

Supergravity solution-generating techniques and canonical transformations of σ -models from $O(D, D)$

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ABSTRACT: Within the framework of the flux formulation of Double Field Theory (DFT) we employ a generalised Scherk-Schwarz ansatz and discuss the classification of the twists that in the presence of the strong constraint give rise to constant generalised fluxes interpreted as gaugings. We analyse the various possibilities of turning on the fluxes H_{ijk} , $F_{ij}{}^k$, $Q_i{}^{jk}$ and R^{ijk} , and the solutions for the twists allowed in each case. While we do not impose the DFT (or equivalently supergravity) equations of motion, our results provide solution-generating techniques in supergravity when applied to a background that does solve the DFT equations. At the same time, our results give rise also to canonical transformations of 2-dimensional σ -models, a fact which is interesting especially because these are integrability-preserving transformations on the worldsheet. Both the solution-generating techniques of supergravity and the canonical transformations of 2-dimensional σ -models arise as maps that leave the generalised fluxes of DFT and their flat derivatives invariant. These maps include the known abelian/non-abelian/Poisson-Lie T-duality transformations, Yang-Baxter deformations, as well as novel generalisations of them.

KEYWORDS: Sigma Models, String Duality, Supergravity Models

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1 Introduction

An important concept in physics is that of symmetry. Symmetries are powerful tools to constrain the form of solutions of physical theories, and they may be used as a guiding principle to construct new physical models. Certain symmetries are particularly useful because they connect physical solutions as well as different theories that are at first sight unrelated. A prominent example is T-duality in string theory, which provides such a connection between strings on backgrounds with very different geometry [1, 2].¹ In this paper we will employ generalised notions of T-duality that go beyond the case of reduction on a torus implemented by Buscher’s rules. When necessary we will refer to the latter case as “abelian” T-duality, to distinguish it from the “non-abelian” T-duality (NATD) transformation of [4–6]. NATD may be viewed as a generalisation where a set of *non-abelian* isometries of the string background are dualised to give rise to a new solution of the low-energy (super)gravity equations of motion, modulo a known anomalous-free condition [7, 8]. The transformation can be implemented by a Buscher-like gauging procedure of the initial isometries, similar to the abelian case. Global issues of NATD still need to be understood [6] and currently the transformation is viewed as a solution-generating technique rather than a fully-fledged symmetry of string theory [9, 10]. Similar comments apply to a further generalisation of NATD that goes under the name of “Poisson-Lie” T-duality (PLTD). In this case there is still a notion of dualisation of a symmetry algebra, but the symmetry does not need to be realised as an isometry of the initial background [11, 12]. This advantage puts the original and dual models on an equal footing at the expense of losing the interpretation of the gauging procedure. All these generalised notions of T-duality transformations were recently shown to admit α' -corrections that promote them to solution-generating techniques in the bosonic and heterotic strings at least to 2 loops (first order in α') [13–15].

Double Field Theory (DFT) [16–19] is an attempt to make (abelian) T-duality manifest at the level of the low-energy action, at the expense of doubling the coordinates x^m of the D -dimensional target-space by pairing them with a set of “dual” coordinates \tilde{x}_m in $X^M = (\tilde{x}_m, x^m)$. Diffeomorphisms, B-field gauge transformations and T-duality maps are combined into the $O(D, D)$ group, defined by the matrices $O_M{}^N$ that leave the $O(D, D)$ metric η_{MN} invariant

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^m{}_n \\ \delta_m{}^n & 0 \end{pmatrix}, \quad (1.1)$$

meaning that $O_M{}^P O_N{}^Q \eta_{PQ} = \eta_{MN}$. The action of DFT is manifestly invariant under constant $O(D, D)$ transformations of the coordinates X^M and of the dynamical fields of the theory, i.e. the generalised metric \mathcal{H}_{MN} and the generalised dilaton d , where the latter is an $O(D, D)$ scalar. The DFT action reduces to the low-energy effective action of the string when constraining the fields to depend only on the physical coordinates x^m , also known as the “strong constraint”. By relaxing the strong constraint DFT may also be viewed as a way to go beyond supergravity to provide a description of backgrounds that are locally non-geometric. Nevertheless, in this paper we will simply use DFT as a convenient

¹Recently a notion of “T-duality” for point particles was introduced in [3].

$O(D, D)$ -covariant rewriting of (super)gravity, and thus the strong constraint is always assumed. We will use in particular the so-called “flux formulation” of DFT [20] where the dynamical fields are the generalised dilaton d and a generalised vielbein E_A^M for the generalised metric, and where the equations of motion are written exclusively in terms of the generalised fluxes \mathcal{F}_{ABC} and \mathcal{F}_A and their (flat) derivatives.

(Super)gravity solution-generating techniques. One motivation of our work is the study and possibly the classification of solution-generating techniques in (super)gravity, where we focus on those that admit a description in terms of $O(D, D)$. In this case we assume that the starting point is a (super)gravity solution, say with Neveu-Schwarz-Neveu-Schwarz (NSNS) fields G_{mn}, B_{mn}, ϕ for the metric, Kalb-Ramond field and dilaton. The question is whether it is possible to construct a map from these fields to a new set of G'_{mn}, B'_{mn}, ϕ' that also give rise to a (super)gravity solution. Our strategy is to rewrite the D -dimensional fields in terms of the doubled fields of DFT, and demand that G_{mn}, B_{mn}, ϕ and G'_{mn}, B'_{mn}, ϕ' give rise to the same generalised fluxes and flat derivatives $\mathcal{F} = \mathcal{F}', \partial\mathcal{F} = \partial\mathcal{F}'$. This is a sufficient condition to have again a (super)gravity solution, and it is the mechanism that applies also for the generalised T-duality transformations.² As remarked, we are interested in classifying solution-generating techniques in (super)gravity rather than the actual (super)gravity solutions, and for this reason at no point we need to impose the DFT equations of motion.

Canonical transformations of σ -models. Another motivation of our work is the classification of canonical transformations of 2-dimensional σ -models, see [27] for a similar discussion and the recent [28] for the relation between the symplectomorphism approach to T-duality and geometric quantisation. Consider a σ -model $S = \int d^2\sigma \mathcal{L}$ whose Lagrangian can be put in the form³ $\mathcal{L} = \frac{1}{2}G_{mn}\dot{x}^m\dot{x}^n - \frac{1}{2}G_{mn}x'^m x'^n - B_{mn}\dot{x}^m x'^n$, which is the Polyakov action in conformal gauge. Introducing momenta p_m conjugate to x^m and going to the first-order formalism, the Lagrangian is $\mathcal{L} = p_m\dot{x}^m - \mathcal{H}$ with the Hamiltonian $\mathcal{H} = \frac{1}{2}\Psi^M \mathcal{H}_{MN} \Psi^N$, where one has $\Psi^M \equiv (p_m, x'^m)$ and

$$\mathcal{H}_{MN} \equiv \begin{pmatrix} G^{mn} & -(G^{-1}B)^m_n \\ (BG^{-1})^n_m & G_{mn} - (BG^{-1}B)_{mn} \end{pmatrix}, \tag{1.2}$$

which coincides with the known parametrisation commonly used for the generalised metric of DFT.⁴ Canonical Poisson brackets for x^m and p_m translate into Poisson

²Invariance of the generalised fluxes under NATD and PLTD was observed and used already in [13, 14, 21–24], and in the case of the Yang-Baxter deformations that we discuss later in [25, 26]. Ref. [15] advocated more general local $O(D, D)$ transformations leaving the generalised fluxes invariant.

³We are ignoring the overall string tension. A dot denotes derivatives with respect to the worldsheet time τ , and a prime with respect to the worldsheet spatial coordinate σ . In this case we limit the discussion to the classical level, and we therefore ignore the dilaton.

⁴If we further identify the momenta with the spatial derivatives of a dual set of coordinates $p_m = \tilde{x}'_m$ then $X^M = (\tilde{x}_m, x^m)$ and we have the action of Tseytlin’s double σ -model [29] $S = \int d^2\sigma \left(\frac{1}{2}\partial_1 X^M (\eta_{MN} + \Omega_{MN}) \partial_0 X^N - \frac{1}{2}\partial_1 X^M \mathcal{H}_{MN} \partial_1 X^N \right)$, where the matrix $\Omega_{MN} \equiv \begin{pmatrix} 0 & \delta^n_m \\ -\delta^n_m & 0 \end{pmatrix}$ denotes the topological term used also in [30]. In this discussion we want to look at the standard σ -model in the Hamiltonian formalism and we therefore refrain from going to the doubled σ -model.

brackets⁵ for Ψ_M , and redefining the phase-space variables as $\Psi_A = E_A^M \Psi_M$ via a generalised vielbein E_A^M for the generalised metric \mathcal{H}_{MN} , one has

$$\{\Psi_A(\sigma_1), \Psi_B(\sigma_2)\} = -\mathcal{F}_{AB}{}^C(\sigma_1)\Psi_C(\sigma_1)\delta_{12} + \eta_{AB}\partial_{\sigma_1}\delta(\sigma_1 - \sigma_2), \quad (1.3)$$

where the generalised fluxes \mathcal{F}_{ABC} appear again. Therefore when two different σ -models admit a rewriting of the phase-space variables in terms of Ψ_A and Ψ'_A respectively, and when they give rise to the same generalised fluxes \mathcal{F}_{ABC} when computing the Poisson brackets, the two σ -models are related by a canonical transformation — *possibly up to zero modes*. This last remark is a consequence of the fact that only the *spatial derivative* of the coordinates x^m appear in Ψ_M , and therefore with the above argument one is not able to claim that the zero-mode contribution to the Poisson brackets remains invariant under the map. To use a uniform terminology throughout the paper we will use the term “solution-generating techniques” notwithstanding that we have in mind canonical transformations of σ -models as well.

Integrable σ -models. Worldsheet (classical) integrability is a statement about the on-shell 2-dimensional σ -model, whose equations of motion can be put in the form of a flatness condition for an object known as the “Lax connection”.⁶ Canonical transformations provide on-shell identifications of the two σ -models, so that if one of the two models admits a formulation in terms of a Lax connection, one can use the canonical transformation to construct the Lax connection for the other σ -model, and argue integrability also in that case. Integrability has played an important role in the understanding of the AdS_5/CFT_4 correspondence [31], and much progress was reached also in lower dimensional holographic examples. Being able to generate new supergravity solutions (for example starting from $AdS_5 \times S^5$) that retain worldsheet integrability gives hope of applying exact methods to non-maximally supersymmetric backgrounds. Additionally, at least when the transformations can be understood as deformations of the original model, this motivates also the search of the corresponding transformation of the holographically-dual conformal field theory (in this example $\mathcal{N} = 4$ super Yang-Mills). Yang-Baxter (YB) deformations are integrability-preserving solution-generating techniques that recently have been extensively studied from various points of view, and that started to be relevant for the superstring in [32, 33]. At least the so-called “homogeneous” YB deformations can be interpreted as solution-generating techniques in the sense of this paper, since they leave the generalised fluxes invariant. The original formulation, the so-called “ η -model” or “inhomogeneous” YB-model

⁵The Poisson brackets are $\{\Psi_M(\sigma_1), \Psi_N(\sigma_2)\} = \frac{1}{2}\eta_{MN}(\partial_1 - \partial_2)\delta_{12} - \frac{1}{2}\Omega_{MN}(\partial_1 + \partial_2)\delta_{12}$ where we use the shorthand notation $\partial_{1,2} = \partial_{\sigma_{1,2}}$, $\delta_{12} = \delta(\sigma_1 - \sigma_2)$, and Ω_{MN} was defined in footnote 4. With appropriate boundary conditions $(\partial_1 + \partial_2)\delta_{12}$ vanishes in the sense of distributions and the Poisson brackets can be rewritten as $\{\Psi_M(\sigma_1), \Psi_N(\sigma_2)\} = \eta_{MN}\partial_{\sigma_1}\delta(\sigma_1 - \sigma_2)$, which is the expression that is commonly used and the one we use in the text. On the line ($\sigma \in \mathbb{R}$) test functions should be smooth functions with compact support, and on the circle ($\sigma \in S^1$) they should be periodic. In the case of the circle one cannot drop terms with $(\partial_1 + \partial_2)\delta_{12}$ if the boundary conditions for test functions allow for non-trivial winding. We refer to [27] for a discussion that uses the original form of the Poisson brackets.

⁶This is a weak notion of integrability, since in general one should argue that the full tower of conserved charges obtained from the monodromy matrix are in involution.

constructed in [34, 35] will also be discussed later. The homogeneous YB deformations are based on solutions of the classical YB equation on the algebra of isometries of the original background, and they can be applied to generic isometric backgrounds [36] including the integrable $AdS_5 \times S^5$ background, giving rise to generalisations of TsT transformations [37]. Preliminary proposals for the corresponding deformations of $\mathcal{N} = 4$ super Yang-Mills were put forward in [38, 39]. As remarked, we are interested in classifying integrability-preserving transformations of 2-dimensional σ -models, rather than the actual integrable models, and for this reason at no point we need to impose the existence of a Lax connection.

α' -corrections. The low-energy effective action of the string has higher-derivative α' -corrections that admit an $O(D, D)$ -covariant formulation at least to 2-loop order.⁷ In particular, as at leading order in α' , the α' -corrected equations of motion of DFT may still be written in the flux formulation in terms of the generalised fluxes and their flat derivatives exclusively. This observation can be used to extend the solution-generating techniques that we classify to higher orders in α' . This strategy was first employed in [26] to obtain the first α' -correction for homogeneous YB deformations, and later for NATD and PLTD⁸ in [13–15]. As argued in [15] the same methods can be applied to more general $O(D, D)$ -covariant solution-generating techniques, as the ones that we consider in this paper.

Summary of the paper. Starting with section 2, we will employ an ansatz for the generalised vielbeins that is known in the literature as “generalised Scherk-Schwarz reductions” [44–46]. Under this ansatz we identify a d -dimensional subspace of the full D -dimensional spacetime, so that we can discuss the more general case of solution-generating techniques acting non-trivially in $d \leq D$ dimensions. Our discussion will be local and we will not discuss global issues of the solutions. We will further restrict ourselves to the case of *constant* generalised fluxes in d dimensions. This set-up encompasses the generalised T-dualities and the prominent integrable deformations so far considered in the literature. Subsequently, we discuss the $O(d, d)$ parametrisation of the “twist” used in the reduction, as well as the constraints and the redundancies that arise. We finally explain methods that will turn out to be useful to treat in particular cases with non-vanishing H -flux. In section 3 we present the classification of the “orbits”, namely the possibilities of turning on the different components of the generalised flux \mathcal{F}_{IJK} (usually denoted as $H_{ijk}, F_{ij}{}^k, Q_i{}^{jk}, R^{ijk}$), and their “representatives”, i.e. the solutions for the twists that they allow. While we initially focus on the fields of the NSNS sector, in section 4 we discuss the Ramond-Ramond (RR) fields of the type II superstring as well. We finish in section 5 with conclusions and an outlook. In appendix A we collect our conventions on notation, in appendix B we give a brief recap on some aspects of DFT and gauged DFT that are relevant for our discussion, in appendix C we discuss how to obtain the parametrisation of the twist that we use, in appendix D we give more details on the formulations to include the RR fields of type II, in appendix E we review the DFT equations of motion in the flux

⁷Recently a tension was identified at 4 loops for the quartic-Riemann terms multiplied by $\zeta(3)$ that appear for the (super)string [40].

⁸For applications of PLTD to obtain string solutions and for different approaches to discuss the α' -corrections when applying PLTD see for example [41–43] and references therein.

formulation, and in appendix F we report on other attempts we made to treat orbits with non-vanishing H -flux.

2 Reduction ansatz and constant fluxes

We start by discussing a specific ansatz for the generalised vielbein of DFT. In turn this implies a specific ansatz for the NSNS fields (metric, B-field and dilaton) of the class of backgrounds that we consider. We refer to appendix B for some definitions and details that may be helpful for readers not familiar with DFT.

2.1 Generalised Scherk-Schwarz ansatz

To be more general, we assume that our backgrounds are parametrised by coordinates $\mathbf{x}^m = (\dot{x}^{\dot{\mu}}, y^\mu)$, where we take $m = 0, \dots, D - 1$ and μ takes $d \leq D$ values. In principle y^μ may include the time direction. In this splitting of coordinates $\dot{x}^{\dot{\mu}}$ will play the role of “spectators” — we will never specify the $\dot{x}^{\dot{\mu}}$ dependence of the solution and $\dot{x}^{\dot{\mu}}$ will not participate in the solution-generating technique. The interesting discussion will therefore involve only the coordinates y^μ . From now on, we will use a boldface notation for coordinates and fields of the full D -dimensional spacetime, to distinguish them from the coordinates and fields of the d -dimensional spaces. We do not make any assumptions regarding their global properties, in fact our discussion will be valid only in local patches. We refer to [47] for a discussion on how to construct generalised Leibniz parallelisable spaces from generalised Scherk-Schwarz uplifts of gauged supergravities.

The ansatz we take for the generalised vielbein and generalised dilaton is [44–46]

$$\mathbf{E}_A{}^M(\mathbf{x}) = \dot{\mathbf{E}}_A{}^I(\dot{x}) \mathbf{U}_I{}^M(y), \quad \mathbf{d}(\mathbf{x}) = \dot{d}(\dot{x}) + \lambda(y). \quad (2.1)$$

The factorisation of the dependence on the coordinates is what plays a crucial role. We will use a dot for fields depending only on spectator coordinates. In the following we will distinguish between M, N, \dots indices and I, J, \dots indices, and similarly for the boldface version, and the reason will be clarified in section 2.3. See also appendix A for a recap of our conventions on notation. The matrix \mathbf{U} is in general an element of $O(D, D)$, and \mathbf{U} and λ are usually called *twists*. Because of the role of spectators of the coordinates $\dot{x}^{\dot{\mu}}$, it is natural to take \mathbf{U} of block form, not mixing $\dot{\mu}$ and μ directions, and acting as the identity in the spectator block, so that⁹

$$\mathbf{U} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}, \quad (2.2)$$

⁹Equation (2.1) is rather natural and it is hard to imagine a more general ansatz if we want to exploit $O(D, D)$. A possible generalisation of (2.2) is to replace the identity matrix in the spectator block with another $O(D - d, D - d)$ element. The simplest example is

$$U_I{}^M = \begin{pmatrix} e^{-\gamma(y)} \delta^{\dot{\mu}}{}_{\dot{\nu}} & 0 \\ 0 & e^{\gamma(y)} \delta_{\dot{\mu}}{}^{\dot{\nu}} \end{pmatrix},$$

where γ plays the role of a generically y -dependent warping factor in front of $\mathbf{G}_{\dot{\mu}\dot{\nu}}$ and $\mathbf{B}_{\dot{\mu}\dot{\nu}}$, see [45, 48].

where we assume that the indices are placed as U_I^M and U_I^M , and

$$\mathbf{a} = \begin{pmatrix} 1_{D-d} & 0 \\ 0 & a \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1_{D-d} & 0 \\ 0 & d \end{pmatrix}. \quad (2.3)$$

In the rest of this paper we will therefore work with the twist matrix $U_I^M \in O(d, d) \subset O(D, D)$. This set-up is in fact known as “generalised Scherk-Schwarz compactifications” and used also in gauged DFT, see appendix B for a short recap of certain aspects. Let us also mention that if we define the generalised metric $\mathcal{H}^{MN} = E_A^M \mathcal{H}^{AB} E_B^N$ and parametrise it as in (1.2) in terms of G_{mn} and B_{mn} (and similarly for the dotted fields) then equation (2.1) implies that $M \equiv G - B$ is of the form

$$M = (\dot{M}\mathbf{b} + \mathbf{d})^{-1}(\dot{M}\mathbf{a} + \mathbf{c}), \quad (2.4)$$

where $\dot{M} \equiv \dot{G} - \dot{B}$. This is the known transformation of M under $O(D, D)$ transformations. In other words the generalised Scherk-Schwarz ansatz is selecting a class of backgrounds with metric, B-field (and dilaton) of a specific form.

