Remarks on holographic models of the Kerr-AdS₅ geometry

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ABSTRACT: We study the low-temperature limit of scalar perturbations of the Kerr-AdS₅ black-hole for generic rotational parameters. We motivate the study by considering real-time holography of small black hole backgrounds. Using the isomonodromic technique, we show that corrections to the extremal limit can be encoded in the monodromy parameters of the Painlevé V transcendent, whose expansion is given in terms of irregular chiral conformal blocks. After discussing the contribution of the intermediate states to the quasinormal modes, we perform a numerical analysis of the low-lying frequencies. We find that the fundamental mode is perturbatively stable at low temperatures for small black holes and that excited perturbations are superradiant, as expected from thermodynamical considerations. We close by considering the holographic interpretation of the unstable modes and the decaying process.

KEYWORDS: AdS-CFT Correspondence, Black Holes, Black Holes in String Theory

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1 Introduction

Holography has been considered a tool to study strongly-coupled phenomena in high energy physics for some time now. In particular, the description of strongly-coupled plasma put forward by [1, 2] and [3–5] laid ground to a very precise procedure to study systems such as finite temperature deformations of conformal field theories based on $\mathcal{N} = 4$ SYM from gravitational physics. It is particularly successful in dealing with the hydrodynamical limit of such systems, and quantities like the entropy to viscosity ratio [1] and energy loss calculations [6, 7].

More recently, it has been stressed the necessity of modeling rotation and vorticity into the holographic picture [8, 9], which depends on the holographic consideration of the generic rotating, vacuum black hole in anti-de Sitter (AdS) backgrounds — Kerr-AdS₅ black hole for short [10, 11]. Usually, the study relies on thermodynamical arguments about the behavior of fluctuations in the black hole background. For instance, in [12] the authors analyzed the propagation of a string in this background to estimate energy losses in a

rotating fluid. Hydrodynamical features and the effect of rotation in large five-dimensional Anti-de Sitter black holes were also studied in [13].

For a given field theory, the holographic description is based on the Ward identities associated to the conformal group currents [14]. The departure from a given ultraviolet fixed point in the renormalization group flow is implemented by the (inwards) radial evolution on the currents using Einstein equations from the asymptotic boundary AdS structure. In this paper, we study the validity of the description for generic rotation parameters at low temperatures. The motivation for this is threefold.

First and foremost, the usefulness of holography to study the infrared (IR) behavior of the associated field theories depends on the gravitational perturbations at low temperatures. As we will revisit below, large black holes have a lower bound in the temperature, so, this study will focus on small black holes in the low temperature regime. At high temperatures, these black holes were studied by using the hydrodynamical approach in [4], which, among other things, recovers (flat space) Navier-Stokes equations by assuming that the mean free path of the components of the fluid is small when compared to the AdS scale. This hints at the fact that the dual field theory is in the strong coupling regime. This construction is at odds with holography, which starts from the assumption that the field theory has small corrections to the ultraviolet (UV) free or weakly interacting conformal point. More specifically, the available examples of holography usually come from theories that are weakly coupled at the UV, and become strongly coupled at the IR. We will make an unwarranted — although not unprecedented — assumption that the description based on conformal currents is still valid in the latter regime. We also remark that it is an open question whether this generic view of holography can be applied to the particular case of QCD-like asymptotically free theories with a mass gap. In the best case scenario, the holographic interpretation in the rest of this paper will refer to generic asymptotically conformal systems, not necessarily subject to the analysis in [15].

Secondly, a more comprehensive study of real-time holography [16] for the Kerr-AdS₅ black hole at low temperatures is desirable. Rotation mixes time and angular coordinates, and induces a “non-flat” asymptotic structure of the black hole background. The first fact poses problems for the analytic continuation of quantities computed in Euclidean signature and the second gives rise to trace anomalies which are heavily dependent on the particular field theory dual considered. We note however, that in a “classical limit” where the AdS radius is much larger than the Planck length, the quantities computed at different asymptotic structures, in our case essentially the one arising from global AdS $\mathbb{R} \times S^3$ and the one arising from the Poincaré patch $\mathbb{R}^{1,3}$ are relatable by a conformal transformation. Also, the repercussions of the AdS instabilities found in [17], of which the scalar field linear unstable modes found by the authors in [18] are a particular example, are expected to be of fundamental relevance to the understanding of holography, and the study presented here seemed like a natural step further.

The third reason is somewhat technical. As argued in [19], black hole perturbations should be described by a particular limit of the four-dimensional conformal block [20]. As argued in [21] (see also [22]), the pertinent perturbation can be computed using *two-dimensional* chiral conformal blocks. Semi-classical aspects of the latter which are relevant

to the discussion were anticipated in [23], and a full fledged solution was presented in the seminal work of [24], also uncovering a relation between semiclassical and $c = 1$ conformal blocks, both related to the Painlevé VI tau function.

The low temperature black hole considered here necessitates a careful consideration of the confluence limit, which is given in terms of the Painlevé V tau function [25]. Albeit in this paper we are focused on the study of the quasi-normal modes (QNMs) for small black holes $r_+ \ll 1$, the resulting relation holds for generic five-dimensional Kerr black holes in the near-extremal limit. On the one hand, this is to our knowledge a new application of the Painlevé V transcendent — see also [26, 27].¹ On the other hand, the appearance of the two-dimensional conformal blocks may be seen from one point of view as a calculational tool, but, as we will argue, the connection between these and four-dimensional blocks merits further investigation.

While the writing of this work was in course, [29] came out with a similar study of (four-dimensional) quasi-normal modes of black holes using the properties of conformal blocks. The authors of [29] employ the usual relation between Fuchsian equations and semiclassical conformal blocks as the starting point, and their relation to Seiberg-Witten theory [30] has been once more studied in [31]. We note, however, that there are advantages and drawbacks to each approach. The expansion in terms of semiclassical conformal blocks is more direct and leads directly to an asymptotic expansion of the accessory parameter (called K_0 below) of the associated differential equation (3.1). The $c = 1$ expansion used here implements an explicit solution to the Riemann-Hilbert problem of finding *both* accessory parameters t_0 and K_0 in terms of monodromy data. As we saw in an earlier paper [18], an asymptotic expansion for the quasi-normal modes in the five-dimensional Kerr-AdS black hole is more effectively obtained using $c = 1$ conformal blocks.

With all this in mind, we have structured the paper as follows. In section 2 we review the parameters of the Kerr-AdS₅ black hole, as well as its asymptotic structure and conserved charges — mass and angular momenta. We spend some time analyzing the holographic interpretation of the asymptotic structure, in particular the interplay between the $\mathbb{R} \times S^3$ (global structure) and $\mathbb{R}^{1,3}$ (Poincaré patch), in a “near the ultraviolet” description which allows us to identify the state in the putative field theory corresponding to the black hole and to digress about the fate of perturbations within the scope of linear analysis.

Section 3 comprises the bulk of the results. We consider scalar perturbations of the Kerr-AdS₅ black hole and show that the low temperature limit of the QNMs is given by zeros of the Painlevé V tau function. Using asymptotic expansions of the Painlevé V transcendent for small parameter, we compute the QNMs frequencies for small black holes. We then turn to discuss the CFT description of the perturbations, and find that the contributions to the fundamental QNM are indeed given by the vacuum conformal block, at least for small black holes. The higher modes are correspondingly given by special blocks with internal dimensions given by degenerate “heavy” operators, in a chiral version of the analysis in [32]. We use the expansions of the Painlevé tau function and the

¹See [28] for the parallel story in four-dimensional Kerr-de Sitter black holes, using the Painlevé VI transcendent.

accessory parameter to derive the expansion of the fundamental QNM frequency for small black holes, including the first correction for non-zero temperature and we compare the asymptotic expression with the numerical result. Higher modes are studied numerically in the same regime, and we find indeed the unstable modes of [33], provided the superradiance condition is met.

In section 4 we discuss the holographic consequences of the analysis, focussing on the unstable modes. We see that the naive extrapolation of the near-ultraviolet result, pointing to the irrelevance of the perturbation, is misguided, and that the instabilities arising from the same perturbations as treated in section 3 may change considerably the fate of the state as one approaches the infrared. We also find that the spatial dependence of the decaying modes may be confined to small angles as the mass of the state increases. We close in section 5 with some remarks on the results and their interpretation.

There are two appendices. Appendix A gives a self-contained derivation of the conserved charges for pure gravity in asymptotically AdS spaces and emphasizes the role of the conformal structure, as a special subcase of [34]. In appendix B we review the Fredholm determinant formulation of the Painlevé VI transcendent derived from the generic isomonodromic tau function proposed in [35], and its dependence on the relevant monodromy parameters is made explicit.

2 The Kerr-AdS₅ black hole

The Kerr-AdS₅ black hole is represented by the metric [10]

$$\begin{aligned}
 ds^2 = & -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a_1 \sin^2 \theta}{1 - a_1^2} d\phi_1 - \frac{a_2 \cos^2 \theta}{1 - a_2^2} d\phi_2 \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a_1 dt - \frac{(r^2 + a_1^2)}{1 - a_1^2} d\phi_1 \right)^2 \\
 & + \frac{1 + r^2}{r^2 \rho^2} \left(a_1 a_2 dt - \frac{a_2 (r^2 + a_1^2) \sin^2 \theta}{1 - a_1^2} d\phi_1 - \frac{a_1 (r^2 + a_2^2) \cos^2 \theta}{1 - a_2^2} d\phi_2 \right)^2 \\
 & + \frac{\Delta_\theta \cos^2 \theta}{\rho^2} \left(a_2 dt - \frac{(r^2 + a_2^2)}{1 - a_2^2} d\phi_2 \right)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2,
 \end{aligned} \tag{2.1}$$

where we set the AdS radius to one and

$$\begin{aligned}
 \Delta_r &= \frac{1}{r^2} (r^2 + a_1^2)(r^2 + a_2^2)(1 + r^2) - 2M, \\
 \Delta_\theta &= 1 - a_1^2 \cos^2 \theta - a_2^2 \sin^2 \theta, \quad \rho^2 = r^2 + a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta,
 \end{aligned} \tag{2.2}$$

and a_1 and a_2 are two independent rotation parameters. The mass relative to the AdS₅ vacuum solution and the angular momenta were the subject of some discussion [34, 36–38]. To add to the confusion, we include our own derivation in appendix A. The values obtained using our procedure match those of [37]

$$\mathcal{M} = \frac{\pi M (2(1 - a_1^2) + 2(1 - a_2^2) - (1 - a_1^2)(1 - a_2^2))}{4(1 - a_1^2)^2(1 - a_2^2)^2}, \tag{2.3}$$

$$\mathcal{J}_1 = \frac{\pi M}{2(1 - a_2^2)} \frac{a_1}{(1 - a_1^2)^2}, \quad \mathcal{J}_2 = \frac{\pi M}{2(1 - a_1^2)} \frac{a_2}{(1 - a_2^2)^2}. \tag{2.4}$$

In the following it will be useful to redefine the black hole parameters in terms of the roots of Δ_r

$$\Delta_r = \frac{1}{r^2}(r^2 - r_0^2)(r^2 - r_-^2)(r^2 - r_+^2) \quad (2.5)$$

which, for a regular, finite charged, dressed horizon black hole satisfy $r_0^2 < -1 < 0 < r_-^2 < r_+^2$. Each non-zero root r_i can be associated to a Killing horizon, although only those of r_- and r_+ are physical. To each horizon we can associate the temperature T_k and angular velocities $\Omega_{1,k}$ and $\Omega_{2,k}$, $k = 0, +, -$, given by

$$\begin{aligned} \Omega_{1,k} &= \frac{a_1(1 - a_1^2)}{r_k^2 + a_1^2} & \Omega_{2,k} &= \frac{a_2(1 - a_2^2)}{r_k^2 + a_2^2} \\ T_k &= \frac{r_k^2 \Delta'_r(r_k)}{4\pi(r_k^2 + a_1^2)(r_k^2 + a_2^2)} = \frac{r_k}{2\pi} \frac{(r_k^2 - r_i^2)(r_k^2 - r_j^2)}{(r_k^2 + a_1^2)(r_k^2 + a_2^2)}, \end{aligned} \quad (2.6)$$

where the prime stands for the derivative with respect to r . We note that $T_+ > 0$, $T_- < 0$ and T_0 is purely imaginary.

2.1 Asymptotic structure

The asymptotic structure of the metric as $r \rightarrow \infty$ is involved for non-zero rotation parameters a_1 and a_2 . By the suitable change of coordinates

$$(1 - a_1^2)\bar{r}^2 \sin^2 \bar{\theta} = (r^2 + a_1^2) \sin^2 \theta, \quad (2.7)$$

$$(1 - a_2^2)\bar{r}^2 \cos^2 \bar{\theta} = (r^2 + a_2^2) \cos^2 \theta, \quad (2.8)$$

$$\bar{t} = t, \quad \bar{\phi}_1 = \phi_1 + a_1 t, \quad \bar{\phi}_2 = \phi_2 + a_2 t, \quad (2.9)$$

we arrive, after some manipulation, at the useful formulas

$$1 + \bar{r}^2 = \frac{1 - a_1^2 \cos^2 \theta - a_2^2 \sin^2 \theta}{(1 - a_1^2)(1 - a_2^2)} (1 + r^2), \quad (2.10)$$

$$\bar{r}^2(1 - a_1^2 \sin^2 \bar{\theta} - a_2^2 \cos^2 \bar{\theta}) = r^2 + a_1^2 \sin^2 \theta + a_2^2 \cos^2 \theta, \quad (2.11)$$

$$\frac{d\bar{r}^2}{1 + \bar{r}^2} + \bar{r}^2 d\bar{\theta}^2 = \rho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right), \quad (2.12)$$

so that, for large $\bar{r}$, the metric becomes

$$\begin{aligned} ds^2 &= -(1 + \bar{r}^2) d\bar{t}^2 + \frac{d\bar{r}^2}{1 + \bar{r}^2} + \bar{r}^2 (d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\phi}_1^2 + \cos^2 \bar{\theta} d\bar{\phi}_2^2) \\ &+ \frac{2M}{\bar{\Delta}_\theta^2 \bar{r}^2} \left(\frac{d\bar{r}^2}{\bar{r}^4} + \frac{1}{\bar{\Delta}_\theta} (d\bar{t} - a_1 \sin^2 \bar{\theta} d\bar{\phi}_1 - a_2 \cos^2 \bar{\theta} d\bar{\phi}_2)^2 \right) + \dots \end{aligned} \quad (2.13)$$

where the ellipsis denotes subleading terms in $\bar{r}$ — see the appendix in [12], and

$$\bar{\Delta}_\theta = 1 - a_1^2 \sin^2 \bar{\theta} - a_2^2 \cos^2 \bar{\theta}. \quad (2.14)$$

In the first line of (2.13) we recognize the metric of vacuum global AdS.

