RECEIVED: March 5, 2021

ACCEPTED: May 12, 2021

PUBLISHED: May 25, 2021

Soft matters, or the recursions with massive spinors

Adam Falkowski^a and Camila S. Machado^b

^a*Université Paris-Saclay, CNRS/IN2P3, IJCLab,
91405 Orsay, France*

^b*PRISMA⁺ Cluster of Excellence Mainz Institute for Theoretical Physics,
Johannes Gutenberg-Universität Mainz,
55099 Mainz, Germany*

E-mail: adam.falkowski@cern.ch, camila.samachado@gmail.com

ABSTRACT: We discuss recursion relations for scattering amplitudes with massive particles of any spin. They are derived via a two-parameter shift of momenta, combining a BCFW-type spinor shift with the soft limit of a massless particle involved in the process. The technical innovation is that spinors corresponding to *massive* momenta are also shifted. Our recursions lead to a reformulation of the soft theorems. The well-known Weinberg's soft factors are recovered and, in addition, the subleading factors appear reshaped such that they are directly applicable to massive amplitudes in the modern on-shell language. Moreover, we obtain new results in the context of non-minimal interactions of massive matter with photons and gravitons. These soft theorems are employed for practical calculations of Compton and higher-point scattering. As a by-product, we introduce a convenient representation of the Compton scattering amplitude for any mass and spin.

KEYWORDS: Scattering Amplitudes, Effective Field Theories

ARXIV EPRINT: [2005.08981](https://arxiv.org/abs/2005.08981)

Contents

1	Introduction	1
2	Spinor conventions	4
3	Soft recursions	5
3.1	Momentum shift	6
3.2	Recursion	6
3.3	Spinor shift	7
4	Soft theorems	9
4.1	Minimal gravity	9
4.2	Non-minimal gravity	12
4.3	Photons	13
5	Applications	15
5.1	Compton scattering	15
5.2	Gravitational Compton scattering	18
5.3	Higher-point	19
6	Exponential representation	20
6.1	Photon exponentiation	21
6.2	Soft exponentiation	22
7	Conclusions	23
A	Summary of spinor shifts	24
B	UV terms in Compton scattering	26
B.1	Electromagnetic Compton scattering	26
B.2	Gravitational Compton scattering	28

1 Introduction

Soft theorems describe general properties of scattering amplitudes in the infrared (IR) regime where the (four-)momentum of one massless particle approaches zero, $p^\mu \rightarrow \epsilon p^\mu$ with $\epsilon \rightarrow 0$. In this limit it is possible to write down a recursion relation connecting

the amplitude to a lower-point one with the soft particle removed. For example, for soft photons with helicity $h = \pm 1$ and soft gravitons with $h = \pm 2$ the recursions take the form

$$\mathcal{M}_{n+1}^{\pm 1} = \left(\frac{S_{\pm 1}^{(0)}}{\epsilon^2} + \frac{S_{\pm 1}^{(1)}}{\epsilon} \right) \mathcal{M}_n, \quad \mathcal{M}_{n+1}^{\pm 2} = \left(\frac{S_{\pm 2}^{(0)}}{\epsilon^3} + \frac{S_{\pm 2}^{(1)}}{\epsilon^2} + \frac{S_{\pm 2}^{(2)}}{\epsilon} \right) \mathcal{M}_n. \quad (1.1)$$

$S_{\pm h}^{(i)}$ are called the soft factors. They are *universal*, in the sense that they do not depend on what other “hard” particles are involved in the scattering.

Since the seminal works of Low [1] and Weinberg [2], which explored the leading term in the soft momentum expansion, the importance of the soft theorems has been revealed through applications that span very different fields. In particular, soft theorems turn out to be intricately related to the consistency conditions of the S-matrix such as gauge invariance, locality and unitarity [3, 4]. Furthermore, it was shown that they can be understood as Ward identities of asymptotic symmetries [5]. They have also been relevant to understanding the landscape of effective field theories (EFTs) (see e.g. [6–10]).

A new life into the study of soft theorems was brought by the emergence of on-shell methods for calculating scattering processes. These are founded on the spinor techniques, which make manifest the little group transformation properties of the amplitudes. An important step in this development was ref. [11] by Cachazo and Strominger. Using the BCFW recursion relations [12] they were able to extend the graviton soft theorems up to sub-subleading order at tree level.¹ Subsequently, the work by Elvang et al. [16] extended on-shell soft theorems to a broad class of theories with massless particles. This is achieved thanks to a modified recursion relation that combines double BCFW and soft shifts. This approach leads to an elegant proof that various S-matrix consistency conditions, such as charge conservation and the equivalence principle, follow directly from Poincaré invariance, locality, and unitarity, without ever mentioning gauge invariance. Moreover, ref. [16] systematized the classes of effective operators that modify the soft theorems at the subleading order for photons and the sub-subleading order for gravitons.

Despite these rapid developments, all of the results derived with on-shell techniques are valid only for massless theories.² Meanwhile, a convenient spinor formalism for dealing with *massive* particles was introduced in [22]. There are several motivations to rewrite the soft theorems such that they can be directly applied to theories using the on-shell language and massive spinors. Perhaps the most pressing is the growing importance of the on-shell program in the context of black hole computations [23–26]. Another is the progress in on-shell formulation of massive EFTs, such as e.g. massive gravity [27–30], or the Standard Model EFT [31–33]. More generally, evolution of on-shell tools to deal with massive theories may reveal unexpected simplicity of their amplitudes, much as it happened for massless theories.

In this paper, we establish soft recursion relations that are valid for general theories with massive particles of any spin, extending the results of [16]. This is achieved by introducing a new kind of holomorphic spinor shift, where also *massive* spinors are shifted in

¹In this paper we restrict the discussion to tree level. Although the leading (Weinberg’s) soft factor is not modified at loop level, the story for the sub-leading orders is more subtle [13–15].

²For soft theorems with massive particles using the effective action formalism, see e.g. [17–21].

addition to massless ones. This allows us to derive a general recursion relation that remains valid even when the soft particle is the only massless particle in the scattering process. Our formula can be applied to describe emission of a soft particle of any helicity $|h| \leq 2$. In this paper we work out the resulting soft theorems for photons and gravitons. These have the same general structure as in the standard derivation, however they can be directly applied to amplitudes written in term of massive and massless spinors. In particular, the subleading soft factors become differential operators acting on the massive spinors and on their little group indices. We explicitly connect the subleading soft theorems to multipole expansion of the 3-point interaction between massive matter and photons/gravitons. As a corollary, we obtain a compact and elegant proof that gravitational dipole interactions of fundamental particles of any spin are inconsistent with the assumption of Poincaré invariance, locality, and unitarity.

We also discuss a couple of concrete applications of the soft recursion relations in massive theories. One is for the simple 4-body process of Compton scattering of photons and gravitons. For any spin of the matter particle, the recursion correctly captures the IR part of the amplitude, that is the one containing the physical poles in the Mandelstam variables. The result agrees with the one in ref. [22] obtained by requiring that the residues of the physical poles factorize into 3-point amplitudes. As discussed in [22, 34], for higher spins the IR part inevitably develops a spurious pole that is not associated with an exchange of a physical particle. Therefore, to make it consistent with unitarity, the Compton amplitude has to be augmented by a UV term that cancels the spurious pole without affecting the physical ones. In our derivation, this UV term arises as a boundary term in the recursion relation. As an aside, we present a systematic algorithm to build the UV term in Compton scattering, alternative to the one in [34]. We also make some baby steps toward higher-point amplitudes of any spin, by calculating specific 5-point amplitudes with 3 photons or 3 gravitons.

Finally, we touch on the subject of exponential representation of soft amplitudes. Exponentiated soft factors play an important role in the matching between quantum and classical limits. In particular, they are instrumental in certain calculations of black hole observables [24, 35]. They can also be used to write scattering amplitudes in the Mellin space, which can be connected with CFT correlators in the celestial sphere (see e.g. [36–38]). Our recursion relations lead to a simple proof of the exponentiation of Compton amplitudes [24] for any mass and spin. We also remark that the photon and graviton soft factors can be written in terms of an exponential operator in the scattering of massive particles of generic spin, which previously was only done for the minimal-helicity violating (MHV) sector of massless gravity/Yang-Mills [14].

Our paper is organized as follows. The (massless and massive) spinor formalism is briefly reviewed in section 2. The new soft recursions, valid for particles of any mass and spin, are derived in section 3, from which the soft theorems in section 4 directly follow. Some applications for calculating concrete amplitudes are presented in section 5. Section 6 discusses the exponential representation of soft amplitudes and Compton scattering. Section 7 is reserved for conclusions and future directions. Appendix A lists the possible spinor shifts realizing our momentum shift, some of which are not employed in this paper but may be useful for other applications. Appendix B describes in detail our construction of UV/boundary terms in Compton scattering.

2 Spinor conventions

We begin with a brief overview of our conventions (readers well-versed in the massless and massive spinor formalism are invited to skip this section). We work in four dimensions with the mostly-minus metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The Lorentz algebra can be decomposed into $\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$. Holomorphic and antiholomorphic spinors ψ_α and $\psi_{\dot{\alpha}}$ transform under the respective $\text{SL}(2, \mathbb{C})$ factors with indices being raised and lowered by the antisymmetric epsilon tensor:

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta, \quad (2.1)$$

and idem for dotted indices. Vector and spinor Lorentz indices can be traded with the help of the sigma matrices $(\sigma^\mu)_{\alpha\dot{\beta}} = (\mathbb{I}, \vec{\sigma})$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = (\mathbb{I}, -\vec{\sigma})$, where $\vec{\sigma}$ are the Pauli matrices. Given the momentum³ p_μ , we can construct the 2×2 matrices

$$p_\mu (\sigma^\mu)_{\alpha\dot{\alpha}} \equiv (p\sigma)_{\alpha\dot{\alpha}}, \quad p_\mu (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \equiv (p\bar{\sigma})^{\dot{\alpha}\alpha}. \quad (2.2)$$

These naturally act on the spinor indices: $(p\bar{\sigma})\psi \equiv (p\bar{\sigma})^{\dot{\alpha}\beta} \psi_\beta$, $(p\sigma)\tilde{\psi} \equiv (p\sigma)_{\alpha\dot{\beta}} \tilde{\psi}^{\dot{\beta}}$, $\tilde{\psi}(p\bar{\sigma}) \equiv \tilde{\psi}_{\dot{\beta}}(p\bar{\sigma})^{\dot{\beta}\alpha}$, $\psi(p\sigma) \equiv \psi^\beta(p\sigma)_{\beta\dot{\alpha}}$, where repeating spinor indices are implicitly summed over. We will often omit the spinor indices, as with these rules the contractions are always unambiguous.

Massless momentum can be represented by a pair of (commuting) spinors $\lambda_\alpha, \tilde{\lambda}_{\dot{\beta}}$:

$$(p\sigma)_{\alpha\dot{\beta}} = \lambda_\alpha \tilde{\lambda}_{\dot{\beta}}, \quad (p\bar{\sigma})^{\dot{\alpha}\beta} = \tilde{\lambda}^{\dot{\alpha}} \lambda^\beta, \quad (2.3)$$

such that $p^2 = 0$ is automatic for any $\lambda, \tilde{\lambda}$ thanks to the identity $\lambda^\alpha \lambda_\alpha = 0 = \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}}$. It follows that the massless spinors satisfy the Weyl equations: $\lambda(p\sigma) = (p\sigma)\tilde{\lambda} = 0$, $(p\bar{\sigma})\lambda = \tilde{\lambda}(p\bar{\sigma}) = 0$. For real p_μ we have the additional constraint $\tilde{\lambda} = \bar{\lambda}$, but in general momenta are allowed to be complex, in which case $\tilde{\lambda}$ and λ are independent. The *little group* transformations (i.e. for a fixed p_μ , the subset of Lorentz transformations leaving p_μ invariant) correspond to $U(1)$ acting as $\lambda \rightarrow t^{-1}\lambda$, $\tilde{\lambda} \rightarrow t\tilde{\lambda}$. The Lorentz-invariant and little group covariant building blocks are commonly represented by the bra-ket notation:

$$\langle ij \rangle \equiv \lambda_i^\alpha \lambda_{j\alpha} = \epsilon^{\beta\alpha} \lambda_{i\alpha} \lambda_{j\beta} = (\lambda_i \lambda_j), \quad [ij] \equiv \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_j^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_{j\dot{\beta}} = (\tilde{\lambda}_i \tilde{\lambda}_j), \quad (2.4)$$

and the Lorentz contraction of momenta can be written as $2p_i p_j = \langle ij \rangle [ji]$.⁴ The bra-ket notation can be naturally extended to more complicated contractions, e.g. $(\lambda_i p_k \sigma \tilde{\lambda}_j) \equiv \langle i p_k j \rangle$, $(\lambda_i p_k \sigma p_l \bar{\sigma} \tilde{\lambda}_j) \equiv \langle i p_k p_l j \rangle$, etc. In the following, we choose to write uncontracted spinors explicitly as in eq. (2.1), and reserve the bra-ket notation for Lorentz contractions only; e.g. we write $\langle ij \rangle \lambda_k \equiv (\lambda_i \lambda_j) \lambda_k$. The massless spinors can be related to polarization

³In this paper “momentum” always means four-momentum.

⁴We always follow the conventions of ref. [39]. For the holomorphic contraction $\langle \cdot \rangle$ this differs by a sign from some of the on-shell literature.

tensors in the standard Lagrangian formalism. In particular, the polarization vectors of a massless spin-1 particle can be written in terms of the spinors as

$$\epsilon_{\mu}^{-} = \frac{(\lambda \sigma_{\mu} \tilde{\zeta})}{\sqrt{2}[\lambda \zeta]}, \quad \epsilon_{\mu}^{+} = \frac{(\zeta \sigma_{\mu} \tilde{\lambda})}{\sqrt{2}\langle \lambda \zeta \rangle}, \quad (2.5)$$

where $\tilde{\zeta}$ and ζ are arbitrary reference spinors representing the gauge freedom. Similarly, for a massless spin-2 particle, the polarization tensors are $\epsilon_{\mu\nu}^{\pm} = \epsilon_{\mu}^{\pm} \epsilon_{\nu}^{\pm}$, where now the freedom of choosing the reference spinors represents general coordinate invariance.

Massive momentum satisfying the on-shell condition $p^2 = m^2$ can be represented by four spinors $\chi_{\alpha}^1, \chi_{\alpha}^2, \tilde{\chi}_{\dot{\beta}1}, \tilde{\chi}_{\dot{\beta}2}$. They can be collected into χ^J and $\tilde{\chi}_J$, where $J = 1, 2$ is identified with the SU(2) little group index which, in complete analogy to spinor indices, can be raised and lowered by epsilon tensors. The spinors are related to the momentum by the formula

$$(p\sigma)_{\alpha\dot{\beta}} = \chi_{\alpha}^J \tilde{\chi}_{\dot{\beta}J}, \quad (p\bar{\sigma})^{\alpha\dot{\beta}} = \tilde{\chi}^{\dot{\alpha}J} \chi^{\beta J}, \quad (2.6)$$

where summation over the little group indices is implicit. This is a natural and convenient generalization of the spinor helicity formalism to massive particles [22] (for earlier works, see e.g. [40, 41]). The massive spinors are normalized as

$$(\chi^J \chi_K) = \delta_K^J m, \quad (\tilde{\chi}_J \tilde{\chi}^K) = \delta_J^K m, \quad (2.7)$$

from which follow the Dirac equations

$$(p\sigma)\tilde{\chi}^J = m\chi^J, \quad (p\bar{\sigma})\chi^J = m\tilde{\chi}^J, \quad \chi^J(p\sigma) = -m\tilde{\chi}^J, \quad \tilde{\chi}^J(p\bar{\sigma}) = -m\chi^J. \quad (2.8)$$

These allow one to trade χ for $\tilde{\chi}$ and vice-versa. An amplitude describing scattering of n massive incoming particles with spin S_i , $i = 1 \dots n$, for each i will contain exactly $2S_i$ spinors $\chi_i^{J_k}$ or $\tilde{\chi}_i^{J_k}$ with *uncontracted* little group indices. Massive *outgoing* particles can be represented by spinors χ_{iJ_k} or $\tilde{\chi}_{iJ_k}$ with lowered little group indices, however in this paper we treat all external particles as incoming. The little group indices corresponding to the same particle are always implicitly symmetrized over. To reduce clutter, in the following we will often omit the little group indices whenever it does not lead to ambiguities. Lorentz contractions of massive spinors will be shortened via the bra-ket notation:

$$\langle \mathbf{i} \mathbf{j} \rangle \equiv \chi_i^{\alpha} \chi_{j\alpha} = (\chi_i \chi_j), \quad [\mathbf{i} \mathbf{j}] \equiv \tilde{\chi}_{i\dot{\alpha}} \tilde{\chi}_j^{\dot{\alpha}} = (\tilde{\chi}_i \tilde{\chi}_j), \quad (2.9)$$

where we adhere to the **bold** notation introduced in ref. [22].

3 Soft recursions

In this section we derive a formula connecting an $n + 1$ -point amplitude with at least one massless particle to n -point amplitudes. The formula picks up the leading terms in the limit where the momentum of a massless particle is continuously taken to zero. That massless particle is called *soft*, and the formula is referred to as the *soft recursion*. The derivation follows similar steps as the one in ref. [16], however it is adapted to allow for massive particles in the scattering process.

