

Modular invariant dynamics and fermion mass hierarchies around $\tau = i$

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ABSTRACT: We discuss fermion mass hierarchies within modular invariant flavour models. We analyse the neighbourhood of the self-dual point $\tau = i$, where modular invariant theories possess a residual Z_4 invariance. In this region the breaking of Z_4 can be fully described by the spurion $\epsilon \approx \tau - i$, that flips its sign under Z_4 . Degeneracies or vanishing eigenvalues of fermion mass matrices, forced by the Z_4 symmetry at $\tau = i$, are removed by slightly deviating from the self-dual point. Relevant mass ratios are controlled by powers of $|\epsilon|$. We present examples where this mechanism is a key ingredient to successfully implement an hierarchical spectrum in the lepton sector, even in the presence of a non-minimal Kähler potential.

KEYWORDS: Beyond Standard Model, Neutrino Physics, Supersymmetric Standard Model, Compactification and String Models

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1 Introduction

Revealing the origin(s) of fermion masses and explaining the structure of quark and lepton mixing are among the deepest long-standing questions in particle physics. Looking for an organising principle behind the observed patterns of fermion masses and mixing, flavour symmetries have been proposed and extensively studied in the last several decades (see ref. [1] for a recent review). In spite of a significant theoretical effort resulting in many models able to describe certain pieces of the flavour puzzle, there is arguably no fundamental theory of flavour. Still, the concept of symmetry is admittedly among the best tools we have to search for such a theory.

While in the pure bottom-up approach flavour symmetries are introduced as a new ingredient, in the top-down perspective they may arise from a UV completion of the Standard Model (SM), perhaps string theory. Today a theory of flavour fully derived from string theory represents a formidable unsolved problem, due to the huge number of possible solutions, and we might be led to consider a more modest approach where the large freedom related to a bottom-up procedure is mitigated by some guiding principle. In particular, in ref. [2], modular invariance arising in many string compactifications has been proposed as a candidate for flavour symmetry in the lepton sector.

In the simplest case, modular invariance arises from the compactification of a higher dimensional theory on a torus or an orbifold. Size and shape of the compact space are parametrised by a modulus τ living in the upper-half complex plane, up to modular transformations. These can be interpreted as discrete gauge transformations, related to the redundancy of the description. The low-energy effective theory, relevant to the known

particle species, has to obey modular invariance and Yukawa couplings become functions of τ . The framework has a big conceptual advantage. In a generic bottom-up approach, realistic flavour symmetries require an ad-hoc symmetry breaking sector, with Vacuum Expectation Values (VEVs) of scalar multiplets — the flavons — carefully tailored in size and orientation. In minimal schemes based on modular invariance, flavons are not needed and the scalar sector can be completely replaced by the moduli space. Moreover the action of modular invariance in generation space occurs through a well-defined set of finite groups. Continuous groups and many discrete groups are not allowed, thus reducing the arbitrariness of the construction.

The proposal has been accompanied by several significant developments and activity in model building (for a review, see [1] and references therein). Recently, the proposed framework of modular invariant supersymmetric theories of flavour has been extended to incorporate (in a non-trivial way) several moduli [3]. The latter might be needed to describe different sectors of a theory, i.e. quarks, charged leptons and neutrinos. Moreover, the role of modular invariance as flavour symmetry has been intensively investigated in the last two years from a top-down perspective [4–7], resulting, in particular, in the concept of eclectic flavour symmetries [8–12]. A top-down construction incorporating several moduli has been very recently formulated in refs. [13, 14].

Despite the appealing aspects, realistic realisations of the framework have still to face several difficulties. In the vast majority of phenomenologically viable modular invariant models constructed so far, the observed hierarchies, in particular those between the charged lepton masses, are achieved by tuning free parameters entering the superpotential. Moreover, even if the requirement of modular invariance represents a severe constraint at the level of the Yukawa couplings of the (supersymmetric) theory, it allows in general a much larger freedom at the level of kinetic terms [2, 15]. Non-minimal kinetic terms are allowed, with corresponding free parameters that affect the prediction of fermion masses and mixing angles. Finally, so far the modulus is mainly treated as a free parameter, varied to optimise the agreement with the data. In concrete models, the preferred value of τ often occurs in the vicinity of the self-dual point $\tau = i$ (see, e.g. refs. [2, 16–26]), where one of the generators of the modular group remains unbroken. In CP invariant realisations, it has been realised that CP violation can be explained by a small departure of the modulus from this special point, where CP is unbroken [2, 19]. It remains to be understood how the dynamics drives the modulus in the vicinity of, but not exactly on, the self-dual point.

In this article, we aim to address the question of whether the lepton mass hierarchies can originate from modular invariance alone, with the Lagrangian parameters being $\mathcal{O}(1)$ quantities. Some attempts to explain mass hierarchies avoiding fine tuning have been made in refs. [27, 28]. In the former study, new fields singlet under both the SM gauge group and finite modular group, but carrying non-trivial modular weights, have been introduced, whereas in the latter work expansions of modular forms around fixed points $\tau = i$, $\tau = \omega \equiv e^{2\pi i/3}$ and $\tau = i\infty$ have been employed, and a semi-analytic study of the fermion mass matrices has been performed.¹ We go beyond previous works discussing in

¹Models of lepton masses and mixing angles with τ sitting exactly at these fixed points have been discussed in refs. [18, 29–33].

detail the structure of the modular invariant theory near $\tau = i$, which is motivated by phenomenological models.

At this point, the theory has a residual Z_4 symmetry (cf. ref. [24]). We show that the action of this symmetry can be realised linearly, even when $\tau \neq i$. In particular, $\epsilon \approx \tau - i$ behaves as a (small) spurion with Z_4 charge $+2$. Thus, in a vicinity of $\tau = i$, we have a Z_4 symmetric theory broken by ϵ . Similar considerations apply to other fixed points as well. We show that the residual Z_4 symmetry can be exploited to reduce the rank of the charged lepton mass matrix at $\tau = i$, with small non-vanishing masses arising from a small departure from the self-dual point.

As we will see in explicit models, when a minimal Kähler potential is adopted, the constraints coming from modular invariance realised near $\tau = i$ can be too restrictive to allow full agreement with the experimental data. We are thus led to discuss non-minimal candidates of Kähler potential, their pattern around $\tau = i$ and their impact on the predictions for masses, mixing angles and CP-violating (CPV) phases. The presence of extra inputs related to a more general Kähler potential is expected to reduce the predictability of our setup. Nevertheless, our analysis suggests that the arbitrariness coming from the Kähler potential can be partially tamed precisely by mass matrices of reduced rank. Moreover, in the models discussed in this paper, a non-minimal Kähler potential allows to obtain full agreement with the data, without being the dominant source of the observed hierarchies. Further constraints on the Kähler potential and consequent improvement in predictability could come by extending the modular group to an eclectic flavour symmetry, where modular invariance is enhanced by a traditional finite flavour symmetry [9].

The article is organised as follows. In section 2, we review the formalism of supersymmetric modular invariant flavour theories. Next, in section 3, we zoom in on their structure near $\tau = i$ and discuss linear realisation of the associated residual Z_4 symmetry. Further, in section 4, we present examples of modular invariant models at level 3 where flavour hierarchies are generated by a small departure from $\tau = i$. Finally, we draw our conclusions in section 5. Appendix A discusses the most general form of the Kähler potential quadratic in the modular forms of level 3 and weight 2.

2 Modular invariant models

In this section we shortly review the formalism of supersymmetric modular invariant theories [34, 35] applied to flavour physics [2]. The theory depends on a set of chiral supermultiplets φ comprising the dimensionless modulus $\tau \equiv \varphi_0/\Lambda$ ($\text{Im } \tau > 0$) and other superfields φ_i ($i \geq 1$). Here Λ represents the cut-off of our effective theory, and can be interpreted as the relevant mass scale of an underlying fundamental theory. In the case of rigid supersymmetry, the Lagrangian \mathcal{L} ,² is fully specified by the Kähler potential $K(\varphi, \bar{\varphi})$, a real gauge-invariant function of the chiral multiplets and their conjugates, by the superpotential $W(\varphi)$, a holomorphic gauge-invariant function of the chiral multiplets, and by the gauge kinetic function $f(\varphi)$, a dimensionless holomorphic gauge-invariant function of the chiral

²Up to terms with at most two derivatives in the bosonic fields.

superfields. Neglecting gauge interactions, we have:

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} K(\varphi, \bar{\varphi}) + \int d^2\theta W(\varphi) + \int d^2\bar{\theta} \bar{W}(\bar{\varphi}). \quad (2.1)$$

The Lagrangian is invariant under transformations γ of the homogeneous modular group $\Gamma = \text{SL}(2, Z)$:

$$\tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \varphi_i \rightarrow (c\tau + d)^{-k_i} \rho(\tilde{\gamma})_{ij} \varphi_j \quad (i, j \geq 1), \quad (2.2)$$

where a, b, c, d are integers obeying $ad - bc = 1$. Such transformations are generated by the two elements of Γ :

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.3)$$

The matrix $\rho(\tilde{\gamma})$ is a unitary representation of the group $\Gamma_N = \Gamma/\Gamma(N)$, obtained as a quotient between the group Γ and a principal congruence subgroup $\Gamma(N)$, the positive integer N being the level of the representation. The level N is kept fixed in the construction, and $\tilde{\gamma}$ represents the equivalence class of γ in Γ_N . In general $\rho(\tilde{\gamma})$ is a reducible representation and all superfields belonging to the same irreducible component should have the same weight k_i , here assumed to be integer.³ In the following, we denote by (φ_i, ψ_i) the spin-(0, 1/2) components of the chiral superfields φ_i ($i \geq 1$).⁴ The terms bilinear in the fermion fields read [37]:

$$\mathcal{L}_F = \mathcal{L}_{F,K} + \mathcal{L}_{F,2}, \quad (2.4)$$

with:⁵

$$\mathcal{L}_{F,K} = i K_i^j \bar{\psi}_j \bar{\sigma}^\mu D_\mu \psi^i, \quad \mathcal{L}_{F,2} = -\frac{1}{2} [W_{ij} - W_l (K^{-1})_m^l K_{ij}^m] \psi^i \psi^j + \text{h.c.}, \quad (2.5)$$

where lower (upper) indices in K and W stand for derivatives with respect to holomorphic (anti-holomorphic) fields. When the scalar fields in eq. (2.5) take their VEVs, we can move to the basis where matter fields are canonically normalised, through a transformation:

$$\psi^i \rightarrow (z^{-1/2})_{ij} \psi^j, \quad (2.6)$$

where the matrix $(z^{1/2})_{ij}$ satisfies: $K_i^j = [(z^{1/2})^\dagger]^{jl} (z^{1/2})_{li}$.⁶ We can identify the fermion mass matrix as:

$$m_{kn} = [W_{ij} - W_l (K^{-1})_m^l K_{ij}^m] (z^{-1/2})_{ik} (z^{-1/2})_{jn}, \quad (2.7)$$

where VEVs are understood. In the previous equation, the second term in the square bracket vanishes when supersymmetry is unbroken and the VEV of W_l is zero. When we

³We restrict to integer modular weights. Fractional weights are in general allowed, but require a suitable multiplier system [8, 36].

