}$ is positive semi-definite and not null.

Case 𝔡. In the last case, we first have $\langle\psi|\eta|\psi\rangle < 0$ for any $|\psi\rangle \neq 0$, which leads to $\tilde{\mathcal{T}}_A$ (2.18) being negative semi-definite and not null. Then $\eta_A^{1/2}\tilde{\mathcal{T}}_A\eta_A^{1/2} = (-\eta_A)^{1/2}(-\tilde{\mathcal{T}}_A)(-\eta_A)^{1/2}$ is positive semi-definite and not null.

In summary, we show that $\eta_A^{1/2}\tilde{\mathcal{T}}_A\eta_A^{1/2}$ is always positive semi-definite and not null.

C A proof of Theorem 4

The result of Theorem 4 follows from the spectral decomposition of pseudo-Hermitian matrices. For any diagonalizable η-pseudo-Hermitian matrix M, we can write the spectral decomposition of M as

$$M = \sum_i \lambda_{0,i}|\psi_{0,i}\rangle\langle\phi_{0,i}| + \sum_j \left(\lambda_{+,j}|\psi_{+,j}\rangle\langle\phi_{+,j}| + \lambda_{-,j}|\psi_{-,j}\rangle\langle\phi_{-,j}|\right), \qquad \text{(C.1)}$$

where λ, $|\psi\rangle$ and $\langle\phi|$ represent the eigenvalue, right eigenvector, and left eigenvector of M, respectively.[13] Since we can always choose the biorthonormal eigenbasis such that

$$|\phi_{0,i}\rangle = \eta|\psi_{0,i}\rangle, \quad |\phi_{\pm,j}\rangle = \eta|\psi_{\mp,j}\rangle, \quad \langle\phi_{\alpha,i}|\psi_{\beta,j}\rangle = \delta_{\alpha\beta}\delta_{ij}, \ (\alpha,\beta \in \{0,\pm\}) \qquad \text{(C.2)}$$

hold [40], the spectrum decomposition becomes

$$
\begin{aligned}
M &= \sum_i \lambda_{0,i}|\psi_{0,i}\rangle\langle\psi_{0,i}|\eta + \sum_j (\lambda_{+,j}|\psi_{+,j}\rangle\langle\psi_{-,j}|\eta + \lambda_{-,j}|\psi_{-,j}\rangle\langle\psi_{+,j}|\eta) \\
&= \sum_i \lambda_{0,i}|\psi_{0,i}\rangle\langle\psi_{0,i}|\eta \\
&\quad + \sum_j \lambda_{+,j}^R\Big[(|\psi_{+,j}\rangle + |\psi_{-,j}\rangle)(\langle\psi_{-,j}| + \langle\psi_{+,j}|)\eta - |\psi_{-,j}\rangle\langle\psi_{-,j}|\eta - |\psi_{+,j}\rangle\langle\psi_{+,j}|\eta\Big] \\
&\quad + \sum_j \lambda_{+,j}^I\Big[(|\psi_{+,j}\rangle - i|\psi_{-,j}\rangle)(\langle\psi_{+,j}| + i\langle\psi_{-,j}|)\eta - |\psi_{-,j}\rangle\langle\psi_{-,j}|\eta - |\psi_{+,j}\rangle\langle\psi_{+,j}|\eta\Big] \\
&= \sum_i \lambda_{0,i}|\psi_{0,i}\rangle\langle\psi_{0,i}|\eta + \sum_j \lambda_{+,j}^R(|\psi_{+,j}\rangle + |\psi_{-,j}\rangle)(\langle\psi_{-,j}| + \langle\psi_{+,j}|)\eta \\
&\quad - \sum_j (\lambda_{+,j}^R + \lambda_{+,j}^I)|\psi_{-,j}\rangle\langle\psi_{-,j}|\eta - \sum_j (\lambda_{+,j}^R + \lambda_{+,j}^I)|\psi_{+,j}\rangle\langle\psi_{+,j}|\eta \\
&\quad + \sum_j \lambda_{+,j}^I(|\psi_{+,j}\rangle - i|\psi_{-,j}\rangle)(\langle\psi_{+,j}| + i\langle\psi_{-,j}|)\eta, \qquad \text{(C.3)}
\end{aligned}
$$

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[13]We use the subscript 0 to stand for real eigenvalues and the corresponding basis eigenvectors and the subscript $\pm$ to stand for the complex eigenvalues with $\pm$ imaginary part and the corresponding basis eigenvectors.

where $\lambda_{+,j}^R$ and $\lambda_{+,j}^I$ are the real and imaginary part of $\lambda_{+,j}$, respectively. Note that every term in summations is η-pseudo-Hermitian.

D Brief review of modular theory in QFTs

For any given open subsystem A in spacetimes, the local algebra $\mathcal{R}_A$ consists of all the operators supported in A. The algebra can also be associated with the domain of dependence of A, denoted by $\mathcal{D}(A)$. The reason is that the operators located in $\mathcal{D}(A)$ can be determined by the ones in A according to the dynamical time evolution of the theory. If A' is another subsystem that is spacelike with A, we expect the operators in A' would commute with the ones in A, that is $[\mathcal{R}_A, \mathcal{R}_{A'}] = 0$.

Denote the algebra associated with the whole spacetime as $\mathcal{R}$. The full Hilbert space $\mathcal{H}_0$ could be constructed by acting the operators in $\mathcal{R}$ on the vacuum state $|0\rangle$. The Reeh-Schlieder theorem says that the set $\{a|0\rangle, a \in \mathcal{R}_A\}$ is also dense in $\mathcal{H}_0$. For any given state $|\psi\rangle$, the theorem means that there exist operator $a \in \mathcal{R}_A$ such that $a|0\rangle$ can be arbitrarily close to $|\psi\rangle$. Thus we could construct the transition matrix $\mathcal{T}^{\psi|\phi}$ only by using the operators located in a subsystem. The above results can also be generalized to any cyclic state $|\Psi\rangle$.