3.1 Momentum shift

Consider a set of $n + 1$ particles with momenta $p_0 \dots p_n$ satisfying $\sum_{i=0}^n p_i = 0$ and the on-shell conditions $p_i^2 = m_i^2$. The particle labeled as 0 is always massless, $p_0^2 = 0$, and it will be our soft particle. We perform a two-parameter deformation of this kinematics. One parameter z controls complex deformation of the momenta of the particle 0 and of two other particles, below labeled as j and k . The other parameter ϵ controls the soft limit of the particle 0. Specifically, the shift is defined as⁵

$$\hat{p}_0 = \epsilon p_0 - z q_0, \quad \hat{p}_j = p_j - \frac{(\epsilon - 1)(p_0 p_k) - z(q_0 p_k)}{q_j p_k} q_j, \quad \hat{p}_k = p_k - \frac{(\epsilon - 1)(p_0 p_j) - z(q_0 p_j)}{q_k p_j} q_k, \quad (3.1)$$

where q_0, q_j, q_k are four-vectors satisfying

$$q_0^2 = q_j^2 = q_k^2 = q_0 p_0 = q_j p_j = q_k p_k = 0, \quad q_j p_0 = q_k p_0 = q_j q_0 = q_k q_0 = q_j q_k = 0. \quad (3.2)$$

The first set of equations in eq. (3.2) ensures that the shift does not change the particles' masses, $\hat{p}_i^2 = m_i^2$, while the role of the second is to ensure shifted momentum conservation, $\sum_{i=0}^n \hat{p}_i = 0$, for any z and ϵ . One advantage of the two-parameter shift in eq. (3.1) compared to simpler BCFW-like one-parameter shifts is that it will directly lead to soft expansion of recursion relations in powers of $1/\epsilon$. As a bonus, it will allow for more control over the recursions' boundary terms, as the latter will start at higher orders in $1/\epsilon$.

3.2 Recursion

Using the momentum shift in eq. (3.1) one can relate, in a quite model-independent way, $n + 1$ -point amplitudes in the soft limit to n -point amplitudes. Consider an $n + 1$ point amplitude $\mathcal{M}_{n+1} \equiv \mathcal{M}(1 \dots n 0)$. We denote $\hat{\mathcal{M}}_{n+1}^{z, \epsilon}$ its shifted version, which is $\mathcal{M}_{n+1}$ evaluated for the “hatted” kinematics in eq. (3.1). Similarly, we denote $\mathcal{M}_n \equiv \mathcal{M}(1 \dots n)$, and its shifted version by $\hat{\mathcal{M}}_n^{z, \epsilon}$. Note that $\hat{\mathcal{M}}^{0, 1} = \mathcal{M}$. At tree level $\hat{\mathcal{M}}^{z, \epsilon}$ is a meromorphic function of z , with all singularities given by a finite set of simple poles at $z = z_i$. Applying the Cauchy formula,

$$\hat{\mathcal{M}}_{n+1}^{0, \epsilon} = - \sum_i \frac{\text{Res}_{z \rightarrow z_i} \hat{\mathcal{M}}_{n+1}^{z, \epsilon}}{z_i} + B_\infty, \quad (3.3)$$

where the boundary term is $B_\infty = \frac{1}{2\pi i} \int_{C_\infty} \frac{\hat{\mathcal{M}}_{n+1}^{z, \epsilon}}{z}$, and C_∞ is the circle at infinity. If $\hat{\mathcal{M}}_{n+1}^{z, \epsilon}$ goes to zero as $z \rightarrow \infty$ then $B_\infty = 0$. The recursion relation is most powerful when $B_\infty = 0$, but this will not always be true in the cases of interest.

We only focus on a subset of the poles related to the emission of the particle zero from an external leg the n -point amplitude $\mathcal{M}_n$. In the limit $\epsilon \rightarrow 0$, singular $1/\epsilon^n$ terms in $\hat{\mathcal{M}}_{n+1}^{z, \epsilon}$ can only come from these residues. There is n such poles z_l related to the solutions

⁵The original reference [16] defines the shift in a slightly different way: $\hat{p}_j = p_j - [\epsilon \alpha_j - z \beta_j] q_j$, $\hat{p}_k = p_k + [\epsilon \alpha_k - z \beta_k] q_k$, such that $\hat{p}_{j,k} = p_{j,k}$ for $\epsilon = z = 0$. In this version the unshifted kinematics is constrained as $\sum_{i=1}^n p_i = 0$ (as opposed to $\sum_{i=0}^n p_i = 0$ in our case). For this reason we find it more convenient to work with the version in eq. (3.1).

of the equations $\hat{P}_{0l}^2|_{z=z_l} = m_l^2$, where $\hat{P}_{0l} \equiv \hat{p}_0 + \hat{p}_l$ and it is understood that $\hat{p}_l = p_l$ for $l \neq j, k$. The poles are located at

$$z_l = \epsilon \frac{p_0 p_l}{q_0 p_l}, \quad l = 1 \dots n, \quad (3.4)$$

and one can rewrite $\hat{P}_{0l}^2 - m_l^2 = -2q_0 p_l(z - z_l)$. This shows that the poles at $\hat{P}_{0l}^2 \rightarrow m_l^2$ are in one-to-one correspondence with the poles at $z \rightarrow z_l$. Unitarity requires that at every pole the amplitude factorizes:

$$\begin{aligned} \hat{\mathcal{M}}^{z, \epsilon}(1 \dots n 0)|_{\hat{P}_{0l}^2 \rightarrow m_l^2} &= -\frac{\hat{\mathcal{M}}^{z_l, \epsilon}(1 \dots P_{0l} \dots n) \hat{\mathcal{M}}^{z_l, \epsilon}((-P_{0l})l 0)}{\hat{P}_{0l}^2 - m_l^2} \\ &= \frac{\hat{\mathcal{M}}^{z_l, \epsilon}(1 \dots P_{0l} \dots n) \hat{\mathcal{M}}^{z_l, \epsilon}((-P_{0l})l 0)}{2q_0 p_l(z - z_l)}. \end{aligned} \quad (3.5)$$

Thus

$$\frac{\text{Res}_{z \rightarrow z_l} \hat{\mathcal{M}}^{z, \epsilon}(0 \dots n)}{z_l} = \frac{1}{\epsilon} \frac{\hat{\mathcal{M}}^{z_l, \epsilon}(1 \dots P_{0l} \dots n) \hat{\mathcal{M}}^{z_l, \epsilon}((-P_{0l})l 0)}{2p_0 p_l}. \quad (3.6)$$

Plugging this back into the Cauchy formula in eq. (3.3) one obtains the soft recursion:

$$\hat{\mathcal{M}}^{0, \epsilon}(1 \dots n 0) = -\frac{1}{\epsilon} \sum_{l=1}^n \frac{\hat{\mathcal{M}}^{z_l, \epsilon}(1 \dots P_{0l} \dots n) \hat{\mathcal{M}}^{z_l, \epsilon}((-P_{0l})l 0)}{2p_0 p_l} + \mathcal{O}(\epsilon^0). \quad (3.7)$$

This formula will be the starting point for deriving a new on-shell incarnation of soft theorems which is valid in the presence of massive particles. We will often refer to the n -point amplitude above as the *hard factor*. The boundary term in eq. (3.3) is absorbed in $\mathcal{O}(\epsilon^0)$ above, as it cannot produce singular terms in the $\epsilon \rightarrow 0$ limit.

3.3 Spinor shift

We turn to discussing spinor shifts that realize the momentum shift in eq. (3.1). The discussion depends on whether the shifted particles j and k are massive or massless. In the following we first work out the case when $m_j > 0$ and $m_k > 0$, and later comment on other possible configurations. Note that m_j and m_k can be different. Recall also that particle 0 is always massless in this discussion, however other particles with $l \neq 0, j, k$ can be massive or massless, as one pleases.

One possible choice of the shift vectors q in eq. (3.1) is

$$q_0 \sigma = y \tilde{\lambda}_0, \quad q_j \sigma = p_j \sigma \tilde{\lambda}_0 \tilde{\lambda}_0, \quad q_k \sigma = p_k \sigma \tilde{\lambda}_0 \tilde{\lambda}_0, \quad (3.8)$$

where y is an arbitrary spinor satisfying $\langle y 0 \rangle \neq 0$. One can verify that these satisfy the conditions in eq. (3.2). We can decompose the shifted momenta into spinors: $\hat{p}_0 \sigma = \lambda_0^z \tilde{\lambda}_0$, $\hat{p}_j \sigma = \chi_j^J \tilde{\chi}_j^{\tilde{z}} J$, $\hat{p}_k \sigma = \chi_k^J \tilde{\chi}_k^{\tilde{z}} J$. At the spinor level, the shift is realized as⁶

$$\begin{aligned} \{0 \bar{m} \bar{m}\} : \quad \lambda_0^z &= \epsilon \lambda_0 - z y, \\ \tilde{\chi}_j^z &= \tilde{\chi}_j + \frac{(\epsilon - 1) \langle 0 p_k 0 \rangle - z \langle y p_k 0 \rangle}{[0 p_j p_k 0]} [j 0] \tilde{\lambda}_0, \\ \tilde{\chi}_k^z &= \tilde{\chi}_k - \frac{(\epsilon - 1) \langle 0 p_j 0 \rangle - z \langle y p_j 0 \rangle}{[0 p_j p_k 0]} [k 0] \tilde{\lambda}_0, \end{aligned} \quad (3.9)$$

⁶In a different context, single parameter shifts of massive spinors were previously considered in refs. [32, 42].

while all other spinors remain unshifted. Since we shift the holomorphic spinor corresponding to the particle 0 and the antiholomorphic spinors corresponding to the particles j, k , we label this shift as $\{0\bar{m}\bar{m}\}$. The parity-reversed version, labelled as $\{\bar{0}mm\}$, is defined in appendix A. Note that our shift is non-trivial for $z = \epsilon = 0$; in this limit the spinors $\tilde{\chi}_{j,k}$ must shift to absorb the original momentum of the particle 0. This “zero-order” shift will play some role in the following. On the other hand, the shift is trivial for $z = 0$ and $\epsilon = 1$, but that is away from the soft limit $\epsilon \rightarrow 0$ on which we are focused in this paper.

We end this section with a number of identities that will be useful in the following (impatient readers are encouraged to skip to the next section). For the $\{0\bar{m}\bar{m}\}$ shift, the location of the poles in eq. (3.4) can be recast as

$$z_l = \epsilon \frac{\langle 0p_l 0 \rangle}{\langle yp_l 0 \rangle}; \quad m_l = 0 \Rightarrow z_l = \epsilon \frac{\langle 0l \rangle}{\langle yl \rangle}. \quad (3.10)$$

At the poles, the shifted spinors can be expressed as

$$\begin{aligned} \lambda_0^{z_l} &= \epsilon \frac{\langle y0 \rangle}{\langle yp_l 0 \rangle} p_l \sigma \tilde{\lambda}_0, \\ \tilde{\chi}_j^{z_l} &= \tilde{\chi}_j - \frac{2p_0 p_k}{[0p_j p_k 0]} [\mathbf{j}0] \tilde{\lambda}_0 + \epsilon \frac{\langle y0 \rangle [0p_k p_l 0]}{\langle yp_l 0 \rangle [0p_j p_k 0]} [\mathbf{j}0] \tilde{\lambda}_0, \\ \tilde{\chi}_k^{z_l} &= \tilde{\chi}_k + \frac{2p_0 p_j}{[0p_j p_k 0]} [\mathbf{k}0] \tilde{\lambda}_0 - \epsilon \frac{\langle y0 \rangle [0p_j p_l 0]}{\langle yp_l 0 \rangle [0p_j p_k 0]} [\mathbf{k}0] \tilde{\lambda}_0. \end{aligned} \quad (3.11)$$

These identities allow us to rewrite $\hat{P}_{0l}$ at the poles. For $l \neq j, k$, $\hat{P}_{0l} = \hat{p}_0 + p_l$, thus

$$\hat{P}_{0l} \sigma|_{z_l} = p_l \sigma + \epsilon \frac{\langle y0 \rangle}{\langle yp_l 0 \rangle} p_l \sigma \tilde{\lambda}_0 \tilde{\lambda}_0, \quad l \neq j, k. \quad (3.12)$$

Since $\hat{P}_{0l}|_{z=z_l}$ is on shell by definition, we can factorize it into spinors:

$$\begin{aligned} m_l > 0 &\Rightarrow \hat{P}_{0l} \sigma|_{z_l} = \chi_{0l}^L \tilde{\chi}_{0l} L, \quad \chi_{0l} = \chi_l, \quad \tilde{\chi}_{0l} = \tilde{\chi}_l + \epsilon \frac{\langle y0 \rangle}{\langle yp_l 0 \rangle} [\mathbf{l}0] \tilde{\lambda}_0, \quad l \neq j, k, \\ m_l = 0 &\Rightarrow \hat{P}_{0l} \sigma|_{z_l} = \lambda_{0l} \tilde{\lambda}_{0l}, \quad \lambda_{0l} = \lambda_l, \quad \tilde{\lambda}_{0l} = \tilde{\lambda}_l + \epsilon \frac{\langle y0 \rangle}{\langle yl \rangle} \tilde{\lambda}_0, \quad l \neq j, k. \end{aligned} \quad (3.13)$$

Similarly at z_j and z_k we have the decomposition

$$\begin{aligned} \hat{P}_{0j} \sigma|_{z_j} &= p_j \sigma - \frac{2p_0 p_k}{[0p_j p_k 0]} p_j \sigma \tilde{\lambda}_0 \tilde{\lambda}_0 \Rightarrow \chi_{0j} = \chi_j, \quad \tilde{\chi}_{0j} = \tilde{\chi}_j - \frac{2p_0 p_k}{[0p_j p_k 0]} [\mathbf{j}0] \tilde{\lambda}_0, \\ \hat{P}_{0k} \sigma|_{z_k} &= p_k \sigma + \frac{2p_0 p_j}{[0p_j p_k 0]} p_k \sigma \tilde{\lambda}_0 \tilde{\lambda}_0, \Rightarrow \chi_{0k} = \chi_k, \quad \tilde{\chi}_{0k} = \tilde{\chi}_k + \frac{2p_0 p_j}{[0p_j p_k 0]} [\mathbf{k}0] \tilde{\lambda}_0. \end{aligned} \quad (3.14)$$

In our story it will be important that these are independent of the soft parameter ϵ . Similarly, the shifts $\tilde{\chi}_j^{z_k}$ and $\tilde{\chi}_k^{z_j}$ are independent of ϵ .

The case where one of the j, k particles is massive and the other is massless is covered in appendix A. We also comment on the limit where all involved particles are massless. This can be easily obtained from eq. (3.9) by “unbolding” the massive spinors, which

corresponds to replacing $\chi_i^J \rightarrow \lambda_i$, $\tilde{\chi}_i^J \rightarrow \tilde{\lambda}_i$ while dropping the SU(2) little group index. Simplifying the resulting expression one obtains

$$\begin{aligned} \{0\bar{0}\bar{0}\} : \quad & \lambda_0^z = \epsilon \lambda_0 - zy, \\ & \tilde{\lambda}_j^z = \tilde{\lambda}_j - \frac{(\epsilon - 1)\langle 0k \rangle - z\langle yk \rangle}{\langle jk \rangle} \tilde{\lambda}_0, \\ & \tilde{\lambda}_k^z = \tilde{\lambda}_k + \frac{(\epsilon - 1)\langle 0j \rangle - z\langle yj \rangle}{\langle jk \rangle} \tilde{\lambda}_0, \end{aligned} \quad (3.15)$$

This reproduces the shift in ref. [16] up to the replacement $\epsilon \rightarrow \epsilon - 1$ which is due to the different prescription for taking the soft limit.

4 Soft theorems

In this section we apply the recursion formula in eq. (3.7) to derive the soft factors for amplitudes containing particles of any mass and spin. This generalizes the discussion in ref. [16]. To our knowledge, this is the first derivation using the on-shell methods with massive spinors. The soft factors that we obtain are applicable to more general physical situations than those found in the previous literature. In the following we will deal with the soft particle of spin 2 (graviton) and spin 1 (photon), however eq. (3.7) goes just as well with massless particles of spin 0 (Goldstone bosons), spin 1/2 (Goldstone fermions), or spin 3/2 (gravitinos). We restrict to single soft particle emission, leaving multiple soft limits for future publications.

4.1 Minimal gravity

We begin with a gravity theory where the massless spin-2 graviton is minimally coupled to matter and to itself. The soft theorems in this case have been studied back and forth, both in the standard Lagrangian and in the on-shell formalisms. Nevertheless, here we provide the soft theorems in a novel form that will be particularly convenient for practical calculations with massive spinors.