Because of the above ansatz, the Weitzenböck connection constructed out of U_I^M is non-vanishing only when the IJK indices are of the type IJK , i.e. they are vector indices of the $O(d, d) \subset O(D, D)$ subgroup. The generalised fluxes then become

$$\begin{aligned} \mathcal{F}_{ABC} &= \dot{\mathcal{F}}_{ABC} + \dot{E}_A^I \dot{E}_B^J \dot{E}_C^K \mathcal{F}_{IJK}, \\ \mathcal{F}_A &= \dot{\mathcal{F}}_A + \dot{E}_A^I \mathcal{F}_I, \end{aligned} \quad (2.5)$$

where $\dot{\mathcal{F}}_{ABC} = 3\dot{\Omega}_{[ABC]}$, $\dot{\Omega}_{ABC} = \dot{E}_A^\mu \partial_\mu \dot{E}_B^J \dot{E}_C^K \mathcal{F}_{IJK}$, $\dot{\mathcal{F}}_A = \dot{\Omega}^B{}_{BA} + 2\dot{E}_A^\mu \partial_\mu \dot{d}$, and

$$\mathcal{F}_{IJK} = 3\Omega_{[IJK]}, \quad \mathcal{F}_I = \Omega^J{}_{JI} + 2U_I^\mu \partial_\mu \lambda, \quad \Omega_{IJK} = U_I^\mu \partial_\mu U_J^N U_{KN}. \quad (2.6)$$

After defining the flat derivative $\partial_A \equiv E_A^M \partial_M$ one finds that

$$\begin{aligned} \partial_A \mathcal{F}_{BCD} &= \dot{E}_A^\mu \partial_\mu \mathcal{F}_{BCD} + \dot{E}_A^I \dot{E}_B^J \dot{E}_C^K \dot{E}_D^L U_I^\mu \partial_\mu \mathcal{F}_{JKL}, \\ \partial_A \mathcal{F}_B &= \dot{E}_A^\mu \partial_\mu \mathcal{F}_B + \dot{E}_A^I \dot{E}_B^J U_I^\mu \partial_\mu \mathcal{F}_J. \end{aligned} \quad (2.7)$$

We will now further restrict the assumptions behind our calculations, and present the constraints coming from Bianchi identities.

2.2 Constant fluxes and pre-Roytenberg algebra

In general $\mathcal{F}_{IJK}, \mathcal{F}_I$ may be y -dependent but in this paper we assume them to be constant. One motivation comes from interpreting the components of the 3-form generalised flux as the coefficients defining the brackets of a Courant algebroid [49–51] on the Roytenberg algebra [52, 53]. In general the Jacobiator of these brackets is non-vanishing, as can be checked in an $O(d, d)$ -covariant way by calculating the Jacobiator of the C-bracket (see appendix B for its definition) of the generalised vielbeins (here the twists U) [20]. In fact

$$\left[[U_I, U_J]_{(C)}, U_K \right]_{(C)}^M = \left[\mathcal{F}_{IJ}{}^L U_L, U_K \right]_{(C)}^M = \left(\mathcal{F}_{IJ}{}^H \mathcal{F}_{KLH} - \partial_K \mathcal{F}_{IJL} + \frac{1}{2} \partial_L \mathcal{F}_{IJK} \right) U^{LM}, \quad (2.8)$$

and the Jacobiator is

$$\text{Jac}(U_I, U_J, U_K)^M = \left(-\mathcal{Z}_{IJKL} + \frac{1}{2} \partial_L \mathcal{F}_{IJK} \right) U^{LM}. \quad (2.9)$$

On the strong constraint the Bianchi identities for the generalised fluxes imply $\mathcal{Z}_{IJKL} = 0$ (see appendix B). If we in addition want to deal with the structure of a Lie algebra rather than the one of a Courant algebroid, it is natural to take the generalised fluxes constant, $\partial_L \mathcal{F}_{IJK} = 0$, such that the Jacobiator vanishes. The assumption of constant fluxes (both \mathcal{F}_{IJK} and \mathcal{F}_I) will simplify considerably our calculations to identify the solution-generating techniques, and ultimately this is the main motivation for this assumption. In particular it will be easier to find different twists (U, λ) and (U', λ') that give rise to the same generalised fluxes $\mathcal{F}_{IJK} = \mathcal{F}'_{IJK}$, $\mathcal{F}_I = \mathcal{F}'_I$, and their constancy immediately implies that not only \mathcal{F}_{ABC} , \mathcal{F}_A but also their flat derivatives remain the same, see (2.7).¹⁰ Let us remark that demanding the invariance of \mathcal{F}_{IJK} , \mathcal{F}_I is a stronger condition compared to the invariance of \mathcal{F}_{ABC} , \mathcal{F}_A , and relaxing this assumption may lead to interesting generalisations on which we will comment in the conclusions.

Let us now discuss the constraints coming from Bianchi identities for the generalised fluxes \mathcal{F}_{IJK} and \mathcal{F}_I . When these are constant the Bianchi identities (on the strong constraint) reduce to

$$\mathcal{F}_{[IJ}{}^H \mathcal{F}_{K]LH} = 0, \quad \mathcal{F}^K \mathcal{F}_{IJK} = 0, \quad \mathcal{F}^I \mathcal{F}_I - \frac{1}{6} \mathcal{F}^{IJK} \mathcal{F}_{IJK} = 0. \quad (2.10)$$

The first can be understood as the Jacobi identity for a Lie algebra, while the rest are constraints involving also \mathcal{F}_I . We introduce Lie algebra generators $\mathbb{T}_I = (\tilde{T}^i, T_i)$ so that $[\mathbb{T}_I, \mathbb{T}_J] = \mathcal{F}_{IJ}{}^K \mathbb{T}_K$, where indices are raised and lowered with the ad-invariant metric,

$$\eta_{IJ} = \begin{pmatrix} 0 & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix}, \quad (2.11)$$

that corresponds to the pairing $\langle\langle \mathbb{T}_I, \mathbb{T}_J \rangle\rangle = \eta_{IJ}$. We adopt the usual notation for the components of the 3-form flux \mathcal{F}_{IJK} when choosing upper or lower indices $\mathcal{F}_{ijk} = H_{ijk}$, $\mathcal{F}_{ij}{}^k = F_{ij}{}^k$, $\mathcal{F}_i{}^{jk} = Q_i{}^{jk}$, $\mathcal{F}^{ijk} = R^{ijk}$. Notice that such an algebra is unimodular $\mathcal{F}_{IJ}{}^J = 0$. The commutation relations of the ‘‘pre-Roytenberg algebra’’¹¹ spanned by \mathbb{T}_I , and that we denote by \mathfrak{t} , read

$$[T_i, T_j] = F_{ij}{}^k T_k + H_{ijk} \tilde{T}^k, \quad [T_i, \tilde{T}^j] = Q_i{}^{jk} T_k - F_{ik}{}^j \tilde{T}^k, \quad [\tilde{T}^i, \tilde{T}^j] = Q_k{}^{ij} \tilde{T}^k + R^{ijk} T_k. \quad (2.12)$$

The Lie algebra of the Drinfel’d double is obtained by setting $H = R = 0$ and keeping F and Q . When only $R = 0$ the above Lie algebra is known as *quasi-Manin* triple. In terms

¹⁰In the context of gauged supergravities the constancy of the generalised fluxes is imposed to make sure that the $(D-d)$ -dimensional theory does not depend on the coordinates y . Our motivation is different and at least in principle there would be nothing wrong with taking \mathcal{F}_{IJK} , \mathcal{F}_I to be y -dependent.

¹¹We follow the terminology usually employed in the literature, so that ‘‘Roytenberg algebra’’ is used in the case of the Courant algebroid where \mathcal{F}_{IJK} are generically non-constant, while ‘‘pre-Roytenberg algebra’’ is just a Lie algebra with a pairing η_{IJ} that splits generators as $\mathbb{T}_I = (\tilde{T}^i, T_i)$.

2.3 $O(d, d)$ parametrisation of the twist U

Under mild assumptions (see [54] and appendix C) the most general $O(d, d)$ parametrisation of the twist U is

$$U_I^M = \begin{pmatrix} \delta_j^i & \beta^{ij} \\ 0 & \delta_i^j \end{pmatrix} \begin{pmatrix} \delta_k^j & 0 \\ b_{jk} & \delta_j^k \end{pmatrix} \begin{pmatrix} (\rho^t)^k{}_\mu & 0 \\ 0 & (\rho^{-1})_k{}^\mu \end{pmatrix}, \quad (2.20)$$

where β^{ij} and b_{ij} are antisymmetric ($\beta^{ij} = -\beta^{ji}$, $b_{ij} = -b_{ji}$) and ρ_μ^j are components of the matrix $\rho \in \text{GL}(d)$. From now on we will use the shorthand notation $\rho_i{}^\mu \equiv (\rho^{-1})_i{}^\mu$, so that the placing of indices will indicate if we use ρ or its inverse. Our motivation is to find *different* U, U' and λ, λ' that give rise to the same generalised fluxes $\mathcal{F}_{IJK}, \mathcal{F}_I$. Notice however that twists that are related by the group of allowed gauge transformations — i.e. $\text{GL}(d)$ diffeomorphisms and gauge transformations of the B-field in d dimensions — should not be considered as genuinely different. We will call H_{geom} this group. Therefore, rather than $O(d, d)$, we should take U to belong to the coset $O(d, d) \setminus H_{\text{geom}}$. Under a $\text{GL}(d)$ diffeomorphism one has the transformation $U_I^M \rightarrow U_I^N R_N^M$ where [55]

$$R_M^N = \begin{pmatrix} (r^t)^\mu{}_\nu & 0 \\ 0 & (r^{-1})_\mu{}^\nu \end{pmatrix} \quad \text{with} \quad (r^{-1})_\mu{}^\nu = \partial_\mu V^\nu. \quad (2.21)$$

Therefore, we see that the action of $\text{GL}(d)$ diffeomorphisms is all reabsorbed in the transformation of ρ in (2.20), meaning $\rho_i{}^\mu \rightarrow \rho_i{}^\nu \partial_\nu V^\mu$. This also implies that β and b in (2.20), while they may be y -dependent, they must transform as scalars under $\text{GL}(d)$ diffeomorphisms. This is the reason why we prefer to distinguish between i, j and μ, ν indices, as well as I, J and M, N for the double indices. Gauge transformations of the B-field in d dimensions are obtained by the transformations $U_I^M \rightarrow U_I^N J_N^M$ where [55]

$$J_M^N = \begin{pmatrix} \delta^\mu{}_\nu & 0 \\ x_{\mu\nu} & \delta_\mu{}^\nu \end{pmatrix}, \quad \text{with} \quad x_{\mu\nu} = \partial_{[\mu} \xi_{\nu]}. \quad (2.22)$$

It is easy to see that this transformation is completely reabsorbed just by the redefinition of b in (2.20) as $b_{ij} \rightarrow b_{ij} + \rho_i{}^\mu \rho_j{}^\nu \partial_{[\mu} \xi_{\nu]}$. To conclude, when we say that U and U' must be different we mean that they must be different elements of the coset $O(d, d) \setminus H_{\text{geom}}$, or in other words that there exists no element $h \in H_{\text{geom}}$ such that $U = U'h$.

An important feature in our discussion is that the ansatz (2.1) is invariant under the redefinition

$$\dot{E}_A^I \rightarrow \dot{E}_A^J V_J^I, \quad U_I^M \rightarrow V^J{}_I U_J^M, \quad (2.23)$$

with $V_I^J \in O(D, D)$ constant and decomposed in terms of a constant matrix $V_I^J \in O(d, d)$ similar as U in equations (2.2) and (2.3). This redefinition by V_J^I does not give rise to new backgrounds, because it signals only the redundancy of the freedom in the decomposition between \dot{E} and U in (2.1). However, under the rotation by V the generalised fluxes will be rotated as $\mathcal{F}'_{IJK} = V_I^U V_J^V V_K^W \mathcal{F}_{UVW}$ and $\mathcal{F}'_I = V_I^U \mathcal{F}_U$. This shows that different forms of the commutation relations of the pre-Roytenberg algebra \mathfrak{r} may be in fact related by rigid $O(d, d)$ transformations, and therefore they need to be considered physically equivalent. Hence, orbits are classified by *equivalent classes*, where the equivalence is given by a rigid

$O(d, d)$ change of basis. Nevertheless, in section 3 we will list them in the intuitive way of figure 1, since the grouping into equivalent classes may require additional knowledge on the pre-Roytenberg algebra. On the other hand, given a background with $\dot{\mathbf{E}}_A^I$ and U_I^M , if we are able to find a U_I^M that yields the same generalised fluxes *up to* a rigid $O(d, d)$ transformation, then we can apply a compensating transformation on $\dot{\mathbf{E}}_A^I$ to make sure that $\mathcal{F}_{ABC}, \mathcal{F}_A$ remain invariant.

Another interesting rigid $O(d, d)$ transformation, that now can give rise to new inequivalent representatives for the twist U , is

$$U_I^M \rightarrow \tilde{U}_I^M = W_I^J U_J^M, \tag{2.24}$$

where again W_I^J is decomposed in terms of a constant matrix $W_I^J \in O(d, d)$ similar as U in equations (2.2) and (2.3), and W_I^J implements an automorphism of the pre-Roytenberg algebra \mathfrak{r} . Notice that here we are not transforming $\dot{\mathbf{E}}_A^I$. In particular, the fact that W is an automorphism, i.e. it satisfies

$$W_I^{I'} W_J^{J'} W_K^{K'} \mathcal{F}_{I'J'K'} = \mathcal{F}_{IJK}, \tag{2.25}$$

guarantees that the transformation (2.24) leaves the generalised fluxes \mathcal{F}_{IJK} invariant. When studying (super)gravity solution-generating techniques we additionally require the automorphism to satisfy $W_I^J \mathcal{F}_J = \mathcal{F}_I$. In this case, if U_I^M is a representative, then the transformation (2.24) generates a new representative $\tilde{U}_I^M = W_I^J U_J^M$. Let us point out that — since $W \in O(d, d)$ — after this transformation the twist can still be parametrized as in (2.20). When viewing the transformation as a solution-generating technique, it will be important to understand in which cases W gives rise to gauge transformations in H_{geom} . This may be complicated to discuss in general, but in certain explicit examples particular conclusions can be made. For instance in the (F) -orbit, modding out by $GL(d)$ transformations requires the automorphism W to be *outer*. Interestingly, when \mathfrak{r} is not semisimple outer automorphisms may also involve continuous parameters. For example, given a 2-cocycle $\omega : \mathfrak{r} \rightarrow \mathfrak{r}$ (see section 3.2 for the definition), then $W = \exp(\zeta\omega)$ is an automorphism of the algebra with $\zeta \in \mathbb{R}$ a continuous parameter. In order to avoid having W inner we have to impose that ω is not coboundary. Notice finally that using (2.23), the transformation by W in (2.24) may equally be seen as leaving the twist U invariant but transforming the spectator contribution $\dot{\mathbf{E}}_A^I$ of the background as $\dot{\mathbf{E}}_A^I \rightarrow \dot{\mathbf{E}}_A^J W^I_J$.

In general $O(d, d)$ transformations with “off-diagonal” components will reshuffle the types of fluxes in complicated ways. Given the commutation relations (2.12), an $O(d, d)$ redefinition $\mathbb{T}'_I = h_I^J \mathbb{T}_J$ with¹³

$$V_I^J = \begin{pmatrix} \delta^i_j + (\beta b)^i_j & \beta^{ij} \\ b_{ij} & \delta_i^j \end{pmatrix}, \tag{2.26}$$

¹³This is (2.20) where we take $\rho = 1$ since it does not rotate the fluxes. However here b and β are *constant* antisymmetric matrices.

gives rise to the rotated structure constants $[T'_i, T'_j] = F'^{jk}T'_k + H'_{ijk}\tilde{T}'^k$, $[T'_i, \tilde{T}'^j] = Q'^{jk}T'_k - F'^{jk}\tilde{T}'^k$, $[\tilde{T}'^i, \tilde{T}'^j] = Q'^{ij}\tilde{T}'^k + R'^{ijk}T'_k$, where

$$\begin{aligned}
 H'_{ijk} &= H_{ijk} + 3F_{[ij}{}^l b_{k]l} - 3b_{l[i} b_{j]m} Q_{|k]}{}^{lm} + b_{il} b_{jm} b_{kn} R^{lmn}, \\
 F'^{jk} &= F_{ij}{}^k - 3F_{[ij}{}^l b_{m]l} \beta^{mk} - H_{ijl} \beta^{lk} + 2b_{l[i} Q_{j]}{}^{lk} + 3b_{l[i} b_{j]m} Q_{|n]}{}^{lm} \beta^{nk} - b_{il} b_{jm} b_{pn} R^{lmn} \beta^{pk} \\
 &\quad + b_{il} b_{jm} R^{lmk}, \\
 Q'^{jk} &= Q_i{}^{jk} + 4b_{l[i} Q_{m]}{}^{lk} \beta^j{}^m + 3b_{l[i} b_{m]p} Q_{|n]}{}^{lp} \beta^{nj} \beta^{mk} - 2F_{il}{}^{[j} \beta^{k]l} + 3F_{[in}{}^l b_{m]l} \beta^{nj} \beta^{mk} \\
 &\quad + H_{imn} \beta^{jm} \beta^{kn} + b_{il} R^{ljk} - 2b_{il} b_{mn} R^{ln[j} \beta^{k]m} + \beta^{l[j} \beta^{k]n} b_{lm} b_{np} R^{mpq} b_{qi}, \\
 R'^{ijk} &= R^{ijk} - 3b_{lm} \beta^{l[i} R^{j]k}{}^m - 3\beta^{l[i} R^{j]mp} \beta^{k]q} b_{lm} b_{pq} - \beta^{il} \beta^{jn} \beta^{kr} b_{lm} b_{np} b_{qr} R^{mpq} \\
 &\quad + 3Q_n{}^{[ij} \beta^{k]n} - 6Q_n{}^{l[i} \beta^{j]n} \beta^{k]m} b_{lm} - 3\beta^{l[i} \beta^{j]n} \beta^{k]q} b_{lm} b_{pq} Q_n{}^{mp} \\
 &\quad + 3\beta^{l[i} F_{lm}{}^j \beta^{k]m} + 3\beta^{l[i} \beta^{j]m} \beta^{k]p} b_{np} F_{lm}{}^n + \beta^{il} \beta^{jm} \beta^{kn} H_{lmn}.
 \end{aligned} \tag{2.27}$$

Given a certain representative of a specific orbit, one may want to apply such an $O(d, d)$ transformation to see if it can be rotated to a different orbit with less (or just different) types of fluxes turned on. In section 3 we will often use this possibility to go from one orbit to another by means of rigid $O(d, d)$ transformations. Alternatively, the above expressions may be used to classify the automorphisms of \mathfrak{t} implemented by b - and β -shifts. Another important case is to take $U' = TU$ related by the rigid T -transformation¹⁴

$$T_I{}^J = \begin{pmatrix} 0 & \delta^{ij} \\ \delta_{ij} & 0 \end{pmatrix}, \tag{2.28}$$

which leads to the usual T-duality-like relations among the fluxes

$$\begin{aligned}
 \mathcal{F}'^i &= \delta^{ij} \mathcal{F}_j, & H'_{ijk} &= \delta_{il} \delta_{jm} \delta_{kn} R^{lmn}, & Q'^{jk} &= \delta_{il} \delta^{jm} \delta^{kn} F_{mn}{}^l, \\
 \mathcal{F}'_i &= \delta_{ij} \mathcal{F}^j, & R'^{ijk} &= \delta^{il} \delta^{jm} \delta^{kn} H_{lmn}, & F'_{ij}{}^k &= \delta_{il} \delta_{jm} \delta^{kn} Q_n{}^{lm}.
 \end{aligned} \tag{2.29}$$

This mechanism is what allows us to describe solution-generating techniques that involve also rigid T -transformations, like abelian/non-abelian/Poisson-Lie T-duality.

Let us remark that the concept of ‘‘Poisson-Lie plurality’’ — namely the possibility of decomposing the same Drinfel’d double \mathfrak{d} in terms of different choices of subalgebras $\mathfrak{d} = \mathfrak{g} \oplus \tilde{\mathfrak{g}} = \mathfrak{g}' \oplus \tilde{\mathfrak{g}}'$ — can be understood as the transformation of U by a constant $O(d, d)$ matrix V as in (2.23). What we have here is a generalisation of the traditional definition of PL plurality in the sense that we do not require \mathfrak{t} to have the structure of a Drinfel’d double, and any $O(d, d)$ transformation may in principle be considered, even those relating different orbits.¹⁵

Let us now look at the explicit expressions for the components of the fluxes in the parametrisation (2.20) used for the twist U . One has

$$\begin{aligned}
 \mathcal{F}_{IJK} &= \delta_{[I}{}^i \delta_{J}{}^j \delta_{K]}{}^k H_{ijk} + 3\delta_{[I}{}^i \delta_{J}{}^j \delta_{K]}{}^k F_{ij}{}^k + 3\delta_{[I}{}^i \delta_{J}{}^j \delta_{K]}{}^k Q_i{}^{jk} + \delta_{[I}{}^i \delta_{J}{}^j \delta_{K]}{}^k R^{ijk}, \\
 \mathcal{F}_I &= \delta_{Ii} \mathcal{F}^i + \delta_I{}^i \mathcal{F}_i,
 \end{aligned} \tag{2.30}$$

¹⁴The matrix form of $T_I{}^J$ and η_{IJ} coincide, but notice the different position of indices in the two definitions.