In order to apply the procedure in appendix A to compute the mass and angular momenta we take $\tilde{\Omega} = \Omega = 1/\sqrt{1+\bar{r}^2}$, and write (2.13) as follows

$$ds^2 = \frac{1}{\Omega^2} \left(-d\bar{t}^2 + \frac{d\Omega^2}{1-\Omega^2} + (1-\Omega^2)d\Omega_3 \right) + 2M \frac{\Omega^2}{\bar{\Delta}_\theta^2} \left(d\Omega^2 + \frac{1}{\bar{\Delta}_\theta} (d\bar{t} - a_1 \sin^2 \bar{\theta} d\bar{\phi}_1 - a_2 \cos^2 \bar{\theta} d\bar{\phi}_2)^2 \right) + \dots, \quad (2.15)$$

where $d\Omega_3 = d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\phi}_1^2 + \cos^2 \bar{\theta} d\bar{\phi}_2^2$ is the usual metric of the 3-sphere. Again, the ellipsis at the end of (2.15) refers to subleading terms, now of order $\mathcal{O}(\Omega^3)$. Like in Fefferman-Graham development [39], the structure of the expansion can be explained by the equations of motion. The induced metric $\bar{g}_{ab}$ at the conformal infinity $\Omega = 0$ is given by the first line of (2.15), without the Ω^{-2} factor. The Ω dependence of this term fixes, up to order Ω^4 , the conformal structure at infinity — see (A.14). The term in the second line of (2.15) defines the asymptotic charges and thus $h_{ab} = \Omega^2 \gamma_{ab}$. Its form is explained in terms of $n^a = \bar{g}^{ab} \nabla_b \Omega$, the unit normal to the boundary, and the vector field

$$\chi^a = \frac{\partial}{\partial \bar{t}} + a_1 \frac{\partial}{\partial \bar{\phi}_1} + a_2 \frac{\partial}{\partial \bar{\phi}_2} = \frac{\partial}{\partial \bar{t}}. \quad (2.16)$$

This vector field generates an isometry of $\bar{g}_{ab}$ and of the AdS metric. It is actually an isometry of the Kerr-AdS metric, and the Killing vector becomes null at the horizons $r = r_0, r_-, r_+$. In terms of the induced metric $\bar{g}_{ab}$, one has

$$\bar{g}_{ab} \chi^a \chi^b = -1 + (1 - \Omega^2)(a_1^2 \sin^2 \bar{\theta} + a_2^2 \cos^2 \bar{\theta}) = -\bar{\Delta}_\theta + \mathcal{O}(\Omega^2). \quad (2.17)$$

We can thus write

$$h_{ab} = 2M \frac{\Omega^2}{\bar{\Delta}_\theta^2} \left(n_a n_b + \frac{1}{\bar{\Delta}_\theta} \bar{\chi}_a \bar{\chi}_b \right) + \mathcal{O}(\Omega^3) \quad (2.18)$$

with $\bar{\chi}_a = \bar{g}_{ab} \chi^b$. The leading contribution to the metric correction h_{ab} is actually invariant under the conformal structure chosen. Indeed, if we take $\Omega' = \omega \Omega$, with $\omega \neq 0$, then the quantities above transform as

$$n'_a = \omega n_a + \Omega \nabla_a \omega, \quad \bar{g}'_{ab} = \omega^2 \bar{g}_{ab}, \quad \bar{\chi}'_a = \omega^2 \bar{\chi}_a, \quad \bar{\Delta}' = \omega^2 \bar{\Delta}_\theta, \quad (2.19)$$

and it can be checked that $h'_{ab} = h_{ab}$ in (2.18), up to terms of order Ω^3 . In the following, we will refer to the leading part in (2.18) as just h_{ab} . It is traceless with respect to the pure AdS metric, and transverse

$$\nabla_a h^{ab} = 0, \quad (2.20)$$

where the covariant derivative is the one compatible with the pure AdS metric.

The true Fefferman-Graham parameter ϱ , defined in [39], can be computed as an expansion in Ω :

$$\varrho = \Omega + \frac{1}{4} \Omega^3 + \frac{1}{4} \left(\frac{1}{2} + \frac{M}{\bar{\Delta}_\theta^2} \right) \Omega^5 + \mathcal{O}(\Omega^7), \quad (2.21)$$

not to be confused with ρ in (2.2). We have the following structure for the metric in terms of ϱ , following the procedure in [14],

$$ds^2 = \frac{d\varrho^2}{\varrho^2} + \frac{1}{\varrho^2} (-d\bar{t}^2 + d\Omega_3) + \frac{1}{2} (-d\bar{t}^2 - d\Omega_3) + \varrho^2 \left(\left(\frac{1}{16} + \frac{M}{2\bar{\Delta}_\theta^2} \right) (-d\bar{t}^2 + d\Omega_3) + \frac{2M}{\bar{\Delta}_\theta^3} (d\bar{t} - a_1^2 \sin^2 \bar{\theta} d\bar{\phi}_1 - a_2^2 \cos^2 \bar{\theta} d\bar{\phi}_2)^2 \right) + \dots, \quad (2.22)$$

again up to $\mathcal{O}(\varrho^3)$ terms. We recognize the induced metric $\bar{g}_{ab}$, corresponding to the induced line element $-d\bar{t}^2 + d\Omega_3$. The particular conformal structure of $\mathbb{R} \times S^3$ is not invariant under the transformation $\Omega \rightarrow \omega\Omega$. By a suitable choice of ω (see (2.30)), one could change the induced metric to the flat Minkowski one $\mathbb{R}^{3,1}$. In the following, we will refer to the $\mathbb{R} \times S^3$ induced metric as $\bar{g}_{ab}$, or *global AdS structure*, and to the flat $\mathbb{R}^{3,1}$ as $\hat{g}_{ab}$, or *flat AdS structure*.

2.2 Holography and hydrodynamical issues

In global AdS, the holographic stress-energy tensor can be read directly from (2.22) above and has the expression [12]

$$T_{ab}^h = \frac{1}{4\pi G_{N,5}} \left(\frac{1}{16} \bar{g}_{ab} + \frac{2M}{\bar{\Delta}_\theta^2} \left(u_{(a} u_{b)} + \frac{1}{4} \bar{g}_{ab} \right) \right) \quad (2.23)$$

whose M -dependent term satisfies the conformally invariant, perfect fluid relation with energy density $\bar{\rho}$ and pressure $\bar{P}$

$$4\pi G_{N,5} \bar{\rho} = 12\pi G_{N,5} \bar{P} = 3M/2\bar{\Delta}_\theta^2, \quad (2.24)$$

hence $\bar{P} = \bar{\rho}/3$, and the holographic stress-energy tensor is traceless. The normalized fluid velocity is $u^a = \chi^a / \sqrt{\bar{\Delta}_\theta}$ and it is divergence-free $\bar{\nabla}_a u^a = 0$, where $\bar{\nabla}_a$ is the covariant derivative associated to the metric $\bar{g}_{ab}$. Despite depicting a classical energy profile, and being dependent on the choice of conformal structure, (2.23) is by construction a valid approximation to the stress-energy tensor of the state in the field theory in the UV limit, which holographically corresponds to $\bar{r} \rightarrow \infty$.

The acceleration of u^a is

$$u^a \bar{\nabla}_a u^b = -\frac{a_1^2 - a_2^2}{\bar{\Delta}_\theta} \sin \bar{\theta} \cos \bar{\theta} \frac{\partial}{\partial \bar{\theta}} = -\frac{1}{4\rho} \bar{\nabla}^b \rho = \frac{1}{2\bar{\Delta}_\theta} \bar{\nabla}^b \bar{\Delta}_\theta, \quad (2.25)$$

and it follows that T_{ab}^h in (2.23) is conserved with respect to the metric $\bar{g}_{ab}$, i.e. $\bar{\nabla}^a T_{ab}^h = 0$. The term independent of M in (2.23) corresponds to the four-dimensional gravitational conformal anomaly [14], and it can be disregarded for large enough M . The total energy associated to the fluid is

$$4\pi G_{N,5} E = 4\pi G_{N,5} \int_{S^3} d\Omega_3 \bar{\rho} = \frac{3\pi^2 M}{(1 - a_1^2)(1 - a_2^2)}, \quad (2.26)$$

which can be compared with the total ADM mass $\mathcal{M}$ in (2.4).

The perfect fluid form of the holographic stress-energy tensor for the Kerr-AdS black hole in (2.23) mirrors the result for four-dimensional backgrounds from [40], and is in fact expected on general grounds. The conformally invariant nature of (2.23) hints at an underlying conformally invariant field theory dual. Following the holographic renormalization group flow towards the IR, the corrections to the metric will in general break this conformal invariance, which is interpreted as the departure from a conformal fixed point in the field theory. If we think of the field theory as free or weakly coupled at the UV, the flow towards the IR may bring the theory to a strongly coupled regime, resulting in, among other things, corrections to the perfect fluid form of its stress-energy tensor (2.23). In the spirit of holographic renormalization group flow, we will assume that the radial evolution given by Einstein equations will give the proper picture for the dual field theory as the energy scale is brought down.

For high-temperatures, one can interpret the results of [41] and [4] as the equivalence between the general relativity equations and the Navier-Stokes equations for the fluid mechanics. In our study, we want to consider low temperatures, outside the scope of their analysis. This necessarily leads us to consider small black holes, because large black holes in AdS are hot. To see this latter point, let us recall that, when written in terms of r_+ , a_1 and a_2 , the temperature of the black hole outer horizon (2.6) is

$$2\pi T_+ = r_+ \left(\frac{r_+^2 + 1}{r_+^2 + a_1^2} + \frac{r_+^2 + 1}{r_+^2 + a_2^2} - \frac{1}{r_+^2} \right), \quad (2.27)$$

which, for constant r_+ , is a monotonically decreasing function of a_1 and a_2 . However, for $a_1, a_2 < 1$, T_+ is always larger than $T_{\min} = (2r_+^2 - 1)/(2\pi r_+)$. We can therefore conclude that there cannot be a zero-temperature black hole for $r_+^2 > 1/2$.

There are basically two ways of dealing with this issue. One is to forgo our misgivings and simply take the calculations we will be performing as definitions of the field theory system in the infrared regime for small T_+ . The other is to view the correspondence between the gravitational and the field theoretical system as dynamical, and give up on the equilibrium view of the fluid, from which the equivalence between mean free path and r_+ was deduced. Either choice will have interesting consequences.

Dwelling in the second choice for a moment, if the fluid is no longer in equilibrium, there is no reason to restrict the analysis to time-independent configurations. The conformal transformation to conformally flat boundary coordinates mixes time and space and induces time-dependence for the dynamics of the state. Using the (classical) conformal symmetry of the fluid, let us then perform the transformation of $\bar{g}_{ab}$ to conformally flat coordinates. The transformation

$$\hat{t} = \frac{1}{2} \tan \frac{\bar{t} + \bar{\chi}}{2} + \frac{1}{2} \tan \frac{\bar{t} - \bar{\chi}}{2}, \quad \hat{r} = \frac{1}{2} \tan \frac{\bar{t} + \bar{\chi}}{2} - \frac{1}{2} \tan \frac{\bar{t} - \bar{\chi}}{2}, \quad (2.28)$$

where

$$\cos \bar{\chi} = \sin \bar{\theta} \cos \bar{\phi}_1, \quad \tan \hat{\theta} = \frac{\cos \bar{\theta}}{\sin \bar{\theta} \sin \bar{\phi}_1} \quad (2.29)$$

and $\hat{\phi} = \bar{\phi}_2$, turns the metric into

$$d\hat{s}^2 = \frac{-d\hat{t}^2 + d\hat{r}^2 + \hat{r}^2(d\hat{\theta}^2 + \sin^2 \hat{\theta} d\hat{\phi}^2)}{(1 + \frac{1}{4}(\hat{t} + \hat{r})^2)(1 + \frac{1}{4}(\hat{t} - \hat{r})^2)}. \quad (2.30)$$

We recall that we use bars to refer to coordinates in the $\mathbb{R} \times S^3$ manifold $(\{\bar{t}, \bar{\theta}, \bar{\phi}_1, \bar{\phi}_2\})$ and hatted coordinates for Minkowski space $\mathbb{R}^{1,3}$ $(\{\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi}\})$. The flat metric will be henceforth referred to as $\hat{g}_{ab}$.

We can use the conformal symmetry of the leading order perturbation h_{ab} in (2.19) to derive the Poincaré coordinates, flat AdS version of the holographic stress-energy tensor, in the same approximation as (2.23). The perturbation is explicitly given by the same formula as (2.18)

$$h_{ab} = 2M \frac{z^2}{\hat{\Delta}^2} \left(n_a n_b + \frac{1}{\hat{\Delta}} \hat{\chi}_a \hat{\chi}_b \right) \quad (2.31)$$

where $n_a = (dz)_a$, $\hat{\chi}_a = \hat{g}_{ab} \chi^b$, and

$$\begin{aligned} \hat{\Delta} &= -\hat{g}_{ab} \chi^a \chi^b \\ &= \left(1 + \frac{1}{4}(\hat{t} + \hat{r})^2 \right) \left(1 + \frac{1}{4}(\hat{t} - \hat{r})^2 \right) - \left(1 + \frac{1}{4}(\hat{t}^2 - \hat{r}^2) \right)^2 a_1^2 - \hat{r}^2 \left(a_1^2 \cos^2 \hat{\theta} + a_2^2 \sin^2 \hat{\theta} \right). \end{aligned} \quad (2.32)$$

The flat-space version of the stress-energy tensor $\hat{T}_{ab}^h = \omega^{-2} T_{ab}^h$, with

$$\omega^2 = \left(1 + \frac{1}{4}(\hat{t} + \hat{r})^2 \right) \left(1 + \frac{1}{4}(\hat{t} - \hat{r})^2 \right) \quad (2.33)$$

is conserved with respect to the flat metric $\hat{g}_{ab}$ (see appendix D in [42]), and it reads

$$\hat{T}_{ab}^h = \hat{\rho} \hat{u}_a \hat{u}_b + \hat{P}(\hat{g}_{ab} + \hat{u}_a \hat{u}_b), \quad \hat{\rho} = 3\hat{P} = \frac{1}{4\pi G_{N,5}} \frac{3M}{2\hat{\Delta}^2}, \quad (2.34)$$

and $\hat{u}^a = \chi^a / \sqrt{\hat{\Delta}}$ is the normalized velocity of the fluid with respect to $\hat{g}_{ab}$. The energy profile as a function of the flat coordinates is shown in figure 1.

In order to study the near ultraviolet behavior in the presence of the perturbation, we turn to the effect of the asymptotic form of the correction h_{ab} . The pure AdS metric encodes the conformal structure — at least of its conformal part. One can then interpret the effect of h_{ab} as encoding the departure from the ultraviolet fixed point as one turns on the renormalization group flow towards the infrared.

One should note at this point that there is an ambiguity in choosing $\hat{\phi} = \bar{\phi}_2$ in passing from the $\mathbb{R} \times S^3$ conformal structure to Minkowski space $\mathbb{R}^{3,1}$. The Kerr-AdS black hole background can be interpreted as the resulting CFT (thermal) state obtained by turning expectation values for the operators comprising the generators of the Cartan subalgebra of the conformal group $SO(4, 2)$

$$\frac{\partial}{\partial \hat{t}} \sim M_{01}, \quad \frac{\partial}{\partial \hat{\phi}_1} \sim M_{23}, \quad \frac{\partial}{\partial \hat{\phi}_2} \sim M_{45} \quad (2.35)$$

Figure 1. Three snapshots of the energy density profile (2.34), as sagittal projection. The free propagation shows two “blobs” of radiation traveling along the z axis and meeting close to the origin as $t \rightarrow 0$. The two parameters a_1 and a_2 control the spread of each blob away from the symmetry axis and the astigmatism as the two blobs meet.

where the right-hand side makes use of an explicit representation of the conformal group as six-dimensional “Lorentz” generators of $\mathbb{R}^{2,4}$. In choosing $\hat{\phi} = \bar{\phi}_2$, we explicitly chose M_{45} as the angular momentum generator in the four-dimensional picture. One could as well choose M_{23} , which would amount to setting $\hat{\phi} = \bar{\phi}_1$. The effect of this choice would be simply to interchange a_1 with a_2 in quantities such as (2.32) and (2.34). We will come back to this point in section 4.