In the on-shell formalism, the minimal coupling of the graviton h to a matter particle X of mass $m > 0$ and any spin S corresponds to the on-shell 3-point amplitudes [22]

$$\mathcal{M}(1_X 2_X 3_h^-) = -\frac{1}{M_{\text{Pl}}} \frac{\langle 3p_1 \zeta \rangle^2 [\mathbf{21}]^{2S}}{[3\zeta]^2 m^{2S}}, \quad \mathcal{M}(1_X 2_X 3_h^+) = -\frac{1}{M_{\text{Pl}}} \frac{\langle \zeta p_1 3 \rangle^2 \langle \mathbf{21} \rangle^{2S}}{\langle 3\zeta \rangle^2 m^{2S}}, \quad (4.1)$$

where $M_{\text{Pl}} = (8\pi G)^{-1/2} \approx 2.4 \times 10^{18} \text{ GeV}$ is the Planck constant, and $\zeta, \tilde{\zeta}$ are arbitrary reference spinors satisfying $[3\zeta] \neq 0$ and $\langle 3\zeta \rangle \neq 0$ (the amplitude is independent of their precise choice). In the standard QFT language, the on-shell amplitude in eq. (4.1) corresponds to graviton coupling to matter via the energy-momentum tensor. Meanwhile, the minimal self-coupling of the graviton in the standard Einstein-Hilbert Lagrangian corresponds in the on-shell language to

$$\mathcal{M}(1_h^- 2_h^- 3_h^+) = -\frac{1}{M_{\text{Pl}}} \frac{\langle 12 \rangle^6}{\langle 13 \rangle^2 \langle 23 \rangle^2}, \quad \mathcal{M}(1_h^+ 2_h^+ 3_h^-) = -\frac{1}{M_{\text{Pl}}} \frac{[12]^6}{[13]^2 [23]^2}. \quad (4.2)$$

We focus on emission of a soft graviton with plus helicity. The starting point is the recursion in eq. (3.7), with the amplitudes deformed by the $\{0\bar{m}\bar{m}\}$ spinor shift in eq. (3.9). We first need an expression for the shifted 3-point amplitude at the z_l -poles:

$$\hat{\mathcal{M}}^{z_l, \epsilon}((-P_{0l})l0_h^+) = -\frac{1}{M_{\text{Pl}}} \frac{\langle yp_l 0 \rangle^2}{\epsilon^2 \langle 0y \rangle^2}. \quad (4.3)$$

This expression is independent of the identity, mass, or spin of the l -th particle (except that for massive particles the identity operator in the little group indices is implicit), which is a realization of the equivalence principle. Plugging this back into eq. (3.7) we obtain a raw version of the soft theorem:

$$\hat{\mathcal{M}}^{0, \epsilon}(1 \dots n 0_h^+) = \frac{1}{\epsilon^3 M_{\text{Pl}}} \sum_{l=1}^n \frac{\langle yp_l 0 \rangle^2}{(2p_0 p_l) \langle 0y \rangle^2} \hat{\mathcal{M}}^{z_l, \epsilon}(1 \dots P_{0l} \dots n) + \mathcal{O}(\epsilon^0). \quad (4.4)$$

To proceed we need to expand the n -point amplitude above in powers of ϵ . Using the identities in eq. (3.11) we find

$$\hat{\mathcal{M}}^{z_l, \epsilon}(1 \dots P_{0l} \dots n) = \left\{ 1 + \epsilon \frac{\langle 0y \rangle}{\langle yp_l 0 \rangle} \tilde{\mathcal{D}}_l + \frac{\epsilon^2}{2} \frac{\langle 0y \rangle^2}{\langle yp_l 0 \rangle^2} \tilde{\mathcal{D}}_l^2 \right\} \hat{\mathcal{M}}^{0,0}(1 \dots n) + \mathcal{O}(\epsilon^3), \quad (4.5)$$

where the differential operator $\tilde{\mathcal{D}}_l$ is

$$\tilde{\mathcal{D}}_l = [0l][0\partial_l] + \frac{[0p_k p_l 0]}{[0p_j p_k 0]} [0j][0\partial_j] - \frac{[0p_j p_l 0]}{[0p_j p_k 0]} [0k][0\partial_k], \quad l \neq j, k, \quad (4.6)$$

and $\tilde{\mathcal{D}}_l = 0$ for $l = j, k$. Here, $[0l][0\partial_l] \equiv (\tilde{\lambda}_0 \tilde{\chi}_l^L) \left(\tilde{\lambda}_0 \frac{\partial}{\partial \tilde{\chi}_l^L} \right)$ for a massive particle l , where for once we explicitly displayed the SU(2) little group index L , which is implicitly summed over. For a massless particle l one should instead use $[0l][0\partial_l] \equiv (\tilde{\lambda}_0 \tilde{\lambda}_l) \left(\tilde{\lambda}_0 \frac{\partial}{\partial \tilde{\lambda}_l} \right)$. Plugging the expanded $\hat{\mathcal{M}}^{z_l, \epsilon}$ back into eq. (4.4) we obtain the soft theorem in the standard form:

$$\hat{\mathcal{M}}^{0, \epsilon}(1 \dots n 0_h^+) = \left\{ \frac{1}{\epsilon^3} S_{+2}^{(0)} + \frac{1}{\epsilon^2} S_{+2}^{(1)} + \frac{1}{\epsilon} S_{+2}^{(0)} \right\} \hat{\mathcal{M}}^{0,0}(1 \dots n) + \mathcal{O}(\epsilon^0). \quad (4.7)$$

The leading soft factor is given by

$$S_{+2}^{(0)} = \frac{1}{M_{\text{Pl}}} \sum_{l=1}^n \frac{\langle yp_l 0 \rangle^2}{(2p_0 p_l) \langle 0y \rangle^2} = \frac{1}{M_{\text{Pl}}} \epsilon_{\mu\nu}^{0+} \sum_{l=1}^n \frac{p_l^\mu p_l^\nu}{p_0 p_l}. \quad (4.8)$$

In the second we step we rewrote the soft factor in the more familiar form as it appears in ref. [2], with the polarization tensor of the soft graviton defined below eq. (2.5) and the gauge parameter ζ is identified with the shift spinor y . This leading soft factor is independent of y as a consequence of the equivalence principle and momentum conservation [43]:

$$\frac{\partial}{\partial y} S_{+2}^{(0)} \sim \sum_{l=1}^n \frac{\langle yp_l 0 \rangle}{\langle y 0 \rangle^3} y = -\frac{\langle yp_0 0 \rangle}{\langle y 0 \rangle^3} y = 0. \quad (4.9)$$

Conversely, the condition that $S_{+2}^{(0)}$ must be independent of the shift spinor y can be employed to *prove* the equivalence principle without any reference to general coordinate invariance.⁷ Indeed, in our derivation, the soft theorems are a direct consequence of Poincaré invariance, locality, and unitarity. In a consistent theory they must lead to unambiguous soft factors for any choice of y . However, that would not be the case if the graviton coupling strength to matter in eq. (4.1) were not universal and precisely correlated with the graviton self-coupling in eq. (4.2).

The subleading soft factor in eq. (4.7) is a differential operator:

$$S_{+2}^{(1)} = \frac{1}{M_{\text{Pl}}} \sum_{l=1}^n \frac{\langle y p_l 0 \rangle}{(2p_0 p_l) \langle 0 y \rangle} \tilde{\mathcal{D}}_l. \quad (4.10)$$

It is independent of the shift spinor y as a consequence of angular momentum conservation. Indeed,

$$\frac{\partial}{\partial y} S_{+2}^{(1)} \sim \sum_{l=1}^n \frac{\langle y 0 \rangle p_l \sigma \tilde{\lambda}_0 - \langle y p_l 0 \rangle \lambda_0}{(2p_0 p_l) \langle y 0 \rangle^2} \tilde{\mathcal{D}}_l = - \sum_{l=1}^n \frac{\langle 0 p_l 0 \rangle y}{(2p_0 p_l) \langle y 0 \rangle^2} \tilde{\mathcal{D}}_l = - \frac{y}{\langle y 0 \rangle^2} \sum_{l=1}^n \tilde{\mathcal{D}}_l. \quad (4.11)$$

Now, $\sum_{l=1}^n \tilde{\mathcal{D}}_l$ can be related to the angular momentum operator $J^{\mu\nu}$ introduced in [44] and generalized to massive particles in [24]:

$$J_l^{\mu\nu} = J_{\chi_l}^{\mu\nu} + J_{\tilde{\chi}_l}^{\mu\nu}, \quad J_{\chi_l}^{\mu\nu} \equiv \frac{i}{4} \left(\chi_l \sigma^\mu \bar{\sigma}^\nu \frac{\partial}{\partial \chi_l} - \chi_l \sigma^\nu \bar{\sigma}^\mu \frac{\partial}{\partial \chi_l} \right), \quad J_{\tilde{\chi}_l}^{\mu\nu} \equiv \frac{i}{4} \left(\tilde{\chi}_l \bar{\sigma}^\mu \sigma^\nu \frac{\partial}{\partial \tilde{\chi}_l} - \tilde{\chi}_l \bar{\sigma}^\nu \sigma^\mu \frac{\partial}{\partial \tilde{\chi}_l} \right). \quad (4.12)$$

For a massless particle l one should replace $\chi \rightarrow \lambda$. Trading $\tilde{\mathcal{D}}_l$ for $J^{\mu\nu}$ we find

$$\frac{\partial}{\partial y} S_{+2}^{(1)} \sim i \frac{(y \sigma^\mu \tilde{\lambda}_0) y}{\langle y 0 \rangle^3} p_0^\nu \sum_{l=0}^n J_l^{\mu\nu} = 0. \quad (4.13)$$

Finally, the sub-subleading soft factor in eq. (4.7) is a double-differential operator:

$$S_{+2}^{(2)} = \frac{1}{4 M_{\text{Pl}}} \sum_{l=1}^n \frac{1}{p_0 p_l} \tilde{\mathcal{D}}_l^2. \quad (4.14)$$

This one is manifestly independent of the shift spinor y .

Following the same steps, but starting instead with the $\{\bar{0}mm\}$ shift defined in eq. (A.12), one can obtain the soft factors for emission of a minus helicity graviton:

$$S_{-2}^{(0)} = \frac{1}{M_{\text{Pl}}} \sum_{l=1}^n \frac{\langle 0 p_l y \rangle^2}{(2p_0 p_l) [0 y]^2}, \quad S_{-2}^{(1)} = \frac{1}{M_{\text{Pl}}} \sum_{l=1}^n \frac{\langle 0 p_l y \rangle}{(2p_0 p_l) [0 y]} \mathcal{D}_l, \quad S_{-2}^{(2)} = \frac{1}{4 M_{\text{Pl}}} \sum_{l=1}^n \frac{1}{p_0 p_l} \mathcal{D}_l^2, \quad (4.15)$$

⁷One might however argue that, in the on-shell approach to soft theorems, the role of the shift spinor y is exactly the same as that of the gauge parameter in the standard approach. From this point of view, gauge invariance thrown out of the door, comes back in through the window. On the other hand, ref. [16] derives the same conclusion by demanding the absence of double poles in z , in which case the parallel with gauge invariance is avoided.

where

$$\mathcal{D}_l = \langle 0\mathbf{l} \rangle \langle 0\partial_{\mathbf{l}} \rangle + \frac{\langle 0p_k p_l 0 \rangle}{\langle 0p_j p_k 0 \rangle} \langle 0\mathbf{j} \rangle \langle 0\partial_{\mathbf{j}} \rangle - \frac{\langle 0p_j p_l 0 \rangle}{\langle 0p_j p_k 0 \rangle} \langle 0\mathbf{k} \rangle \langle 0\partial_{\mathbf{k}} \rangle, \quad l \neq j, k \text{ and } \mathcal{D}_l = 0, \quad l = j, k. \quad (4.16)$$

Let us briefly comment on the relation of our results with the previous literature. It is a simple exercise to show that in the limit where all particles (also j and k) are massless, the soft factors we derived reduce to those found in [16]. In particular, the differential operator $\tilde{\mathcal{D}}_l$ reduces to

$$\tilde{\mathcal{D}}_l \rightarrow [0l] \left\{ [0\partial_l] - \frac{\langle lk \rangle}{\langle jk \rangle} [0\partial_j] + \frac{\langle lj \rangle}{\langle jk \rangle} [0\partial_k] \right\} = [0l] \nabla_{0l}, \quad (4.17)$$

where ∇_{0l} is defined in eq. (19) of [16]. On the other hand, our soft theorems do not exactly reduce to those in [11], the obstruction being the last two terms in the curly bracket in eq. (4.17). As discussed in [16], this difference is due to working with “stripped” amplitudes $\mathcal{M}$, rather than with S matrix elements $T \sim \delta^4(p_0 + \dots + p_N) \mathcal{M}$ as in [11]. The price to pay for us is that the soft theorems look somewhat more complicated, and that they depend on the exact manner in which the soft limit is taken. The advantage is that for practical purposes one works with the stripped amplitudes $\mathcal{M}$. The expressions we provide are ready to use out of the box with massive amplitudes written using the modern on-shell spinor formalism.

4.2 Non-minimal gravity

We move to consider theories where gravitational interactions of matter are non-minimal. Let us start by assuming that a massive particle X with spin $S > 0$ and mass $m > 0$ has an anomalous gravitomagnetic dipole moment:

$$\mathcal{M}(\mathbf{1}_X \mathbf{2}_X \mathbf{3}_h^+) \stackrel{?}{=} -\frac{1}{M_{\text{Pl}} m^{2S}} \frac{\langle \zeta p_1 3 \rangle^2}{\langle 3 \zeta \rangle^2} \left[\langle \mathbf{21} \rangle^{2S} + \frac{d_X}{m^2} \frac{\langle \zeta p_1 3 \rangle}{\langle 3 \zeta \rangle} \langle \mathbf{31} \rangle \langle \mathbf{32} \rangle \langle \mathbf{21} \rangle^{2S-1} \right]. \quad (4.18)$$

It was argued in ref. [34] that general coordinate invariance forbids $d_X \neq 0$. But let us press on to see what happens. It turns out that the dipole affects the subleading soft factor, which is modified as

$$S_{+2}^{(1)} = \frac{1}{M_{\text{Pl}}} \sum_{l=1}^n \frac{\langle y p_l 0 \rangle}{(2p_0 p_l) \langle 0y \rangle} \left[\delta_L^{\tilde{J}} \tilde{\mathcal{D}}_l + \frac{d_l}{m_l^2} \delta_{L_2 \dots L_{2S_l}}^{J_2 \dots J_{2S_l}} (\tilde{\chi}_l^{J_1} \tilde{\lambda}_0) (\tilde{\chi}_{l L_1} \tilde{\lambda}_0) \right]. \quad (4.19)$$

The first term in the bracket comes from the minimal coupling, and is the same as in eq. (4.10), except this time we explicitly display the identity operator acting on the little group indices of the l -th particle. The second factor is due to the dipole, where, as always, symmetrization of $J_1 \dots J_{2S_l}$ and $L_1 \dots L_{2S_l}$ is implicit. The problem is that, for $d_l \neq 0$, this soft factor is *not* independent of the shift spinor:

$$\frac{\partial}{\partial y} S_{+2}^{(1)} \sim \frac{y}{\langle y 0 \rangle^2} \delta_{L_2 \dots L_{2S_l}}^{J_2 \dots J_{2S_l}} \sum_{l=1}^n \frac{d_l}{m_l^2} (\tilde{\chi}_l^{J_1} \tilde{\lambda}_0) (\tilde{\chi}_{l L_1} \tilde{\lambda}_0) \neq 0. \quad (4.20)$$

This provides a simple and general proof that, for a particle of any mass and spin, the presence of an anomalous gravitomagnetic dipole moment is in conflict with the assumptions of Poincaré invariance, locality and unitarity.

Given the dipole is forbidden, for $S \geq 1$ the leading deformation of the minimal coupling is the quadrupole:

$$\begin{aligned}\mathcal{M}(\mathbf{1}_X \mathbf{2}_X 3_h^+) &= -\frac{1}{M_{\text{Pl}} m^{2S}} \frac{\langle \zeta p_1 3 \rangle^2}{\langle 3\zeta \rangle^2} \left[\langle \mathbf{21} \rangle^{2S} + \frac{Q}{2m^4} \frac{\langle \zeta p_1 3 \rangle^2}{\langle 3\zeta \rangle^2} \langle \mathbf{31} \rangle^2 \langle \mathbf{32} \rangle^2 \langle \mathbf{21} \rangle^{2S-2} + \dots \right], \\ \mathcal{M}(\mathbf{1}_X \mathbf{2}_X 3_h^-) &= -\frac{1}{M_{\text{Pl}} m^{2S}} \frac{\langle 3p_1 \zeta \rangle^2}{[3\zeta]^2} \left[[\mathbf{21}]^{2S} + \frac{Q}{2m^4} \frac{\langle 3p_1 \zeta \rangle^2}{[3\zeta]^2} [\mathbf{31}]^2 [\mathbf{32}]^2 [\mathbf{21}]^{2S-2} + \dots \right],\end{aligned}\quad (4.21)$$

where the dots stand for eventual higher multipoles, which do not affect the soft theorems.⁸ The quadrupole does arise from local Lagrangians, roughly speaking, when gravity interacts with matter via the Riemann tensor; for example, for $S = 1$ it corresponds to the coupling $\mathcal{L} \sim Q X_{\mu\nu} X_{\alpha\beta} R_{\mu\nu\alpha\beta}$.

We find that the quadrupole enters into the sub-subleading soft factor:

$$\begin{aligned}S_{+2}^{(2)} &= \frac{1}{4 M_{\text{Pl}}} \sum_{l=1}^n \frac{1}{p_0 p_l} \left[\tilde{\mathcal{D}}_l^2 \delta_{\vec{L}}^{\vec{J}} + \frac{Q_l}{m_l^2} (\tilde{\chi}_l^{J_1} \tilde{\lambda}_0) (\tilde{\chi}_l^{J_2} \tilde{\lambda}_0) (\tilde{\chi}_{l L_1} \tilde{\lambda}_0) (\tilde{\chi}_{l L_2} \tilde{\lambda}_0) \delta_{L_3 \dots L_{2S_l}}^{J_3 \dots J_{2S_l}} \right], \\ S_{-}^{(2)} &= \frac{1}{4 M_{\text{Pl}}} \sum_{l=1}^n \frac{1}{p_0 p_l} \left[\mathcal{D}_l^2 \delta_{\vec{L}}^{\vec{J}} + \frac{Q_l}{m_l^2} (\chi_l^{J_1} \lambda_0) (\chi_l^{J_2} \lambda_0) (\chi_{l L_1} \lambda_0) (\chi_{l L_2} \lambda_0) \delta_{L_3 \dots L_{2S_l}}^{J_3 \dots J_{2S_l}} \right],\end{aligned}\quad (4.22)$$

where, as always, symmetrization over the little group indices is understood. The double-differential operator due to the minimal coupling is now supplemented by an algebraic operator originating from the quadrupole. Note that the latter acts non-trivially on the little group indices: the l -th term in the sum acts on the upper indices $\vec{L} \equiv L_1 \dots L_{2S_l}$ of the l -th particle in $\hat{\mathcal{M}}^{0,0}(1 \dots l^{\vec{L}} \dots n)$.