The Tomita operator S_Ψ for the state $|\Psi\rangle$ is antilinear and satisfies

$$S_\Psi a|\Psi\rangle = a^\dagger|\Psi\rangle, \tag{D.1}$$

for any $a \in \mathcal{R}_A$. By definition it is obvious that $S_\Psi^2 = 1$. S_Ψ has a unique polar decomposition

$$S_\Psi = J_\Psi \Delta_\Psi^{1/2}, \tag{D.2}$$

where J_Ψ is antiunitary, $\Delta_\Psi^{1/2}$ is a positive Hermitian operator. J_Ψ is called the modular conjugation satisfying $J_\Psi^2 = 1$ and $J_\Psi^\dagger = J_\Psi$. Δ_Ψ is the modular operator associated with $\mathcal{R}_A$ and $|\Psi\rangle$. Similarly, one could define the modular operator $\bar{S}_\Psi$ associated with $\mathcal{R}_{\bar{A}}$. By using $S_\Psi^2 = 1$ we have

$$J_\Psi \Delta_\Psi^{1/2} J_\Psi = \Delta_\Psi^{-1/2}. \tag{D.3}$$

It can be shown that

$$\bar{S}_\Psi = S_\Psi^\dagger = \Delta_\Psi^{1/2} J_\Psi = J_\Psi \Delta^{-1/2}. \tag{D.4}$$

Consider the d-dimensional Minkowski spacetime. The metric is $ds^2 = -dt^2 + dx^2 + d\vec{y}^2$, where $\vec{y}$ are coordinates of $(d-2)$-dimensional Euclidean space. Let the subsystem A be $x > 0$. The domain of dependence of A is known as the Rindler wedge $\mathcal{W}_A$. For the vacuum state $|0\rangle$ the modular conjugation J_Ω is given by

$$J_\Omega = \mathrm{CRT}, \tag{D.5}$$

which is first proved by Bisognano and Wichmann [50]. The modular operator Δ_Ω can be formly written as

$$\Delta_\Omega = \rho_A \otimes \rho_{\bar{A}}^{-1}, \tag{D.6}$$

where $\rho_A := e^{-2\pi K_A}$ and $\rho_{\bar{A}} := e^{-2\pi K_{\bar{A}}}$ are the reduced density matrices of A and $\bar{A}$. K_A and $K_{\bar{A}}$ are known as the modular Hamiltonian of A and $\bar{A}$. The density matrices are
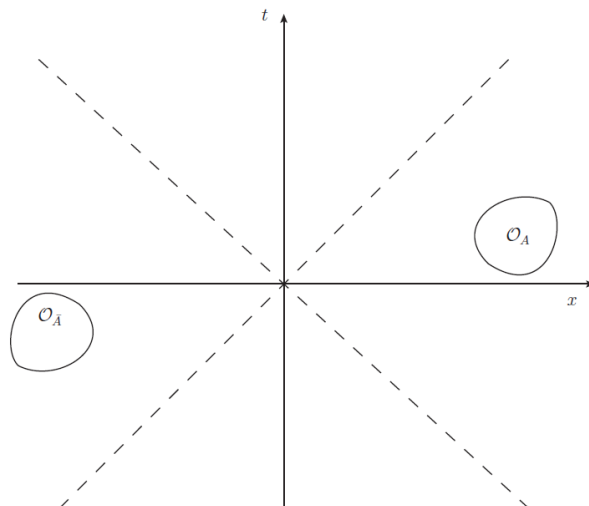
Figure 2. Illustration of the operators $\mathcal{O}_A$ and $\mathcal{O}_{\bar{A}}$.

positive operators. For any positive function $f(x)$, the operators $f(\rho_A)$ or $f(\rho_{\bar{A}})$ are also positive. For example, one could define the operator $\rho_A^{1/2} = e^{-\pi K_A}$, $\rho_A^{1/4} = e^{-\pi K_A/2}$. It is obvious the modular operator $\Delta_\Omega = e^{-2\pi(K_A - K_{\bar{A}})}$ is a positive Hermitian operator.

For the Rindler wedge K_A and $K_{\bar{A}}$ are associated with the Lorentz boost generators

$$
\begin{aligned}
K_A &= \int_{t=0, x \geq 0} dx d^{d-2} y x T_{00}, \\
K_{\bar{A}} &= - \int_{t=0, x \leq 0} dx d^{d-2} y x T_{00}.
\end{aligned}
\tag{D.7}
$$

For any Hermitian operator $\mathcal{O}(t, x, \vec{y})$, according to the definition of S_Ω we have

$$
S_\Omega \mathcal{O}(t, x, \vec{y})|0\rangle = \mathcal{O}(t, x, \vec{y})|0\rangle,
\tag{D.8}
$$

which leads to

$$
\Delta_\Omega^{1/2} \mathcal{O}(t, x, \vec{y})|0\rangle = J_\Omega \mathcal{O}(t, x, \vec{y})|0\rangle|0\rangle = \mathcal{O}(-t, -x, \vec{y})|0\rangle,
\tag{D.9}
$$

where we use the fact $J_\Omega^2 = 1$ and $J_\Omega|0\rangle = |0\rangle$.

E   Details of the example in QFTs

The transition matrix (3.8) is related to the operators $\mathcal{O}_A$ and $\mathcal{O}_{\bar{A}}$. In figure 2, we show the positions of the two operators. Taking $\mathcal{O}_{\bar{A}}$ into (3.8) we obtain

$$
\mathcal{T}^{\mathcal{O}_A} = \frac{\mathcal{O}_A|0\rangle\langle 0|\mathcal{O}_A J_\Omega}{\langle 0|\mathcal{O}_A J_\Omega \mathcal{O}_A|0\rangle}.
\tag{E.1}
$$

By the definition of Tomita operator we have

$$
\begin{aligned}
S_\Omega \mathcal{O}_A^\dagger|0\rangle &= J_\Omega \Delta_\Omega^{1/2} \mathcal{O}_A|0\rangle \\
&= J_\Omega \Delta_\Omega^{1/2} J_\Omega J_\Omega \mathcal{O}_A^\dagger|0\rangle \\
&= \Delta_\Omega^{-1/2} J_\Omega \mathcal{O}_A^\dagger|0\rangle,
\end{aligned}
\tag{E.2}
$$

where in the second step we use the fact $J_\Omega^2 = 1$, in the third step we use (D.3). Therefore, we have

$$J_\Omega \mathcal{O}_A^\dagger |0\rangle = \Delta_\Omega^{1/2} \mathcal{O}_A |0\rangle. \tag{E.3}$$

The transition matrix (E.1) is reduced to

$$\mathcal{T}^{\mathcal{O}_A} = \frac{\mathcal{O}_A |0\rangle \langle 0| \mathcal{O}_A^\dagger \Delta_\Omega^{1/2}}{\langle 0| \mathcal{O}_A^\dagger \Delta_\Omega^{1/2} \mathcal{O}_A |0\rangle}. \tag{E.4}$$

In the main text, we only discuss the special case (3.5), for which the eigenvalues of $\mathcal{T}_A^{\mathcal{O}_A}$ are positive real. More generally, one could choose $\mathcal{O}_A = \sum_j C_j \mathcal{O}_j(x_1, t_1)$, where $\mathcal{O}_j$ are Hermitian operators (not necessarily be primary), C_j are arbitrary constants. By $\mathcal{O}_{\bar{A}} = J_\Omega \mathcal{O}_{\bar{A}} J_\Omega$ we have $\mathcal{O}_{\bar{A}} = \sum_j C_j^* \mathcal{O}_j(-x_1, -t_1)$. It is expected that the pseudo Rényi entropy for the transition matrix associated with these operators will also be real.