¹⁵For explicit examples on how to relate representatives of the (F, H) and of the (F, Q) orbits see e.g. [56].

where

$$\begin{aligned} H_{ijk} &\equiv \mathcal{F}_{ijk} = \mathcal{F}_{[ijk]}, & F_{ij}{}^k &\equiv \mathcal{F}_{ij}{}^k = \mathcal{F}_{[ij]}{}^k, \\ Q_i{}^{jk} &\equiv \mathcal{F}_i{}^{jk} = \mathcal{F}_i{}^{[jk]}, & R^{ijk} &\equiv \mathcal{F}^{ijk} = \mathcal{F}^{[ijk]}, \end{aligned} \quad (2.31)$$

are¹⁶

$$\begin{aligned} H_{ijk} &= 3(\partial_{[i}b_{jk]} + w_{[ij}{}^l b_{k]l}), \\ F_{ij}{}^k &= w_{ij}{}^k + \beta^{kl} H_{ijl} \\ Q_i{}^{jk} &= \partial_i \beta^{jk} - 2\beta^{l[j} F_{il}{}^{k]} + \beta^{l[j} \beta^{k]m} H_{ilm} \\ R^{ijk} &= -3\beta^{l[i} Q_l{}^{jk]} + 3\beta^{l[i} \beta^{j]m} F_{lm}{}^{k]} - \beta^{l[i} \beta^{j]m} \beta^{k]n} H_{lmn}. \end{aligned} \quad (2.32)$$

Here we defined

$$\partial_i \equiv \rho_i{}^\mu \partial_\mu, \quad w_{ij}{}^k \equiv -2\rho_{[i}{}^\mu \rho_{j]}{}^\nu \partial_\mu \rho_\nu{}^k. \quad (2.33)$$

To make $w_{ij}{}^k$ appear also in the generalised flux with one index, we rewrite $\lambda = \bar{\lambda} - \frac{1}{2} \log \det \rho$ so that using $d \log \det \rho = \text{Tr}(d\rho \rho^{-1})$ we find

$$\mathcal{F}_i = w_{ij}{}^j + 2\partial_i \bar{\lambda}, \quad \mathcal{F}^i = \partial_j \beta^{ji} + \beta^{ik} (w_{kj}{}^j + 2\partial_k \bar{\lambda}) = \partial_j \beta^{ji} + \beta^{ik} \mathcal{F}_k. \quad (2.34)$$

Notice that if $w_{ij}{}^k$ can be interpreted as structure constants of a Lie algebra, then its definition is simply the Maurer-Cartan (MC) identity for ρ , which can then be taken of MC form $\rho = g^{-1} dg = dy^\mu \rho_\mu{}^i t_i$, where $t_i \in \text{Lie}(G)$ are the generators of the Lie algebra with structure constants $w_{ij}{}^k$ and $g \in G$. This happens for example for all orbits that have $H_{ijk} = 0$, since in this case the Bianchi identities imply that $F_{ij}{}^k$ solves the Jacobi identity and the above equations give $F_{ij}{}^k = w_{ij}{}^k$. We will see that the discussion for orbits with non-vanishing H -flux is more complicated, and in the next section we present some methods that we will use in this case.

2.4 Twist ansatz for orbits with H -flux

The methods explained in this section may be useful in general, but later we will actually use them only for orbits which have non-vanishing H_{ijk} , so this section is not necessary for reading the discussion of the other orbits.¹⁷ The particular issue with non-vanishing H -flux is that $w_{ij}{}^k$ may not be interpreted as structure constants of a Lie algebra (it may not even be constant) and therefore ρ can not be taken of MC form. Nevertheless, we can maintain a geometrical interpretation by exploiting the fact that the *doubled* manifold — spanned locally by the coordinates y^μ and their T-duals $\tilde{y}_{\tilde{\mu}}$ — can be interpreted as a group manifold \mathbb{R} associated to the pre-Roytenberg algebra \mathfrak{r} , equipped with the bilinear ad-invariant form η of split signature. This is in fact the set-up also known as DFT_{WZW} of [57]

¹⁶See for example [20] for similar expressions. The last two equations could be rewritten also as

$$\begin{aligned} Q_i{}^{jk} &= \partial_i \beta^{jk} - 2\beta^{l[j} w_{il}{}^{k]} - \beta^{m[j} \beta^{k]n} H_{imn} \\ R^{ijk} &= -3\beta^{l[i} \partial_l \beta^{jk]} - 3\beta^{l[i} \beta^{j]m} w_{lm}{}^{k]} - \beta^{l[i} \beta^{j]m} \beta^{k]n} H_{lmn}. \end{aligned}$$

¹⁷For a discussion on dealing with orbits with $H \neq 0$ without appealing to the methodology of this section, see appendix F.

and employed also in [22]. We decompose the group elements $g(Y^M) \in \mathbb{R}$ specifically as $g(Y) = \tilde{m}(\tilde{y})m(y)$, where $\tilde{m} = e^{\tilde{m}_i(\tilde{y})\tilde{T}^i}$ and $m = e^{m^i(y)T_i}$, which is always possible locally. Notice that the subspaces spanned by T_i and \tilde{T}^i are not necessarily Lie subalgebras of \mathfrak{r} .¹⁸ Let us introduce in \mathfrak{r} the adjoint action by m and the right-invariant one-form dmm^{-1} as

$$m\mathbb{T}_I m^{-1} = M_I^J \mathbb{T}_J = e^{\text{ad}_{m^i T_i}} \mathbb{T}_I, \quad dmm^{-1} = dy^M V_M^I \mathbb{T}_I = dy^\mu V_\mu^i T_i + dy^\mu V_{\mu i} \tilde{T}^i. \quad (2.35)$$

We will denote the inverse of V_μ^i as V_i^μ , i.e. $V_\mu^i V_i^\nu = \delta_\mu^\nu$ and $V_i^\mu V_\mu^j = \delta_i^j$. The MC identity for the one-form dmm^{-1} in \mathfrak{r} projected to the subspaces is

$$\begin{aligned} 2\partial_{[\mu} V_{\nu]}^i &= V_\mu^j V_\nu^k F_{jk}^i + 2V_{[\mu}^j V_{\nu]k} Q_j^{ki} + V_{\mu j} V_{\nu k} R^{ijk}, \\ 2\partial_{[\mu} V_{\nu]i} &= V_{\mu j} V_{\nu k} Q_i^{jk} + 2V_{[\mu}^j V_{\nu]k} F_{ij}^k + V_\mu^j V_\nu^k H_{ijk}. \end{aligned} \quad (2.36)$$

In addition, to first order in the expansion $m = e^{m^i T_i}$, we have $M_i^j = \delta_i^j + m^l F_{li}^j + \mathcal{O}(m^2)$, $M^i_j = \delta_j^i - m^l F_{lj}^i + \mathcal{O}(m^2)$, $M_{ij} = m^l H_{lij} + \mathcal{O}(m^2)$ and $M^{ij} = m^l Q_l^{ij} + \mathcal{O}(m^2)$, so that we can take M_i^j and M^i_j invertible at least in a certain neighborhood around the identity, while M_{ij} and M^{ij} may not be invertible.

When the H -flux is non-vanishing, depending on the orbit in consideration, we will consider two possible parametrisations for the twist U_I^M that are equivalent to further specifications of the more general parametrisation given in (2.20). Essentially they encompass an ansatz for the ρ -twist and a rewriting of the β - and b -twist of (2.20). In particular, we will consider¹⁹

$$U_I^{(1)M} = \begin{pmatrix} \delta_j^i + \xi^{il}\omega_{lj} & \xi^{ij} \\ \omega_{ij} & \delta_i^j \end{pmatrix}_I {}^J M^J K \begin{pmatrix} (V^t)^k{}_\mu & 0 \\ 0 & V_k{}^\mu \end{pmatrix}_K^M, \quad (2.37)$$

or

$$U_I^{(2)M} = M_I^J \begin{pmatrix} \delta_k^j + \xi^{jl}\omega_{lk} & \xi^{jk} \\ \omega_{jk} & \delta_j^k \end{pmatrix}_J {}^K \begin{pmatrix} (V^t)^k{}_\mu & 0 \\ 0 & V_k{}^\mu \end{pmatrix}_K^M, \quad (2.38)$$

where the ξ^{ij}, ω_{ij} that we use in $U^{(1)}$ and $U^{(2)}$ are different because of the different position of the adjoint action M_I^J in the above expressions. In general a twist given by $U^{(1)}$ may not be rewritten in the form of $U^{(2)}$, and viceversa, so generically speaking the two twists should not be viewed as equivalent.²⁰ When, respectively, $M^{ij} = 0$ ($M_{ij} = 0$) we will consider

¹⁸The interpretation given in [22] is that the physical manifold is understood as a coset $\mathbb{R} \backslash \tilde{\mathcal{M}}$, where $\tilde{\mathcal{M}}$ is parametrised by the \tilde{y}_μ . This is possible only when the R -flux vanishes, since in this case the generators \tilde{T}^i span a subalgebra of \mathfrak{r} . We will not need to appeal to this coset interpretation. The parametrisations for the twists that we use manifestly satisfy the strong constraint.

¹⁹If we relaxed the strong constraint we could simply take for the twist the components of $g^{-1}dg$, since the MC identity would ensure that they satisfy the correct algebra relations. When imposing the strong constraint this is not possible because of the non-invertibility of the would-be frame field. Therefore one has to look for other solutions for the twists. Notice that $U^{(2)}$ is a generalisation of the twist used in [22], which is necessary if one wants to be able to describe, for example, the so-called Yang-Baxter deformations.

²⁰By equating the four block components of $U^{(1)}(\tilde{\xi}, \tilde{\omega})$ and $U^{(2)}(\xi, \omega)$ we will have for $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$ the relations $\tilde{\omega} = (M_4 + M_3\xi)\omega M_1^{-1}$ and $\tilde{\xi} = (M_4 + M_3\xi)^{-1}M_1\xi$ constraint with the consistency relations $\omega M_1^{-1}M_2 = (M_4 + M_3\xi)^{-1}M_3\xi$ and $M_1\xi(M_4 + M_3\xi)^{-1}M_3M_1^{-1}M_2 = M_2(M_4 + M_3\xi)^{-1}M_3\xi$. As we will see these are consistent when M_2 or M_3 are vanishing. In general it may not be possible to solve these constraints since M_2 or M_3 may not be invertible.

$U^{(1)}$ ($U^{(2)}$) since the inclusion of the adjoint action realises a simple shift²¹ of ω_{ij} (ξ^{ij}) in terms of M_{ij} (M^{ij}). Notice that both twists are automatically elements of $O(d, d)$ since ξ^{ij} and ω_{ij} are antisymmetric matrices and the bilinear form η is ad-invariant. Furthermore ξ^{ij} and ω_{ij} depend only on the coordinates y^μ such that the twists satisfy manifestly the strong constraint.

In the case that the R -flux R^{ijk} is vanishing and $\xi^{ij} = 0$ the twist $U_I^{(2)M}$ coincides with the generalised frame fields of [22] in which a particular solution for ω was found for general (F, H, Q) -flux. In fact, in the parametrisation $U_I^{(2)M}$ we can generalise this solution to include non-vanishing R^{ijk} and ξ^{ij} as well. In order to do so we follow [22] and observe that the flux equations can be written as

$$\mathcal{F}_{IJK} = M_I^{I'} M_J^{J'} M_K^{K'} (T_{I'J'K'} + S_{I'J'K'}) = T_{IJK} + S_{IJK}, \quad (2.39)$$

where,

$$\begin{aligned} T_{IJK} &= 3\hat{U}_{[I}^M \partial_M (\hat{U}_{J}^N) \hat{U}_{K]N}, & \hat{U}_I^M &= \begin{pmatrix} (\delta_j^i + \xi^{il}\omega_{lj})(V^t)^j{}_\mu & \xi^{ij}V_j{}^\mu \\ \omega_{ij}(V^t)^j{}_\mu & V_i{}^\mu \end{pmatrix}, \\ S_{IJK} &= 3\Lambda_{[I}^L \mathcal{F}_{JK]L}, & \Lambda_I^J &= \hat{U}_I^M V_M^J = \begin{pmatrix} \xi^{ik}V_k{}^\mu V_{\mu j} & \xi^{ij} \\ V_i{}^\mu V_{\mu j} & \delta_i^j \end{pmatrix}, \end{aligned} \quad (2.40)$$

and we have used the identity

$$\partial_M M_I^J = V_M^K M_I^L \mathcal{F}_{KL}^J, \quad (2.41)$$

as well as the fact that the adjoint action is an automorphism of \mathfrak{t} ,

$$\mathcal{F}_{IJK} = M_I^{I'} M_J^{J'} M_K^{K'} \mathcal{F}_{I'J'K'}. \quad (2.42)$$

While the equations for $F_{ij}{}^k$, $Q_i{}^{jk}$ and R^{ijk} become involved due to the presence of ξ^{ij} , the equation for H_{ijk} is independent of ξ^{ij} (and the inclusion of R^{ijk}). In particular it reads

$$3V_i{}^\mu V_j{}^\nu V_k{}^\rho \partial_{[\mu} \omega_{\nu\rho]} = -2H_{ijk} - 3V_{[i}{}^\mu F_{jk]}{}^l V_{\mu l}, \quad (2.43)$$

where we have defined $\omega_{\mu\nu} = V_\mu{}^i \omega_{ij} V_\nu{}^j$. One can verify using (2.36) that a particular solution for this (inhomogeneous) differential equation is given by $\omega^{\text{inhom.}} = \bar{\omega}^{(2)} - \Omega^{(2)}$ where,

$$\bar{\omega}^{(2)} = \frac{1}{2} V_\mu{}^i V_\nu{}^j dy^\mu \wedge dy^\nu, \quad (2.44)$$

which satisfies

$$\partial_{[\mu} \bar{\omega}_{\nu\rho]}^{(2)} = \frac{1}{2} \left(-V_\mu{}^i V_\nu{}^j V_\rho{}^k H_{ijk} - V_{[\mu}{}^i V_\nu{}^j V_{\rho]k} F_{ij}{}^k + V_{[\mu}{}^i V_\nu{}^j V_{\rho]k} Q_i{}^{jk} + V_{\mu i} V_{\nu j} V_{\rho k} R^{ijk} \right) \quad (2.45)$$

²¹If we considered instead $U^{(2)}$ ($U^{(1)}$), the ρ -twist would receive a contribution from the (unknown) ω_{ij} (ξ^{ij}) which would further complicate the discussion.

and $\Omega^{(2)}$ chosen such that²²

$$d\Omega^{(2)} = \frac{1}{12} \mathcal{F}_{IJK} V^I \wedge V^J \wedge V^K + \frac{1}{6} R^{ijk} V_i \wedge V_j \wedge V_k. \quad (2.46)$$

Notice that $\Omega^{(2)}$ may not exist globally and an explicit expression can only be found by choosing particular coordinates in a local patch. The most general solution to (2.43) is then given by $\omega_{ij} = \omega_{ij}^{\text{inhom.}} + \omega_{ij}^{\text{hom.}}$ where $\omega_{ij}^{\text{hom.}}$ is a closed two-form. Let us point out that this is a solution for the ω form of $U^{(2)}$, and it is not a solution for the ω of $U^{(1)}$ in (2.37). Furthermore, within parametrisation $U^{(2)}$, it is still necessary to solve the other flux equations for $F_{ij}{}^k$, $Q_i{}^{jk}$ and R^{ijk} in terms of the unknown ξ^{ij} . Using the equations (2.39) and (2.40) we find in particular

$$\begin{aligned} S_{ij}{}^k &= 2F_{ij}{}^k - 2Q_{[i}{}^{kl} \Lambda_{j]l} + \xi^{kl} H_{ijl} + \xi^{kn} \Lambda_{nl} F_{ij}{}^l, \\ S_i{}^{jk} &= Q_i{}^{jk} + \Lambda_{il} R^{ljk} - 2F_{il}{}^{[j} \xi^{k]l} - 2\Lambda_{ml} Q_i{}^{[lj} \xi^{k]m}, \\ S^{ijk} &= 3Q_l{}^{[ij} \xi^{k]l} + 3\Lambda_{ml} R^{[ij} \xi^{k]m}, \end{aligned} \quad (2.47)$$

with $\Lambda_{ij} = V_i{}^\mu V_{\mu j}$, and

$$\begin{aligned} T_{ij}{}^k &= \hat{w}_{ij}{}^k + \xi^{kl} H_{ijl}, \\ T_i{}^{jk} &= \hat{\partial}_i \xi^{jk} - 2\xi^{l[j} \hat{w}_{il}{}^{k]} - \xi^{m[j} \xi^{k]n} H_{imn}, \\ T^{ijk} &= -3\xi^{l[i} \hat{\partial}_l \xi^{jk]} - 3\xi^{l[i} \xi^{j]m} \hat{w}_{lm}{}^{[k]} - \xi^{l[i} \xi^{j]m} \xi^{[k]n} H_{lmn}, \end{aligned} \quad (2.48)$$

where we have used eqs. (2.32) (by replacing β^{ij} with ξ^{ij} and $\rho_i{}^\mu$ with $V_i{}^\mu$) and defined $\hat{\partial}_i = V_i{}^\mu \partial_\mu$ and $\hat{w}_{ij}{}^k = -2V_{[i}{}^\mu V_{j]}{}^\nu \partial_\mu V_\nu{}^k$. Using the MC identity (2.36) we have

$$\hat{w}_{ij}{}^k = -F_{ij}{}^k + 2Q_{[i}{}^{kl} \Lambda_{j]l} - \Lambda_{il} \Lambda_{jm} R^{lmk}. \quad (2.49)$$

The equations for ξ^{ij} describing the (F, H, Q, R) -orbit can then be rewritten as

$$2\xi^{kl} H_{ijl} + \xi^{km} \Lambda_{ml} F_{ij}{}^l - \Lambda_{il} \Lambda_{jm} R^{lmk} = 0, \quad (2.50)$$

$$\hat{\partial}_i \xi^{jk} + 2\Lambda_{il} \xi^{m[j} Q_m{}^{k]l} + \Lambda_{il} R^{ljk} + 2\Lambda_{il} \Lambda_{mn} \xi^{m[j} R^{k]ln} - \xi^{m[j} \xi^{k]n} H_{imn} = 0, \quad (2.51)$$

$$3Q_l{}^{[ij} \xi^{k]l} + 3\xi^{l[i} \xi^{j]m} F_{lm}{}^{[k]} - 3\xi^{l[i} \xi^{j]m} R^{[k]pq} \Lambda_{lp} \Lambda_{mq} + 2\xi^{l[i} \xi^{j]m} \xi^{[k]n} H_{lmn} = R^{ijk}, \quad (2.52)$$

where in the latter equation we have used (2.51). Using (2.50) we can rewrite (2.52) also as

$$3Q_l{}^{[ij} \xi^{k]l} + 3\xi^{l[i} \xi^{j]m} F_{lm}{}^{[k]} - 2\Lambda_{lp} \Lambda_{mq} \xi^{l[i} \xi^{j]m} R^{[k]pq} - \xi^{l[i} \xi^{j]m} \xi^{[k]n} \Lambda_{np} F_{lm}{}^p = R^{ijk}. \quad (2.53)$$

We will not solve these equations in general, but rather use them when studying sub-orbits with $H \neq 0$. Notice however that for a non-vanishing R -flux they do not allow a trivial solution $\xi = 0$.

While we will use these methods to look at orbits with non-vanishing H -flux, let us point out that the framework described here can cover the most general representatives of the (F) , (Q) , (F, Q) and (F, R) orbits derived later.

²²When $R^{ijk} = 0$, the three-form $d\Omega^{(2)}$ takes a nice expression in terms of the group elements m , in particular [22]

$$d\Omega^{(2)} = \frac{1}{12} \langle dmm^{-1}, [dmm^{-1}, dmm^{-1}] \rangle.$$

Comments on other possible ansatze. The methodology described above to find a suitable twist when $H \neq 0$ is far from the most general parametrisation considered in (2.20): there are still several possibilities to generalise the two ansatze considered in (2.37) and (2.38).

