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The $\mathcal{N}=2$ supersymmetric $w_{1+\infty}$ symmetry in the two-dimensional SYK models

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Abstract: We identify the rank $(q_{\text{syk}}+1)$ of the interaction of the two-dimensional $\mathcal{N}=(2,2)$ SYK model with the deformation parameter λ in the Bergshoeff, de Wit and Vasiliev(in 1991)'s linear $W_\infty[\lambda]$ algebra via $\lambda=\frac{1}{2(q_{\text{syk}}+1)}$ by using a matrix generalization. At the vanishing λ (or the infinity limit of q_{syk}), the $\mathcal{N}=2$ supersymmetric linear $W_\infty^{N,N}[\lambda=0]$ algebra contains the matrix version of known $\mathcal{N}=2$ W_∞ algebra, as a subalgebra, by realizing that the N-chiral multiplets and the N-Fermi multiplets in the above SYK models play the role of the same number of $\beta\,\gamma$ and $b\,c$ ghost systems in the linear $W_\infty^{N,N}[\lambda=0]$ algebra. For the nonzero λ, we determine the complete $\mathcal{N}=2$ supersymmetric linear $W_\infty^{N,N}[\lambda]$ algebra where the structure constants are given by the linear combinations of two different generalized hypergeometric functions having the λ dependence. The weight-1, $\frac{1}{2}$ currents occur in the right hand sides of this algebra and their structure constants have the λ factors. We also describe the $\lambda=\frac{1}{4}$ (or $q_{\text{syk}}=1$) case in the truncated subalgebras by calculating the vanishing structure constants.

Keywords: Conformal and W Symmetry, Higher Spin Symmetry

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1 Introduction

The celestial holography [1] connects the gravitational scattering in asymptotically flat spacetimes with the conformal field theory which lives on the celestial sphere. By using the low energy scattering problems, the symmetry algebra of the conformal field theory for flat space has been found in [2]. Furthermore, in [3], the group of symmetries on the celestial sphere plays the role of the wedge subalgebra of $w_{1+\infty}$ algebra [4]. We should understand the unknown structures behind these findings in order to convince the above duality. In [5, 6], the supersymmetric $w_{1+\infty}$ algebra has been identified with the corresponding soft current algebra in the supersymmetric Einstein-Yang-Mills theory. Recently, in [7], the holographic map from two-dimensional SYK models to the conformally soft sector of gravity in four-dimensional asymptotically flat spacetimes is studied. One of the motivations in this paper is to consider other types of SYK models and to check whether we have similar $w_{1+\infty}$ symmetry or not. See the review papers [1, 8–10] on the celestial holography.[1]

In $\mathcal{N}=(2,2)$ SYK models [19–21], the two U(1) symmetries of the $\mathcal{N}=(0,2)$ SYK models can be combined with U(1) R symmetry and the chiral and Fermi multiplets are also combined into $\mathcal{N}=(2,2)$ chiral multiplet. This implies that their charges are related to each other. It turns out that the stress energy tensor takes simple form and the coefficients of the stress energy tensor are related to the rank of the interaction of the SYK models. The standard $\mathcal{N}=2$ superconformal algebra is realized by the chiral multiplets and Fermi multiplets in quadratic form with various powers of (antiholomorphic) derivatives.

In the $\mathcal{N}=2$ supersymmetric linear $W_{\infty}[\lambda]$ algebra [22, 23], by so-called $\beta\,\gamma$ and $b\,c$ ghost systems, the higher spin currents with one-parameter are determined by the quadratic forms of these bosonic and fermionic operators.[2] In this case, the standard $\mathcal{N}=2$ superconformal algebra can be written in terms of the currents of low weights. Moreover, the so-called $\mathcal{N}=2$ scalar multiplet can be described by the lowest bosonic

[1] The relevant works on the celestial holography in the connection with the $w_{1+\infty}$ symmetry can be found in [11–18].
[2] This is not the asymptotic symmetry algebra introduced in [24].

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and fermionic currents. As a subalgebra, the bosonic algebra contains $W_\infty[\lambda]$ algebra and $W_\infty[\lambda+\frac{1}{2}]$ algebra. They claim that the $\mathcal{N}=2$ supersymmetric linear $W_\infty[\lambda]$ algebra is isomorphic to the $\mathcal{N}=2$ supersymmetric linear $W_\infty[\frac{1}{2}-\lambda]$ algebra because there exist some transformations between the above $\beta\,\gamma$ and $b\,c$ ghost systems by introducing two real anticommuting parameters.[3]

In this paper, by realizing that the above two models have their own one parameter, i) the rank $(q_{\rm syk}+1)$ of the interaction of the SYK models, and ii) the λ parameter and the fundamental building blocks are characterized by chiral and Fermi multiplets on the one hand and by $\beta\,\gamma$ and $b\,c$ ghost systems on the other hand, we would like to study the precise relation between the $\mathcal{N}=(2,2)$ SYK models and the $\mathcal{N}=2$ $\beta\,\gamma$ and $b\,c$ ghost systems.

> At first, we make a generalization of [22, 23] by introducing the multiple $\beta\,\gamma$ and $b\,c$ ghost systems. Then we can compare with each stress energy tensor (or the generators of $\mathcal{N}=2$ superconformal algebra) described above. This will provide the exact correspondence between the two parameters mentioned before. At $\lambda=0$, we identify the free field realization in [6] with the ones from $\beta\,\gamma$ and $b\,c$ ghost systems. This implies that the realization of $\mathcal{N}=2$ supersymmetric linear $W_\infty^{N,N}[\lambda=0]$ algebra is described by the above $\mathcal{N}=(2,2)$ SYK models together with the infinity limit of the rank of the interaction.
>
> At nonzero λ, by using the higher spin currents of the matrix generalized $\beta\,\gamma$ and $b\,c$ ghost systems [22, 23], we determine the complete $\mathcal{N}=2$ supersymmetric linear $W_\infty^{N,N}[\lambda]$ algebra in terms of various (anti)commutator relations. The structure constants originate from the oscillator construction in the AdS_3 Vasiliev higher spin theory [25]. At $\lambda=\frac{1}{4}$ (corresponding to the rank $(q_{\rm syk}+1)=2$ of the interaction of the $\mathcal{N}=(2,2)$ SYK models), we show how the truncated subalgebra arises by calculating the vanishing structure constants. Finally, we also describe the relation with celestial holography briefly.

In section 2, we review both $\mathcal{N}=(2,2)$ SYK models and the $\mathcal{N}=2$ supersymmetric linear $W_\infty[\lambda]$ algebra.

In section 3, by matrix generalization of the $\mathcal{N}=2$ supersymmetric linear $W_\infty[\lambda]$ algebra, the realization of $\mathcal{N}=2$ supersymmetric linear $W_\infty^{N,N}[\lambda=0]$ algebra in the $\mathcal{N}=(2,2)$ SYK models is described. For nonzero λ, starting from the λ dependent higher spin currents, we construct the (anti)commutator relations by checking the structure constants explicitly. The realization of $\mathcal{N}=2$ supersymmetric linear $W_\infty^{N,N}[\lambda=\frac{1}{4}]$ algebra in the $\mathcal{N}=(2,2)$ SYK models is studied. The relation with celestial holography is obtained.

In section 4, we summarize what we have obtained in this paper and further directions are also described.

In appendices, some detailed calculations in section 3 are explained.

We are using the Thielemans package [26] with a mathematica [27].

[3]The notations for the subscript ∞ will be clearer when we discuss about the algebra itself in section 3.

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2 Review

2.1 Two-dimensional $\mathcal{N} = (2,2)$ SYK models

In the two-dimensional SYK model [19], there are N chiral multiplets $\Phi^a(a = 1, 2, \cdots, N)$ and M Fermi multiplets $\Lambda^i(i = 1, 2, \cdots, M)$ with a random coupling. This random coupling of the interaction of the SYK model has a rank of $(q_{\text{syk}}+1)$. The model with $N = M$ has an enhanced $\mathcal{N} = (2,2)$ supersymmetry and reduces to the one studied in [19, 20].[4] The lowest components of these superfields satisfy the following operator product expansions (OPEs)

$$\bar{\partial}\,\bar{\phi}^a(\bar{z})\,\phi^b(\bar{w}) = \frac{\delta^{ab}}{(\bar{z}-\bar{w})} + \cdots, \qquad \frac{i}{\sqrt{2}}\,\bar{\lambda}^a(\bar{z})\,\frac{i}{\sqrt{2}}\,\lambda^b(\bar{w}) = \frac{\delta^{ab}}{(\bar{z}-\bar{w})} + \cdots. \tag{2.1}$$

For the fermions in the second equation of (2.1), the proper normalization is performed, compared to the one in [21]. The conformal weights for ϕ^a, $\bar{\partial}\,\bar{\phi}^a$, λ^a and $\bar{\lambda}^a$ in the anti-holomorphic sector are given by $\frac{1}{2(q_{\text{syk}}+1)}$, $1 - \frac{1}{2(q_{\text{syk}}+1)}$, $\frac{1}{2} + \frac{1}{2(q_{\text{syk}}+1)}$ and $\frac{1}{2} - \frac{1}{2(q_{\text{syk}}+1)}$.

The lowest supermultiplet contains the weight-1 operator, two supercharges and the stress energy tensor. Then the standard $\mathcal{N} = 2$ superconformal algebra is realized by [21, 33]

$$\begin{aligned}
J &= \frac{q_{\text{syk}}}{2(q_{\text{syk}}+1)}\bar{\lambda}^a\,\lambda^a - \frac{1}{(q_{\text{syk}}+1)}\phi^a\,\bar{\partial}\,\bar{\phi}^a, \\
G^+ &= \frac{1}{\sqrt{2}}\,\bar{\partial}\,\bar{\phi}^a\,\lambda^a, \\
G^- &= -\frac{q_{\text{syk}}}{\sqrt{2}\,(1+q_{\text{syk}})}\left[\bar{\partial}\,\phi^a\,\bar{\lambda}^a - \frac{1}{q_{\text{syk}}}\,\phi^a\,\bar{\partial}\,\bar{\lambda}^a\right], \\
T &= \frac{1}{(2q_{\text{syk}}+2)}\left[(2q_{\text{syk}}+1)\,\bar{\partial}\,\phi^a\,\bar{\partial}\,\bar{\phi}^a - \phi^a\,\bar{\partial}^2\,\bar{\phi}^a - \frac{(q_{\text{syk}}+2)}{2}\,\bar{\partial}\,\bar{\lambda}^a\,\lambda^a + \frac{q_{\text{syk}}}{2}\,\bar{\lambda}^a\,\bar{\partial}\,\lambda^a\right].
\end{aligned} \tag{2.2}$$

The central charge in (2.2), where the fourth order pole in the OPE $T(\bar{z})\,T(\bar{w})$ is $\frac{c}{2}$, is given by

$$c = 3N\,\frac{(q_{\text{syk}}-1)}{(q_{\text{syk}}+1)}. \tag{2.3}$$

Each independent term in the stress energy tensor contributes to its own central term and the overall factor N appears in (2.3). Some typo in [21] is corrected in (2.2). Note that we can multiply any (pure imaginary) numerical number into the G^+ and its inverse into the G^- without changing the definition of the $\mathcal{N} = 2$ superconformal algebra. The central terms of the OPEs, $J(\bar{z})\,J(\bar{w})$ and $G^+(\bar{z})\,G^-(\bar{w})$, are given by $\frac{c}{3}$ and $\frac{c}{3}$ respectively.[5]

[4] From now on, we use the terminology of $\mathcal{N} = 2$ rather than $\mathcal{N} = (2,2)$ for simplicity. See also relevant works in [28–32]. More literatures can be found in [21].

[5] Note that in the $\mathcal{N} = (0,2)$ SYK model, the stress energy tensor takes the more general form [21] and the condition for the $\mathcal{N} = 2$ supersymmetry enables us to have simpler form for the stress energy tensor.

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2.2 The $\mathcal{N}=2$ supersymmetric linear $W_\infty[\lambda]$ algebra

In [22, 23], the $\mathcal{N}=2$ supersymmetric linear $W_\infty[\lambda]$ algebra[6] is realized by the following $\beta\,\gamma$ and $b\,c$ ghost systems which satisfy the OPEs

$$\gamma(\bar{z})\,\beta(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})} + \cdots\,, \qquad c(\bar{z})\,b(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})} + \cdots\,. \tag{2.4}$$

The conformal weights for β, γ, b and c are given by λ, $1-\lambda$, $\frac{1}{2}+\lambda$ and $\frac{1}{2}-\lambda$ respectively.[7] Note that the normalizations in the right hand sides of (2.4) are given by $+1$.

Then the higher spin currents are given by [22, 23][8]

$$\begin{aligned}
V_\lambda^{(s)+} &= \sum_{i=0}^{s-1} a^i(s,\lambda)\,\bar{\partial}^{s-1-i}\,((\bar{\partial}^i\,\beta)\,\gamma) + \sum_{i=0}^{s-1} a^i(s,\lambda+\frac{1}{2})\,\bar{\partial}^{s-1-i}\,((\bar{\partial}^i\,b)\,c),\\
V_\lambda^{(s)-} &= -\frac{(s-1+2\lambda)}{(2s-1)}\sum_{i=0}^{s-1} a^i(s,\lambda)\,\bar{\partial}^{s-1-i}\,((\bar{\partial}^i\,\beta)\,\gamma)\\
&\quad + \frac{(s-2\lambda)}{(2s-1)}\sum_{i=0}^{s-1} a^i(s,\lambda+\frac{1}{2})\,\bar{\partial}^{s-1-i}\,((\bar{\partial}^i\,b)\,c),\\
Q_\lambda^{(s)+} &= \sum_{i=0}^{s-1} \alpha^i(s,\lambda)\,\bar{\partial}^{s-1-i}\,((\bar{\partial}^i\,\beta)\,c) - \sum_{i=0}^{s-2} \beta^i(s,\lambda)\,\bar{\partial}^{s-2-i}\,((\bar{\partial}^i\,b)\,\gamma),\\
Q_\lambda^{(s)-} &= \sum_{i=0}^{s-1} \alpha^i(s,\lambda)\,\bar{\partial}^{s-1-i}\,((\bar{\partial}^i\,\beta)\,c) + \sum_{i=0}^{s-2} \beta^i(s,\lambda)\,\bar{\partial}^{s-2-i}\,((\bar{\partial}^i\,b)\,\gamma).
\end{aligned} \tag{2.5}$$

The λ-dependent coefficients appearing in (2.5) are given by

$$\begin{aligned}
a^i(s,\lambda) &\equiv \binom{s-1}{i}\,\frac{(-2\lambda-s+2)_{s-1-i}}{(s+i)_{s-1-i}}, & 0\leq i\leq (s-1),\\
\alpha^i(s,\lambda) &\equiv \binom{s-1}{i}\,\frac{(-2\lambda-s+2)_{s-1-i}}{(s+i-1)_{s-1-i}}, & 0\leq i\leq (s-1),\\
\beta^i(s,\lambda) &\equiv \binom{s-2}{i}\,\frac{(-2\lambda-s+2)_{s-2-i}}{(s+i)_{s-2-i}}, & 0\leq i\leq (s-2).
\end{aligned} \tag{2.6}$$

The λ-dependent coefficients (2.6) are not independent. Some properties of these coefficients are given by appendix A of [22]. The binomial coefficients for parentheses are used and the rising Pochhammer symbol $(a)_n \equiv a(a+1)\cdots(a+n-1)$ is also used here. We

[6] In this paper, we are considering the linear algebra where the corresponding OPE does not have any quadratic or higher order terms in the currents of the right hand side although the currents are quadratic in the operators. See also the relevant work in [34] where the nonlinear structures occur in the context of AdS_3 higher spin theory.

[7] In terms of the parameter λ_{hs} of the higher spin algebra $hs[\lambda_{hs}]$, there exists a relation $\lambda = \frac{1}{2}\,\lambda_{hs}$.

[8] We thank M. Vasiliev for pointing out that these are quasiprimary operators under the stress energy tensor $V_\lambda^{(2)+}$ ten years ago.

can check that the $\mathcal{N}=2$ superconformal generators are given by[9]

$$\begin{aligned}
J &= V_\lambda^{(1)-}, \\
G^+ &= -\frac{\sqrt{2}}{2}\,(Q_\lambda^{(2)+}-Q_\lambda^{(2)-}), \\
G^- &= \frac{\sqrt{2}}{4}\,(Q_\lambda^{(2)+}+Q_\lambda^{(2)-}), \\
T &= V_\lambda^{(2)+}.
\end{aligned} \tag{2.7}$$

The lowest s value for the bosonic currents $V_\lambda^{(s)\pm}$ is given by $s=1$. One of them plays the role of the weight-1 current of the $\mathcal{N}=2$ superconformal algebra in (2.7). The lowest s value for the fermionic currents $Q_\lambda^{(s)\pm}$ is given by $s=1$ also. In [22, 23], the $\mathcal{N}=2$ scalar multiplet is denoted by $(Q_\lambda^{(1)+}=Q_\lambda^{(1)-}, V_\lambda^{(1)+})$. We can easily see that the weights for the composite operators $\beta\,\gamma$, $b\,c$, $\beta\,c$ and $b\,\gamma$ are given by 1, 1, $\frac{1}{2}$ and $\frac{3}{2}$ respectively and all the λ dependence is gone. This means that their weights for the bosonic currents $V_\lambda^{(s)\pm}$ are given by s while the weights for the fermionic currents $Q_\lambda^{(s)\pm}$ are given by $(s-\frac{1}{2})$.[10]

3 The $\mathcal{N}=2$ supersymmetric linear $W_\infty^{N,N}[\lambda]$ algebra

3.1 The matrix generalization of $\mathcal{N}=2$ supersymmetric linear $W_\infty[\lambda]$ algebra

In order to describe the multiple number of the chiral multiplets (or the Fermi multiplets), we need to introduce the multiple number of $\beta\,\gamma$ and $b\,c$ systems [36] satisfying the following defining OPEs

$$\gamma^{i,\bar{a}}(\bar{z})\,\beta^{\bar{j},b}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})}\,\delta^{i\bar{j}}\,\delta^{\bar{a}b}+\cdots\,, \qquad c^{i,\bar{a}}(\bar{z})\,b^{\bar{j},b}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})}\,\delta^{i\bar{j}}\,\delta^{\bar{a}b}+\cdots\,. \tag{3.1}$$

The fundamental indices $a,b,\cdots$ of SU(N) in (3.1) runs over $a,b,\cdots=1,2,\cdots,N$ and the antifundamental indices $\bar{a},\bar{b},\cdots$ of SU(N) runs over $\bar{a},\bar{b},\cdots=1,2,\cdots,N$. Similarly, we can associate the indices $i,j,\cdots$ and the indices $\bar{i},\bar{j},\cdots$ with the corresponding fundamental and antifundamentals of SU(L).[11]

[9] In next section we will present their explicit forms in terms of the composite operators in the ghost systems. When we take $N=1$ over there, then we obtain the exact results [22, 23].

[10] In terms of the ghost systems, we have $V_\lambda^{(1)+}=\beta\,\gamma+b\,c$ and $Q_\lambda^{(1)+}=Q_\lambda^{(1)-}=\beta c$. In order to calculate some OPEs between the ghost systems and the currents in (2.5), some partial results on the highest order poles between them in [22, 35] are helpful. For example, for the calculation of the various central charges in the given OPEs, we have to consider the highest order poles only.

[11] We consider SU(L)-singlet currents in this paper.