The result of Theorem 2 can be used to construct the η-pseudo-Hermitian transition matrix in QFTs. Assume $|\Psi\rangle$ is a cyclic state for the algebra $\mathcal{R}_A$. The general η-pseudo-Hermitian transition matrices in QFTs are

$$\mathcal{T}^{\mathcal{O}_A} = \frac{\mathcal{O}_A |\Psi\rangle \langle \Psi| \mathcal{O}_A^\dagger \eta}{\langle \Psi| \mathcal{O}_A^\dagger \eta \mathcal{O}_A |\Psi\rangle}. \tag{E.5}$$

If η is taken to be identity, the transition matrix reduces to the Hermitian case. By using (D.1) one could rewrite the above formula as

$$
\begin{aligned}
\mathcal{T}^{\mathcal{O}_A} &= \frac{\mathcal{O}_A |\Psi\rangle \langle \Psi| \mathcal{O}_A S_\Psi^\dagger \eta}{\langle \Psi| \mathcal{O}_A^\dagger \eta \mathcal{O}_A |\Psi\rangle} \\
&= \frac{\mathcal{O}_A |\Psi\rangle \langle \Psi| \mathcal{O}_A J_\Omega \Delta_\Psi^{-1/2} \eta}{\langle \Psi| \mathcal{O}_A^\dagger \eta \mathcal{O}_A |\Psi\rangle}
\end{aligned} \tag{E.6}
$$

This paper only focuses on the vacuum state $|0\rangle$. Our example (E.1) is a special case $\eta = \Delta_\Omega^{1/2}$. In general, one could choose $\eta = \eta_A \otimes \eta_{\bar{A}}$, where η_A and $\eta_{\bar{A}}$ are invertible positive operators. Using Theorem 3, one could show that the pseudo Rényi entropy is also real in this case.

F Finite dimension example

Assume the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, the dimension of $\mathcal{H}_{A(\bar{A})}$ is d. Eq. (3.3) provides us a way to generate the η-pseudo-Hermitian transition matrices with $\eta = \eta_A \otimes \eta_{\bar{A}}$. One could arbitrarily choose the reference state $|\Psi\rangle$, i.e., the coefficients c_k and the operators $a, \eta_{A(\bar{A})}$. With a given basis $|k\rangle_{A(\bar{A})}$, we have the expansion

$$
\begin{aligned}
a &= \sum_{ij} a_{ij} |i\rangle_A {}_A\langle j|, \\
\eta_A &= \sum_{m,n} \eta_{mn} |m\rangle_A {}_A\langle n|, \\
\eta_{\bar{A}} &= \sum_{m,n} \bar{\eta}_{mn} |m\rangle_{\bar{A}} {}_{\bar{A}}\langle n|.
\end{aligned} \tag{F.1}
$$

The matrices η_{mn} and $\bar{\eta}_{mn}$ should be Hermitian and invertible.

In finite dimension, it is easy to show the Reeh-Schlieder theorem. Any state $|\psi\rangle$ can be constructed by only local operations on A or $\bar{A}$. The reference state $|\Psi\rangle := \sum_k c_k |k\rangle_A \otimes |k\rangle_{\bar{A}}$ is cyclic if the coefficients c_k are all non-vanishing. For any given state $|\psi\rangle$, we can expand it as

$$|\psi\rangle = \sum_{i,j} \psi_{ij} |i\rangle_A \otimes |j\rangle_{\bar{A}}. \tag{F.2}$$

It is enough to show that the basis $|i\rangle_A \otimes |j\rangle_{\bar{A}}$ of $\mathcal{H}$ can be obtained only by local operations on $|\Psi\rangle$. One could achieve this by acting an operator $|i\rangle_A{}_A\langle j|$ on $|\Psi\rangle$.

Taking (F.1) into (3.3) one could obtain the transition matrix $\mathcal{T}^a$. Eq. (3.4) can be obtained by partial trace $tr_{\bar{A}} \mathcal{T}^a := \sum_k {}_{\bar{A}}\langle k| \mathcal{T}^a |k\rangle_{\bar{A}}$. One could generate random matrices a_{ij}, η_{mn} and $\bar{\eta}_{mn}$ by software, e.g., Mathematica. Then we can construct the matrices $\mathcal{T}_A^a$ (3.4) and evaluate the eigenvalues of them. According to Corollary 2, the transition matrices by linear combinations of $\mathcal{T}^a$ can also have positive eigenvalues.

We have the following three different cases.

Case I: $\eta_{A(\bar{A})}$ is Hermitian and invertible matrices. Generally, in this case, the eigenvalues are expected to come in complex conjugate pairs or be real.

Case II: $\eta_{A(\bar{A})} = \mathcal{O}_{A(\bar{A})} \mathcal{O}_{A(\bar{A})}^\dagger$. $\mathcal{O}_{A(\bar{A})}$ is an arbitrary invertible operator. The eigenvalues, in this case, are expected to be real and positive. By considering the normalization of $\mathcal{T}_A^a$ the eigenvalues should belong to $[0,1]$. Thus the pseudo Rényi entropy should be real.

Case III: the linear combinations of $\mathcal{T}^{a^I}$,

$$\mathcal{T} := \sum_I x_I \mathcal{T}^{a^I}, \tag{F.3}$$

where x_I are positive numbers satisfying $\sum_I x_I = 1$, $\mathcal{T}^{a^I}$ is $\eta_A \otimes \eta_{\bar{A}}$-pseudo-Hermitian transition matrices with $\eta_{A(\bar{A})} = \mathcal{O}_{A(\bar{A})} \mathcal{O}_{A(\bar{A})}^\dagger$. In this case, the eigenvalues of $\mathcal{T}_A$ are positive.

F.1 Numerical result with $d = 3$

We show examples for these three cases in the main text, obtained by randomly choosing the matrices. In the following, we would like to show an example with $d = 3$.