⁴The distinction between superfields and their scalar components should be clear from the context.

⁵The covariant derivative is $D_\mu \psi^i = \partial_\mu \psi^i + (K^{-1})_m^i K_{ki}^m \partial_\mu \varphi^k \psi^l$.

⁶Notice that this transformation mixes holomorphic and anti-holomorphic indices, and there is no more fundamental distinction between upper and lower components of the matrix $(z^{1/2})$.

turn on supersymmetry breaking effects, the first term is expected to dominate over the second one, provided there is a sufficient gap between the sfermion masses m_{SUSY} and the messenger/cutoff scale M . This holds both for vector-like and for chiral fermions. Indeed, up to loop factors or other accidental factors, the VEVs of W_l , W_{ij} and K_{ij}^m are of the order of $m_{\text{SUSY}}M$, M and $1/M$, respectively, when fermions are vector-like. When chiral fermions are considered, W_{ij} and K_{ij}^m are both depleted by v/M with respect to the vector-like case, v denoting the gauge symmetry breaking scale. Thus we have a relative suppression between the two contributions of order m_{SUSY}/M , which can be made tiny (cf. ref. [17]). If we work under this assumption, the mass matrix is well approximated by:

$$m_{kn} = W_{ij} (z^{-1/2})_{ik} (z^{-1/2})_{jn} . \tag{2.8}$$

The supersymmetry breaking terms neglected here can be useful to give masses to light fermions, which otherwise would remain massless in the exact supersymmetry limit. We will come back to this point when discussing concrete models, in section 4. Due to the conservation of the electric charge, the equality of eq. (2.8) holds separately in any charge sector. By focusing on the lepton sector (E^c, L) and by assuming that the neutrino masses arise from the Weinberg operator, we have:

$$W = -E_i^c \mathcal{Y}_{ij}^e(\tau) L_j H_d - \frac{1}{2\Lambda_L} L_i \mathcal{C}_{ij}^\nu(\tau) L_j H_u H_u , \tag{2.9}$$

where $H_{u,d}$ are the Higgs chiral multiplets and Λ_L is the scale where lepton number is broken. The general relation (2.8) specialises into:

$$m_e = \left(z_{E^c}^{-1/2}\right)^T \mathcal{Y}^e(\tau) \left(z_L^{-1/2}\right) v_d , \quad m_\nu = \left(z_L^{-1/2}\right)^T \mathcal{C}^\nu(\tau) \left(z_L^{-1/2}\right) v_u^2 / \Lambda_L , \tag{2.10}$$

where we have absorbed the renormalisation factors for $H_{u,d}$ in the definition of their VEVs. In section 4, we will also comment on the special limit where $z_{E^c, L}^{-1/2}$ are universal, i.e. proportional to the unit matrix. The mass matrices obtained in this case will be referred to as “bare” matrices and denoted by $m_{e,\nu}^{(0)}$. An important consequence of modular invariance is the special functional dependence of $\mathcal{Y}^e(\tau)$ and $\mathcal{C}^\nu(\tau)$ on the modulus τ . Under a transformation of Γ , the chiral multiplets ($E_i^c, L_i, H_{u,d}$) transform as in eq. (2.2), with weights $(k_{E_i^c}, k_{L_i}, k_{H_{u,d}})$ and representations $(\rho_{E^c}(\tilde{\gamma}), \rho_L(\tilde{\gamma}), \mathbb{1})$. For the superpotential W to be modular invariant, $\mathcal{Y}^e(\tau)$ and $\mathcal{C}^\nu(\tau)$ should obey:

$$\mathcal{Y}^e(\gamma\tau) = (c\tau + d)^{k_e} \rho_{E^c}^*(\tilde{\gamma}) \mathcal{Y}^e(\tau) \rho_L^\dagger(\tilde{\gamma}), \quad \mathcal{C}^\nu(\gamma\tau) = (c\tau + d)^{k_\nu} \rho_L^*(\tilde{\gamma}) \mathcal{C}^\nu(\tau) \rho_L^\dagger(\tilde{\gamma}), \tag{2.11}$$

where the weights $k_{e,\nu}$ are matrices satisfying: $(k_e)_{ij} = k_{E_i^c} + k_{L_j} + k_{H_d}$ and $(k_\nu)_{ij} = k_{L_i} + k_{L_j} + 2k_{H_u}$. Thus $\mathcal{Y}^e(\tau)$ and $\mathcal{C}^\nu(\tau)$ are modular forms of given level and weight. Since the linear space of such modular forms is finite dimensional, the choices for $\mathcal{Y}^e(\tau)$ and $\mathcal{C}^\nu(\tau)$ are limited. If neutrino masses originate from a type I seesaw mechanism, eqs. (2.9) and (2.10) hold with the identification:

$$\frac{\mathcal{C}^\nu(\tau)}{\Lambda_L} = -(\mathcal{Y}^\nu(\tau))^T \mathcal{M}(\tau)^{-1} \mathcal{Y}^\nu(\tau), \tag{2.12}$$

where $\mathcal{Y}^\nu(\tau)$ and $\mathcal{M}(\tau)$ denote the matrix of neutrino Yukawa couplings and the mass matrix of the heavy electroweak singlets N^c , respectively. Notice that there is no dependence on the renormalisation factor $(z_{N^c}^{-1/2})$ of the heavy modes. In some cases $\mathcal{Y}^e(\tau)$ and/or $\mathcal{C}^\nu(\tau)$ are completely determined as a function of τ up to an overall constant, thus providing a strong potential constraint on the mass spectrum, eq. (2.10).

Unfortunately, such property does not extend to the Kähler potential K and to the renormalisation factors $(z_{E^c,L}^{-1/2})$. Minimal choices of K , appropriate for a perturbative regime, can receive large non-perturbative corrections in the region of the moduli space we will consider. Without a control over the non-perturbative dynamics, in a generic point of the moduli space the factors $(z_{E^c,L}^{-1/2})$ remain unknown. If we allowed for completely arbitrary $(z_{E^c,L}^{-1/2})$, under mild conditions any mass matrix could be predicted. From eq. (2.10) we see that, given $\mathcal{Y}^e(\tau)$ and $(z_L^{-1/2})$, we could reproduce any desired matrix m_e , by selecting a particular $(z_{E^c}^{-1/2})^T$:

$$\left(z_{E^c}^{-1/2}\right)^T = m_e \left(\mathcal{Y}^e(\tau) v_d \left(z_L^{-1/2}\right)\right)^{-1}. \tag{2.13}$$

An arbitrary matrix m_e would result in a completely unconstrained lepton mixing matrix. Similar considerations would apply to the neutrino mass matrix m_ν .

The loss of predictability associated to the Kähler corrections may however be less severe than eq. (2.13) might suggest, for two reasons. First, note that the above solution requires a non-singular $\mathcal{Y}^e(\tau)$. A singular $\mathcal{Y}^e(\tau)$ can only give rise to a singular m_e . Correspondingly, a hierarchical $\mathcal{Y}^e(\tau)$ can only correspond to a hierarchical m_e , unless the eigenvalues of the matrix $(z_{E^c}^{-1/2})^T$ in eq. (2.13) come in very large ratios. Although we cannot exclude the latter possibility, here we focus on the class of models where the corrections associated to the Kähler potential do not alter the “bare” limit by more than about one order of magnitude. Hence a singular or nearly singular $\mathcal{Y}^e(\tau)$ will tame the loss of predictability associated with the Kähler potential. Needless to say, a hierarchical $\mathcal{Y}^e(\tau)$ is needed to reproduce the mass spectrum in the charged lepton sector. Different considerations apply to the neutrino sector, where a singular $\mathcal{C}^\nu(\tau)$ might not be a good first order approximation of the data.

A second constraint on the effect of the Kähler corrections arises in the vicinity of the fixed points of Γ , $\tau = i$, $\tau = -1/2 + i\sqrt{3}/2$ and $\tau = i\infty$, invariant under the action of the elements S , ST and T , respectively. In the following, we will assume the modulus to be in the vicinity of the point $\tau = i$, as suggested by several models to correctly reproduce the data. The invariance under S provides a constraint on the possible form of the Kähler potential at $\tau = i$ and in its vicinity.