A comparatively simple and preliminary calculation is to study the effect on the conformal dimension Δ , as read from the two-point function of two scalar primary operators as we depart from the conformal point in the ultraviolet. In general, one expects a number of these primary operators to share the same conformal dimension, and the presence of the background does not lift this degeneracy. As it is well-known, the primary operators satisfy a conformal Casimir Ward identity, which can be seen as the scalar wave equation in AdS. To give the calculation a purpose independent of the background conformal structure chosen at infinity, we will introduce the $SO(4,2)$ invariant quantity σ :

$$\sigma = \frac{z_i^2 + z_j^2 + |\mathbf{x}_i - \mathbf{x}_j|^2}{2z_i z_j}, \quad (2.36)$$

which is related to the invariant AdS geodesic distance ℓ by $\sigma = \cosh \ell$. As it is well-known, the pure AdS scalar propagator

$$G(\sigma) = \langle \Phi(z_i, \mathbf{x}_i) \Phi(z_j, \mathbf{x}_j) \rangle = 2^\Delta K \sigma^{-\Delta} {}_2F_1 \left(\frac{1}{2}\Delta, \frac{1}{2}\Delta + \frac{1}{2}; \Delta - 1; \sigma^{-2} \right), \quad (2.37)$$

reproduces the conformal two-point function for two local primary operators with conformal dimension Δ sitting at the same value of the cut-off parameter $z = z_i = z_j$. As one considers large values of z , however, the simple CFT expression $|\mathbf{x}_i - \mathbf{x}_j|^{-2\Delta}$ no longer holds, since keeping the coordinate z fixed (and large) is akin to keeping fixed the length of the flux tube between the insertions [43], which is not usually achieved in QFT calculations. At any rate, as seen in [44] — see also [21], this dependence can be trivialized for pure AdS by a suitable choice of the z coordinate.

The correction h_{ab} to the pure AdS metric g_{ab} at spatial infinity modifies this dependence (indices are raised with the vacuum metric):

$$\bar{\nabla}^2 G = \nabla^2 G - g^{ab} g^{cd} (\nabla_a h_{db} - \nabla_d h_{ab}) \nabla_c G - h^{ab} \nabla_a \nabla_b G = \Delta(\Delta - 4)G, \quad (2.38)$$

where g_{ab} and ∇_a are the vacuum AdS metric and covariant derivative, and we are assuming non-coincident insertion points. The last term $\Delta(\Delta - 4)$ is equal to the AdS mass squared, which is equal to the $SO(4, 2)$ quadratic Casimir. Continuing with the calculation of (2.38), we have, due to the tracelessness of h_{ab} ,

$$g^{ab} g^{cd} (\nabla_a h_{db} - \nabla_d h_{ab}) = \nabla_a h^{ac}, \quad (2.39)$$

where indices are raised with the pure AdS₅ metric g^{ab} . By means of the formulas (A.2) and (A.3), with the choice of $\tilde{\Omega} = z$ so that $\tilde{g}_{ab} = \hat{g}_{ab}$ in appendix A, we relate this divergence to

$$\nabla_a h^{ac} = z^4 \left(\hat{\nabla}_a \hat{h}^{ac} - \frac{4Mz}{\hat{\Delta}^2} \hat{n}^c \right) = 0, \quad (2.40)$$

which vanishes owing to the tracelessness of h_{ab} and the relation between the acceleration of u^a and the gradient of $\hat{\Delta}$. The term $h^{ab} \nabla_a \nabla_b G$ in (2.38) gives less trouble, and the resulting equation is

$$\left(\sigma^2 - 1 - z^4 \hat{h}^{ab} \nabla_a \sigma \nabla_b \sigma \right) \frac{d^2 G}{d\sigma^2} + 5\sigma \frac{dG}{d\sigma} = \Delta(\Delta - 4)G, \quad (2.41)$$

where $\hat{h}^{ab} = \hat{g}^{ac} h_{cd} \hat{g}^{db}$ has its indices raised by the flat metric $\hat{g}_{ab}$. Using the formula for σ given in terms of the flat coordinates and z ,

$$z^4 \hat{h}^{ab} \nabla_a \sigma \nabla_b \sigma = \frac{2M}{\hat{\Delta}^2(\mathbf{x}_i)} |\mathbf{x}_{ij}|^4 + \mathcal{O}(z^2, |\mathbf{x}_{ij}|^6), \quad (2.42)$$

the simple Ansatz $G \propto \sigma^{-\Delta'}$ then captures the asymptotic behavior of the solution, with

$$\begin{aligned} \Delta' &= \Delta + \frac{\Delta(\Delta + 1)}{2(\Delta - 2)} \left(\frac{z^4}{|\mathbf{x}_{ij}|^4} + \frac{2Mz^4}{\hat{\Delta}^2(\mathbf{x}_i)} \right) + \dots \\ &= \Delta_{\text{AdS}} \left(1 + \frac{1}{4\pi G_{N,5}} \frac{2(\Delta + 1)}{3(\Delta - 2)} \hat{\rho} z^4 \right) + \dots \end{aligned} \quad (2.43)$$

where we can read the anomalous dimension as the correction to the pure AdS value Δ_{AdS} , defined by the expansion of the vacuum propagator (2.37), that is, in the absence of the black hole.

Note that the effect of the background, at least perturbatively, is to *increase* the conformal dimension of the perturbation for $\Delta > 2$, which applies in particular to the marginal case $\Delta = 4$. For z infinitesimally close to zero and Δ sufficiently close to 4, (2.43) tells us that, as the correction to the AdS value Δ_{AdS} increases, Δ' crosses the marginal value of 4 and the corresponding perturbation becomes irrelevant, and as such should not alter substantially the fate of the flow as one approaches the infrared limit. On the other hand, the fact that corrections to the bare AdS value do become large at $z^4 \sim M^{-1}$ can be seen as hint that the theory is becoming increasingly strongly coupled towards the IR. As we will see in the following, the remark about the fluctuations not substantially altering the fate of the state as it flows will have to be revised.

3 Scalar perturbations

A complete holographic interpretation of a scalar perturbation requires the analysis of the field in the exact Kerr-AdS₅ background, which we will consider in the low temperature regime. Let us start with generic considerations on the scalar field. The Klein-Gordon equation in Kerr-AdS₅ is separable in terms of two (Fuchsian) ordinary differential equations, which can be cast in the standard form:

$$\begin{aligned} \frac{d^2 y}{dz^2} + p(z) \frac{dy}{dz} + q(z)y(z) &= 0, \\ p(z) &= \frac{1-\theta_0}{z} + \frac{1-\theta_t}{z-t_0} + \frac{1-\theta_1}{z-1}, \\ q(z) &= \frac{(\theta_0 + \theta_t + \theta_1 + \theta_\infty - 2)(\theta_0 + \theta_t + \theta_1 - \theta_\infty - 2)}{4z(z-1)} - \frac{t_0(t_0-1)K_0}{z(z-1)(z-t_0)}, \end{aligned} \quad (3.1)$$

where the set $\{\theta_0, \theta_t, \theta_1, \theta_\infty\}$ are called the *single monodromy parameters* and $\{t_0, K_0\}$ are called the *accessory parameters*. For the Klein-Gordon field mode Φ_{n,ℓ,m_1,m_2} , with

$$\Phi_{n,\ell,m_1,m_2}(t, r, \theta, \phi_1, \phi_2) = e^{-i\omega t + im_1 \phi_1 + im_2 \phi_2} R_{n,\ell,m_1,m_2}(r) S_{\ell,m_1,m_2}(\theta), \quad (3.2)$$

the radial system's single monodromy parameters are

$$\theta_{\text{Rad},0} = \theta_-, \quad \theta_{\text{Rad},t} = \theta_+, \quad \theta_{\text{Rad},1} = 2 - \Delta, \quad \theta_{\text{Rad},\infty} = \theta_0$$

where

$$\theta_k = \frac{i}{2\pi} \left(\frac{\omega - m_1 \Omega_{k,1} - m_2 \Omega_{k,2}}{T_k} \right), \quad k = \pm, 0, \quad \theta_\infty = 2 - \Delta, \quad (3.3)$$

and the respective accessory parameters are

$$t_{\text{Rad},0} = z_0 = \frac{r_+^2 - r_-^2}{r_+^2 - r_0^2}, \quad (3.4)$$

$$\begin{aligned} 4z_0(z_0-1)K_{\text{Rad},0} &= -\frac{C_\ell + r_-^2 \Delta(\Delta-4) - \omega^2}{r_+^2 - r_0^2} - (z_0-1)[(\theta_- + \theta_+ - 1)^2 - \theta_0^2 - 1] \\ &\quad - z_0 \left[2(\theta_+ - 1)(1 - \Delta) + (2 - \Delta)^2 - 2 \right]. \end{aligned} \quad (3.5)$$

The angular system's single monodromy parameters are given by

$$\theta_{\text{Ang},0} = m_1, \quad \theta_{\text{Ang},t} = m_2, \quad \theta_{\text{Ang},1} = 2 - \Delta, \quad \theta_{\text{Ang},\infty} \equiv \varsigma = \omega + a_1 m_1 + a_2 m_2, \quad (3.6)$$

and the accessory parameters are

$$t_{\text{Ang},0} = u_0 = \frac{a_2^2 - a_1^2}{a_2^2 - 1}, \quad (3.7)$$

$$\begin{aligned} 4u_0(u_0-1)K_{\text{Ang},0} &= -\frac{C_\ell - \omega^2 - a_1^2 \Delta(\Delta-4)}{1 - a_2^2} - u_0 \left[(m_2 + \Delta - 1)^2 - m_2^2 - 1 \right] \\ &\quad - (u_0 - 1) \left[(m_1 + m_2 + 1)^2 - \varsigma^2 - 1 \right]. \end{aligned} \quad (3.8)$$

In previous work [18, 21], the authors stressed how the eigenvalue problem for the angular equation and the QNM problem can be solved in terms of the *composite monodromy parameters* associated with the differential equation (3.1). In particular, the transformation between the accessory parameters $\{t_0, K_0\}$ and the composite monodromy parameters $\{\sigma_{0t}, \sigma_{t1}\}$ is defined in terms of the Painlevé VI transcendent tau function:

$$\tau(\theta_0, \theta_t, \theta_1, \theta_\infty; \sigma_{0t}, \sigma_{t1}; t_0) = 0, \quad (3.9)$$

$$\frac{\partial}{\partial t} \log \tau(\theta_0, \theta_t - 1, \theta_1, \theta_\infty + 1; \sigma_{0t} - 1, \sigma_{t1} - 1; t_0) - \frac{(\theta_t - 1)\theta_1}{2(t_0 - 1)} - \frac{(\theta_t - 1)\theta_0}{2t_0} = K_0. \quad (3.10)$$

The expansion of the tau function for small t_0 is given in appendix B, and the reader is referred to [35] for further details.

As derived in [21], the quantization condition for the angular eigenfunctions can be cast in terms of the monodromy parameters:

$$\sigma_{\text{Ang}, 0t} = -\ell, \quad \ell \in \mathbb{N}, \quad (3.11)$$

from which we can compute the small u_0 expansion of the separation constant C_ℓ :

$$\begin{aligned} C_\ell = & \omega^2 + \ell(\ell + 2) - \varsigma^2 - \frac{a_1^2 + a_2^2}{2} \left(\ell(\ell + 2) - \varsigma^2 - \Delta(\Delta - 4) \right) \\ & - \frac{(a_1^2 - a_2^2)(m_1^2 - m_2^2)}{2\ell(\ell + 2)} \left(\ell(\ell + 2) - \varsigma^2 + (\Delta - 2)^2 \right) \\ & - \frac{(a_1^2 - a_2^2)^2}{1 - a_2^2} \left[\frac{(\ell(\ell + 2) + m_2^2 - m_1^2)(\ell(\ell + 2) + (\Delta - 2)^2 - \varsigma^2)}{2\ell(\ell + 2)} \right. \\ & - \frac{13}{32} \ell(\ell + 2) + \frac{1}{32} (5 + 14(m_1^2 + \varsigma^2) - 18(m_2^2 + (\Delta - 2)^2)) \\ & - \frac{((m_1 + 1)^2 - m_2^2)((1 - m_1)^2 - m_2^2)((\Delta - 1)^2 - \varsigma^2)((\Delta - 3)^2 - \varsigma^2)}{32(\ell - 1)(\ell + 3)} \\ & + \frac{((m_1^2 - m_2^2)((\Delta - 2)^2 - \varsigma^2) + 8)^2 - 64 - 2(m_1^2 + m_2^2)((\Delta - 2)^2 - \varsigma^2)^2}{32\ell(\ell + 2)} \\ & \left. - \frac{2(m_1^2 - m_2^2)^2((\Delta - 2)^2 + \varsigma^2)}{32\ell(\ell + 2)} - \frac{(m_1^2 - m_2^2)^2((\Delta - 2)^2 - \varsigma^2)^2}{64} \left(\frac{1}{(\ell + 2)^3} - \frac{1}{\ell^3} \right) \right] \\ & + \mathcal{O} \left(\left(\frac{a_1^2 - a_2^2}{1 - a_2^2} \right)^3 \right). \end{aligned} \quad (3.12)$$

The radial QNMs can also be determined with similar techniques. As seen in [18], the purely ingoing boundary condition at infinity requires that the composite monodromy parameter satisfies:

$$R_{n, \ell, m_1, m_2}(r) \sim \begin{cases} (r - r_+)^{\frac{1}{2}\theta_+}, & r \rightarrow r_+; \\ r^{2-\Delta}, & r \rightarrow \infty, \end{cases} \implies \cos \pi \sigma_{1t} = \cos \pi(\theta_t + \theta_1). \quad (3.13)$$

This condition can be conveniently written in terms of the expansion parameter s that enters the tau function expansion (B.6). From the explicit representation for the monodromy

matrices (see also [45]), we have

$$\begin{aligned} \sin^2 \pi \sigma_{0t} \cos \pi \sigma_{1t} &= \cos \pi \theta_0 \cos \pi \theta_\infty + \cos \pi \theta_t \cos \pi \theta_1 \\ &\quad - \cos \pi \sigma_{0t} (\cos \pi \theta_0 \cos \pi \theta_1 + \cos \pi \theta_t \cos \pi \theta_\infty) \\ &\quad - \frac{1}{2} (\cos \pi \theta_\infty - \cos \pi (\theta_1 - \sigma_{0t})) (\cos \pi \theta_0 - \cos \pi (\theta_t - \sigma_{0t})) s \\ &\quad - \frac{1}{2} (\cos \pi \theta_\infty - \cos \pi (\theta_1 + \sigma_{0t})) (\cos \pi \theta_0 - \cos \pi (\theta_t + \sigma_{0t})) s^{-1}. \end{aligned} \quad (3.14)$$

Note that this equation is of the form $A - Bs - Cs^{-1} = 0$, and by direct calculation it can be checked that

$$A = B + C + \sin^2 \pi \sigma_{0t} (\cos \pi (\theta_t + \theta_1) - \cos \pi \sigma_{1t}).$$

Then, due to the quantization condition (3.13), the roots are readily seen to be $s = 1$ and $s = C/B$, with the last one compatible with the parametrization of the monodromy matrices. The parameter s can be written explicitly in terms of the monodromy parameters as

$$s = \frac{\sin \frac{\pi}{2} (\theta_t + \theta_0 + \sigma) \sin \frac{\pi}{2} (\theta_t - \theta_0 + \sigma) \sin \frac{\pi}{2} (\theta_1 + \theta_\infty + \sigma) \sin \frac{\pi}{2} (\theta_1 - \theta_\infty + \sigma)}{\sin \frac{\pi}{2} (\theta_t + \theta_0 - \sigma) \sin \frac{\pi}{2} (\theta_t - \theta_0 - \sigma) \sin \frac{\pi}{2} (\theta_1 + \theta_\infty - \sigma) \sin \frac{\pi}{2} (\theta_1 - \theta_\infty - \sigma)}. \quad (3.15)$$

This expression for s , when substituted in the tau function with the θ_k parameters of the radial equation, will allow us through (3.10) to constrain the frequencies ω , and thus determine the QNMs of the scalar perturbation.