4.3 Photons

Photons can be tackled by exactly the same methods as gravitons, except that the calculations are simpler. Below we summarize the soft theorems for photon emission. Electromagnetic interactions of a particle X with mass m and spin S are described by the following 3-point amplitudes [22, 34]:

$$\begin{aligned}\mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^+) &= \frac{\sqrt{2}e}{m^{2S}} \left\{ q \frac{\langle \zeta p_1 3 \rangle}{\langle 3\zeta \rangle} \langle \mathbf{21} \rangle^{2S} + \frac{S a}{m^2} \frac{\langle \zeta p_1 3 \rangle^2}{\langle 3\zeta \rangle^2} \langle \mathbf{31} \rangle \langle \mathbf{32} \rangle \langle \mathbf{21} \rangle^{2S-1} + \dots \right\}, \\ \mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^-) &= \frac{\sqrt{2}e}{m^{2S}} \left\{ q \frac{\langle 3p_1 \zeta \rangle}{[3\zeta]} [\mathbf{21}]^{2S} + \frac{S \bar{a}}{m^2} \frac{\langle 3p_1 \zeta \rangle^2}{[3\zeta]^2} [\mathbf{31}] [\mathbf{32}] [\mathbf{21}]^{2S-1} + \dots \right\},\end{aligned}\quad (4.23)$$

where e is the electromagnetic coupling constant, q is the electric charge, $(a + \bar{a})/2 = (g - 2)/2$ is the anomalous magnetic moment in the standard normalization, $a \neq \bar{a}$ gives the electric dipole moment, and the dots stand for eventual higher multipoles which are irrelevant for the soft theorems. The first term in each bracket describes the minimal coupling, while the second describes non-minimal dipole interactions for $S \geq 1/2$. In the

⁸The n -th multipole enters the recursion in eq. (3.7) with the $\frac{(\lambda_0^{z_1} \chi_1)^n (\lambda_0^{z_2} \chi_2)^n}{(\lambda_0^{z_l} \zeta)^n}$ factor, which scales as ϵ^n , given $\lambda_0^{z_l} \sim \epsilon$.

Lagrangian parlance, the former arises when photon interacts via $D_\mu = \partial_\mu - ieqA_\mu$ in the kinetic term, while the higher multipoles correspond to photon interacting via $F_{\mu\nu}$. For example, for real a , the $S = 1/2$ Lagrangian would be $\mathcal{L} \supset eq\bar{\psi}\gamma_\mu\psi A_\mu + \frac{eq}{4m}\bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu}$, while for $S = 1$ it would be $\mathcal{L} \supset -iqe(V_{\mu\nu}\bar{V}_\nu - \bar{V}_{\mu\nu}V_\nu)A_\mu + ie(q+2a)V_\mu\bar{V}_\nu F_{\mu\nu}$.

Plugging the plus helicity amplitude in eq. (4.23) into the recursion in eq. (3.7) one finds the raw version of the soft theorem:

$$\hat{\mathcal{M}}^{0,\epsilon}(1\dots n0^+) = -\frac{e}{\sqrt{2}}\sum_{l=1}^n\frac{1}{p_0p_l}\left[\frac{q_l\langle yp_l0\rangle}{\epsilon^2\langle 0y\rangle}\delta_{\vec{L}}^{\vec{J}} + \frac{S_la_l}{m_l\epsilon}(\tilde{\lambda}_0\tilde{\chi}_l^{J_1})(\tilde{\lambda}_0\tilde{\chi}_{lL_1})\delta_{L_2\dots L_{2S_l}}^{J_2\dots J_{2S_l}}\right]\hat{\mathcal{M}}^{z_l,\epsilon}(1\dots P_0^{\vec{L}}\dots n) + \mathcal{O}(\epsilon^0). \quad (4.24)$$

The first term in the bracket originates from the minimal coupling, while the second describes a correction from the dipole. The latter has a non-trivial action on the little group indices of $\hat{\mathcal{M}}$. Using the expansion of $\hat{\mathcal{M}}$ in powers of ϵ , cf. eq. (4.5), one derives the soft theorem:

$$\hat{\mathcal{M}}^{0,\epsilon}(1\dots n0^\pm) = -\left\{\frac{1}{\epsilon^2}S_{\pm 1}^{(0)} + \frac{1}{\epsilon}S_{\pm 1}^{(1)}\right\}\hat{\mathcal{M}}^{0,0}(1\dots n) + \mathcal{O}(\epsilon^0). \quad (4.25)$$

The leading soft factor for emission of a plus helicity photon is

$$S_{+1}^{(0)} = \frac{e}{\sqrt{2}}\sum_{l=1}^n\frac{q_l\langle yp_l0\rangle}{\langle 0y\rangle\langle p_0p_l\rangle} = e\epsilon_\mu^{0+}\sum_{l=1}^n\frac{q_lp_l^\mu}{p_0p_l}, \quad (4.26)$$

where in the second step we introduced the polarization vector of the soft photon defined in eq. (2.5), so as to make contact with the Weinberg's formula. The independence of $S^{(0)}$ on the shift spinor y follows from charge conservation [43]:

$$\frac{\partial}{\partial y}S_{+1}^{(0)} \sim \frac{y}{\langle y0\rangle^2}\sum_{l=1}^nq_l = 0; \quad (4.27)$$

or the other way around: given that the soft recursion must be valid for any y proves charge conservation in any Poincaré invariant, local, and unitary theory.

The subleading soft factor is given by

$$S_{+1}^{(1)} = \frac{e}{\sqrt{2}}\sum_{l=1}^n\frac{1}{p_0p_l}\left[q_l\tilde{\mathcal{D}}_l\delta_{\vec{L}}^{\vec{J}} + \frac{S_la_l}{m_l}(\tilde{\lambda}_0\tilde{\chi}_l^{J_1})(\tilde{\lambda}_0\tilde{\chi}_{lL_1})\delta_{L_2\dots L_{2S_l}}^{J_2\dots J_{2S_l}}\right], \quad (4.28)$$

where $\tilde{\mathcal{D}}_l$ is defined in eq. (4.6). This one is manifestly independent of y , which in particular confirms what everybody knows that arbitrary electromagnetic dipoles are allowed. Here, we display the action on little group indices of a massive l -th particle, which is non-trivial when dipoles are involved. If on the other hand the l -th particle is massless, we should unbold the l -th term above, that is trade $\chi_l^L \rightarrow \lambda_l$, dropping the $SU(2)$ little group indices. Finally, the soft factors for emission of a minus helicity photon are

$$S_{-1}^{(0)} = \frac{e}{\sqrt{2}}\sum_{l=1}^n\frac{q_l\langle 0p_ly\rangle}{[0y]\langle p_0p_l\rangle} = e\epsilon_\mu^{0-}\sum_{l=1}^n\frac{q_lp_l^\mu}{p_0p_l},$$

$$S_{-1}^{(1)} = \frac{e}{\sqrt{2}}\sum_{l=1}^n\frac{1}{p_0p_l}\left[q_l\mathcal{D}_l\delta_{\vec{L}}^{\vec{J}} + \frac{S_l\bar{a}_l}{m_l}(\lambda_0\chi_l^{J_1})(\lambda_0\chi_{lL_1})\delta_{L_2\dots L_{2S_l}}^{J_2\dots J_{2S_l}}\right]. \quad (4.29)$$

5 Applications

The goal of this section is to demonstrate that the recursion relations and soft theorems we obtained previously allow one to efficiently calculate amplitudes with massive particles of arbitrary spin. Below we will reproduce some known results, as well as derive new ones not encountered in the literature so far.

5.1 Compton scattering

We start with the Compton scattering. We are interested in the amplitude $\mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^\pm 0_\gamma^+)$, where 0_γ is a soft photon⁹ of plus helicity, and X is a particle of mass m , unit charge, and arbitrary spin S . We will use the $\{0\bar{m}\bar{m}\}$ shift in eq. (3.9) with $j = 1$ and $k = 2$. We first assume minimal coupling, that is the absence of the dipole and higher multipoles in eq. (4.23) (the dipole case will be discussed shortly). The soft theorem in eq. (4.25) reduces to

$$\hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^\pm 0_\gamma^+) = -\frac{1}{\epsilon^2} S_{+1}^{(0)} \hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^\pm) + B_\infty. \quad (5.1)$$

In this case the subleading soft factor $S_{+1}^{(1)}$ is void for the simple reason that, for Compton scattering, we only have two factorization poles $z_{1,2}$, and $\tilde{\mathcal{D}}_l$ in eq. (4.6) is zero for $l = j, k$. Somewhat pedantically, we replaced $\mathcal{O}(\epsilon^0)$ in eq. (4.25) by the boundary term B_∞ , which is also $\mathcal{O}(\epsilon^0)$. The point is that, for 4-body processes, photon emission from external legs accounts for all factorization poles. Therefore, the recursion relation in eq. (3.7) should reproduce the full Compton amplitude (not just the leading terms for $\epsilon \rightarrow 0$) up to the boundary term introduced in eq. (3.3). The leading soft factor for Compton scattering is easily calculated as

$$S_{+1}^{(0)} = \sqrt{2}e \frac{\langle 3p_1 0 \rangle [03]}{(2p_1 p_0)(2p_2 p_0)}. \quad (5.2)$$

For minimal coupling, the hard 3-point amplitude on the right-hand side of eq. (5.1) evaluates to

$$\begin{aligned} \hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^-) &= \sqrt{2}e \frac{\langle 3p_1 0 \rangle}{[30]} \left[-\frac{\langle 13 \rangle [20] + \langle 23 \rangle [10]}{\langle 3p_1 0 \rangle} \right]^{2S}, \\ \hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^+) &= \sqrt{2}e \frac{[30]}{\langle 3p_1 0 \rangle} \frac{\langle 21 \rangle^{2S}}{m^{2S-2}}. \end{aligned} \quad (5.3)$$

It is important to stress that the shift is not void in $\hat{\mathcal{M}}^{0,0}$. That is because the shift we defined in eq. (3.9) is non-trivial for $z = \epsilon = 0$ (it is trivial if $\epsilon = 1$ instead). For this reason $\hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^\pm)$ above is not simply a copy of $\mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^\pm)$ in eq. (5.1). This zero-order shift is absolutely crucial to recover the correct result. In particular, the non-trivial spinor structure in $\hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^-)$ appears as a consequence of the identity

⁹In the Mandelstam variables the soft photon limit $p_0 \rightarrow 0$ corresponds to $t \rightarrow m^2$, $u \rightarrow m^2$, $s \rightarrow 0$. For $m > 0$ this of course cannot be realized with real kinematics, but the limit does make sense in complex kinematics.

$(\tilde{\chi}_1^z \tilde{\chi}_2^z)|_{z=\epsilon=0} = m \frac{\langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}]}{\langle 3p_1 0 \rangle}$. All in all we reconstruct

$$\begin{aligned}\hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^- 0_\gamma^+) &= \frac{2e^2 \langle 3p_1 0 \rangle^2}{\epsilon^2 (2p_1 p_0)(2p_2 p_0)} \left[-\frac{\langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}]}{\langle 3p_1 0 \rangle} \right]^{2S} + B_\infty, \\ \hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^+ 0_\gamma^+) &= \frac{2e^2 [30]^2}{\epsilon^2 (2p_1 p_0)(2p_2 p_0)} \frac{\langle \mathbf{21} \rangle^{2S}}{m^{2S-2}} + B'_\infty.\end{aligned}\quad (5.4)$$

On general grounds, this correctly describes leading terms of the Compton amplitude in the limit when the momentum of 0_γ^+ is soft. The full amplitude valid for *any* momentum of 0_γ^+ is recovered by setting $\epsilon = 1$. In this limit, the same-helicity amplitude above is a fully consistent expression for any S , also when $B'_\infty = 0$.¹⁰ A non-zero B'_∞ corresponds to a contact term (that is to say, without any singularities in the kinematic variables); for example we can have $B'_\infty = \frac{[\mathbf{21}]^{2S}}{m^{2S}} \frac{[30]^2}{\Lambda^2}$. On the other hand, the opposite-helicity amplitude is healthy as it stands only for $S \leq 1$. As discussed in [22], for $S > 1$ the first term in eq. (5.4) develops an unphysical pole at $\langle 3p_1 0 \rangle \rightarrow 0$. That means the boundary term cannot vanish for $S > 1$; it must be non-zero to cancel the unphysical pole, and at the same time it cannot have any singularities as $p_0 p_1 \rightarrow 0$ or $p_0 p_2 \rightarrow 0$. Therefore it must be of the form $B_\infty = \sum_{k=1}^{2S-2} \frac{C_k}{\langle 3p_1 0 \rangle^k}$, where C_k are pure contact terms.

One possible form of the boundary term is

$$\begin{aligned}B_\infty &= -\frac{2e^2}{(2m^2)^{2S} (2p_1 p_0)(2p_2 p_0)} \sum_{k=3}^{2S} \binom{2S}{k} \frac{\mathcal{Y}(\mathcal{Y}^{k-1} - \{\mathcal{Y}||\mathcal{Z}\} \mathcal{Z}^{k-2})(-\mathcal{X})^{2S-k}}{\langle 3p_1 0 \rangle^{k-2}}, \\ \mathcal{Y} &\equiv (2p_1 p_0) \langle \mathbf{13} \rangle [\mathbf{20}] + (2p_2 p_0) \langle \mathbf{23} \rangle [\mathbf{10}], \quad \mathcal{Z} \equiv (2p_1 p_0) \langle \mathbf{13} \rangle [\mathbf{20}] - (2p_2 p_0) \langle \mathbf{23} \rangle [\mathbf{10}], \\ \mathcal{X} &\equiv m \langle \mathbf{12} \rangle + m[\mathbf{12}] - \langle \mathbf{1} p_0 \mathbf{2} \rangle - \langle \mathbf{2} p_3 \mathbf{1} \rangle,\end{aligned}\quad (5.5)$$

where $\{\mathcal{Y}||\mathcal{Z}\} = \mathcal{Y}$ for even k , and $\{\mathcal{Y}||\mathcal{Z}\} = \mathcal{Z}$ for odd k . Any other legal boundary term will differ from the one above only by contact terms. The absence of physical poles in eq. (5.5) is not manifest, but it immediately follows from $\mathcal{Y}^{2n} - \mathcal{Z}^{2n} = (\mathcal{Y}^2 - \mathcal{Z}^2)(\mathcal{Y}^{2n-2} + \dots + \mathcal{Z}^{2n-2})$, given that $(\mathcal{Y}^2 - \mathcal{Z}^2) = 16(p_1 p_0)(p_2 p_0) \langle \mathbf{13} \rangle [\mathbf{20}] \langle \mathbf{23} \rangle [\mathbf{10}]$. A rationale for this boundary term is postponed to appendix B. For the present discussion we notice that B_∞ is a polynomial in z after applying our $\{0\bar{m}\bar{m}\}$ shift, which is due to the fact that the shift enters only via the $p_1 p_0$ and $p_2 p_0$ factors in $\mathcal{Y}$ and $\mathcal{Z}$. More precisely, the boundary term is of the form $B_\infty = \sum_{n=1}^{2S-2} \frac{P_n(z)}{\langle 3p_1 0 \rangle^n}$, where $P_n(z)$ is the n -th order polynomial in z . Furthermore, $P_n(z) \sim \epsilon^n$ in the soft limit. Thus the boundary term does not show up in the soft theorems, but still it contributes to the recursion in eq. (3.7) via the integral over the circle at infinity. Overall $B_\infty \sim \mathcal{O}(\epsilon^1)$, better than $\mathcal{O}(\epsilon^0)$ deduced on general grounds. The boundary term we propose is arguably more compact than the one quoted in [34]. It also makes manifest that the UV behavior of Compton scattering for $S > 1$ can be $\mathcal{O}(E^{4S}/m^{4S})$. In fact, one can further soften it down to $\mathcal{O}(E^{4S-2}/m^{4S-2})$ by subtracting from it a judiciously chosen contact term. See appendix B for more details.

We turn to another example where the subleading soft factor is no longer moot. We consider Compton scattering for arbitrary $S \geq 1/2$ when the dipole a in eq. (4.23) is

¹⁰For $S > 1$, the same-helicity Compton amplitude diverges as $(p_1 + p_2)^2 \rightarrow \infty$, in which case it is consistent in the sense of an effective theory amplitude below some cutoff scale.

switched on. In order to keep this example simple, we restrict to the case where $q = 0$, however a more general formula can be easily worked out by the same methods. The hard amplitude with a minus helicity photon is given by

$$\hat{\mathcal{M}}^{0,0}(1_X 2_{\bar{X}} 3_\gamma^-) = (-1)^{2S+1} \frac{\sqrt{2}eS\bar{a}}{m} \langle \mathbf{13} \rangle \langle \mathbf{23} \rangle \left[\frac{\langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}]}{\langle 3p_1 0 \rangle} \right]^{2S-1}. \quad (5.6)$$

The subleading soft factor for a plus helicity photon takes the form

$$S_{+1}^{(1)} = \frac{\sqrt{2}Se a}{m} \left[\frac{[01](\tilde{\lambda}_0 \tilde{\chi}_1 L_1)}{2p_0 p_1} - \frac{[02](\tilde{\lambda}_0 \tilde{\chi}_2 K_1)}{2p_0 p_2} \right]. \quad (5.7)$$

Only those little group indices that act non-trivially on the hard factor are displayed. With a little help of spinor algebra one can show that this action amounts to

$$\begin{aligned} & [01](\tilde{\lambda}_0 \tilde{\chi}_1 L_1) \left\{ \langle \mathbf{13} \rangle \langle \mathbf{23} \rangle \left(\langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}] \right)^{2S-1} \right\} \\ &= -\langle 3p_1 0 \rangle \langle \mathbf{23} \rangle [\mathbf{10}] \left(\langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}] \right)^{2S-2} \\ & \quad \times \left[\langle \mathbf{13} \rangle [\mathbf{20}] + \frac{1}{2S} \langle \mathbf{23} \rangle [\mathbf{10}] \right], \\ & [02](\tilde{\lambda}_0 \tilde{\chi}_2 K_1) \left\{ \langle \mathbf{13} \rangle \langle \mathbf{23} \rangle \left(\langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}] \right)^{2S-1} \right\} \\ &= \langle 3p_1 0 \rangle \langle \mathbf{13} \rangle [\mathbf{20}] \left(\langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}] \right)^{2S-2} \\ & \quad \times \left[\frac{1}{2S} \langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}] \right]. \end{aligned} \quad (5.8)$$

The $1/2S$ factors appear as a consequence of symmetrization of the little group indices. Putting it all together:

$$\begin{aligned} \hat{\mathcal{M}}^{0,\epsilon}(1_X 2_{\bar{X}} 3_\gamma^- 0_\gamma^+) &= -\frac{1}{\epsilon} S_{+1}^{(1)} \hat{\mathcal{M}}^{0,0}(1_X 2_{\bar{X}} 3_\gamma^-) + B_\infty \\ &= |a|^2 \frac{(-1)^{2S} S e^2}{\epsilon m^2 (2p_1 p_0) (2p_2 p_0)} \left[\frac{\langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}]}{\langle 3p_1 0 \rangle} \right]^{2S-2} \\ & \quad \times \left\{ (2p_1 p_0) \langle \mathbf{13} \rangle [\mathbf{20}] (\langle \mathbf{13} \rangle [\mathbf{20}] + 2S \langle \mathbf{23} \rangle [\mathbf{10}]) \right. \\ & \quad \left. + (2p_2 p_0) \langle \mathbf{23} \rangle [\mathbf{10}] (2S \langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}]) \right\} + B_\infty. \end{aligned} \quad (5.9)$$

This describes the leading effect of the dipole in the limit where the momentum p_0 is soft. Away from this limit, the expression makes sense for $B_\infty = 0$ as long as $S \leq 1$, whereas for $S \geq 3/2$ one needs to adjust B_∞ so as to cancel the unphysical pole at $\langle 3p_1 0 \rangle \rightarrow 0$. A possible boundary term can be constructed using the techniques discussed in appendix B, and is displayed in full glory in eq. (B.11).