Case I: the matrices a_{ij}, η_{mn} and $\bar{\eta}_{mn}$ are randomly generated by Mathematica,

$$\eta_A = \begin{pmatrix} -12.7085 & 24.1113 + 2.50006i & 12.752 - 7.64134i \\ 24.1113 - 2.50006i & -34.9796 & 16.159 + 12.3798i \\ 12.752 + 7.64134i & 16.159 - 12.3798i & 6.06277 \end{pmatrix}$$

$$\eta_{\bar{A}} = \begin{pmatrix} -18.2979 & -5.89479 - 25.5118i & 5.61273 + 21.1508i \\ -5.89479 + 25.5118i & -32.9428 & 12.504 - 10.931i \\ 5.61273 - 21.1508i & 12.504 + 10.931i & -25.1785 \end{pmatrix}$$

$$a = \begin{pmatrix} 17.5055 - 19.3962i & 8.29301 + 13.4073i & -13.3458 + 5.79992i \\ -2.34212 + 12.1545i & -19.0161 + 8.64625i & 17.1027 + 20.3801i \\ -5.46605 - 24.0534i & -0.924333 + 21.9112i & -19.6201 + 22.0798i \end{pmatrix}$$

The reference state $|\Psi\rangle = \frac{1}{\sqrt{3}}\sum_{k=1}^{3}|k\rangle_A|k\rangle_{\bar{A}}$. One could evaluate the reduced transition matrix $\mathcal{T}_A^a$ by using (3.4), it is given by

$$\mathcal{T}_A^a = \begin{pmatrix} 1.06475 + 0.82173i & -2.58944 - 1.89593i & -0.414598 + 1.47507i \\ 5.81552 + 1.36514i & -2.51542 - 3.45208i & 3.90801 + 1.03913i \\ 4.96095 + 0.0703348i & -5.61733 - 0.740827i & 2.45067 + 2.63035i \end{pmatrix}$$

It is obvious that $\mathcal{T}_A^a$ is non-Hermitian. The eigenvalues of it are

$$\lambda_1 = 0.7053 - 6.27836i, \ \lambda_2 = 0.7053 + 6.27836i, \ \lambda_3 = -0.410601. \tag{F.4}$$

The pseudo Rényi entropy may not be real in this case. e.g., $S^{(2)}(\mathcal{T}_A^a) = -4.3525 + 3.14159i$.

Case III: we take $x_1 = 0.932007$, $x_2 = 0.0679932$. η_{mn}, $\bar{\eta}_{mn}$ and a_{ij} are given by

$$\eta_A = \begin{pmatrix} 2241.0 & -1009.73 + 735.915i & 286.517 + 572.134i \\ -1009.73 - 735.915i & 1007.3 & 58.5703 - 617.441i \\ 286.517 - 572.134i & 58.5703 + 617.441i & 1399.02 \end{pmatrix}$$

$$\eta_{\bar{A}} = \begin{pmatrix} 967.287 & -307.565 - 126.497i & -129.349 + 149.126i \\ -307.565 + 126.497i & 1336.39 & 209.52 + 1520.81i \\ -129.349 - 149.126i & 209.52 - 1520.81i & 2269.12 \end{pmatrix}$$

$$a^1 = \begin{pmatrix} -12.9325 + 0.0289028i & -7.24499 + 4.48426i & -14.4313 + 15.8304i \\ -4.09857 - 26.938i & -14.2456 - 2.55161i & 10.7265 - 5.71364i \\ 4.30869 + 11.8775i & -19.1378 + 9.46391i & 1.32846 + 4.07899i \end{pmatrix}$$

$$a^2 = \begin{pmatrix} -9.48366 + 25.7059i & -6.14031 + 23.5242i & -13.0021 - 20.8661i \\ -3.87512 + 5.57805i & 4.9788 - 6.5475i & 1.21723 + 7.54634i \\ -10.6898 + 13.5806i & 11.563 - 1.35289i & -14.61 + 21.6139i \end{pmatrix}$$

The reference state $|\Psi\rangle = \frac{1}{\sqrt{3}}\sum_{k=1}^{3}|k\rangle_A|k\rangle_{\bar{A}}$. We have the reduced transition matrix $\mathcal{T}_A := x_1 T_A^{a^1} + x_2 T_A^{a^2}$

$$\mathcal{T}_A = \begin{pmatrix} 0.530706 - 0.0443678i & -0.249067 + 0.220004i & 0.16275 + 0.257015i \\ -0.0842444 - 0.152933i & 0.129589 + 0.0451508i & 0.0460981 - 0.13474i \\ 0.213683 - 0.402251i & 0.0403375 + 0.312084i & 0.339705 - 0.000782992i \end{pmatrix} \tag{F.5}$$

The eigenvalues are

$$\lambda_1 = 0.938253, \ \lambda_2 = 0.0533309, \ \lambda_3 = 0.00841637. \tag{F.6}$$

The pseudo Rényi entropy is real. The result is shown in figure 3.

F.2 Example: $S^{(n)}(\mathcal{T}_A) > 0$, η_A is not positive definite

Theorem 3 only gives a sufficient condition for $S^{(n)}(\mathcal{T}_A) > 0$. In this section, we would like to use a numerical example to show it is not a necessary condition. We will focus on
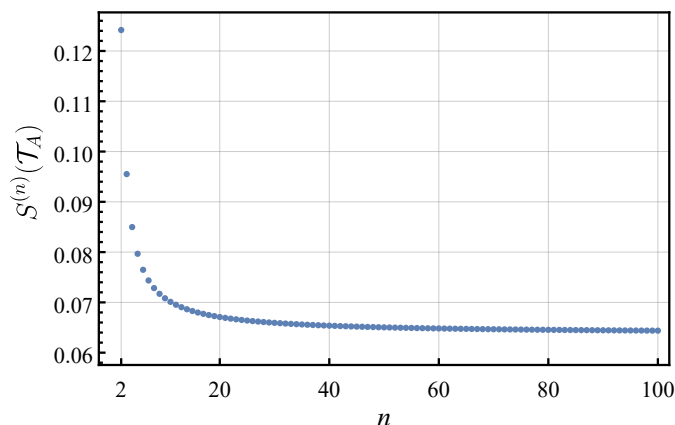
Figure 3. The plot of $S^{(n)}(\mathcal{T}_A)$.