3.1 The low-temperature limit

At small r_+ , one can check from (2.27) that the requirement of positive temperature limits the range of available a_1 and a_2 , which will be taken to be of order r_+ . Bearing this in mind, we define the parameters

$$\epsilon = \frac{r_+^2 - r_-^2}{2r_+^2}, \quad \alpha_1 = \frac{a_1}{r_+}, \quad \alpha_2 = \frac{a_2}{r_+}, \quad \alpha_+^2 = \frac{a_1^2 + a_2^2}{2r_+^2}, \quad \alpha_-^2 = \frac{a_1^2 - a_2^2}{2r_+^2}, \quad (3.16)$$

with the understanding that α_+ will be of order 1. We will see below that the value of ϵ will be constrained by temperature requirements, whereas α_- is completely determined from ϵ , α_+ and r_+ .

The radii and mass parameters are in turn given in terms of ϵ , and $\alpha_\pm$ by

$$r_-^2 = (1 - 2\epsilon)r_+^2, \quad -r_0^2 = 1 + 2(1 + \alpha_+^2 - \epsilon)r_+^2, \quad (3.17)$$

$$2M = -(1 + r_0^2)(1 + r_+^2)(1 + r_-^2) = 2r_+^2(1 + r_+^2)(1 + \alpha_+^2 - \epsilon)(1 + (1 - 2\epsilon)r_+^2). \quad (3.18)$$

Given that the black hole charges are determined by three parameters, the following relation between r_+ , ϵ , α_+ and α_- holds

$$(\alpha_+^2 - (1 - 2\epsilon)r_+^2)^2 - \alpha_-^4 = (1 - 2\epsilon)(1 + r_+^2)(1 + (1 - 2\epsilon)r_+^2), \quad (3.19)$$

so that, for fixed r_+ and ϵ , the set of available α_+^2 and α_-^2 spans a hyperbola. The requisite of positive temperature will truncate the hyperbola well before $\alpha_+^2 < 1/r_+^2$, which corresponds to $a_1, a_2 < 1$. We will assume that $a_1 > a_2$, i.e. α_-^2 positive.

Let us now use $z_0 = (r_+^2 - r_-^2)/(r_+^2 - r_0^2)$ as extremality parameter. We can see that it is proportional to the temperature by employing (2.6)

$$2\pi T_+ = \frac{r_+(r_+^2 - r_-^2)(r_+^2 - r_0^2)}{(r_+^2 + a_1^2)(r_+^2 + a_2^2)} = z_0 \frac{(1 + (3 + 2\alpha_+^2 - 2\epsilon)r_+^2)^2}{2r_+^3(1 + \alpha_+^2 - \epsilon)(1 + (1 - 2\epsilon)r_+^2)}, \quad (3.20)$$

and it vanishes with ϵ :

$$z_0 = \frac{r_+^2 - r_-^2}{r_+^2 - r_0^2} = \frac{2\epsilon r_+^2}{1 + (3 + 2\alpha_+^2 - 2\epsilon)r_+^2}, \quad \epsilon = z_0 \frac{1 + (3 + 2\alpha_+^2)r_+^2}{2(1 + z_0)r_+^2}. \quad (3.21)$$

The parametrization of T_+ in terms of z_0 is relevant for the near-extremal limit of the monodromy parameters (3.3), to be discussed below. Dwelling on this a little further, consider the parametrization of T_+ in terms of ϵ from (3.20)

$$2\pi T_+ = \frac{\epsilon}{r_+} \frac{1 + (3 + 2\alpha_+^2 - 2\epsilon)r_+^2}{(1 + \alpha_+^2 - \epsilon)(1 + (1 - 2\epsilon)r_+^2)}. \quad (3.22)$$

We find that the temperature is generically proportional to ϵ , but small temperatures are defined relative to r_+ . This means that, for small black holes, it is not sufficient to require small ϵ , but also that $\epsilon \ll r_+$. In addition, we will take $\alpha_{\pm}$ to be of order one, following the argument at the beginning of the section.

With the parametrization above, the $\theta_{\pm}$ parameters defined by (3.3) have the asymptotic form:

$$\theta_{\pm} = \pm \frac{i\Lambda}{z_0} + \frac{\theta_{\star}}{2}, \quad (3.23)$$

where Λ and $\theta_{\star}$ will have their expansions in r_+ given below. Low temperature ($\epsilon \ll 1$), small r_+ expansions of the relevant quantities are

$$\theta_0 = \omega + (m_1\alpha_1 + m_2\alpha_2)r_+ - 3(1 + \alpha_+^2 - \epsilon)\omega r_+^2 + \dots \quad (3.24)$$

$$\begin{aligned} \Lambda = & -(2(1 + \alpha_2^2) - \epsilon(1 - \alpha_2^2))\frac{m_1\alpha_1}{2}r_+^2 - (2(1 + \alpha_1^2) - \epsilon(1 - \alpha_1^2))\frac{m_2\alpha_2}{2}r_+^2 \\ & + (2(1 + \alpha_+^2) - (3 + \alpha_+^2)\epsilon)\omega r_+^3 + \dots \end{aligned} \quad (3.25)$$

$$\begin{aligned} \theta_{\star} = & -i(2(1 - \alpha_2^2) - \epsilon(1 - 3\alpha_2^2))\frac{m_1\alpha_1}{4} - i(2(1 - \alpha_1^2) - \epsilon(1 - 3\alpha_1^2))\frac{m_2\alpha_2}{4} \\ & + i(2 + \epsilon)(1 + \alpha_+^2)\frac{\omega r_+}{2} + \dots \end{aligned} \quad (3.26)$$

where $\alpha_i = a_i/r_+$, and the ellipsis stands for terms of higher order in r_+ and ϵ , which will not be relevant for the following analysis. Note that the leading behavior of Λ and $\theta_{\star}$ with r_+ depends on whether m_1 and m_2 vanish or not. In particular, if $m_1 = m_2 = 0$, then Λ vanishes as r_+^3 and $\theta_{\star}$ vanishes as r_+ . Otherwise, they vanish as r_+^2 and have finite limits, respectively. This behavior does not depend on whether ϵ , which is proportional to the temperature, is zero or not.

The quantization conditions (3.10) can be applied, along with (3.15) to find the QNMs' frequencies. For generic temperatures, the study was performed in [18]. The low-temperature limit is a confluence limit of the Heun equation, given that both θ_+ and θ_- diverge as $z_0 \rightarrow 0$. The tau function defined in (B.1) will have to be studied in this limit.

3.1.1 The confluent limit of the tau function

The Plemelj operator D which composes the Fredholm determinant (B.1) has a fairly straightforward confluence limit. We will assume that the σ and κ parameters, defined in (B.6), have finite limits as $z_0 \rightarrow 0$, which will be verified *a posteriori*. In this case, the hypergeometric functions entering the parametrices D can be expanded in an asymptotic series for small z_0 yielding confluent hypergeometric functions

$$\Psi(-\sigma, \theta_+, \theta_-; z_0/z) = \Psi_c^{(0)}(-\sigma, \theta_*; i\Lambda/z) + z_0 \Psi_c^{(1)}(-\sigma, \psi_+, \psi_-; i\Lambda/z) + \dots \quad (3.27)$$

where the confluent parametrix is, defining $\theta_* = \psi_+ + \psi_-$:

$$\Psi_c^{(0)}(-\sigma, \theta_*; i\Lambda/z) = \begin{pmatrix} \phi_c(-\sigma, \theta_*; i\Lambda/z) & \chi_c(-\sigma, \theta_*; i\Lambda/z) \\ \chi_c(\sigma, \theta_*; i\Lambda/z) & \phi_c(\sigma, \theta_*; i\Lambda/z) \end{pmatrix},$$

$$\phi_c(\pm\sigma, \theta_*; i\Lambda/z) = {}_1F_1\left(\frac{-\theta_* \pm \sigma}{2}; \pm\sigma; -i\Lambda/z\right), \quad (3.28)$$

$$\chi_c(\theta_*, \pm\sigma; -i\Lambda/z) = \pm \frac{-\theta_* \pm \sigma}{2\sigma(1 \pm \sigma)} \frac{i\Lambda}{z} {}_1F_1\left(1 + \frac{-\theta_* \pm \sigma}{2}, 2 \pm \sigma; -i\Lambda/z\right). \quad (3.29)$$

It follows that the operator D is analytic in z_0 and Λ

$$D(z_0, i\Lambda) = D_0(i\Lambda) + z_0 D_1(i\Lambda) + z_0^2 D_2(i\Lambda) + \dots, \quad (3.30)$$

with each term computed recursively in terms of confluent hypergeometric functions. The first term $D_0(i\Lambda)$ defines the Fredholm determinant expression for the Painlevé V transcendent defined in [25]. As we will see, we will not need the corrections $D_1, D_2, \dots$ for our analysis.

Let us now study the behavior of the s parameter appearing in (B.1) as $z_0 \rightarrow 0$. The expansion of the Painlevé VI tau function has the schematic structure (B.9)

$$\tau = z_0^{A_0} (1 - z_0)^{A_1} (1 + A_{1,-1} Y^{-1} z_0 + A_{1,0} z_0 + A_{1,1} Y z_0 + \mathcal{O}(z_0^2)), \quad (3.31)$$

where $A_0, A_1, A_{1,0}, A_{1,\pm 1}$ are monodromy-dependent coefficients and $Y = \kappa z_0^\sigma$, with κ defined in (B.6). The series, save from the $z_0^{A_0}$ term, is analytic in z_0 and meromorphic in Y . Finding the non-trivial zero for Y — the first condition in (3.10) — can then be achieved perturbatively (assuming $\Re\sigma > 0$)

$$Y = \kappa z_0^\sigma = -A_{1,-1} z_0 \chi_{VI}(z_0), \quad (3.32)$$

where $\chi_{VI}(z_0)$ is given as a series in z_0 . The left hand side can be written as

$$\kappa z_0^\sigma = s \Pi z_0^\sigma \frac{\Gamma(1 + \frac{i}{2} \Lambda z_0^{-1} + \frac{\sigma}{2})}{\Gamma(1 + \frac{i}{2} \Lambda z_0^{-1} - \frac{\sigma}{2})} \quad (3.33)$$

with Π defined as the confluent limit of the product of Gamma functions in (B.6)

$$\Pi = \frac{\Gamma^2(1 - \sigma) \Gamma(1 + \frac{1}{2}(\theta_* + \sigma)) \Gamma(1 + \frac{1}{2}(\theta_1 - \theta_\infty + \sigma)) \Gamma(1 + \frac{1}{2}(\theta_1 + \theta_\infty + \sigma))}{\Gamma^2(1 + \sigma) \Gamma(1 + \frac{1}{2}(\theta_* - \sigma)) \Gamma(1 + \frac{1}{2}(\theta_1 - \theta_\infty - \sigma)) \Gamma(1 + \frac{1}{2}(\theta_1 + \theta_\infty - \sigma))}. \quad (3.34)$$

The right hand-side of (3.32) can now be expanded

$$A_{1,-1} z_0 \chi_{VI}(z_0) = A_{1,-1}^V z_0 \left(\frac{\sigma}{2} + \frac{i\Lambda}{z_0} \right) (\chi_V(i\Lambda) + \mathcal{O}(\Lambda^n z_0^m)) \quad (3.35)$$

where $\chi_V(i\Lambda)$ is the series analogous to (3.32) that we find from the Painlevé V tau function. The correction terms to the series are, as argued above, of the form $\Lambda^n z_0^m$ and hence subleading. Using the property of the gamma function in the left hand side to cancel the $\sigma/2 + i\Lambda/z_0$ term, we have the expansion

$$\frac{\Gamma(\frac{i}{2}\Lambda z_0^{-1} + \frac{\sigma}{2})}{\Gamma(1 + \frac{i}{2}\Lambda z_0^{-1} - \frac{\sigma}{2})} = \left(\frac{i\Lambda}{z_0} \right)^{\sigma-1} \left(1 + \mathcal{O}\left(\frac{z_0^2}{\Lambda^2} \right) \right), \quad (3.36)$$

for the argument of the Gamma function not close to the negative real axis — which should be expected for real positive frequencies. Combining the expansions, we find that in the confluent limit, (3.32) is replaced by

$$\kappa z^\sigma = s \Pi(i\Lambda)^\sigma \left(1 + \mathcal{O}(z_0^2 \Lambda^{-2}) \right). \quad (3.37)$$

In conclusion, we have that in order to compute the first non-trivial z_0 correction, it suffices to consider the Painlevé V expansion. Corrections to the monodromy parameter are of order z_0^2 , and hence less relevant for the confluent limit.

Let us now review the type of correction we have for the tau function away from the confluence point, where it coincides with the Painlevé V transcendent. The relevant expansions in Λ and z_0 are:

1. Expansion of the s parameter, which is given in terms of powers of $-iz_0/\Lambda$.
2. Expansion of the parametrices $\Psi_c^{(i)}$ (3.27), which are given in terms of hypergeometric functions with monomials in $z_0^n (i\Lambda)^m$.

It is not difficult to see that, in the small Λ limit, the first type of expansion will dominate over the second. Disregarding the contributions of the second type, we have exactly the same type of expansion as the Painlevé V. This means that the first non-trivial correction to the accessory parameter problem of the confluent Heun equation comes in the monodromy parameter, not in the tau function expansion. In short, the first non-trivial correction is obtained by expanding the κ parameter.

The final step needed before the calculation can be performed is solving for the quantization condition for the radial equation, expressed in terms of the monodromy parameters in (3.15). It approaches, as $z_0 \rightarrow 0$,

$$s = e^{\mp i\pi\sigma} \frac{\sin \frac{\pi}{2}(\theta_\star + \sigma)}{\sin \frac{\pi}{2}(\theta_\star - \sigma)} \frac{\sin \frac{\pi}{2}(\theta_1 + \theta_\infty + \sigma)}{\sin \frac{\pi}{2}(\theta_1 + \theta_\infty - \sigma)} \frac{\sin \frac{\pi}{2}(\theta_1 - \theta_\infty + \sigma)}{\sin \frac{\pi}{2}(\theta_1 - \theta_\infty - \sigma)} + \dots, \quad (3.38)$$

for $\Re\Lambda > 0$ or $\Re\Lambda < 0$, respectively. In this case the convergence is exponential, up to terms of order $e^{-2\pi|\Lambda|/z_0}$. Again, corrections are (exponentially) suppressed in terms of $|\Lambda|$.