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By multiplying the generators of SU(N) into the previous relations (2.5), we obtain the following matrix generalization of the work in [22, 23]

$$
\begin{aligned}
V^{(s)+}_{\lambda} &= \sum_{i=0}^{s-1} a^i(s,\lambda)\bar{\partial}^{s-1-i}((\bar{\partial}^i\beta^{\bar{l}b})\delta_{l\bar{l}}\delta_{\bar{a}b}\gamma^{l\bar{a}}) + \sum_{i=0}^{s-1} a^i\left(s,\lambda+\frac{1}{2}\right)\bar{\partial}^{s-1-i}((\bar{\partial}^i b^{\bar{l}b})\delta_{l\bar{l}}\delta_{\bar{a}b}c^{l\bar{a}}),\\
V^{(s)+}_{\lambda,\hat{A}} &= \sum_{i=0}^{s-1} a^i(s,\lambda)\bar{\partial}^{s-1-i}((\bar{\partial}^i\beta^{\bar{l}b})\delta_{l\bar{l}}t^{\hat{A}}_{b\bar{a}}\gamma^{l\bar{a}}) + \sum_{i=0}^{s-1} a^i\left(s,\lambda+\frac{1}{2}\right)\bar{\partial}^{s-1-i}((\bar{\partial}^i b^{\bar{l}b})\delta_{l\bar{l}}t^{\hat{A}}_{b\bar{a}}c^{l\bar{a}}),\\
V^{(s)-}_{\lambda} &= -\frac{(s-1+2\lambda)}{(2s-1)}\sum_{i=0}^{s-1} a^i(s,\lambda)\bar{\partial}^{s-1-i}((\bar{\partial}^i\beta^{\bar{l}b})\delta_{l\bar{l}}\delta_{\bar{a}b}\gamma^{l\bar{a}})\\
&\quad+\frac{(s-2\lambda)}{(2s-1)}\sum_{i=0}^{s-1} a^i\left(s,\lambda+\frac{1}{2}\right)\bar{\partial}^{s-1-i}((\bar{\partial}^i b^{\bar{l}b})\delta_{l\bar{l}}\delta_{\bar{a}b}c^{l\bar{a}}),\\
V^{(s)-}_{\lambda,\hat{A}} &= -\frac{(s-1+2\lambda)}{(2s-1)}\sum_{i=0}^{s-1} a^i(s,\lambda)\bar{\partial}^{s-1-i}((\bar{\partial}^i\beta^{\bar{l}b})\delta_{l\bar{l}}t^{\hat{A}}_{b\bar{a}}\gamma^{l\bar{a}})\\
&\quad+\frac{(s-2\lambda)}{(2s-1)}\sum_{i=0}^{s-1} a^i\left(s,\lambda+\frac{1}{2}\right)\bar{\partial}^{s-1-i}((\bar{\partial}^i b^{\bar{l}b})\delta_{l\bar{l}}t^{\hat{A}}_{b\bar{a}}c^{l\bar{a}}),\\
Q^{(s)+}_{\lambda} &= \sum_{i=0}^{s-1} \alpha^i(s,\lambda)\bar{\partial}^{s-1-i}((\bar{\partial}^i\beta^{\bar{l}b})\delta_{l\bar{l}}\delta_{\bar{a}b}c^{l\bar{a}}) - \sum_{i=0}^{s-2} \beta^i(s,\lambda)\bar{\partial}^{s-2-i}((\bar{\partial}^i b^{\bar{l}b})\delta_{l\bar{l}}\delta_{\bar{a}b}\gamma^{l\bar{a}}),\\
Q^{(s)+}_{\lambda,\hat{A}} &= \sum_{i=0}^{s-1} \alpha^i(s,\lambda)\bar{\partial}^{s-1-i}((\bar{\partial}^i\beta^{\bar{l}b})\delta_{l\bar{l}}t^{\hat{A}}_{b\bar{a}}c^{l\bar{a}}) - \sum_{i=0}^{s-2} \beta^i(s,\lambda)\bar{\partial}^{s-2-i}((\bar{\partial}^i b^{\bar{l}b})\delta_{l\bar{l}}t^{\hat{A}}_{b\bar{a}}\gamma^{l\bar{a}}),\\
Q^{(s)-}_{\lambda} &= \sum_{i=0}^{s-1} \alpha^i(s,\lambda)\bar{\partial}^{s-1-i}((\bar{\partial}^i\beta^{\bar{l}b})\delta_{l\bar{l}}\delta_{\bar{a}b}c^{l\bar{a}}) + \sum_{i=0}^{s-2} \beta^i(s,\lambda)\bar{\partial}^{s-2-i}((\bar{\partial}^i b^{\bar{l}b})\delta_{l\bar{l}}\delta_{\bar{a}b}\gamma^{l\bar{a}}),\\
Q^{(s)-}_{\lambda,\hat{A}} &= \sum_{i=0}^{s-1} \alpha^i(s,\lambda)\bar{\partial}^{s-1-i}((\bar{\partial}^i\beta^{\bar{l}b})\delta_{l\bar{l}}t^{\hat{A}}_{b\bar{a}}c^{l\bar{a}}) + \sum_{i=0}^{s-2} \beta^i(s,\lambda)\bar{\partial}^{s-2-i}((\bar{\partial}^i b^{\bar{l}b})\delta_{l\bar{l}}t^{\hat{A}}_{b\bar{a}}\gamma^{l\bar{a}}).
\end{aligned}
\tag{3.2}
$$

The adjoint index $\hat{A}$ runs over $\hat{A} = 1, 2, \cdots, (N^2-1)$.[12]

The central charge of the stress energy tensor is given by[13]

$$
c = 3\,N\,(1-4\lambda). \tag{3.3}
$$

By comparing (2.3) with (3.3), the deformation parameter in [22, 23] plays the role of the rank of the random coupling of the SYK model and it is given by[14]

$$
\lambda = \frac{1}{2(q_{\text{syk}}+1)}. \tag{3.4}
$$

[12] Note that in [36], there appear the extra factors $\pm(-1)^s$ or $\pm(-1)^{s-\frac{1}{2}}$ in various places in the coefficients of (3.2).

[13] We consider the $L=1$ case. If we consider the general L, then this L factor appears in the central charge.

[14] It has been conjectured in [37] from the $\mathcal{N}=(0,2)$ SYK models that the parameter λ_{hs} (See also the footnote 7) is related to the q_{syk} and is given by $\lambda_{hs} = \frac{1}{\frac{M}{N}\,q_{\text{syk}}}$.

We can write down the generators of the $\mathcal{N}=2$ superconformal algebra for matrix generalization from (2.7) as follows:

$$\begin{aligned}
J &= (-1+2\lambda)\, c^a\, b^a - 2\lambda\, \beta^a\, \gamma^a, \\
G^+ &= \sqrt{2}\, \gamma^a\, b^a, \\
G^- &= -\sqrt{2}\, \lambda\, \beta^a\, \bar{\partial}\, \bar{\lambda}^a - \frac{(-1+2\lambda)}{\sqrt{2}}\, \bar{\partial}\, \beta^a\, \bar{\lambda}^a, \\
T &= (1-\lambda)\, \bar{\partial}\, \beta^a \gamma^a - \lambda\, \beta^a\, \bar{\partial}\, \gamma^a + \frac{1}{2}\, (1+2\lambda)\, \bar{\partial}\, c^a\, b^a + \frac{1}{2}\, (-1+2\lambda)\, c^a\, \bar{\partial}\, b^a.
\end{aligned} \tag{3.5}$$

For $N=1$, we observe that the above relations (3.5) are reduced to the ones in [22]. By realizing the following relations with (3.4),[15]

$$\gamma^a \leftrightarrow \bar{\partial}\, \bar{\phi}^a, \qquad \beta^a \leftrightarrow \phi^a, \qquad c^a \leftrightarrow \frac{i}{\sqrt{2}}\, \bar{\lambda}^a, \qquad b^a \rightarrow \frac{i}{\sqrt{2}}\, \lambda^a, \tag{3.6}$$

we observe that the relations (3.5) can be identified with the ones in (2.2) together with a factor $\sqrt{2}i$ in G^+ and a factor $-\frac{i}{\sqrt{2}}$ in G^- (The OPE $G^+(\bar{z})\, G^-(\bar{w})$ does not change with these factors). The conformal weights for both sides in (3.6) are consistent with each other. We expect that there is a one-to one correspondence between the $\mathcal{N}=2$ SYK model and the $\beta\,\gamma$ and $b\,c$ ghost systems in the $\mathcal{N}=2$ supersymmetric linear $W_{\infty}^{N,N}[\lambda]$ algebra.

3.2 The $\mathcal{N}=2$ supersymmetric linear $W_{\infty}^{N,N}[\lambda=0]$ algebra

From the exact correspondence between the chiral multiplets and the Fermi multiplets of the $\mathcal{N}=2$ $(2,2)$ SYK model and the $\beta\,\gamma$ and $b\,c$ ghost systems in (3.6), we expect that there exist the precise relations for the higher spin currents between them. By linear combinations among the higher spin currents in (3.2) we can write down the higher spin currents of [6] in terms of (3.2) at $\lambda=0$ and it turns out that for SU(N)-singlet currents we have[16]

$$\begin{aligned}
W_{F,h} &= \frac{n_{W_{F,h}}}{q^{h-2}}\, \frac{(-1)^h}{\sum_{i=0}^{h-1}\, a^i(h,\lambda+\frac{1}{2}=\frac{1}{2})} \left[\frac{(h-1+2\lambda)}{(2h-1)}\, V_{\lambda}^{(h)+} + V_{\lambda}^{(h)-} \right]_{\lambda=0}, \\
W_{B,h} &= \frac{n_{W_{B,h}}}{q^{h-2}}\, \frac{(-1)^h}{\sum_{i=0}^{h-1}\, a^i(h,\lambda=0)} \left[\frac{(h-2\lambda)}{(2h-1)}\, V_{\lambda}^{(h)+} - V_{\lambda}^{(h)-} \right]_{\lambda=0}, \\
Q_{h+\frac{1}{2}} &= \frac{1}{2}\, \frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}}\, \frac{(-1)^{h+1}\, h}{\sum_{i=0}^{h-1}\, \beta^i(h+1,\lambda=0)} \left[Q_{\lambda}^{(h+1)-} - Q_{\lambda}^{(h+1)+} \right]_{\lambda=0}, \\
\bar{Q}_{h+\frac{1}{2}} &= \frac{1}{2}\, \frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}}\, \frac{(-1)^{h+1}}{\sum_{i=0}^{h}\, \alpha^i(h+1,\lambda=0)} \left[Q_{\lambda}^{(h+1)-} + Q_{\lambda}^{(h+1)+} \right]_{\lambda=0}.
\end{aligned} \tag{3.7}$$

For $h=1$ with $\lambda=0$, the coefficient of the first term of $W_{F,h=1}$ in (3.7) vanishes and the $W_{F,h=1}$ is proportional to $V_{\lambda=0}^{(1)-} = -c^a\, b^a$. See also (2.7) and (3.5). On the other hand, the

[15] In [21], we used the different terminology for the bosons. Note that when we change the ordering in the first OPE of (3.1), there is a minus sign in the right hand side while there is no minus sign in the second OPE of (3.1) after this change. We should also make sure that they have the correct weights in terms of deformation parameter.

[16] The parameter q here is the same as the λ in [6]. We can associate the $\psi^{\bar{i},a}$ and $\bar{\psi}^{j,\bar{b}}$ of [6] with $b^{\bar{i},a}$ and $c^{j,\bar{b}}$.

coefficient of the first term of $W_{B,h=1}$ in (3.7) does not vanish and the $W_{B,h=1}$ with $\lambda = 0$ is proportional to $-\gamma^a\,\beta^a$ (which holds for nonzero λ). Then the current $W_{B,h=1}$ arises only in the $\beta\,\gamma$ and $b\,c$ ghost systems. For $h = 0$, the $Q_{\frac{1}{2}}$ vanishes and the $\bar{Q}_{\frac{1}{2}}$ is proportional to $-\beta^a\,c^a$ which does not occur in the construction of [6]. See also the footnote 21. Therefore, we expect that there appears the presence of the current $W_{B,h=1}$ and the current $\bar{Q}_{\frac{1}{2}}$ in the $\mathcal{N} = 2$ supersymmetric linear $W_\infty^{N,N}[\lambda = 0]$ algebra.[17]

Furthermore, we can compare with each coefficient appearing in the free field realization in [6] and the one in (3.2) at $\lambda = 0$. In order to do this, we should act the antiholomorphic partial derivatives on the composite operators fully. Then the binomial coefficients appear. It turns out that there appear the following identities

$$\begin{aligned}
\frac{n_{W_{F,h}}}{q^{h-2}}(-1)^k \binom{h-1}{k}^2 &= \frac{n_{W_{F,h}}}{q^{h-2}} \frac{1}{\sum_{i=0}^{h-1} a^i(h,\frac{1}{2})} \sum_{i=0}^{h-1} a^i\left(h,\frac{1}{2}\right) \binom{h-1-i}{k}, \\
\frac{n_{W_{B,h}}}{q^{h-2}} \frac{(-1)^k}{(h-1)} \binom{h-1}{k}\binom{h-1}{k+1} &= \frac{n_{W_{B,h}}}{q^{h-2}} \frac{1}{\sum_{i=0}^{h-1} a^i(h,0)} \sum_{i=0}^{h-1} a^i(h,0) \binom{h-1-i}{k}, \qquad (3.8) \\
\frac{n_{W_{Q,h}}(-1)^k}{q^{h-\frac{3}{2}}} \binom{h-\frac{3}{2}}{k}\binom{h-\frac{1}{2}}{k} &= \frac{n_{W_{Q,h-\frac{1}{2}}}}{q^{h-2}} \frac{(-1)^h (h-1)}{\sum_{i=0}^{h-2} \beta^i(h,0)} \sum_{i=0}^{h-2} \beta^i(h,0) \binom{h-2-i}{k}, \\
\frac{n_{W_{Q,h}}(-1)^{h-\frac{3}{2}+k}}{q^{h-\frac{3}{2}}} \binom{h-\frac{3}{2}}{k}\binom{h-\frac{1}{2}}{k} &= \frac{n_{W_{Q,h-\frac{1}{2}}}}{q^{h-2}} \frac{(-1)^h}{\sum_{i=0}^{h-1} \alpha^i(h,0)} \sum_{i=0}^{h-1} \alpha^i(h,0) \binom{h-1-i}{k}.
\end{aligned}$$

It is rather nontrivial to check these relations for generic h and k, but we can try to do this for several values for these quantities. Note that in the right hand sides of (3.8), the additional binomial coefficients occur by expanding the antiholomorphic partial derivatives fully as described before.[18]

Therefore, the $\mathcal{N} = 2$ SYK model has $\mathcal{N} = 2$ supersymmetric linear $W_\infty^{N,N}[\lambda = 0]$ algebra where the higher spin currents are given by (3.2) and by using the relations (3.7), the explicit (anti)commutator relations can be read off from the previous results in [6](See also [38]). The relation between the parameters is given by (3.4). Of course, as explained before, in the (anti)commutators relations, we observe that there appear the currents $W_{B,h=1}$ and $\bar{Q}_{\frac{1}{2}}$. Moreover, their OPEs with other higher spin currents $W_{F,h\geq 1}$, $W_{B,h\geq 1}$, $Q_{h+\frac{1}{2}\geq\frac{1}{2}}$ and $\bar{Q}_{h+\frac{1}{2}\geq\frac{1}{2}}$ will appear in general. In next section, we will present the (anti)commutator relations for nonzero λ. Therefore, once we put the λ to be zero in these equations, we obtain the final results.

3.2.1 The realization of $\mathcal{N} = 2$ supersymmetric linear $W_\infty^{N,N}[\lambda = 0]$ algebra in the $\mathcal{N} = 2$ SYK model

That is, in the limit of

$$q_{\rm syk} \to \infty, \qquad (3.9)$$

[17] The SU(N)-adjoint currents are given by appendix A.

[18] Note that the various summations in the denominators of the right hand sides of (3.8) can be written in terms of the fractional forms of the various gamma functions at nonzero λ [35].

the $\mathcal{N}=2$ SYK models reveal the $\mathcal{N}=2$ supersymmetric linear $W_{\infty}^{N,N}[\lambda=0]$ algebra. The generators are given by

$$\begin{array}{llllll} i) & W_{F,h\geq 1}, & W_{B,h\geq 2}, & Q_{h+\frac{3}{2}\geq\frac{3}{2}}, & \bar{Q}_{h+\frac{1}{2}\geq\frac{3}{2}}, & \\ ii) & W_{B,h=1}, & \bar{Q}_{h+\frac{1}{2}=\frac{1}{2}}. & & & \end{array} \tag{3.10}$$

The algebra between the currents in the first line of (3.10) is closed and the explicit form is given by the ones in [6]. In the right hand sides of these (anti)commutator relations we can see only the operators in the first line of (3.10). Due to the presence of the operators in the second line of (3.10), we should calculate the OPEs between these weight-1, $\frac{1}{2}$ currents and the remaining ones in the first line of (3.10) as well as their own OPEs in order to describe the full algebra if we do not decouple these currents.[19] As we will see next section, once the λ becomes nonzero value (a deviation from (3.9)), then this does not hold any more because the right hand sides of these (anti)commutator relations possess the operators in the second line of (3.10).

3.3 The $\mathcal{N}=2$ supersymmetric linear $W_{\infty}^{N,N}[\lambda]$ algebra

3.3.1 The higher spin currents for nonzero λ

Let us consider the nonzero λ case. We take the previous expressions (3.7) by considering the λ dependence explicitly. Then we have the following SU(N)-singlet currents[20]

$$\begin{aligned} W_{F,h}^{\lambda} &= \frac{n_{W_{F,h}}}{q^{h-2}} \, \frac{(-1)^h}{\sum_{i=0}^{h-1} \, a^i(h,\lambda+\frac{1}{2}=\frac{1}{2})} \left[\frac{(h-1+2\lambda)}{(2h-1)} \, V_{\lambda}^{(h)+} + V_{\lambda}^{(h)-} \right], \\ W_{B,h}^{\lambda} &= \frac{n_{W_{B,h}}}{q^{h-2}} \, \frac{(-1)^h}{\sum_{i=0}^{h-1} \, a^i(h,\lambda=0)} \left[\frac{(h-2\lambda)}{(2h-1)} \, V_{\lambda}^{(h)+} - V_{\lambda}^{(h)-} \right], \\ Q_{h+\frac{1}{2}}^{\lambda} &= \frac{1}{2} \, \frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}} \, \frac{(-1)^{h+1} \, h}{\sum_{i=0}^{h-1} \, \beta^i(h+1,\lambda=0)} \left[Q_{\lambda}^{(h+1)-} - Q_{\lambda}^{(h+1)+} \right], \\ \bar{Q}_{h+\frac{1}{2}}^{\lambda} &= \frac{1}{2} \, \frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}} \, \frac{(-1)^{h+1}}{\sum_{i=0}^{h} \, \alpha^i(h+1,\lambda=0)} \left[Q_{\lambda}^{(h+1)-} + Q_{\lambda}^{(h+1)+} \right]. \end{aligned} \tag{3.11}$$

In particular, $V_{\lambda}^{(1)-}$ has $\gamma^a \, \beta^a$ term also for nonzero λ. See also the weight-1 current in (3.5). We would like to obtain the algebra generated by these currents in (3.11).

---

[19] We will see that we have their explicit forms as $W_{B,1} = -\frac{1}{4} \, \gamma^a \, \beta^a$ and $\bar{Q}_{\frac{1}{2}} = -\frac{1}{\sqrt{2}} \, \beta^a \, c^a$. Even they do not depend on the λ parameter from the footnote 21. By construction of [6], there is no $\phi^{\bar{i},a}$ dependence and its derivative $\bar{\partial} \, \phi^{\bar{i},a}$ appears only.

[20] The SU(N)-adjoint currents are given by appendix A. Each λ independent coefficient of bosonic current is the same and each λ independent coefficient of fermionic current is the same: $\frac{n_{W_{F,h}}}{q^{h-2}} \, \frac{(-1)^h}{\sum_{i=0}^{h-1} \, a^i(h,\lambda+\frac{1}{2}=\frac{1}{2})} = \frac{n_{W_{B,h}}}{q^{h-2}} \, \frac{(-1)^h}{\sum_{i=0}^{h-1} \, a^i(h,\lambda=0)}$ and $\frac{1}{2} \, \frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}} \, \frac{(-1)^{h+1} \, h}{\sum_{i=0}^{h-1} \, \beta^i(h+1,\lambda=0)} = \frac{1}{2} \, \frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}} \, \frac{(-1)^{h+1}}{\sum_{i=0}^{h} \, \alpha^i(h+1,\lambda=0)}$.

We present the currents for low weights as follows:

$$\begin{aligned}
W^{\lambda}_{F,1} &= -\frac{1}{4}\Big(V^{(1)-}_{\lambda}+2\lambda\,V^{(1)+}_{\lambda}\Big), & W^{\lambda}_{F,2} &= \Big(V^{(2)-}_{\lambda}+\frac{1}{3}\,(1+2\lambda)\,V^{(2)+}_{\lambda}\Big),\\
W^{\lambda}_{F,3} &= -4\Big(V^{(3)-}_{\lambda}+\frac{1}{5}\,(2+2\lambda)\,V^{(3)+}_{\lambda}\Big), & W^{\lambda}_{F,4} &= 16\Big(V^{(4)-}_{\lambda}+\frac{1}{7}\,(3+2\lambda)\,V^{(4)+}_{\lambda}\Big),\\
W^{\lambda}_{B,1} &= -\frac{1}{4}\Big(-V^{(1)-}_{\lambda}+(1-2\lambda)\,V^{(1)+}_{\lambda}\Big), & W^{\lambda}_{B,2} &= \Big(-V^{(2)-}_{\lambda}+\frac{1}{3}\,(2-2\lambda)\,V^{(2)+}_{\lambda}\Big),\\
W^{\lambda}_{B,3} &= -4\Big(-V^{(3)-}_{\lambda}+\frac{1}{5}\,(3-2\lambda)\,V^{(3)+}_{\lambda}\Big), & W^{\lambda}_{B,4} &= 16\Big(-V^{(4)-}_{\lambda}+\frac{1}{7}\,(4-2\lambda)\,V^{(4)+}_{\lambda}\Big),\\
Q^{\lambda}_{\frac{3}{2}} &= \frac{1}{\sqrt{2}}\Big(Q^{(2)-}_{\lambda}-Q^{(2)+}_{\lambda}\Big), & Q^{\lambda}_{\frac{5}{2}} &= -2\sqrt{2}\Big(Q^{(3)-}_{\lambda}-Q^{(3)+}_{\lambda}\Big),\\
Q^{\lambda}_{\frac{7}{2}} &= 8\sqrt{2}\Big(Q^{(4)-}_{\lambda}-Q^{(4)+}_{\lambda}\Big), & Q^{\lambda}_{\frac{9}{2}} &= -32\sqrt{2}\Big(Q^{(5)-}_{\lambda}-Q^{(5)+}_{\lambda}\Big),\\
\bar{Q}^{\lambda}_{\frac{1}{2}} &= -\frac{1}{2\sqrt{2}}\Big(\bar{Q}^{(1)-}_{\lambda}+\bar{Q}^{(1)+}_{\lambda}\Big), & \bar{Q}^{\lambda}_{\frac{3}{2}} &= \frac{1}{\sqrt{2}}\Big(\bar{Q}^{(2)-}_{\lambda}+\bar{Q}^{(2)+}_{\lambda}\Big),\\
\bar{Q}^{\lambda}_{\frac{5}{2}} &= -2\sqrt{2}\Big(\bar{Q}^{(3)-}_{\lambda}+\bar{Q}^{(3)+}_{\lambda}\Big), & \bar{Q}^{\lambda}_{\frac{7}{2}} &= 8\sqrt{2}\Big(\bar{Q}^{(4)-}_{\lambda}+\bar{Q}^{(4)+}_{\lambda}\Big),\\
\bar{Q}^{\lambda}_{\frac{9}{2}} &= -32\sqrt{2}\Big(\bar{Q}^{(5)-}_{\lambda}+\bar{Q}^{(5)+}_{\lambda}\Big), & & \cdots .
\end{aligned} \tag{3.12}$$

According to the normalization in (3.11), as we increase the spin, we simply multiply by -4 (except $\bar{Q}^{\lambda}_{\frac{1}{2}}$). We will calculate the various OPEs by using these explicit expressions (3.12).