- (i) The most obvious generalisation is to replace V_μ^i in (2.37) and (2.38) as $V_\mu^i \rightarrow A_\mu^\nu V_\nu^i$ for some matrix $A_\mu^\nu(y)$ which, to exclude diffeomorphisms, is not of the form $\partial_\mu a^\nu$ for some (vector) $a^\nu(y)$. Up to a rewriting, this is now equivalent to the general parametrisation (2.20), which will be more convenient to use in practice.
- (ii) Another possibility is to consider a different parametrisation of the group elements $g \in \mathbb{R}$. In particular we can take instead $g(Y) = m(\tilde{y})\tilde{m}(y)$, with $\tilde{m} = e^{\tilde{m}_i(y)\tilde{T}^i}$ and $m = e^{m^i(\tilde{y})T_i}$, and define all the quantities of the twist U in terms of the group elements \tilde{m} (so that, again, U manifestly satisfies the strong constraint). We define $\tilde{M}_I^J \mathbb{T}_J = \tilde{m} \mathbb{T}_I \tilde{m}^{-1}$ and $\tilde{V} = d\tilde{m}\tilde{m}^{-1} = dy^\mu \tilde{V}_{\mu i} \tilde{T}^i + dy^\mu \tilde{V}_\mu^i T_i$. When trying to find non-trivial solutions for the equivalent of ξ^{ij} , this parametrisation could be convenient when the generators \tilde{T}^i span a subalgebra of \mathfrak{r} such that $\tilde{V}_\mu^i = 0$ and $\tilde{M}^{ij} = 0$. In that case, we can consider the following twists

$$\tilde{U}_I^{(3)M} = \begin{pmatrix} \delta_k^j + \xi^{jl}\omega_{lk} & \xi^{jk} \\ \omega_{jk} & \delta_j^k \end{pmatrix}_I \tilde{M}_J^K \begin{pmatrix} \delta^{kl}(\tilde{V}^t)_{l\mu} & 0 \\ 0 & \delta_{kl}\tilde{V}^{l\mu} \end{pmatrix}_K^M, \tag{2.54}$$

$$\tilde{U}_I^{(4)M} = \tilde{M}_I^J \begin{pmatrix} \delta_k^j + \xi^{jl}\omega_{lk} & \xi^{jk} \\ \omega_{jk} & \delta_j^k \end{pmatrix}_J^K \begin{pmatrix} \delta^{kl}(\tilde{V}^t)_{l\mu} & 0 \\ 0 & \delta_{kl}\tilde{V}^{l\mu} \end{pmatrix}_K^M, \tag{2.55}$$

where we have included a rigid T -transformation, and where we have denoted the inverse of $\tilde{V}_{\mu i}$ as $\tilde{V}^{i\mu}$. Because $\tilde{M}^{ij} = 0$ these two parametrisations are actually equivalent. When $\text{span}(\tilde{T}^i)$ does not form a subalgebra, however, we expect that not much is gained compared to the parametrisations of (2.37) and (2.38).

Although we will not consider these generalisations further, it would certainly be interesting to systematically understand whether non-trivial and inequivalent solutions for ξ can be found in specific orbits within these other ansatze. See also appendix F for some further comments.

3 Classification of orbits

In the following we will discuss the orbits found by turning on all combinations of the fluxes F, H, Q and R as in figure 1. Let us remark that when looking at solution-generating techniques in supergravity we want to impose that both $\mathcal{F}_{IJK} = \mathcal{F}'_{IJK}$ and $\mathcal{F}_I = \mathcal{F}'_I$, while when looking at canonical transformations of σ -models it is enough to impose $\mathcal{F}_{IJK} = \mathcal{F}'_{IJK}$. That means that there may be certain (classical) canonical transformations of the σ -model that cannot be interpreted as supergravity solution-generating techniques. Examples of this kind are non-unimodular homogeneous Yang-Baxter deformations, that will be discussed in the (F)-orbit (see [58] for some exceptions in this class).

3.1 \emptyset

The simplest orbit is the one where all fluxes are zero. From the equation for $F_{ij}{}^k$ one gets $w_{ij}{}^k = 0$, which can be solved up to $GL(d)$ transformations by $\rho_\mu{}^i = \delta_\mu^i$. The equation for H_{ijk} can be solved at least locally by taking b_{ij} constant up to gauge transformations. Finally, the equation for $Q_i{}^{jk}$ is solved by taking also β^{ij} constant, and the equation for R^{ijk} is automatically satisfied. If the space is compact,²³ this is the example of the d -dimensional torus with no flux, decorated with constant b - and β -shifts.

3.2 F

In this orbit we assume that the only non-vanishing flux is the F -flux. First let us discuss the restrictions imposed by the Bianchi identities. They imply that $F_{ij}{}^k$ satisfies the Jacobi identity. We will call \mathfrak{g} the corresponding Lie algebra with generators $T_i \in \mathfrak{g}$. The remaining generators \tilde{T}^i of the pre-Roytenberg algebra \mathfrak{r} form an abelian algebra $\tilde{\mathfrak{g}}$, so that we have $\mathfrak{r} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$. Moreover $F_{ij}{}^k \mathcal{F}_k = 0$ and $F_{ik}{}^j \mathcal{F}^k = 0$ imply that

$$\mathcal{F}_k = 0 \text{ if } T_k \in [\mathfrak{g}, \mathfrak{g}], \quad \mathcal{F}^k = 0 \text{ if } T_k \notin Z(\mathfrak{g}), \quad (3.1)$$

where $[\mathfrak{g}, \mathfrak{g}]$ is the derived algebra and $Z(\mathfrak{g})$ is the center of the algebra. Notice that for *semisimple* algebras this means that $\mathcal{F}_i = 0$ and $\mathcal{F}^i = 0$. Finally we also have the orthogonality condition $\mathcal{F}^i \mathcal{F}_i = 0$.

Let us now turn to solving the equations (2.32) for the fluxes in terms of the functions ρ, b, β appearing in the parametrisation of the twist U . First we have $F_{ij}{}^k = w_{ij}{}^k$ and, therefore, the definition of $w_{ij}{}^k$ in (2.33) reads like the (left) MC identity. Although we can take ρ of MC form, this solution can be generalised by taking $\rho_i{}^\mu = W_i{}^j \bar{\rho}_j{}^\mu$ with $\bar{\rho} = \bar{g}^{-1} d\bar{g}$ of MC form for a $\bar{g} \in G$ and W an automorphism of the Lie algebra, $W_i{}^l W_j{}^m F_{lm}{}^n = F_{ij}{}^k W_k{}^n$. To mod out by $GL(d)$ diffeomorphisms, we have to take W to be an *outer* automorphism. The dressing by the automorphism W will burden the following expressions, but we prefer to keep W explicitly because different outer automorphisms W will correspond to inequivalent representatives.

Solving for $H_{ijk} = 0$ globally is a question about the second de Rham cohomology of the manifold. It is well known that if \mathfrak{g} is the Lie algebra of the compact and connected Lie group G , then the n -th cohomology group $H^n(G)$ with real coefficients is isomorphic to $H^n(\mathfrak{g}, \mathbb{R})$, the n -th Chevalley-Eilenberg Lie algebra cohomology with coefficients in \mathbb{R} [59]. In other words, one has to impose $H_{ijk} = 0$ with b_{ij} constant, which leads to the equation $F_{[ij}{}^l b_{k]l} = 0$ implying that b_{ij} is a constant 2-cocycle. It is useful to rewrite this in operatorial form. Given a 2-cocycle ω_{ij} solving $F_{[ij}{}^l \omega_{k]l} = 0$ one can construct a linear operator $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $\omega T_i = \omega_{ij} T^j$, where algebra indices are raised²⁴ and lowered with a symmetric invariant bilinear form on the algebra, $\kappa_{ij} = \langle T_i, T_j \rangle$. Then the 2-cocycle condition in operatorial form reads

$$\omega[x, y] = [\omega x, y] + [x, \omega y], \quad x, y \in \mathfrak{g}. \quad (3.2)$$

²³Remember that for \mathbb{R}^d the second de Rham cohomology is trivial, and the exact b_{ij} can be gauged away.

²⁴Notice that when we write T^i we mean $T^i = \kappa^{ij} T_j$, which is different from the other generators \tilde{T}^i of \mathfrak{r} .

Because we want to mod out by gauge transformations of the B-field, we have to impose that b_{ij} is not coboundary, in other words $b_{ij} \neq F_{ij}{}^k x_k$ for some constants x_k , or equivalently in operatorial form $b \neq \text{ad}_x$ for some constant $x \in \mathfrak{g}$. If \mathfrak{g} is semisimple, all 2-cocycles are 2-coboundaries — the second Lie algebra cohomology is trivial. Therefore, in the compact case interesting solutions to $H_{ijk} = 0$ are possible only for \mathfrak{g} non-semisimple. In the non-compact case there is no general theorem relating the Lie algebra and de Rham cohomologies.²⁵ Putting global issues aside, the equation $H_{ijk} = 0$ may be solved for example by $b_{ij} = \hat{b}_{ij} + \hat{\hat{b}}_{ij}$ with $\hat{b}_{ij} = \langle \hat{\omega} T_i, T_j \rangle$ and $\hat{\hat{b}}_{ij} = \langle \hat{\hat{\omega}}_g T_i, T_j \rangle$, where $\hat{\omega}_g = W^{-1} \circ \hat{\omega}_{\bar{g}} \circ W$, $\hat{\hat{\omega}}_{\bar{g}} = \text{Ad}_{\bar{g}}^{-1} \circ \hat{\omega} \circ \text{Ad}_{\bar{g}}$ and both $\hat{\omega}$ and $\hat{\hat{\omega}}$ are constant 2-cocycles that are not 2-coboundaries.

The equation for β^{ij} coming from $Q_i{}^{jk} = 0$ is

$$\partial_i \beta^{jk} - 2\beta^{l[j} F_{il}{}^{k]} = 0, \tag{3.3}$$

and its most generic solution is

$$\beta^{ij} = \langle r_g(T^i), (T^j) \rangle, \tag{3.4}$$

with $r_g = W^{-1} \circ r_{\bar{g}} \circ W$, $r_{\bar{g}} = \text{Ad}_{\bar{g}}^{-1} \circ r \circ \text{Ad}_{\bar{g}}$ and r a constant and antisymmetric $r^t = -r$ linear operator on the algebra. Transposition is understood with respect to the ad-invariant symmetric bilinear form of \mathfrak{g} , i.e. $\langle x, ry \rangle = \langle r^t x, y \rangle$. It is easy to argue that the above is the most general solution. In fact, we may view $r \equiv \text{Ad}_{\bar{g}} W \circ \beta \circ W^{-1} \text{Ad}_{\bar{g}}^{-1}$ just as a redefinition of the variables for which we want to solve the equation. But then, by using $d \text{Ad}_g = \text{Ad}_g \text{ad}_{g^{-1}dg}$, one finds that the differential equation for β is equivalent to $\partial_\mu r^{ij} = 0$, and therefore r must be constant.

The equation $R^{ijk} = 0$ reads

$$\beta^{l[i} \beta^{j|m} F_{lm}{}^{k]} = 0, \tag{3.5}$$

which is known as the classical Yang-Baxter equation (CYBE) on \mathfrak{g} . Therefore both β^{ij} and r^{ij} must solve the CYBE. To rewrite it in operatorial form we may introduce the linear operator $r : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $rT^i = r^{ij}T_j$. Then the CYBE is

$$[rx, ry] - r([rx, y] + [x, ry]) = 0, \quad \forall x, y \in \mathfrak{g}. \tag{3.6}$$

Notice that r may come with an overall continuous parameter — in fact its entries may depend on several independent parameters — so that we can interpret it as a deformation of a representative with $\beta = 0$.

It is simple to discuss the rigid $O(d, d)$ automorphisms that leave this orbit invariant. From (2.27) one sees that the algebra is invariant if and only if $F_{[ij}{}^l b_{k]l} = 0$ and $F_{il}{}^{[j} \beta^{k]l} = 0$. The former is a constant b -shift by a 2-cocycle, while the latter a constant β -shift where β commutes with the adjoint action on the algebra. Both are included in the above solutions.²⁶

²⁵For example \mathbb{R}^2 is de Rham trivial, but the 2-dimensional abelian algebra has non-trivial Lie algebra cohomology. In fact it admits a non-trivial 2-cocycle giving rise to the three-dimensional Heisenberg-Weyl algebra.

²⁶Notice that for such β one has $r_{\bar{g}} = r$ because the adjoint action commutes with β .

One also finds

$$\mathcal{F}_i = F_{ij}{}^j + 2\partial_i\bar{\lambda}, \quad \mathcal{F}^i = -\beta^{jk}F_{jk}{}^i + 2\beta^{ij}\partial_j\bar{\lambda}. \quad (3.7)$$

Notice that for consistency $\partial_i\bar{\lambda}$ must be constant which means $\partial_\mu(\rho_i{}^\nu\partial_\nu\bar{\lambda}) = 0$. This in turn implies $\partial_\mu\partial_\nu\bar{\lambda} = \partial_\mu\rho_\nu{}^i\rho_i{}^\epsilon\partial_\epsilon\bar{\lambda}$. Antisymmetrising in μ, ν and using the invertibility of ρ one gets the condition

$$F_{ij}{}^k\rho_k{}^\mu\partial_\mu\bar{\lambda} = 0. \quad (3.8)$$

If an algebra \mathfrak{g} is such that its derived algebra is strictly a subalgebra $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$, i.e. there are certain generators of \mathfrak{g} not contained in $[\mathfrak{g}, \mathfrak{g}]$, then for those generators $\partial_i\bar{\lambda}$ is not set to zero by (3.8). Solvable algebras are examples of this kind of algebras, and in this case one may have $\partial_i\bar{\lambda} = \text{constant}$ but $\bar{\lambda}$ not constant. If the algebra is semisimple, then the above equation (and the invertibility of ρ) imply $\bar{\lambda}$ constant. Notice in addition that β must be such that \mathcal{F}^i is constant.

To summarise, the (F) -orbit contains representatives that are invariant under a group G of isometries. They are found by taking ρ of MC form and setting $\beta = 0$. From these representatives one may construct the so-called Yang-Baxter (YB) deformations. They are found by switching on β of the above form, in general multiplied by a deformation parameter. We refer to the section on the (F, R) -orbit for a discussion on the relation of YB-deformations to PL-plurality. Finally we have also included the possibility of outer automorphisms in this (F) -orbit.

Comments on Yang-Baxter deformations. Notice that if we start from an isometric background ($\beta = 0$) then $\mathcal{F}^i = 0$, and if we turn on a YB deformation ($\beta \neq 0$) then to have invariance of \mathcal{F} we must have $\beta^{jk}F_{jk}{}^i = 2\beta^{ij}\partial_j\bar{\lambda}$. If the original dilaton is isometric ($\bar{\lambda}$ is constant), then this implies the unimodularity condition for β and r [60]. Otherwise it looks like a generalisation of this condition. We can use YB deformations to give an example of the discussion on the geometric interpretation of the backgrounds as in appendix B.1, and see how the fluxes in *curved* indices change under a YB deformation of an isometric background. We refer to [61] for a comprehensive review on non-geometric backgrounds in string theory. As the starting point ($\beta = b = 0$) we take a background that has only non-vanishing $F_{\mu\nu}{}^\rho$. From (B.15) with

$$h_M{}^N = \begin{pmatrix} \delta^\mu{}_\nu & -\beta'^{\mu\nu} \\ -b'_{\mu\nu} & \delta_\mu{}^\nu + b'_{\mu\rho}\beta'^{\rho\nu} \end{pmatrix}, \quad \text{where } \beta'^{\mu\nu} = \rho_i{}^\mu\rho_j{}^\nu\beta'^{ij}, \quad b'_{\mu\nu} = \rho_\mu{}^i\rho_\nu{}^jb'_{ij}, \quad (3.9)$$

one finds that the fluxes in curved indices after the transformation are

$$\begin{aligned} H'_{\mu\nu\rho} &= 6b'_{\delta[\mu}\beta'^{\delta\alpha}F_{\alpha|\nu}{}^\gamma b'_{\rho]\gamma}, \\ F'_{\mu\nu}{}^\rho &= F_{\mu\nu}{}^\rho + 4b'_{\delta[\mu}\beta'^{\alpha[\rho}F_{\nu]\alpha}{}^{\delta]}, \\ Q'_{\mu}{}^{\nu\rho} &= 2\beta'^{\beta[\nu}F_{\mu\beta}{}^{\rho]}, \\ R'^{\mu\nu\rho} &= 0, \end{aligned} \quad (3.10)$$

where we used the 2-cocycle condition for b and the CYBE for β . Notice that we have the relations

$$F'_{\mu\nu}{}^\rho = F_{\mu\nu}{}^\rho + 2b'_{\delta[\mu}Q'_{\nu]}{}^{\rho\delta}, \quad H'_{\mu\nu\rho} = \frac{3}{2}b'_{\alpha[\rho}F'_{\mu\nu]}{}^\alpha. \quad (3.11)$$

To conclude, $R^{\mu\nu\rho}$ remains vanishing, but it is possible to shift $F_{\mu\nu}{}^\rho$ and generate $H_{\mu\nu\rho}$ and $Q_\mu{}^{\nu\rho}$. In general YB backgrounds have therefore an interpretation as T-folds, see also [62].

3.3 Q

The (Q)-orbit is related to the previous one by a rigid T -transformation. Because the (F)-orbit contains isometric backgrounds, the (Q)-orbit will contain their non-abelian T-duals. Similarly to the previous case, Bianchi identities imply the Jacobi identity for $Q_i{}^{jk}$. Now the generators \tilde{T}^i span a non-abelian algebra $\tilde{\mathfrak{g}}$, while the generators T_i span an abelian algebra \mathfrak{g} so that $\mathfrak{r} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$.

The conditions $F_{ij}{}^k = w_{ij}{}^k = 0$ and $H_{ijk} = 0$ may be solved as in the (\emptyset)-orbit by taking $\rho_i{}^\mu = \delta_i{}^\mu$ and, at least locally, b constant up to gauge transformations. The equation $Q_i{}^{jk} = \partial_i\beta^{jk}$ is the interesting one in this orbit and it is easily integrated to

$$\beta^{ij} = y^\mu \delta_\mu{}^k Q_k{}^{ij} + \tilde{\omega}^{ij}, \tag{3.12}$$

with $\tilde{\omega}^{ij}$ the constants of integration. Finally, imposing $R^{ijk} = 0$ at each order in y and using the Jacobi identity for $Q_i{}^{jk}$, we find that $\tilde{\omega}$ must be a 2-cocycle of $\tilde{\mathfrak{g}}$

$$Q_l{}^{[ij}\tilde{\omega}{}^{k]l} = 0. \tag{3.13}$$

Notice that in this case the role of upper and lower indices is exchanged compared to the previous (F)-orbit. The 2-cocycle $\tilde{\omega}$ may be multiplied by an overall continuous parameter — in fact its entries may depend on several independent parameters — so that we can think of it as a deformation of the $\tilde{\omega} = 0$ case. Also in this discussion the above solutions are already including the rigid $O(d, d)$ transformations of the pre-Roytenberg algebra \mathfrak{r} under consideration.

For the generalised flux with one index we have

$$\mathcal{F}_i = 2\partial_i\bar{\lambda}, \quad \mathcal{F}^i = Q_j{}^{ji} + 2\tilde{\omega}{}^{ik}\partial_k\bar{\lambda} + 2y^j Q_j{}^{ik}\partial_k\bar{\lambda}. \tag{3.14}$$

Constancy of the fluxes implies $\bar{\lambda} = \alpha_i y^i + \bar{\lambda}_0$ and $Q_j{}^{ik}\alpha_k = 0$, for all i, j , for some constants α_i and $\bar{\lambda}_0$. Therefore the fluxes become $\mathcal{F}_i = 2\alpha_i$, and $\mathcal{F}^i = Q_j{}^{ji} + 2\tilde{\omega}{}^{ik}\alpha_k$. Bianchi identities imply

$$Q_i{}^{jk}\mathcal{F}_k = 0, \quad Q_k{}^{ij}\mathcal{F}^k = 0, \quad \mathcal{F}^i\mathcal{F}_i = 0, \tag{3.15}$$

which again imply that $\mathcal{F}_k = 0$ if $\tilde{T}^k \notin Z(\tilde{\mathfrak{g}})$ and $\mathcal{F}^k = 0$ if $\tilde{T}^k \in [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$. Notice that the first and third conditions above hold thanks to $Q_j{}^{ik}\alpha_k = 0$ and antisymmetry of $\tilde{\omega}$. The second condition reads $Q_k{}^{ij}Q_l{}^{lk} + 2Q_k{}^{ij}\tilde{\omega}{}^{kl}\alpha_l = 0$. Recall that in the case of $\tilde{\mathfrak{g}}$ semisimple we have the stronger conditions $\mathcal{F}_i = \mathcal{F}^i = 0$.

The fact that the (F)- and (Q)-orbits are related by a rigid T -transformation means that starting with an $F_{ij}{}^k$ we can relate it to a $Q_i{}^{jk}$ as $F_{ij}{}^k = \delta_{il}\delta_{jm}\delta^{kn}Q_n{}^{lm}$. Similarly, $\mathcal{F}_i = \delta_{ij}\mathcal{F}'^j$ and $\mathcal{F}^i = \delta_{ij}\mathcal{F}'_j$, and the generators are identified as $T_i = \delta_{ij}\tilde{T}'^j, T^i = \delta^{ij}\tilde{T}'_j$ so that the roles of \mathfrak{g} and $\tilde{\mathfrak{g}}$ are exchanged.