a three-dimensional example. Choosing the matrices

$$
\eta_A = \begin{pmatrix} 13.9359 & -17.8554 + 8.22163i & 0.740751 - 0.860494i \\ -17.8554 - 8.22163i & 11.7561 & 3.87722 + 0.527719i \\ 0.740751 + 0.860494i & 3.87722 - 0.527719i & 4.32501 \end{pmatrix},
$$

$$
\eta_{\bar{A}} = \begin{pmatrix} 2.68826 + 0.i & -2.76297 + 6.09204i & -13.4254 - 5.89942i \\ -2.76297 - 6.09204i & 23.4288 + 0.i & 2.24652 - 1.6307i \\ -13.4254 + 5.89942i & 2.24652 + 1.6307i & 6.07729 + 0.i \end{pmatrix},
$$

$$
a = \begin{pmatrix} 2.79442 + 26.2305i & 14.4042 - 1.54735i & 1.27623 + 2.29185i \\ 17.0343 + 21.4595i & 6.13678 - 4.72818i & -6.82378 + 24.1677i \\ -6.55401 + 2.08772i & -6.0073 - 29.8274i & -7.59207 - 24.0165i \end{pmatrix}. \quad (F.7)
$$

The reference state $|\Psi\rangle = \frac{1}{\sqrt{3}} \sum_{k=1}^{3} |k\rangle_A |k\rangle_{\bar{A}}$. The eigenvalues of η_A and $\eta_{\bar{A}}$ are

$$
\eta_A \rightarrow \{32.6819, -7.87014, 5.20516\},
$$
$$
\eta_{\bar{A}} \rightarrow \{26.3549, 17.3493, -11.5099\}. \quad (F.8)
$$

Thus they are not positive operators. The eigenvalues of $\mathcal{T}_A$ are

$$
\lambda_1 = 0.849706, \quad \lambda_2 = 0.075147 - 0.106401i, \quad \lambda_3 = 0.075147 + 0.106401i. \quad (F.9)
$$

The pseudo Rényi entropy is positive in this example as shown in figure 4
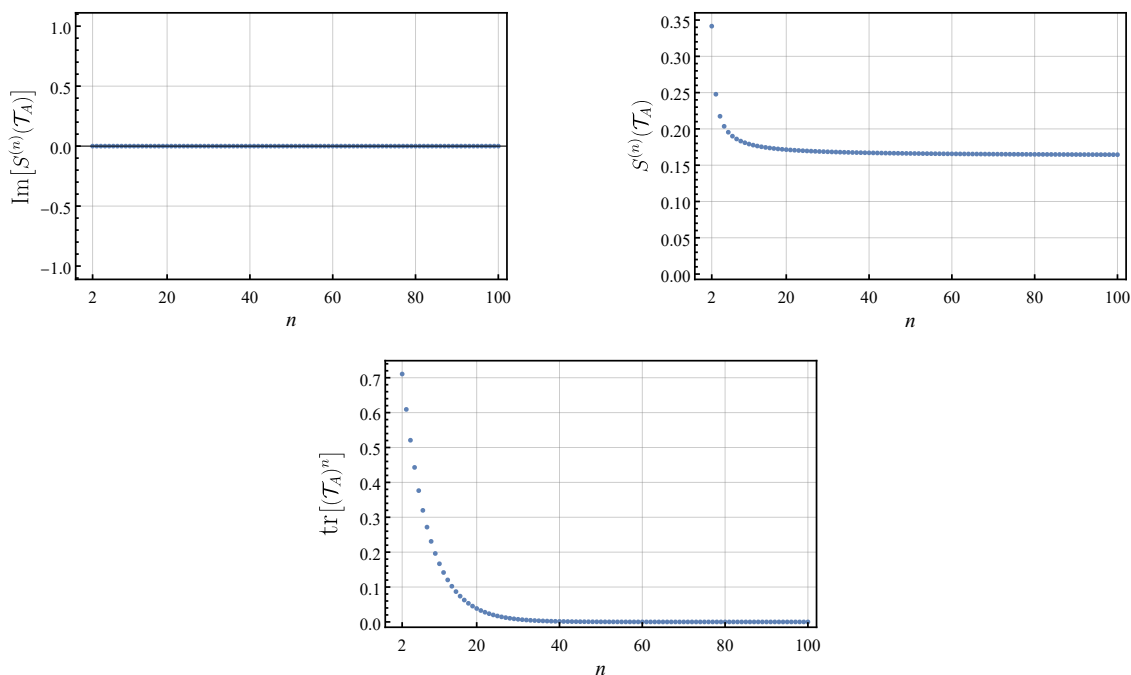
Figure 4. The plot of $S^{(n)}(\mathcal{T}_A)$ and $\text{tr}[(\mathcal{T}_A)^n]$. The upper left plot shows the imaginary part of $S^{(n)}(\mathcal{T}_A)$, which are vanishing. The upper right plot shows $S^{(n)}(\mathcal{T}_A)$. The lower plot shows $\text{tr}[(\mathcal{T}_A)^n]$, which are in the region $(0, 1)$.

F.3   Example: $S^{(n)}(\mathcal{T}_A) < 0$

In this section we show an example for which $S^{(n)}(\mathcal{T}_A) < 0$. Choosing the matrices

$$
\eta_A = \begin{pmatrix} 13.9359 & -17.8554 + 8.22163i & 0.740751 - 0.860494i \\ -17.8554 - 8.22163i & 11.7561 & 3.87722 + 0.527719i \\ 0.740751 + 0.860494i & 3.87722 - 0.527719i & 4.32501 \end{pmatrix},
$$

$$
\eta_{\bar{A}} = \begin{pmatrix} 2.68826 & -2.76297 + 6.09204i & -13.4254 - 5.89942i \\ -2.76297 - 6.09204i & 23.4288 & 2.24652 - 1.6307i \\ -13.4254 + 5.89942i & 2.24652 + 1.6307i & 6.07729 \end{pmatrix},
$$

$$
a = \begin{pmatrix} 2.79442 + 26.2305i & 14.4042 - 1.54735i & 1.27623 + 2.29185i \\ 17.0343 + 21.4595i & 6.13678 - 4.72818i & -6.82378 + 24.1677i \\ -6.55401 + 2.08772i & -6.0073 - 29.8274i & -7.59207 - 24.0165i \end{pmatrix}. \tag{F.10}
$$