The generators in (3.5) in terms of the currents can be written as

$$\begin{aligned}
J &= V^{(1)-}_{\lambda} = -4\Big((1-2\lambda)\,W_{F,1}-2\lambda\,W_{B,1}\Big),\\
G^{+} &= Q^{\lambda}_{\frac{3}{2}} = \frac{1}{\sqrt{2}}\Big(Q^{(2)-}_{\lambda}-Q^{(2)+}_{\lambda}\Big),\\
G^{-} &= \frac{1}{2}\,\bar{Q}^{\lambda}_{\frac{3}{2}} = \frac{1}{2\sqrt{2}}\Big(\bar{Q}^{(2)-}_{\lambda}+\bar{Q}^{(2)+}_{\lambda}\Big),\\
T &= V^{(2)+}_{\lambda} = \Big(W^{\lambda}_{F,2}+W^{\lambda}_{B,2}\Big).
\end{aligned} \tag{3.13}$$

At $\lambda=0$, the weight-1 current J in (3.13) of the $\mathcal{N}=2$ superconformal algebra does not depend on the bosonic $\beta\,\gamma$ operators.[21]

3.3.2 The structure constants for nonzero λ

Let us introduce the generalized hypergeometric function

$$\phi^{h_1,h_2}_{r}(\Lambda,a) \equiv \;{}_4F_3\left[\begin{matrix}\frac{1}{2}+\Lambda\,,\,\frac{1}{2}-\Lambda\,,\,\frac{1+a-r}{2}\,,\,\frac{a-r}{2}\\ \frac{3}{2}-h_1\,,\,\frac{3}{2}-h_2\,,\,\frac{1}{2}+h_1+h_2-r\end{matrix}\middle|\,1\right]. \tag{3.14}$$

In general, the sum of upper four elements plus 1 ($=\frac{5}{2}+a-r$) is not equal to the sum of lower three elements ($=\frac{7}{2}-r$) for generic $a\neq 1$.[22] Furthermore, we introduce the mode

[21]Note that we have $W^{\lambda}_{B,1}=-\frac{1}{4}\,\gamma^a\,\beta^a$ and $\bar{Q}^{\lambda}_{\frac{1}{2}}=-\frac{1}{\sqrt{2}}\,\beta^a\,c^a$.

[22]For $\lambda=0$, we introduce the generalized hypergeometric function

$$\hat{\phi}^{h_1,h_2}_{h}(x,y)\equiv {}_4F_3\left[\begin{matrix}-\frac{1}{2}-x-2y,\frac{3}{2}-x+2y,-\frac{h+1}{2}+x,-\frac{h}{2}+x\\ -h_1+\frac{3}{2},-h_2+\frac{3}{2},h_1+h_2-h-\frac{3}{2}\end{matrix};1\right].$$

dependent function

$$N_h^{h_1,h_2}(m,n) \equiv \sum_{l=0}^{h+1} (-1)^l \begin{pmatrix} h+1 \\ l \end{pmatrix} [h_1-1+m]_{h+1-l}[h_1-1-m]_l \\ \times [h_2-1+n]_l [h_2-1-n]_{h+1-l}. \tag{3.15}$$

The falling Pochhammer symbol $[a]_n \equiv a(a-1)\cdots(a-n+1)$ in (3.15) is used.

We have found three different kinds of structure constants in the context of the matrix generalization of AdS_3 Vasiliev higher spin theory as follows [25]:

$$\begin{aligned}
\mathrm{BB}_{r,\pm}^{h_1,h_2}(m,n;\mu) &\equiv -\frac{1}{(r-1)!} N_{r-2}^{h_1,h_2}(m,n) \Bigg[\phi_r^{h_1,h_2}(\mu,1) \pm \phi_r^{h_1,h_2}(1-\mu,1)\Bigg], \\
\mathrm{BF}_{r,\pm}^{h_1,h_2+\frac{1}{2}}(m,\rho;\mu) &\equiv -\frac{1}{(r-1)!} N_{r-2}^{h_1,h_2+\frac{1}{2}}(m,\rho) \Bigg[\phi_{r+1}^{h_1,h_2+1}\left(\mu,\frac{3\pm1}{2}\right) \\
&\qquad \pm \phi_{r+1}^{h_1,h_2+1}\left(1-\mu,\frac{3\pm1}{2}\right)\Bigg], \\
\mathrm{FF}_{r,\pm}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(\rho,\omega;\mu) &\equiv -\frac{1}{(r-1)!} N_{r-2}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(\rho,\omega) \Bigg[\phi_{r+1}^{h_1+1,h_2+1}\left(\mu,\frac{3\pm1}{2}\right) \\
&\qquad \pm \phi_{r+1}^{h_1+1,h_2+1}\left(1-\mu,\frac{3\pm1}{2}\right)\Bigg],
\end{aligned} \tag{3.16}$$

where the relations (3.14) and (3.15) are needed.[23]

From the lesson of [25] where the mode dependent structure constants for vanishing λ can be written in terms of the linear combinations of (3.16), we do expect that for nonzero λ, they satisfy as follows:

$$\begin{aligned}
p_{F,h}^{h_1,h_2}(m,n,\lambda) &= -\frac{1}{4}\Bigg[\mathrm{BB}_{h+2,+}^{h_1,h_2} + \mathrm{BB}_{h+2,-}^{h_1,h_2}\Bigg]_{\mu=2\lambda}, \\
p_{B,h}^{h_1,h_2}(m,n,\lambda) &= -\frac{1}{4}\Bigg[\mathrm{BB}_{h+2,+}^{h_1,h_2} - \mathrm{BB}_{h+2,-}^{h_1,h_2}\Bigg]_{\mu=2\lambda},
\end{aligned}$$

By using the notation of (3.14), we have

$$\hat{\phi}_{r-1-a}^{h_1,h_2}(0,\frac{1}{2}(-1-\Lambda)) \equiv {}_4F_3 \left[\begin{matrix} \frac{1}{2}+\Lambda\,,\,\frac{1}{2}-\Lambda\,,\,\frac{1+a-r}{2}\,,\,\frac{a-r}{2} \\ \frac{3}{2}-h_1\,,\,\frac{3}{2}-h_2\,,\,(\frac{1}{2}+h_1+h_2-r)+a-1 \end{matrix} \middle| 1 \right].$$

The last of lower elements contains the additional $(a-1)$ which is nonzero for $a \neq 1$. We check that for $a=1$, the expression of (3.14) reduces to the one in [39] where their s, r, i and j in (3.14) correspond to our $-\frac{1}{2}(1-\Lambda)$, $\frac{1}{2}(r-2)$, (h_1-2) and (h_2-2) respectively.

[23] We have the following symmetry between the structure constants [25] under the transformation $\mu \leftrightarrow 1-\mu$

$$\begin{aligned}
\mathrm{BB}_{r,\pm}^{h_1,h_2}(m,n;\mu) &= \pm\mathrm{BB}_{r,\pm}^{h_1,h_2}(m,n;1-\mu), \\
\mathrm{BF}_{r,\pm}^{h_1,h_2+\frac{1}{2}}(m,\rho;\mu) &= \pm\mathrm{BF}_{r,\pm}^{h_1,h_2+\frac{1}{2}}(m,\rho;1-\mu), \\
\mathrm{FF}_{r,\pm}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(\rho,\omega;\mu) &= \pm\mathrm{FF}_{r,\pm}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(\rho,\omega;1-\mu).
\end{aligned}$$

This implies that the half of the structure constants vanishes at $\mu=\frac{1}{2}$ ($\lambda=\frac{1}{4}$ or $q_{\mathrm{syk}}=1$).

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$$q_{F,2h}^{h_1,h_2+\frac{1}{2}}(m,n,\lambda) = \left[-\frac{1}{8}\mathrm{BF}_{2h+2,+}^{h_1,h_2+\frac{1}{2}} + \frac{(2h_1-2h-3)}{16(h+1)}\,\mathrm{BF}_{2h+2,-}^{h_1,h_2+\frac{1}{2}}\right]_{\mu=2\lambda},$$

$$q_{F,2h+1}^{h_1,h_2+\frac{1}{2}}(m,n,\lambda) = \left[\frac{1}{8}\mathrm{BF}_{2h+3,+}^{h_1,h_2+\frac{1}{2}} - \frac{(h_1-h-2)}{4(2h+3)}\,\mathrm{BF}_{2h+3,-}^{h_1,h_2+\frac{1}{2}}\right]_{\mu=2\lambda},$$

$$q_{B,2h}^{h_1,h_2+\frac{1}{2}}(m,n,\lambda) = \left[-\frac{1}{8}\mathrm{BF}_{2h+2,+}^{h_1,h_2+\frac{1}{2}} - \frac{(2h_1-2h-3)}{16(h+1)}\,\mathrm{BF}_{2h+2,-}^{h_1,h_2+\frac{1}{2}}\right]_{\mu=2\lambda},$$

$$q_{B,2h+1}^{h_1,h_2+\frac{1}{2}}(m,n,\lambda) = \left[-\frac{1}{8}\mathrm{BF}_{2h+3,+}^{h_1,h_2+\frac{1}{2}} - \frac{(h_1-h-2)}{4(2h+3)}\,\mathrm{BF}_{2h+3,-}^{h_1,h_2+\frac{1}{2}}\right]_{\mu=2\lambda},$$

$$o_{F,2h}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(m,n,\lambda) = \left[-\mathrm{FF}_{2h+1,+}^{h_1+\frac{1}{2},h_2+\frac{1}{2}} - \frac{2(h_1+h_2-h)}{(2h+1)}\,\mathrm{FF}_{2h+1,-}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}\right]_{\mu=2\lambda},$$

$$o_{F,2h+1}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(m,n,\lambda) = \left[\mathrm{FF}_{2h+2,+}^{h_1+\frac{1}{2},h_2+\frac{1}{2}} + \frac{2(h_1+h_2-h)-1}{2(h+1)}\,\mathrm{FF}_{2h+2,-}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}\right]_{\mu=2\lambda},$$

$$o_{B,2h}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(m,n,\lambda) = \left[-\mathrm{FF}_{2h+1,+}^{h_1+\frac{1}{2},h_2+\frac{1}{2}} + \frac{2(h_1+h_2-h)}{(2h+1)}\,\mathrm{FF}_{2h+1,-}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}\right]_{\mu=2\lambda},$$

$$o_{B,2h+1}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(m,n,\lambda) = \left[-\mathrm{FF}_{2h+2,+}^{h_1+\frac{1}{2},h_2+\frac{1}{2}} + \frac{2(h_1+h_2-h)-1}{(2h+2)}\,\mathrm{FF}_{2h+2,-}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}\right]_{\mu=2\lambda}. \tag{3.17}$$

We have checked that the above relations for several h_1, h_2 and h are satisfied in the specific OPE examples. That is, the structure constants are indeed the right hand sides of (3.17).

3.3.3 The example of the explicit OPE $W_{F,4}^{\lambda}(\bar{z})\,W_{F,4}^{\lambda}(\bar{w})$ for nonzero λ

For example, for $h_1 = 4$ and $h_2 = 4$, we can calculate the OPE $W_{F,4}^{\lambda}(\bar{z})\,W_{F,4}^{\lambda}(\bar{w})$ by using (3.11), (3.2) and (3.1) and reexpressing each pole in terms of $W_{F,h}^{\lambda}(\bar{w})$ with $h = 2, 4, 6$ and their derivatives as follows:[24]

$$
\begin{aligned}
W_{F,4}^{\lambda}(\bar{z})\,W_{F,4}^{\lambda}(\bar{w}) &= \frac{1}{(\bar{z}-\bar{w})^8}\left[-\frac{768}{5}(112\lambda^6-280\lambda^4+147\lambda^2-9)\right] \\
&+\frac{1}{(\bar{z}-\bar{w})^6}\left[\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)\right]W_{F,2}^{\lambda}(\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^5}\,\frac{1}{2}\left[\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)\right]\bar{\partial}\,W_{F,2}^{\lambda}(\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^4}\left[\frac{3}{20}\,\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}^2\,W_{F,2}^{\lambda}-\frac{96}{5}(4\lambda^2-19)\,W_{F,4}^{\lambda}\right](\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^3}\left[\frac{1}{30}\,\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}^3\,W_{F,2}^{\lambda}-\frac{1}{2}\,\frac{96}{5}(4\lambda^2-19)\,\bar{\partial}\,W_{F,4}^{\lambda}\right](\bar{w})
\end{aligned}
$$

[24] In the OPEs we are considering in this paper, the number N is fixed by $N = 3$ which is the smallest number having the nontrivial d symbol of $SU(N)$. The number L is fixed by $L = 1$ which is introduced around the equation (3.1).

$$
+\frac{1}{(\bar{z}-\bar{w})^2}\left[\frac{1}{168}\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}^4\,W^{\lambda}_{F,2}-\frac{5}{36}\frac{96}{5}(4\lambda^2-19)\,\bar{\partial}^2\,W^{\lambda}_{F,4}\right.
$$

$$
\left.+6\,W^{\lambda}_{F,6}\right](\bar{w})+\frac{1}{(\bar{z}-\bar{w})}\left[\frac{1}{1120}\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}^5\,W^{\lambda}_{F,2}\right.
$$

$$
\left.-\frac{1}{36}\frac{96}{5}(4\lambda^2-19)\,\bar{\partial}^3\,W^{\lambda}_{F,4}+\frac{1}{2}\,6\,\bar{\partial}\,W^{\lambda}_{F,6}\right](\bar{w})+\cdots
$$

$$
=\frac{1}{(\bar{z}-\bar{w})^8}\left[-\frac{768}{5}(112\lambda^6-280\lambda^4+147\lambda^2-9)\right]-p^{4,4}_{F,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,2}(\bar{w})}{(\bar{z}-\bar{w})}\right]
$$

$$
-p^{4,4}_{F,2}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,4}(\bar{w})}{(\bar{z}-\bar{w})}\right]-p^{4,4}_{F,0}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,6}(\bar{w})}{(\bar{z}-\bar{w})}\right]+\cdots. \tag{3.18}
$$

In the second relation of (3.18), we reexpress the structure constants in terms of the differential operators $p^{4,4}_{F,6-h}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)$ with $h=2,4,6$. By using the first equation of (3.17) for fixed $h_1=h_2=4$, we obtain

$$
\begin{aligned}
p^{4,4}_{F,4}(m,n,\lambda) \;=\; & \frac{64}{525}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3) \\
& \times(m-n)(3m^4-2m^3n+4m^2n^2-39m^2-2mn^3+20mn+3n^4-39n^2+108).
\end{aligned} \tag{3.19}
$$

From this (3.19), we can read off the corresponding differential operator by taking the terms having a degree $(h+1)=5$ as follows:

$$
\begin{aligned}
p^{4,4}_{F,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda) \;=\; & \frac{64}{525}\times 9\times\frac{1}{9}\,(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3) \\
& \times\Big(3\bar{\partial}^5_{\bar{z}}-5\bar{\partial}^4_{\bar{z}}\bar{\partial}_{\bar{w}}+6\bar{\partial}^3_{\bar{z}}\bar{\partial}^2_{\bar{w}}-6\bar{\partial}^2_{\bar{z}}\bar{\partial}^3_{\bar{w}}+5\bar{\partial}_{\bar{z}}\bar{\partial}^4_{\bar{w}}-3\bar{\partial}^5_{\bar{w}}\Big).
\end{aligned} \tag{3.20}
$$

Then we can calculate the quantity $-p^{4,4}_{F,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,2}(\bar{w})}{(\bar{z}-\bar{w})}\right]$ by acting (3.20) on the operator $\left[\frac{W^{\lambda}_{F,2}(\bar{w})}{(\bar{z}-\bar{w})}\right]$ and this will lead to the corresponding terms in (3.18). Similarly, we can calculate $p^{4,4}_{F,2}(m,n,\lambda)=-\frac{8}{15}\,(-19)\,(-\frac{1}{19})\,(4\lambda^2-19)(m-n)(m^2-mn+n^2-7)$. Then we can determine $p^{4,4}_{F,2}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)=-\frac{8}{15}\,(-19)\,(-\frac{1}{19})\,(4\lambda^2-19)\Big(\bar{\partial}^3_{\bar{z}}-2\bar{\partial}^2_{\bar{z}}\bar{\partial}_{\bar{w}}+2\bar{\partial}_{\bar{z}}\bar{\partial}^2_{\bar{w}}-\bar{\partial}^3_{\bar{w}}\Big)$ and this leads to the current $W^{\lambda}_{F,4}(\bar{w})$ and its derivatives in (3.18) by performing $-p^{4,4}_{F,2}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,4}(\bar{w})}{(\bar{z}-\bar{w})}\right]$. Finally, after calculating $p^{4,4}_{F,0}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)=3(m-n)$, the result of $-p^{4,4}_{F,0}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,6}(\bar{w})}{(\bar{z}-\bar{w})}\right]$ provides the corresponding terms in (3.18).[25]

In appendix B we present other various OPEs between the SU(N)-singlet currents for fixed h_1 and h_2. In this way, we make sure that the structure constants in (3.17) are correct ones.

[25]The corresponding commutator relation can be obtained by using the formula in [40, 41]. For example, we can obtain that $p_{ijk}(m,n)$ in (2.54) of [40] is given by $\frac{1}{3360}\,(m-n)\,(3m^4-2m^3n+4m^2n^2-39m^2-2mn^3+20mn+3n^4-39n^2+108)$. After multiplying the coefficient $\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)$, we obtain the above $p^{4,4}_{F,4}(m,n,\lambda)$ in (3.19).

3.3.4 The complete (anti)commutator relations between the SU(N)-singlet currents for nonzero λ

From the analysis of previous section together with the similar descriptions in appendix B, we conclude that the final complete (anti)commutator relations between the SU(N)-singlet currents for nonzero λ with the insertion of q dependence appropriately can be summarized by[26]

$$
\begin{aligned}
[(W^{\lambda}_{\mathrm{F},h_1})_m,(W^{\lambda}_{\mathrm{F},h_2})_n] &= \sum_{h=0,\mathrm{even}}^{h_1+h_2-3} q^h\, p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda}_{\mathrm{F},h_1+h_2-2-h})_{m+n} \\
&\quad + N\, c_{W_{\mathrm{F},h_1}}(m,\lambda)\,\delta^{h_1h_2}\,q^{2(h_1-2)}\,\delta_{m+n}, \\
[(W^{\lambda}_{\mathrm{B},h_1})_m,(W^{\lambda}_{\mathrm{B},h_2})_n] &= \sum_{h=0,\mathrm{even}}^{h_1+h_2-4} q^h\, p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda}_{\mathrm{B},h_1+h_2-2-h})_{m+n} \\
&\quad + \Big[q^h\, p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\Big]_{h=h_1+h_2-3} \\
&\quad + N\, c_{W_{\mathrm{B},h_1}}(m,\lambda)\,\delta^{h_1h_2}\,q^{2(h_1-2)}\,\delta_{m+n}, \\
[(W^{\lambda}_{\mathrm{F},h_1})_m,(Q^{\lambda}_{h_2+\frac{1}{2}})_r] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\,(Q^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\,, \\
[(W^{\lambda}_{\mathrm{B},h_1})_m,(Q^{\lambda}_{h_2+\frac{1}{2}})_r] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\,(Q^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\,, \\
[(W^{\lambda}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda}_{h_2+\frac{1}{2}})_r] &= \sum_{h=-1}^{h_1+h_2-3} q^h\,(-1)^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\,(\bar{Q}^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&\quad + \Big[q^h\,(-1)^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\,(\bar{Q}^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big]_{h=h_1+h_2-2}\,, \\
[(W^{\lambda}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda}_{h_2+\frac{1}{2}})_r] &= \sum_{h=-1}^{h_1+h_2-3} q^h\,(-1)^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\,(\bar{Q}^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&\quad + \Big[q^h\,(-1)^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\,(\bar{Q}^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big]_{h=h_1+h_2-2}\,, \\
\{(Q^{\lambda}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda}_{h_2+\frac{1}{2}})_s\} &= \sum_{h=0}^{h_1+h_2-1} q^h\, o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\,(W^{\lambda}_{\mathrm{F},h_1+h_2-h})_{r+s} \\
&\quad + \sum_{h=0}^{h_1+h_2-2} q^h\, o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\,(W^{\lambda}_{\mathrm{B},h_1+h_2-h})_{r+s} \\
&\quad + \Big[q^h\, o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\,(W^{\lambda}_{\mathrm{B},h_1+h_2-h})_{r+s}\Big]_{h=h_1+h_2-1} \\
&\quad + N\, c_{Q\bar{Q}_{h_1+\frac{1}{2}}}(r,\lambda)\,\delta^{h_1h_2}\,q^{2(h_1+\frac{1}{2}-1)}\delta_{r+s}.
\end{aligned}
\tag{3.21}
$$

[26] In this paper, we do not present the explicit form for the central charges $c_{W_{\mathrm{F},h_1}}(m,\lambda)$, $c_{W_{\mathrm{B},h_1}}(m,\lambda)$ and $c_{Q\bar{Q}_{h_1+\frac{1}{2}}}(r,\lambda)$. We calculated them for fixed h_1 and h_2 in previous section, and appendix B. The calculations can be performed by taking the procedure in [42]. Or we can use the partial results in [35] to obtain them explicitly.