Comments on non-abelian T-duality. If the starting point is an isometric background with only $F_{ij}{}^k \neq 0$, the above β with $\tilde{\omega}^{ij} = 0$ is in fact the β -twist of non-abelian T-duality (NATD), as can be easily seen by comparing to (2.4). Starting from $\mathcal{F}_i = F_{ij}{}^j + 2\partial_i\bar{\lambda}$ and $\mathcal{F}^i = 0$ in the (F) -orbit, then invariance of the fluxes implies $\mathcal{F}'_i = 2\alpha_i = 0$ in the Q' -orbit, and also $Q'_j{}^{ji} = \delta^{ik}(F_{kj}{}^j + 2\partial_k\bar{\lambda})$ which means $\partial_i\bar{\lambda} = -F_{ij}{}^j$. For an isometric dilaton $\bar{\lambda}$ is constant, and one gets also in this case a unimodularity condition $F_{ij}{}^j = 0$ which is the anomaly-free condition of [7, 8]. Let us use the example of NATD to see how the fluxes in *curved* indices can transform, following the discussion of appendix B.1. We start again with an isometric background where only $F_{\mu\nu}{}^\rho \neq 0$. From (B.15) we have

$$h_M{}^N = \begin{pmatrix} -\delta^{\tilde{\mu}}{}_{\tilde{i}}\tilde{\beta}^{ij}\delta_{jk}\rho_\nu{}^k & \delta^{\tilde{\mu}}{}_{\tilde{i}}\delta^{ij}\rho_j{}^\nu \\ \delta_{\tilde{\mu}}{}^i(1 + \tilde{b}\tilde{\beta})_i{}^j\delta_{jk}\rho_\nu{}^k & -\delta_{\tilde{\mu}}{}^i\tilde{b}_{ij}\delta^{jk}\rho_k{}^\nu \end{pmatrix} = \begin{pmatrix} -\tilde{\beta}^{\tilde{\mu}}{}_{\tilde{\nu}} & \rho^{\tilde{\mu}\nu} \\ \rho_{\tilde{\mu}\nu}^t + (\tilde{b}\tilde{\beta})_{\tilde{\mu}\nu} & -\tilde{b}_{\tilde{\mu}}{}^\nu \end{pmatrix}, \quad (3.16)$$

where $\tilde{\beta}^{\tilde{\mu}}{}_{\tilde{\nu}} = \delta^{\tilde{\mu}}{}_{\tilde{i}}\tilde{\beta}^{ij}\delta_{jk}\rho_\nu{}^k$ and $\tilde{b}_{\tilde{\mu}}{}^\nu = \delta_{\tilde{\mu}}{}^i\tilde{b}_{ij}\delta^{jk}\rho_k{}^\nu$. We are being explicit in writing all tensors and indices, and we prefer to use a tilde (rather than a prime) both on the functions and on the coordinate indices when they refer to the NATD representative of the (Q) -orbit. After the NATD transformation the fluxes are

$$\begin{aligned} \tilde{H}_{\tilde{\mu}\tilde{\nu}\tilde{\rho}} &= 3\delta_{[\tilde{\mu}}{}^i\delta_{\tilde{\nu}}{}^j\delta_{\tilde{\rho}]}{}^k\tilde{b}_i{}^l\tilde{b}_j{}^m F_{lm}{}^k \\ \tilde{F}_{\tilde{\mu}\tilde{\nu}}{}^{\tilde{\rho}} &= 2\delta_{[\tilde{\mu}}{}^i\delta_{\tilde{\nu}]}{}^j\delta^{\tilde{\rho}k}\tilde{b}_i{}^l F_{lk}{}^j \\ \tilde{Q}_{\tilde{\mu}}{}^{\tilde{\nu}\tilde{\rho}} &= \delta_{\tilde{\mu}i}\delta^{\tilde{\nu}j}\delta^{\tilde{\rho}k} F_{jk}{}^i, \\ \tilde{R}^{\tilde{\mu}\tilde{\nu}\tilde{\rho}} &= 0. \end{aligned} \quad (3.17)$$

where we used that $\tilde{\beta}$ is a 2-cocycle for the Lie algebra with structure constants $Q_i{}^{jk}$. Notice that $\tilde{Q}_{\tilde{\mu}}{}^{\tilde{\nu}\tilde{\rho}}$ is always non-zero when we dualise a non-abelian algebra, and the background has the interpretation of a T-fold. If we turn on \tilde{b} we can also generate geometric fluxes. Finally we have the relations $\tilde{F}_{\mu\nu}{}^\rho = 2\tilde{b}_{\delta[\mu}\tilde{Q}_{\nu]}{}^{\rho\delta}$, $\tilde{H}_{\mu\nu\rho} = -\frac{3}{2}\tilde{b}_{\alpha[\rho}\tilde{F}_{\mu\nu]}{}^\alpha$, notice the different sign in the last equation compared to the YB case.

Comments on “deformed T-duals”. When in the (Q) -orbit we turn on $\tilde{\omega}^{ij}$ (which may include an overall deformation parameter) we generate representatives that can be understood as *deformations* of NATD, and that in [63] were called “deformed T-duals” (DTD). See also [36, 64]. It turns out that YB-deformations are actually related to DTD models [63, 65], and here we rephrase this fact in $O(d, d)$ language. In order to do that we will take the point of view of the YB-deformation, and for this reason we will have structure constants $F_{ij}{}^k$ for the non-abelian Lie algebra \mathfrak{g} . This also means that we will need to lower the indices of $\tilde{\omega}$ with deltas $\tilde{\omega}_{ij} = \delta_{ik}\delta_{jl}\tilde{\omega}^{kl}$ to take care of the duality relation. First let us point out that in the above discussion \mathfrak{g} does not need to be semisimple. Actually, if we want $\tilde{\omega}_{ij}$ to generate a non-trivial deformation from the point of view of DTD, then we must take \mathfrak{g} non semisimple.²⁷ Therefore in the following we will not assume semisimplicity.

²⁷In fact, given that the second Lie algebra cohomology of semisimple algebras is trivial, any $\tilde{\omega}_{ij}$ would be a coboundary (i.e. $\tilde{\omega}_{ij} = F_{ij}{}^k c_k$ for some c_k). Therefore we could remove $\tilde{\omega}$ by redefining the coordinates. It may still be interesting to consider $\tilde{\omega}$ coboundary because we can still relate to it YB deformations, as we are about to see. These will be equivalent to NATD only when the deformation parameter takes *finite* values, the limit $\eta \rightarrow 0$ being degenerate. Prominent examples are Jordanian deformations, see [33, 66].

Nevertheless, we will assume that \mathfrak{g} is a subalgebra of an algebra \mathfrak{f} that admits a non-degenerate symmetric invariant bilinear form κ . If T_i are generators of \mathfrak{g} , we generate the subalgebra \mathfrak{g}^* of \mathfrak{f} by T^i , where we use κ^{-1} to raise the indices of T_i . We can think of $\tilde{\omega}$ as a linear operator in \mathfrak{f} if we restrict it to the subalgebras $\tilde{\omega} : \mathfrak{g} \rightarrow \mathfrak{g}^*$. Let us now assume that $\tilde{\omega}$ is invertible in this restriction and let us call $r : \mathfrak{g}^* \rightarrow \mathfrak{g}$ the inverse of $\tilde{\omega}$. This means that $r\tilde{\omega} = P$ and $\tilde{\omega}r = P^T$, where P, P^T project on $\mathfrak{g}, \mathfrak{g}^*$ respectively. Because $\tilde{\omega}$ is a 2-cocycle in \mathfrak{g} , r satisfies the CYBE (on the whole \mathfrak{f}). Now, given the coordinates y used in the DTD model, let us consider the following change of coordinates

$$y = \zeta P^T \frac{1 - \text{Ad}_{\bar{g}}^{-1}}{\log \text{Ad}_{\bar{g}}} \tilde{\omega} \log \bar{g}, \quad \bar{g} \in G. \quad (3.18)$$

Here $y = y^\mu \delta_{\mu i} T^i \in \mathfrak{g}^*$. One can prove that this change of coordinates implies [64]

$$P^T(\text{ad}_y + \zeta \tilde{\omega})P = \zeta P^T(\tilde{\omega}_{\bar{g}})P, \quad dy = \zeta P^T(\tilde{\omega}_{\bar{g}} \bar{g}^{-1} d\bar{g}), \quad (3.19)$$

where $\tilde{\omega}_{\bar{g}} = \text{Ad}_{\bar{g}}^{-1} \tilde{\omega} \text{Ad}_{\bar{g}}$, which in components read²⁸

$$y^\mu \delta_{\mu k} f_{ij}{}^k + \zeta \tilde{\omega}_{ij} = \zeta (\tilde{\omega}_{\bar{g}})_{ij}, \quad dy^\mu = \zeta d\bar{y}^\nu \bar{\rho}_\nu{}^i (\tilde{\omega}_{\bar{g}})_{ij} \delta^{j\mu}, \quad (3.20)$$

where we define $\bar{\rho}$ such that $d\bar{y}^\nu \bar{\rho}_\nu{}^i T_i = \bar{g}^{-1} d\bar{g}$. We can take into account the above change of coordinates in the twist \tilde{U} under consideration by using the above substitution and Jacobian, so that

$$\tilde{U} = \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} \begin{pmatrix} 1 & \tilde{\beta} \\ 0 & 1 \end{pmatrix} \rightarrow \bar{U} = \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} \begin{pmatrix} 1 & \zeta \delta \omega_{\bar{g}} \delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta (\bar{\rho} \omega_{\bar{g}} \delta)^t & 0 \\ 0 & \zeta^{-1} (\bar{\rho} \omega_{\bar{g}} \delta)^{-1} \end{pmatrix} \quad (3.21)$$

where the last matrix implements the Jacobian, and we are writing explicitly the δ^{ij} or δ_{ij} while omitting the indices. A straightforward calculation gives

$$\bar{U} = \begin{pmatrix} 0 & \eta r_{\bar{g}} \bar{\rho}^{-1} \\ -\zeta \omega_{\bar{g}} \bar{\rho}^t & \bar{\rho}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \eta r_{\bar{g}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\zeta \omega_{\bar{g}} & 1 \end{pmatrix} \begin{pmatrix} \bar{\rho}^t & 0 \\ 0 & \bar{\rho}^{-1} \end{pmatrix} \quad (3.22)$$

where we used that r is the inverse of ω and $\eta = \zeta^{-1}$. Thanks to a diffeomorphism, we were therefore able to rewrite the twist of DTD in terms of the one of a YB-deformation, plus a shift of b so that $H_{ijk} = 0$. Interestingly, this \bar{U} is not included in the discussion of appendix C.

3.4 R

Having $H = F = Q = 0$ immediately implies that $R = 0$. Indeed, it is well-known that the (R) -orbit is not realisable when imposing the strong constraint. Notice that therefore we cannot include the rigid T -transformation of the (H) -orbit as a solution-generating technique.

²⁸When writing the equations in components we are automatically implementing the projectors, because the indices i, j were restricted to \mathfrak{g} from the beginning.

3.5 F, Q

In the (F, Q) -orbit the Bianchi identities imply that both the F - and Q -flux satisfy the Jacobi identity on their own, and in addition there is an identity mixing them

$$F_{ij}{}^k Q_k{}^{lm} - 4Q_{[i}{}^{k[l} F_{j]k}{}^{m]} = 0. \quad (3.23)$$

Hence, F and Q can be interpreted as the structure constants of Lie algebras \mathfrak{g} and $\tilde{\mathfrak{g}}$ respectively, with generators T_i and \tilde{T}^i . Together with the ad-invariant pairing η the structure $\mathfrak{r} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$ is known as a Drinfel'd double. Furthermore the Bianchi's for \mathcal{F}_I read

$$F_{ij}{}^k \mathcal{F}_k = 0, \quad Q_k{}^{ij} \mathcal{F}^k = 0, \quad Q_i{}^{jk} \mathcal{F}_k - F_{ik}{}^j \mathcal{F}^k = 0, \quad \mathcal{F}_k \mathcal{F}^k = \frac{1}{2} F_{ij}{}^k Q_k{}^{ij}. \quad (3.24)$$

The first two identities imply

$$\mathcal{F}_i = 0 \text{ if } T_i \in [\mathfrak{g}, \mathfrak{g}], \quad \mathcal{F}^i = 0 \text{ if } T^i \in [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]. \quad (3.25)$$

Finally notice that by tracing the mixed Jacobi identity (3.23) over $i = l$ and $j = m$ we can also write the last identity as $\mathcal{F}_i \mathcal{F}^i = Q_i{}^{ji} F_{jk}{}^k$.

Let us now solve the flux equations (2.32) for the twist $U(\rho, b, \beta)$. The solution for the ρ - and b -twist found from the F - and H -flux equation, respectively, are equivalent to the solutions derived and explained in the (F) -orbit, see section 3.2. Recall that, up to diffeomorphisms, we have $\rho_i{}^\mu = W_i{}^j \bar{\rho}_j{}^\mu$ with W an outer automorphism of \mathfrak{g} and $\bar{\rho} = \bar{g}^{-1} d\bar{g}$ of (left-invariant) MC form. The β -twist, however, must now solve

$$Q_i{}^{jk} = \partial_i \beta^{jk} + 2\beta^{[j|l} F_{il}{}^{k]}. \quad (3.26)$$

Notice that this is a linear inhomogeneous partial differential equations (PDE). The most general solution to such equations is found by adding the most general homogeneous solution to a particular solution $\beta_{\text{inhom.}}$ of (3.26). The β -twist of Poisson-Lie symmetric backgrounds [11, 12, 67], appropriately dressed by the automorphism W , is an example of such a particular solution. It is given by

$$\beta_{\text{inhom.}}^{ij} = \langle \text{Ad}_{\bar{g}^{-1}} \cdot P \cdot \text{Ad}_{\bar{g}} \widetilde{W}(\tilde{T}^i), \widetilde{W}(\tilde{T}^j) \rangle, \quad (3.27)$$

where $\widetilde{W} = W^{-t}$ is an automorphism of $\tilde{\mathfrak{g}}$, P projects on \mathfrak{g} , and recall that we have defined the bracket $\langle \mathbb{T}_I, \mathbb{T}_J \rangle = \eta_{IJ}$.²⁹ The most general homogeneous solution is known from the discussion in the (F) -orbit, i.e. given by (3.4), which we denote here as

$$\beta_{\text{hom.}}^{ij} = \eta \langle r_{\bar{g}}(T^i), T^j \rangle, \quad (3.28)$$

where, recall, $r_{\bar{g}} \equiv W^{-1} \circ \text{Ad}_{\bar{g}}^{-1} \circ r \circ \text{Ad}_{\bar{g}} \circ W$, $r^t = -r$ and $T^i = \kappa^{ij} T_j$ with $\kappa_{ij} = \langle T_i, T_j \rangle$. Notice also that here we explicitly introduce a *deformation* parameter η . At this point this is not necessary, but we can do it because we are solving a homogeneous equation, and it will be useful to solve for the R -flux equation. Therefore, turning on $\beta_{\text{hom.}}$ can be seen as

²⁹The property $\widetilde{W} = W^{-t}$ follows from requiring that W and \widetilde{W} form an automorphism in the Drinfel'd double as $W(\mathfrak{r}) = \widetilde{W}(\tilde{\mathfrak{g}}) \oplus W(\mathfrak{g})$ which preserves the bilinear form η_{IJ} .

a deformation of the PL-symmetric background at $\eta = 0$. Concluding, the most general solution to (3.26) is $\beta^{ij} = \beta_{\text{inhom.}}^{ij} + \beta_{\text{hom.}}^{ij}$. Solving for $R^{ijk} = 0$ gives additional algebraic conditions on the operator r . In particular, by expanding order by order in η , it must satisfy the equations³⁰

$$2\beta_{\text{inhom.}}^{l[i} \beta_{\text{hom.}}^{j]m} F_{lm}{}^{|k]} - \beta_{\text{hom.}}^{l[i} Q_l{}^{jk]} = 0, \tag{3.29}$$

$$\beta_{\text{hom.}}^{l[i} \beta_{\text{hom.}}^{j]m} F_{lm}{}^{|k]} = 0, \tag{3.30}$$

where we have used the identity $\beta_{\text{inhom.}}^{l[i} \beta_{\text{inhom.}}^{j]m} F_{lm}{}^{|k]} - \beta_{\text{inhom.}}^{l[i} Q_l{}^{jk]} = 0$ (see e.g. [67]). Therefore, from the second condition we find again the requirement that r must satisfy the CYBE (3.6).³¹ The first condition, on the other hand, should be viewed as a compatibility condition between $\beta_{\text{inhom.}}$ and $\beta_{\text{hom.}}$. In order to interpret it in a field-independent way it will be convenient to write out the components of the adjoint action by \bar{g} in the Drinfel'd double \mathfrak{r} ,

$$\bar{g}T_i\bar{g}^{-1} = (\text{Ad}_{\bar{g}})_i{}^j T_j, \quad \bar{g}\tilde{T}^i\bar{g}^{-1} = (\text{Ad}_{\bar{g}})^{ij} T_j + (\text{Ad}_{\bar{g}})_i{}^j \tilde{T}^j. \tag{3.31}$$

From $\text{Ad}_{\bar{g}} \in O(d, d)$ and $\text{Ad}_{\bar{g}} \circ \text{Ad}_{\bar{g}^{-1}} = 1$ we can derive the relations $(\text{Ad}_{\bar{g}})^i{}_j = (\text{Ad}_{\bar{g}^{-1}})_j{}^i$, $(\text{Ad}_{\bar{g}^{-1}})^{ij} = (\text{Ad}_{\bar{g}})^{ji}$ and $(\text{Ad}_{\bar{g}})^{ij} = -(\text{Ad}_{\bar{g}^{-1}})_l{}^i (\text{Ad}_{\bar{g}})^{ml} (\text{Ad}_{\bar{g}})_m{}^j$. The solution for β^{ij} can now be written as

$$\beta^{ij} = (W^{-1})_l{}^i \bar{\beta}^{lm} (W^{-1})_m{}^j, \quad \bar{\beta}^{ij} = (\text{Ad}_{\bar{g}})^{il} (\text{Ad}_{\bar{g}^{-1}})_l{}^j + \eta (\text{Ad}_{\bar{g}^{-1}})_l{}^i r^{lm} (\text{Ad}_{\bar{g}^{-1}})_m{}^j. \tag{3.32}$$

Using the previous relations, the automorphism properties for W and those for $\text{Ad}_{\bar{g}}$ in \mathfrak{r} , in particular using

$$Q_i{}^{jk} = (\text{Ad}_{\bar{g}})_i{}^l (\text{Ad}_{\bar{g}^{-1}})_m{}^j (\text{Ad}_{\bar{g}^{-1}})_n{}^k Q_l{}^{mn} - 2(\text{Ad}_{\bar{g}})_i{}^l (\text{Ad}_{\bar{g}^{-1}})_m{}^j (\text{Ad}_{\bar{g}})^{kn} F_{ln}{}^m, \tag{3.33}$$

we find that the compatibility condition (3.29) takes the following simple form

$$r^{l[i} Q_l{}^{jk]} = 0, \tag{3.34}$$

which is the condition of r^{ij} being a 2-cocycle of $\tilde{\mathfrak{g}}$.

It is interesting to see in which cases we are genuinely in the (F, Q) -orbit modulo rigid $O(d, d)$ transformations. From (2.27) we find that we can turn off the Q -flux — and thus describe only the (F) -orbit — when it is of the form $Q_i{}^{jk} = 2F_{il}{}^{[j} \beta^{k]l}$ for some constant antisymmetric β^{ij} which satisfies the CYBE $\beta^{l[i} F_{lm}{}^j \beta^{k]m} = 0$ on \mathfrak{g} . Equivalently, the F -flux can be turned off by a rigid $O(d, d)$ when it is of the form $F_{ij}{}^k = -2b_{l[i} Q_j]{}^{lk}$ for some constant antisymmetric b that satisfies the CYBE $b_{l[i} b_{jm} Q_k]{}^{lm} = 0$ on $\tilde{\mathfrak{g}}$.

For the generalised flux \mathcal{F}_I we have

$$\mathcal{F}_i = F_{ij}{}^j + 2\partial_i \bar{\lambda}, \quad \mathcal{F}^i = Q_j{}^{ji} - \beta^{jl} F_{jl}{}^i + 2\beta^{ij} \partial_j \bar{\lambda}. \tag{3.35}$$

³⁰This is where the explicit parameter η becomes useful. Because of the presence of η , the R -flux equation gives two equations for $\beta_{\text{inhom.}}$ and $\beta_{\text{hom.}}$ which have a nice interpretation. However this splitting is not necessary, and in general one gets one single algebraic constraint on β .

³¹Let us point out that when $\tilde{\mathfrak{g}}$ is abelian, and thus $Q_i{}^{jk}$ is vanishing, then $\beta_{\text{inhom.}}^{ij} = 0$, and we safely reduce to the most general representative of the (F) -orbit in which r must satisfy the CYBE.

Similarly to the (F) -orbit we must have that $F_{ij}{}^k \partial_k \bar{\lambda} = 0$ such that $\partial_i \bar{\lambda} = \alpha_i$, and thus \mathcal{F}_i , is constant. Additionally, notice that the β -twist must be such that \mathcal{F}^i is constant. One can verify that this is already implied by using the ordinary Jacobi identities, the trace of the mixed Jacobi identity (3.23) over j, m and the mixed Bianchi identity for \mathcal{F}^i and \mathcal{F}_i .