3 Residual symmetry near $\tau = i$

The residual symmetry of the theory at $\tau = i$ is the cyclic group Z_4 generated by the element S , whose action on the chiral multiplets φ_i in $\tau = i$ can be read from eq. (2.2):

$$\varphi_i \rightarrow \sigma_{ij} \varphi_j, \quad \sigma_{ij} = i^{ki} \rho(\tilde{S})_{ij} \quad (i, j > 0), \tag{3.1}$$

where σ is unitary, σ^2 is a parity operator and $\sigma^4 = \mathbb{1}$. To analyse the neighbourhood of $\tau = i$, we expand both the Kähler potential and the superpotential in powers of the matter fields φ_i ($i > 0$):⁷

$$\begin{aligned} W &= \sum_{i_1, \dots, i_n} Y_{i_1, \dots, i_n}(\tau) \varphi_{i_1} \dots \varphi_{i_n} + \dots \\ K &= \bar{\varphi}_i z^i_j(\tau, \bar{\tau}) \varphi_j + \dots \end{aligned} \tag{3.2}$$

In the vicinity of $\tau = i$, it is possible to cast the theory as an ordinary Z_4 invariant theory, where the symmetry acts linearly on the fields, slightly broken by the spurion $(\tau - i)$. When we depart from $\tau = i$, the S elements acts on the fields as:

$$\tau \rightarrow -\frac{1}{\tau}, \quad \varphi_i \rightarrow (-\tau)^{-k_i} \rho(\tilde{S})_{ij} \varphi_j \quad (i, j > 0). \tag{3.3}$$

We perform the field redefinition:

$$\begin{cases} \tau = i \frac{i + \epsilon/2}{i - \epsilon/2} \\ \tilde{\varphi}_j = \left(1 - i \frac{\epsilon}{2}\right)^{-k_j} \varphi_j \end{cases} \tag{3.4}$$

mapping the upper-half complex plane into the disk $|\epsilon| < 2$. In the linear approximation:

$$\epsilon = (\tau - i) + O\left((\tau - i)^2\right). \tag{3.5}$$

Under the S transformation in (3.3), the new fields transform as:

$$\begin{cases} \epsilon \rightarrow -\epsilon \\ \tilde{\varphi}_i \rightarrow \sigma_{ij} \tilde{\varphi}_j \end{cases} \tag{3.6}$$

We see that the action of the Z_4 symmetry is linear in the new field basis, even when $\tau \neq i$. In particular ϵ behaves as a spurion with Z_4 charge +2. In the new field basis, the coefficients of the field expansion (3.2) read:

$$\begin{aligned} \tilde{Y}_{i_1, \dots, i_n} &= \left(1 - i \frac{\epsilon}{2}\right)^{k_{i_1} + \dots + k_{i_n}} Y_{i_1, \dots, i_n} \\ \tilde{z}^i_j &= \left(1 + i \frac{\bar{\epsilon}}{2}\right)^{k_i} z^i_j \left(1 - i \frac{\epsilon}{2}\right)^{k_j} \end{aligned} \tag{3.7}$$

The invariance of the theory under Z_4 requires $\tilde{Y}_{i_1, \dots, i_n}(\epsilon)$ and $\tilde{z}^i_j(\epsilon, \bar{\epsilon})$ to satisfy:

$$\begin{aligned} \tilde{Y}_{i_1, \dots, i_n}(\epsilon) &= \sigma_{j_1 i_1} \dots \sigma_{j_n i_n} \tilde{Y}_{j_1, \dots, j_n}(-\epsilon) \\ \tilde{z}^i_j(\epsilon, \bar{\epsilon}) &= \sigma^{\dagger ik} \tilde{z}^k_l(-\epsilon, -\bar{\epsilon}) \sigma_{lj} \end{aligned} \tag{3.8}$$

In particular, setting $\epsilon = 0$, the above equations express the necessary conditions for the invariance of the theory at the symmetric point $\tau = i$. By expanding $\tilde{z}^i_j(\epsilon, \bar{\epsilon})$ in powers of ϵ

⁷Electrically neutral multiplets whose scalar component acquires a VEV, like $H_{u,d}$, might mix in the kinetic term with the modulus τ . The mixing is parametrically suppressed by v/Λ and will be ignored in the following.

Observable	Best-fit value with 1σ error	
m_e/m_μ	$0.0048^{+0.0002}_{-0.0002}$	
m_μ/m_τ	$0.0565^{+0.0045}_{-0.0045}$	
	NO	IO
$\delta m^2 [10^{-5} \text{ eV}^2]$	$7.42^{+0.21}_{-0.20}$	
$\Delta m^2 [10^{-3} \text{ eV}^2]$	$2.480^{+0.026}_{-0.028}$	$-2.461^{+0.028}_{-0.028}$
$r \equiv \delta m^2/ \Delta m^2 $	$0.0299^{+0.0009}_{-0.0009}$	$0.0301^{+0.0009}_{-0.0009}$
$\sin^2 \theta_{12}$	$0.304^{+0.012}_{-0.012}$	$0.304^{+0.013}_{-0.012}$
$\sin^2 \theta_{13}$	$0.02219^{+0.00062}_{-0.00063}$	$0.02238^{+0.00063}_{-0.00062}$
$\sin^2 \theta_{23}$	$0.573^{+0.016}_{-0.020}$	$0.575^{+0.016}_{-0.019}$
δ/π	$1.09^{+0.15}_{-0.13}$	$1.57^{+0.14}_{-0.17}$

Table 1. Best-fit values of the charged lepton mass ratios and the neutrino oscillation parameters with the corresponding 1σ errors. For the charged lepton mass ratios we have used the values given in ref. [38], averaged over $\tan\beta$ as described in the text, whereas for the neutrino parameters we have used the results obtained in refs. [39, 40] (with Super-Kamiokande atmospheric data).

we see that the terms of first order vanish, up to possible non-diagonal terms relating fields with opposite value of σ . We conclude that in a neighbourhood of the fixed point $\tau = i$, and in the absence of any information about the Kähler potential, the theory reduces to a linearly realised Z_4 flavour symmetric theory, in the presence of a (small) spurion with charge $+2$.

4 Models

In this section, we present two models making use of the results of the previous section to account for the observed hierarchies in the lepton spectrum, namely the smallness of the charged lepton mass ratios, m_e/m_τ and m_μ/m_τ and of the neutrino mass ratio $r \equiv \delta m^2/|\Delta m^2|$, where $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ (with the standard neutrino labelling). The hierarchies will be naturally accounted for by the small breaking of Z_4 , $|\epsilon| \ll 1$, i.e. by the closeness of the modulus τ to the Z_4 symmetric point $\tau = i$, while the parameters in the superpotential will be $\mathcal{O}(1)$, and the corrections to the minimal Kähler will not be larger than $\mathcal{O}(1)$. In table 1, we collect the best-fit values of the leptonic parameters with the corresponding 1σ uncertainties. For the charged lepton mass ratios we use the results of ref. [38], where for m_μ/m_τ we take an average between the values obtained for $\tan\beta = 10$ and $\tan\beta = 38$. For the neutrino oscillation parameters we employ the results of the global analysis performed in refs. [39, 40]. In what follows, when fitting models to the data, we use five dimensionless observables that have been mea-

	L	E_1^c	E_2^c	E_3^c	H_u	H_d
$SU(2)_L \times U(1)_Y$	$(\mathbf{2}, -1/2)$	$(\mathbf{1}, +1)$	$(\mathbf{1}, +1)$	$(\mathbf{1}, +1)$	$(\mathbf{2}, +1/2)$	$(\mathbf{2}, -1/2)$
Γ_3	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}$	$\mathbf{1}$
k	1	3	3	3	0	0

Table 2. Assignment of representations and modular weights in Model 1.

	\tilde{L}_1	\tilde{L}_2	\tilde{L}_3	\tilde{E}_1^c	\tilde{E}_2^c	\tilde{E}_3^c	\tilde{H}_u	\tilde{H}_d	ϵ
Z_4	1	-1	-1	-1	-1	-1	0	0	2

Table 3. Z_4 charges (mod 4) in Model 1.

sured with a good precision, i.e. two mass ratios,⁸ m_μ/m_τ and r , and three leptonic mixing angles, $\sin^2 \theta_{12}$, $\sin^2 \theta_{13}$, $\sin^2 \theta_{23}$. Regarding the Dirac CPV phase, δ , values between π and 2π (approximately) are currently allowed at 3σ for both neutrino mass spectrum with normal ordering (NO) and that with inverted ordering (IO). Moreover, in ref. [18], it has been shown that under the transformation $\tau \rightarrow -\tau^*$ and complex conjugation of couplings present in the superpotential, CPV phases change their signs, whereas masses and mixing angles remain the same. In fact, this reflects CP properties of modular invariant models [19] (see also [41]). As a consequence, the Dirac phase δ is not particularly constraining for our fits, and we do not include it in the list of input observables, regarding the obtained values as predictions.

4.1 Model 1: Weinberg operator and inverted ordering

We work at level 3, and the relevant finite modular group is Γ_3 . In this subsection, we assume that neutrino masses are generated by the Weinberg operator. The field content of the model along with the assignment of Γ_3 representations and modular weights k is shown in table 2. The corresponding charges under Z_4 , obtained using eq. (3.1), are shown in table 3. We work in a real basis for the elements of Γ_3 where $\rho(\tilde{S}) = \text{diag}(+1, -1, -1)$ for the irreducible three-dimensional representation.

The quantum number assignments have immediate consequences for the charged lepton mass spectrum:

1. At $\tau = i$, the charged lepton mass matrix m_e has rank one. This follows from the Z_4 charges in table 3, forcing

$$m_e = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix}. \tag{4.1}$$

⁸In the models presented below, $m_e = 0$ by construction, so we do not include the ratio m_e/m_μ here. See subsection 4.3 for possible ways of generating non-zero m_e .

2. For a generic $\tau \neq i$, m_e has rank two. While Z_4 alone would allow m_e to have rank three, the underlying modular invariance forces the coefficients of the first and second rows of m_e to be proportional, thus reducing the rank. In fact, modular invariance requires the coupling of E_1^c and E_2^c to L to be proportional to the same modular form multiplet, namely, the triplet of weight four. The Kähler corrections cannot modify the rank condition. Thus, in the considered model, the electron has zero mass.