The reference state $|\Psi\rangle = \frac{1}{\sqrt{3}} \sum_{k=1}^{3} |k\rangle_A |k\rangle_{\bar{A}}$. The eigenvalues of η_A, $\eta_{\bar{A}}$ and $\mathcal{T}_A$ are given by

$$
\begin{aligned}
\eta_A &\to \{85.7965, -45.7377, -0.637431\}, \\
\eta_{\bar{A}} &\to \{-51.9884, -40.48, 28.8633\}, \\
\mathcal{T}_A &\to \{1.37237, -0.368265, -0.00410468\}. \tag{F.11}
\end{aligned}
$$

The pseudo Rényi entropy is negative in this example. The result is shown in figure 5.
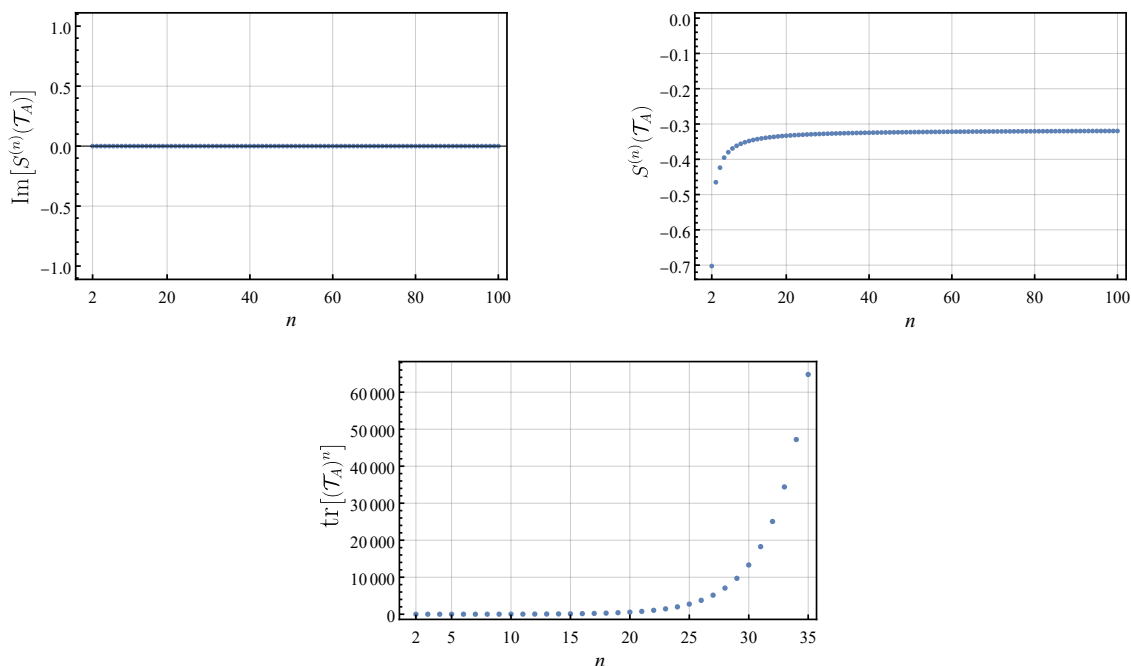
Figure 5. The plot of $S^{(n)}(\mathcal{T}_A)$ and $\mathrm{tr}[(\mathcal{T}_A)^n]$. The upper left plot shows the imaginary part of $S^{(n)}(\mathcal{T}_A)$, which are vanishing. The upper right plot shows $S^{(n)}(\mathcal{T}_A)$, which are negative. The lower plot shows $\mathrm{tr}[(\mathcal{T}_A)^n]$.

F.4 Example with $d = 2$

Let the reference state be $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_{\bar{A}} + |1\rangle_A|1\rangle_{\bar{A}})$. Let the operators η_A and $\eta_{\bar{A}}$ be diagonal, a be arbitrary. They are given by

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}, \quad \eta_{\bar{A}} = \begin{pmatrix} \bar{\eta}_1 & 0 \\ 0 & \bar{\eta}_2 \end{pmatrix}. \tag{F.12}$$

Assume η_A and $\eta_{\bar{A}}$ to be positive, thus $\eta_{1(2)} > 0$, $\bar{\eta}_{1(2)} > 0$. One could construct the transition matrix $\mathcal{T}^a$ with these operators. According to Theorem 3 we know the eigenvalues of $\mathcal{T}_A^a$ are positive. With some calculations, we have

$$\mathcal{T}_A^a = \begin{pmatrix} \frac{|a_{11}|^2\eta_1\bar{\eta}_1 + |a_{12}|^2\eta_1\bar{\eta}_2}{|a_{11}|^2\eta_1\bar{\eta}_1 + |a_{12}|^2\eta_1\bar{\eta}_2 + |a_{21}|^2\bar{\eta}_1\eta_2 + |a_{22}|^2\eta_2\bar{\eta}_2} & \frac{a_{11}a_{21}^*\eta_2\bar{\eta}_1 + a_{12}a_{22}^*\eta_2\bar{\eta}_2}{|a_{11}|^2\eta_1\bar{\eta}_1 + |a_{12}|^2\eta_1\bar{\eta}_2 + |a_{21}|^2\bar{\eta}_1\eta_2 + |a_{22}|^2\eta_2\bar{\eta}_2} \\ \frac{a_{11}^*a_{21}\eta_1\bar{\eta}_1 + a_{12}^*a_{22}\eta_1\bar{\eta}_2}{|a_{11}|^2\eta_1\bar{\eta}_1 + |a_{12}|^2\eta_1\bar{\eta}_2 + |a_{21}|^2\bar{\eta}_1\eta_2 + |a_{22}|^2\eta_2\bar{\eta}_2} & \frac{|a_{21}|^2\bar{\eta}_1\eta_2 + |a_{22}|^2\eta_2\bar{\eta}_2}{|a_{11}|^2\eta_1\bar{\eta}_1 + |a_{12}|^2\eta_1\bar{\eta}_2 + |a_{21}|^2\bar{\eta}_1\eta_2 + |a_{22}|^2\eta_2\bar{\eta}_2} \end{pmatrix}. \tag{F.13}$$

The pseudo Rényi entropy of the 2-qubit system is studied in [18]. They claim the eigenvalues of $\mathcal{T}_A^a$ are positive if and only if $0 \leq det[\mathcal{T}_A^a] \leq 1/4$. With some calculations, we have

$$\begin{aligned} \det[\mathcal{T}_A^a] &= \frac{|a_{12}a_{21} - a_{11}a_{22}|^2\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2}{(|a_{11}|^2\eta_1\bar{\eta}_1 + |a_{12}|^2\eta_1\bar{\eta}_2 + |a_{21}|^2\bar{\eta}_1\eta_2 + |a_{22}|^2\eta_2\bar{\eta}_2)^2} \\ &\leq \frac{|a_{12}a_{21} - a_{11}a_{22}|^2\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2}{(2|a_{11}||a_{22}|\sqrt{\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2} + 2|a_{12}||a_{21}|\sqrt{\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2})^2} \leq \frac{1}{4}. \end{aligned} \tag{F.14}$$