Summarising, in this orbit we identified the possibility of a generalisation of a homogeneous YB-deformation — compatible as a 2-cocycle with the Q -flux — which deforms the ordinary PL symmetric backgrounds [11, 12] described previously in the literature.³² Interestingly these deformations can be implemented even when the initial background has no isometry, in contrast to the usual homogeneous YB-deformations described in the (F) -orbit.

3.6 F, R

In the (F, R) -orbit the Bianchi identities imply that $F_{ij}{}^k$ are the structure constants of the Lie algebra \mathfrak{g} generated by T_i and that the R -flux satisfies $F_{nm}{}^{[k} R^{ij]n} = 0$. In turn the latter implies the (weaker) condition that $R_{ijk} = \kappa_{il} \kappa_{jm} \kappa_{kn} R^{lmn}$ with $\kappa_{ij} = \langle T_i, T_j \rangle$ is a 3-cocycle of \mathfrak{g} . For the fluxes with one-index we must impose that $F_{ij}{}^k \mathcal{F}_k = F_{ik}{}^j \mathcal{F}^k = R^{ijk} \mathcal{F}_k = 0$ as well as the orthogonality $\mathcal{F}_i \mathcal{F}^i = 0$. Hence we must have the same conditions as given in (3.1) together with $R^{ijk} \mathcal{F}_k = 0$.

The solutions for (ρ, b, β) to the flux equations (2.32) are equivalent to the solutions of the (F) -orbit apart from the condition that follows from the R -flux equation. Hence, the expression for the β -twist is given in (3.4) and the solution for ρ is given by $\rho_i{}^\mu = W_i{}^j \bar{\rho}_j{}^\mu$, where W is an outer automorphism and $\bar{\rho} = \bar{g}^{-1} d\bar{g}$ is a left-invariant MC of \mathfrak{g} . The R -flux equation now reads by using $W, \text{Ad}_{\bar{g}} \in \text{Aut}(\mathfrak{g})$

$$(W_{\bar{g}})_l{}^i (W_{\bar{g}})_m{}^j (W_{\bar{g}})_n{}^k R^{lmn} = 3r^{l[i} \bar{r}^{j]m} F_{lm}{}^{[k]}, \quad W_{\bar{g}} \equiv W \cdot \text{Ad}_{\bar{g}}. \quad (3.36)$$

Using the fact that in \mathfrak{r} the adjoint action by \bar{g} is of block-diagonal form, i.e. $(\text{Ad}_{\bar{g}})_{ij} = (\text{Ad}_{\bar{g}})^{ij} = 0$, and that $\text{Ad}_{\bar{g}}$ is an automorphism of \mathfrak{r} , one can derive the following identity $(\text{Ad}_{\bar{g}})_l{}^i (\text{Ad}_{\bar{g}})_m{}^j (\text{Ad}_{\bar{g}})_n{}^k R^{lmn} = R^{ijk}$, so that the condition (3.36) is in reality field-independent and can be written as

$$R^{ijk} = 3\bar{r}^{l[i} \bar{r}^{j]m} F_{lm}{}^{[k]}, \quad \bar{r}^{ij} \equiv (W^{-1})_l{}^i r^{lm} (W^{-1})_m{}^j. \quad (3.37)$$

A natural choice is to take the R -flux of the form

$$R^{ijk} = \alpha \kappa^{il} \kappa^{jm} F_{lm}{}^k, \quad (3.38)$$

which is a 3-cocycle of \mathfrak{g} , and where α is a real constant. This is of course always a possible choice. Assuming it from now on, (3.37) requires that r solves the modified CYBE (mCYBE), that is

$$[rx, ry] - r([rx, y] + [x, ry]) = -c^2[x, y], \quad \forall x, y \in \mathfrak{g}, \quad (3.39)$$

with, up to redefinitions of the r operator, $c^2 = \alpha = \{-1, +1\}$. Here $c^2 = 1$ is known as a split r -matrix while $c^2 = -1$ as a non-split r -matrix. Notice that when r satisfies the

³²As in the case of the standard homogeneous YB-deformation of the (F) -orbit, it should be possible to understand also this case as a version of PL plurality, and it would be interesting to see this explicitly.

mCYBE then so does \bar{r} . Turning on an outer automorphism W can therefore be seen as mapping different solutions of the mCYBE to each other. When \mathfrak{g} is a semisimple (and bosonic) Lie algebra, the canonical solution of the mCYBE is known as the Drinfel'd-Jimbo r -matrix [68, 69] which is unique [70] up to a $GL(l, \mathbb{C})$ freedom on the Cartan subalgebra (CSA) directions of \mathfrak{g} with l the rank of \mathfrak{g} . In particular the canonical Drinfel'd-Jimbo r -matrix is given in a Cartan-Weyl basis of the complexified algebra $\mathfrak{g}_{\mathbb{C}}$ and annihilates the Cartan generators while multiplying positive and negative roots with $\mp c$ respectively. When the real form $\mathfrak{g}_{\mathbb{C}}$ is compact, no split solutions exist and one can only consider a non-split r -matrix. When the real form is non-compact, however, both possibilities can exist. Since an outer automorphism of semisimple algebras maps the CSA and the set of simple roots to itself, the inclusion of W will not affect the canonical r -matrix.

To truly sit in the (F, R) -orbit we must take $c^2 \neq 0$ such that the R -flux does not vanish. This means that “turning on” r (equivalently, β), e.g. by means of a deformation parameter, can not be seen as a solution-generating technique in the (F, R) -orbit. Hence, in the (F, R) -orbit, and assuming the flux configuration of eq. (3.38), we can describe only the split ($c^2 = 1$) and the non-split ($c^2 = -1$) inhomogeneous Yang-Baxter models (see e.g. [34, 35, 71]). One can verify that in these cases we are genuinely in (F, R) under rigid $O(d, d)$ equivalence relations. In other words, we cannot turn off the R -flux and F -flux as can be seen by rewriting (2.27) for $H = Q = H' = Q' = 0$ such that

$$R^{ijk} = 3\bar{r}^{l[i}\bar{r}^{j]m}F_{lm}^{k]} + 3\beta^{l[i}\beta^{j]m}F_{lm}^{k]} \tag{3.40}$$

for some constant antisymmetric matrix β^{ij} . In general there may be no real solution for β^{ij} that sets R^{ijk} to zero.³³ When we cannot turn off R^{ijk} , we can also conclude that we cannot turn off $F_{ij}{}^k$. Otherwise, we would be in the (R) -orbit and recall that this orbit cannot be realised under the strong constraint.

However, an interesting observation is that these (F, R) -representatives can be mapped, using a particular rigid $O(d, d)$ transformation, to the self-dual (F, Q) -orbit of Poisson-Lie symmetric backgrounds. From (2.27) and taking $b_{ij} = 0$ and $\beta^{ij} = \langle \bar{r}T^i, T^j \rangle$ we find that³⁴

$$H'_{ijk} = 0, \quad F'_{ij}{}^k = F_{ij}{}^k, \quad Q'_i{}^{jk} = -2F_{il}{}^{[j}\beta^{k]l}, \quad R'^{ijk} = 0, \tag{3.41}$$

for generic c^2 , so that this discussion applies also to the homogeneous YB-deformations of the (F) -orbit. Here $Q'_i{}^{jk}$ are the structure constants of \mathfrak{g}_r whose Lie bracket is defined as

$$[x, y]_r = [rx, y] + [x, ry], \quad \forall x, y \in \mathfrak{g}, \tag{3.42}$$

and which precisely underlies the bi-algebra structure corresponding to the Drinfel'd double $\mathfrak{r} = \mathfrak{g} \oplus \mathfrak{g}_r$ that in the $c^2 = -1$ case gives rise to the PL-duality between the inhomogeneous YB-model and the λ^* -deformation [72–74]. Therefore, when assuming the expression for R^{ijk} given in (3.38), the discussion of the solution-generating techniques in this orbit (as

³³Interestingly, it would be possible to remove the R -flux if the algebra \mathfrak{g} admitted both split and non-split solutions of the mCYBE. It would also be interesting to understand the consequences of removing R by relaxing the reality of β .

³⁴Notice that here we do not require $Q' = 0$ so that we do not have the expression (3.40) for R' .

well as the discussion on the \mathcal{F}_I -flux equations) is implicitly captured by the discussion given in section 3.5. As already remarked, the above rigid $O(d, d)$ transformation that sends us to the (F, Q) -orbit can be understood as a notion of a PL-plurality transformation.

3.7 Q, R

In the (Q, R) -orbit the Bianchi identities imply that $Q_i{}^{jk}$ can be interpreted as structure constants of a Lie algebra and that R^{ijk} is a 3-cocycle. Solving the flux equations (2.32) gives, for the same reasons as in section 3.3 for the (Q) -orbit, that $\rho_\mu{}^i = \delta_\mu^i$ and that β^{ij} is given by (3.12). Using the Bianchi identity the R -flux equation becomes

$$R^{ijk} = -3\tilde{\omega}^{[i}Q_l{}^{jk]}, \tag{3.43}$$

where recall $\tilde{\omega}^{ij}$ is an antisymmetric constant matrix. Importantly, a rigid $O(d, d)$ transformation as in (2.27) can undo this R -flux contribution at no other expense, and therefore we are effectively describing the (Q) -orbit.³⁵ Hence, the genuine (Q, R) -orbit with non-vanishing R -flux can not be realised when imposing the strong constraint, and therefore we do not expect to be able to apply a rigid T -transformation from the (F, H) -orbit when the H -flux is non-vanishing. This is reminiscent of the known anomaly obstructions to gauging global symmetries in WZW models [75], which are part of the (F, H) -orbit.³⁶

3.8 F, Q, R

In the (F, Q, R) -orbit the $F_{ij}{}^k$ flux can still be interpreted as the structure constants of a Lie algebra \mathfrak{g} spanned by the generators T_i , while in general the $Q_i{}^{jk}$ flux satisfies Bianchi identities mixed with R^{ijk} and $F_i{}^{jk}$, see (2.15)—(2.17).

To solve for the twist functions (ρ, b, β) of U parametrised as in (2.20) we can observe that the flux equations for $H_{ijk}, F_{ij}{}^k$ and $Q_i{}^{jk}$ are identical to the equations in the (F, Q) -orbit of section 3.5. Therefore, the b -twist is found from the second de Rham cohomology of the manifold while the ρ - and β -twist take the form³⁷

$$\rho = g^{-1}dg, \quad \beta^{ij} = \langle\langle \text{Ad}_{g^{-1}} \cdot P \cdot \text{Ad}_g \tilde{T}^i, \tilde{T}^j \rangle\rangle + \eta \langle r_g(T^i), T^j \rangle, \tag{3.44}$$

with $g \in \exp \mathfrak{g}$ and $r_g = \text{Ad}_{g^{-1}} \circ r \circ \text{Ad}_g$. For simplicity we have dropped here the possibility of automorphisms of \mathfrak{t} but recall from the general discussion around eq. (2.24) that this is always possible.³⁸ What does change in this orbit, however, is the algebraic R -flux

³⁵In section 3.3 for the (Q) -orbit, $\tilde{\omega}^{ij}$ was constrained to be a 2-cocycle in order to have vanishing R -flux. In that discussion it was assumed that the symmetry in (2.23) had been fixed, or in other words that \tilde{E} is not allowed to transform. Here we do not need to impose any constraint on $\tilde{\omega}^{ij}$ because the R -flux is not required to vanish at the start, it is rather removed by the transformation (2.23) under which \tilde{E} is allowed to transform.

³⁶Notice that in our framework we can only deform/dualise one copy of the symmetry group of the WZW model, e.g. the left one.

³⁷See section 3.5 for the definitions of the various objects, which are not affected by the presence of the R -flux.

³⁸For example, recall that $\rho_i{}^\mu = W_i{}^j \bar{\rho}_j{}^\mu$ with $W_i{}^j \in \text{Out}(\mathfrak{g})$ and $\bar{\rho} = \bar{g}^{-1}d\bar{g}$, $\bar{g} \in \exp \mathfrak{g}$, is also a solution for ρ . When we take for instance the automorphism $W_I{}^L \in \text{Aut}(\mathfrak{t})$ to be block-diagonal, i.e. $W_{ij} = W^{ij} = 0$ then we will have $\widetilde{W}^i{}_j \equiv W^i{}_j = (W^{-1})_j{}^i$ as well as the relations $Q_i{}^{jk} = W_i{}^l W^j{}_m W^k{}_n Q_l{}^{mn}$ and $R^{ijk} = W^i{}_l W^j{}_m W^k{}_n R^{lmn}$, so that the solution of the Q -flux equation for β^{ij} given in (3.27) still holds.

equation. We find

$$R^{ijk} = 3\beta_{\text{hom.}}^{l[i} \beta_{\text{hom.}}^{j]m} F_{lm}{}^{k]} - 3\beta_{\text{hom.}}^{l[i} Q_l^{jk]} + 6\beta_{\text{inhom.}}^{l[i} \beta_{\text{hom.}}^{j]m} F_{lm}{}^{k]}, \quad (3.45)$$

or in terms of the adjoint action by g in \mathfrak{r} ,

$$(\text{Ad}_g)_l^i (\text{Ad}_g)_m^j (\text{Ad}_g)_n^k R^{lmn} = 3\eta^2 r^{l[i} r^{j]m} F_{lm}{}^{k]} - 3\eta r^{l[i} Q_l^{jk]} \quad (3.46)$$

where we have used the identity (3.33) (which we note is not affected by the presence of R -flux). In the (F, Q) orbit, in order to have vanishing R -flux, r was required to be a solution of the CYBE for $F_{ij}{}^k$ and to satisfy the 2-cocycle condition with $Q_i{}^{jk}$. These conditions are now relaxed by the presence of the non-trivial R -flux. Notice that the right-hand-side of the above equation is constant (i.e. g -independent), which puts a strong constraint on the left-hand-side. Taking $g = e^x$ with $x \in \mathfrak{g}$ and expanding order by order in x we have respectively at leading and first order

$$R^{ijk} = 3\eta^2 r^{l[i} r^{j]m} F_{lm}{}^{k]} - 3\eta r^{l[i} Q_l^{jk]}, \quad F_{lm}{}^{[i} R^{jk]m} = 0. \quad (3.47)$$

The first condition then implies that we can remove the R -flux using a rigid $O(d, d)$ transformation. Indeed taking in (2.27) $b = 0$ and $\beta = -\eta r$ we can turn off the R -flux at the expense of shifting the Q -flux and, therefore, we are effectively describing here the (F, Q) -orbit. Therefore we do not expect to be able to apply a rigid T -transformation from the (F, H, Q) -orbit of quasi-Manin triples to the (F, Q, R) -orbit.

3.9 H

We now start the discussion of orbits with non-vanishing H -flux, following the framework explained in section 2.4. The first example is the (H) -orbit. Notice that the Bianchi identities for \mathcal{F}_{IJK} are trivially satisfied while those for \mathcal{F}_I require $H_{ijk}\mathcal{F}^k = 0$ and the orthogonality $\mathcal{F}_i\mathcal{F}^i = 0$.

Using the notation of section 2.4, for the adjoint action by m and the one-form V^I we simply find in this orbit that

$$M_I{}^J = \begin{pmatrix} \delta_j^i & 0 \\ m^k H_{ijk} & \delta_i^j \end{pmatrix}, \quad V_\mu{}^i = \partial_\mu m^i, \quad V_{\mu i} = -\frac{1}{2} V_\mu{}^j m^k H_{ijk}. \quad (3.48)$$

As argued, since $M^{ij} = 0$, we continue with the twist $U_I^{(1)M}$ to solve the flux equations (2.32). In terms of the functions ρ, b, β of (2.20) we have $\rho_\mu{}^i = V_\mu{}^i$, $\beta^{ij} = \xi^{ij}$ and $b_{ij} = \omega_{ij} + 2V_{[i}{}^\mu V_{\mu j]}$. In fact this $\rho_\mu{}^i$ can be gauged away by a diffeomorphism to get $\rho_\mu{}^i = \delta_\mu^i$, which we understand as a consequence of the ansatz $U_I^{(1)M}$, and thus we have $w_{ij}{}^k = 0$. Consequently, solving for $F_{ij}{}^k = 0$ constrains β^{ij} to satisfy

$$\beta^{kl} H_{ijl} = 0. \quad (3.49)$$

Solving for $Q_i{}^{jk} = 0$ gives that β^{ij} must be constant, and $R^{ijk} = 0$ is solved automatically. From (2.27) one sees that this constant β -transformation satisfying (3.49) is precisely an automorphism of the pre-Roytenberg algebra, and therefore it can be removed by a rigid

$O(d, d)$ transformation as in (2.23). Then we are describing the example of the torus with H -flux, see e.g. [76]. To complete the discussion we can also solve for the b -twist, or equivalently ω_{ij} . By defining $b_{\mu\nu} = \rho_\mu{}^i b_{ij} \rho_\nu{}^j$ and observing that $b_{\mu\nu} = \omega_{\mu\nu} - 2\bar{\omega}_{\mu\nu}$, with $\bar{\omega}$ defined as in (2.44), we find that

$$3\partial_{[\mu}\omega_{\nu\rho]} = -2V_\mu{}^i V_\nu{}^j V_\rho{}^k H_{ijk} \tag{3.50}$$

which has a particular solution $\omega_{\mu\nu}^{\text{inhom.}} = \frac{4}{3}V_\mu{}^i V_\nu{}^j$, so that we have an explicit expression at least locally. The most general solution is $\omega_{\mu\nu} = \omega_{\mu\nu}^{\text{inhom.}} + \omega_{\mu\nu}^{\text{hom.}}$ with $\omega^{\text{hom.}}$ a closed two-form.

For the generalised flux \mathcal{F}_I we have $\mathcal{F}_i = 2\partial_i\bar{\lambda}$ and $\mathcal{F}^i = 0$ (after setting $\beta = 0$ by the rigid $O(d, d)$ transformation) so that the above Bianchi identities are automatically satisfied.

Another class of representatives. Let us now present another class of representatives within the (H) -orbit that are not captured by the above discussion and the methods of section 2.4.³⁹ We will try to look for representatives with non-vanishing $w_{ij}{}^k$. In particular we will restrict to the case of *constant* $w_{ij}{}^k$ and we will want to interpret them as structure constants of a Lie algebra \mathfrak{g} , so that we will prefer to use the notation $f_{ij}{}^k = w_{ij}{}^k$. This can be achieved simply by taking $\rho = g^{-1}dg$ of MC form, with g a group element of G such that $\mathfrak{g} = \text{Lie}(G)$. Then from $F_{ij}{}^k = 0$ we have

$$f_{ij}{}^k = -\beta^{kl} H_{ijl}. \tag{3.51}$$

For \mathfrak{g} non-abelian this is possible only if we turn on a certain β , and in general it is consistent only if $\beta^{kl} H_{ijl}$ is constant. The Jacobi identity for the structure constants $f_{ij}{}^k$ now implies

$$\beta^{nl} H_{n[ij} H_{k]lp} \beta^{mp} = 0. \tag{3.52}$$

While a non-constant β may be still possible, let us take a constant β to simplify the discussion further. Then from $Q_i{}^{jk} = 0$ it follows

$$\beta^{lj} f_{il}{}^k = 0, \quad \implies \quad \beta^{jl} \beta^{km} H_{ilm} = 0, \tag{3.53}$$

which is a weaker condition compared to (3.49). Notice that when this condition holds, both $R^{ijk} = 0$ and the Jacobi identity for $f_{ij}{}^k$ are automatically satisfied. For the fluxes \mathcal{F}_I we have that $\mathcal{F}_i = -H_{ijl}\beta^{jl} + 2\partial_i\bar{\lambda}$ and $\mathcal{F}^i = 2\beta^{jk}\partial_k\bar{\lambda}$, where (3.53) was used, and the Bianchi identities involving \mathcal{F}_I are automatically satisfied. Constancy of the fluxes imposes also the condition $f_{ij}{}^k\partial_k\bar{\lambda} = 0$, which is equivalent to the Bianchi identity $H_{ijk}\mathcal{F}^k = 0$.

One can check that in $d = 3$ there is no solution to (3.53) but already in $d = 4$ there are several. Given a basis e_i we first write the 3-form H -flux as

$$H = 4h_{[l}e_i \wedge e_j \wedge e_k] = h_1e_2 \wedge e_3 \wedge e_4 - h_2e_1 \wedge e_3 \wedge e_4 + h_3e_1 \wedge e_2 \wedge e_4 - h_4e_1 \wedge e_2 \wedge e_3. \tag{3.54}$$

³⁹For more comments on solving the (H) -orbit without relying on section 2.4 see appendix F.