3.1.2 Quasi-normal modes

With these premises, the solution for the QNMs follows from the radial quantization condition, expressed in terms of s in (3.38), and the analysis is actually closely related to that of [18]. In the $z_0 \rightarrow 0$ limit, the conditions to solve become

$$\tau_V(\theta_*, \theta_1, \theta_\infty; \sigma, s; i\Lambda) = 0, \quad (3.39)$$

$$c_0 = i\Lambda \frac{d}{du} \log \hat{\tau}(\theta_* - 1, \theta_1, \theta_\infty + 1; \sigma - 1, s; i\Lambda) + \frac{1}{4}((\sigma - 1)^2 - (\theta_* - 1)^2) \quad (3.40)$$

with $\hat{\tau} = \det(\mathbb{1} - A\Phi D_0 \Phi^{-1})$ and $\tau_V(i\Lambda) = \tilde{C}(i\Lambda)^{-\frac{1}{4}(\sigma^2 - \frac{1}{2}\theta_*^2)} e^{-\frac{i}{2}\Lambda\theta_1\hat{\tau}}$ is the Painlevé V tau function. The Plemelj operator A is defined in appendix B and D_0 is in (3.30). The accessory parameter c_0 is the limit as $z_0 \rightarrow 0$ of the radial accessory parameter $z_0 K_0$, given by (3.5) in our application.

The procedure is now to tackle the condition in (3.39), which can be inverted for the zero $\tau_V = 0$ in order to solve for κz_0^σ as a series in Λ . The result is

$$s\Pi(i\Lambda)^\sigma \left(1 + \mathcal{O}(z_0^2) + \dots\right) = i \frac{(\sigma + \theta_*)((\sigma + \theta_1)^2 - \theta_\infty^2)}{8\sigma^2(\sigma - 1)^2} \Lambda \chi(i\Lambda) \quad (3.41a)$$

where $\chi(i\Lambda)$ is given as a series in Λ

$$\chi(i\Lambda) = 1 + \chi_1(i\Lambda) + \chi_2(i\Lambda)^2 + \dots, \quad (3.41b)$$

with

$$\chi_1 = (\sigma - 1) \frac{\theta_*(\theta_1^2 - \theta_\infty^2)}{\sigma^2(\sigma - 2)^2}, \quad (3.41c)$$

and

$$\begin{aligned} \chi_2 = & \frac{\theta_*^2(\theta_1^2 - \theta_\infty^2)^2}{64} \left(\frac{5}{\sigma^4} - \frac{1}{(\sigma - 2)^4} - \frac{2}{(\sigma - 2)^2} + \frac{2}{\sigma(\sigma - 2)} \right) \\ & - \frac{(\theta_1^2 - \theta_\infty^2)^2 + 2\theta_*^2(\theta_1^2 + \theta_\infty^2)}{64} \left(\frac{1}{\sigma^2} - \frac{1}{(\sigma - 2)^2} \right) \\ & + \frac{(1 - \theta_*^2)(\theta_1^2 - (\theta_\infty - 1)^2)(\theta_1^2 - (\theta_\infty + 1)^2)}{128} \left(\frac{1}{(\sigma + 1)^2} - \frac{1}{(\sigma - 3)^2} \right). \end{aligned} \quad (3.41d)$$

The higher order terms can be computed recursively. For our application, we use the quantization condition (3.38) and, after using $\sin(\pi z)\Gamma(z)\Gamma(1-z) = \pi$ a number of times, (3.41a) now reads

$$\chi(i\Lambda) = e^{-\frac{i}{2}\pi\sigma} \Theta \Lambda^{\sigma-1} \left(1 + \mathcal{O}(z_0^2) + \dots\right) \quad (3.42)$$

where

$$\Theta = i \frac{\Gamma^2(2 - \sigma)\Gamma(\frac{1}{2}(\sigma - \theta_*))\Gamma(\frac{1}{2}(\sigma - \theta_1 + \theta_\infty))\Gamma(\frac{1}{2}(\sigma - \theta_1 - \theta_\infty))}{\Gamma^2(\sigma)\Gamma(\frac{1}{2}(2 - \sigma - \theta_*))\Gamma(\frac{1}{2}(2 - \sigma - \theta_1 + \theta_\infty))\Gamma(\frac{1}{2}(2 - \sigma - \theta_1 - \theta_\infty))}. \quad (3.43)$$

The treatment of the second condition (3.40) is analogous, substituting the value of $\chi(i\Lambda)$ into the logarithmic derivative and solving for c_0 as a function of Λ . The result, assuming $\Re\sigma > 0$, is

$$c_0 = k_0 + k_1(i\Lambda) + k_2(i\Lambda)^2 + \dots + k_n(i\Lambda)^n + \dots, \quad (3.44a)$$

with the first three terms in the expansion given by

$$k_0 = \frac{(\sigma - 1)^2 - (\theta_\star - 1)^2}{4}, \quad k_1 = \frac{\theta_1 - 1}{2} + \frac{\theta_\star}{2} + \frac{\theta_\star(\theta_1^2 - \theta_\infty^2)}{4\sigma(\sigma - 2)}, \quad (3.44b)$$

$$k_2 = \frac{1}{32} + \frac{\theta_\star^2(\theta_1^2 - \theta_\infty^2)^2}{64} \left(\frac{1}{\sigma^3} - \frac{1}{(\sigma - 2)^3} \right) + \frac{(1 - \theta_\star^2)(\theta_\infty^2 - \theta_1^2)^2 + 2\theta_\star^2(\theta_\infty^2 + \theta_1^2)}{32\sigma(\sigma - 2)} - \frac{(1 - \theta_\star^2)((\theta_\infty - 1)^2 - \theta_1^2)((\theta_\infty + 1)^2 - \theta_1^2)}{32(\sigma + 1)(\sigma - 3)}. \quad (3.44c)$$

Each term in the expansion of χ and c_0 is a meromorphic function of σ , symmetric under $\sigma \rightarrow 2 - \sigma$. There are single poles at $\sigma = 3, 4, 5, \dots$, with the pole of order $\sigma = n$ only seen at k_{n-1} term, and a pole of higher order near $\sigma = 2$, which is present at all orders save for k_0 . The asymptotics of the χ_n (3.41a) and k_n (3.44) terms as $\sigma \simeq 2$, reads:

$$(\sigma - 2)\chi_n \simeq k_n = -\mathbf{C}_{n-1} \frac{\theta_\star^n(\theta_1^2 - \theta_\infty^2)^n}{8^n(\sigma - 2)^{2n-1}} + \dots, \quad n \geq 2, \quad (3.45)$$

where the terms left out are of lower order in $\sigma - 2$. The relation $k_n = (\sigma - 2)\chi_n$ no longer holds as one considers less divergent terms $(\sigma - 2)^{-m}$, $m < 2n - 1$. Finally, $\mathbf{C}_n$ is the n -th Catalan number:

$$\mathbf{C}_n = \frac{1}{n+1} \binom{2n}{n} = 1, 1, 2, 5, 14, \dots \quad (3.46)$$

This remark will be useful when we discuss the CFT description and the fundamental QNM in the following.

3.2 QNMs and CFT description of perturbations

Before delving into the QNMs, let us discuss the conformal aspects of the perturbations, both in the four-dimensional picture, and the “two-dimensional” point of view encoded in the preceding analysis. If we take the massless scalar field (at $\Delta = 4$) to represent the scalar sector of gravitational perturbations in AdS, then the gravitational QNMs can be understood as resonances created by perturbations of the CFT state represented by the stress-energy tensor (2.23). The holographic view of the scattering process has been considered in a great number of papers over the years, with the ideas listed here discussed in [19] and [22]. The starting point is that scattering amplitudes are encoded in the 4-point function

$$\langle \mathcal{V}_{\text{Heavy}}(\infty) \mathcal{V}_{\text{Light}}(\mathbf{x}_1) \mathcal{V}_{\text{Light}}(\mathbf{x}_2) \mathcal{V}_{\text{Heavy}}(0) \rangle \quad (3.47)$$

with $\mathcal{V}_{\text{Heavy}}$ vertex operators associated with the background charges — the mass and angular momenta of the black hole in our case — and $\mathcal{V}_{\text{Light}}$ associated with the local insertion of the scalar perturbations. In a Euclidean setting the positions of the heavy operators are unambiguous, but in our Lorentz setting we will take 0 and ∞ to denote past and future infinity respectively. The correlation functions such as in (3.47) have a rich history in the early development of CFTs, notably the seminal work of [44, 46–48]. The “partial wave” decomposition studied there can be thought of as a higher dimensional conformal block, in which the quantum numbers of the intermediate states mirror the decomposition of the (scalar) wave equation in terms of $\text{SO}(4, 2)$ quantum numbers associated to the generators (2.35).

For pure AdS, it has been known for some time that these higher dimensional conformal blocks can be factorized in terms of “two-dimensional” classical conformal blocks (hypergeometric functions) [20, 49, 50]. Now, we want to view the conditions (3.10) relating the parameters of the differential equation to monodromy data as a deformed version of this factorization, valid for the case with non-zero background charges (2.35). Given that the tau function expansion is given in terms of conformal blocks [51], there are indeed some analogies, although in this case the two-dimensional CFT is chiral and has unit central charge $c = 1$. Furthermore, z_0 and u_0 are independent, and in principle complex, variables.

The latter point may be a consequence of Liouville exponentiation [52], and indeed there is a semiclassical view also relating the accessory parameter of the Heun equation to the Painlevé transcendent [23]. In the latter view, the accessory parameter also appears as the logarithmic derivative of the two-dimensional conformal block:

$$\langle V_{\Delta_1}(\infty) V_{\Delta_2}(1) \Pi_\sigma V_{\Delta_3}(t) V_{\Delta_4}(0) \rangle \quad (3.48)$$

where $V_{\Delta_i}(z_i)$ denotes each vertex operator, and Π_σ the projection onto the Verma module constructed on the primary state, with Liouville momentum parametrized by σ . The conformal dimension Δ_i of each vertex operator $V_{\Delta_i}(z)$ is given in terms of the Liouville momentum P_i and the Liouville coupling constant b as

$$\Delta_i = \frac{c-1}{24} + P_i^2, \quad c = 1 + 6Q^2 = 1 + 6(b + b^{-1})^2. \quad (3.49)$$

In this picture, all operators appearing in (3.10) are “heavy” in the sense that they have large Liouville momenta:

$$P_0 = \frac{\theta_-}{b}, \quad P_t = \frac{\theta_+}{b}, \quad P_1 = \frac{2 - \Delta}{b}, \quad P_\infty = \frac{\theta_0}{b}, \quad (3.50)$$

which not only provide the Liouville interpretation for the single monodromy parameters θ_i , but also hint at a deeper connection. If the black hole absorbs a quantum of energy and angular momenta given by ω , m_1 and m_2 , respectively, then the increase in its entropy is given by

$$\delta S = \frac{\omega - m_1 \Omega_{1,+} - m_2 \Omega_{2,+}}{b^2 T_+} = 2\pi \frac{\theta_+}{b^2}, \quad (3.51)$$

where we recovered Liouville theory’s $\hbar = b^2$. Thus, the entropy gained in the process is, up to the Liouville coupling, given by the Liouville momentum. This gives support to the interpretation of V_{Δ_i} , at least for the inner and outer horizon, as associated to a *thermal state* in the underlying dual theory. By assuming the latter to be unitary and modular invariant, we can use Cardy’s formula [53]

$$\frac{S}{2\pi} \simeq \sqrt{\frac{c}{6} \left(L_0 - \frac{c}{24} \right)} \quad (3.52)$$

for the entropy of such CFT at conformal dimension $L_0 = \Delta_i$. Assuming the classical limit $b \rightarrow 0$, and solving for $L_0 = \Delta_i$, we have again the interpretation of $\delta S_i / 2\pi b$ as the Liouville momentum.

The construction outlined above is suitable for the generic case considered in [18], but it works in a completely analogous way for the confluence limit we considered in section 3.1.1. In the latter case, the two primary vertex operators associated to the inner and outer horizons merge into a Whittaker operator [54]. The associated conformal block is then an irregular one [55], instead of (3.48). The argument presented here about the nature of the intermediate states does have a direct parallel, though, and for further details we refer to the discussion in [27].

3.3 Fundamental QNM

The calculation of the fundamental mode is technically very similar to the corresponding one in [18], where one solves for the first eigenfrequency $\omega_{1,0,0,0}$ by applying the conditions (3.10) given the quantization of the angular and radial monodromy parameters (3.11) and (3.13). From the discussion above, the corresponding task at low temperature involves solving the equations (3.42) and (3.44).

As suggested by figure 2, for generic values of M , a_1 and a_2 a smooth zero-temperature limit for the fundamental QNM frequency exists. At small r_+ , we can work out asymptotic formulas, with

$$i\Lambda = i\lambda r_+^3, \quad \theta_\star = i\phi_\star r_+, \quad \sigma = 2 - \nu r_+^2, \quad (3.53)$$

where the values of λ and $\phi_\star$ can be read from (3.25) and (3.26), by setting $m_1 = m_2 = 0$. We find

$$\phi_\star = \frac{1}{2}(2 + \epsilon)(1 + \alpha_+^2)\Delta + \dots, \quad \lambda = (2(1 + \alpha_+^2) - (3 + \alpha_+^2)\epsilon)\Delta + \dots, \quad (3.54)$$

where we used the fact that ω is approximately Δ in the $r_+ \rightarrow 0$ limit. As a matter of fact, we will assume that

$$\theta_0 = \omega - 3(1 + \alpha_+^2 - \epsilon)\omega r_+^2 + \dots = \Delta - \beta r_+^2, \quad (3.55)$$

where we introduced the β parameter to measure the difference between Δ and the monodromy parameter θ_0 . Note also that the expansion of θ_0 implies that β is related to the correction to the QNM frequency at small r_+

$$\omega = \Delta - (\beta - 3(1 + \alpha_+^2 - \epsilon)\Delta)r_+^2 + \mathcal{O}(\epsilon^2, r_+^4). \quad (3.56)$$

Finally, we remark that, at low temperature we have $\epsilon \ll r_+ \ll 1$, and (3.19) implies that α_1 and α_2 are related to r_+ ,

$$\alpha_1^2 \alpha_2^2 = (1 - 2\epsilon)((1 + r_+^2)^2 - 2r_+^2 \alpha_+^2) + \mathcal{O}(\epsilon^2). \quad (3.57)$$

We will now consider the conditions (3.39) and (3.40), encoded in the equations (3.42) and (3.44), as equations determining β and ν for small r_+ . First, the Θ parameter in (3.42) has the small r_+ expansion for $\ell = 0$:

$$\Theta = \frac{\phi_\star}{2r_+^3 \nu^2} \frac{(\nu + \beta)}{(\nu - \beta)} (\Delta - 1) \left(1 + i \frac{\nu}{\phi_\star} r_+ + \mathcal{O}(\epsilon r_+^2, r_+^2) \right). \quad (3.58)$$

Figure 2. The fundamental QNM frequency $\omega_{1,0,0,0}$ as a function of the outer horizon temperature T_+ for a Kerr-AdS₅ black hole with fixed $r_+ = 0.03$ and $\alpha_+^2 = \sqrt{2}$. We note that the temperature has the effect of increasing both the real and imaginary parts of the frequency, up to the maximal value $2\pi(T_+)_{\max} \simeq 8.7$, corresponding to $r_- = 0$, or $\epsilon = 1/2$ in (3.22).

As anticipated by the argument in the preceding sections, the ϵ corrections are accompanied by a factor of r_+^2 and are always subdominant in the small r_+ limit.