The above result can be generalized to arbitrary positive $\tilde{\eta}_A$ and $\tilde{\eta}_{\bar{A}}$. Since they are Hermitian operators, there exists unitary operator U_A and $U_{\bar{A}}$ such that

$$\tilde{\eta}_A = U_A \eta_A U_A^\dagger, \quad \tilde{\eta}_{\bar{A}} = U_{\bar{A}} \eta_{\bar{A}} U_{\bar{A}}^\dagger, \tag{F.15}$$

where η_A and $\eta_{\bar{A}}$ are digonal. The transition matrix $\mathcal{T}^a$ with a given reference state $|\Psi'\rangle$ is given by

$$\mathcal{T}^a \propto a|\Psi'\rangle\langle\Psi'|a^\dagger U_A \eta_A U_A^\dagger U_{\bar{A}} \eta_{\bar{A}} U_{\bar{A}}^\dagger. \tag{F.16}$$

Taking partial trace we have

$$\mathcal{T}_A^a = tr_{\bar{A}} \mathcal{T}^a \propto a(tr_{\bar{A}} U_{\bar{A}}^\dagger |\Psi'\rangle\langle\Psi'|U_{\bar{A}} \eta_{\bar{A}}) a^\dagger U_A \eta_A U_A^\dagger. \tag{F.17}$$

It is always possible to make the operator $tr_{\bar{A}} U_{\bar{A}}^\dagger |\Psi'\rangle\langle\Psi'|U_{\bar{A}} \eta_{\bar{A}} = tr_{\bar{A}}|\Psi\rangle\langle\Psi|\eta_{\bar{A}}$ by choosing suitable $|\Psi'\rangle$. With this choice one can show $det[\mathcal{T}_A^a]$ is equal to (F.14). Therefore, the transition matrix $\mathcal{T}_A^a$ having positive eigenvalues satisfies that $det[\mathcal{T}_A^a] \leq 1/4$, which is consistent with the result in [18].

G   Details of the example of free scalar with $\theta \neq 0$

In the main text we discuss the transition matrix (3.17), which can be written as the form

$$\mathcal{T}^{\psi|\phi} \propto |\Psi\rangle\langle\Psi|U, \tag{G.1}$$

where U is a unitary operator, it seems the above transition matrix is not like the general form (2.9) for the pure pseudo-Hermitian transition matrix. In this section, we will show the transition matrix (G.1) can be pseudo-Hermitian for some particular unitary operator U and pure state $|\Psi\rangle$.

To satisfy the pseudo Hermitian condition we should require

$$\eta|\Psi\rangle\langle\Psi|U\eta^{-1} = U^\dagger|\Psi\rangle\langle\Psi|. \tag{G.2}$$

Define the operator $\eta' := U\eta$. This condition is given by

$$\eta'|\Psi\rangle = |\Psi\rangle, \quad \langle\Psi|(\eta'^{-1})^\dagger = \langle\Psi|. \tag{G.3}$$

which can be transformed to the operator relation

$$\eta' = \eta'^{-1} + \alpha P_\perp^\Psi, \tag{G.4}$$

where α is some constant, $P_\perp^\Psi$ satisfies the condition $P_\perp^\Psi|\Psi\rangle = 0$. One special case is taking $\alpha = 0$. One would have the following relations:

$$(\eta')^2 = 1, \quad \eta' = (\eta')^{-1}. \tag{G.5}$$

One could check the above two qubits example satisfies the constraints (G.3) and (G.5). For the example of free scalar theory with $\theta \neq 0$, one could show the transition matrix is pseudo-Hermitian by proving the existence of the operator η' which satisfies the conditions (G.3), (G.4), or (G.5).

H Calculation of pseudo Rényi entropy by replica method

We outline the replica method in QFTs to compute the pseudo Rényi entropy in this appendix. In particular, we focus on 2D CFTs as the correlation functions in the replica manifold are easy to obtain by conformal mapping. Let's consider a 2D CFT with Lagrangian $\mathcal{L}(\phi, \partial\phi)$ dwells on a Euclidean plane Σ_1 ($ds^2 = dw d\bar{w}$, $(w, \bar{w}) = (x + i\tau, x - i\tau)$) and a transition matrix generated by a local operator $\mathcal{O}(w, \bar{w}) \equiv e^{\tau H} \mathcal{O}(x, 0) e^{-\tau H}$,

$$\mathcal{T}_E^{\mathcal{O}} = \frac{\mathcal{O}(w_1, \bar{w}_1)|0\rangle\langle 0|\mathcal{O}^\dagger(w_2, \bar{w}_2)}{\langle 0|\mathcal{O}^\dagger(w_2, \bar{w}_2)\mathcal{O}(w_1, \bar{w}_1)|0\rangle}, \tag{H.1}$$

where $w_1 = x_1 - i\tau_1$ and $w_2 = x_2 + i\tau_2$, $(\tau_1, \tau_2 > 0)$. The reduced transition matrix of a subsystem A, $\mathcal{T}_{E,A}^{\mathcal{O}} := \mathrm{tr}_{\bar{A}}\mathcal{T}_E^{\mathcal{O}}$, can be expressed by path integral with operators inserted at $(w_1, \bar{w}_1)$ and $(w_2, \bar{w}_2)$ on the w-plane with a cut on A

$$\langle\phi_{A_-}|\mathcal{T}_{E,A}^{\mathcal{O}}|\phi_{A_+}\rangle = \frac{\int_{\phi(x\in A, \tau=0_-)=\phi_{A_-}(x)}^{\phi(x\in A, \tau=0_+)=\phi_{A_+}(x)} [d\phi]\mathcal{O}^\dagger(w_2, \bar{w}_2)\mathcal{O}(w_1, \bar{w}_1)\exp\left\{-\int_{\mathbb{R}^2}\mathcal{L}(\phi, \partial\phi)\right\}}{\int [d\phi]\mathcal{O}^\dagger(w_2, \bar{w}_2)\mathcal{O}(w_1, \bar{w}_1)\exp\left\{-\int_{\mathbb{R}^2}\mathcal{L}(\phi, \partial\phi)\right\}}$$