Then one possible solution to (3.53) is found by setting for example all $\beta^{ij} = 0$ except β^{23} , and $h_1 = h_4 = 0$. The only non-vanishing components of the structure constants are then

$$f_{14}{}^2 = -h_2\beta^{23}, \quad f_{14}{}^3 = -h_3\beta^{23}, \quad (3.55)$$

which define a Heisenberg algebra

$$[t_1, t_4] = -\beta^{23}(h_2t_2 + h_3t_3). \quad (3.56)$$

Interestingly we may view this class of representatives as a deformation of the representatives described previously. It would be nice to extend the methods of section 2.4 to include these also.

3.10 H, R

In the (H, R) -orbit the Bianchi identities read

$$H_{ijk}R^{klm} = 0, \quad H_{ijk}\mathcal{F}^k = 0, \quad R^{ijk}\mathcal{F}_k = 0, \quad \mathcal{F}_i\mathcal{F}^i = 0. \quad (3.57)$$

Following section 2.4, one can verify that, compared to the (H) -orbit, the adjoint action by m and the one-form $dm m^{-1}$ do not change with the presence of R -flux and thus are given in (3.48). Using the Bianchi identities notice that we therefore have that $V_{\mu l}R^{ijl} = 0$.⁴⁰ Continuing with the parametrisation $U_I^{(1)M}$ and solving for the flux equations with $\rho_{\mu}{}^i = V_{\mu}{}^i$, $\beta^{ij} = \xi^{ij}$ and $b_{ij} = \omega_{ij} + 2V_i{}^{\mu}V_{\mu j}$ we ultimately find that R^{ijk} must be vanishing and, therefore, that we reduce to the (H) -orbit. Hence, we must conclude that the (H, R) -orbit (with non-vanishing R -flux) can not be realised within the framework of section 2.4.

3.11 F, H

In the (F, H) -orbit the Bianchi identities imply that $F_{ij}{}^k$ satisfies the Jacobi identity and can be interpreted as the structure constants of a Lie algebra that we will call \mathfrak{f} . We should however point out that the generators are not T_i (since their commutation relations in \mathfrak{r} are of the form $[T, T] \sim FT + H\tilde{T}$), and we will denote them by $t_i \in \mathfrak{f}$. Moreover, the Bianchi identities imply that H_{ijk} is a 3-cocycle of \mathfrak{f} and that $H_{ijk}\mathcal{F}^k + F_{ij}{}^k\mathcal{F}_k = 0$, $F_{ik}{}^j\mathcal{F}^k = 0$ and $\mathcal{F}_i\mathcal{F}^i = 0$, and in particular $\mathcal{F}^k = 0$ if $t_k \notin Z(\mathfrak{f})$. When \mathfrak{f} is semisimple the Bianchi identities imply $\mathcal{F}_i = \mathcal{F}^i = 0$.

Let us now ask when we are genuinely inside this orbit, and not in simpler sub-orbits, or in other words when there exists a transformation of the pre-Roytenberg algebra that sets either F or H (or both) to zero. From (2.27) one sees that H can be removed by a rigid $O(d, d)$ if it is of the form $H_{ijk} = -3F_{[ij}{}^l b_{k]l}$ for some constant b , i.e. if H is a coboundary for \mathfrak{f} . We therefore want to restrict ourselves to H in the third Lie algebra cohomology of \mathfrak{f} . For \mathfrak{f} semisimple $H^3(\mathfrak{f}, \mathbb{R}) = \mathbb{R}^n$ [77], where n labels the number of simple factors in \mathfrak{f} , such that one can take $H_{ijk} = \sum_{a=1}^n \alpha_a F_{ij}{}^l \kappa_{lk}$ with $\kappa_{ij} = \langle t_i, t_j \rangle$, and α_a , $a = 1, \dots, n$ real constants. Viceversa, starting from (F, H) and using (2.27) one sees that F can be

⁴⁰This is consistent with the MC identity (2.36) for $V_{\mu}{}^i$.

removed if it is of the form $F_{ij}{}^k = H_{ijl}\beta^{lk}$ for some constant β which additionally satisfies $H_{ilm}\beta^{jl}\beta^{km} = 0$ such that the Q and R -fluxes are to remain vanishing.

Following section 2.4, the adjoint action by m now takes the form

$$M_I{}^J = \begin{pmatrix} M^i{}_j & 0 \\ M_{ij} & M_i{}^j \end{pmatrix}, \quad (3.58)$$

where, using ad-invariance of η_{IJ} , we have $M^i{}_j = (M^{-1})_j{}^i$ and $M_{ij} = -M_i{}^l M_{kl} (M^{-1})_j{}^k$. Furthermore, using that $M \in \text{Aut}(\mathfrak{t})$ we have the following identities

$$\begin{aligned} F_{ij}{}^l M_l{}^k &= M_i{}^l M_j{}^m F_{lm}{}^k, \\ F_{ij}{}^l M_{lk} + H_{ijl} (M^{-1})_k{}^l &= 2M_{[i|l} M_{j]}{}^m F_{mk}{}^l + M_i{}^l M_j{}^m H_{lmk}, \end{aligned} \quad (3.59)$$

of which the last identity can be rewritten as

$$M_i{}^l M_j{}^m M_k{}^n H_{lmn} = H_{ijk} + 3F_{[ij}{}^l M_k]{}^n M_{ln}. \quad (3.60)$$

Notice in addition that the components $M_i{}^j$ give an automorphism of \mathfrak{f} .

Since $M^{ij} = 0$ we prefer the parametrisation $U_I^{(1)M}$ for the twist, that corresponds to

$$\rho_i{}^\mu = M_i{}^j V_j{}^\mu, \quad \beta^{ij} = \xi^{ij}, \quad b_{ij} = \omega_{ij} + M_{ik} M_j{}^k. \quad (3.61)$$

Using the MC identity (2.36), the expression for the derivatives of $M_I{}^J$ (2.41) and the automorphism properties, we find $w_{ij}{}^k = F_{ij}{}^k$. Hence we might as well take the ρ -twist as $\rho_i{}^\mu = W_i{}^j \bar{\rho}_j{}^\mu$ with $\bar{\rho} = \bar{g}^{-1} d\bar{g}$, $\bar{g} \in \exp \mathfrak{f}$ and $W \in \text{Out}(\mathfrak{f})$. Solving for the F -flux equation now implies that

$$\beta^{kl} H_{ijl} = 0. \quad (3.62)$$

Solving for $Q_i{}^{jk} = 0$ and $R^{ijk} = 0$ then simply gives the homogeneous YB solution for $\beta^{ij} = \langle r_g(t^i), t^j \rangle$ — as in (3.4) but defined with different generators — in which the constant antisymmetric operator r solves the CYBE on \mathfrak{f} . Notice that β is, however, constrained by the condition (3.62). Finally, to solve for the b -twist we rewrite the H -flux equation as $3\partial_{[\mu} b_{\nu\rho]} = \rho_\mu{}^i \rho_\nu{}^j \rho_\rho{}^k H_{ijk}$ with $b_{\mu\nu} = \rho_\mu{}^i (\omega_{ij} + M_{ik} M_j{}^k) \rho_\nu{}^j$, and observe that for $\tilde{b}_{\mu\nu} \equiv \rho_\mu{}^i M_{ik} M_j{}^k \rho_\nu{}^j$ we find

$$3\partial_{[\mu} \tilde{b}_{\nu\rho]} = \rho_\mu{}^i \rho_\nu{}^j \rho_\rho{}^k H_{ijk} + 2V_\mu{}^i V_\nu{}^j V_\rho{}^k H_{ijk} + 3V_{[\mu}{}^i V_\nu{}^j V_{\rho]k} F_{ij}{}^k, \quad (3.63)$$

where we have used (2.36), (2.41), (3.59) and (3.60). From the definitions for $\bar{\omega}^{(2)}$ and $\Omega^{(2)}$ given in (2.44) and (2.46) we can solve the H -flux equation for $b_{\mu\nu}$ by

$$b_{\mu\nu} = \bar{\omega}_{\mu\nu}^{(2)} - \Omega_{\mu\nu}^{(2)} + \tilde{b}_{\mu\nu} = \bar{\omega}_{\mu\nu}^{(2)} - \Omega_{\mu\nu}^{(2)} + \rho_\mu{}^i M_{ik} M_j{}^k \rho_\nu{}^j. \quad (3.64)$$

Notice that the most general solution adds closed two-forms admitted by the manifold to $b_{\mu\nu}$ but also that this solution might not exist globally.

Finally for the equations for the generalised flux \mathcal{F}_I in this orbit we have

$$\mathcal{F}_i = F_{ij}{}^j + 2\alpha_i, \quad \mathcal{F}^i = -\beta^{jl} F_{jl}{}^i + 2\beta^{il} \alpha_l \quad (3.65)$$

where we have used $\beta^{kl}H_{lij} = 0$ and where, as before, $\alpha_i = \partial_i \bar{\lambda}$ is a constant, which implies $F_{ij}{}^k \alpha_k = 0$. The Bianchi identities for \mathcal{F}_I therefore read

$$F_{ij}{}^l F_{lm}{}^m - H_{ijl} \beta^{mn} F_{mn}{}^l = 0, \quad 2F_{il}{}^j \beta^{lm} \alpha_m - F_{il}{}^j \beta^{mn} F_{mn}{}^l = 0. \quad (3.66)$$

One can verify that the second identity guarantees also the orthogonality condition $\mathcal{F}^i \mathcal{F}_i = 0$. Together with the second identity and the Jacobi identity for $F_{ij}{}^k$ one can verify that all these conditions guarantee also the constancy of \mathcal{F}^i . Notice that if the algebra \mathfrak{f} is semisimple and the dilaton is isometric ($\alpha_i = 0$) these conditions reduce to the requirement that \mathfrak{f} as well as β and the r -operator are unimodular [60].

An example of a σ -model in the (F, H) -orbit is the Principal Chiral Model (PCM) with the addition of a Wess-Zumino (WZ) term (in the absence of spectators in fact we describe the whole σ -model). When fixing the correct normalisation between the coupling in front of the PCM and the WZ actions, this becomes the WZW model [78], which is in addition a conformal field theory. The pure NSNS bosonic-string background $AdS_3 \times S^3 \times T^4$ can be realised as an $SL(2, \mathbb{R}) \times SU(2)$ WZW model plus four free bosons parametrising the torus.

Inhomogeneous and homogeneous YB-deformations of PCM + WZ models were discussed in [79]. The homogenous-type deformations leave the generalised fluxes invariant and therefore can be included in our discussion. However we would like to point out that the homogeneous YB-deformations of [79] are always such that the image of the r -matrix (which is a subalgebra \mathfrak{f} of the full Lie algebra of the group used to construct the models) is a *solvable* algebra. As a consequence of Cartan's criterion, the components of the H -flux computed from the WZ term (e.g. $H_{ijk} = \alpha f_{ij}{}^l \kappa_{kl}$ for some constant α) are zero when all three legs are along \mathfrak{f} . Selecting a solvable subalgebra is equivalent to splitting the coordinates \mathbf{x}^m of the full target-space into spectator coordinates $\dot{x}^{\dot{\mu}}$ and coordinates y^μ such that $H_{\mu\nu\rho} = 0$. Therefore in our language the homogeneous deformations of [79] should be viewed as a deformation of the simpler (F) -orbit. Notice that the solvability condition can be seen as a possible solution to (3.62). New deformations of the PCM + WZ model correspond to solving (3.62) whilst relaxing the solvability condition.

Let us now turn to the generically inhomogeneous YB-deformation of the PCM + WZ model of [79–81]. To compare to those results let us consider the $O(d, d)$ matrix

$$\bar{U}_I{}^J = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} \gamma^{-1} \cosh(\zeta r \kappa) & \gamma^{-1} \kappa^{-1} \sinh(\zeta \kappa r) \\ \gamma \kappa \sinh(\zeta r \kappa) & \gamma \cosh(\zeta \kappa r) \end{pmatrix}, \quad (3.67)$$

where $\gamma, \zeta \in \mathbb{R}$ are parameters, κ_{ij} is a symmetric matrix that we will interpret as the Killing metric of an algebra \mathfrak{f} , and r^{ij} is an antisymmetric matrix. No further restriction on r is required for \bar{U} to be in $O(d, d)$. However, to have an integrable σ -model further conditions are necessary, see [79–81]. If we take $\bar{\mathcal{H}}^{IJ} = \bar{U}_K{}^I \dot{\mathcal{H}}^{KL} \bar{U}_L{}^J$, where simply

$$\dot{\mathcal{H}}^{IJ} = \begin{pmatrix} \kappa_{ij} & 0 \\ 0 & \kappa^{ij} \end{pmatrix}, \quad (3.68)$$

and in the absence of spectators, then using (2.4) it is easy to check that the combination of metric and B-field $\bar{M} = \bar{G} - \bar{B}$ parametrising $\bar{\mathcal{H}}$ will be

$$\bar{M} = \left(\frac{1 + \gamma^2}{1 - \gamma^2} - e^{-2\zeta\kappa r} \right)^{-1} \left(\frac{1 + \gamma^2}{1 - \gamma^2} + e^{-2\zeta\kappa r} \right) \kappa. \quad (3.69)$$

Comparing to equation (1.2) of [79] we see that in order to rewrite the σ -model action in an $O(d, d)$ covariant form it is sufficient to take for example the twist⁴¹

$$U = \begin{pmatrix} 1 & 0 \\ \hat{b} & 1 \end{pmatrix} \bar{U} \begin{pmatrix} v^t & 0 \\ 0 & v^{-1} \end{pmatrix}, \quad (3.70)$$

with $v = dgg^{-1}$ and $g \in \exp \mathfrak{f}$, and where the \hat{b} -shift of the B-field takes care of the WZ term. Let us mention that \bar{U} is of the form of (2.20) with

$$\bar{\rho} = \gamma^{-1}(\cosh(\zeta\kappa r))^{-1}, \quad \bar{\beta} = \gamma^{-2}\kappa^{-1} \sinh(\zeta\kappa r)(\cosh(\zeta\kappa r))^{-1}, \quad \bar{b} = \gamma^2\kappa \sinh(\zeta r\kappa) \cosh(\zeta r\kappa). \quad (3.71)$$

Let us now ignore the WZ term for a moment ($\hat{b} = 0$). The v -twist is very simple and when taken alone it gives rise to F -flux only, in particular $F_{ij}{}^k = -f_{ij}{}^k$ in terms of the structure constants of \mathfrak{f} . The multiplication by \bar{U} from the left can be seen as a rigid $O(d, d)$ rotation, and for this reason (when $\hat{b} = 0$) we can still interpret the model as being in the (F) -orbit. On the other hand, if we include \bar{U} when computing the fluxes we obtain

$$\begin{aligned} H_{ijk} &= -3\gamma^3(\cosh \zeta\kappa r)_{[i}{}^l(\cosh \zeta\kappa r)_j{}^m(\sinh \zeta\kappa r)_{k]}{}^n f_{lmn}, \\ F_{ij}{}^k &= -\gamma[(\cosh \zeta\kappa r)_i{}^l(\cosh \zeta\kappa r)_j{}^m(\cosh \zeta r\kappa)_n{}^k \\ &\quad + 2(\cosh \zeta\kappa r)_{[i}{}^l(\sinh \zeta\kappa r)_{j]}{}^m(\sinh \zeta r\kappa)_n{}^k] f_{lm}{}^n, \\ Q_i{}^{jk} &= -\gamma^{-1}[(\sinh \zeta\kappa r)_i{}^l(\sinh \zeta r\kappa)_m{}^j(\sinh \zeta r\kappa)_n{}^k \\ &\quad + 2(\cosh \zeta\kappa r)_i{}^l(\sinh \zeta r\kappa)_m{}^j(\cosh \zeta r\kappa)_n{}^k] f_l{}^{mn}, \\ R^{ijk} &= -3\gamma^{-3}(\sinh \zeta r\kappa)^{[i}{}^l(\sinh \zeta r\kappa)_m{}^j(\cosh \zeta r\kappa)_{k]}{}^n f^{lmn}, \end{aligned} \quad (3.72)$$

where the indices of $f_{ij}{}^k$ are raised and lowered with κ^{-1} and κ . Notice that all the fluxes above are non-zero. While, as already mentioned, at $\hat{b} = 0$ this is just a rigid $O(d, d)$ rotation of a representative of the (F) -orbit, when including now \hat{b} the new fluxes written in terms of the above ones are

$$\begin{aligned} H'_{ijk} &= H_{ijk} + 3F_{[ij}{}^l \hat{b}_{k]l} - 3\hat{b}_{l[i} \hat{b}_{j]m} Q_{|k]}{}^{lm} + \hat{b}_{il} \hat{b}_{jm} \hat{b}_{kn} R^{lmn} + \hat{H}_{ijk}, \\ F'_{ij}{}^k &= F_{ij}{}^k + 2\hat{b}_{l[i} Q_{j]}{}^{lk} + \hat{b}_{il} \hat{b}_{jm} R^{lmk}, \\ Q'_i{}^{jk} &= Q_i{}^{jk} + \hat{b}_{il} R^{ljk}, \\ R'^{ijk} &= R^{ijk}, \end{aligned} \quad (3.73)$$

where \hat{H}_{ijk} is the shift of the H -flux produced by \hat{b} . The requirement is that in the undeformed case ($\zeta = 0, \gamma = 1$) the H -flux is proportional to the structure constants of

⁴¹To relate to the parameters in [79] we have to identify $\frac{1+\gamma^2}{1-\gamma^2} = e^x$ and $\zeta = -\rho/2$.

f, namely $-3f_{[ij}^l \hat{b}_{k]l} + \hat{H}_{ijk} = \alpha f_{ijk}$ for some coefficient α . It would be interesting to see if there is a different parametrisation of the twist of the inhomogeneous YB-deformation of PCM + WZ that gives rise to simpler expressions for the fluxes, and if there is a parametrisation in which the fluxes are manifestly constant. This would help understand what is the simplest orbit in which one can describe this model. Notice that we have not used the mCYBE for r above, and it will likely play a role in that computation.

3.12 H, Q

In the (H, Q) -orbit the Bianchi identities imply that the Q -flux represents the structure constants of a Lie algebra $\tilde{\mathfrak{g}}$ with generators \tilde{T}^i and that the H -flux satisfies $H_{k[ij} Q_l]^{mk} = 0$. The latter in turn implies that $H^{ijk} = \tilde{\kappa}^{il} \tilde{\kappa}^{jm} \tilde{\kappa}^{kn} H_{lmn}$, with $\tilde{\kappa}^{ij} = \langle \tilde{T}^i, \tilde{T}^j \rangle$ the Killing form, must be a 3-cocycle of $\tilde{\mathfrak{g}}$. Moreover $Q_i{}^{jk} \mathcal{F}_k = 0$ and $Q_k{}^{ij} \mathcal{F}^k = 0$ imply that $\mathcal{F}_k = 0$ if $\tilde{T}^k \notin Z(\tilde{\mathfrak{g}})$ and $\mathcal{F}^k = 0$ if $\tilde{T}^k \in [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$. We also have the orthogonality condition $\mathcal{F}_i \mathcal{F}^i = 0$ as well as $H_{ijk} \mathcal{F}^k = 0$.

Since in this orbit the H -flux is non-vanishing we turn to a particular parametrisation of the twist U explained in section 2.4. As all of the components of the adjoint action $M_I^J = \langle m T_I m^{-1}, T^J \rangle$ are turned on, we prefer to continue with $U_I^{(2)M}$ given in (2.38), where we know in general the solution for ω_{ij} . In terms of the functions (ρ, b, β) of (2.20) we have in general

$$\begin{aligned} \rho_i{}^\mu &= (M_{ij} \xi^{jk} + M_i{}^j \delta_j^k) V_k{}^\mu, \\ \beta^{ij} \rho_j{}^\mu &= (M^i{}_j \xi^{jk} + M^{ij} \delta_j^k) V_k{}^\mu, \\ b_{ij} \rho_\mu{}^j &= (M_{ij} (\delta + \xi \omega)^j{}_k + M_i{}^j \omega_{jk}) V_\mu{}^k. \end{aligned} \quad (3.74)$$

Recall from section 2.4 that the H -flux equation is solved, up to closed two-forms, by $\omega = \bar{\omega} - \Omega^{(2)}$. Instead of trying to solve the other flux equations given in eq. (2.32) directly, it will be more convenient to use the expressions given in (2.50)–(2.53). In this orbit they read

$$\xi^{kl} H_{ijl} = 0, \quad (3.75)$$

$$\partial_\mu \xi^{ij} + 2V_\mu{}^l \xi^{m[i} Q_m{}^{j]l} = 0, \quad (3.76)$$

$$Q_l{}^{[ij} \xi^{k]l} = 0. \quad (3.77)$$

Notice that $\xi = 0$ is a trivial possible solution. To solve these equations in general, it will be important to consider the series expansion of the one-form $dm m^{-1}$. We find

$$\partial_\mu m m^{-1} = V_\mu{}^i T_i + V_{\mu i} \tilde{T}^i = \sum_{N=0}^{\infty} \frac{1}{(N+1)!} \text{ad}_{m^i T_i}^N \partial_\mu m^j T_j, \quad (3.78)$$

with

$$\begin{aligned} V_\mu{}^i &= \sum_{N=0}^{\infty} \frac{1}{(2N+1)!} \partial_\mu m^j ((mH \cdot mQ)^N)_j{}^i, \\ V_{\mu i} &= \sum_{N=0}^{\infty} \frac{1}{(2N+2)!} \partial_\mu m^j (mH)_{jl} ((mQ \cdot mH)^N)^l{}_i, \end{aligned} \quad (3.79)$$

where we have defined $(mH)_{ij} \equiv m^l H_{lij}$ and $(mQ)^{ij} \equiv m^l Q_l^{ij}$, and which can be proven by induction. Notice that we can always write $V_{\mu i} = A_{\mu}^{jk} H_{jki}$ for some tensor A which carries all the field dependence. With this observation, it is now easy to show that using eq. (3.75) and the Bianchi identity $H_{k[ij}Q_l]^{mk} = 0$ that

$$V_{\mu l} \xi^{m[i} Q_m^{j]l} = 0. \tag{3.80}$$

Hence eq. (3.76) simply requires that ξ^{ij} must be a constant matrix. Then the remaining conditions are algebraic: eq. (3.77) implies that ξ must be a 2-cocycle for $\tilde{\mathfrak{g}}$ while eq. (3.75) can be seen as a compatibility condition requiring that ξ and H must be orthogonal. Therefore, turning on such a particular ξ from $\xi = 0$ can be interpreted as a deformation which is a (novel) solution-generating technique.