The accessory parameter c_0 as in (3.44) and the χ function defined in (3.42) both receive contributions from all orders in z_0 in the small r_+ limit. This is due to the proximity of σ to the critical value 2, as further clarified in the next subsection. Contributions to c_0 and χ can be resummed using the generating formula for the Catalan numbers

$$1 + x + 2x^2 + 5x^3 + 14x^4 + \dots = \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}. \quad (3.59)$$

The result for χ can then be written as

$$\chi = \frac{1}{2} + \frac{\phi_*(\theta_0^2 - \theta_1^2)}{8\nu^2} \lambda + \frac{1}{2} \sqrt{1 + \frac{\phi_*(\theta_0^2 - \theta_1^2)}{2\nu^2}} \lambda - \frac{(\theta_0^2 - \theta_1^2)^2 \lambda^2}{64\nu^2 \sqrt{1 + \frac{\phi_*(\theta_0^2 - \theta_1^2)}{2\nu^2}} \lambda} r_+^2 + \mathcal{O}(r_+^4), \quad (3.60)$$

where $\theta_1 = 2 - \Delta$. At this order in r_+ , we can substitute $\theta_0^2 - \theta_1^2 = 4(\Delta - 1) + \mathcal{O}(r_+^2)$. The first non-trivial correction in ϵ can be obtained by just considering the first correction in the parameters λ and ϕ_* . The expansion for the accessory parameter c_0 in the right-hand side of (3.44a) can also be resummed in the same way:

$$c_0 = \frac{i}{2} \phi_* r_+ + \frac{1}{4} \phi_*^2 r_+^2 - \frac{\nu r_+^2}{2} \sqrt{1 + \frac{\phi_*(\theta_0^2 - \theta_1^2)}{2\nu^2}} \lambda + \frac{i}{2} (\theta_1 - 1) \lambda r_+^3 + \mathcal{O}(r_+^4). \quad (3.61)$$

The same parameter c_0 is defined as $\lim_{z_0 \rightarrow 0} z_0 K_0$ and can be computed from (3.5) and (3.12) by taking the appropriate $r_- \rightarrow r_+$ limit. In the $\epsilon \ll r_+ \ll 1$ limit and for

$\ell = 0$, this provides for the left-hand side of equation (3.44a),

$$c_0 = \frac{i}{2}\phi_\star r_+ + \frac{1}{4}\phi_\star^2 r_+^2 - \frac{1+\alpha_+^2}{2}\Delta(\Delta+2)r_+^2 + \frac{1}{2}\Delta(\Delta+2)\epsilon r_+^2 + \frac{i}{2}(\theta_1 - 1)\lambda r_+^3 + \mathcal{O}(\epsilon^2, \epsilon r_+^4, r_+^4), \quad (3.62)$$

which, equated to the right-hand side of (3.61), leads to an equation for ν as in (3.53)

$$\sqrt{\nu^2 + 2\lambda\phi_\star(\Delta-1)} = (1+\alpha_+^2)\Delta(\Delta+2) - \Delta(\Delta+2)\epsilon + \dots, \quad (3.63)$$

where the terms left out scale with ϵr_+^2 and ϵ^2 . Taking into account (3.57), this equation can be solved to yield, at first non-trivial order in ϵ and r_+ ,

$$\nu = (1+\alpha_+^2)\Delta\sqrt{\Delta^2+8} \left(1 - \frac{\Delta^2+2\Delta+6}{(1+\alpha_+^2)(\Delta^2+8)}\epsilon\right) + \mathcal{O}(\epsilon^2, r_+^2, \epsilon r_+^2) \quad (3.64)$$

and likewise for χ

$$\chi = \frac{(\Delta+2+\sqrt{\Delta^2+8})^2}{4(\Delta^2+8)} \left(1 - \frac{(\Delta+2)(\sqrt{\Delta^2+8}-\Delta-2)}{(1+\alpha_+^2)(\Delta^2+8)}\epsilon\right) + \mathcal{O}(\epsilon^2, r_+^2, \epsilon r_+^2). \quad (3.65)$$

Finally, by substituting (3.64) and (3.65) into (3.42), we can compute β

$$\beta = (1+\alpha_+^2)(\Delta(\Delta+2) + 2i\Delta(\Delta-1)r_+) - \Delta(\Delta+2)\epsilon - i(3+\alpha_+^2)\Delta(\Delta-1)\epsilon r_+ + \mathcal{O}(r_+^2, r_+^2 \log r_+, \epsilon r_+^2, \epsilon^2). \quad (3.66)$$

We remark that now, due to the presence of the $\Lambda^{\sigma-1}$ term in (3.42), there will be non-analytical corrections of the type $r_+^2 \log r_+$ which, however, are subdominant for small r_+ . Finally, using the relation between β and ω given by (3.56), we find the first corrections to the fundamental QNM frequency as a function of r_+ and ϵ ,

$$\begin{aligned} \omega_{1,0,0,0} &= \Delta - (1+\alpha_+^2)\Delta(\Delta-1)r_+^2 - 2i(1+\alpha_+^2)\Delta(\Delta-1)r_+^3 \\ &\quad + \Delta(\Delta-1)\epsilon r_+^2 + i(3+\alpha_+^2)\Delta(\Delta-1)\epsilon r_+^3 \\ &\quad + \mathcal{O}(r_+^4, r_+^4 \log r_+, \epsilon r_+^4, \epsilon^2 r_+^2) \end{aligned} \quad (3.67)$$

where the subscript stands for n, ℓ, m_1 and m_2 quantum numbers.

The imaginary part of the frequency controls the relaxation times of the perturbation in the dual CFT. Equation (3.67) is valid for $\epsilon \ll r_+ \ll 1$, with α_+ constant and of order $\mathcal{O}(1)$. In this limit, the effect of non-zero temperature is accounted for by the ϵ -dependent terms, which, in agreement with figure 2, induce an increase of the real and imaginary parts of the frequency, as ϵ increases. Despite having a “reinforcing” effect, making perturbations longer-lived, the temperature does not have a strong enough influence to induce instabilities. For perturbations above the unitarity bound $\Delta > 1$, the fundamental mode is stable at zero or small enough temperature. These results complement the finite-temperature result of [18]. The analytical prediction in (3.67) is compared with the numerical results in figure 3.

Figure 3. The expression for the fundamental mode $\omega_{1,0,0,0}$ given by (3.67) compared with numerical results for $\epsilon = 10^{-6}$ and $\alpha_+^2 = \sqrt{2}$.

3.3.1 Catalan numbers and intermediate CFT levels

The two-dimensional version of (3.47), depicting the perturbation of the black hole background, was studied in [32], where it was shown that the vacuum block, where the internal dimension is zero, was given by the generating function for the Catalan numbers in the heavy-light semiclassical limit, where two of the insertions have high conformal dimensions $\Delta_i \propto c \rightarrow \infty$, with c being the central charge.

There is indeed a clear parallel to what is being counted in the fundamental mode calculations in the last subsection. The finite-temperature version of the calculation outlined in [18] does have this hierarchy between the vertex operators, since the ones associated with the inner and outer horizons do become “light” in the small black hole regime, and the residue at $\sigma = 2$ does indeed correspond to the identity operator as intermediate state in (3.47).

Yet, there are subtleties. We have already remarked that (3.48) is a chiral conformal block. A difference from [32] is that in principle all operators in (3.48) are “heavy”, in the sense that their conformal dimensions scaling with c . The parametric hierarchy between operators arises from the parameter r_+ , which can be made small. It is a combined effect of the position of the insertion and the scaling of the dimensions of the “light” operators. The corresponding calculation using the Virasoro algebra seems straightforward, but lies outside the scope of this work.

Another difference comes from the confluence limit taken as the temperature goes to zero. In this case the limiting form of the Liouville momenta for the inner and outer horizon is given by (3.23), and the conformal dimensions of the operators associated to $\theta_{\pm}$ are in fact much larger than c . The “light” operator is the resulting operator coming from the leading term of the OPE for the operators related to the inner and outer horizons:

$$V_{P_3}(t)V_{P_4}(0) \sim V_P(0) + \dots \quad (3.68)$$

where we label the operators by the Liouville momentum, and $P = P_3 + P_4$. As the first descendant of the resulting module $[V_P]$ one finds the Whittaker operator [54, 55]

$$V_{\theta_*/b, i\Lambda}(0) \sim: e^{2\theta_*/b\phi_L + i\Lambda\partial\phi_L} : \quad (3.69)$$

which is labelled by the confluence parameters. We used the Feigin-Fuchs parametrization of the Liouville field ϕ_L (see [55] or [27]) in the right-hand side of (3.69) to compare our notation with the literature. In this case, the parameters in the regime investigated here do warrant the “light” requisite of [32] since they are proportional to r_+ and r_+^3 , respectively, as can be checked in (3.25) and (3.26). In light of this, the results obtained in section 3.3 point to an analogous simplification of the conformal block in terms of the Catalan number generating function when the two light operators in the analysis of [32] are substituted by an irregular light operator. For more on the relation between irregular conformal blocks and the Painlevé V tau function we recommend [25].

3.4 Higher quasi-normal modes

Higher QNMs, with $\ell \geq 1$, allow for non-zero values of m_1 and m_2 . In turn, these multiply the angular velocities $\Omega_{i,+}$, which are given in our parametrization by

$$\Omega_{i,+} = \frac{a_i(1 - a_i^2)}{r_+^2 + a_i^2} = \frac{\alpha_i(1 - \alpha_i^2 r_+^2)}{r_+(1 + \alpha_i^2)}, \quad (3.70)$$

with $\alpha_i = a_i/r_+$, $i = 1, 2$. As explained before, at small r_+ the range of allowed a_i is capped by r_+ , so it makes sense to consider fixed values of $\alpha_i \lesssim 1$. Then in the small r_+ regime, the velocities $\Omega_{i,\pm} \sim 1/r_+$ become very large and the monodromy parameter (3.3)

$$\theta_+ = \frac{i}{2\pi} \left(\frac{\omega - m_1\Omega_{1,+} - m_2\Omega_{2,+}}{T_+} \right) \quad (3.71)$$

varies so that its imaginary part can become negative. Note that r_+ can alternatively be thought of as the ratio between the outer horizon and the AdS radius.

As a matter of fact, it has been long established from heuristic arguments that combinations of ω, m_1 and m_2 such that $\Im\theta_+ < 0$ display stimulated emission — the Zeldovich-Starobinski effect — see [56] for a detailed account. When $\Im\theta_+$ is negative, we have the *superradiance window* and unstable modes are bound to appear. The instability was anticipated by the thermodynamical analysis [11] and first observed numerically for four-dimensional black holes in [57]. The effect is, however, tiny and hard to study using conventional numerical methods, even at finite temperature [58]. Our results in [18] provided supporting evidence for the positivity of the imaginary part of the $\ell = 1, m_1 = 1, m_2 = 0$ QNM frequency in the small r_+ limit, based on the quantization conditions (3.11) and (3.13).

The low temperature version of the calculation for QNMs in [18] follows an analogous strategy, based on (3.42) and (3.44). Since the final analytical expression for the imaginary part of the $\ell \geq 1$ QNM frequencies is not particularly illuminating, we will limit ourselves to present some numerical results and discuss the qualitative features of the analyzed modes. The numerical analysis was conducted using an arbitrary-precision implementation of the

Figure 4. The $\ell = 1$ modes for $a_{12} = a_1^2 - a_2^2 = 0.001$ as a function of r_+ and fixed, very small temperature $T_+ \sim 10^{-8}$. The modes $m_1 = 1, m_2 = 0$ (dark red) and $m_1 = 0, m_2 = 1$ (light red) display a small positive imaginary part for values where $\Im\theta_+ < 0$ (inset).

Painlevé VI and V tau functions with a Fourier truncation of the operators A, D and D_0 at $N = 32$ levels, using the [Arb](#) library at 96-digit precision.² The library is designed to control numerical uncertainties by performing calculations with intervals of complex numbers. In all of the analysis the intervals were too small to be of significance, of order 10^{-12} , and they will be omitted from the analysis.

In figure 4 we display a typical plot of QNMs frequencies as a function of r_+ for $\ell = 1$. One sees that modes with either m_1 or m_2 positive are unstable for small values of r_+ and moderate values of a_1 and a_2 , of which the example shown is characteristic. From the superradiant condition $\Im\theta_+ < 0$, one sees that there are two competing factors determining whether the mode will be unstable or not: we have the positive contribution from the real part of the frequency and the negative contribution from the angular velocity. Now, as one turns on the rotation parameters a_1 and a_2 , the real part of the frequency gets negative corrections that overwhelm the growth in angular velocity, effectively reducing the superradiant window. The “kink” in the zoomed in inset as $\Im\theta_+$ crosses to positive values comes from the change of sign of Λ in (3.23). In the zero temperature limit, this change of sign will induce a discontinuity in the s value obtained from the quantization condition (3.38).

From the conformal block point of view, the calculation of the QNMs is truly perturbative in the CFT levels, with the internal momentum centered at $\sigma = 2 + \ell - \nu_\ell r_+^2$. It is interesting to note that these should correspond in the $r_+ \rightarrow 0$ limit to degenerate

²The implementation of the Painlevé VI and V tau functions in [Julia](#) programming language can be obtained in <https://github.com/strings-ufpe/painleve>. The authors thank O. Lisovsky for clarification on the details of the truncation in a private communication.

Figure 5. Stable (top, blue) and unstable (bottom, red) modes for $\ell = 2$ and $a_1^2 - a_2^2 = 0.001$ as functions of r_+ . Like in the $\ell = 1$ case, the positive imaginary part is very small and restricted to the condition $\Im \theta_+ < 0$.

“heavy” Liouville operators at integer $\sigma \simeq 2 + \ell$. As a matter of fact, the latter values for σ do indeed correspond to poles in the conformal block expansion at level $t_0^{\ell+1}$. It would be interesting to investigate whether this alternative approach could lead to a simplified analytic study of these modes.

In figure 5 we display the $\ell = 2$ case. Again, the unstable modes correspond to positive m_1 and/or m_2 , and the region of instability is determined by the superradiant window $\Im \theta_+ < 0$. The comparison to $\ell = 1$ — see also figure 6 — shows that the decay rate of the most unstable mode [11], i.e., the one with largest $m_1 \Omega_{1,+} + m_2 \Omega_{2,+}$ decreases with ℓ . Indeed, figure 7 shows that the maximum of the decay rate as a function of r_+ does decrease from $\ell = 1$ to $\ell = 4$, making those higher ℓ states longer-lived. If this trend continues for larger ℓ , it would mean that the decaying profile of the scalar emissions from

Figure 6. The frequencies of the unstable modes for $\ell = 1$ as a function of r_+ for various values of $a_{12} = a_1^2 - a_2^2$. Note that the superradiant window induced by the condition $\Im \theta_+ < 0$ becomes smaller with increasing a_{12} due to the larger reduction of the real part of the eigenfrequency.

Figure 7. The most unstable mode $m_1 = \ell$ as a function of r_+ for $\ell = 1, 2, 3, 4$. Note that the smaller decay rate for larger values of ℓ is accompanied by a larger width of the superradiant window. The black hole is expected to decay through different values of ℓ , m_1 and m_2 depending on the values of its charges.

the black hole is a combined effect of the angular dependence of the coupling between the black hole and the scalar perturbations on the one hand and the half-life of the perturbation as a function of ℓ , and the radial eigenvalue n on the other.