5.2 Gravitational Compton scattering

Calculation of gravitational Compton scattering amplitudes $\mathcal{M}(\mathbf{1}_X \mathbf{2}_X 3_h 0_h)$ follows the steps laid out in the previous subsection, with only minor modifications. We again work with the $\{0\bar{m}\bar{m}\}$ shift of the 012 legs. Consider matter particles minimally coupled to gravity as in eq. (4.1). The leading soft factor $S_{+2}^{(0)}$ in eq. (4.8) is easily calculated as

$$S_{+2}^{(0)} = -\frac{\langle 3p_1 0 \rangle^2 [03]^2}{M_{\text{Pl}}(2p_1 p_0)(2p_2 p_0)(2p_3 p_0)}. \quad (5.10)$$

The absence of subleading soft factors for minimal coupling is just a tad less trivial to demonstrate than in the photon case. Starting from the expression in eq. (4.10) one notes that for our shift $\tilde{\mathcal{D}}_1 = \tilde{\mathcal{D}}_2 = 0$, while the remaining one reduces to

$$\tilde{\mathcal{D}}_3 = [01] \left(\tilde{\lambda}_0 \frac{\partial}{\partial \chi_1} \right) + [02] \left(\tilde{\lambda}_0 \frac{\partial}{\partial \chi_2} \right) + [03] \left(\tilde{\lambda}_0 \frac{\partial}{\partial \lambda_3} \right). \quad (5.11)$$

Thus, $\tilde{\mathcal{D}}_3$ is proportional to the total angular momentum operator, and it annihilates 3-point amplitudes. As in the photon case, the IR behavior of gravitational Compton amplitudes is fully determined by the leading soft factor, as long as the graviton is minimally coupled to matter. Given eq. (4.1) and the zero-order shift in eq. (3.9), the relevant hard factors are evaluated as

$$\begin{aligned} \hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_X 3_h^-) &= -\frac{1}{M_{\text{Pl}}} \frac{\langle 3p_1 0 \rangle^2}{[30]^2} \left[-\frac{\langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}]}{\langle 3p_1 0 \rangle} \right]^{2S}, \\ \hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_X 3_h^+) &= -\frac{1}{M_{\text{Pl}}} \frac{[30]^2}{\langle 3p_1 0 \rangle^2} \frac{\langle \mathbf{21} \rangle^{2S}}{m^{2S-4}}. \end{aligned} \quad (5.12)$$

Thus, the gravitational Compton scattering amplitudes are reconstructed as

$$\begin{aligned} \hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_X 3_h^- 0_h^+) &= \frac{S_{+2}^{(0)}}{\epsilon^3} \hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_X 3_h^-) \\ &= \frac{(-1)^{2S} \langle 3p_1 0 \rangle^4}{\epsilon^3 M_{\text{Pl}}^2 (2p_1 p_0)(2p_2 p_0)(2p_3 p_0)} \left[\frac{\langle \mathbf{13} \rangle [\mathbf{20}] + \langle \mathbf{23} \rangle [\mathbf{10}]}{\langle 3p_1 0 \rangle} \right]^{2S} + B_\infty, \\ \hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_X 3_h^+ 0_h^+) &= \frac{S_{+2}^{(0)}}{\epsilon^3} \hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_X 3_h^+) \\ &= \frac{[30]^4}{\epsilon^3 M_{\text{Pl}}^2 (2p_1 p_0)(2p_2 p_0)(2p_3 p_0)} \frac{\langle \mathbf{21} \rangle^{2S}}{m^{2S-4}} + B'_\infty. \end{aligned} \quad (5.13)$$

The IR part agrees with the expression derived in [22] by matching the residues at the kinematic poles. The amplitude away from the soft limit is obtained using $\mathcal{M} = \hat{\mathcal{M}}^{0,1}$. Much as for photons, it is consistent to set $B'_\infty = 0$ for any spin, however for $S > 2$ the boundary term B_∞ has to be non-zero so as to cancel the unphysical $\langle 3p_1 0 \rangle$ pole, see appendix B.2 for an explicit expression.

Let us also work out a simple example beyond the leading soft factor. Consider a massive particle with spin $S \geq 1$ and non-minimal quadrupole interactions with the graviton,

corresponding to $Q \neq 0$ in eq. (4.21). In this case, the plus-helicity sub-subleading soft factor in eq. (4.22) for gravitational Compton scattering reduces to

$$S_{+2}^{(2)} = \frac{Q}{4M_{\text{Pl}}m^2} \left\{ \frac{[\mathbf{10}]^2(\tilde{\chi}_{L_1 L_1} \tilde{\lambda}_0)(\tilde{\chi}_{L_2 L_2} \tilde{\lambda}_0)}{p_0 p_1} + \frac{[\mathbf{20}]^2(\tilde{\chi}_{2 K_1} \tilde{\lambda}_0)(\tilde{\chi}_{2 K_2} \tilde{\lambda}_0)}{p_0 p_2} \right\}, \quad (5.14)$$

where we displayed the little group indices acting non-trivially on the hard factor. The latter for plus helicity photons is generalized as

$$\hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_X 3_h^+) = -\frac{1}{M_{\text{Pl}}m^{2S-4}} \left\{ \frac{[30]^2}{\langle 3p_1 0 \rangle^2} \langle \mathbf{21} \rangle^{2S} + \frac{Q}{2} \frac{[30]^4}{\langle 3p_1 0 \rangle^4} \langle \mathbf{13} \rangle^2 \langle \mathbf{23} \rangle^2 \langle \mathbf{21} \rangle^{2S-2} \right\}. \quad (5.15)$$

Applying this to derive, for example, the same-helicity Compton amplitude we find

$$\begin{aligned} \hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_X 3_h^+ 0_h^+) &= \left[\frac{S_{+2}^{(0)}}{\epsilon^3} + \frac{S_{+2}^{(2)}}{\epsilon} \right] \hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_X 3_h^+) = \frac{[30]^4}{\epsilon^3 M_{\text{Pl}}^2 (2p_1 p_0)(2p_2 p_0)(2p_3 p_0)} \frac{\langle \mathbf{21} \rangle^{2S}}{m^{2S-4}} \\ &+ \frac{Q}{2} \frac{[30]^6}{\epsilon^3 M_{\text{Pl}}^2 (2p_1 p_0)(2p_2 p_0)(2p_3 p_0)} \frac{\langle \mathbf{13} \rangle^2 \langle \mathbf{23} \rangle^2 \langle \mathbf{21} \rangle^{2S-2}}{\langle 3p_1 0 \rangle^2 m^{2S-4}} \\ &- \frac{Q}{2} \frac{[30]^2}{\epsilon M_{\text{Pl}}^2 \langle 3p_1 0 \rangle^2} \left[\frac{[\mathbf{10}]^2 \langle \mathbf{2p}_1 0 \rangle^2}{2p_0 p_1} + \frac{[\mathbf{20}]^2 \langle \mathbf{1p}_2 0 \rangle^2}{2p_0 p_2} \right] \frac{\langle \mathbf{21} \rangle^{2S-2}}{m^{2S-2}} \\ &+ \mathcal{O}(Q^2) + B'_\infty. \end{aligned} \quad (5.16)$$

Unlike in the minimal case, in the presence of the quadrupole the unphysical $\langle 3p_1 0 \rangle$ pole appears also in the same-helicity amplitude, and it is present already for $S = 1$. If one uses this formula away from the soft limit, one needs to include an appropriate boundary term B'_∞ to cancel this pole for any $S \geq 1$.

5.3 Higher-point

To conclude this section, we discuss a couple of examples of 5-point amplitude calculations using the soft theorems of section 4. The goal is to provide an illustration how these formulas should be employed in the case of higher-point amplitudes. The examples involve a matter particle X with mass m , arbitrary spin S , and the minimal coupling to photons and gravitons.

Our first example is the amplitude describing the production of three photons in the $X\bar{X}$ annihilation: $\mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma 4_\gamma 0_\gamma)$. We use the $\{0\bar{m}\bar{m}\}$ ($\{\bar{0}mm\}$) shift on the 012 legs to calculate emission of a soft photon 0_γ with plus (minus) helicity. The leading soft factors for emission of 0_γ are given by

$$S_{+1}^{(0)} = \frac{\sqrt{2}e[0p_1 p_2 0]}{(2p_1 p_0)(2p_2 p_0)}, \quad S_{-1}^{(0)} = \frac{\sqrt{2}e\langle 0p_1 p_2 0 \rangle}{(2p_1 p_0)(2p_2 p_0)}. \quad (5.17)$$

These expressions are in fact true for $\mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma \dots n_\gamma 0_\gamma)$ with any $n - 2 \geq 1$ number of hard photons. The subleading factors are moot for the minimal coupling. That is because

the only charged particles are at the positions 1 and 2, however $\mathcal{D}_{1,2} = 0$ for the shift we have chosen. Given eq. (5.4), the hard factor in the soft recursion is given by

$$\hat{\mathcal{M}}^{0,0}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^+ 4_\gamma^+) = \frac{2e^2[34]^2}{(2\tilde{p}_1 p_3)(2\tilde{p}_1 p_4)} \frac{\langle \mathbf{21} \rangle^{2S}}{m^{2S-2}}, \quad (5.18)$$

where

$$2\tilde{p}_1 p_l \equiv 2\tilde{p}_1 p_l|_{z=0, \epsilon=0} = 2p_1 p_l + 2p_2 p_0 \frac{[0p_1 p_l 0]}{[0p_1 p_2 0]}. \quad (5.19)$$

Putting it all together,

$$\begin{aligned} \hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^+ 4_\gamma^+ 0_\gamma^+) &= \frac{2\sqrt{2}e^3[34]^2[0p_1 p_2 0]}{\epsilon^2(2p_1 p_0)(2p_2 p_0)(2\tilde{p}_1 p_3)(2\tilde{p}_1 p_4)} \frac{\langle \mathbf{21} \rangle^{2S}}{m^{2S-2}} + \mathcal{O}(\epsilon^0), \\ \hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^+ 4_\gamma^+ 0_\gamma^-) &= \frac{2\sqrt{2}e^3[34]^2\langle 0p_1 p_2 0 \rangle}{\epsilon^2(2p_1 p_0)(2p_2 p_0)(2\tilde{p}_1 p_3)(2\tilde{p}_1 p_4)} \frac{\langle \mathbf{21} \rangle^{2S}}{m^{2S-2}} + \mathcal{O}(\epsilon^0). \end{aligned} \quad (5.20)$$

The similar approach can be used when photons are replaced with gravitons: $\mathcal{M}(\mathbf{1}_X \mathbf{2}_X 3_h 4_h 0_h)$. Below we sketch the derivation, however the final formulas are too lengthy to quote here. The leading soft factor for emission of a plus helicity graviton is

$$S_{+2}^{(0)} = \frac{1}{M_{\text{Pl}} \langle 03 \rangle^2} \left\{ \frac{\langle 3p_1 0 \rangle^2}{2p_0 p_1} + \frac{\langle 3p_2 0 \rangle^2}{2p_0 p_2} + \frac{\langle 34 \rangle^2 [40]^2}{2p_0 p_4} \right\}. \quad (5.21)$$

Here we set $y = \lambda_3$ in eq. (4.8), since the expression is true for an arbitrary spinor y . A major difference compared to the photon calculation before is that now the subleading soft factors are non-trivial. Indeed, picking again $y = \lambda_3$ in eq. (4.10), the subleading soft factor can be written as

$$S_{+2}^{(1)} = \frac{\langle 34 \rangle [40]}{M_{\text{Pl}} (2p_0 p_4) \langle 03 \rangle} \tilde{\mathcal{D}}_4, \quad \tilde{\mathcal{D}}_4 = [04][0\partial_4] + \frac{\langle 4p_2 0 \rangle [40]}{[0p_1 p_2 0]} [01][0\partial_1] - \frac{\langle 4p_1 0 \rangle [40]}{[0p_1 p_2 0]} [02][0\partial_2]. \quad (5.22)$$

The soft expansion of the 5-point amplitude $\mathcal{M}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_X 3_h^+ 4_h^+ 0_h^+)$ is calculated using eq. (4.7) with the soft factors above acting on $\mathcal{M}^{0,0}(\mathbf{1}_X \mathbf{2}_X 3_h^+ 4_h^+)$, which in turn can be obtained from eq. (5.13). One can explicitly verify that $\tilde{\mathcal{D}}_4 \mathcal{M}^{0,0}(\mathbf{1}_X \mathbf{2}_X 3_h^+ 4_h^+) \neq 0$, thus the subleading soft factor is not moot in this case.

6 Exponential representation

It has been known for a long time that the multipole expansion of classical spinning bodies, such as a Kerr black hole, can be resummed in a simple exponential form [45]. However, the corresponding description in terms of on-shell amplitudes was obtained just recently in [24]. That reference noticed that the on-shell 3-point amplitude describing minimal graviton interactions with matter of any spin can be recast in a compact exponential form, with the angular momentum operator in the exponent. Subsequently, they demonstrated that also the 4-point gravitational Compton scattering amplitude can be recast in an exponential form. This formal rephrasing turns out to be useful to compute the scattering angle of two spinning black holes [24]. As we are going to show, our soft recursion leads to a simple proof

of the exponentiation of the Compton amplitudes for arbitrary spin of the matter particle. In the following, we demonstrate it for the photon amplitudes, however the derivation for gravity is totally analogous. We also discuss the exponential representation of the soft theorems for any (integer) helicity of the soft particle.

6.1 Photon exponentiation

Let us first rewrite the photon's minimal coupling as

$$\mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^-) = \frac{2eq(p_1 \epsilon_3^-)}{m^{2S}} [\mathbf{21}]^{2S}, \quad \mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^+) = \frac{2eq(p_1 \epsilon_3^+)}{m^{2S}} \langle \mathbf{21} \rangle^{2S}. \quad (6.1)$$

This is obtained from the leading term in eq. (4.23) by replacing the ζ terms with the polarization vectors defined in eq. (2.5).¹¹ Eq. (6.1) can be equivalently recast as

$$\begin{aligned} \mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^-) &= \frac{2eq(p_1 \epsilon_3^-)}{m^{2S}} \exp \left(-i \frac{\epsilon_3^-{}^\mu p_3^\nu J_1^{\mu\nu}}{(p_1 \epsilon_3^-)} \right) \langle \mathbf{21} \rangle^{2S}, \\ \mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^+) &= \frac{2eq(p_1 \epsilon_3^+)}{m^{2S}} \exp \left(-i \frac{\epsilon_3^+{}^\mu p_3^\nu J_1^{\mu\nu}}{(p_1 \epsilon_3^+)} \right) [\mathbf{21}]^{2S}, \end{aligned} \quad (6.2)$$

where $J_i^{\mu\nu}$ are the angular momentum operators introduced in eq. (4.12). Note that we could equivalently replace $1 \rightarrow 2$ in the exponent. The proof relies on the identities $\langle \mathbf{12} \rangle - [\mathbf{12}] = \frac{\langle 31|\zeta]}{m^2[3\zeta]} [\mathbf{31}][\mathbf{32}] = \frac{\langle \zeta 13]}{m^2\langle 3\zeta]} \langle \mathbf{31} \rangle \langle \mathbf{32} \rangle$ and

$$\epsilon_3^{-\mu} p_3^\nu J_{\chi_l}^{\mu\nu} = \frac{i}{\sqrt{2}} \langle 3\mathbf{l} \rangle \left(\lambda_3 \frac{\partial}{\partial \chi_l} \right), \quad \epsilon_3^{+\mu} p_3^\nu J_{\tilde{\chi}_l}^{\mu\nu} = \frac{i}{\sqrt{2}} [\mathbf{3}\mathbf{l}] \left(\tilde{\lambda}_3 \frac{\partial}{\partial \tilde{\chi}_l} \right), \quad \epsilon_3^{-\mu} p_3^\nu J_{\tilde{\chi}_l}^{\mu\nu} = \epsilon_3^{+\mu} p_3^\nu J_{\chi_l}^{\mu\nu} = 0, \quad (6.3)$$

from which it follows $\chi_2 \left(1 - i \frac{\epsilon_3^{-\mu} p_3^\nu J_1^{\mu\nu}}{(p_1 \epsilon_3^-)} \right) \chi_1 = [\mathbf{21}]$, $\tilde{\chi}_2 \left(1 - i \frac{\epsilon_3^{+\mu} p_3^\nu J_1^{\mu\nu}}{(p_1 \epsilon_3^+)} \right) \tilde{\chi}_1 = \langle \mathbf{21} \rangle$.