For $\tau \approx i + \epsilon$, with $|\epsilon| \ll 1$, we obtain the prediction

$$m_e : m_\mu : m_\tau = 0 : \mathcal{O}(\epsilon) : 1. \tag{4.2}$$

Concerning the neutrino mass spectrum, from the charges of the lepton doublets L_i in table 3, we deduce that m_ν in $\tau = i$ takes the following general form:

$$m_\nu = \begin{pmatrix} 0 & a & b \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}. \tag{4.3}$$

This matrix has rank two and two degenerate non-zero eigenvalues. Notice that, while a generic Z_4 model would not account for the values of the parameters a and b , here the underlying modular invariance fixes the relative values, before Kähler corrections. With the Z_4 assignment of table 3, we are implicitly using the basis where S is diagonal for the irreducible triplet of Γ_3 and we find $a/b = Y_3(i)/Y_2(i)$, where $Y^{(2)}(\tau) \equiv (Y_1(\tau), Y_2(\tau), Y_3(\tau))^T$ denotes the weight-two triplet of modular forms. On the other hand, generic Kähler corrections could mix L_2 and L_3 , as they have the same Z_4 charge (see eq. (3.8)), leading to arbitrary a/b , as in generic Z_4 models. For $\tau \approx i + \epsilon$, the rank of m_ν becomes three, and we obtain the neutrino mass spectrum with inverted ordering of the form

$$m_1 : m_2 : m_3 = 1 : (1 + \mathcal{O}(\epsilon)) : \mathcal{O}(\epsilon), \tag{4.4}$$

and, in particular,

$$r = \mathcal{O}(\epsilon). \tag{4.5}$$

Clearly, both qualitative relations (4.2) and (4.5) are phenomenologically intriguing. They are consequences of modular invariance alone, and are thus independent from the parameters in the superpotential or the Kähler potential (provided these are non-hierarchical by themselves). In subsection 4.3, we discuss two possible mechanisms to generate a naturally small electron mass.

While this model successfully accounts for the observed mass hierarchies (with a non-vanishing electron mass still to be generated), it is not satisfactory when it comes to the mixing angles. The point is that in order for eq. (4.3) to lead to a reasonable leading order approximation, the tau lepton should correspond to a linear combination of L_2 and L_3 , while eq. (4.1) forces the tau lepton to be mainly L_1 . Indeed, the prediction for the mixing angles at $\tau = i$ is

$$\sin^2 \theta_{13} = \cos^2 \theta_{12}^e, \quad \sin^2 \theta_{12} = \frac{1}{2}, \quad \sin^2 \theta_{23} = 1, \tag{4.6}$$

where θ_{12}^e is an arbitrary angle related to the presence of two vanishing eigenvalues in m_e , to be fixed by the Z_4 breaking. These predictions imply that in order to generate the correct mixing angles $\sin^2 \theta_{23} \approx 0.6$ and $\sin^2 \theta_{12} \approx 0.3$, large hierarchical deviations from the minimal Kähler metrics are required,⁹ as $|\epsilon| \ll 1$ cannot give rise to such large corrections. This is clearly an unpleasant feature, since it introduces a source of hierarchy in the Lagrangian parameters. We carried out a full numerical study of the model, after adding a non-minimal Kähler potential depending on four new real parameters. The outcome confirms the above qualitative considerations. More precisely, we gauge the degree of hierarchy related to a non-canonical Kähler potential K by means of the condition number

$$\kappa(K) = \lambda_{\max}(z)/\lambda_{\min}(z), \quad (4.7)$$

the ratio between the maximum and minimum eigenvalues of z^i_j at the best-fit point. We find that all Kähler metrics providing a good fit near $\tau = i$ turn out to have $\kappa(K_{L, E^c})$ very large, typically in the range $10^3 \div 10^4$. We discuss in the next subsection a seesaw variant of the present model which allows to mitigate the need of hierarchical Kähler metrics.

4.2 Model 2: seesaw mechanism and normal ordering

The main phenomenological obstructions in the model discussed above are the leading order predictions for the mixing angles. In this subsection, we show how to evade them by introducing electroweak singlet neutrinos N^c and generating the Weinberg operator through the type I seesaw mechanism. This widens the class of possible neutrino mass matrices that can be obtained, if the singlet neutrino mass matrix becomes singular in the limit $\tau \rightarrow i$. In this case, for the standard analysis of the seesaw mechanism to be valid, singlet neutrino masses are required to be large compared to the electroweak scale. In the example discussed below, this is easily achieved outside of a neighbourhood of $\tau = i$, provided the overall singlet neutrino mass scale is large enough.

To be concrete, we augment the field content of table 2 with electroweak singlets $N^c \sim \mathbf{3}$ under Γ_3 , with weight $k_{N^c} = 1$. As before, we denote by $Y^{(2)}$ the weight 2 modular form triplet, and by $Y^{(4)} \equiv (Y^{(2)}Y^{(2)})_{\mathbf{3}_S}$ the weight 4 triplet of modular forms. We denote by $\mathbf{3}_S$ and $\mathbf{3}_A$ the symmetric and antisymmetric triplet contractions of two Γ_3 triplets, respectively. The superpotential $W = W_e + W_\nu$ of the lepton sector reads:

$$W_e = - \left[\alpha E_1^c \left(LY^{(4)} \right)_1 + \beta E_2^c \left(LY^{(4)} \right)_1 + \gamma E_3^c \left(LY^{(4)} \right)_{1''} \right] H_d, \quad (4.8)$$

$$W_\nu = -\kappa \left(\left[(N^c L)_{\mathbf{3}_S} + g (N^c L)_{\mathbf{3}_A} \right] Y^{(2)} \right)_1 H_u - \Lambda \left(N^c N^c Y^{(2)} \right)_1. \quad (4.9)$$

The parameters κ and Λ can be made real without loss of generality, whereas g is complex in general. In the real basis for $\mathbf{3}$ of Γ_3 , this superpotential leads to the following matrices

⁹The need for non-minimal Kähler metrics stems not only from the leading order predictions for the mixing angles, but also from the mass spectrum of the model. In the vicinity of $\tau = i$ we found, both numerically and through an approximate analytical study, that m_μ/m_τ is smaller than r , while data require the opposite.

$\mathcal{Y}^e(\tau)$, $\mathcal{Y}^\nu(\tau)$ and $\mathcal{M}(\tau)$:

$$\mathcal{Y}^e(\tau) = 2 \begin{pmatrix} \alpha Y_2 Y_3 & \alpha Y_1 Y_3 & \alpha Y_1 Y_2 \\ \beta Y_2 Y_3 & \beta Y_1 Y_3 & \beta Y_1 Y_2 \\ \gamma Y_2 Y_3 & \gamma \omega Y_1 Y_3 & \gamma \omega^2 Y_1 Y_2 \end{pmatrix}, \quad (4.10)$$

$$\mathcal{Y}^\nu(\tau) = \kappa \left[\begin{pmatrix} 0 & Y_3 & Y_2 \\ Y_3 & 0 & Y_1 \\ Y_2 & Y_1 & 0 \end{pmatrix} + g \begin{pmatrix} 0 & Y_3 & -Y_2 \\ -Y_3 & 0 & Y_1 \\ Y_2 & -Y_1 & 0 \end{pmatrix} \right], \quad (4.11)$$

$$\mathcal{M}(\tau) = 2\Lambda \begin{pmatrix} 0 & Y_3 & Y_2 \\ Y_3 & 0 & Y_1 \\ Y_2 & Y_1 & 0 \end{pmatrix}. \quad (4.12)$$

The matrix $\mathcal{C}^\nu(\tau)$ of eq. (2.9) is now given by the seesaw formula of eq. (2.12).

Some analytical considerations easily follow from the previous equations for the “bare” quantities, i.e. those corresponding to the minimal Kähler potential. We make use of the following ϵ -expansion of $Y^{(2)}$:¹⁰

$$Y_1 = -ix\epsilon, \quad Y_2 = y(1+i\epsilon), \quad Y_3 = y^*(1+i\epsilon), \quad (4.13)$$

where, up to an overall constant, $x \approx 1.49087$ and $y = \sqrt{3}/2 + i(3/2 - \sqrt{3})$. To first order in ϵ we obtain:

$$\frac{\mathcal{Y}^\nu(\tau)}{\kappa} = \begin{pmatrix} 0 & (1+g)y^* & (1-g)y \\ (1-g)y^* & 0 & 0 \\ (1+g)y & 0 & 0 \end{pmatrix} + i\epsilon \begin{pmatrix} 0 & (1+g)y^* & (1-g)y \\ (1-g)y^* & 0 & -(1+g)x \\ (1+g)y & -(1-g)x & 0 \end{pmatrix}, \quad (4.14)$$

$$\frac{\mathcal{M}(\tau)}{2\Lambda} = \begin{pmatrix} 0 & y^* & y \\ y^* & 0 & 0 \\ y & 0 & 0 \end{pmatrix} + i\epsilon \begin{pmatrix} 0 & y^* & y \\ y^* & 0 & -x \\ y & -x & 0 \end{pmatrix}. \quad (4.15)$$

Notice that the bare Majorana mass matrix has one eigenvalue proportional to ϵ , thus vanishing in the limit $\tau \rightarrow i$. This corresponds to the case of single right-handed neutrino dominance, in which one of the electroweak singlet neutrinos is massless in the symmetric limit.¹¹ Inverting the matrix in eq. (4.15) and using the seesaw relation for the bare light neutrino mass matrix $m_\nu^{(0)}$ we find to $\mathcal{O}(\epsilon)$:

$$m_\nu^{(0)} = \begin{pmatrix} \frac{2ig^2|y|^2}{x} \frac{1}{\epsilon} - \frac{4g^2|y|^2}{x} & -(1+g^2)y^* & -(1+g^2)y \\ -(1+g^2)y^* & 0 & 0 \\ -(1+g^2)y & 0 & 0 \end{pmatrix} + \mathcal{O}(\epsilon). \quad (4.16)$$

The leading order form of the charged lepton mass matrix is as in eq. (4.1).