One should also note from figure 6 that the superradiant window appears to decrease as $a_{12} = a_1^2 - a_2^2$ increases, indicating that, as far as the contributions to θ_+ are concerned, the negative correction to the real part of the eigenfrequency in (3.71) dominates over the effect of the increased angular rotation. So, as a_{12} increases, the real part of $\Im\theta_+$ turns positive. In fact, around the value $a_{12} \simeq 0.045$ there is no superradiance for $\ell = 1$, as can be inferred from figure 6. This is in contrast with the fragmentation phenomenon, which is expected at sufficiently distorted — i.e., highly rotating — five-dimensional black holes as argued in [59].

4 Holographic decay

Given the results of the preceeding sections for the QNMs and their instability, we may reflect on the fate of the corresponding state in the putative dual CFT. The instabilities signal that the state associated to the black hole will decay. The particular features of the decay, such as its rate and final products, will of course depend on the coupling between the black hole and the perturbation fields, which in holography can be read from the stress-energy tensor. For the scalar type of perturbations considered here we can deduce an interaction Hamiltonian of the sort $\mathcal{H}_{\text{int}} = \lambda h^{ab} T_{ab}$. As we learned from the discussion above, the imaginary part of the QNMs frequencies remains fairly constant with $\ell > 0$, with the most unstable mode at given ℓ being the one with the largest $m_1\Omega_{1,+} + m_2\Omega_{2,+}$.

Let us first consider the case where the field theory lives in the global boundary $\mathbb{R} \times S^3$. The energy density profile can be read from (2.23)

$$4\pi G_{N,5}\bar{\rho} = \frac{3M}{2\bar{\Delta}_\theta^2} = \frac{3M}{2} \left(1 + 2(a_1^2 \sin^2 \bar{\theta} + a_2^2 \cos^2 \bar{\theta}) + \dots \right). \quad (4.1)$$

We interpret the expansion in the right-hand side to be the contribution of the higher S^3 spherical harmonics. From (4.1) we conclude that the generation of the ℓ -th mode will be dampened by a factor of a_i^ℓ . Since a_1 and a_2 are parametrically small, of order r_+ , the corresponding five-dimensional spheroidal harmonics can be approximated by their zero rotation counterparts, i.e., the three-dimensional spherical harmonics [60]

$$Y_\ell^{m_1, m_2}(\bar{\theta}, \bar{\phi}_1, \bar{\phi}_2) = \sqrt{\frac{\ell+1}{2\pi^2}} \sqrt{\frac{((\ell+m_1+m_2)/2)!((\ell-m_1-m_2)/2)!}{((\ell+m_1-m_2)/2)!((\ell-m_1+m_2)/2)!}} \times \\ (\sin \bar{\theta})^{m_1} (\cos \bar{\theta})^{m_2} P_{\frac{1}{2}(\ell-m_1-m_2)}^{(m_1, m_2)}(\cos 2\bar{\theta}) e^{im_1\bar{\phi}_1 + im_2\bar{\phi}_2}, \quad (4.2)$$

where $P_n^{(a,b)}(z)$ are the Jacobi polynomials. At given ℓ , the most unstable mode has the energy dependence proportional to the tt -component of the scalar stress-energy tensor, in turn roughly proportional to the absolute value squared of the field itself

$$\bar{\rho}_{\text{fluc.}} \propto |\Phi_{\ell, m_1=\ell, m_2=0}|^2 = \left| e^{-i\omega(\ell)\bar{t}} Y_\ell^{\ell, 0} \right|^2 \propto e^{2\Im\omega(\ell)\bar{t}} \sin^{2\ell} \bar{\theta}, \quad (4.3)$$

which favors localization at the $\bar{\theta} \simeq \pi/2$ plane. In this expression $\Im\omega(\ell)$ corresponds to the imaginary part of the eigenfrequency $\omega_{1,\ell,\ell,0}$, which is assumed positive for the unstable mode.

In the case where the dual theory lives in flat coordinates, we can in principle just apply the transformation (2.29) to translate from the global results above. However, our analysis in section 3 was made by assuming $a_1 > a_2$, which in global coordinates is not a restriction because one can interchange the azimuthal variables $\bar{\phi}_1$ and $\bar{\phi}_2$. When we transform to flat coordinates, we choose one of them to map to the coordinate $\hat{\phi}$, and we have to consider the two choices separately.

For the case $a_1 > a_2$, we have $\hat{\phi} = \bar{\phi}_2$ as in section 3, and the energy profile for the fluctuation reads

$$\bar{\rho}_{\text{fluc.}} \propto e^{2\Im\omega(\ell)\bar{t}} \left(\frac{(1 + \frac{1}{4}(\hat{t}^2 - \hat{r}^2))^2 + \hat{r}^2 \cos^2 \hat{\theta}}{(1 + \frac{1}{4}(\hat{t} + \hat{r})^2)(1 + \frac{1}{4}(\hat{t} - \hat{r})^2)} \right)^{\ell/2} \quad (4.4)$$

where

$$\bar{t} = \arctan \frac{\hat{t} + \hat{r}}{2} + \arctan \frac{\hat{t} - \hat{r}}{2}. \quad (4.5)$$

The case $a_1 < a_2$, can be realized by setting $\hat{\phi} = \bar{\phi}_1$, and the coordinate transformation changes from (2.29) to

$$\sin \bar{\theta} = \sin \hat{\chi} \sin \hat{\theta}, \quad (4.6)$$

which corresponds to interchanging $a_1 \leftrightarrow a_2$ in the profile (2.32). The energy dissipated in the perturbation mode is now

$$\bar{\rho}_{\text{fluc.}} \propto e^{2\Im\omega(\ell)\bar{t}} \left(\frac{\hat{r}^2 \sin^2 \hat{\theta}}{(1 + \frac{1}{4}(\hat{t} + \hat{r})^2)(1 + \frac{1}{4}(\hat{t} - \hat{r})^2)} \right)^{\ell/2}, \quad (4.7)$$

with $\bar{t}$ given by (4.5).

We note that, with the second choice, there is a tendency for the decay to occur in the $\hat{x} - \hat{y}$ plane ($\sin^2 \hat{\theta} \simeq 1$), following the maximum of $\bar{\rho}_{\text{fluc.}}$. As we discussed above, the specific value of ℓ which will be preferred by the decay will depend on the details of the state and the coupling. In figure 7 we see that, even though the maximum of the imaginary part of the eigenfrequency decreases with ℓ , larger values of r_+ tend to favor larger values of ℓ for the decay. This leads us to infer that, at low temperatures, the localization effect of the decay products resulting from (4.7) increases with the mass $\mathcal{M}$. We also point out the difference between the time dependence of the decay modes depending whether we consider global coordinates $\mathbb{R} \times S^3$ (bar coordinates) or conformally flat coordinates $\mathbb{R}^{3,1}$ (hatted coordinates). Whereas in global coordinates the fluctuations grow exponentially without bound, at least in the linear analysis we do here, the growth in conformally flat coordinates is capped as $\hat{t} \rightarrow \infty$ due to (4.5).

5 Discussion

In this paper we analyzed the holographic aspects of the generically rotating Kerr-AdS₅ black hole in Lorentzian signature, focussing on the low-temperature limit $T_+ \simeq 0$ and

small black holes $r_+ \ll 1$. We reviewed the definition of asymptotic charges, discussed the geometrical meaning of the regularization involved in the definition of mass, and determined the holographic stress-energy tensor associated to the black hole in the dual field theory — whose thermal state is characterized by non-zero vacuum expectation values for all generators of the Weyl subgroup of the conformal group $SO(4, 2)$. Finally, we computed the first correction to the boundary-to-boundary scalar propagator due to the presence of the black hole and found that it makes scalar perturbations with dimension close to the marginal value ($\Delta = 4$) irrelevant, thus suggesting that such perturbations should not substantially change the fate of the background as the theory flows to the IR.

In order to further analyze this question, we set out to treat scalar perturbations by means of the exact solution constructed in [18], adapted to the low-temperature limit. We found instabilities in the QNMs for all values of the angular eigenvalue $\ell > 0$, as anticipated by [33], using thermodynamical arguments. For $\ell = 0$, the first order correction in the temperature has an enhancement effect which increases the decay time of the perturbation, but it is not strong enough to induce instabilities. We have also studied numerically the dependence of the $\ell > 0$ instabilities on the black hole parameters as we vary ℓ . These instabilities of AdS space are expected from general arguments [17], and we attempted a holographic interpretation by studying the instabilities of the thermal field theory state corresponding to the black hole and the spatial dependence of its decay products. We found qualitative hints that the ejecta tend to collimate at the axis of highest rotation as the black hole mass increases.

We have also shown that the general procedure of [18], which solved for the non-local boundary conditions related to QNMs in terms of monodromy data of the Heun differential equations involved, can be used effectively to find QNMs in the zero temperature limit. This limit led us to consider the monodromy parameters to be determined by the Painlevé V transcendent, obtained from the confluence limit of the Painlevé VI. The importance of the expansion of the Painlevé transcendents in terms of $c = 1$ Virasoro conformal blocks, as studied in [25], provides us with the same interesting factorization of the *four-dimensional* conformal blocks in terms of (semi-classical) *two-dimensional* ones, as first anticipated by [20]. The new ingredient here is that the two-dimensional conformal blocks are of the irregular type, as also appeared in the black hole perturbation context in [27]. The semiclassical conformal blocks associated to the radial perturbations seem to arise from a unitary theory, and the Vertex operators associated to the inner and outer horizons can be interpreted as thermal states, with their degeneracy equating the entropy of the scalar perturbation with quantum numbers given by ω, m_1 and m_2 as it is absorbed by the black hole. Finally, the study of QNMs at small values of r_+ showed the relevance of short Virasoro representations for the intermediate states of the conformal blocks, again anticipated in a different context by [32].

The motivation for considering low-temperature black holes stemmed from the necessity for pushing the hydrodynamical analogy of [41], if we want to understand the holographic interpretation of instabilities in AdS space. We have seen that the phase diagram of these states in the dual theory has a richer set of associated phenomena than what is expected from the “naive” near-UV considerations, with instabilities in the higher modes and

collimation of decay products. Tools usually used in holographic renormalization-group flow are not really suitable to determine the fate of the state as one flows to the infrared. We can use the same methods proposed here to study what happens at larger values of r_+ , but the appearance of instabilities as well as fragmentation issues [59] — the latter absent here — can further contribute to complicate the issue of determining the fate of the decay process.

Finally, we stress that the isomonodromic method used here for the numerical analysis overcomes a major hurdle arising in usual Frobenius matching. At low temperatures, the latter method suffers from large spatial oscillations of the modes near the outer horizon r_+ . The isomonodromic method has no such hindrance. Moreover, our investigation also shows that the mode expansion used in Frobenius matching is not suitable to study the fundamental mode. Indeed, in the finite temperature case, we have shown that the monodromy parameters, and thus the actual value of the QNM frequency, receive contributions from all levels in the Frobenius expansion. These contributions can be interpreted, in the two-dimensional conformal block attribution, as coming from all descendants of the CFT primaries associated to the intermediate states. We verified that these contributions can be resummed in terms of the generating function of the Catalan numbers. Also, short representations of the CFT Virasoro algebra are relevant for the calculation of QNMs at higher levels. We defer the study of other interesting black hole limits, as well as higher spin perturbations [61] to future work.

Acknowledgments

We apologize for any omission in this important and long standing field of research. The authors would like to thank Dmitry Melnikov, Monica Guica, Oleg Lisovsky for comments and suggestions along the way. Finally, we thank the referee for pointing out the necessity of further comparison between (3.67) and our numerical results, motivating a revision of the fundamental mode frequency in a previous version of this paper.

A Asymptotic charges of AdS spaces

Let the asymptotic metric be given by

$$g_{ab} = \frac{1}{\tilde{\Omega}^2} \tilde{g}_{ab} + h_{ab} \quad (\text{A.1})$$

where h_{ab} is understood to be small with respect to the first term in the $\tilde{\Omega} \rightarrow 0$ limit, in a manner that will become precise at the end of the appendix. As we will see, for asymptotically AdS spaces, $\tilde{\Omega}$ has spacelike gradient and vanishes at the conformal boundary. The boundary structure is dependent on the particular choice of $\tilde{\Omega}$, and the two choices considered in this paper are $\tilde{g}_{ab}$ is either the flat metric (Poincaré patch), called $\hat{g}_{ab}$ in the main text, or the $\mathbb{R} \times S^3$ metric (global coordinates), referred to by $\bar{g}_{ab}$ in the main text. Let us introduce the covariant derivatives ∇_a , associated with g_{ab} and $\tilde{\nabla}_a$, associated with $\tilde{g}_{ab}$. If χ^a is a vector field, we can use Leibniz rule to show that the difference $\nabla_a \chi^b - \tilde{\nabla}_a \chi^b$ is a linear operator on χ^a , therefore, for any two covariant derivatives, we have

$$\nabla_a \chi^b = \tilde{\nabla}_a \chi^b + C^b_{ac} \chi^c, \quad (\text{A.2})$$

with the difference between the connections given by the conditions that ∇_a is compatible with g_{ab} and $\tilde{\nabla}_a$ with $\tilde{g}_{ab}$. To first order in h_{ab} :

$$\begin{aligned} C^b_{ac} &= \frac{1}{2} g^{bd} \left(\tilde{\nabla}_a g_{dc} + \tilde{\nabla}_c g_{ad} - \tilde{\nabla}_d g_{ac} \right) \\ &= \frac{1}{2} \tilde{\Omega}^2 \left(\tilde{g}^{bd} - \tilde{\Omega}^2 h^{bd} \right) \left(\tilde{\nabla}_a \left(\frac{1}{\tilde{\Omega}^2} \tilde{g}_{dc} + h_{dc} \right) + \tilde{\nabla}_c \left(\frac{1}{\tilde{\Omega}^2} \tilde{g}_{ad} + h_{ad} \right) - \tilde{\nabla}_d \left(\frac{1}{\tilde{\Omega}^2} \tilde{g}_{ac} + h_{ac} \right) \right), \end{aligned} \quad (\text{A.3})$$

with indices in the right hand side now and hereafter raised with $\tilde{g}^{ab}$, so that $h^{ab} \equiv \tilde{g}^{ac} h_{cd} \tilde{g}^{db}$. Expanding up to quadratic terms in h_{ab} ,

$$\begin{aligned} C^b_{ac} &= -\frac{1}{\tilde{\Omega}} \left(\delta_c^b \tilde{\nabla}_a \tilde{\Omega} + \delta_a^b \tilde{\nabla}_c \tilde{\Omega} - \tilde{g}_{ac} \tilde{\nabla}^b \tilde{\Omega} \right) + \tilde{\Omega} \left(h_c^b \tilde{\nabla}_a \tilde{\Omega} + h_a^b \tilde{\nabla}_c \tilde{\Omega} - \tilde{g}_{ac} h^{bd} \tilde{\nabla}_d \tilde{\Omega} \right) \\ &\quad + \frac{1}{2} \tilde{\Omega}^2 \left(\tilde{\nabla}_a h_c^b + \tilde{\nabla}_c h_a^b - \tilde{\nabla}^b h_{ac} \right). \end{aligned} \quad (\text{A.4})$$

Conserved quantities associated with vector fields ξ^a are defined as the integral of the $(d-2)$ -form

$$Q[\xi]_{a_1 \dots a_{d-2}} = -\frac{1}{2\kappa} \epsilon[g]_{aba_1 \dots a_{d-2}} g^{ac} \nabla_c \xi^b, \quad (\text{A.5})$$

with $\epsilon[g]$ the volume form associated with g_{ab} .³ The contravariant derivative can also be computed to first order in h_{ab} :

$$g^{ac} \nabla_c \xi^b = \tilde{\Omega}^2 (\tilde{g}^{ac} - \tilde{\Omega}^2 h^{ac}) (\tilde{\nabla}_c \xi^b + C^b_{cd} \xi^d) \quad (\text{A.6})$$

or

$$\begin{aligned} g^{ac} C^b_{cd} &= -\tilde{\Omega} (\delta_d^b \tilde{\nabla}^a \tilde{\Omega} + \tilde{g}^{ab} \tilde{\nabla}_d \tilde{\Omega} - \delta_d^a \tilde{\nabla}^b \tilde{\Omega}) + \tilde{\Omega}^3 (h_d^b \tilde{\nabla}^a \tilde{\Omega} + h^{ab} \tilde{\nabla}_d \tilde{\Omega} - h_d^a \tilde{\nabla}^b \tilde{\Omega}) \\ &\quad + \tilde{\Omega}^3 (\delta_d^b h^{ac} \tilde{\nabla}_c \tilde{\Omega} + h^{ab} \tilde{\nabla}_d \tilde{\Omega} - \delta_d^a h^{bc} \tilde{\nabla}_c \tilde{\Omega}) + \frac{1}{2} \tilde{\Omega}^4 (\tilde{\nabla}^a h_d^b + \tilde{\nabla}_d h^{ab} - \tilde{\nabla}^b h_d^a), \end{aligned} \quad (\text{A.7})$$

Now, the vector fields we are interested in are Killing vector fields of the induced metric at the conformal boundary $\tilde{\Omega} = 0$. These will satisfy:

$$\xi^d \tilde{\nabla}_d \tilde{\Omega} = 0, \quad \tilde{\nabla}^a \xi^b + \tilde{\nabla}^b \xi^a = 0, \quad (\text{A.8})$$

denoting the vanishing Lie derivatives with respect to ξ^a of $\tilde{\Omega}$ and $\tilde{g}^{ab}$, respectively. Note that we are assuming that ξ^a and $\tilde{\nabla}^a \tilde{\Omega}$ are orthogonal, which is suitable for the asymptotically AdS case.