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In the right hand sides of (3.21), we emphasize that the additional weights $\frac{1}{2}, 1$ currents appear by taking the square brackets.[27] Of course we assume that the possible lowest weights for $W^{\lambda}_{\mathrm{F},h}$, $W^{\lambda}_{\mathrm{B},h}$, $Q^{\lambda}_{h+\frac{1}{2}}$ and $\bar{Q}^{\lambda}_{h+\frac{1}{2}}$ are given by $h=1$, $h=2$, $h+\frac{1}{2}=\frac{3}{2}$ and $h+\frac{1}{2}=\frac{3}{2}$ respectively. In other words, among the field contents in [42], the above weights $\frac{1}{2}, 1$ currents occur in the right hand sides of the (anti)commutator relations at nonzero λ. In order to fully describe the complete structure of the $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda]$ algebra, we need to calculate the OPEs between the additional weights $\frac{1}{2}, 1$ currents and the remaining currents. Also their own OPEs should be calculated.

In appendix C, we present some OPEs containing the additional currents $\bar{Q}^{\lambda}_{\frac{1}{2}}$ or $W^{\lambda}_{\mathrm{B},1}$ for fixed h_1 and h_2. It turns out that the OPEs containing the weight-$\frac{1}{2}$ current $\bar{Q}^{\lambda}_{\frac{1}{2}}$ have the previous known structure constants while the OPEs containing the weight-1 current $W^{\lambda}_{\mathrm{B},1}$, at first sight, do not have their structure constants which can be written in terms of the known expressions appearing in (3.21), although there are explicit λ-dependent terms in their OPEs.

In particular, we can check the following relations[28]

$$\begin{aligned}
p_{\mathrm{B}}^{h_1,h_2,h=h_1+h_2-3}(m,n,\lambda=0) &= q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h=h_1+h_2-2}(m,r,\lambda=0) \\
&= q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h=h_1+h_2-2}(m,r,\lambda=0) \\
&= o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h=h_1+h_2-1}(r,s,\lambda=0) \\
&= 0. \qquad (3.22)
\end{aligned}$$

Then according to (3.22), the square brackets in the above (anti)commutator relations (3.21) vanish at $\lambda=0$ and we reproduce the subalgebra of the $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda=0]$ algebra [6]. The bosonic subalgebra is given by $W^{N}_{1+\infty}[\lambda=0]$ generated by $W^{\lambda=0}_{F,h}$ and $W^{N}_{\infty}[\lambda=0]$ generated by $W^{\lambda=0}_{B,h}$. Then by the appropriate limit for the parameter q, we will obtain the $w_{1+\infty}$ algebra from the former. If the decoupling of $W^{\lambda=0}_{F,h=1}$ in the former occurs like as the footnote 31, then this bosonic subalgebra becomes $W^{N}_{\infty}[\lambda=0]$.

In appendix D, the remaining (anti)commutator relations of $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda]$ algebra are summarized, by considering the SU(N) adjoint index $\hat{A}$ properly.

[27] We can still use the (anti)commutator relations for λ=0 by allowing the corresponding upper limits in the summation over the dummy variable h properly at the four places. Each single term can combine with each summation term because the λ and mode dependent structure constants in each single term can be written in terms of the same structure constants in each summation term.

[28] Similar relations can be checked as follows: $p_{\mathrm{F}}^{h_1,h_2,h=h_1+h_2-3}(m,n,\lambda=\frac{1}{2}) = q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h=h_1+h_2-2}(m,r,\lambda=\frac{1}{2}) = q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h=h_1+h_2-2}(m,r,\lambda=\frac{1}{2}) = o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h=h_1+h_2-1}(r,s,\lambda=\frac{1}{2}) = 0$. We reproduce the subalgebra of the $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda=\frac{1}{2}]$ algebra which is isomorphic to the $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda=0]$ algebra as in the introduction. In this case, the bosonic subalgebra is given by $W^{N}_{\infty}[\lambda=\frac{1}{2}]$ generated by $W^{\lambda=\frac{1}{2}}_{F,h}$ and $W^{N}_{1+\infty}[\lambda=\frac{1}{2}]$ generated by $W^{\lambda=\frac{1}{2}}_{B,h}$. Then by the contraction limit for the parameter q, we obtain the $w_{1+\infty}$ algebra from the latter. If the decoupling of $W^{\lambda=\frac{1}{2}}_{B,h=1}$ in the latter occurs as in the subsection 3.3.5, then this bosonic subalgebra becomes $W^{N}_{\infty}[\lambda=\frac{1}{2}]$.

3.3.5 The decoupling of $\bar{Q}^{\lambda}_{\frac{1}{2}}$ and $W^{\lambda}_{B,1}$

At nonzero λ, the (anti)commutator relations imply that the weight-1, $\frac{1}{2}$ currents occur in the right hand sides. Now we can try to decouple them. Let us consider the fifth equation of (3.21) by taking $h_1 = 1$ and $h_2 = 1$. Then we can calculate the OPE $W^{\lambda}_{F,1}(\bar{z})\, \bar{Q}^{\lambda}_{\frac{3}{2}}(\bar{w})$. By requiring that the new weight-$\frac{3}{2}$ current should remove the unwanted current $\bar{Q}^{\lambda}_{\frac{1}{2}}$,[29] we have

$$\bar{Q}^{\lambda}_{new,\frac{3}{2}} = \bar{Q}^{\lambda}_{\frac{3}{2}} - 4\lambda\, \bar{\partial}\, \bar{Q}^{\lambda}_{\frac{1}{2}}. \tag{3.23}$$

Now we go to the sixth equation of (3.21) and substitute (3.23) into that equation in order to obtain the new weight-2 current. It turns out that by taking

$$W^{\lambda}_{new,B,2} = W^{\lambda}_{B,2} - 4\,\lambda\, \bar{\partial}\, W^{\lambda}_{B,1}, \tag{3.24}$$

we can remove the unwanted current $\bar{Q}^{\lambda}_{\frac{1}{2}}$. We can check that the $W^{\lambda}_{B,1}$ dependence disappears when we calculate the OPE $Q^{\lambda}_{\frac{3}{2}}(\bar{z})\, \bar{Q}^{\lambda}_{new,\frac{3}{2}}(\bar{w})$. Similarly, we can calculate

$$\bar{Q}^{\lambda}_{new,\frac{5}{2}} = \bar{Q}^{\lambda}_{\frac{5}{2}} - \frac{8}{3}\,\lambda\,(1+2\lambda)\, \bar{\partial}^2\, \bar{Q}^{\lambda}_{\frac{1}{2}}, \tag{3.25}$$

by considering the OPE $W^{\lambda}_{F,1}(\bar{z})\, \bar{Q}^{\lambda}_{\frac{5}{2}}(\bar{w})$ and removing the unwanted current $\bar{Q}^{\lambda}_{\frac{1}{2}}$. Moreover we can obtain

$$W^{\lambda}_{new,B,3} = W^{\lambda}_{B,3} - \frac{8}{3}\,\lambda\,(1+2\lambda)\; \bar{\partial}^2\, W^{\lambda}_{B,1}, \tag{3.26}$$

by considering the OPE $W^{\lambda}_{B,3}(\bar{z})\, \bar{Q}^{\lambda}_{\frac{5}{2}}(\bar{w})$ and removing the unwanted current $\bar{Q}^{\lambda}_{\frac{1}{2}}$. In this way we determine the new currents, (3.23), (3.24), (3.25) and (3.26). We expect that we can continue to perform this procedure and remove the above weight-1, $\frac{1}{2}$ currents.[30]

Therefore, in principle, eventually we obtain the complete (anti)commutator relations with modified λ-dependent known structure constants, as a subalgebra, where the unwanted weight-1, $\frac{1}{2}$ currents disappear completely.[31]

3.3.6 The realization of $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda=\frac{1}{4}]$ algebra in the $\mathcal{N}=2$ SYK model

As noted in the footnote 23, at $\lambda = \frac{1}{4}$,[32] all the second terms in the right hand sides of the structure constants (3.17) vanish. In the $\mathcal{N}=2$ SYK models, this is equivalent to take the following limit

$$q_{\mathrm{syk}} \to 1. \tag{3.27}$$

[29] We need to admit that the new currents are not quasiprimary operators.

[30] As described before, by using the partial results in [35], we can obtain the new currents for any weight h by calculating the highest order pole in any OPE in order to remove the weigh-$\frac{1}{2}$ current.

[31] We can decouple the $W^{\lambda}_{F,1}$ and $\bar{Q}^{\lambda}_{\frac{1}{2}}$ by introducing the new currents $\bar{Q}^{\lambda}_{new,\frac{3}{2}} = \bar{Q}^{\lambda}_{\frac{3}{2}} - 2(2\lambda-1)\, \bar{\partial}\, \bar{Q}^{\lambda}_{\frac{1}{2}}$ and $W^{\lambda}_{new,F,2} = W^{\lambda}_{F,2} - 2(2\lambda-1)\, \bar{\partial}\, W^{\lambda}_{F,1}$ in the context of the $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda=\frac{1}{2}]$ algebra together with the footnote 28. The higher weight currents can be constructed similarly.

[32] At this point, $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda=\frac{1}{4}]$ algebra is self isomorphic because the solution of $\lambda = \frac{1}{2} - \lambda$ provides the $\lambda = \frac{1}{4}$, as in the introduction.

The interaction is quadratic. As observed in [22], there exists a subalgebra generated by[33]

$$
\begin{aligned}
&V_\lambda^{(s),+}, s=2,4,6,\cdots, && V_\lambda^{(s),-}, s=1,3,5,\cdots, \\
&Q_\lambda^{(s),+}, s=1,3,5,\cdots, && Q_\lambda^{(s),-}, s=1,3,5,\cdots,
\end{aligned} \tag{3.28}
$$

or by

$$
\begin{aligned}
&V_\lambda^{(s),+}, s=2,4,6,\cdots, && V_\lambda^{(s),-}, s=1,3,5,\cdots, \\
&Q_\lambda^{(s),+}, s=2,4,6,\cdots, && Q_\lambda^{(s),-}, s=2,4,6,\cdots.
\end{aligned} \tag{3.29}
$$

There is no supersymmetry in the first case (3.28) [22].[34] We calculate some OPEs for fixed h_1 and h_2 in order to see this behavior explicitly in appendix E. In the basis of [22], we obtain the following relations from (3.11)

$$
\begin{aligned}
V_\lambda^{(h),+} &= \frac{1}{\left[\frac{n_{W_{F,h}}}{q^{h-2}} \frac{(-1)^h}{\sum_{i=0}^{h-1} a^i(h,\lambda+\frac{1}{2}=\frac{1}{2})}\right]} W^\lambda_{F,h} + \frac{1}{\left[\frac{n_{W_{B,h}}}{q^{h-2}} \frac{(-1)^h}{\sum_{i=0}^{h-1} a^i(h,\lambda=0)}\right]} W^\lambda_{B,h}, \\
V_\lambda^{(h),-} &= \frac{\frac{(h-2\lambda)}{(2h-1)}}{\left[\frac{n_{W_{F,h}}}{q^{h-2}} \frac{(-1)^h}{\sum_{i=0}^{h-1} a^i(h,\lambda+\frac{1}{2}=\frac{1}{2})}\right]} W^\lambda_{F,h} - \frac{\frac{(h-1+2\lambda)}{(2h-1)}}{\left[\frac{n_{W_{B,h}}}{q^{h-2}} \frac{(-1)^h}{\sum_{i=0}^{h-1} a^i(h,\lambda=0)}\right]} W^\lambda_{B,h}, \\
Q_\lambda^{(h+1),+} &= \frac{1}{2}\left(- \frac{1}{\left[\frac{1}{2}\frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}} \frac{(-1)^{h+1}h}{\sum_{i=0}^{h-1} \beta^i(h+1,\lambda=0)}\right]} Q^\lambda_{h+\frac{1}{2}} + \frac{1}{\left[\frac{1}{2}\frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}} \frac{(-1)^{h+1}}{\sum_{i=0}^{h} \alpha^i(h+1,\lambda=0)}\right]} \bar{Q}^\lambda_{h+\frac{1}{2}} \right), \\
\bar{Q}_\lambda^{(h+1),-} &= \frac{1}{2}\left(\frac{1}{\left[\frac{1}{2}\frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}} \frac{(-1)^{h+1}h}{\sum_{i=0}^{h-1} \beta^i(h+1,\lambda=0)}\right]} Q^\lambda_{h+\frac{1}{2}} + \frac{1}{\left[\frac{1}{2}\frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}} \frac{(-1)^{h+1}}{\sum_{i=0}^{h} \alpha^i(h+1,\lambda=0)}\right]} \bar{Q}^\lambda_{h+\frac{1}{2}} \right).
\end{aligned} \tag{3.30}
$$

Then we can calculate the commutator relation $[(V_\lambda^{(h_1),+})_m, (V_\lambda^{(h_2),+})_n]$. By substituting the first equation of (3.30) into this commutator, we obtain the following coefficient

[33]We thank M. Vasiliev for discussion on this matter further.

[34]It would be interesting to study this case in the context of the SYK models because there is no supersymmetry.

function of $(V_\lambda^{(h_1+h_2-2-h),-})_{m+n}$:

$$\left[\frac{1}{\frac{n_{W_{F,h_1}}}{q^{h_1-2}}\frac{(-1)^{h_1}}{\sum_{i=0}^{h_1-1}a^i(h_1,\frac{1}{2})}}\right]\left[\frac{1}{\frac{n_{W_{F,h_2}}}{q^{h_2-2}}\frac{(-1)^{h_2}}{\sum_{i=0}^{h_2-1}a^i(h_2,\frac{1}{2})}}\right]$$
$$\times\left[\frac{n_{W_{F,h_1+h_2-2-h}}}{q^{h_1+h_2-2-h-2}}\frac{(-1)^{h_1+h_2-2-h}}{\sum_{i=0}^{h_1+h_2-2-h-1}a^i(h_1+h_2-2-h,\frac{1}{2})}\right]q^h\,p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda)$$
$$-\left[\frac{1}{\frac{n_{W_{B,h_1}}}{q^{h_1-2}}\frac{(-1)^{h_1}}{\sum_{i=0}^{h_1-1}a^i(h_1,0)}}\right]\left[\frac{1}{\frac{n_{W_{B,h_2}}}{q^{h_2-2}}\frac{(-1)^{h_2}}{\sum_{i=0}^{h_2-1}a^i(h_2,0)}}\right]$$
$$\times\left[\frac{n_{W_{B,h_1+h_2-2-h}}}{q^{h_1+h_2-2-h-2}}\frac{(-1)^{h_1+h_2-2-h}}{\sum_{i=0}^{h_1+h_2-2-h-1}a^i(h_1+h_2-2-h,0)}\right]q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda). \quad (3.31)$$

Here $p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda)$ and $p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)$ are given by the first terms in (3.17). Now we can check the above coefficient (3.31) vanishes at $\lambda=\frac{1}{4}$ (corresponding to (3.27) in the $\mathcal{N}=2$ SYK models) implying that we can decouple the currents $V_\lambda^{(h),-}$ with even h and therefore, we do not have these currents in (3.28) or (3.29). For even h_1 and h_2, the combination (h_2+h_2-2) is even.

Similarly, the decoupling of the currents $V_\lambda^{(h),+}$ with odd h can be analyzed as follows. The commutator relation $[(V_\lambda^{(h_1),+})_m,(V_\lambda^{(h_2),-})_n]$ can be obtained by substituting the first and second equations of (3.30) into this commutator, we obtain the following coefficient function of $(V_\lambda^{(h_1+h_2-2-h),+})_{m+n}$ as follows:

$$\left[\frac{(h_2-2\lambda)}{(2h_2-1)}\right]\left[\frac{(h_1+h_2-2-h-1+2\lambda)}{2(h_1+h_2-2-h)-1}\right]\left[\frac{1}{\frac{n_{W_{F,h_1}}}{q^{h_1-2}}\frac{(-1)^{h_1}}{\sum_{i=0}^{h_1-1}a^i(h_1,\frac{1}{2})}}\right]\left[\frac{1}{\frac{n_{W_{F,h_2}}}{q^{h_2-2}}\frac{(-1)^{h_2}}{\sum_{i=0}^{h_2-1}a^i(h_2,\frac{1}{2})}}\right]$$
$$\times\left[\frac{n_{W_{F,h_1+h_2-2-h}}}{q^{h_1+h_2-2-h-2}}\frac{(-1)^{h_1+h_2-2-h}}{\sum_{i=0}^{h_1+h_2-2-h-1}a^i(h_1+h_2-2-h,\frac{1}{2})}\right]q^h\,p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda)$$
$$-\left[\frac{(h_2-1+2\lambda)}{(2h_2-1)}\right]\left[\frac{(h_1+h_2-2-h-2\lambda)}{2(h_1+h_2-2-h)-1}\right]\left[\frac{1}{\frac{n_{W_{B,h_1}}}{q^{h_1-2}}\frac{(-1)^{h_1}}{\sum_{i=0}^{h_1-1}a^i(h_1,0)}}\right]\left[\frac{1}{\frac{n_{W_{B,h_2}}}{q^{h_2-2}}\frac{(-1)^{h_2}}{\sum_{i=0}^{h_2-1}a^i(h_2,0)}}\right]$$
$$\times\left[\frac{n_{W_{B,h_1+h_2-2-h}}}{q^{h_1+h_2-2-h-2}}\frac{(-1)^{h_1+h_2-2-h}}{\sum_{i=0}^{h_1+h_2-2-h-1}a^i(h_1+h_2-2-h,0)}\right]q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda). \quad (3.32)$$

The above coefficient (3.32) vanishes at $\lambda=\frac{1}{4}$ and we can decouple the currents $V_\lambda^{(h),+}$ with odd h and therefore, we do not have these currents in (3.28) or (3.29). For even h_1 and odd h_2, the combination (h_2+h_2-2) is odd. In appendix E, we will see more details on this matter.

3.3.7 The relation with celestial holography

We have found the matrix generalization of the $\mathcal{N}=2$ supersymmetric W_∞ algebra [42] by adding the additional parameter λ. Then we can follow the procedure of [43] by using the topological twisting [44, 45]. The bosonic SU(N)-singlet current of weight h can be given by $W^{\lambda}_{B,h}$, $W^{\lambda}_{F,h}$, $\bar{\partial}\,W^{\lambda}_{B,h-1}$ and $\bar{\partial}\,W^{\lambda}_{F,h-1}$. The corresponding SU($N$)-adjoint current can be constructed by multiplying the SU(N) generators into the above four kinds of operators. For the fermionic currents, we take $Q^{\lambda}_{h+\frac{1}{2}}$ and $Q^{\lambda,\hat{A}}_{h+\frac{1}{2}}$. Then the seven OPEs between these currents (or the corresponding (anti)commutator relations) can be determined explicitly. The structure constants found in [6, 43] can be generalized to λ dependent ones where the explicit expressions are given by (3.17). When we apply the two-dimensional algebra to the $\mathcal{N}=1$ supersymmetric Einstein-Yang-Mills theory, it is crucial to realize that the mode dependent function (3.15) is obtained by performing the nontrivial contour integrals [14]. Then the OPEs between the graviton, the gravitino, the gluon and the gluino can be obtained and the corresponding structure constants are given in (3.17) with λ dependence.

4 Conclusions and outlook

We derived that the parameter of $\mathcal{N}=2$ SYK models can be realized by the one in the $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda]$ algebra through the equation (3.4). The complete results for the $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda]$ algebra are summarized by (3.21) and appendix (D.1).

It is an open problem to compare the present results with the ones in [22, 23] and to observe how they coincide with each other analytically. So far, we have considered the $\mathcal{N}=2$ SYK models and it is interesting to study the $\mathcal{N}=(0,2)$ SYK models and check whether there exists a higher spin realization or not. There is a partial work in [21] on the limit of $q_{\mathrm{syk}} \rightarrow \frac{N}{M}$ in this direction. It is also interesting problem to generalize the work of [46] to the case having the above $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda]$ algebra.