¹⁰One can prove, in general, that $\frac{d}{d\tau}\Big|_i Y_{2,3} = iY_{2,3}(i)$. Moreover, we can rephase $Y^{(2)}$ in such a way that $(Y_3(i))^* = Y_2(i) \equiv y$. In this basis, we find that $\frac{d}{d\tau}\Big|_i Y_1 \in -i\mathbb{R}^+$.

¹¹As already observed, in order for the standard seesaw analysis to be valid, we must require the product $|\epsilon|\Lambda$ (that is the order of magnitude of the lightest right-handed neutrino mass) to be large with respect to the electroweak scale. For the present model this does not pose any practical restriction, since the best-fit region (see below) is achieved for values of $|\epsilon| \sim 10^{-2} \div 10^{-1}$.

The leading order predictions for neutrino masses and mixing angles strongly depend upon the parameter g .

- A neutrino mass spectrum with IO can be realised when $|g|^2 \ll |\epsilon|$. Then, $m_\nu^{(0)}$ has approximately the same form as in the model with the Weinberg operator considered in subsection 4.1. We get the neutrino mass spectrum with IO:

$$m_1 : m_2 : m_3 = 1 : \approx 1 : \mathcal{O}(\epsilon) \tag{4.17}$$

and the predictions for the mixing angles reported in eq. (4.6), in particular, $\sin^2 \theta_{23} = 1$.

- A neutrino mass spectrum with NO can be realised when $|\epsilon| \ll |g|^2$. In this case,

$$m_\nu^{(0)} = \begin{pmatrix} c(g^2/\epsilon) & a & b \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} + \mathcal{O}(\epsilon), \tag{4.18}$$

with $|a|, |b|, |c|$ being $\mathcal{O}(1)$ numbers. Therefore, we have the neutrino mass spectrum with NO:

$$m_1 : m_2 : m_3 = \mathcal{O}(\epsilon^2/g^2) : \mathcal{O}(\epsilon^2/g^4) : \mathcal{O}(1) \tag{4.19}$$

implying $r = \mathcal{O}(\epsilon^4/g^4)$. The mixing angles are:

$$\sin^2 \theta_{12} = \mathcal{O}(1), \quad \sin^2 \theta_{23} \approx \sin^2 \theta_{13} \approx r^{1/2}. \tag{4.20}$$

Again, the leading order prediction for $\sin^2 \theta_{23}$ is far away from its measured value and requires significant corrections from the Kähler potential.

To verify the viability of the model we have performed a full numerical study, also allowing for a non-minimal form of the Kähler potential for the matter fields. In general, modular invariance allows many terms in the Kähler potential [2, 15]. In the considered bottom-up approach, there seems to be no way of reducing the number of these terms. However, this may change if modular symmetry is augmented by a traditional finite flavour symmetry [9] or, perhaps, if some other top-down principle is in action. In what follows, to be concrete, we adopt three simplifying assumptions:

- The new terms in K are quadratic in $Y^{(2)}$. This is sufficient to illustrate our results.
- The minimal form (up to overall normalisation) is restored at $\text{Im} \tau \rightarrow \infty$. This assumption is inspired by the minimal form of the Kähler potential arising in certain string theory compactifications in the large volume limit, corresponding to $\text{Im} \tau \rightarrow \infty$ (see, e.g. [15, 42] and references therein).
- The diagonal entries, already controlled by the minimal Kähler potential, are not affected by the new terms.

Under these assumptions and with the assignment of representations and weights given in table 2, we find:¹²

$$K = L^\dagger K_L L + E^{c\dagger} K_{E^c} E^c, \quad (4.21)$$

where

$$K_L = \frac{1}{2 \operatorname{Im} \tau} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2 \operatorname{Im} \tau \begin{pmatrix} 0 & (\alpha_5 + i\alpha_6) X_{12} & (\alpha_5 - i\alpha_6) X_{13} \\ (\alpha_5 - i\alpha_6) X_{12}^* & 0 & (\alpha_5 + i\alpha_6) X_{23} \\ (\alpha_5 + i\alpha_6) X_{13}^* & (\alpha_5 - i\alpha_6) X_{23}^* & 0 \end{pmatrix}. \quad (4.22)$$

Here α_5 and α_6 are real coefficients and

$$X_{12} = Y_1^* Y_2 - Y_1 Y_2^*, \quad X_{13} = Y_1^* Y_3 - Y_1 Y_3^*, \quad X_{23} = Y_2^* Y_3 - Y_2 Y_3^*. \quad (4.23)$$

In the E^c sector, we obtain:

$$K_{E^c} = \frac{1}{8 (\operatorname{Im} \tau)^3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2 \operatorname{Im} \tau} \begin{pmatrix} 0 & 0 & c_{13} X \\ 0 & 0 & c_{23} X \\ c_{13}^* X^* & c_{23}^* X^* & 0 \end{pmatrix}, \quad (4.24)$$

where c_{13} and c_{23} are complex coefficients and

$$X = Y_1^* Y_1 + \omega Y_2^* Y_2 + \omega^2 Y_3^* Y_3 \sim \mathbf{1}'' . \quad (4.25)$$

Noteworthy, the seesaw formula (2.12) does not depend on the renormalisation factor $z_{N^c}^{-1/2}$ of the heavy fields N^c , so that we will not need to specify the Kähler metric of N^c in what follows.

The inclusion of a non-minimal Kähler potential, even within the above restrictive assumptions, brings in several additional free parameters: $\alpha_{5,6}$, $\operatorname{Re}(c_{13,23})$ and $\operatorname{Im}(c_{13,23})$.¹³ Adding them to β/α , γ/α , $\operatorname{Re}(g)$, $\operatorname{Im}(g)$, $\operatorname{Re}(\tau)$ and $\operatorname{Im}(\tau)$, we have a total of 12 dimensionless input parameters, more than the number of observables. Thus the focus of our analysis cannot be on predictability. Rather, we are interested in accounting for the mass hierarchies in terms of the Z_4 parameter ϵ , in the context of a model reproducing all lepton masses and mixings. While the mass hierarchies alone can be easily accommodated without the need of hierarchical Lagrangian parameters, some degree of hierarchy turns out to be required by the need to fix the mixing parameters. Useful parameters to estimate such hierarchies in the Kähler potential are the condition numbers of eq. (4.7). To establish the possibility to reproduce all the relevant observables, and the role of Z_4 breaking in setting the mass hierarchies, we have selected several benchmark points with slightly different features. We show the results of two (pairs) of the benchmark points in tables 4 and 5. In all such benchmark points, all five dimensionless observables take exactly their experimental best-fit values (for the time being we set $m_e = 0$). In addition, the model predicts a normal ordered neutrino mass spectrum and the values of the CPV phases. Interestingly, for both pairs of the benchmark points, the predicted value of δ (the one with minus sign) matches

¹²We present the full expressions for K_L and K_{E^c} quadratic in $Y^{(2)}$ in appendix A.

¹³In our numerical analysis, we have set $\operatorname{Im}(c_{13,23}) = 0$.

Input parameters		Observables		Predictions	
$\text{Re } \tau$	± 0.0235	m_e/m_μ	0	m_1 [eV]	0.0062
$\text{Im } \tau$	1.080	m_μ/m_τ	0.0565	m_2 [eV]	0.0106
β/α	0.1459	r	0.0299	m_3 [eV]	0.0506
γ/α	5.955	$\sin^2 \theta_{12}$	0.304	δ/π	± 0.92
$\text{Re } g$	-0.1494	$\sin^2 \theta_{13}$	0.02219	α_{21}/π	± 0.97
$\text{Im } g$	∓ 0.3169	$\sin^2 \theta_{23}$	0.573	α_{31}/π	± 0.93
α_5	-0.2071	δm^2 [10^{-5} eV ²]	7.42	$ m_{ee} $ [eV]	0
α_6	-0.1437	Δm^2 [10^{-3} eV ²]	2.480	$\sum_i m_i$ [eV]	0.0673
c_{13}	-0.2656			Ordering	NO
c_{23}	0.0145			M_1/Λ	0.225
$v_u^2 \kappa^2/\Lambda$ [eV]	0.0189			M_2/Λ	2.298
$ \epsilon \approx \tau - i $	0.0830			M_3/Λ	2.524

Table 4. First pair of best-fit points in a vicinity of $\tau = i$ found considering the Kähler potential in eqs. (4.22)–(4.25).