We move to the calculation of the opposite helicity Compton amplitude using the soft recursion in eq. (3.7) with the $\{0\bar{m}\bar{m}\}$ shift on the 012 legs. For the soft 3-point amplitude we use the exponentiated form in eq. (6.2), while for the hard factor we use the minimal form in eq. (6.1). Then the recursion reduces to

$$\begin{aligned} \hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_\gamma^- 0_\gamma^+) &= \frac{(-1)^{2S} 4q^2 c^2}{\epsilon m^{2S}} \left\{ \frac{(\hat{p}_2 \epsilon_3^-)(\hat{p}_1 \hat{\epsilon}_0^+)}{u - m^2} \exp \left(-i \frac{\epsilon_3^{-\mu} p_3^\nu J_2^{\mu\nu}}{(\hat{p}_2 \epsilon_3^-)} \right) \langle \mathbf{21} \rangle^{2S} \Big|_{z=z_1} \right. \\ &\quad \left. + \frac{(\hat{p}_1 \epsilon_3^-)(\hat{p}_2 \hat{\epsilon}_0^+)}{t - m^2} \exp \left(-i \frac{\epsilon_3^{-\mu} p_3^\nu J_1^{\mu\nu}}{(\hat{p}_1 \epsilon_3^-)} \right) \langle \mathbf{21} \rangle^{2S} \Big|_{z=z_2} \right\} + B_\infty, \end{aligned} \quad (6.4)$$

where we already contracted the χ_{0l} spinors using eq. (2.7). The crucial identities for this calculation are

$$\begin{aligned} (\hat{p}_1 \hat{\epsilon}_0^+)|_{z=z_1} &= \frac{\langle y p_1 0 \rangle}{\epsilon \sqrt{2} \langle 0y \rangle} = \frac{1}{\epsilon} (p_1 \epsilon_0^+)|_{\zeta=y}, & (\hat{p}_2 \hat{\epsilon}_0^+)|_{z=z_2} &= \frac{\langle y p_2 0 \rangle}{\epsilon \sqrt{2} \langle 0y \rangle} = \frac{1}{\epsilon} (p_2 \epsilon_0^+)|_{\zeta=y}, \\ (\hat{p}_2 \epsilon_3^-)|_{z=z_1} &= \frac{\langle 3 p_2 0 \rangle}{\sqrt{2} [30]} = (p_2 \epsilon_3^-)|_{\tilde{\zeta}=\tilde{\lambda}_0}, & (\hat{p}_1 \epsilon_3^-)|_{z=z_2} &= \frac{\langle 3 p_1 0 \rangle}{\sqrt{2} [30]} = (p_1 \epsilon_3^-)|_{\tilde{\zeta}=\tilde{\lambda}_0}. \end{aligned} \quad (6.5)$$

¹¹While polarization vectors themselves depends on the gauge (or the reference spinor ζ), the 3-point function is gauge-independent because $p_1 p_3 = 0$ on the 3-body kinematics.

One should stress that these are valid for any reference spinors ζ and $\tilde{\zeta}$; no “gauge-fixing” was performed along the way. Then eq. (6.4) can be written as

$$\hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_{\gamma}^- 0_{\gamma}^+) = \frac{(-1)^{2S} 2q^2 e^2 \langle 3p_1 0 \rangle}{\epsilon^2 m^{2S} [30] \langle 0y \rangle} \left\{ -\frac{\langle yp_1 0 \rangle}{u - m^2} \exp \left(-i \frac{\epsilon_3^-{}^\mu p_3^\nu J_2^{\mu\nu}}{(p_2 \epsilon_3^-)|_{\tilde{\zeta}=\tilde{\lambda}_0}} \right) \langle \mathbf{21} \rangle^{2S} \right. \\ \left. + \frac{\langle yp_2 0 \rangle}{t - m^2} \exp \left(-i \frac{\epsilon_3^-{}^\mu p_3^\nu J_1^{\mu\nu}}{(p_1 \epsilon_3^-)|_{\tilde{\zeta}=\tilde{\lambda}_0}} \right) \langle \mathbf{21} \rangle^{2S} \right\} + B_\infty. \quad (6.6)$$

In the exponentials, the numerators do not depend on the reference spinors thanks to the antisymmetry of $J^{\mu\nu}$. On the other hand, in the denominators, $(p_i \epsilon_3^-)$ do depend on the reference spinors. Our calculation unambiguously dictates that these have to be evaluated at a particular reference spinor, $\tilde{\zeta} = \tilde{\lambda}_0$.¹² The two exponentials are in fact equal, so we can pull them in front of the curly bracket. Then, summing the t - and u -poles, the dependence on the shift spinor y cancels out and we obtain the formula for the Compton amplitude in the exponential form [26]:

$$\hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_{\gamma}^- 0_{\gamma}^+) = (-1)^{2S} \frac{2q^2 e^2 \langle 3p_1 0 \rangle^2}{\epsilon^2 m^{2S} (t - m^2)(u - m^2)} \exp \left(-i \frac{\epsilon_3^-{}^\mu p_3^\nu J_1^{\mu\nu}}{(p_1 \epsilon_3^-)|_{\tilde{\zeta}=\tilde{\lambda}_0}} \right) \langle \mathbf{21} \rangle^{2S} + B_\infty. \quad (6.7)$$

This proves exponentiation of IR terms in Compton scattering for any matter spin. Recall that B_∞ starts at $\mathcal{O}(\epsilon^1)$, and thus qualifies as a UV term. Given its explicit form in eq. (5.5), it does not appear to have a simple exponential representation. Finally, notice that the exponential can be rewritten in a more explicit form as

$$\hat{\mathcal{M}}^{0,\epsilon}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_{\gamma}^- 0_{\gamma}^+) = (-1)^{2S} \frac{2q^2 e^2 \langle 3p_1 0 \rangle^2}{\epsilon^2 m^{2S} (t - m^2)(u - m^2)} \exp \left(\frac{[30] \langle 31 \rangle (\lambda_3 \partial_{\chi_1})}{\langle 3p_1 0 \rangle} \right) \langle \mathbf{21} \rangle^{2S} + B_\infty. \quad (6.8)$$

6.2 Soft exponentiation

Ref. [14] showed that the soft theorem for gravitons and gluons could be written in terms of an exponential operator acting on lower point amplitudes. However, the proof is only valid for the minimal-helicity-violating (MHV) sector, as it relies on the vanishing of the boundary terms discussed in [11]. Using the soft recursion we can show that the exponential form is actually valid beyond the MHV sector. In fact, the exponential operator can be extended for amplitudes with massive particles of any spin and with a soft particle of any integer helicity $|h| \leq 2$. We begin by rewriting the minimal 3-point amplitude for $h \geq 0$ as

$$\mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3^{-h}) = g_X \left(\frac{\langle 3p_1 \zeta \rangle}{\langle 3\zeta \rangle} \right)^h \frac{[\mathbf{21}]^{2S}}{m^{2S-1+h}}, \quad \mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3^h) = g_X \left(\frac{\langle \zeta p_1 3 \rangle}{[3\zeta]} \right)^h \frac{\langle \mathbf{21} \rangle^{2S}}{m^{2S-1+h}}, \quad (6.9)$$

where $g_X = \sqrt{2}eq$ for $|h| = 1$, and $g_X = m/M_{\text{Pl}}$ for $|h| = 2$. The soft expansion in eq. (4.4) and eq. (4.24) can be collectively written as:

$$\hat{\mathcal{M}}^{0,\epsilon}(1 \dots n 0^+) = \frac{1}{\epsilon^{1+h} m^{1+h}} \sum_{l=1}^n \frac{g_l \langle yp_l 0 \rangle^h}{(2p_0 p_l) \langle 0y \rangle^h} \hat{\mathcal{M}}^{z_l,\epsilon}(1 \dots P_{0l} \dots n) + \mathcal{O}(\epsilon^0). \quad (6.10)$$

¹²This illuminates one tiny detail in the derivation of ref. [24], where the choice of the reference spinor was made ad-hoc.

Now, the Taylor expansion in eq. (4.5) can be (trivially) recast in the exponential form:

$$\hat{\mathcal{M}}^{z_l, \epsilon}(1 \dots P_{0l} \dots n) = \exp \left(\epsilon \frac{\langle 0y \rangle}{\langle y p_l 0 \rangle} \tilde{\mathcal{D}}_l \right) \hat{\mathcal{M}}^{0,0}(1 \dots n) + \mathcal{O}(\epsilon^3). \quad (6.11)$$

This way, for minimal coupling, we obtain a similar form of the soft theorem as the one in ref. [14]:

$$\mathcal{M}^{0, \epsilon}(1 \dots n 0^+) = \frac{1}{\epsilon^{1+h} m^{1+h}} \sum_{l=1}^n \frac{g_X \langle y p_l 0 \rangle^h}{(2p_0 p_l) \langle 0y \rangle^h} \exp \left(\epsilon \frac{\langle 0y \rangle}{\langle y p_l 0 \rangle} \tilde{\mathcal{D}}_l \right) \hat{\mathcal{M}}^{0,0}(1 \dots n) + \mathcal{O}(\epsilon^0). \quad (6.12)$$

where the differential operator $\tilde{\mathcal{D}}_l$ is given in eq. (4.6). This form may be useful to study amplitudes in the Mellin space along the lines of ref. [46]. Moreover, our results in eq. (4.28) for the electromagnetic dipole suggest that a generalization (for soft photons) consists in replacing

$$\tilde{\mathcal{D}}_l \rightarrow \tilde{\mathcal{D}}_l + \frac{S a_l}{q_l m} [0l]^2, \quad (6.13)$$

where for massive l the suppressed little group indices can be read off from eq. (4.28). Similarly, for gravitons, eq. (4.22) can be reproduced via $\tilde{\mathcal{D}}_l \rightarrow \tilde{\mathcal{D}}_l + \epsilon Q_l \frac{\langle y p_l 0 \rangle}{2 \langle 0y \rangle} [0l]^4$.

7 Conclusions

This paper provides a new toolbox to study theories with particles of arbitrary mass and spin. We introduced a class of complex momentum shifts combining a BCFW-like spinor shift with a soft limit of a massless particle present in the process. The important technical novelty is that our shifts are compatible with the massive spinors introduced by Arkani-Hamed, Huang, and Huang in ref. [22]. Via the usual road of the Cauchy theorem, these shifts allow one to derive novel recursion relations for scattering amplitudes, even when all but one particles involved are massive.

This formal development will find multiple practical applications, we hope. In this paper we applied it to derive a new incarnation of the soft theorems for emission of a soft photon or a graviton. They function, out of the box, with amplitudes expressed in the on-shell spinor formalism, also with massive spinors, thus generalizing the massless results of ref. [16]. This allows us to systematically explore soft theorems for general theories, and make a direct connection with the multipole expansion of the on-shell 3-point interaction between matter and photons/gravitons. In particular, our soft theorems lead to an elegant proof that gravitational dipole interactions are incompatible with the assumption of locality, unitarity and Poincaré invariance. We also provide a concise expression for the modification of the soft theorems due to the quadrupole (for gravitons) and dipole (for photons) non-minimal interactions with matter.

Generally, soft theorems only capture the leading terms in the limit where the massless particle momentum goes to zero, because they take into account only the soft particle emission from external legs. However, for Compton scattering, emission from external legs

accounts for all factorization channels. Therefore, Compton amplitudes derived via soft theorems reflect the complete structure of physical poles required by unitarity. A subtlety appears for spin $S \geq 3/2$ (for photons) and $S \geq 5/2$ (for gravitons) when these amplitudes develop a spurious pole, which has to be subtracted by adding a judiciously designed UV term. Our approach leads to a new perspective on these UV terms: they arise as boundary terms in our recursion relations. We laid out an algorithm for constructing the UV term for Compton scattering of photons, which leads to a more compact and transparent formula than the one in ref. [34]. We also derived the UV term for Compton scattering via the electromagnetic dipole. These results may be useful for practical applications, for example in nuclear or atomic physics. Furthermore, our soft recursion leads by the shortest route to recasting the Compton amplitude into a form where the increasing matter spin corresponds to an action of an exponentiated angular momentum operator [24, 26]. We were also able to rewrite more general soft recursions in the exponentiated form, extending the results of ref. [14].

Our soft recursions may help with calculation of multi-leg amplitudes — the topic we only fleetingly touched in this paper. That could in turn boost the amplitude methods of classical calculations of electromagnetic and gravitational radiation along the lines of ref. [23]. These recursions should also be useful to derive more general soft theorems for Goldstone fermions [47] and gravitinos [48]. Furthermore, it would be interesting to generalize our recursions to multiple soft particles [49] (as recently explored with the CHY formalism [50]). More speculatively, the tools we provide may open up new areas of exploration of massive amplitudes. One direction to mention in this context is the link with the asymptotic symmetries [5, 51], another is the double copy structure of the soft theorems for massive particles [52].

Acknowledgments

AF is partially supported by the French Agence Nationale de la Recherche (ANR) under grant ANR-19-CE31-0012 (project MORA). The work of C.S.M. is supported by the Alexander von Humboldt Foundation, in the framework of the Sofja Kovalevskaja Award 2016, endowed by the German Federal Ministry of Education and Research and also supported by the Cluster of Excellence “Precision Physics, Fundamental Interactions, and Structure of Matter” (PRISMA⁺ EXC 2118/1) funded by the German Research Foundation (DFG) within the German Excellence Strategy (Project ID 39083149).

A Summary of spinor shifts

In this appendix we list distinct spinor shifts realizing the momentum shift in eq. (3.1). We label them as $\{0\bar{A}\bar{B}\}$ or $\{\bar{0}AB\}$, where $A(B) = 0$ when the j -th (k -th) particle is massless, and $A(B) = m$ when the j -th (k -th) particle is massive. The label $\{0\bar{A}\bar{B}\}$ ($\{\bar{0}AB\}$) corresponds to shifting the holomorphic(antiholomorphic) spinor of the massless particle 0, and the antiholomorphic(holomorphic) spinors of the j -th and k -th particles.

$\{0\bar{0}\bar{0}\}$.

$$q_0\sigma = y\tilde{\lambda}_0, \quad q_j\sigma = \lambda_j\tilde{\lambda}_0, \quad q_k\sigma = \lambda_k\tilde{\lambda}_0. \quad (\text{A.1})$$

$$\begin{aligned} \lambda_0^z &= \epsilon\lambda_0 - zy, \\ \tilde{\lambda}_j^z &= \tilde{\lambda}_j - \frac{(\epsilon-1)\langle 0k \rangle - z\langle yk \rangle}{\langle jk \rangle} \tilde{\lambda}_0, \\ \tilde{\lambda}_k^z &= \tilde{\lambda}_k + \frac{(\epsilon-1)\langle 0j \rangle - z\langle yj \rangle}{\langle jk \rangle} \tilde{\lambda}_0. \end{aligned} \quad (\text{A.2})$$

$\{\bar{0}00\}$.

$$q_0\sigma = \lambda_0\tilde{y}, \quad q_j\sigma = \lambda_0\tilde{\lambda}_j, \quad q_k\sigma = \lambda_0\tilde{\lambda}_k. \quad (\text{A.3})$$

$$\begin{aligned} \tilde{\lambda}_0^z &= \epsilon\tilde{\lambda}_0 - z\tilde{y}, \\ \lambda_j^z &= \lambda_j - \frac{(\epsilon-1)[0k] - z[yk]}{[jk]} \lambda_0, \\ \lambda_k^z &= \lambda_k + \frac{(\epsilon-1)[0j] - z[yj]}{[jk]} \lambda_0. \end{aligned} \quad (\text{A.4})$$

$\{0\bar{m}\bar{0}\}$.

$$q_0\sigma = y\tilde{\lambda}_0, \quad q_j\sigma = p_j\sigma\tilde{\lambda}_0\tilde{\lambda}_0, \quad q_k\sigma = \lambda_k\tilde{\lambda}_0. \quad (\text{A.5})$$

$$\begin{aligned} \lambda_0^z &= \epsilon\lambda_0 - zy, \\ \tilde{\chi}_j^z &= \tilde{\chi}_j - \frac{(\epsilon-1)\langle k0 \rangle - z\langle ky \rangle}{\langle kp_j0 \rangle} [\mathbf{j}\tilde{0}] \tilde{\lambda}_0, \\ \tilde{\lambda}_k^z &= \tilde{\lambda}_k - \frac{(\epsilon-1)\langle 0p_j0 \rangle - z\langle yp_j0 \rangle}{\langle kp_j0 \rangle} \tilde{\lambda}_0. \end{aligned} \quad (\text{A.6})$$

$\{\bar{0}m0\}$.

$$q_0\sigma = \lambda_0\tilde{y}, \quad q_j\sigma = \lambda_0\lambda_0p_j\bar{\sigma}, \quad q_k\sigma = \lambda_0\tilde{\lambda}_k. \quad (\text{A.7})$$

$$\begin{aligned} \tilde{\lambda}_0^z &= \epsilon\tilde{\lambda}_0 - z\tilde{y}, \\ \chi_j^z &= \chi_j - \frac{(\epsilon-1)[k0] - z[ky]}{\langle 0p_jk \rangle} [\mathbf{j}0] \lambda_0, \\ \lambda_k^z &= \lambda_k - \frac{(\epsilon-1)\langle 0p_j0 \rangle - z\langle 0p_jy \rangle}{\langle 0p_jk \rangle} \lambda_0. \end{aligned} \quad (\text{A.8})$$

$\{0\bar{m}\bar{m}\}$.

$$q_0\sigma = y\tilde{\lambda}_0, \quad q_j\sigma = p_j\sigma\tilde{\lambda}_0\tilde{\lambda}_0, \quad q_k\sigma = p_k\sigma\tilde{\lambda}_0\tilde{\lambda}_0. \quad (\text{A.9})$$

$$\begin{aligned} \lambda_0^z &= \epsilon\lambda_0 - zy, \\ \tilde{\chi}_j^z &= \tilde{\chi}_j + \frac{(\epsilon-1)\langle 0p_k0 \rangle - z\langle yp_k0 \rangle}{[0p_jp_k0]} [\mathbf{j}0] \tilde{\lambda}_0, \\ \tilde{\chi}_k^z &= \tilde{\chi}_k - \frac{(\epsilon-1)\langle 0p_j0 \rangle - z\langle yp_j0 \rangle}{[0p_jp_k0]} [\mathbf{k}0] \tilde{\lambda}_0. \end{aligned} \quad (\text{A.10})$$

$\{\bar{0}mm\}$.

$$q_0\sigma = \lambda_0\tilde{y}, \quad q_j\sigma = \lambda_0 p_j \bar{\sigma} \lambda_0, \quad q_k\sigma = \lambda_0 p_k \bar{\sigma} \lambda_0. \quad (\text{A.11})$$

$$\begin{aligned} \tilde{\lambda}_0^z &= \epsilon \tilde{\lambda}_0 - z \tilde{y}, \\ \chi_j^z &= \chi_j + \frac{(\epsilon - 1)\langle 0p_k 0 \rangle - z\langle 0p_k y \rangle}{\langle 0p_j p_k 0 \rangle} \langle \mathbf{j}0 \rangle \lambda_0, \\ \chi_k^z &= \chi_k - \frac{(\epsilon - 1)\langle 0p_j 0 \rangle - z\langle 0p_j y \rangle}{\langle 0p_j p_k 0 \rangle} \langle \mathbf{k}0 \rangle \lambda_0. \end{aligned} \quad (\text{A.12})$$

B UV terms in Compton scattering

We have seen in section 5 that, when spin S of the matter particle is larger than one, the IR part of the Compton amplitude obtained via soft theorems develops an unphysical pole at $\langle 3p_1 4 \rangle \rightarrow 0$. The same happens for $S > 2$ in gravitational Compton scattering. The unphysical pole has to be canceled by the boundary term in the recursion relation, which we refer to as the *UV term* in the discussion below. In this appendix we present an algorithm to construct a correct UV term, alternative to the one in ref. [34].