(H.2)

Building on (H.2), $\mathrm{tr}[(\mathcal{T}_{E,A}^{\mathcal{O}})^n]$ is given by a $2n$-point correlation function on a n-sheet Riemann surface Σ_n,

$$\mathrm{tr}[(\mathcal{T}_{E,A}^{\mathcal{O}})^n] = \frac{\mathcal{Z}_n}{\mathcal{Z}_1^n} \cdot \frac{\langle\mathcal{O}(w_1, \bar{w}_1)\mathcal{O}^\dagger(w_2, \bar{w}_2)\ldots\mathcal{O}(w_{2n-1}, \bar{w}_{2n-1})\mathcal{O}^\dagger(w_{2n}, \bar{w}_{2n})\rangle_{\Sigma_n}}{\langle\mathcal{O}^\dagger(w_2, \bar{w}_2)\mathcal{O}(w_1, \bar{w}_1)\rangle_{\Sigma_1}^n} \tag{H.3}$$

where $\mathcal{Z}_1$ and $\mathcal{Z}_n$ are partition functions on Σ_1 and Σ_n, respectively, and $\mathcal{O}(w_{2k-1}, \bar{w}_{2k-1})$ and $\mathcal{O}^\dagger(w_{2k}, \bar{w}_{2k})$ denote the operators inserted at kth sheet. The nth pseudo Rényi entropy of $\mathcal{T}_{E,A}^{\mathcal{O}}$ turns out to be

$$S^{(n)}(\mathcal{T}_{E,A}^{\mathcal{O}}) = S_{A;vac}^{(n)} + \Delta S^{(n)}(\mathcal{T}_{E,A}^{\mathcal{O}}), \tag{H.4}$$

where $S_{A;vac}^{(n)} \equiv \frac{1}{1-n} \log \frac{\mathcal{Z}_n}{\mathcal{Z}_1^n}$ is the nth Rényi entropy of A when the total system is in the vacuum, and $\Delta S^{(n)}(\mathcal{T}_{E,A}^{\mathcal{O}})$ we refer to as the excess of nth pseudo Rényi entropy of A,

$$\Delta S^{(n)}(\mathcal{T}_{E,A}^{\mathcal{O}}) = \frac{1}{1-n} \log \frac{\langle \mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^{\dagger}(w_2,\bar{w}_2)\dots\mathcal{O}(w_{2n-1},\bar{w}_{2n-1})\mathcal{O}^{\dagger}(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n}}{\langle \mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^{\dagger}(w_2,\bar{w}_2)\rangle_{\Sigma_1}^n}. \tag{H.5}$$

For our purposes, we only focus on the 2nd pseudo Rényi entropy,

$$\Delta S^{(2)}(\mathcal{T}_{E,A}^{\mathcal{O}}) = -\log \frac{\langle \mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^{\dagger}(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^{\dagger}(w_4,\bar{w}_4)\rangle_{\Sigma_2}}{\langle \mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^{\dagger}(w_2,\bar{w}_2)\rangle_{\Sigma_1}^2}. \tag{H.6}$$

Meanwhile, $\mathcal{O}$ is assumed to be a primary with chiral and anti-chiral conformal dimension $\Delta_{\mathcal{O}}$. By conformal symmetry, the 2- and 4-point function of $\mathcal{O}$ on Σ_1 can be expressed as

$$\langle \mathcal{O}(z_1,\bar{z}_1)\mathcal{O}^{\dagger}(z_2,\bar{z}_2)\rangle_{\Sigma_1} = \frac{c_{12}}{|z_{12}|^{4\Delta_{\mathcal{O}}}}, \tag{H.7}$$

$$\langle \mathcal{O}(z_1,\bar{z}_1)\mathcal{O}^{\dagger}(z_2,\bar{z}_2)\mathcal{O}(z_3,\bar{z}_3)\mathcal{O}^{\dagger}(z_4,\bar{z}_4)\rangle_{\Sigma_1} = |z_{13}z_{24}|^{-4\Delta_{\mathcal{O}}}G(\eta,\bar{\eta}), \tag{H.8}$$

respectively, where $\eta := \frac{z_{12}z_{34}}{z_{13}z_{24}}$ and $\bar{\eta} := \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}$ are the cross ratios. Since there are conformal mappings

$$z = \begin{cases} w^{1/n}, & A = [0,\infty), \\ \left(\frac{w+L}{w-L}\right)^{1/n}, & A = [-L,L], \end{cases} \tag{H.9}$$

from Σ_n to Σ_1, the 4-point function on Σ_2 is obtained by applying the above conformal mappings with $n=2$

$$\langle \mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^{\dagger}(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^{\dagger}(w_4,\bar{w}_4)\rangle_{\Sigma_2} = \begin{cases} \left|\frac{64L^2 z_1^2 z_2^2}{(z_1^2-1)^2(z_2^2-1)^2}\right|^{-4\Delta_{\mathcal{O}}}G(\eta,\bar{\eta}), & A=[-L,L], \\ \\ |16z_1^2 z_2^2|^{-4\Delta_{\mathcal{O}}}G(\eta,\bar{\eta}), & A=[0,+\infty). \end{cases} \tag{H.10}$$

Substituting (H.7) and (H.10) into (H.6) and after some algebra, we obtain

$$\Delta S^{(2)}(\mathcal{T}_{E,A}^{\mathcal{O}}) = \log \frac{c_{12}^2}{|\eta(1-\eta)|^{4\Delta_{\mathcal{O}}}G(\eta,\bar{\eta})}, \tag{H.11}$$

which only depends on the cross ratios η and $\bar{\eta}$. The 2nd pseudo Rényi entropy with regard to the real-time dependent transition matrix can be obtained by applying the analytic continuation to τ_1 and τ_2 in the above result. When $\tau_1 \to \epsilon + it$ and $\tau_2 \to \epsilon - it$, we meet the case studied in [27]. As we mentioned in the previous section, we would like to focus on the case of $\mathcal{T}^{\mathcal{O}} = \frac{\mathcal{O}(x,t)|0\rangle\langle 0|\mathcal{O}(-x,-t)}{\langle 0|\mathcal{O}(-x,-t)\mathcal{O}(x,t)|0\rangle}$. Thus we have the analytic continuation $\tau_1 = \tau_2 \to \epsilon - it$. An infinitesimally small regularization parameter ϵ is introduced to suppress the high energy modes [55].

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