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A The SU(N)-adjoint higher spin currents

As done in (3.7), we can check that the SU(N)-adjoint higher spin currents at vanishing λ can be obtained as follows:

$$
\begin{aligned}
W^{\hat{A}}_{F,h} &= \frac{n_{W_{F,h}}}{q^{h-2}}\,\frac{(-1)^h}{\sum_{i=0}^{h-1}\,a^i(h,\lambda+\frac{1}{2}=\frac{1}{2})}\left[\frac{(h-1+2\lambda)}{(2h-1)}\,V^{(h)+}_{\lambda,\hat{A}}+V^{(h)-}_{\lambda,\hat{A}}\right]_{\lambda=0},\\
W^{\hat{A}}_{B,h} &= \frac{n_{W_{B,h}}}{q^{h-2}}\,\frac{(-1)^h}{\sum_{i=0}^{h-1}\,a^i(h,\lambda=0)}\left[\frac{(h-2\lambda)}{(2h-1)}\,V^{(h)+}_{\lambda,\hat{A}}-V^{(h)-}_{\lambda,\hat{A}}\right]_{\lambda=0},\\
Q^{\hat{A}}_{h+\frac{1}{2}} &= \frac{1}{2}\,\frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}}\,\frac{(-1)^{h+1}\,h}{\sum_{i=0}^{h-1}\,\beta^i(h+1,\lambda=0)}\left[Q^{(h+1)-}_{\lambda,\hat{A}}-Q^{(h+1)+}_{\lambda,\hat{A}}\right]_{\lambda=0},\\
\bar{Q}^{\hat{A}}_{h+\frac{1}{2}} &= \frac{1}{2}\,\frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}}\,\frac{(-1)^{h+1}}{\sum_{i=0}^{h}\,\alpha^i(h+1,\lambda=0)}\left[Q^{(h+1)-}_{\lambda,\hat{A}}+Q^{(h+1)+}_{\lambda,\hat{A}}\right]_{\lambda=0}.
\end{aligned}
\tag{A.1}
$$

Similarly, for nonzero λ, we take the following higher spin currents together with (3.2)

$$
\begin{aligned}
W^{\lambda,\hat{A}}_{F,h} &= \frac{n_{W_{F,h}}}{q^{h-2}} \frac{(-1)^h}{\sum_{i=0}^{h-1} a^i(h,\lambda+\frac{1}{2}=\frac{1}{2})} \left[\frac{(h-1+2\lambda)}{(2h-1)} V^{(h)+}_{\lambda,\hat{A}} + V^{(h)-}_{\lambda,\hat{A}}\right], \\
W^{\lambda,\hat{A}}_{B,h} &= \frac{n_{W_{B,h}}}{q^{h-2}} \frac{(-1)^h}{\sum_{i=0}^{h-1} a^i(h,\lambda=0)} \left[\frac{(h-2\lambda)}{(2h-1)} V^{(h)+}_{\lambda,\hat{A}} - V^{(h)-}_{\lambda,\hat{A}}\right], \\
Q^{\lambda,\hat{A}}_{h+\frac{1}{2}} &= \frac{1}{2} \frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}} \frac{(-1)^{h+1}\, h}{\sum_{i=0}^{h-1} \beta^i(h+1,\lambda=0)} \left[Q^{(h+1)-}_{\lambda,\hat{A}} - Q^{(h+1)+}_{\lambda,\hat{A}}\right], \\
\bar{Q}^{\lambda,\hat{A}}_{h+\frac{1}{2}} &= \frac{1}{2} \frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}} \frac{(-1)^{h+1}}{\sum_{i=0}^{h} \alpha^i(h+1,\lambda=0)} \left[Q^{(h+1)-}_{\lambda,\hat{A}} + Q^{(h+1)+}_{\lambda,\hat{A}}\right].
\end{aligned}
\tag{A.2}
$$

In the normalization of appendix (A.2), we take the same normalization of appendix (A.1). That is, the overall factor does not depend on the λ explicitly.

B The partial OPEs in the $\mathcal{N}=2$ supersymmetric linear $W^{N,N}_{\infty}[\lambda]$ algebra

We present six examples of (3.21) for fixed h_1 and h_2.

B.1 The OPE $W^{\lambda}_{B,4}(\bar{z})\, W^{\lambda}_{B,4}(\bar{w})$

From (3.11), (3.2) and (3.1), we can calculate the OPE between the weight-4 currents as follows:

$$
\begin{aligned}
& W^{\lambda}_{B,4}(\bar{z})\, W^{\lambda}_{B,4}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^8}\left[\frac{3072}{5}(28\lambda^6-84\lambda^5+35\lambda^4+70\lambda^3-42\lambda^2-7\lambda+3)\right] \\
&+\frac{1}{(\bar{z}-\bar{w})^6}\left[\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+1)\right] W^{\lambda}_{B,2}(\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^5}\frac{1}{2}\left[\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+1)\right] \bar{\partial}\, W^{\lambda}_{B,2}(\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^4}\left[\frac{3}{20}\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+1)\bar{\partial}^2\, W^{\lambda}_{B,2}\right. \\
&\left.-\frac{192}{5}(2\lambda^2-2\lambda-9)W^{\lambda}_{B,4}\right](\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^3}\left[\frac{1}{30}\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+1)\bar{\partial}^3 W^{\lambda}_{B,2}\right. \\
&\left.-\frac{1}{2}\frac{192}{5}(2\lambda^2-2\lambda-9)\bar{\partial}W^{\lambda}_{B,4}\right](\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^2}\left[\frac{1}{168}\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^4\, W^{\lambda}_{B,2}\right. \\
&\left.-\frac{5}{36}\frac{192}{5}(2\lambda^2-2\lambda-9)\,\bar{\partial}^2\, W^{\lambda}_{B,4}+6\, W^{\lambda}_{B,6}\right](\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})}\left[\frac{1}{1120}\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^5\, W^{\lambda}_{B,2}\right.
\end{aligned}
$$

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$$-\frac{1}{36}\,\frac{192}{5}(2\lambda^2-2\lambda-9)\,\bar{\partial}^3\,W^{\lambda}_{B,4}+\frac{1}{2}\,6\,\bar{\partial}\,W^{\lambda}_{B,6}\Bigg](\bar{w})+\cdots$$

$$=\frac{1}{(\bar{z}-\bar{w})^8}\left[\frac{3072}{5}(28\lambda^6-84\lambda^5+35\lambda^4+70\lambda^3-42\lambda^2-7\lambda+3)\right]$$

$$-p^{4,4}_{B,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,2}(\bar{w})}{(\bar{z}-\bar{w})}\right]-p^{4,4}_{B,2}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,4}(\bar{w})}{(\bar{z}-\bar{w})}\right]-p^{4,4}_{B,0}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,6}(\bar{w})}{(\bar{z}-\bar{w})}\right]$$

$$+\cdots. \tag{B.1}$$

It is straightforward to calculate this result because we are considering the linear algebra and collect each pole in terms of the various descendant terms and new quasiprimary operator inside the Thielemans package [26]. Then all the structure constants can be determined and depend on the λ explicitly as above. In appendix (B.1), we also present the structure constants in terms of (3.17) after inserting the derivatives. As we expect, up to minus sign, we observe that the above OPE behaves as the first and third terms of the second equation in (3.21) in the sense that there are three terms with correct mode dependent structure constants. Due to the $(-1)^{h-1}$ factor when we change from the second equation of (3.21) to the above OPE and there are $h=0,2,4$ cases which are even, we have all the minus signs in the above OPE.

Then how we can see the existence of the second term of the second equation of (3.21)? We can read off the weight from the condition $h=h_1+h_2-3$ with $h_1+h_2-2-h=1$ and we take $h_1=4$ and $h_2=3$. Then we observe the weight for the $W^{\lambda}_{B,h_1+h_2-2-h}$ is equal to 1. Then we can calculate the following OPE

$$W^{\lambda}_{B,4}(\bar{z})\,W^{\lambda}_{B,3}(\bar{w})=\frac{1}{(\bar{z}-\bar{w})^7}\Big[384(\lambda-1)\lambda(2\lambda-3)(2\lambda-1)(2\lambda+1)\Big]$$

$$+\frac{1}{(\bar{z}-\bar{w})^6}\Big[512(\lambda-1)\lambda(2\lambda-3)(2\lambda+1)\,W^{\lambda}_{B,1}\Big](\bar{w})$$

$$+\frac{1}{(\bar{z}-\bar{w})^5}\Big[512(\lambda-1)\lambda(2\lambda-3)(2\lambda+1)\,\bar{\partial}\,W^{\lambda}_{B,1}\Big](\bar{w})$$

$$+\frac{1}{(\bar{z}-\bar{w})^4}\left[\frac{1}{2}\,512(\lambda-1)\lambda(2\lambda-3)(2\lambda+1)\,\bar{\partial}^2\,W^{\lambda}_{B,1}-\frac{384}{5}(\lambda-2)(\lambda+1)\,W^{\lambda}_{B,3}\right](\bar{w})$$

$$+\frac{1}{(\bar{z}-\bar{w})^3}\left[\frac{1}{6}\,512(\lambda-1)\lambda(2\lambda-3)(2\lambda+1)\,\bar{\partial}^3\,W^{\lambda}_{B,1}-\frac{2}{3}\,\frac{384}{5}(\lambda-2)(\lambda+1)\,\bar{\partial}\,W^{\lambda}_{B,3}\right](\bar{w})$$

$$+\frac{1}{(\bar{z}-\bar{w})^2}\Bigg[\frac{1}{24}\,512(\lambda-1)\lambda(2\lambda-3)(2\lambda+1)\,\bar{\partial}^4\,W^{\lambda}_{B,1}-\frac{5}{21}\,\frac{384}{5}(\lambda-2)(\lambda+1)\,\bar{\partial}^2\,W^{\lambda}_{B,3}$$

$$+5\,W^{\lambda}_{B,5}\Bigg](\bar{w})$$

$$+\frac{1}{(\bar{z}-\bar{w})}\Bigg[\frac{1}{120}\,512(\lambda-1)\lambda(2\lambda-3)(2\lambda+1)\,\bar{\partial}^5\,W^{\lambda}_{B,1}-\frac{5}{84}\,\frac{384}{5}(\lambda-2)(\lambda+1)\,\bar{\partial}^3\,W^{\lambda}_{B,3}$$

$$+\frac{3}{5}\,5\,\bar{\partial}\,W^{\lambda}_{B,5}\Bigg](\bar{w})+\cdots$$

$$=\frac{1}{(\bar{z}-\bar{w})^7}\Big[384(\lambda-1)\lambda(2\lambda-3)(2\lambda-1)(2\lambda+1)\Big]$$

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$$-p^{4,3}_{B,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,1}(\bar{w})}{(\bar{z}-\bar{w})}\right]-p^{4,3}_{B,2}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,3}(\bar{w})}{(\bar{z}-\bar{w})}\right]-p^{4,3}_{B,0}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,5}(\bar{w})}{(\bar{z}-\bar{w})}\right]$$
$$+\cdots. \tag{B.2}$$

Therefore, in this example, we observe the three terms of the second equation of (3.21). Note that all the structure constants associated with the weight-1 current $W^{\lambda}_{B,1}(\bar{w})$ in appendix (B.2) contain the λ factor explicitly. This implies that when we take the vanishing limit for the λ, the weight-1 current and its descendant terms disappear.

B.2 The OPE $W^{\lambda}_{F,4}(\bar{z})\,Q^{\lambda}_{\frac{7}{2}}(\bar{w})$

Let us consider the third equation of (3.21) with $h_1=4$ and $h_2=3$. Then we obtain the following result as done before

$$\begin{aligned}
&W^{\lambda}_{F,4}(\bar{z})\,Q^{\lambda}_{\frac{7}{2}}(\bar{w})=\frac{1}{(\bar{z}-\bar{w})^6}\left[\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\right]Q^{\lambda}_{\frac{3}{2}}(\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^5}\left[\frac{2}{3}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\,\bar{\partial}\,Q^{\lambda}_{\frac{3}{2}}\right.\\
&\left.-\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,Q^{\lambda}_{\frac{5}{2}}\right](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^4}\left[\frac{1}{4}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\,\bar{\partial}^2\,Q^{\lambda}_{\frac{3}{2}}\right.\\
&\left.-\frac{3}{5}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}\,Q^{\lambda}_{\frac{5}{2}}-\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,Q^{\lambda}_{\frac{7}{2}}\right](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^3}\left[\frac{1}{15}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\,\bar{\partial}^3\,Q^{\lambda}_{\frac{3}{2}}\right.\\
&-\frac{1}{5}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}^2\,Q^{\lambda}_{\frac{5}{2}}-\frac{4}{7}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\bar{\partial}\,Q^{\lambda}_{\frac{7}{2}}\\
&\left.+\frac{4}{5}(2\lambda^2-5\lambda-27)\,Q^{\lambda}_{\frac{9}{2}}\right](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^2}\left[\frac{1}{72}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\,\bar{\partial}^4\,Q^{\lambda}_{\frac{3}{2}}\right.\\
&-\frac{1}{21}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}^3\,Q^{\lambda}_{\frac{5}{2}}-\frac{5}{28}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\bar{\partial}^2\,Q^{\lambda}_{\frac{7}{2}}\\
&\left.+\frac{5}{9}\frac{4}{5}(2\lambda^2-5\lambda-27)\,\bar{\partial}\,Q^{\lambda}_{\frac{9}{2}}+\frac{1}{10}(2\lambda+27)\,Q^{\lambda}_{\frac{11}{2}}\right](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})}\left[\frac{1}{420}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\,\bar{\partial}^5\,Q^{\lambda}_{\frac{3}{2}}\right.\\
&-\frac{1}{112}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}^4\,Q^{\lambda}_{\frac{5}{2}}\\
&-\frac{5}{126}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\bar{\partial}^3\,Q^{\lambda}_{\frac{7}{2}}\\
&\left.+\frac{1}{6}\frac{4}{5}(2\lambda^2-5\lambda-27)\,\bar{\partial}^2\,Q^{\lambda}_{\frac{9}{2}}+\frac{6}{11}\frac{1}{10}(2\lambda+27)\,\bar{\partial}\,Q^{\lambda}_{\frac{11}{2}}-\frac{1}{4}\,Q^{\lambda}_{\frac{13}{2}}\right](\bar{w})+\cdots
\end{aligned}$$

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$$
\begin{aligned}
&= -q_{F,4}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{3}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] + q_{F,3}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{5}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] - q_{F,2}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{7}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] \\
&+q_{F,1}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{9}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] - q_{F,0}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{11}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] + q_{F,-1}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{13}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] \\
&+\cdots . \qquad\qquad (\text{B.3})
\end{aligned}
$$

As we expect, there are six nontrivial terms with $h = -1$, 0, 1, 2, 3 and 4. Due to the previous overall factor $(-1)^{h-1}$, we have minus signs for the even h. For odd h we have plus signs in the above OPE in appendix (B.3).

B.3 The OPE $W^{\lambda}_{B,4}(\bar{z})\, Q^{\lambda}_{\frac{7}{2}}(\bar{w})$

Let us move on the fourth equation of (3.21) and we can calculate the corresponding OPE for $h_1 = 4$ and $h_2 = 3$ in our notation and this leads to the following OPE

$$
\begin{aligned}
&W^{\lambda}_{B,4}(\bar{z})\, Q^{\lambda}_{\frac{7}{2}}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^6}\left[-\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\right] Q^{\lambda}_{\frac{3}{2}}(\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^5}\left[-\frac{2}{3}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}\, Q^{\lambda}_{\frac{3}{2}}\right. \\
&\left.+\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\, Q^{\lambda}_{\frac{5}{2}}\right](\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^4}\left[-\frac{1}{4}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^2\, Q^{\lambda}_{\frac{3}{2}}\right. \\
&\left.+\frac{3}{5}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,\bar{\partial}\, Q^{\lambda}_{\frac{5}{2}} + \frac{32}{25}(\lambda-2)\left(4\lambda^2-22\lambda-51\right) Q^{\lambda}_{\frac{7}{2}}\right](\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^3}\left[-\frac{1}{15}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^3\, Q^{\lambda}_{\frac{3}{2}}\right. \\
&+\frac{1}{5}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,\bar{\partial}^2\, Q^{\lambda}_{\frac{5}{2}} + \frac{4}{7}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\bar{\partial}\, Q^{\lambda}_{\frac{7}{2}} \\
&\left.-\frac{4}{5}(2\lambda^2+3\lambda-29)\, Q^{\lambda}_{\frac{9}{2}}\right](\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^2}\left[-\frac{1}{72}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^4\, Q^{\lambda}_{\frac{3}{2}}\right. \\
&+\frac{1}{21}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,\bar{\partial}^3\, Q^{\lambda}_{\frac{5}{2}} + \frac{5}{28}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\bar{\partial}^2\, Q^{\lambda}_{\frac{7}{2}} \\
&\left.-\frac{5}{9}\frac{4}{5}(2\lambda^2+3\lambda-29)\,\bar{\partial}\, Q^{\lambda}_{\frac{9}{2}} + \frac{1}{5}(-\lambda+14)\, Q^{\lambda}_{\frac{11}{2}}\right](\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})}\left[-\frac{1}{420}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^5\, Q^{\lambda}_{\frac{3}{2}}\right. \\
&+\frac{1}{112}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,\bar{\partial}^4\, Q^{\lambda}_{\frac{5}{2}} + \frac{5}{126}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\bar{\partial}^3\, Q^{\lambda}_{\frac{7}{2}} \\
&\left.-\frac{1}{6}\frac{4}{5}(2\lambda^2+3\lambda-29)\,\bar{\partial}^2\, Q^{\lambda}_{\frac{9}{2}} + \frac{6}{11}\frac{1}{5}(-\lambda+14)\,\bar{\partial}\, Q^{\lambda}_{\frac{11}{2}} + \frac{1}{4}\, Q^{\lambda}_{\frac{13}{2}}\right](\bar{w}) + \cdots
\end{aligned}
$$

$$= -q_{B,4}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{3}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] + q_{B,3}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{5}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] - q_{B,2}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{7}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]$$
$$+q_{B,1}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{9}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] - q_{B,0}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{11}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] + q_{B,-1}^{4,\frac{7}{2}}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda)\left[\frac{Q^{\lambda}_{\frac{13}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]$$
$$+\cdots . \tag{B.4}$$

Again, the alternating signs appear due to the previous analysis in appendix (B.4). For even h, there is a minus sign.

B.4 The OPE $W^{\lambda}_{F,4}(\bar{z})\,\bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w})$

Let us continue to calculate the OPE associated with the fifth equation of (3.21) and it turns out that

$$W^{\lambda}_{F,4}(\bar{z})\,\bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^7}\left[-\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\right]\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^6}\left[-\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{1}{2}}\right.$$
$$\left.+\frac{512}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\,\bar{Q}^{\lambda}_{\frac{3}{2}}\right](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^5}\left[-\frac{1}{2}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{1}{2}}\right.$$
$$+\frac{2}{3}\frac{512}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{3}{2}}$$
$$\left.+\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{Q}^{\lambda}_{\frac{5}{2}}\right](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^4}\left[-\frac{1}{6}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}^3\,\bar{Q}^{\lambda}_{\frac{1}{2}}\right.$$
$$\frac{1}{4}\frac{512}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{3}{2}}$$
$$\left.+\frac{3}{5}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{5}{2}} - \frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\bar{Q}^{\lambda}_{\frac{7}{2}}\right](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^3}\left[-\frac{1}{24}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}^4\,\bar{Q}^{\lambda}_{\frac{1}{2}}\right.$$
$$+\frac{1}{15}\frac{512}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\,\bar{\partial}^3\,\bar{Q}^{\lambda}_{\frac{3}{2}}$$
$$+\frac{1}{5}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{5}{2}} - \frac{4}{7}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{7}{2}}$$
$$\left.-\frac{4}{5}(2\lambda^2-5\lambda-27)\,\bar{Q}^{\lambda}_{\frac{9}{2}}\right](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^2}\left[-\frac{1}{120}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}^5\,\bar{Q}^{\lambda}_{\frac{1}{2}}\right.$$
$$+\frac{1}{72}\frac{512}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\,\bar{\partial}^4\,\bar{Q}^{\lambda}_{\frac{3}{2}}$$
$$+\frac{1}{21}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}^3\,\bar{Q}^{\lambda}_{\frac{5}{2}} - \frac{5}{28}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{7}{2}}$$

$$
\begin{aligned}
&-\frac{5}{9}\frac{4}{5}(2\lambda^2-5\lambda-27)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{9}{2}}+\frac{1}{10}(2\lambda+27)\,\bar{Q}^{\lambda}_{\frac{11}{2}}\Bigg](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})}\Bigg[-\frac{1}{720}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}^6\,\bar{Q}^{\lambda}_{\frac{1}{2}}\\
&+\frac{1}{420}\frac{512}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)\,\bar{\partial}^5\,\bar{Q}^{\lambda}_{\frac{3}{2}}\\
&+\frac{1}{112}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\bar{\partial}^4\,\bar{Q}^{\lambda}_{\frac{5}{2}}-\frac{5}{126}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\bar{\partial}^3\,\bar{Q}^{\lambda}_{\frac{7}{2}}\\
&-\frac{1}{6}\frac{4}{5}(2\lambda^2-5\lambda-27)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{9}{2}}+\frac{6}{11}\frac{1}{10}(2\lambda+27)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{11}{2}}+\frac{1}{4}\,\bar{Q}^{\lambda}_{\frac{13}{2}}\Bigg](\bar{w})+\cdots\\
&=-q^{4,\frac{7}{2}}_{F,5}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]-q^{4,\frac{7}{2}}_{F,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{3}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]-q^{4,\frac{7}{2}}_{F,3}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{5}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]\\
&-q^{4,\frac{7}{2}}_{F,2}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]-q^{4,\frac{7}{2}}_{F,1}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{9}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]-q^{4,\frac{7}{2}}_{F,0}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{11}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]\\
&-q^{4,\frac{7}{2}}_{F,-1}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{13}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]+\cdots. \qquad \text{(B.5)}
\end{aligned}
$$

Note that in the fifth equation of (3.21), there exists $(-1)^h$ factor. So when we write down the OPE as above, this factor combines with the previous factor $(-1)^{h-1}$. This implies that there are no h dependence in the (-1) factor. We are left with the final (-1) factor which appears in appendix (B.5). As described before, the structure constants associated with the weight-$\frac{1}{2}$ current contain the λ factor and this leads to the fact that this weight-$\frac{1}{2}$ current disappear when we take the vanishing λ limit.[35]

B.5 The OPE $W^{\lambda}_{B,4}(\bar{z})\,\bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w})$

For the sixth equation of (3.21), we can calculate the following OPE

$$
\begin{aligned}
&W^{\lambda}_{B,4}(\bar{z})\,\bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w})=\frac{1}{(\bar{z}-\bar{w})^7}\left[\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\right]\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^6}\Bigg[\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{1}{2}}\\
&-\frac{1024}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{Q}^{\lambda}_{\frac{3}{2}}\Bigg](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^5}\Bigg[\frac{1}{2}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{1}{2}}\\
&-\frac{2}{3}\frac{1024}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{3}{2}}\\
&-\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,\bar{Q}^{\lambda}_{\frac{5}{2}}\Bigg](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^4}\Bigg[\frac{1}{6}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}^3\,\bar{Q}^{\lambda}_{\frac{1}{2}}
\end{aligned}
$$

[35] There is a factor $(2\lambda-1)$ which vanishes at $\lambda=\frac{1}{2}$. See also the footnote 28.