For this discussion we find it convenient to introduce the auxiliary objects:

$$\begin{aligned} \mathcal{A} &\equiv \langle 13 \rangle [24], & \mathcal{B} &\equiv \langle 23 \rangle [14], & \mathcal{X} &\equiv m\langle 12 \rangle + m[12] - \langle 1p_4 2 \rangle - \langle 2p_3 1 \rangle \\ \mathcal{Y} &\equiv (u - m^2)\mathcal{A} + (t - m^2)\mathcal{B}, & \mathcal{Z} &\equiv (u - m^2)\mathcal{A} - (t - m^2)\mathcal{B}. \end{aligned} \quad (\text{B.1})$$

They satisfy the important identities

$$\langle 3p_1 4 \rangle \mathcal{X} - \mathcal{Y} = 2m^2 (\mathcal{A} + \mathcal{B}), \quad \mathcal{Z} = -\langle 3p_1 4 \rangle (\langle 1p_4 2 \rangle + \langle 2p_3 1 \rangle). \quad (\text{B.2})$$

B.1 Electromagnetic Compton scattering

The opposite helicity Compton scattering amplitude for matter particle X of mass m and spin S can be written as $\mathcal{M}(\mathbf{1}_X \mathbf{2}_{\bar{X}} 3_{\gamma}^- 4_{\gamma}^+) = \mathcal{M}_{\text{IR}} + \mathcal{M}_{\text{UV}}$. The IR term takes the form [22]:

$$\mathcal{M}_{\text{IR}} = (-1)^{2S} \frac{2e^2 \langle 3p_1 4 \rangle^2}{(t - m^2)(u - m^2)} \left[\frac{\mathcal{A} + \mathcal{B}}{\langle 3p_1 4 \rangle} \right]^{2S}, \quad (\text{B.3})$$

where $t = (p_1 + p_3)^2$, $u = (p_1 + p_4)^2$. As discussed in [22], the IR term fully accounts for the kinematic t - and u -channel poles required by unitarity. The UV term can be zero or a pure contact term for $S \leq 1$. However, for $S > 1$ the IR term develops an unphysical pole at $\langle 3p_1 4 \rangle \rightarrow 0$. Therefore the UV term must be necessarily non-zero to cancel the unphysical pole, without screwing up the physical t - and u -poles.

To understand the general formula for arbitrary spin S , it is useful start with a simpler example of $S = 2$. The first identity in eq. (B.2) allows one to reshuffle the IR term, so as to rewrite the residue of the unphysical pole in a more convenient form [34]. Plugging it into eq. (B.3), the IR term for $S = 2$ can be rewritten as

$$\begin{aligned} \mathcal{M}_{\text{IR}} &= \frac{2e^2 \langle 3p_1 4 \rangle^2}{(2m^2)^4 (t - m^2)(u - m^2)} \left[\frac{\langle 3p_1 4 \rangle \mathcal{X} - \mathcal{Y}}{\langle 3p_1 4 \rangle} \right]^4 \\ &= \frac{2e^2}{(2m^2)^4 (t - m^2)(u - m^2)} \left[\frac{\mathcal{Y}^4}{\langle 3p_1 4 \rangle^2} - \frac{4\mathcal{X}\mathcal{Y}^3}{\langle 3p_1 4 \rangle} + \dots \right], \end{aligned} \quad (\text{B.4})$$

where the dots stand for terms with no unphysical poles. One can cancel the double pole in $\langle 3p_1 4 \rangle$ by including in the $\mathcal{M}_{\text{UV}}$ the term $-\frac{2e^2}{(2m^2)^4(t-m^2)(u-m^2)}\frac{\mathcal{Y}^4}{\langle 3p_1 4 \rangle^2}$. This would however tamper with the t - and u -poles, which we want to avoid. The trick is to compensate powers of $\mathcal{Y}$ with powers of $\mathcal{Z}$ which, by the second identity in eq. (B.2), do not affect the residue at $\langle 3p_1 4 \rangle \rightarrow 0$. The crucial observation is that

$$\frac{\mathcal{Y}^2 - \mathcal{Z}^2}{(t-m^2)(u-m^2)} = 4\mathcal{A}\mathcal{B} \quad (\text{B.5})$$

does not have a pole at $t \rightarrow m^2$ or $u \rightarrow m^2$. Therefore, a UV term proportional to $\frac{\mathcal{Y}^2 - \mathcal{Z}^2}{(t-m^2)(u-m^2)}$ will affect the $\langle 3p_1 4 \rangle$ poles without destroying the correct structure of physical poles provided by the IR term. In our case, the double pole at $\langle 3p_1 4 \rangle \rightarrow 0$ is canceled if the UV term contains $\mathcal{M}_{\text{UV}} \supset -\frac{2e^2}{(2m^2)^4(t-m^2)(u-m^2)}\frac{\mathcal{Y}^4 - \mathcal{Z}^4}{\langle 3p_1 4 \rangle^2}$. In exactly the same way we can deal with the single $\langle 3p_1 4 \rangle$ pole in eq. (B.4). All in all, a legal UV term canceling all $\langle 3p_1 4 \rangle$ poles in the $S = 2$ Compton amplitude is given by

$$\begin{aligned} \mathcal{M}_{\text{UV}} &= -\frac{2e^2}{(2m^2)^4(t-m^2)(u-m^2)} \left\{ \frac{\mathcal{Y}^2(\mathcal{Y}^2 - \mathcal{Z}^2)}{\langle 3p_1 4 \rangle^2} - \frac{4\mathcal{X}\mathcal{Y}(\mathcal{Y}^2 - \mathcal{Z}^2)}{\langle 3p_1 4 \rangle} \right\} + C \\ &= -\frac{e^2\mathcal{A}\mathcal{B}}{2m^8} \left\{ \frac{\mathcal{Y}^2}{\langle 3p_1 4 \rangle^2} - \frac{4\mathcal{X}\mathcal{Y}}{\langle 3p_1 4 \rangle} \right\} + C, \end{aligned} \quad (\text{B.6})$$

where C is any pure contact term. This construction guarantees that $\mathcal{M}_{\text{IR}} + \mathcal{M}_{\text{UV}}$ has no unphysical poles, and that it has the same physical poles as $\mathcal{M}_{\text{IR}}$. The UV term is ultraviolet in two ways. One, it scales as $\mathcal{O}(\epsilon^1)$ in the (complex kinematics) limit where $t \rightarrow m^2$, $u \rightarrow m^2$, and $s \rightarrow 0$ which is the limit where the photons become soft. This is unlike the IR term which scales as $\mathcal{O}(\epsilon^{-2})$ in the soft limit. Two, $\mathcal{M}_{\text{UV}}$ grows for $E \gg m$ as $\mathcal{O}(E^8/m^8)$, unlike the IR term, which is regular at high energies. This growth leads to a loss of perturbative unitarity at the scale Λ close to the mass m of the particle X , which is what ultimately prevents the possibility of fundamental spin-2 massive particles. In fact, we can further massage eq. (B.6) so as to improve the UV behavior by one notch, so as to (slightly) delay the onset of strong coupling. Using the first identity in eq. (B.2) to eliminate $\mathcal{Y}$, and absorbing the resulting $\mathcal{X}^2$ terms into C , we end up with

$$\mathcal{M}_{\text{UV}} = -\frac{2e^2\mathcal{A}\mathcal{B}}{m^6} \left\{ \frac{m^2(\mathcal{A} + \mathcal{B})^2}{\langle 3p_1 4 \rangle^2} + \frac{\mathcal{X}(\mathcal{A} + \mathcal{B})}{\langle 3p_1 4 \rangle} \right\} + C'. \quad (\text{B.7})$$

This has a better UV behavior, $\mathcal{M}_{\text{UV}} \sim \mathcal{O}(E^6/m^6)$. Since the UV terms in eq. (B.6) and eq. (B.7) differ only by a contact term, they have exactly the same pole structure, and they both cancel the unphysical pole in $\mathcal{M}_{\text{IR}}$.

Following exactly the same steps one can construct $\mathcal{M}_{\text{UV}}$ for arbitrary $S > 1$:

$$\mathcal{M}_{\text{UV}} = -\frac{2e^2}{(2m^2)^{2S}(t-m^2)(u-m^2)} \sum_{k=3}^{2S} \binom{2S}{k} \frac{(-\mathcal{X})^{2S-k} \mathcal{Y} [\mathcal{Y}^{k-1} - \{\mathcal{Y}||\mathcal{Z}\} \mathcal{Z}^{k-2}]}{\langle 3p_1 4 \rangle^{k-2}}, \quad (\text{B.8})$$

where $\{\mathcal{Y}||\mathcal{Z}\} = \mathcal{Y}(\mathcal{Z})$ for even (odd) k . Note that $\mathcal{X}$ and $\mathcal{Z}$ are odd under $1 \leftrightarrow 2$, while $\mathcal{Y}$ is even. Therefore $\mathcal{M}_{\text{UV}}$ has the same symmetry under $1 \leftrightarrow 2$ as the IR term: it is

even (odd) for integer (half-integer) spin. The absence of physical poles in $\mathcal{M}_{\text{UV}}$ follows from $\mathcal{Y}^n - \mathcal{Z}^n = (\mathcal{Y}^2 - \mathcal{Z}^2)(\mathcal{Y}^{n-2} + \mathcal{Y}^{n-4}\mathcal{Z}^2 + \dots + \mathcal{Z}^{n-2})$ for any even $n \geq 2$. This is the most compact form of the UV term, but not the one with the best possible UV behavior. As in the case of $S = 2$, we can always soften the UV behavior from $\mathcal{O}(E^{4S}/m^{4S})$ down to $\mathcal{O}(E^{4S-2}/m^{4S-2})$ by using the first identity in eq. (B.2) and isolating the resulting $\mathcal{O}(E^{4S}/m^{4S})$ contact term. The consequence is that the maximum cutoff scale of an effective theory describing a spin- $S > 1$ particle with mass m and electric charge q satisfies

$$\Lambda_{\text{max}} \lesssim \left(\frac{4\pi}{|q|e} \right)^{\frac{1}{2S-1}}. \quad (\text{B.9})$$

This bound was obtained by other methods in ref. [53].

For Compton scattering via the electromagnetic dipole the IR amplitude can be read off from eq. (5.9):

$$\mathcal{M}_{\text{IR}} = |a|^2 \frac{(-1)^{2S} S e^2}{2m^2(t-m^2)(u-m^2)} \left[\frac{\mathcal{A} + \mathcal{B}}{\langle 3p_1 4 \rangle} \right]^{2S-2} \left\{ (\mathcal{Y} + \mathcal{Z})(\mathcal{A} + 2S\mathcal{B}) + (\mathcal{Y} - \mathcal{Z})(2S\mathcal{A} + \mathcal{B}) \right\}. \quad (\text{B.10})$$

The corresponding UV term can be easily constructed by the same techniques:

$$\begin{aligned} \mathcal{M}_{\text{UV}} = & -|a|^2 \frac{S e^2}{(2m^2)^{2S-1}(t-m^2)(u-m^2)} \sum_{k=1}^{2S-2} \binom{2S-2}{k} \frac{(-X)^{2S-2-k}}{\langle 3p_1 4 \rangle^k} \\ & \times \left\{ (\mathcal{Y}^k - \mathcal{Z}^k)(\mathcal{Y} + \mathcal{Z})(\mathcal{A} + 2S\mathcal{B}) + (\mathcal{Y}^k - (-\mathcal{Z})^k)(\mathcal{Y} - \mathcal{Z})(2S\mathcal{A} + \mathcal{B}) \right\}. \end{aligned} \quad (\text{B.11})$$

B.2 Gravitational Compton scattering

We turn to the analogue of the Compton process in gravity, where photons are replaced by massless spin-2 gravitons. We assume minimal gravitational interactions of a spin- S massive particle X . We focus on the opposite helicity Compton amplitude and again separate the UV and IR terms: $\mathcal{M}(\mathbf{1}_X \mathbf{2}_X \mathbf{3}_h^- \mathbf{4}_h^+) = \mathcal{M}_{\text{IR}} + \mathcal{M}_{\text{UV}}$. The IR term takes the form [22]:

$$\begin{aligned} \mathcal{M}_{\text{IR}} &= \frac{(\mathcal{A} + \mathcal{B})^4}{M_{\text{Pl}}^2 s(t-m^2)(u-m^2)} \left[\frac{\mathcal{A} + \mathcal{B}}{\langle 3p_1 4 \rangle} \right]^{2S-4} \\ &= \frac{\mathcal{A} + \mathcal{B}}{M_{\text{Pl}}^2 (2m^2)^{2S-2}} \sum_{k=0}^{2S-3} (-1)^k \binom{2S-3}{k} X^{2S-3-k} I_k, \end{aligned} \quad (\text{B.12})$$

where we abbreviated

$$I_k \equiv \frac{2m^2(\mathcal{A} + \mathcal{B})^2}{s(t-m^2)(u-m^2)} \frac{\mathcal{Y}^k}{\langle 3p_1 4 \rangle^{k-1}}, \quad (\text{B.13})$$

and $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}$ are defined in eq. (B.1). For $S > 2$ the IR term in eq. (B.12) develops a pole at $\langle 3p_1 4 \rangle \rightarrow 0$, therefore $\mathcal{M}_{\text{UV}}$ has to be non-zero for $S > 2$ so as to cancel the unphysical pole. Below we construct a befitting UV term. We will first find a set of functions U_k such

that $U_k + I_k$ is regular at $\langle 3p_14 \rangle \rightarrow 0$ and has the same kinematic poles as I_k . Given such a set, a legal UV term for $S > 2$ is simply

$$\mathcal{M}_{\text{UV}} = \frac{\mathcal{A} + \mathcal{B}}{M_{\text{Pl}}^2 (2m^2)^{2S-2}} \sum_{k=2}^{2S-3} (-1)^k \binom{2S-3}{k} X^{2S-3-k} U_k. \quad (\text{B.14})$$

We will construct U_k recursively. In this endeavour the following identity is instrumental:

$$(\mathcal{A} + \mathcal{B})^2 = \frac{1}{m^2} \left\{ s\mathcal{A}\mathcal{B} + m\langle 3p_14 \rangle (\mathcal{A} + \mathcal{B}) (\langle \mathbf{12} \rangle + [\mathbf{12}]) - \langle 3p_14 \rangle^2 \langle \mathbf{12} \rangle [\mathbf{12}] \right\}. \quad (\text{B.15})$$

This allows us to rewrite I_k as

$$I_k = \frac{2\mathcal{A}\mathcal{B}}{(t-m^2)(u-m^2)} \frac{\mathcal{Y}^k}{\langle 3p_14 \rangle^{k-1}} + c_k (\langle \mathbf{12} \rangle + [\mathbf{12}]) - d_k \langle \mathbf{12} \rangle [\mathbf{12}], \quad (\text{B.16})$$

where once again we introduced some auxiliary notation:

$$c_k \equiv \frac{2m(\mathcal{A} + \mathcal{B})}{s(t-m^2)(u-m^2)} \frac{\mathcal{Y}^k}{\langle 3p_14 \rangle^{k-2}}, \quad d_k \equiv \frac{2}{s(t-m^2)(u-m^2)} \frac{\mathcal{Y}^k}{\langle 3p_14 \rangle^{k-3}}. \quad (\text{B.17})$$

We can rearrange c_k as

$$c_k = \frac{2m(\mathcal{A} + \mathcal{B})}{s(t-m^2)(u-m^2)} \frac{\mathcal{Y}^{k-1}}{\langle 3p_14 \rangle^{k-2}} [-2m^2(\mathcal{A} + \mathcal{B}) + \mathcal{X}\langle 3p_14 \rangle] = -2mI_{k-1} + \mathcal{X}c_{k-1}. \quad (\text{B.18})$$

This recursive equation is solved by

$$c_k = -2m \sum_{l=1}^{k-2} I_{k-l} \mathcal{X}^{l-1} + \mathcal{X}^{k-2} c_2. \quad (\text{B.19})$$

We can handle d_k in a similar way

$$d_k \equiv \frac{2}{s(t-m^2)(u-m^2)} \frac{\mathcal{Y}^{k-2}}{\langle 3p_14 \rangle^{k-3}} [-2m^2(\mathcal{A} + \mathcal{B}) + \mathcal{X}\langle 3p_14 \rangle]^2 = 4m^2 I_{k-2} - 4m^2 \mathcal{X} c_{k-2} + \mathcal{X}^2 d_{k-2}. \quad (\text{B.20})$$

The solution is

$$\begin{aligned} d_k &= 4m^2 \sum_{l=1}^{(k-2)/2} (I_{k-2l} - \mathcal{X} c_{k-2l}) \mathcal{X}^{2(l-1)} + \mathcal{X}^{k-2} d_2, & \text{even } k, \\ d_k &= 4m^2 \sum_{l=1}^{(k-3)/2} (I_{k-2l} - \mathcal{X} c_{k-2l}) \mathcal{X}^{2(l-1)} + \mathcal{X}^{k-3} d_3, & \text{odd } k. \end{aligned} \quad (\text{B.21})$$

Plugging the solution back into the formula for I_k we obtain

$$\begin{aligned} I_k &= \frac{2\mathcal{A}\mathcal{B}}{(t-m^2)(u-m^2)} \frac{\mathcal{Y}^k}{\langle 3p_14 \rangle^{k-1}} - 2m(\langle \mathbf{12} \rangle + [\mathbf{12}]) \sum_{l=1}^{k-2} I_{k-l} \mathcal{X}^{l-1} \\ &\quad - 4m^2 \langle \mathbf{12} \rangle [\mathbf{12}] \sum_{l=1}^{(k-\{2|3\})/2} \left(I_{k-2l} + 2m\mathcal{X} \sum_{n=1}^{k-2l-2} I_{k-n} \mathcal{X}^{n-1} \right) \mathcal{X}^{2(l-1)} + \dots \end{aligned} \quad (\text{B.22})$$

where the dots stand for $d_{2,3}$ and c_2 terms, which do not have poles at $\langle 3p_14 \rangle$. We are ready to write down a recursive equation for U_k :

$$U_k = -\frac{2\mathcal{A}\mathcal{B}}{(t-m^2)(u-m^2)} \frac{\mathcal{Y}^k - \{\mathcal{Z}||Y\}\mathcal{Z}^{k-1}}{\langle 3p_14 \rangle^{k-1}} - 2m(\langle \mathbf{12} \rangle + [\mathbf{12}]) \sum_{l=1}^{k-2} U_{k-l} \mathcal{X}^{l-1} \\ - 4m^2 \langle \mathbf{12} \rangle [\mathbf{12}] \sum_{l=1}^{(k-\{2||3\})/2} \left(U_{k-2l} + 2m\mathcal{X} \sum_{n=1}^{k-2l-2} U_{k-n} \mathcal{X}^{n-1} \right) \mathcal{X}^{2(l-1)}, \quad (\text{B.23})$$

where $\{\mathcal{Z}||Y\} = \mathcal{Z}(\mathcal{Y})$ for even (odd) k , and $\{2||3\} = 2(3)$ for even (odd) l . An implicit convention above is that, if the upper limit of a summation is smaller than the lower limit, then the whole sum is void.