Input parameters		Observables		Predictions	
$\text{Re } \tau$	± 0.0328	m_e/m_μ	0	m_1 [eV]	0.0063
$\text{Im } \tau$	1.137	m_μ/m_τ	0.0565	m_2 [eV]	0.0107
β/α	0.2388	r	0.0299	m_3 [eV]	0.0506
γ/α	7.854	$\sin^2 \theta_{12}$	0.304	δ/π	± 0.91
$\text{Re } g$	-0.2234	$\sin^2 \theta_{13}$	0.02219	α_{21}/π	± 0.98
$\text{Im } g$	± 0.4469	$\sin^2 \theta_{23}$	0.573	α_{31}/π	± 0.88
α_5	-0.1865	δm^2 [10^{-5} eV ²]	7.42	$ m_{ee} $ [eV]	0
α_6	-0.1116	Δm^2 [10^{-3} eV ²]	2.480	$\sum_i m_i$ [eV]	0.0675
c_{13}	-0.2405			Ordering	NO
c_{23}	-0.0959			M_1/Λ	0.353
$v_u^2 \kappa^2/\Lambda$ [eV]	0.0191			M_2/Λ	2.130
$ \epsilon \approx \tau - i $	0.1408			M_3/Λ	2.483

Table 5. Second pair of best-fit points in a vicinity of $\tau = i$ found considering the Kähler potential in eqs. (4.22)–(4.25).

its experimental best-fit value. Notice also the interesting result $|m_{ee}| = 0$ which at the leading order can be seen as a simple consequence of the matrix patterns (4.16) and (4.1).¹⁴ Finally, we report in the last column the masses M_i , $i = 1, 2, 3$, of the heavy neutrinos in the units of Λ . Although the value of Λ cannot be uniquely fixed, it can be estimated as (see the first column of the tables) $\Lambda \approx v_u^2 \kappa^2 / (0.02 \text{ eV}) \approx 10^{15} \sin^2 \beta \text{ GeV}$, where we have used $v_u = v \sin \beta$, with $v = 174 \text{ GeV}$, and $\kappa \sim \mathcal{O}(1)$. This implies that for $\tan \beta \gtrsim 1$, the scale $\Lambda \gtrsim 5 \times 10^{14} \text{ GeV}$. Let us stress once again that in the considered model, $M_1 \sim |\epsilon| \Lambda$, and thus, it is generated by a small departure of τ from i .

Our analysis shows that the mass hierarchies are indeed governed by Z_4 breaking, whereas, in general, Kähler corrections reflect on the lepton mass spectrum through $\mathcal{O}(1)$ changes. For example, in the first pair of benchmark points (see table 4), we verified numerically that Kähler corrections only affect the mass ratios by about a factor of 2;¹⁵ on the other hand, at these points, the Kähler metrics are by themselves somewhat hierarchical, as shown by the condition numbers $\kappa(K_L) \approx 12$ and $\kappa(K_{E^c}) \approx 16$. In the second pair of benchmark points shown in table 5, the hierarchies in the Kähler metrics are both reduced (the condition numbers are $\kappa(K_L) \approx 6$ and $\kappa(K_{E^c}) \approx 12$), and points with even milder hierarchies may potentially be found. These observations lead us to conclude that the deviations from the canonical Kähler metric present in the best-fit points, have little to do with the mass spectrum hierarchies; rather, they are necessary in order to reproduce the correct PMNS mixing angles.

We have analysed more in detail the dependence of the fitted observables on the parameters of the Kähler potential. In figures 1 and 2, we plot the values of the five dimensionless observables versus $\alpha_{5,6}$ and $c_{13,23}$, respectively.¹⁶ All other input parameters are fixed to their best-fit values as in table 4. We see that the parameters α_5 and α_6 strongly impact the predictions for the mixing angles and the two mass ratios, r and m_μ/m_τ , whereas c_{13} and c_{23} in K_{E^c} mainly affect the predictions for $\sin^2 \theta_{23}$ and m_μ/m_τ .

In conclusion, we see that a mass matrix of reduced rank at the self-dual point $\tau = i$ can explain the observed mass hierarchies in terms of $\mathcal{O}(1)$ Lagrangian parameters. At the same time, at least in the model considered here, moderately hierarchical Kähler and superpotential parameters are needed to fix the mixing angle predictions. Whether or not this is a general feature of this class of models is a question which definitely requires further investigation, but is beyond the scope of the present work. On the other hand, in order to get a fully realistic description of lepton masses we should still generate a non-vanishing electron mass, without perturbing too much the results achieved so far. We discuss this point in the next subsection.

¹⁴Given the column ordering in eq. (4.1), $|m_{ee}|$ is given at the leading order by $(m_\nu^{(0)})_{33} = 0 + \mathcal{O}(\epsilon)$.

¹⁵For the minimal Kähler potential, i.e. setting $\alpha_5 = \alpha_6 = c_{13} = c_{23} = 0$, we find $m_\mu/m_\tau = 0.0520$ and $r = 0.0637$, whereas the angles $\sin^2 \theta_{12} = 0.228$, $\sin^2 \theta_{13} = 0.03751$ and $\sin^2 \theta_{23} = 0.256$ are far away from their experimental values.

¹⁶For the values of $\alpha_{5,6}$ ($c_{13,23}$) beyond the range displayed in the x -axis, the matrix K_L (K_{E^c}) is not positive definite, and thus, the corresponding Kähler metric is well-defined only for the displayed range of $\alpha_{5,6}$ ($c_{13,23}$).

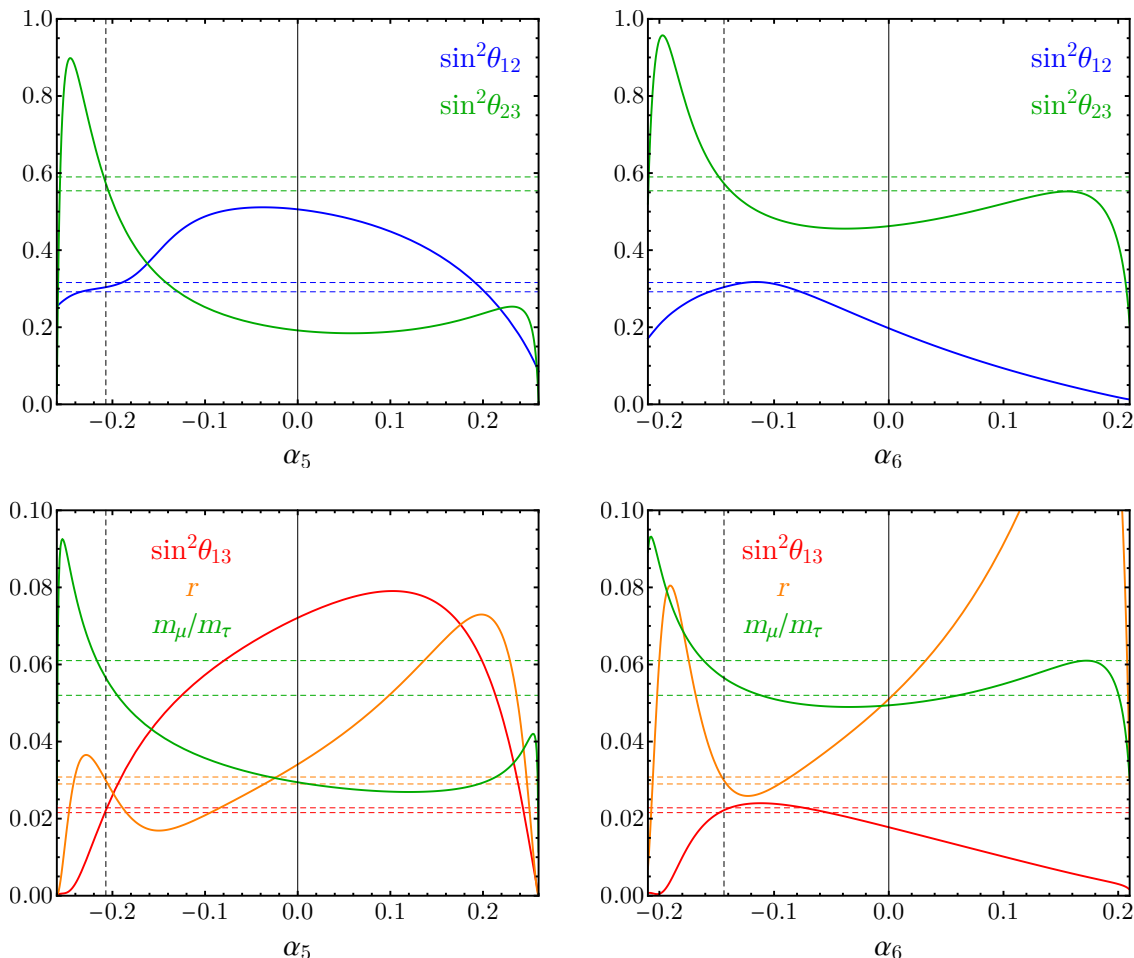


Figure 1. Dependence of the mixing angles and two mass ratios on α_5 (left) and α_6 (right), fixing all other input parameters to their best-fit values from table 4. The *horizontal dashed lines* indicate the boundaries of the respective 1σ ranges from table 1. The *vertical dashed line* in the left (right) panels stands for the best-fit value of α_5 (α_6) from table 4.

4.3 Generating $m_e \neq 0$

Both the models discussed above yield, by construction, $m_e = 0$. One can easily concoct mechanisms to generate the small electron mass without spoiling the other predictions. We give below two examples, where m_e is generated by supersymmetry breaking and by dimension six operators, respectively.