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$$
\begin{aligned}
&-\frac{1}{4}\frac{1024}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{3}{2}} \\
&-\frac{3}{5}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{5}{2}}+\frac{32}{25}(\lambda-2)\left(4\lambda^2-22\lambda-51\right)\,\bar{Q}^{\lambda}_{\frac{7}{2}}\Bigg](\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^3}\Bigg[\frac{1}{24}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}^4\,\bar{Q}^{\lambda}_{\frac{1}{2}} \\
&-\frac{1}{15}\frac{1024}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^3\,\bar{Q}^{\lambda}_{\frac{3}{2}} \\
&-\frac{1}{5}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{5}{2}}+\frac{4}{7}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{7}{2}} \\
&+\frac{4}{5}(2\lambda^2+3\lambda-29)\,\bar{Q}^{\lambda}_{\frac{9}{2}}\Bigg](\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})^2}\Bigg[\frac{1}{120}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}^5\,\bar{Q}^{\lambda}_{\frac{1}{2}} \\
&-\frac{1}{72}\frac{1024}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^4\,\bar{Q}^{\lambda}_{\frac{3}{2}} \\
&-\frac{1}{21}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,\bar{\partial}^3\,\bar{Q}^{\lambda}_{\frac{5}{2}}+\frac{5}{28}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{7}{2}} \\
&+\frac{5}{9}\frac{4}{5}(2\lambda^2+3\lambda-29)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{9}{2}}+\frac{1}{5}(-\lambda+14)\,\bar{Q}^{\lambda}_{\frac{11}{2}}\Bigg](\bar{w}) \\
&+\frac{1}{(\bar{z}-\bar{w})}\Bigg[\frac{1}{720}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)\,\bar{\partial}^6\,\bar{Q}^{\lambda}_{\frac{1}{2}} \\
&-\frac{1}{420}\frac{1024}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^5\,\bar{Q}^{\lambda}_{\frac{3}{2}} \\
&-\frac{1}{112}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,\bar{\partial}^4\,\bar{Q}^{\lambda}_{\frac{5}{2}}+\frac{5}{126}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\bar{\partial}^3\,\bar{Q}^{\lambda}_{\frac{7}{2}} \\
&+\frac{1}{6}\frac{4}{5}(2\lambda^2+3\lambda-29)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{9}{2}}+\frac{6}{11}\frac{1}{5}(-\lambda+14)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{11}{2}}-\frac{1}{4}\,\bar{Q}^{\lambda}_{\frac{13}{2}}\Bigg](\bar{w})+\cdots \\
&=-q^{4,\frac{7}{2}}_{B,5}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]-q^{4,\frac{7}{2}}_{B,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{3}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]-q^{4,\frac{7}{2}}_{B,3}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{5}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] \\
&-q^{4,\frac{7}{2}}_{B,2}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]-q^{4,\frac{7}{2}}_{B,1}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{9}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]-q^{4,\frac{7}{2}}_{B,0}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{11}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] \\
&-q^{4,\frac{7}{2}}_{B,-1}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{13}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]+\cdots\,.
\end{aligned}
\tag{B.6}
$$

In appendix (B.6), there are overall minus signs as mentioned before because the sixth equation of (3.21) has the $(-1)^h$ factor. Again, the presence of the weight-$\frac{1}{2}$ current appears at the nonzero λ.[36]

[36]We observe that there is a factor $(2\lambda-1)$ which vanishes at $\lambda=\frac{1}{2}$. See the footnote 28.

B.6 The OPE $Q^{\lambda}_{\frac{7}{2}}(\bar{z})\,\bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w})$

Now let us look at the final equation of (3.21) and we consider the following OPE for $h_1 = h_2 = 3$ in our notation

$$
\begin{aligned}
& Q^{\lambda}_{\frac{7}{2}}(\bar{z})\bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^7}\Bigg[-\frac{3072}{5}(4\lambda-1)(12\lambda^4-12\lambda^3-13\lambda^2+8\lambda+3)\Bigg] \\
& +\frac{1}{(\bar{z}-\bar{w})^6}\Bigg[\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda-1)W^{\lambda}_{F,1} \\
& +\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)W^{\lambda}_{B,1}\Bigg](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^5}\Bigg[\frac{1}{2}\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda-1)\bar{\partial}W^{\lambda}_{F,1} \\
& +\frac{1}{2}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\bar{\partial}W^{\lambda}_{B,1} \\
& +\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)W^{\lambda}_{F,2} \\
& +\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)W^{\lambda}_{B,2}\Bigg](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^4}\Bigg[\frac{1}{6}\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda-1)\bar{\partial}^2W^{\lambda}_{F,1} \\
& +\frac{1}{6}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\bar{\partial}^2W^{\lambda}_{B,1} \\
& +\frac{1}{2}\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\bar{\partial}W^{\lambda}_{F,2} \\
& +\frac{1}{2}\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\bar{\partial}W^{\lambda}_{B,2} \\
& -\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)W^{\lambda}_{F,3}-\frac{512}{25}(\lambda+1)(4\lambda^2+2\lambda-27)W^{\lambda}_{B,3}\Bigg](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^3}\Bigg[\frac{1}{24}\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda-1)\bar{\partial}^3W^{\lambda}_{F,1} \\
& +\frac{1}{24}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\bar{\partial}^3W^{\lambda}_{B,1} \\
& +\frac{3}{20}\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\bar{\partial}^2W^{\lambda}_{F,2} \\
& +\frac{3}{20}\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\bar{\partial}^2W^{\lambda}_{B,2} \\
& -\frac{1}{2}\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)\bar{\partial}W^{\lambda}_{F,3}-\frac{1}{2}\frac{512}{25}(\lambda+1)(4\lambda^2+2\lambda-27)\bar{\partial}W^{\lambda}_{B,3} \\
& -\frac{64}{25}(4\lambda^2+18\lambda-61)W^{\lambda}_{F,4}-\frac{64}{25}(4\lambda^2-22\lambda-51)W^{\lambda}_{B,4}\Bigg](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^2}\Bigg[\frac{1}{120}\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda-1)\bar{\partial}^4W^{\lambda}_{F,1} \\
& +\frac{1}{120}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\bar{\partial}^4W^{\lambda}_{B,1} \\
& +\frac{1}{30}\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\bar{\partial}^3W^{\lambda}_{F,2}
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{1}{30}\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\bar{\partial}^3 W^{\lambda}_{B,2}\\
&-\frac{1}{7}\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)\bar{\partial}^2 W^{\lambda}_{F,3}-\frac{1}{7}\frac{512}{25}(\lambda+1)(4\lambda^2+2\lambda-27)\bar{\partial}^2 W^{\lambda}_{B,3}\\
&-\frac{1}{2}\frac{64}{25}(4\lambda^2+18\lambda-61)\bar{\partial} W^{\lambda}_{F,4}-\frac{1}{2}\frac{64}{25}(4\lambda^2-22\lambda-51)\bar{\partial} W^{\lambda}_{B,4}\\
&+\frac{8}{5}(2\lambda-13)W^{\lambda}_{F,5}+\frac{16(\lambda+6)}{5}W^{\lambda}_{B,5}\Bigg](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})}\Bigg[\frac{1}{720}\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda-1)\bar{\partial}^5 W^{\lambda}_{F,1}\\
&+\frac{1}{720}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\bar{\partial}^5 W^{\lambda}_{B,1}\\
&+\frac{1}{168}\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\bar{\partial}^4 W^{\lambda}_{F,2}\\
&+\frac{1}{168}\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\bar{\partial}^4 W^{\lambda}_{B,2}\\
&-\frac{5}{168}\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)\bar{\partial}^3 W^{\lambda}_{F,3}-\frac{5}{168}\frac{512}{25}(\lambda+1)(4\lambda^2+2\lambda-27)\bar{\partial}^3 W^{\lambda}_{B,3}\\
&-\frac{5}{36}\frac{64}{25}(4\lambda^2+18\lambda-61)\bar{\partial}^2 W^{\lambda}_{F,4}-\frac{5}{36}\frac{64}{25}(4\lambda^2-22\lambda-51)\bar{\partial}^2 W^{\lambda}_{B,4}\\
&+\frac{1}{2}\frac{8}{5}(2\lambda-13)\bar{\partial} W^{\lambda}_{F,5}+\frac{1}{2}\frac{16(\lambda+6)}{5}\bar{\partial} W^{\lambda}_{B,5}+2W^{\lambda}_{F,6}+2W^{\lambda}_{B,6}\Bigg](\bar{w})+\cdots\\
&=\frac{1}{(\bar{z}-\bar{w})^7}\left[-\frac{3072}{5}(4\lambda-1)(12\lambda^4-12\lambda^3-13\lambda^2+8\lambda+3)\right]\\
&-o^{\frac{7}{2},\frac{7}{2}}_{F,5}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,1}(\bar{w})}{(\bar{z}-\bar{w})}\right]-o^{\frac{7}{2},\frac{7}{2}}_{B,5}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,1}(\bar{w})}{(\bar{z}-\bar{w})}\right]+o^{\frac{7}{2},\frac{7}{2}}_{F,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,2}(\bar{w})}{(\bar{z}-\bar{w})}\right]\\
&+o^{\frac{7}{2},\frac{7}{2}}_{B,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,2}(\bar{w})}{(\bar{z}-\bar{w})}\right]-o^{\frac{7}{2},\frac{7}{2}}_{F,3}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,3}(\bar{w})}{(\bar{z}-\bar{w})}\right]-o^{\frac{7}{2},\frac{7}{2}}_{B,3}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,3}(\bar{w})}{(\bar{z}-\bar{w})}\right]\\
&+o^{\frac{7}{2},\frac{7}{2}}_{F,2}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,4}(\bar{w})}{(\bar{z}-\bar{w})}\right]+o^{\frac{7}{2},\frac{7}{2}}_{B,2}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,4}(\bar{w})}{(\bar{z}-\bar{w})}\right]-o^{\frac{7}{2},\frac{7}{2}}_{F,1}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,5}(\bar{w})}{(\bar{z}-\bar{w})}\right]\\
&-o^{\frac{7}{2},\frac{7}{2}}_{B,1}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,5}(\bar{w})}{(\bar{z}-\bar{w})}\right]+o^{\frac{7}{2},\frac{7}{2}}_{F,0}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{F,6}(\bar{w})}{(\bar{z}-\bar{w})}\right]+o^{\frac{7}{2},\frac{7}{2}}_{B,0}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{W^{\lambda}_{B,6}(\bar{w})}{(\bar{z}-\bar{w})}\right]+\cdots.
\end{aligned}
\tag{B.7}
$$

There appear four different kinds of terms in appendix (B.7). In this case, for even h, there are plus signs. Note that there is a factor $(4\lambda-1)$ which vanishes at $\lambda=\frac{1}{4}$. Moreover, the λ factor appears in the weight-1 current $W^{\lambda}_{B,1}(\bar{w})$ (and its descendant terms).[37]

C Some OPEs containing the $\bar{Q}^{\lambda}_{\frac{1}{2}}$ or the $W^{\lambda}_{B,1}$

We consider the OPEs corresponding to the (anti)commutator relations in (3.21) where the left hand sides contain $\bar{Q}^{\lambda}_{\frac{1}{2}}$ or the $W^{\lambda}_{B,1}$. For the first four cases, we have $\bar{Q}^{\lambda}_{\frac{1}{2}}$ and for the remaining ones we have $W^{\lambda}_{B,1}$.

[37] There is a factor $(2\lambda-1)$ which vanishes at $\lambda=\frac{1}{2}$ in the weight-1 current $W^{\lambda}_{F,1}(\bar{w})$ (and its descendant terms). See also the footnote 28.

C.1 The OPE $W^{\lambda}_{F,4}(\bar{z})\,\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})$

Let us consider the fifth equation of (3.21). We calculate the following OPE by taking the second current as $\bar{Q}^{\lambda}_{\frac{1}{2}}$

$$
\begin{aligned}
& W^{\lambda}_{F,4}(\bar{z})\,\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w}) = -\frac{1}{(\bar{z}-\bar{w})^4}\,\frac{8}{5}(\lambda-1)(2\lambda-3)(2\lambda-1)\,\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^3}\left[-4\,\frac{8}{5}(\lambda-1)(2\lambda-3)(2\lambda-1)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{1}{2}}+\frac{16}{5}(\lambda-1)(2\lambda-3)\,\bar{Q}^{\lambda}_{\frac{3}{2}}\right](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^2}\left[-5\,\frac{8}{5}(\lambda-1)(2\lambda-3)(2\lambda-1)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{1}{2}}+\frac{5}{3}\,\frac{16}{5}(\lambda-1)(2\lambda-3)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{3}{2}}\right. \\
& \left.+(3-2\lambda)\,\bar{Q}^{\lambda}_{\frac{5}{2}}\right](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})}\left[-\frac{10}{3}\,\frac{8}{5}(\lambda-1)(2\lambda-3)(2\lambda-1)\,\bar{\partial}^3\,\bar{Q}^{\lambda}_{\frac{1}{2}}+\frac{5}{4}\,\frac{16}{5}(\lambda-1)(2\lambda-3)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{3}{2}}\right. \\
& \left.+\frac{6}{5}\,(3-2\lambda)\,\bar{Q}^{\lambda}_{\frac{5}{2}}+\frac{1}{2}\,\bar{Q}^{\lambda}_{\frac{7}{2}}\right](\bar{w})+\cdots \\
& = -q^{4,\frac{1}{2}}_{F,2}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]-2q^{4,\frac{1}{2}}_{F,1}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{3}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]-2q^{4,\frac{1}{2}}_{F,0}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{5}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right] \\
& -2q^{4,\frac{1}{2}}_{F,-1}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}},\lambda)\left[\frac{\bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w})}{(\bar{z}-\bar{w})}\right]+\cdots\,. \qquad \text{(C.1)}
\end{aligned}
$$

It turns out that we can express the OPE in appendix (C.1) as the one in appendix (B.5) except the numerical values -2 rather than -1. If we rescale $\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})$ by $\frac{1}{2}$, then there appears 2 in the first term of appendix (B.5) and the above 2's in appendix (C.1) disappear. Even at $\lambda=0$, this OPE arises.

C.2 The OPE $W^{\lambda}_{B,4}(\bar{z})\,\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})$

Let us consider the sixth equation of (3.21). We calculate the following OPE by taking the second current as $\bar{Q}^{\lambda}_{\frac{1}{2}}$

$$
\begin{aligned}
& W^{\lambda}_{B,4}(\bar{z})\,\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^4}\,\frac{16}{5}\lambda(\lambda+1)(2\lambda+1)\,\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^3}\left[4\,\frac{16}{5}\lambda(\lambda+1)(2\lambda+1)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{1}{2}}-\frac{16}{5}(\lambda+1)(2\lambda+1)\,\bar{Q}^{\lambda}_{\frac{3}{2}}\right](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^2}\left[5\,\frac{16}{5}\lambda(\lambda+1)(2\lambda+1)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{1}{2}}-\frac{5}{3}\,\frac{16}{5}\,(\lambda+1)(2\lambda+1)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{3}{2}}\right. \\
& \left.+2(1+\lambda)\,\bar{Q}^{\lambda}_{\frac{5}{2}}\right](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})}\left[\frac{10}{3}\,\frac{16}{5}\lambda(\lambda+1)(2\lambda+1)\,\bar{\partial}^3\,\bar{Q}^{\lambda}_{\frac{1}{2}}-5\,\frac{16}{5}\,(\lambda+1)(2\lambda+1)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{3}{2}}\right. \\
& \left.+\frac{6}{5}\,2(1+\lambda)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{5}{2}}-\frac{1}{2}\,\bar{Q}^{\lambda}_{\frac{7}{2}}\right](\bar{w})+\cdots
\end{aligned}
$$

$$= -q^{4,\frac{1}{2}}_{B,2}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda) \left[\frac{\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})}{(\bar{z}-\bar{w})} \right] - 2q^{4,\frac{1}{2}}_{B,1}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda) \left[\frac{\bar{Q}^{\lambda}_{\frac{3}{2}}(\bar{w})}{(\bar{z}-\bar{w})} \right] - 2q^{4,\frac{1}{2}}_{B,0}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda) \left[\frac{\bar{Q}^{\lambda}_{\frac{5}{2}}(\bar{w})}{(\bar{z}-\bar{w})} \right]$$
$$-2q^{4,\frac{1}{2}}_{B,-1}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda) \left[\frac{\bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w})}{(\bar{z}-\bar{w})} \right] + \cdots . \tag{C.2}$$

The OPE in appendix (C.2) looks similar to the one in appendix (B.6). Again, by the rescaling of the $\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})$, we can do the previous analysis. Due to the λ factor in front of $\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})$, we observe that this term (and its descendant terms) vanishes at $\lambda = 0$.

C.3 The OPE $Q^{\lambda}_{\frac{7}{2}}(\bar{z})\, \bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})$

Let us consider the final equation of (3.21), where the corresponding second current is given by $\bar{Q}^{\lambda}_{\frac{1}{2}}$. It turns out that

$$Q^{\lambda}_{\frac{7}{2}}(\bar{z})\, \bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^4} \left[-\frac{48}{5}(6\lambda^2 - 3\lambda + 1) \right]$$
$$+ \frac{1}{(\bar{z}-\bar{w})^3} \left[\frac{64}{5}(\lambda+1)(2\lambda+1)\, W^{\lambda}_{F,1} + \frac{64}{5}(\lambda-1)(2\lambda-3)\, W^{\lambda}_{B,1} \right](\bar{w})$$
$$+ \frac{1}{(\bar{z}-\bar{w})^2} \left[8\, \frac{64}{5}(\lambda+1)(2\lambda+1)\, \bar{\partial}\, W^{\lambda}_{F,1} + 8\, \frac{64}{5}(\lambda-1)(2\lambda-3)\, \bar{\partial}\, W^{\lambda}_{B,1} \right.$$
$$\left. + \frac{8}{5}\,(1+\lambda)\, W^{\lambda}_{F,2} - \frac{32}{5}(2\lambda-3)\, W^{\lambda}_{B,2} \right](\bar{w})$$
$$+ \frac{1}{(\bar{z}-\bar{w})} \left[\frac{20}{3}\, \frac{64}{5}(\lambda+1)(2\lambda+1)\, \bar{\partial}^2\, W^{\lambda}_{F,1} + \frac{20}{3}\, \frac{64}{5}(\lambda-1)(2\lambda-3)\, \bar{\partial}^2\, W^{\lambda}_{B,1} \right.$$
$$\left. + \frac{5}{4}\, \frac{8}{5}\,(1+\lambda)\, \bar{\partial}\, W^{\lambda}_{F,2} - \frac{5}{4}\, \frac{32}{5}(2\lambda-3)\, \bar{\partial}\, W^{\lambda}_{B,2} + 4\, W^{\lambda}_{F,3} + 4\, W^{\lambda}_{B,3} \right](\bar{w}) + \cdots$$
$$= 2o^{\frac{7}{2},\frac{1}{2}}_{F,2}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda) \left[\frac{W^{\lambda}_{F,1}(\bar{w})}{(\bar{z}-\bar{w})} \right] + 2o^{\frac{7}{2},\frac{1}{2}}_{B,2}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda) \left[\frac{W^{\lambda}_{B,1}(\bar{w})}{(\bar{z}-\bar{w})} \right] - 2o^{\frac{7}{2},\frac{1}{2}}_{F,1}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda) \left[\frac{W^{\lambda}_{F,2}(\bar{w})}{(\bar{z}-\bar{w})} \right]$$
$$-2o^{\frac{7}{2},\frac{1}{2}}_{B,1}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda) \left[\frac{W^{\lambda}_{B,2}(\bar{w})}{(\bar{z}-\bar{w})} \right] + 2o^{\frac{7}{2},\frac{1}{2}}_{F,0}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda) \left[\frac{W^{\lambda}_{F,3}(\bar{w})}{(\bar{z}-\bar{w})} \right] + 2o^{\frac{7}{2},\frac{1}{2}}_{B,0}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda) \left[\frac{W^{\lambda}_{B,3}(\bar{w})}{(\bar{z}-\bar{w})} \right]$$
$$+ \cdots . \tag{C.3}$$

By multiplying $\frac{1}{2}$ both sides, then we can absorb the numerical value 2 in the right hand side of appendix (C.3). Then the behavior of this OPE looks similar to the one in appendix (B.7).