Wrapping it all up,

$$\begin{aligned} g^{ac} \nabla_c \xi^b &= 2\tilde{\Omega} \xi^{[a} \tilde{\nabla}^{b]} \tilde{\Omega} + \tilde{\Omega}^2 \tilde{\nabla}^{[a} \xi^{b]} - 2\tilde{\Omega}^3 (\xi^d h_d^{[a} \tilde{\nabla}^{b]} \tilde{\Omega} + \xi^{[a} h^{b]c} \tilde{\nabla}_c \tilde{\Omega}) \\ &\quad + \tilde{\Omega}^4 \left(\xi^d \tilde{\nabla}^{[a} h_d^{b]} - h^{d[a} \tilde{\nabla}_d \xi^{b]} + \frac{1}{2} \mathcal{L}_\xi h^{ab} \right) + \mathcal{O}(h^2) \end{aligned} \quad (\text{A.9})$$

where $\mathcal{L}_\xi h^{ab}$ is the Lie derivative of h^{ab} with respect to ξ^a . The volume form is readily computed:

$$\epsilon[g] = \frac{1}{\tilde{\Omega}^d} \epsilon[\tilde{g}] \left(1 + \frac{1}{2} \tilde{\Omega}^2 h_a^a + \mathcal{O}(h^2) \right) \quad (\text{A.10})$$

³For convenience, a term equal to the volume of the $d-2$ sphere dividing $Q[\xi]$ is omitted.

and so the charge density can be expanded

$$\begin{aligned}
 Q[\xi]_{a_1 \dots a_{d-2}} = & -\frac{1}{2\kappa} \epsilon[\tilde{g}]_{aba_1 \dots a_{d-2}} \frac{1}{\tilde{\Omega}^{d-1}} \left(2\xi^{[a} \tilde{\nabla}^{b]} \tilde{\Omega} + \tilde{\Omega} \tilde{\nabla}^{[a} \xi^{b]} \right. \\
 & - 2\tilde{\Omega}^2 \left(\xi^d h_d^{[a} \tilde{\nabla}^{b]} \tilde{\Omega} + \xi^{[a} h^{b]c} \tilde{\nabla}_c \tilde{\Omega} - \frac{1}{2} h_c^c \xi^{[a} \tilde{\nabla}^{b]} \tilde{\Omega} \right) \\
 & \left. + \tilde{\Omega}^3 \left(\xi^d \tilde{\nabla}^{[a} h_d^{b]} - h^{d[a} \tilde{\nabla}_d \xi^{b]} + \frac{1}{2} h_c^c \tilde{\nabla}^{[a} \xi^{b]} \right) + \mathcal{O}(h^2) \right) \quad (\text{A.11})
 \end{aligned}$$

By integrating $\mathbf{Q}[\xi]$ in the conformal boundary $\tilde{\Omega} \rightarrow 0$ limit we can find the conserved quantity. Supposing a falloff behavior of the stress-energy tensor, the Einstein equation for negative cosmological constant (and $\ell_{\text{AdS}} = 1$) near infinity is written as:

$$R_{ac} = -(d-1)g_{ac}, \quad (\text{A.12})$$

which, when written in terms of $\tilde{g}_{ab}$, yields, up to terms of order h_{ab} :

$$\tilde{R}_{ac} + (d-2) \frac{\tilde{\nabla}_a \tilde{\nabla}_c \tilde{\Omega}}{\tilde{\Omega}} - (d-1) \tilde{g}_{ac} \tilde{g}^{bd} \frac{\tilde{\nabla}_b \tilde{\Omega} \tilde{\nabla}_d \tilde{\Omega}}{\tilde{\Omega}^2} + \tilde{g}_{ac} \tilde{g}^{bd} \frac{\tilde{\nabla}_b \tilde{\nabla}_d \tilde{\Omega}}{\tilde{\Omega}} = -(d-1) \frac{\tilde{g}_{ac}}{\tilde{\Omega}^2}. \quad (\text{A.13})$$

Requiring that the term proportional to $\tilde{\Omega}^{-2}$ vanishes at the boundary implies that $n^a = \tilde{g}^{ab} \tilde{\nabla}_b \tilde{\Omega}$ is normalized space-like according to $\tilde{g}_{ab}$ in the $\tilde{\Omega} \rightarrow 0$ limit. We note, however, that the order $\mathcal{O}(\tilde{\Omega}^{-1})$ term will set the (conformal, Ricci) geometry at infinity through

$$\tilde{R}_{ac} - \frac{1}{2(d-1)} \tilde{g}_{ac} \tilde{R} = -(d-2) \frac{1}{\tilde{\Omega}} \tilde{\nabla}_a \tilde{\nabla}_c \tilde{\Omega}. \quad (\text{A.14})$$

Therefore the conformal geometry chosen for the four-dimensional manifold at infinity will influence the asymptotics of $\tilde{\Omega}$, and thus the actual value of the asymptotic charges.

We may consider now two cases for $\mathbf{Q}[\xi]$, depending on ξ^a :

1. ξ^a is the time translation operator, and therefore orthogonal to the surface of integration used to define the conserved quantity;
2. ξ^a generates rotations, so it is tangent to the surface of integration;

In the first case, the term $\xi^{[a} \tilde{\nabla}^{b]} \tilde{\Omega}$ is a bi-vector normal to the surface of integration and $\mathbf{Q}[\xi]$ diverges in the $\tilde{\Omega} \rightarrow 0$ limit. One usually deals with this divergence by adding a boundary (Gibbons-Hawking-York) term to the action to regularize it, although the procedure of fixing the conformal structure, much in the spirit of what has been done in this calculation, was defended in [62]. For the perturbed conformal metric, the vector $\tilde{\nabla}^a \tilde{\Omega}$ approaches the unit vector at spatial infinity n^a by:

$$n^a = (1+N)^{-1/2} \tilde{\nabla}^a \tilde{\Omega}, \quad N = \tilde{\Omega}^2 h_{ab} \tilde{\nabla}^a \tilde{\Omega} \tilde{\nabla}^b \tilde{\Omega}. \quad (\text{A.15})$$

The second term in brackets in the first line of (A.11) is $\tilde{\Omega} \tilde{\nabla}^{[a} \xi^{b]}$ which we will also take to vanish, either because ξ^a is covariantly constant with respect to $\tilde{g}_{ab}$, as in the case of time translation, or because the derivative is tangent to the surface of integration, as in

the case of rotations. The next terms will converge in the $\tilde{\Omega} \rightarrow 0$ limit if $h_{ab} = \tilde{\Omega}^{d-3} \gamma_{ab}$. Substituting in (A.11) and taking the limit we have:

$$Q[\xi]_{a_1 \dots a_{d-2}} = \frac{1}{\kappa} \epsilon[\tilde{g}]_{aba_1 \dots a_{d-2}} ((d-1)n^a \gamma_c^b \xi^c - 2\xi^a \gamma_c^b n^c + (\gamma_c^c + N)\xi^a n^b) + \mathcal{O}(\tilde{\Omega}). \quad (\text{A.16})$$

One can then conclude that the value for generic charges in asymptotically AdS spaces stems not only from the conformal structure encoded in $\tilde{g}$, but also from the notion of “normalized direction in RG flow”, encoded by the normalization of n^a .

B The Fredholm determinant formulation of the Painlevé VI transcendent

This is a review of the Fredholm determinant formulation of the Painlevé VI, borrowing heavily from [35]

$$\tau(t) = \text{const} \cdot t^{\frac{1}{4}(\sigma^2 - \theta_0^2 - \theta_t^2)} (1-t)^{-\frac{1}{2}\theta_t \theta_1} \det(\mathbb{1} - \mathbf{A}\Phi(t)\mathbf{D}\Phi(t)^{-1}), \quad (\text{B.1})$$

where the Plemelj operators $\mathbf{A}, \mathbf{D}$ act on the space of pairs of square-integrable functions defined on $\mathcal{C}$, a circle on the complex plane with radius $R < 1$:

$$(\mathbf{A}g)(z) = \oint_{\mathcal{C}} \frac{dz'}{2\pi i} A(z, z') g(z'), \quad (\mathbf{D}g)(z) = \oint_{\mathcal{C}} \frac{dz'}{2\pi i} D(z, z') g(z'), \quad g(z) = \begin{pmatrix} f_+(z) \\ f_-(z) \end{pmatrix}, \quad (\text{B.2})$$

with kernels given, for $|t| < R$, explicitly by

$$\begin{aligned} A(z, z') &= \frac{\Psi(\sigma, \theta_1, \theta_\infty; z) \Psi^{-1}(\sigma, \theta_1, \theta_\infty; z') - \mathbb{1}}{z - z'}, \\ D(z, z') &= \frac{\mathbb{1} - \Psi(-\sigma, \theta_t, \theta_0; t/z) \Psi^{-1}(-\sigma, \theta_t, \theta_0; t/z')}{z - z'}. \end{aligned} \quad (\text{B.3})$$

The operators $\mathbf{A}$ and $\mathbf{D}$ can be thought of as projecting the analytic and principal parts of a generic function defined on the circle $\mathcal{C}$ into the space generated by functions with definite monodromy. The parametrix Ψ and the “gluing” matrix Φ are

$$\Psi(\alpha_1, \alpha_2, \alpha_3; z) = \begin{pmatrix} \phi(\alpha_1, \alpha_2, \alpha_3; z) & \chi(\alpha_1, \alpha_2, \alpha_3; z) \\ \chi(-\alpha_1, \alpha_2, \alpha_3; z) & \phi(-\alpha_1, \alpha_2, \alpha_3; z) \end{pmatrix}, \quad \Phi(\kappa, \sigma; t) = \begin{pmatrix} t^{-\sigma/2} \kappa^{-1/2} & 0 \\ 0 & t^{\sigma/2} \kappa^{1/2} \end{pmatrix}, \quad (\text{B.4})$$

with ϕ and χ given in terms of Gauss’ hypergeometric function — note the overall minus sign with respect to the conventions in [35]:

$$\begin{aligned} \phi(\alpha_1, \alpha_2, \alpha_3; z) &= {}_2F_1\left(\frac{1}{2}(\alpha_1 - \alpha_2 + \alpha_3), \frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3); \alpha_1; z\right) \\ \chi(\alpha_1, \alpha_2, \alpha_3; z) &= \frac{\alpha_3^2 - (\alpha_1 - \alpha_2)^2}{4\alpha_1(1 + \alpha_1)} z {}_2F_1\left(1 + \frac{1}{2}(\alpha_1 - \alpha_2 + \alpha_3), 1 + \frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3); 2 + \alpha_1; z\right). \end{aligned} \quad (\text{B.5})$$

Finally, κ is a known function of the monodromy parameters:

$$\kappa = s \frac{\Gamma^2(1-\sigma) \Gamma(1+\frac{1}{2}(\theta_t+\theta_0+\sigma)) \Gamma(1+\frac{1}{2}(\theta_t-\theta_0+\sigma))}{\Gamma^2(1+\sigma) \Gamma(1+\frac{1}{2}(\theta_t+\theta_0-\sigma)) \Gamma(1+\frac{1}{2}(\theta_t-\theta_0-\sigma))} \times \frac{\Gamma(1+\frac{1}{2}(\theta_1+\theta_\infty+\sigma)) \Gamma(1+\frac{1}{2}(\theta_1-\theta_\infty+\sigma))}{\Gamma(1+\frac{1}{2}(\theta_1+\theta_\infty-\sigma)) \Gamma(1+\frac{1}{2}(\theta_1-\theta_\infty-\sigma))}, \quad (\text{B.6})$$

and, finally, the parameter s is given in terms of the monodromy parameters $\{\sigma_{0t}, \sigma_{t1}\}$:

$$s = \frac{(w_{1t} - 2p_{1t} - p_{0t}p_{01}) - (w_{01} - 2p_{01} - p_{0t}p_{1t}) \exp(\pi i \sigma_{0t})}{(2 \cos \pi(\theta_t - \sigma_{0t}) - p_0)(2 \cos \pi(\theta_1 - \sigma_{0t}) - p_\infty)}, \quad (\text{B.7})$$

where

$$p_i = 2 \cos \pi \theta_i, \quad p_{ij} = 2 \cos \pi \sigma_{ij}, \quad (\text{B.8})$$

$$w_{0t} = p_0 p_t + p_1 p_\infty, \quad w_{1t} = p_1 p_t + p_0 p_\infty, \quad w_{01} = p_0 p_1 + p_t p_\infty.$$

The expansion for the tau function at small t can be computed directly from its definition:

$$\tau(t) = C t^{\frac{1}{4}(\sigma^2 - \theta_0^2 - \theta_t^2)} (1-t)^{\frac{1}{2}\theta_1\theta_t} \left(1 + \left(\frac{\theta_1\theta_t}{2} + \frac{(\theta_0^2 - \theta_t^2 - \sigma^2)(\theta_\infty^2 - \theta_1^2 - \sigma^2)}{8\sigma^2} \right) t - \frac{(\theta_0^2 - (\theta_t - \sigma)^2)(\theta_\infty^2 - (\theta_1 - \sigma)^2)}{16\sigma^2(1+\sigma)^2} \kappa t^{1+\sigma} - \frac{(\theta_0^2 - (\theta_t + \sigma)^2)(\theta_\infty^2 - (\theta_1 + \sigma)^2)}{16\sigma^2(1-\sigma)^2} \kappa^{-1} t^{1-\sigma} + \dots \right). \quad (\text{B.9})$$

The accessory parameter K_0 , defined by (3.10) as essentially the logarithm derivative of τ , has an analogous expansion for small t and can be seen in [18].

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