If supersymmetry is broken by some F -term, fermion masses get corrected by the second term of eq. (2.7), which, as discussed below the same equation, scales as $m_{\text{SUSY}} v/M$ for SM fermion masses (where M is the SUSY breaking messenger scale). For instance, a Kähler interaction of the form:

$$K \supset \frac{1}{\Lambda^2} \chi^\dagger E_i^c \left[a_i(\tau) H_d + b_i(\tau) \tilde{H}_u \right] L + \text{h.c.}, \quad (4.26)$$

where the superfield χ gets a supersymmetry breaking expectation value $\langle \chi \rangle = F\theta^2$, gives

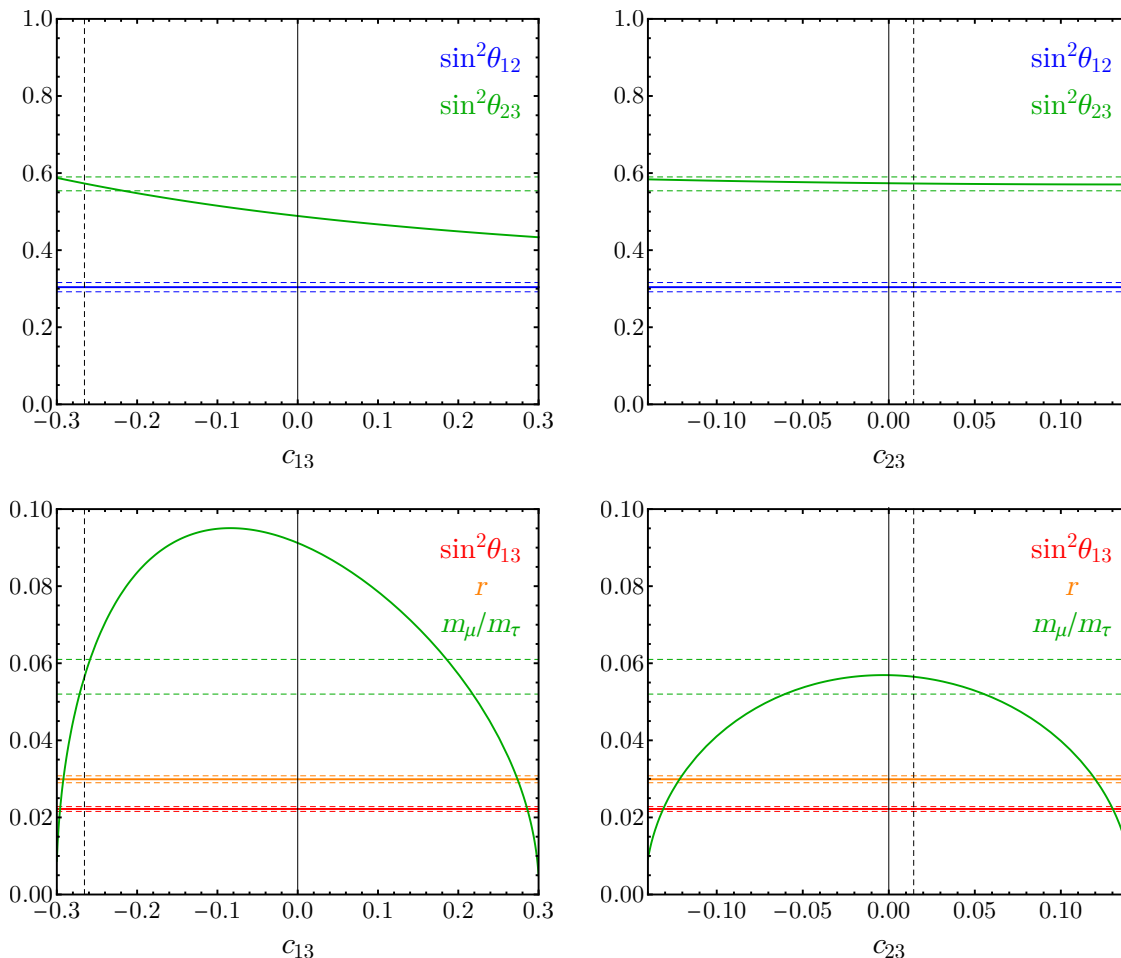


Figure 2. Dependence of the mixing angles and two mass ratios on c_{13} (left) and c_{23} (right), fixing all other input parameters to their best-fit values from table 4. The *horizontal dashed lines* indicate the boundaries of the respective 1σ ranges from table 1. The *vertical dashed line* in the left (right) panels stands for the best-fit value of c_{13} (c_{23}) from table 4.

a contribution to the charged lepton Yukawa matrix proportional to F/Λ^2 , which in turn generically induces an electron Yukawa coupling of the same order.

As a second possibility, one may generate $m_e \neq 0$ through the dimension six operator:

$$(E_i^c L H_d)(H_u H_d), \tag{4.27}$$

whose Wilson coefficient should be a modular form of the appropriate weight. In order for this mechanism to work, we need to generalise the weight assignments in table 2. We make the following requirements:

$$k_L = 1 - k_u, \tag{4.28}$$

$$k_{E^c} = 3 + k_u - k_d, \tag{4.29}$$

$$k_u + k_d \neq 0. \tag{4.30}$$

The first two conditions ensure that the superpotentials discussed in the previous subsections have weight zero; the last condition implies that the operator (4.27) has different weight from the corresponding renormalisable Yukawa term,¹⁷ so that it couples to a functionally independent modular form multiplet (making the resulting charged lepton mass matrix of rank three). Such a mechanism thus generates $m_e \sim v_u v_d^2 / \Lambda^2$, where Λ is the scale at which the operator in eq. (4.27) is generated.

While in some flavour models, the ratios m_e/m_τ and m_μ/m_τ are associated to different powers of the same expansion parameter, we note that here the two ratios are associated to independent parameters.

5 Conclusion

Supersymmetric modular invariant theories offer an attractive framework to address the flavour puzzle. The role of flavour symmetry is played by modular invariance, regarded as a discrete gauge symmetry, thus circumventing the obstruction concerning fundamental global symmetries. The arbitrary symmetry breaking sector of the conventional models based on flavour symmetries is replaced by the moduli space. Yukawa couplings become modular forms, severely restricted by the matter transformation properties. So far this framework has delivered interesting preliminary results especially in the lepton sector, where neutrino masses and lepton mixing parameters can be efficiently described in terms of a limited number of input parameters.

Weak points in most of the existing constructions are the need of independent hierarchical parameters to describe charged lepton masses, the reduced predictability caused by a general form of the Kähler potential, and the absence of a reliable dynamical mechanism to determine the value of τ in the vacuum. As a matter of fact, in several models reproducing lepton masses and mixing parameters, the required value of τ is close to i , the self-dual point where the generator S of the modular group and CP (if the Lagrangian is CP invariant) are unbroken. A small departure of τ from i suffices to generate sizeable CP-violating effects in the lepton sector.

For these reasons, we were led to analyse more in detail the vicinity of $\tau = i$. Our goal was to show that a small deviation from the self-dual point can be responsible for the observed mass hierarchy $m_e \ll m_\mu \ll m_\tau$ and $\delta m^2 \ll \Delta m^2$. At $\tau = i$, the theory has an exact Z_4 symmetry, generated by the element S of the modular group. In the neighbourhood of $\tau = i$, the breaking of Z_4 can be fully described by the (small) spurion $\epsilon \approx \tau - i$, that flips its sign under Z_4 . We explained how to exploit this residual Z_4 symmetry in order to obtain lepton mass matrices having reduced rank at $\tau = i$. This can be easily done with a suitable assignment of modular weights and representations for matter fields. There is a twofold advantage in this strategy. First, mass ratios that are forced to vanish at $\tau = i$ by the Z_4 symmetry are expected to acquire small values $\propto |\epsilon|^n$ ($n > 0$) near the self-dual point. Second, the reduced rank of the mass matrices can tame

¹⁷Curiously, the same condition can be exploited to make the Higgs μ -term vanish at $\tau = i$. The Higgs μ -term, being a Γ_3 singlet modular form of weight $k_u + k_d$, vanishes by eq. (3.3) at $\tau = i$ if $k_u + k_d \neq 0 \pmod{4}$, since all Γ_3 singlets have $\rho(\tilde{S}) = 1$.

the contribution from a non-minimal Kähler potential, provided the metrics of the matter fields do not display large hierarchies.

To see whether this strategy can be successfully realised or not, we built a concrete model at level 3, where neutrinos get masses through the type I seesaw mechanism. The model predicts a normal mass ordering. The number of parameters exceeds the number of fitted observables and we cannot claim predictability. However, with τ being near i , mass ratios and mixing angles are reproduced with input parameters nearly of the same order of magnitude and matter kinetic terms display only a moderate hierarchy. We saw that the main contribution to the mass hierarchy can be induced by the singular mass matrix at the Z_4 symmetric point. In the model we considered, the Kähler potential and the other Lagrangian parameters are crucial in order to correctly reproduce the values of the mixing angles. While the Z_4 symmetry plays a fundamental role in all our discussion, we notice that our model could not have been realised in the context of a Z_4 flavour symmetry alone. In particular, the electron mass vanishes in the models we have considered due to the correlations among generic Z_4 -invariant operators provided by the underlying modular invariance. Also the leading order values of the mixing angles are dictated by Z_4 . We have discussed possible sources of a non-vanishing electron mass. While the models we formulated have clearly room for improvement, we consider them as a good starting point to naturally accommodate the observed fermion mass hierarchies within a modular invariant framework.

Note Added. Few weeks after completion of this work, ref. [43] appeared on the arXiv, in which the authors performed a systematic study of all fixed points, $\tau_S = i$, $\tau_{ST} = e^{2\pi i/3}$, and $\tau_T = i\infty$, assuming a minimal form of the Kähler potential. The points τ_{ST} and τ_T enjoy residual $Z_3 \times Z_2$ and $Z_N \times Z_2$ symmetries, respectively (where N is the level of modular forms used in the construction, $N = 3$ in our work). It is found, in particular, that fermion mass hierarchies crucially depend on the decomposition of field representations under the residual symmetry group.