C.4 The OPE $W^{\lambda}_{B,1}(\bar{z})\, \bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})$

When we take the first current as $W_{B,1}$ further corresponding to the sixth equation of (3.21) then we obtain

$$W^{\lambda}_{B,1}(\bar{z})\, \bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})} \left[-\frac{1}{4}\, \bar{Q}^{\lambda}_{\frac{1}{2}} \right](\bar{w}) + \cdots$$
$$= -q^{1,\frac{1}{2}}_{B,-1}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}, \lambda) \left[\frac{\bar{Q}^{\lambda}_{\frac{1}{2}}(\bar{w})}{(\bar{z}-\bar{w})} \right] + \cdots . \tag{C.4}$$

Now we see that the similar behavior in appendix (C.4) arises, compared to the ones in appendix (B.6) or appendix (C.2).

From now on, we consider that the first current is given by $W^{\lambda}_{B,1}$.

C.5 The OPE $W^{\lambda}_{B,1}(\bar{z})\, W^{\lambda}_{B,1}(\bar{w})$

We obtain the following OPE corresponding to the second equation of (3.21)

$$W^{\lambda}_{B,1}(\bar{z})\, W^{\lambda}_{B,1}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^2}\left[-\frac{3}{16}\right] + \cdots . \tag{C.5}$$

C.6 The OPE $W^{\lambda}_{B,1}(\bar{z})\, W^{\lambda}_{B,4}(\bar{w})$

When the second current is given by $W^{\lambda}_{B,4}(\bar{w})$, we obtain the following OPE

$$\begin{aligned}
& W^{\lambda}_{B,1}(\bar{z})\, W^{\lambda}_{B,4}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^5}\left[-\frac{24}{5}(2\lambda-1)(2\lambda^2-2\lambda+1)\right] \\
& +\frac{1}{(\bar{z}-\bar{w})^4}\left[\frac{96}{5}(2\lambda^2-2\lambda+1)W^{\lambda}_{B,1}\right](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^3}\left[-\frac{96}{5}(2\lambda^2-2\lambda+1)\,\bar{\partial}\,W^{\lambda}_{B,1}+8(2\lambda-1)\,W^{\lambda}_{B,2}\right](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^2}\left[\frac{1}{8}\,\frac{96}{5}(2\lambda^2-2\lambda+1)\,\bar{\partial}^2\,W^{\lambda}_{B,1}-\frac{1}{4}8(2\lambda-1)\,\bar{\partial}\,W^{\lambda}_{B,2}+3\,W^{\lambda}_{B,3}\right](\bar{w}) \\
& +\cdots .
\end{aligned} \tag{C.6}$$

C.7 The OPE $W^{\lambda}_{B,1}(\bar{z})\, Q^{\lambda}_{\frac{7}{2}}(\bar{w})$

For the fourth equation of (3.21), we can calculate the following OPE

$$\begin{aligned}
& W^{\lambda}_{B,1}(\bar{z})\, Q^{\lambda}_{\frac{7}{2}}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^3}\left[-\frac{4}{5}(\lambda+1)(2\lambda+1)\,Q^{\lambda}_{\frac{3}{2}}\right](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^2}\left[\frac{1}{3}\,\frac{4}{5}(\lambda+1)(2\lambda+1)\,\bar{\partial}\,Q^{\lambda}_{\frac{3}{2}}-\frac{4}{5}(\lambda+1)\,Q^{\lambda}_{\frac{5}{2}}\right](\bar{w})+\frac{1}{(\bar{z}-\bar{w})}\left[\frac{1}{4}\,Q^{\lambda}_{\frac{7}{2}}\right](\bar{w}) \\
& +\cdots .
\end{aligned} \tag{C.7}$$

C.8 The OPE $W^{\lambda}_{B,1}(\bar{z})\, \bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w})$

For the sixth equation of (3.21), we can calculate the following OPE

$$\begin{aligned}
& W^{\lambda}_{B,1}(\bar{z})\, \bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^4}\left[-\frac{8}{5}(\lambda-1)(2\lambda-3)(2\lambda-1)\,\bar{Q}^{\lambda}_{\frac{1}{2}}\right](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^3}\left[2\,\frac{8}{5}(\lambda-1)(2\lambda-3)(2\lambda-1)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{1}{2}}-\frac{12}{5}(\lambda-1)(2\lambda-3)\,\bar{Q}^{\lambda}_{\frac{3}{2}}\right](\bar{w}) \\
& +\frac{1}{(\bar{z}-\bar{w})^2}\left[\frac{1}{2}\,\frac{8}{5}(\lambda-1)(2\lambda-3)(2\lambda-1)\,\bar{\partial}^2\,\bar{Q}^{\lambda}_{\frac{1}{2}}+\frac{1}{3}\,\frac{12}{5}(\lambda-1)(2\lambda-3)\,\bar{\partial}\,\bar{Q}^{\lambda}_{\frac{3}{2}}\right. \\
& \left.-\frac{3}{5}(2\lambda-3)\,\bar{Q}^{\lambda}_{\frac{5}{2}}\right](\bar{w})-\frac{1}{(\bar{z}-\bar{w})}\,\frac{1}{4}\,\bar{Q}^{\lambda}_{\frac{7}{2}}(\bar{w})+\cdots .
\end{aligned} \tag{C.8}$$

In appendices (C.5), (C.6), (C.7), and (C.8), we cannot express the structure constants in terms of (3.17).

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D The remaining (anti)commutator relations of $\mathcal{N}=2$ supersymmetric linear $W_\infty^{N,N}[\lambda]$ algebra

The remaining 19 (anti)commutator relations are determined by

$$
\begin{aligned}
[(W^{\lambda}_{\mathrm{F},h_1})_m,(W^{\lambda,\hat{A}}_{\mathrm{F},h_2})_n] =& \sum_{h=0,\mathrm{even}}^{h_1+h_2-3} q^h p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda)(W^{\lambda,\hat{A}}_{\mathrm{F},h_1+h_2-2-h})_{m+n},\\
[(W^{\lambda,\hat{A}}_{\mathrm{F},h_1})_m,(W^{\lambda,\hat{B}}_{\mathrm{F},h_2})_n] =& -\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3} q^h p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{\mathrm{F},h_1+h_2-2-h})_{m+n}\\
&+c_{W_{\mathrm{F},h_1}}(m,\lambda)\delta^{\hat{A}\hat{B}}\delta^{h_1h_2}q^{2(h_1-2)}\delta_{m+n}\\
&+\sum_{h=0,\mathrm{even}}^{h_1+h_2-3} q^h p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda)\Big(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{\mathrm{F},h_1+h_2-2-h})_{m+n}\\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(W^{\lambda}_{\mathrm{F},h_1+h_2-2-h})_{m+n}\Big),\\
[(W^{\lambda}_{\mathrm{B},h_1})_m,(W^{\lambda,\hat{A}}_{\mathrm{B},h_2})_n] =& \sum_{h=0,\mathrm{even}}^{h_1+h_2-4} q^h p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)(W^{\lambda,\hat{A}}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\\
&+\Big[q^h p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)(W^{\lambda,\hat{A}}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\Big]_{h=h_1+h_2-3},\\
[(W^{\lambda,\hat{A}}_{\mathrm{B},h_1})_m,(W^{\lambda,\hat{B}}_{\mathrm{B},h_2})_n] =& -\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-4} q^h p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\\
&+\Big[-q^h p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\Big]_{h_1+h_2-3}\\
&+c_{W_{\mathrm{B},h_1}}(m,\lambda)\delta^{\hat{A}\hat{B}}\delta^{h_1h_2}q^{2(h_1-2)}\delta_{m+n}\\
&+\sum_{h=0,\mathrm{even}}^{h_1+h_2-4} q^h p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\Big(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(W^{\lambda}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\Big)\\
&+\Big[q^h p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\Big(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(W^{\lambda}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\Big)\Big]_{h=h_1+h_2-3},\\
[(W^{\lambda}_{\mathrm{F},h_1})_m,(Q^{\lambda,\hat{A}}_{h_2+\frac{1}{2}})_r] =& \sum_{h=-1}^{h_1+h_2-3} q^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(Q^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
[(W^{\lambda,\hat{A}}_{\mathrm{F},h_1})_m,(Q^{\lambda}_{h_2+\frac{1}{2}})_r] =& \sum_{h=-1}^{h_1+h_2-3} q^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(Q^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
[(W^{\lambda,\hat{A}}_{\mathrm{F},h_1})_m,(Q^{\lambda,\hat{B}}_{h_2+\frac{1}{2}})_r] =& \sum_{h=-1}^{h_1+h_2-3} q^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(Q^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{h=-1}^{h_1+h_2-3} q^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\bigg(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(Q^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(Q^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\bigg),
\end{aligned}
$$

$$
[(W^{\lambda}_{\mathrm{B},h_1})_m,(Q^{\lambda,\hat{A}}_{h_2+\frac{1}{2}})_r]=\sum_{h=-1}^{h_1+h_2-3} q^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(Q^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},
$$

$$
[(W^{\lambda,\hat{A}}_{\mathrm{B},h_1})_m,(Q^{\lambda}_{h_2+\frac{1}{2}})_r]=\sum_{h=-1}^{h_1+h_2-3} q^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(Q^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},
$$

$$
\begin{aligned}
[(W^{\lambda,\hat{A}}_{\mathrm{B},h_1})_m,(Q^{\lambda,\hat{B}}_{h_2+\frac{1}{2}})_r]=&-\sum_{h=-1}^{h_1+h_2-3} q^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(Q^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&+\sum_{h=-1}^{h_1+h_2-3} q^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\bigg(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(Q^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(Q^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\bigg),
\end{aligned}
$$

$$
\begin{aligned}
[(W^{\lambda}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda,\hat{A}}_{h_2+\frac{1}{2}})_r]=&\sum_{h=-1}^{h_1+h_2-3} q^h(-1)^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(\bar{Q}^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&+\bigg[q^h(-1)^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(\bar{Q}^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\bigg]_{h=h_1=h_2-2},
\end{aligned}
$$

$$
\begin{aligned}
[(W^{\lambda,\hat{A}}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda}_{h_2+\frac{1}{2}})_r]=&\sum_{h=-1}^{h_1+h_2-3} q^h(-1)^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(\bar{Q}^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&+\bigg[q^h(-1)^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(\bar{Q}^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\bigg]_{h=h_2+h_2-2},
\end{aligned}
$$

$$
\begin{aligned}
[(W^{\lambda,\hat{A}}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda,\hat{B}}_{h_2+\frac{1}{2}})_r]=&-\sum_{h=-1}^{h_1+h_2-3} q^h(-1)^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(\bar{Q}^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&+\bigg[-q^h(-1)^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(\bar{Q}^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\bigg]_{h=h_1+h_2-2} \\
&+\sum_{h=-1}^{h_1+h_2-3} q^h(-1)^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\bigg(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(\bar{Q}^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(\bar{Q}^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\bigg) \\
&+\bigg[q^h(-1)^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\bigg(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(\bar{Q}^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(\bar{Q}^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\bigg)\bigg]_{h=h_1+h_2-2},
\end{aligned}
$$

$$
[(W^{\lambda}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda,\hat{A}}_{h_2+\frac{1}{2}})_r]=\sum_{h=-1}^{h_1+h_2-3} q^h(-1)^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(\bar{Q}^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}
$$

$$
\begin{aligned}
&+\Big[q^h(-1)^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(\bar{Q}^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big]_{h=h_1+h_2-2},\\
[(W^{\lambda,\hat{A}}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda}_{h_2+\frac{1}{2}})_r] =& \sum_{h=-1}^{h_1+h_2-3} q^h(-1)^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(\bar{Q}^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\\
&+\Big[q^h(-1)^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)(\bar{Q}^{\lambda,\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big]_{h=h_1+h_2-2},\\
[(W^{\lambda,\hat{A}}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda,\hat{B}}_{h_2+\frac{1}{2}})_r] =& \sum_{h=-1}^{h_1+h_2-3} q^h(-1)^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(\bar{Q}^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\\
&+\Big[q^h(-1)^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(\bar{Q}^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big]_{h=h_1+h_2-2}\\
&+\sum_{h=-1}^{h_1+h_2-3} q^h(-1)^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\Big(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(\bar{Q}^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(\bar{Q}^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big)\\
&+\Big[q^h(-1)^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\Big(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(\bar{Q}^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(\bar{Q}^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big)\Big]_{h=h_1=h_2-2},\\
\{(Q^{\lambda}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,\hat{A}}_{h_2+\frac{1}{2}})_s\} =& \sum_{h=0}^{h_1+h_2-1} q^h o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)(W^{\lambda,\hat{A}}_{\mathrm{F},h_1+h_2-h})_{r+s}\\
&+\sum_{h=0}^{h_1+h_2-2} q^h o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)(W^{\lambda,\hat{A}}_{\mathrm{B},h_1+h_2-h})_{r+s}\\
&+\Big[q^h o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)(W^{\lambda,\hat{A}}_{\mathrm{B},h_1+h_2-h})_{r+s}\Big]_{h=h_1+h_2-1},\\
\{(Q^{\lambda,\hat{A}}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda}_{h_2+\frac{1}{2}})_s\} =& \sum_{h=0}^{h_1+h_2-1} q^h o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)(W^{\lambda,\hat{A}}_{\mathrm{F},h_1+h_2-h})_{r+s}\\
&+\sum_{h=0}^{h_1+h_2-2} q^h o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)(W^{\lambda,\hat{A}}_{\mathrm{B},h_1+h_2-h})_{r+s}\\
&+\Big[q^h o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)(W^{\lambda,\hat{A}}_{\mathrm{B},h_1+h_2-h})_{r+s}\Big]_{h=h_1+h_2-1},\\
\{(Q^{\lambda,\hat{A}}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,\hat{B}}_{h_2+\frac{1}{2}})_s\} =& \sum_{h=0}^{h_1+h_2-1} q^h o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{F,h_1+h_2-h})_{r+s}\\
&+\sum_{h=0}^{h_1+h_2-1} q^h o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\Big(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{F,h_1+h_2-h})_{r+s}\\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(W^{\lambda}_{F,h_1+h_2-h})_{r+s}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{h=0}^{h_1+h_2-2} q^h o_{\rm B}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{B,h_1+h_2-h})_{r+s}\\
&+\Big[-q^h o_{\rm B}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\frac{i}{2}f^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{B,h_1+h_2-h})_{r+s}\Big]_{h=h_1+h_2-1}\\
&+\sum_{h=0}^{h_1+h_2-2} q^h o_{\rm B}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\Big(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{B,h_1+h_2-h})_{r+s}\\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(W^{\lambda}_{B,h_1+h_2-h})_{r+s}\Big)\\
&+\Big[q^h o_{\rm B}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\Big(\frac{1}{2}d^{\hat{A}\hat{B}\hat{C}}(W^{\lambda,\hat{C}}_{B,h_1+h_2-h})_{r+s}\\
&+\frac{1}{N}\delta^{\hat{A}\hat{B}}(W^{\lambda}_{B,h_1+h_2-h})_{r+s}\Big)\Big]_{h=h_1+h_2-1}\\
&+c_{Q\bar{Q}_{h_1+\frac{1}{2}}}(r,\lambda)\delta^{\hat{A}\hat{B}}\delta^{h_1h_2}q^{2(h_1+\frac{1}{2}-1)}\delta_{r+s}.
\end{aligned}
\tag{D.1}
$$

As in the section 3, we intentionally make the square brackets in the appendix (D.1) in order to emphasize that the current $\bar{Q}^{\lambda}_{\frac{1}{2}}$ or the current $W^{\lambda}_{B,1}$ occurs inside of the square brackets when we restrict to the operators in the left hand sides which do not have these weight-$\frac{1}{2}$, 1 currents.

E Some OPEs for $\lambda = \frac{1}{4}$

We present some OPEs for the particular value of $\lambda = \frac{1}{4}$ for fixed h_1 and h_2 as follows. We keep the λ dependence without substituting this value in order to see the factor $(1-4\lambda)$. It turns out that