The recursive equation (B.23) supplemented by the boundary condition $U_0 = U_1 = 0$ defines U_k for arbitrary k . Via eq. (B.14), this allows one to construct an appropriate UV term for any spin S of the matter particle. Let us see how it works in some simple examples. For $k = 2$ we get

$$U_2 = -\frac{2\mathcal{A}\mathcal{B}}{(t-m^2)(u-m^2)} \frac{\mathcal{Y}^2 - \mathcal{Z}^2}{\langle 3p_14 \rangle} = -\frac{8\mathcal{A}^2\mathcal{B}^2}{\langle 3p_14 \rangle}. \quad (\text{B.24})$$

By eq. (B.14), the UV term for $S = 5/2$ reads

$$\mathbf{S} = \mathbf{5/2}: \quad \mathcal{M}_{\text{UV}} = \frac{\mathcal{A} + \mathcal{B}}{M_{\text{Pl}}^2(2m^2)^3} U_2 = -\frac{\mathcal{A}^2\mathcal{B}^2(\mathcal{A} + \mathcal{B})}{M_{\text{Pl}}^2 m^6 \langle 3p_14 \rangle}. \quad (\text{B.25})$$

For $k = 3$ eq. (B.23) yields

$$U_3 = -8\mathcal{A}^2\mathcal{B}^2 \left(\frac{\mathcal{Y}}{\langle 3p_14 \rangle^2} - \frac{2m(\langle \mathbf{12} \rangle + [\mathbf{12}])}{\langle 3p_14 \rangle} \right). \quad (\text{B.26})$$

By eq. (B.14), the UV term for $S = 3$ reads

$$\mathcal{M}_{\text{UV}} = \frac{\mathcal{A} + \mathcal{B}}{M_{\text{Pl}}^2(2m^2)^4} (3\mathcal{X}U_2 - U_3) \\ = -\frac{\mathcal{A}^2\mathcal{B}^2(\mathcal{A} + \mathcal{B})}{M_{\text{Pl}}^2 m^8} \left(\frac{\mathcal{X}}{\langle 3p_14 \rangle} + \frac{m(\langle \mathbf{12} \rangle + [\mathbf{12}])}{\langle 3p_14 \rangle} + \frac{m^2(\mathcal{A} + \mathcal{B})}{\langle 3p_14 \rangle^2} \right). \quad (\text{B.27})$$

And so on.

From eq. (B.23) one can read off that U_k scales as $\mathcal{O}(E^{2(k+1)})$ at large energies. For Compton processes with energies much above the particle's mass one thus finds that the UV term grows with energy as

$$\mathcal{M}_{\text{UV}} \sim \mathcal{O} \left(\frac{E^{4S-2}}{m^{4S-4} M_{\text{Pl}}^2} \right), \quad E \gg m, \quad (\text{B.28})$$

for any $S > 2$. This violates perturbative unitarity at some energy scale above the particle's mass. One concludes that the maximum cutoff scale for an effective theory describing a particle of mass m and spin $S > 2$ satisfies

$$\Lambda_{\text{max}} \lesssim \left(4\pi M_{\text{Pl}} m^{2S-2} \right)^{\frac{1}{2S-1}}. \quad (\text{B.29})$$

This bound was conjectured in refs. [28, 54]. The bound is totally general and universal, since every particle has to be coupled to gravity. As the spin increases, the maximum cutoff quickly approaches m , squeezing the validity range of the effective theory to a very narrow energy interval. This imposes severe restrictions on constructing viable effective theories of higher-spin particles (see also refs. [55, 56] for other arguments leading to similar conclusions).

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] F.E. Low, *Bremsstrahlung of very low-energy quanta in elementary particle collisions*, *Phys. Rev.* **110** (1958) 974 [[INSPIRE](#)].
- [2] S. Weinberg, *Infrared photons and gravitons*, *Phys. Rev.* **140** (1965) B516 [[INSPIRE](#)].
- [3] J. Broedel, M. de Leeuw, J. Plefka and M. Rosso, *Constraining subleading soft gluon and graviton theorems*, *Phys. Rev. D* **90** (2014) 065024 [[arXiv:1406.6574](#)] [[INSPIRE](#)].
- [4] L. Rodina, *Scattering amplitudes from soft theorems and infrared behavior*, *Phys. Rev. Lett.* **122** (2019) 071601 [[arXiv:1807.09738](#)] [[INSPIRE](#)].
- [5] A. Strominger, *Lectures on the infrared structure of gravity and gauge theory*, [arXiv:1703.05448](#) [[INSPIRE](#)].
- [6] C. Cheung, K. Kampf, J. Novotny, C.-H. Shen and J. Trnka, *On-shell recursion relations for effective field theories*, *Phys. Rev. Lett.* **116** (2016) 041601 [[arXiv:1509.03309](#)] [[INSPIRE](#)].
- [7] C. Cheung, K. Kampf, J. Novotny, C.-H. Shen and J. Trnka, *A periodic table of effective field theories*, *JHEP* **02** (2017) 020 [[arXiv:1611.03137](#)] [[INSPIRE](#)].
- [8] C. Cheung, K. Kampf, J. Novotny, C.-H. Shen, J. Trnka and C. Wen, *Vector effective field theories from soft limits*, *Phys. Rev. Lett.* **120** (2018) 261602 [[arXiv:1801.01496](#)] [[INSPIRE](#)].
- [9] H. Elvang, M. Haddjantonis, C.R.T. Jones and S. Paranjape, *Soft bootstrap and supersymmetry*, *JHEP* **01** (2019) 195 [[arXiv:1806.06079](#)] [[INSPIRE](#)].
- [10] I. Low and Z. Yin, *Soft bootstrap and effective field theories*, *JHEP* **11** (2019) 078 [[arXiv:1904.12859](#)] [[INSPIRE](#)].
- [11] F. Cachazo and A. Strominger, *Evidence for a new soft graviton theorem*, [arXiv:1404.4091](#) [[INSPIRE](#)].
- [12] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [[hep-th/0501052](#)] [[INSPIRE](#)].
- [13] Z. Bern, L.J. Dixon, M. Perelstein and J.S. Rozowsky, *Multileg one loop gravity amplitudes from gauge theory*, *Nucl. Phys. B* **546** (1999) 423 [[hep-th/9811140](#)] [[INSPIRE](#)].
- [14] S. He, Y.-T. Huang and C. Wen, *Loop corrections to soft theorems in gauge theories and gravity*, *JHEP* **12** (2014) 115 [[arXiv:1405.1410](#)] [[INSPIRE](#)].
- [15] B. Sahoo and A. Sen, *Classical and quantum results on logarithmic terms in the soft theorem in four dimensions*, *JHEP* **02** (2019) 086 [[arXiv:1808.03288](#)] [[INSPIRE](#)].
- [16] H. Elvang, C.R.T. Jones and S.G. Naculich, *Soft photon and graviton theorems in effective field theory*, *Phys. Rev. Lett.* **118** (2017) 231601 [[arXiv:1611.07534](#)] [[INSPIRE](#)].

- [17] M. Bianchi and A.L. Guerrieri, *On the soft limit of closed string amplitudes with massive states*, *Nucl. Phys. B* **905** (2016) 188 [[arXiv:1512.00803](#)] [[INSPIRE](#)].
- [18] S. Atul Bhatkar and B. Sahoo, *Subleading soft theorem for arbitrary number of external soft photons and gravitons*, *JHEP* **01** (2019) 153 [[arXiv:1809.01675](#)] [[INSPIRE](#)].
- [19] A. Sen, *Subleading soft graviton theorem for loop amplitudes*, *JHEP* **11** (2017) 123 [[arXiv:1703.00024](#)] [[INSPIRE](#)].
- [20] A. Laddha and A. Sen, *Sub-subleading soft graviton theorem in generic theories of quantum gravity*, *JHEP* **10** (2017) 065 [[arXiv:1706.00759](#)] [[INSPIRE](#)].
- [21] S. Chakrabarti, S.P. Kashyap, B. Sahoo, A. Sen and M. Verma, *Subleading soft theorem for multiple soft gravitons*, *JHEP* **12** (2017) 150 [[arXiv:1707.06803](#)] [[INSPIRE](#)].
- [22] N. Arkani-Hamed, T.-C. Huang and Y.-T. Huang, *Scattering amplitudes for all masses and spins*, [arXiv:1709.04891](#) [[INSPIRE](#)].
- [23] D.A. Kosower, B. Maybee and D. O’Connell, *Amplitudes, observables, and classical scattering*, *JHEP* **02** (2019) 137 [[arXiv:1811.10950](#)] [[INSPIRE](#)].
- [24] A. Guevara, A. Ochirov and J. Vines, *Scattering of spinning black holes from exponentiated soft factors*, *JHEP* **09** (2019) 056 [[arXiv:1812.06895](#)] [[INSPIRE](#)].
- [25] A. Laddha and A. Sen, *Gravity waves from soft theorem in general dimensions*, *JHEP* **09** (2018) 105 [[arXiv:1801.07719](#)] [[INSPIRE](#)].
- [26] N. Arkani-Hamed, Y.-T. Huang and D. O’Connell, *Kerr black holes as elementary particles*, *JHEP* **01** (2020) 046 [[arXiv:1906.10100](#)] [[INSPIRE](#)].
- [27] J. Bonifacio and K. Hinterbichler, *Bounds on amplitudes in effective theories with massive spinning particles*, *Phys. Rev. D* **98** (2018) 045003 [[arXiv:1804.08686](#)] [[INSPIRE](#)].
- [28] J. Bonifacio and K. Hinterbichler, *Universal bound on the strong coupling scale of a gravitationally coupled massive spin-2 particle*, *Phys. Rev. D* **98** (2018) 085006 [[arXiv:1806.10607](#)] [[INSPIRE](#)].
- [29] J. Bonifacio, K. Hinterbichler and R.A. Rosen, *Constraints on a gravitational Higgs mechanism*, *Phys. Rev. D* **100** (2019) 084017 [[arXiv:1903.09643](#)] [[INSPIRE](#)].
- [30] A. Falkowski and G. Isabella, *Matter coupling in massive gravity*, *JHEP* **04** (2020) 014 [[arXiv:2001.06800](#)] [[INSPIRE](#)].
- [31] Y. Shadmi and Y. Weiss, *Effective field theory amplitudes the on-shell way: scalar and vector couplings to gluons*, *JHEP* **02** (2019) 165 [[arXiv:1809.09644](#)] [[INSPIRE](#)].
- [32] R. Aoude and C.S. Machado, *The rise of SMEFT on-shell amplitudes*, *JHEP* **12** (2019) 058 [[arXiv:1905.11433](#)] [[INSPIRE](#)].
- [33] G. Durieux, T. Kitahara, Y. Shadmi and Y. Weiss, *The electroweak effective field theory from on-shell amplitudes*, *JHEP* **01** (2020) 119 [[arXiv:1909.10551](#)] [[INSPIRE](#)].
- [34] M.-Z. Chung, Y.-T. Huang, J.-W. Kim and S. Lee, *The simplest massive S-matrix: from minimal coupling to black holes*, *JHEP* **04** (2019) 156 [[arXiv:1812.08752](#)] [[INSPIRE](#)].
- [35] A. Guevara, A. Ochirov and J. Vines, *Black-hole scattering with general spin directions from minimal-coupling amplitudes*, *Phys. Rev. D* **100** (2019) 104024 [[arXiv:1906.10071](#)] [[INSPIRE](#)].
- [36] C. Cheung, A. de la Fuente and R. Sundrum, *4D scattering amplitudes and asymptotic symmetries from 2D CFT*, *JHEP* **01** (2017) 112 [[arXiv:1609.00732](#)] [[INSPIRE](#)].

- [37] S. Pasterski and S.-H. Shao, *Conformal basis for flat space amplitudes*, *Phys. Rev. D* **96** (2017) 065022 [[arXiv:1705.01027](#)] [[INSPIRE](#)].
- [38] L. Donnay, A. Puhm and A. Strominger, *Conformally soft photons and gravitons*, *JHEP* **01** (2019) 184 [[arXiv:1810.05219](#)] [[INSPIRE](#)].
- [39] H.K. Dreiner, H.E. Haber and S.P. Martin, *Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry*, *Phys. Rept.* **494** (2010) 1 [[arXiv:0812.1594](#)] [[INSPIRE](#)].
- [40] S. Dittmaier, *Weyl-van der Waerden formalism for helicity amplitudes of massive particles*, *Phys. Rev. D* **59** (1998) 016007 [[hep-ph/9805445](#)] [[INSPIRE](#)].
- [41] E. Conde and A. Marzolla, *Lorentz constraints on massive three-point amplitudes*, *JHEP* **09** (2016) 041 [[arXiv:1601.08113](#)] [[INSPIRE](#)].
- [42] R. Franken and C. Schwinn, *On-shell constructibility of Born amplitudes in spontaneously broken gauge theories*, *JHEP* **02** (2020) 073 [[arXiv:1910.13407](#)] [[INSPIRE](#)].
- [43] S. Weinberg, *Photons and gravitons in S-matrix theory: derivation of charge conservation and equality of gravitational and inertial mass*, *Phys. Rev.* **135** (1964) B1049 [[INSPIRE](#)].
- [44] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [[hep-th/0312171](#)] [[INSPIRE](#)].
- [45] R.O. Hansen, *Multipole moments of stationary space-times*, *J. Math. Phys.* **15** (1974) 46 [[INSPIRE](#)].
- [46] A. Guevara, *Notes on conformal soft theorems and recursion relations in gravity*, [arXiv:1906.07810](#) [[INSPIRE](#)].
- [47] W.-M. Chen, Y.-T. Huang and C. Wen, *New fermionic soft theorems for supergravity amplitudes*, *Phys. Rev. Lett.* **115** (2015) 021603 [[arXiv:1412.1809](#)] [[INSPIRE](#)].
- [48] D. Jain and A. Rudra, *Leading soft theorem for multiple gravitini*, *JHEP* **06** (2019) 004 [[arXiv:1811.01804](#)] [[INSPIRE](#)].
- [49] T. Klose, T. McLoughlin, D. Nandan, J. Plefka and G. Travaglini, *Double-soft limits of gluons and gravitons*, [arXiv:1504.05558](#) [[INSPIRE](#)].
- [50] S. Chakrabarti, S.P. Kashyap, B. Sahoo, A. Sen and M. Verma, *Testing subleading multiple soft graviton theorem for CHY prescription*, *JHEP* **01** (2018) 090 [[arXiv:1709.07883](#)] [[INSPIRE](#)].
- [51] Y.T.A. Law and M. Zlotnikov, *Massive spinning bosons on the celestial sphere*, *JHEP* **06** (2020) 079 [[arXiv:2004.04309](#)] [[INSPIRE](#)].
- [52] H. Johansson and A. Ochirov, *Double copy for massive quantum particles with spin*, *JHEP* **09** (2019) 040 [[arXiv:1906.12292](#)] [[INSPIRE](#)].
- [53] M. Porrati and R. Rahman, *A model independent ultraviolet cutoff for theories with charged massive higher spin fields*, *Nucl. Phys. B* **814** (2009) 370 [[arXiv:0812.4254](#)] [[INSPIRE](#)].
- [54] R. Rahman, *Interacting massive higher spin fields*, Ph.D. thesis, [New York University](#), New York, NY, U.S.A. (2009).
- [55] S. Caron-Huot, Z. Komargodski, A. Sever and A. Zhiboedov, *Strings from massive higher spins: the asymptotic uniqueness of the Veneziano amplitude*, *JHEP* **10** (2017) 026 [[arXiv:1607.04253](#)] [[INSPIRE](#)].
- [56] N. Afkhami-Jeddi, S. Kundu and A. Tajdini, *A bound on massive higher spin particles*, *JHEP* **04** (2019) 056 [[arXiv:1811.01952](#)] [[INSPIRE](#)].