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A Kähler potential quadratic in $Y^{(2)}$

In the real basis for the Γ_3 generators S and T in the 3-dimensional representation we employ in this work, L^* and $Y^{(2)*}$ transform as triplets, i.e. $L^* \rightarrow \rho_{\mathbf{3}}(\tilde{\gamma})L^*$ and $Y^{(2)*} \rightarrow \rho_{\mathbf{3}}(\tilde{\gamma})Y^{(2)*}$. Thus, we can contract first L^* with L and $Y^{(2)*}$ with $Y^{(2)}$, and after

that perform contractions of the obtained multiplets. Proceeding in this way, we obtain:

$$(L^*L)_{\mathbf{1}} = L^\dagger L, \quad (L^*L)_{\mathbf{1}'} = L^\dagger M_{\mathbf{1}'} L, \quad (L^*L)_{\mathbf{1}''} = L^\dagger M_{\mathbf{1}''} L, \quad (\text{A.1})$$

$$(L^*L)_{\mathbf{3}_S} = \begin{pmatrix} L^\dagger M_{\mathbf{3}_S}^{(1)} L \\ L^\dagger M_{\mathbf{3}_S}^{(2)} L \\ L^\dagger M_{\mathbf{3}_S}^{(3)} L \end{pmatrix}, \quad (L^*L)_{\mathbf{3}_A} = \begin{pmatrix} L^\dagger M_{\mathbf{3}_A}^{(1)} L \\ L^\dagger M_{\mathbf{3}_A}^{(2)} L \\ L^\dagger M_{\mathbf{3}_A}^{(3)} L \end{pmatrix}, \quad (\text{A.2})$$

with the matrices $M_{\mathbf{r}}$ being

$$M_{\mathbf{1}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad M_{\mathbf{1}''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad (\text{A.3})$$

$$M_{\mathbf{3}_S}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_{\mathbf{3}_S}^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_{\mathbf{3}_S}^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.4})$$

$$M_{\mathbf{3}_A}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad M_{\mathbf{3}_A}^{(2)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_{\mathbf{3}_A}^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.5})$$

and $\omega = e^{2\pi i/3}$. The same equations hold for $(Y^{(2)*}Y^{(2)})_{\mathbf{r}}$. Taking further invariant contractions of the obtained multiplets, we find

$$\begin{aligned} K_L &= (2 \operatorname{Im} \tau)^{-k_L} \mathbb{1} \\ &+ (2 \operatorname{Im} \tau)^{k_Y - k_L} \left\{ \alpha_1 Y^{(2)\dagger} Y^{(2)} \mathbb{1} + \alpha_2 \left[(Y^{(2)\dagger} M_{\mathbf{1}''} Y^{(2)}) M_{\mathbf{1}'} + (Y^{(2)\dagger} M_{\mathbf{1}'} Y^{(2)}) M_{\mathbf{1}''} \right] \right. \\ &\quad + \alpha_3 i \left[(Y^{(2)\dagger} M_{\mathbf{1}''} Y^{(2)}) M_{\mathbf{1}'} - (Y^{(2)\dagger} M_{\mathbf{1}'} Y^{(2)}) M_{\mathbf{1}''} \right] \\ &\quad + \alpha_4 \sum_{n=1}^3 (Y^{(2)\dagger} M_{\mathbf{3}_S}^{(n)} Y^{(2)}) M_{\mathbf{3}_S}^{(n)} + \alpha_5 \sum_{n=1}^3 (Y^{(2)\dagger} M_{\mathbf{3}_A}^{(n)} Y^{(2)}) M_{\mathbf{3}_A}^{(n)} \\ &\quad \left. + \alpha_6 \sum_{n=1}^3 i (Y^{(2)\dagger} M_{\mathbf{3}_A}^{(n)} Y^{(2)}) M_{\mathbf{3}_S}^{(n)} + \alpha_7 \sum_{n=1}^3 i (Y^{(2)\dagger} M_{\mathbf{3}_S}^{(n)} Y^{(2)}) M_{\mathbf{3}_A}^{(n)} \right\}, \end{aligned} \quad (\text{A.6})$$

where α_j , $j = 1, \dots, 7$, are real coefficients, which accompany hermitian matrices. (We have used the fact that $M_{\mathbf{1}'}^\dagger = M_{\mathbf{1}''}$, $M_{\mathbf{3}_S}^{(n)\dagger} = M_{\mathbf{3}_S}^{(n)}$, and $M_{\mathbf{3}_A}^{(n)\dagger} = -M_{\mathbf{3}_A}^{(n)}$.)

One of our assumptions is that the canonical form of K_L is restored at $\operatorname{Im} \tau \rightarrow \infty$. The q -expansions of Y_i in the real basis read:

$$Y_1(\tau) = \frac{1}{\sqrt{3}} \left(1 - 6q^{1/3} - 18q^{2/3} + 12q + \dots \right), \quad (\text{A.7})$$

$$Y_2(\tau) = \frac{1}{\sqrt{3}} \left(1 - 6\omega q^{1/3} - 18\omega^2 q^{2/3} + 12q + \dots \right), \quad (\text{A.8})$$

$$Y_3(\tau) = \frac{1}{\sqrt{3}} \left(1 - 6\omega^2 q^{1/3} - 18\omega q^{2/3} + 12q + \dots \right), \quad (\text{A.9})$$

where $q = e^{2\pi i\tau}$. Thus, at $\text{Im } \tau \rightarrow \infty$

$$Y_1 = Y_2 = Y_3 \sim \frac{1}{\sqrt{3}}, \quad (\text{A.10})$$

and K_L has the following form:

$$K_L \sim \frac{1}{2 \text{Im } \tau} \mathbb{1} + \frac{2}{3} \text{Im } \tau \begin{pmatrix} \frac{3}{2}\alpha_1 & \alpha_4 + i\alpha_7 & \alpha_4 - i\alpha_7 \\ \alpha_4 - i\alpha_7 & \frac{3}{2}\alpha_1 & \alpha_4 + i\alpha_7 \\ \alpha_4 + i\alpha_7 & \alpha_4 - i\alpha_7 & \frac{3}{2}\alpha_1 \end{pmatrix}, \quad (\text{A.11})$$

where we have used $k_L = 1$ and $k_Y = 2$. To satisfy our assumption of the asymptotic behaviour of K_L , the coefficients $\alpha_1 = \alpha_4 = \alpha_7 = 0$. Thus, the number of free parameters in K_L is reduced from seven to four. Then, the elements of K_L from eq. (A.6) read:

$$(K_L)_{11} = \frac{1}{2 \text{Im } \tau} + 2 \text{Im } \tau \left[2\alpha_2 |Y_1|^2 - (\alpha_2 + \sqrt{3}\alpha_3) |Y_2|^2 - (\alpha_2 - \sqrt{3}\alpha_3) |Y_3|^2 \right], \quad (\text{A.12})$$

$$(K_L)_{22} = \frac{1}{2 \text{Im } \tau} + 2 \text{Im } \tau \left[-(\alpha_2 - \sqrt{3}\alpha_3) |Y_1|^2 + 2\alpha_2 |Y_2|^2 - (\alpha_2 + \sqrt{3}\alpha_3) |Y_3|^2 \right], \quad (\text{A.13})$$

$$(K_L)_{33} = \frac{1}{2 \text{Im } \tau} + 2 \text{Im } \tau \left[-(\alpha_2 + \sqrt{3}\alpha_3) |Y_1|^2 - (\alpha_2 - \sqrt{3}\alpha_3) |Y_2|^2 + 2\alpha_2 |Y_3|^2 \right], \quad (\text{A.14})$$

$$(K_L)_{12} = 2 \text{Im } \tau (\alpha_5 + i\alpha_6) [Y_1^* Y_2 - Y_1 Y_2^*], \quad (\text{A.15})$$

$$(K_L)_{13} = 2 \text{Im } \tau (\alpha_5 - i\alpha_6) [Y_1^* Y_3 - Y_1 Y_3^*], \quad (\text{A.16})$$

$$(K_L)_{23} = 2 \text{Im } \tau (\alpha_5 + i\alpha_6) [Y_2^* Y_3 - Y_2 Y_3^*]. \quad (\text{A.17})$$

For the sake of simplicity, we set further $\alpha_2 = \alpha_3 = 0$. In this case, the diagonal entries of K_L are not affected by the contributions containing modular forms, on the contrary to the off-diagonal elements. Thereby, we arrive at the form of K_L in eqs. (4.22) and (4.23).

What concerns K_{E^c} , with the assignment of representations and weights given in table 2, the most general Kähler potential quadratic in $Y^{(2)}$ reads

$$K_{E^c} = \frac{1}{8 (\text{Im } \tau)^3} \begin{pmatrix} c_{11}^0 & c_{12}^0 & 0 \\ c_{12}^{0*} & c_{22}^0 & 0 \\ 0 & 0 & c_{33}^0 \end{pmatrix} + \frac{1}{2 \text{Im } \tau} \begin{pmatrix} c_{11} Y^{(2)\dagger} Y^{(2)} & c_{12} Y^{(2)\dagger} Y^{(2)} & c_{13} Y^{(2)\dagger} M_{1''} Y^{(2)} \\ c_{12}^* Y^{(2)\dagger} Y^{(2)} & c_{22} Y^{(2)\dagger} Y^{(2)} & c_{23} Y^{(2)\dagger} M_{1''} Y^{(2)} \\ c_{13}^* Y^{(2)\dagger} M_{1'} Y^{(2)} & c_{23}^* Y^{(2)\dagger} M_{1'} Y^{(2)} & c_{33} Y^{(2)\dagger} Y^{(2)} \end{pmatrix}, \quad (\text{A.18})$$

with $c_{ii}^{(0)}$ being real and $c_{ij}^{(0)}$, $i \neq j$, complex coefficients.

Taking into account that at $\text{Im } \tau \rightarrow \infty$, the invariant combination $Y^{(2)\dagger} Y^{(2)} \sim 1$, whereas $X \equiv Y^{(2)\dagger} M_{1''} Y^{(2)}$ and $X^* = Y^{(2)\dagger} M_{1'} Y^{(2)}$ decay exponentially, we find

$$K_{E^c} \sim \frac{1}{8 (\text{Im } \tau)^3} \begin{pmatrix} c_{11}^0 & c_{12}^0 & 0 \\ c_{12}^{0*} & c_{22}^0 & 0 \\ 0 & 0 & c_{33}^0 \end{pmatrix} + \frac{1}{2 \text{Im } \tau} \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{12}^* & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix}. \quad (\text{A.19})$$

In order to restore the canonical form of K_{E^c} in the considered limit, $c_{12}^0 = c_{12} = 0$. Furthermore, we set $c_{ii} = 0$ for simplicity. Finally, we can always make $c_{ii}^0 = 1$ by independent rescalings of E_i^c , $i = 1, 2, 3$. Thus, we recover K_{E^c} given by eqs. (4.24) and (4.25).

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