$$
\begin{aligned}
&V^{(4)+}_{\lambda}(\bar{z})\,V^{(4)+}_{\lambda}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^8}\Big[-\frac{21}{5}(4\lambda-1)(12\lambda^4-12\lambda^3-13\lambda^2+8\lambda+3)\Big]\\
&+\frac{1}{(\bar{z}-\bar{w})^6}\Big[\frac{56}{15}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,V^{(2)+}_{\lambda}\\
&+\frac{8}{5}(\lambda+1)(2\lambda-3)(4\lambda-1)\,V^{(2)-}_{\lambda}\Big](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^5}\Big[\frac{1}{2}\frac{56}{15}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}V^{(2)+}_{\lambda}\\
&+\frac{1}{2}\frac{8}{5}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}V^{(2)-}_{\lambda}\Big](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^4}\Big[\frac{3}{20}\frac{56}{15}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^2V^{(2)+}_{\lambda}\\
&+\frac{3}{20}\frac{8}{5}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}^2V^{(2)-}_{\lambda}-\frac{18}{35}(12\lambda^2-6\lambda-43)\,V^{(4)+}_{\lambda}\\
&-\frac{6}{5}(4\lambda-1)\,V^{(4)-}_{\lambda}\Big](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^3}\Big[\frac{1}{30}\frac{56}{15}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^3V^{(2)+}_{\lambda}
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{1}{30}\,\frac{8}{5}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}^3\,V_\lambda^{(2)-}-\frac{1}{2}\frac{18}{35}(12\lambda^2-6\lambda-43)\,\bar{\partial}\,V_\lambda^{(4)+}\\
&-\frac{1}{2}\,\frac{6}{5}(4\lambda-1)\,\bar{\partial}\,V_\lambda^{(4)-}\Bigg](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^2}\left[\frac{1}{168}\,\frac{56}{15}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^4\,V_\lambda^{(2)+}\right.\\
&+\frac{1}{168}\,\frac{8}{5}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}^4\,V_\lambda^{(2)-}-\frac{5}{36}\,\frac{18}{35}(12\lambda^2-6\lambda-43)\,\bar{\partial}^2\,V_\lambda^{(4)+}\\
&\left.-\frac{5}{36}\,\frac{6}{5}(4\lambda-1)\,\bar{\partial}^2\,V_\lambda^{(4)-}+6\,V_\lambda^{(6)+}\right](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})}\left[\frac{1}{1120}\,\frac{56}{15}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^5\,V_\lambda^{(2)+}\right.\\
&+\frac{1}{1120}\,\frac{8}{5}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}^5\,V_\lambda^{(2)-}-\frac{1}{36}\,\frac{18}{35}(12\lambda^2-6\lambda-43)\,\bar{\partial}^3\,V_\lambda^{(4)+}\\
&\left.-\frac{1}{36}\,\frac{6}{5}(4\lambda-1)\,\bar{\partial}^3\,V_\lambda^{(4)-}+\frac{1}{2}\,6\,\bar{\partial}\,V_\lambda^{(6)+}\right](\bar{w})+\cdots,\\
&V_\lambda^{(4)+}(\bar{z})\,V_\lambda^{(3)-}(\bar{w})=\frac{1}{(\bar{z}-\bar{w})^6}\left[\frac{2}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,V_\lambda^{(1)-}\right](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^5}\left[\frac{2}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}\,V_\lambda^{(1)-}\right](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^4}\left[\frac{1}{2}\,\frac{2}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^2\,V_\lambda^{(1)-}\right.\\
&\left.+\frac{12}{125}(\lambda+1)(2\lambda-3)(4\lambda-1)\,V_\lambda^{(3)+}-\frac{6}{25}\,(12\lambda^2-6\lambda-43)\,V_\lambda^{(3)-}\right](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^3}\left[\frac{1}{6}\,\frac{2}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^3\,V_\lambda^{(1)-}\right.\\
&\left.+\frac{2}{3}\,\frac{12}{125}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}\,V_\lambda^{(3)+}-\frac{2}{3}\,\frac{6}{25}\,(12\lambda^2-6\lambda-43)\,\bar{\partial}\,V_\lambda^{(3)-}\right](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^2}\left[\frac{1}{24}\,\frac{2}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^4\,V_\lambda^{(1)-}\right.\\
&+\frac{5}{21}\,\frac{12}{125}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}^2\,V_\lambda^{(3)+}-\frac{5}{21}\,\frac{6}{25}\,(12\lambda^2-6\lambda-43)\,\bar{\partial}^2\,V_\lambda^{(3)-}\\
&\left.-\frac{2}{9}(4\lambda-1)\,V_\lambda^{(5)+}+5\,V_\lambda^{(5)-}\right](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})}\left[\frac{1}{120}\,\frac{2}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^5\,V_\lambda^{(1)-}\right.\\
&+\frac{5}{84}\,\frac{12}{125}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}^3\,V_\lambda^{(3)+}-\frac{5}{84}\,\frac{6}{25}\,(12\lambda^2-6\lambda-43)\,\bar{\partial}^3\,V_\lambda^{(3)-}\\
&\left.-\frac{3}{5}\,\frac{2}{9}(4\lambda-1)\,\bar{\partial}\,V_\lambda^{(5)+}+\frac{3}{5}\,5\,\bar{\partial}\,V_\lambda^{(5)-}\right](\bar{w})+\cdots,\\
&V_\lambda^{(4)+}(\bar{z})\,Q_\lambda^{(4)+}(\bar{w})=\frac{1}{(\bar{z}-\bar{w})^6}\left[\frac{7}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,Q_\lambda^{(2)+}\right.\\
&+\frac{1}{(\bar{z}-\bar{w})^5}\left[\frac{2}{3}\,\frac{7}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}\,Q_\lambda^{(2)+}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{8}{25}(\lambda+1)(2\lambda-3)(4\lambda-1)\,Q_\lambda^{(3)-}\Big](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^4}\Big[\frac{1}{4}\,\frac{7}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^2\,Q_\lambda^{(2)+}\\
&+\frac{3}{5}\,\frac{8}{25}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}\,Q_\lambda^{(3)-}-\frac{9}{25}\,(12\lambda^2-6\lambda-43)\,Q_\lambda^{(4)+}\Big](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^3}\Big[\frac{1}{15}\,\frac{7}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^3\,Q_\lambda^{(2)+}\\
&+\frac{1}{5}\,\frac{8}{25}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}^2\,Q_\lambda^{(3)-}-\frac{4}{7}\,\frac{9}{25}\,(12\lambda^2-6\lambda-43)\,\bar{\partial}\,Q_\lambda^{(4)+}\\
&-\frac{2}{5}(4\lambda-1)\,Q_\lambda^{(5)-}\Big](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^2}\Big[\frac{1}{72}\,\frac{7}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^4\,Q_\lambda^{(2)+}\\
&+\frac{1}{21}\,\frac{8}{25}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}^3\,Q_\lambda^{(3)-}-\frac{5}{28}\,\frac{9}{25}\,(12\lambda^2-6\lambda-43)\,\bar{\partial}^2\,Q_\lambda^{(4)+}\\
&-\frac{5}{9}\,\frac{2}{5}(4\lambda-1)\,\bar{\partial}\,Q_\lambda^{(5)-}+\frac{11}{2}\,Q_\lambda^{(6)+}\Big](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})}\Big[\frac{1}{420}\,\frac{7}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\bar{\partial}^5\,Q_\lambda^{(2)+}\\
&+\frac{1}{112}\,\frac{8}{25}(\lambda+1)(2\lambda-3)(4\lambda-1)\,\bar{\partial}^4\,Q_\lambda^{(3)-}-\frac{5}{126}\,\frac{9}{25}\,(12\lambda^2-6\lambda-43)\,\bar{\partial}^3\,Q_\lambda^{(4)+}\\
&-\frac{1}{6}\,\frac{2}{5}(4\lambda-1)\,\bar{\partial}^2\,Q_\lambda^{(5)-}+\frac{6}{11}\,\frac{11}{2}\,\bar{\partial}\,Q_\lambda^{(6)+}\Big](\bar{w})+\cdots,\\
&V_\lambda^{(4)+}(\bar{z})\,Q_\lambda^{(3)-}(\bar{w})=\frac{1}{(\bar{z}-\bar{w})^6}\Big[\frac{10}{3}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)\,Q_\lambda^{(1)-}\\
&+\frac{1}{(\bar{z}-\bar{w})^5}\Big[2\,\frac{10}{3}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)\,\bar{\partial}\,Q_\lambda^{(1)-}+\frac{4}{5}(\lambda-1)(2\lambda+1)(4\lambda-1)\,Q_\lambda^{(2)+}\Big](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^4}\Big[\frac{3}{2}\,\frac{10}{3}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)\,\bar{\partial}^2\,Q_\lambda^{(1)-}+\frac{4}{5}(\lambda-1)(2\lambda+1)(4\lambda-1)\,\bar{\partial}\,Q_\lambda^{(2)+}\\
&-\frac{21}{10}\,(\lambda+1)(2\lambda-3)\,Q_\lambda^{(3)-}\Big](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^3}\Big[\frac{2}{3}\,\frac{10}{3}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)\,\bar{\partial}^3\,Q_\lambda^{(1)-}+\frac{1}{2}\,\frac{4}{5}(\lambda-1)(2\lambda+1)(4\lambda-1)\,\bar{\partial}^2\,Q_\lambda^{(2)+}\\
&-\frac{4}{5}\,\frac{21}{10}\,(\lambda+1)(2\lambda-3)\,\bar{\partial}\,Q_\lambda^{(3)-}-\frac{8}{15}\,(4\lambda-1)\,Q_\lambda^{(4)+}\Big](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^2}\Big[\frac{5}{24}\,\frac{10}{3}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)\,\bar{\partial}^4\,Q_\lambda^{(1)-}\\
&+\frac{1}{6}\,\frac{4}{5}(\lambda-1)(2\lambda+1)(4\lambda-1)\,\bar{\partial}^3\,Q_\lambda^{(2)+}\\
&-\frac{5}{12}\,\frac{21}{10}\,(\lambda+1)(2\lambda-3)\,\bar{\partial}^2\,Q_\lambda^{(3)-}-\frac{5}{7}\,\frac{8}{15}\,(4\lambda-1)\,\bar{\partial}\,Q_\lambda^{(4)+}+\frac{9}{2}\,Q_\lambda^{(5)-}\Big](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})}\Big[\frac{1}{20}\,\frac{10}{3}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)\,\bar{\partial}^5\,Q_\lambda^{(1)-}
\end{aligned}
$$

$$
+\frac{1}{24}\frac{4}{5}(\lambda-1)(2\lambda+1)(4\lambda-1)\,\bar{\partial}^4 Q_\lambda^{(2)+}
$$
$$
-\frac{5}{42}\frac{21}{10}(\lambda+1)(2\lambda-3)\,\bar{\partial}^3 Q_\lambda^{(3)-}-\frac{15}{56}\frac{8}{15}(4\lambda-1)\,\bar{\partial}^2 Q_\lambda^{(4)+}+\frac{2}{3}\frac{9}{2}\,\bar{\partial} Q_\lambda^{(5)-}\Big](\bar{w})
$$
$$
+\cdots. \tag{E.1}
$$

From appendix (E.1), we observe that the currents appearing in the right hand sides, $V_\lambda^{(2),-}$, $V_\lambda^{(4),-}$ (for nonzero $V_\lambda^{(h)+}$ with even h), $V_\lambda^{(3),+}$, $V_\lambda^{(5),+}$ (for nonzero $V_\lambda^{(h)-}$ with odd h), $Q_\lambda^{(3),-}$, $Q_\lambda^{(5),-}$ (for nonzero $Q_\lambda^{(h),\pm}$ with even h), $Q_\lambda^{(2),+}$ and $Q_\lambda^{(4),+}$ (for nonzero $Q_\lambda^{(h),\pm}$ with odd h) have the $(1-4\lambda)$ factor. Therefore, these currents are decoupled from the remaining subalgebra generated by (3.28) or (3.29).

We can use the first and the third equations of (3.30) and calculate the commutator relation $[(V_\lambda^{(h_1),+})_m,(Q_\lambda^{(h_2+1),+})_r]$ and focus on the coefficient function in front of $(Q_\lambda^{(h_1+h_2-2-h+1),-})_{m+r}$. Then we obtain the following result

$$
\begin{aligned}
&-\left[\frac{1}{\frac{n_{W_{F,h_1}}}{q^{h_1-2}}\,\frac{(-1)^{h_1}}{\sum_{i=0}^{h_1-1}a^i(h_1,\frac{1}{2})}}\right]\frac{1}{2}\left[\frac{1}{\frac{1}{2}\,\frac{n_{W_{Q,h_2+\frac{1}{2}}}}{q^{h_2-1}}\,\frac{(-1)^{h_2+1}\,h_2}{\sum_{i=0}^{h_2-1}\beta^i(h_2+1,0)}}\right]\\
&\times\Bigg(\left[\frac{1}{2}\,\frac{n_{W_{Q,h_1+h_2-2-h+\frac{1}{2}}}}{q^{h_1+h_2-2-h-1}}\,\frac{(-1)^{h_1+h_2-2-h+1}\,(h_1+h_2-2-h)}{\sum_{i=0}^{h_1+h_2-2-h-1}\beta^i(h_1+h_2-2-h+1,0)}\right]q^h\,q_F^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\\
&\times((Q_\lambda^{(h_1+h_2-2-h+1),-})_{m+r}-(Q_\lambda^{(h_1+h_2-2-h+1),+})_{m+r})\\
&-\left[\frac{1}{2}\,\frac{n_{W_{Q,h_1+h_2-2-h+\frac{1}{2}}}}{q^{h_1+h_2-2-h-1}}\,\frac{(-1)^{h_1+h_2-2-h+1}}{\sum_{i=0}^{h_1+h_2-2-h}\alpha^i(h_1+h_2-2-h+1,0)}\right]q^h\,q_F^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\\
&\times(-1)^h\,((Q_\lambda^{(h_1+h_2-2-h+1),-})_{m+r}+(Q_\lambda^{(h_1+h_2-2-h+1),+})_{m+r})\Bigg)\\
&-\left[\frac{1}{\frac{n_{W_{B,h_1}}}{q^{h_1-2}}\,\frac{(-1)^{h_1}}{\sum_{i=0}^{h_1-1}a^i(h_1,0)}}\right]\frac{1}{2}\left[\frac{1}{\frac{1}{2}\,\frac{n_{W_{Q,h_2+\frac{1}{2}}}}{q^{h_2-1}}\,\frac{(-1)^{h_2+1}}{\sum_{i=0}^{h_2}\alpha^i(h_2+1,0)}}\right]\\
&\times\Bigg(\left[\frac{1}{2}\,\frac{n_{W_{Q,h_1+h_2-2-h+\frac{1}{2}}}}{q^{h_1+h_2-2-h-1}}\,\frac{(-1)^{h_1+h_2-2-h+1}\,(h_1+h_2-2-h)}{\sum_{i=0}^{h_1+h_2-2-h-1}\beta^i(h_1+h_2-2-h+1,0)}\right]q^h\,q_B^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\\
&\times((Q_\lambda^{(h_1+h_2-2-h+1),-})_{m+r}-(Q_\lambda^{(h_1+h_2-2-h+1),+})_{m+r})\\
&-\left[\frac{1}{2}\,\frac{n_{W_{Q,h_1+h_2-2-h+\frac{1}{2}}}}{q^{h_1+h_2-2-h-1}}\,\frac{(-1)^{h_1+h_2-2-h+1}}{\sum_{i=0}^{h_1+h_2-2-h}\alpha^i(h_1+h_2-2-h+1,0)}\right]q^h\,q_F^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\\
&\times(-1)^h\,((Q_\lambda^{(h_1+h_2-2-h+1),-})_{m+r}+(Q_\lambda^{(h_1+h_2-2-h+1),+})_{m+r})\Bigg),
\end{aligned}
\tag{E.2}
$$

which leads to zero at $\lambda=\frac{1}{4}$ for the coefficient of $(Q_\lambda^{(h_1+h_2-2-h+1),-})_{m+r}$ with odd $(h_1+h_2-2-h+1)$. The property appearing in the footnote 20 is used.

Similarly, by using the first and the fourth equations of (3.30) and calculate the commutator relation $[(V_\lambda^{(h_1),+})_m,(Q_\lambda^{(h_2+1),-})_r]$ and focus on the coefficient function in front of $(Q_\lambda^{(h_1+h_2-2-h+1),+})_{m+r}$. We have checked that the coefficient function appearing in the mode $(Q_\lambda^{(h_1+h_2-2-h+1),+})_{m+r}$ with even $(h_1+h_2-2-h+1)$ vanishes at $\lambda=\frac{1}{4}$ from similar equation of appendix (E.2).

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References

[1] S. Pasterski, M. Pate and A.-M. Raclariu, *Celestial holography*, in 2022 *Snowmass summer study*, (2021) [arXiv:2111.11392] [INSPIRE].

[2] A. Guevara, E. Himwich, M. Pate and A. Strominger, *Holographic symmetry algebras for gauge theory and gravity*, *JHEP* **11** (2021) 152 [arXiv:2103.03961] [INSPIRE].

[3] A. Strominger, $w_{1+\infty}$ *and the celestial sphere*, arXiv:2105.14346 [INSPIRE].

[4] I. Bakas, *The large N limit of extended conformal symmetries*, *Phys. Lett. B* **228** (1989) 57 [INSPIRE].

[5] C. Ahn, *Towards a supersymmetric* $w_{1+\infty}$ *symmetry in the celestial conformal field theory*, *Phys. Rev. D* **105** (2022) 086028 [arXiv:2111.04268] [INSPIRE].

[6] C. Ahn, *A deformed supersymmetric* $w_{1+\infty}$ *symmetry in the celestial conformal field theory*, arXiv:2202.02949 [INSPIRE].

[7] S. Pasterski and H. Verlinde, *Mapping SYK to the sky*, arXiv:2201.05054 [INSPIRE].

[8] A.B. Prema, G. Compère, L.P. de Gioia, I. Mol and B. Swidler, *Celestial holography: lectures on asymptotic symmetries*, *SciPost Phys. Lect. Notes* **47** (2022) 1 [arXiv:2109.00997] [INSPIRE].

[9] S. Pasterski, *Lectures on celestial amplitudes*, *Eur. Phys. J. C* **81** (2021) 1062 [arXiv:2108.04801] [INSPIRE].

[10] A.-M. Raclariu, *Lectures on celestial holography*, arXiv:2107.02075 [INSPIRE].

[11] K. Costello and N.M. Paquette, *Celestial holography meets twisted holography: 4d amplitudes from chiral correlators*, arXiv:2201.02595 [INSPIRE].

[12] L. Freidel, D. Pranzetti and A.-M. Raclariu, *Higher spin dynamics in gravity and* $w_{1+\infty}$ *celestial symmetries*, arXiv:2112.15573 [INSPIRE].

[13] W. Bu, *Supersymmetric celestial OPEs and soft algebras from the ambitwistor string worldsheet*, arXiv:2111.15584 [INSPIRE].

[14] J. Mago, L. Ren, A.Y. Srikant and A. Volovich, *Deformed* $w_{1+\infty}$ *algebras in the celestial CFT*, arXiv:2111.11356 [INSPIRE].

[15] A. Ball, S.A. Narayanan, J. Salzer and A. Strominger, *Perturbatively exact* $w_{1+\infty}$ *asymptotic symmetry of quantum self-dual gravity*, *JHEP* **01** (2022) 114 [arXiv:2111.10392] [INSPIRE].

[16] T. Adamo, L. Mason and A. Sharma, *Celestial* $w_{1+\infty}$ *symmetries from twistor space*, *SIGMA* **18** (2022) 016 [arXiv:2110.06066] [INSPIRE].

[17] H. Jiang, *Celestial OPEs and* $w_{1+\infty}$ *algebra from worldsheet in string theory*, *JHEP* **01** (2022) 101 [arXiv:2110.04255] [INSPIRE].

[18] H. Jiang, *Holographic chiral algebra: supersymmetry, infinite Ward identities, and EFTs*, *JHEP* **01** (2022) 113 [arXiv:2108.08799] [INSPIRE].

[19] J. Murugan, D. Stanford and E. Witten, *More on supersymmetric and* 2*d analogs of the SYK model*, *JHEP* **08** (2017) 146 [arXiv:1706.05362] [INSPIRE].

[20] K. Bulycheva, $\mathcal{N}=2$ *SYK model in the superspace formalism*, *JHEP* **04** (2018) 036 [arXiv:1801.09006] [INSPIRE].

[21] C. Ahn and C. Peng, *Chiral algebras of two-dimensional SYK models*, *JHEP* **07** (2019) 092 [arXiv:1812.05106] [INSPIRE].

[22] E. Bergshoeff, B. de Wit and M.A. Vasiliev, *The structure of the super-*$W_\infty(\lambda)$ *algebra*, *Nucl. Phys. B* **366** (1991) 315 [INSPIRE].

[23] E. Bergshoeff, M.A. Vasiliev and B. de Wit, *The super-*$W_\infty(\lambda)$ *algebra*, *Phys. Lett. B* **256** (1991) 199 [INSPIRE].

[24] M.R. Gaberdiel and T. Hartman, *Symmetries of holographic minimal models*, *JHEP* **05** (2011) 031 [arXiv:1101.2910] [INSPIRE].

[25] C. Ahn and M.H. Kim, *The* $\mathcal{N}=4$ *higher spin algebra for generic* μ *parameter*, *JHEP* **02** (2021) 123 [arXiv:2009.04852] [INSPIRE].

[26] K. Thielemans, *A Mathematica package for computing operator product expansions*, *Int. J. Mod. Phys. C* **2** (1991) 787 [INSPIRE].

[27] Wolfram Research Inc., *Mathematica*, version 13.0.0, https://www.wolfram.com/mathematica/, Champaign, IL, U.S.A. (2021).

[28] E. Silverstein and E. Witten, *Global* U(1) *R-symmetry and conformal invariance of* (0, 2) *models*, *Phys. Lett. B* **328** (1994) 307 [hep-th/9403054] [INSPIRE].

[29] E. Witten, *Two-dimensional models with* (0, 2) *supersymmetry: perturbative aspects*, *Adv. Theor. Math. Phys.* **11** (2007) 1 [hep-th/0504078] [INSPIRE].

[30] M. Dedushenko, *Chiral algebras in Landau-Ginzburg models*, *JHEP* **03** (2018) 079 [arXiv:1511.04372] [INSPIRE].

[31] S. Kachru and E. Witten, *Computing the complete massless spectrum of a Landau-Ginzburg orbifold*, *Nucl. Phys. B* **407** (1993) 637 [hep-th/9307038] [INSPIRE].

[32] T. Kawai and K. Mohri, *Geometry of* (0, 2) *Landau-Ginzburg orbifolds*, *Nucl. Phys. B* **425** (1994) 191 [hep-th/9402148] [INSPIRE].

[33] E. Witten, *On the Landau-Ginzburg description of* $N=2$ *minimal models*, *Int. J. Mod. Phys. A* **9** (1994) 4783 [hep-th/9304026] [INSPIRE].

[34] K. Hanaki and C. Peng, *Symmetries of holographic super-minimal models*, *JHEP* **08** (2013) 030 [arXiv:1203.5768] [INSPIRE].

[35] H. Moradi and K. Zoubos, *Three-point functions in* $\mathcal{N}=2$ *higher-spin holography*, *JHEP* **04** (2013) 018 [arXiv:1211.2239] [INSPIRE].

[36] T. Creutzig, Y. Hikida and P.B. Ronne, *Extended higher spin holography and Grassmannian models*, *JHEP* **11** (2013) 038 [arXiv:1306.0466] [INSPIRE].

[37] C. Peng, $\mathcal{N}=(0,2)$ *SYK, chaos and higher-spins*, *JHEP* **12** (2018) 065 [arXiv:1805.09325] [INSPIRE].

[38] S. Odake, *Unitary representations of* W_∞ *algebras*, *Int. J. Mod. Phys. A* **7** (1992) 6339 [hep-th/9111058] [INSPIRE].

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[39] C.N. Pope, L.J. Romans and X. Shen, *W_∞ and the Racah-Wigner algebra*, Nucl. Phys. B **339** (1990) 191 [INSPIRE].

[40] R. Blumenhagen and E. Plauschinn, *Introduction to conformal field theory: with applications to string theory*, Lect. Notes Phys. **779** (2009) 1 [INSPIRE].

[41] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel and R. Varnhagen, *W algebras with two and three generators*, Nucl. Phys. B **361** (1991) 255 [INSPIRE].

[42] C.N. Pope, L.J. Romans and X. Shen, *The complete structure of W_∞*, Phys. Lett. B **236** (1990) 173 [INSPIRE].

[43] C.N. Pope, L.J. Romans, E. Sezgin and X. Shen, *W topological matter and gravity*, Phys. Lett. B **256** (1991) 191 [INSPIRE].

[44] E. Witten, *Topological quantum field theory*, Commun. Math. Phys. **117** (1988) 353 [INSPIRE].

[45] T. Eguchi and S.-K. Yang, *$N = 2$ superconformal models as topological field theories*, Mod. Phys. Lett. A **5** (1990) 1693 [INSPIRE].

[46] A. Fotopoulos, S. Stieberger, T.R. Taylor and B. Zhu, *Extended super BMS algebra of celestial CFT*, JHEP **09** (2020) 198 [arXiv:2007.03785] [INSPIRE].

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