To have the expression of the (ρ, b, β) -twists it will again be convenient to consider a series expansion, now of M_{ij} . One finds in particular $M_{ij} = \sum_{N=0}^{\infty} \frac{1}{(2N+1)!} (mH)_{il} ((mQ \cdot mH)^N)^l_j$ so that upon (3.75) we have $M_{ij} \xi^{jk} = 0$. Hence

$$\rho_i^\mu = M_i^j V_j^\mu, \quad \beta^{ij} = (M^{-1})_l^i \xi^{lk} (M^{-1})_k^j + M^{ik} (M^{-1})_k^j, \quad b_{ij} = M_i^l \omega_{lm} M_j^m, \tag{3.81}$$

where we have used $M_i^l M^j_l + M_{il} M^{jl} = \delta_i^j$ and $M_i^l M_{jl} + M_{il} M_j^l = 0$, from $M_I^L \in O(d, d)$, as well as $M_{ij} \xi^{jk} = 0$. We can now easily calculate the expression for $w_{ij}{}^k$ resulting in

$$w_{ij}{}^k = H_{ijl} M^{ln} (M^{-1})_n{}^k. \tag{3.82}$$

Let us now consider the conditions in which case a rigid $O(d, d)$ transformation by (2.27) describes a simpler orbit in disguise (i.e. H or Q can be turned off at no other cost). We find that we can not turn off the H -flux with a constant b and β transformation since the condition found from $H'_{ijk} = 0$ together with the consistency condition from $F'_{ij}{}^k = 0$ requires H_{ijk} to be vanishing, which is of course not possible. The same holds for the Q -flux where the condition required from $Q_i{}^{jk} = 0$ does not match with the conditions found from $R^{ijk} = 0$ as well as $F_i{}^{jk} = 0$.

To complete the discussion of this orbit in general we have for the generalised fluxes \mathcal{F}_I

$$\mathcal{F}_i = w_{ij}{}^j + 2\partial_i \bar{\lambda}, \quad \mathcal{F}^i = Q_j{}^{ji} + 2\beta^{ij} \partial_j \bar{\lambda}, \tag{3.83}$$

with $w_{ij}{}^k \partial_i \bar{\lambda} = 0$ such that \mathcal{F}_i is constant. From the requirement that \mathcal{F}_i is constant, $\partial_i \mathcal{F}^j = 0$, we therefore also must have that $Q_i{}^{jk} \partial_k \bar{\lambda} = 0$. From the Bianchi $Q_i{}^{jk} \mathcal{F}_k = 0$ we then find the condition $Q_i{}^{jk} w_{kl}{}^l = Q_i{}^{jk} H_{klm} \beta^{ml} = 0$. Furthermore $H_{ijk} \mathcal{F}^k = 0$ implies $H_{ijk} Q_l{}^{lk} = 0$, and $Q_k{}^{ij} \mathcal{F}^k = 0$ implies $Q_k{}^{ij} Q_l{}^{lk} + 2Q_k{}^{ij} \beta^{kl} \alpha_l = 0$. Notice that the orthogonality condition $\mathcal{F}_i \mathcal{F}^i = 0$ will be automatically satisfied because of the previous conditions.

(Asymmetrical) λ -deformations on group manifolds. It is known from [22], when ignoring spectators, that the WZW model and its λ -deformation [82] can be described by the particular twist $U^{(2)}$ with $\xi = 0$ and by a particular choice of fluxes (or, equivalently, a particular pre-Roytenberg algebra). In the following we will show that, generalising [22], also the asymmetrical λ -deformation on group manifolds of [83, 84] fits within the same

pre-Roytenberg algebra. We take $\mathfrak{r} = \mathfrak{k} \oplus \mathfrak{k}$ where \mathfrak{k} is a Lie algebra with generators t_i and structure constants $f_{ij}{}^k$. The bilinear form η_{IJ} is taken to be [22, 74]

$$\langle\langle \{x_1, y_1\}, \{x_2, y_2\} \rangle\rangle = \langle x_1, x_2 \rangle - \langle y_1, y_2 \rangle, \quad (3.84)$$

where $x_i, y_i \in \mathfrak{k}$ and $\kappa_{ij} = \langle t_i, t_j \rangle$ is the Killing form on \mathfrak{k} . The generators \tilde{T}^i with structure constants $Q_i{}^{jk}$ generate the diagonal embedding $\tilde{\mathfrak{g}} = \mathfrak{k}_{\text{diag}}$ in \mathfrak{r} by the map $x \mapsto \{x, x\}/\sqrt{2}$. This subgroup is maximally isotropic. The complementary isotropic subspace, spanned by T_i , is the anti-diagonal embedding by the map $x \mapsto \{x, -x\}/\sqrt{2}$. Hence, in this case we have

$$H_{ijk} = \frac{1}{\sqrt{2}} f_{ij}{}^l \kappa_{lk}, \quad Q_i{}^{jk} = \frac{1}{\sqrt{2}} \kappa_{il} \kappa^{jm} \kappa^{kn} f_{mn}{}^l, \quad F_{ij}{}^k = 0, \quad R^{ijk} = 0. \quad (3.85)$$

Notice that the Bianchi identities for \mathcal{F}_{IJK} are automatically satisfied upon the Jacobi identity for $f_{ij}{}^k$ and the ad-invariance of κ_{ij} . To connect to the λ -deformed background in terms of the metric and B-field there is a subtlety in choosing a good parametrisation of the group element $m = e^{m^i(y)T_i}$. Following [22] we take $m = \{\bar{g}, \bar{g}^{-1}\}$ with $\bar{g} \in \exp(\mathfrak{k})$ and define a group element $\tilde{g} \equiv \bar{g}^2 \in \exp(\mathfrak{k})$. Using this identification in the twist $U_I^{(2)M}$ we find in terms of the general parametrisation (2.20) that⁴²

$$\rho_i{}^\mu = \frac{1}{\sqrt{2}} (1 + \text{Ad}_{\tilde{g}})_i{}^j \tilde{v}_j{}^\mu, \quad \beta = -\kappa^{-1} \frac{1 - \text{Ad}_{\tilde{g}}}{1 + \text{Ad}_{\tilde{g}}}, \quad b = b_0 + \rho^{-1} \omega \rho^{-t}, \quad (3.86)$$

with

$$b_0 = \frac{1}{4} (\text{Ad}_{\tilde{g}} - \text{Ad}_{\tilde{g}}^{-1}) \kappa, \quad (3.87)$$

and $\tilde{v} = d\tilde{g}\tilde{g}^{-1}$. Deviating from [22] we now consider a different parametrisation denoted by $m = \{g, g^{-1}\}$ with $g \in \exp \mathfrak{k}$ and which is related to the previous parametrisation as $\text{Ad}_{\tilde{g}} = \text{Ad}_g \circ W$ and $\tilde{v} = v = dgg^{-1}$ in which W is a constant *outer* automorphism of \mathfrak{k} which preserves the metric κ . In particular this means that the group elements \tilde{g} and g are not related by a trivial field redefinition.⁴³ The twist functions simply become

$$\rho^{-1} = \frac{1}{\sqrt{2}} (1 + \text{Ad}_g W) v^{-1}, \quad \beta = -\kappa^{-1} \frac{1 - \text{Ad}_g W}{1 + \text{Ad}_g W}, \quad (3.88)$$

$$b = \frac{1}{4} (\text{Ad}_g W - W^{-1} \text{Ad}_g^{-1}) \kappa + \rho^{-1} \omega \rho^{-t}. \quad (3.89)$$

A final subtlety in calculating the background from the twist functions is the choice of the matrix $\dot{E}_A{}^I$, which is constant when we turn off the spectator fields. To obtain the (asymmetrical) λ -deformed background we must take $\dot{E}_A{}^I$ such that [22, 74]

$$\dot{\mathcal{H}}^{IJ} = \dot{E}_A{}^I \mathcal{H}^{AB} \dot{E}_B{}^J = \begin{pmatrix} \frac{1+\lambda}{1-\lambda} \kappa & 0 \\ 0 & \frac{1-\lambda}{1+\lambda} \kappa^{-1} \end{pmatrix}, \quad (3.90)$$

⁴²For details of this calculation see [22]. Following the logic of [22] we find that to find the same deformed geometry, we should replace $e = g^{-1}dg$ with $v = dgg^{-1}$ in their frame fields of eq. (5.63), an overall minus sign is missing in entry (1, 2) and (2, 1), and we should take $\lambda \rightarrow -\lambda$ in eq. (5.66). This likely corresponds to a known symmetry of these backgrounds [85] and indeed we agree on the final result.

⁴³A useful trick to get this generalisation $\text{Ad}_{\tilde{g}} = \text{Ad}_g \circ W$ and $\tilde{v} = v = dgg^{-1}$ is to consider a constant $w \in \exp \mathfrak{g}$ such that $\tilde{g} = gw$ and define $W(t_i) = wT_iw^{-1}$. This W would be an *inner* automorphism and it would be removed by a trivial field redefinition, but the dressing by W in the formulas above is the same also in the case of *outer* automorphisms.

in which λ is the eponymous deformation parameter. The metric and the B-field can then be extracted from the generalised metric $\mathcal{H}^{MN} = U_I^{(2)M} \dot{\mathcal{H}}^{IJ} U_J^{(2)N}$ resulting, in the coordinate frame, in

$$G = \frac{1 - \lambda^2}{2} v O_g \kappa O_g^t v^t, \quad B = \frac{1}{2} v (O_g \kappa - \kappa O_g^t) v^t - \omega, \quad (3.91)$$

with $O_g = (1 - \lambda \text{Ad}_g W)^{-1}$ and where $d\omega = -\frac{1}{12} f_{ijk} v^i \wedge v^j \wedge v^k$ gives rise to the well-known WZ term. This coincides precisely with the metric and B-field of the asymmetrical λ -model on the group manifold $G = \exp \mathfrak{k}$ [83] and with the original λ -model when $W = 1$ [82]. Let us point out that here W should not be mistaken with an automorphism of the pre-Roytenberg algebra as given in (2.24).

Curiously this implies that the WZW model background, which can be found by taking the limit $\lambda \rightarrow 0$, is described in the (H, Q) -orbit as well as the (F, H) -orbit. This then begs the question if there is a rigid $O(d, d)$ transformation (2.27) that relates the (H, Q) -orbit (with the H and Q fluxes proportional to each other via the Killing form as in (3.85)) to the (F, H) -orbit (with F and H again proportional). However, in general there is no such rigid $O(d, d)$ transformation, as can be checked for example in the $\mathfrak{su}(2)$ case.

On the other hand, a rigid $O(d, d)$ transformation does allow to describe the (asymmetrical) λ -models in the self-dual (F, Q) -orbit of Poisson-Lie symmetric backgrounds. By taking $\beta^{ij} = 0$ and $b_{ij} = \langle \tilde{r} t_i, t_j \rangle$, with \tilde{r} a constant antisymmetric operator that satisfies the mCYBE (3.39) for $c^2 = 1$ on \mathfrak{k} , we find

$$H'_{ijk} = 0, \quad F'_{ij}{}^k = \sqrt{2} b_{l[i} f_{j]m}{}^k \kappa^{lm}, \quad Q'_i{}^{jk} = \frac{1}{\sqrt{2}} \kappa_{il} \kappa^{jm} \kappa^{kn} f_{mn}{}^l, \quad R'^{ijk} = 0, \quad (3.92)$$

where we have used the ad-invariance of κ . This is now up to analytic continuations (essentially sending $c^2 = 1$ to $c^2 = -1$ or vice versa) the Poisson-Lie dual of the non-split η -deformation [22, 72–74] of which the (F, Q) -fluxes were given in eq. (3.41).

Let us emphasise that the generalised fluxes \mathcal{F}_{IJK} remain invariant when turning on a non-trivial automorphism W from the (original) λ -model representative with $W = 1$. Hence this map can be understood as a solution-generating technique — recall that in the case of supergravities we have to require that also \mathcal{F}_I stays invariant.⁴⁴ Furthermore starting e.g. with the λ -model representative ($\xi = 0$), and turning on a constant ξ which is orthogonal to H as well as a 2-cocycle for Q , is a novel solution-generating technique. In fact, if we transform the twist by a constant β -shift from the left, the fluxes transform as in (2.27), with $\beta = \xi$ and $b = 0$ in this case; then demanding that the fluxes remain invariant produces the two conditions (3.75) and (3.77). Therefore turning on a constant ξ can be understood as the implementation of an automorphism of the pre-Roytenberg algebra as in (2.24). Interestingly this deformation may be viewed as a generalisation of the DTD models (which on its own generalise ordinary NATD) in the (Q) -orbit, but now with the addition of H -flux.

⁴⁴Notice that in this parametrisation λ enters \dot{E} , and for this reason turning on λ is not necessarily a solution-generating technique. It is in fact known that the λ -deformation is not a marginal deformation of the WZW model [82], and in general one has to add RR fluxes to get a supergravity solution [60, 86, 87].

Comments on finding other representatives. Let us illustrate that outside of the ansatz considered in this section, the (H, Q) -orbit may still capture different non-trivial representatives. Since the generators \tilde{T}^i span a subalgebra $\tilde{\mathfrak{g}}$, another interesting ansatz allowing for systematic progress is the twist considered in eq. (2.54) or eq. (2.55). Taking the latter, and assuming that δ_{ij} is an ad-invariant bilinear form of $\tilde{\mathfrak{g}}$, we find for the equivalent of the F -, Q - and R -flux equations of (2.39) and (2.40) that

$$\xi^{kl} H_{ijl} = \delta_{il} Q_j^{lk}, \quad Q_i^{jk} = \hat{\partial}_i \xi^{jk} - 3\delta_{il} \xi^{m[j} Q_m^{k]l}, \quad 3\xi^{l[i} Q_l^{jk]} - \xi^{l[i} \xi^{j]m} Q_m^{[k]n} \delta_{ln} = 0, \quad (3.93)$$

with $\hat{\partial}_i \equiv \delta_{ij} \tilde{V}^{j\mu} \partial_\mu$ and $\hat{w}_{ij}{}^k = \delta_{il} Q_j^{lk}$. In this case notice that the expansion of $\tilde{V}_{\mu i}$ is in terms of Q instead of H , namely $\tilde{V}_{\mu i} = \sum_{N=0} \frac{1}{N+1} \partial_\mu \tilde{m}^j (\tilde{m} Q)^j{}_i$. It is now indeed clear that these equations may hold a genuinely different solution. We leave this problem, and the possibility of other ansätze, open.

3.13 F, H, R

In the (F, H, R) -orbit the Bianchi identities imply that the F -flux represents the structure constants of a Lie algebra \mathfrak{f} whose generators we denote by t_i and which — as in section 3.11 — do not coincide with T_i . Then H_{ijk} and $R_{ijk} = \kappa_{il} \kappa_{jm} \kappa_{kn} R^{lmn}$, with $\kappa_{ij} = \langle t_i, t_j \rangle$, are 3-cocycles of \mathfrak{f} . In fact the R -flux satisfies the stronger condition $F_{ij}{}^{[k} R^{lm]j} = 0$. Additionally we have $H_{ijk} R^{klm} = 0$ as well as the Bianchi identities (2.18) and (2.19) for \mathcal{F}_I .

Since the H -flux is non-vanishing we turn again to methodology of section 2.4. Compared to the (F, H) -orbit of section 3.11, notice that the presence of R -flux does not affect the adjoint action M_I^J given in (3.58). Therefore taking the ansatz $U_I^{(1)M}$ of eq. (2.37) we have in terms of the (ρ, b, β) -twists again that

$$\rho_i{}^\mu = M_i{}^J V_J{}^\mu, \quad \beta^{ij} = \xi^{ij}, \quad b_{ij} = \omega_{ij} + M_{ik} M_j{}^k. \quad (3.94)$$

Now, however, the equation for $w_{ij}{}^k = -2\rho_i{}^\mu \rho_j{}^\nu \partial_{[\mu} \rho_{\nu]}{}^k$ will in principle receive contributions from $V_{\mu i} R^{ijk}$ by using the MC identity (2.36) and the expressions for the derivatives of M_I^J (2.41). Before calculating $w_{ij}{}^k$ let us first derive several useful properties. First notice from the automorphism identities that we have from the vanishing Q -flux that

$$M_{il} R^{lmn} = 0, \quad (3.95)$$

where we have used the fact that $M^i{}_l$ has an inverse. Hence the automorphism property for $F_{ij}{}^k$ becomes simply $F_{ij}{}^k = M_i{}^l M_j{}^m (M^{-1})_n{}^k F_{lm}{}^n$. Additionally it will be useful to calculate the expansion of the one-form $V = dm m^{-1}$ in terms of $m = \exp(m^i T_i)$. In general we have

$$\partial_\mu m m^{-1} = V_\mu{}^i T_i + V_{\mu i} T^i = \sum_{N=0}^{\infty} V_\mu^{(N)}, \quad V_\mu^{(N)} \equiv \frac{1}{(N+1)!} \text{ad}_{m^i T_i}^N \partial_\mu m^j T_j. \quad (3.96)$$

Using the commutation relations in \mathfrak{t} we have the following expressions

$$\begin{aligned} V_\mu{}^i &= \sum_{N=0}^{\infty} \frac{1}{(N+1)!} \partial_\mu m^j (m F^N)_j{}^i \\ V_{\mu i} &= \sum_{N=1}^{\infty} \frac{1}{(N+1)!} \partial_\mu m^j \sum_{K=0}^{N-1} (-)^K (m F^{N-1-K} \cdot m H \cdot (m F^t)^K)_{ji} \end{aligned} \quad (3.97)$$

where we have defined $(mF)_i^j \equiv m^l F_{li}^j$, $(mF^t)_i^j \equiv m^l F_{lj}^i$ and $(mH)_{ij} \equiv m^l H_{lij}$, and which can be proved by induction using the relation $V^{(N+1)} = \frac{1}{N+2} \text{ad}_m V^{(N)}$. An important consequence is that using the Bianchi identities $F_{ij}^{[k} R^{lm]j} = 0$ and $H_{ijk} R^{klm} = 0$ we have

$$V_{\mu i} R^{ijk} = 0. \quad (3.98)$$

In particular with $H_{ijk} R^{klm} = 0$ this is easily seen for the case $K = 0$ in (3.97). When $K \neq 0$ it is sufficient to verify that, upon using $F_{ij}^{[k} R^{lm]j} = 0$ enough times, eventually an index of R^{ijk} will be contracted with an index of H_{ijk} . Using all of the above in the calculation for w_{ij}^k we find that the contributions of the R -flux will vanish

$$\begin{aligned} w_{ij}^k &= F_{ij}^k - \rho_i^\mu \rho_j^\nu V_{\mu l} V_{\nu m} R^{lmn} M_n^k, \\ &= F_{ij}^k. \end{aligned} \quad (3.99)$$

The same is true when trying to solve the H -flux equation in terms of the b -twist: in the calculation of $d\tilde{b}$ with $\tilde{b}_{\mu\nu} = \rho_\mu^i M_{ik} M_j^k \rho_\nu^j$ the contributions received from R^{ijk} are all of the form $V_{\mu i} R^{ijk}$. Hence, also in this orbit the solution for the b -twist is given in eq. (3.64).

From solving the F -flux equation of (2.32) we now have the condition $\beta^{kl} H_{ijl} = 0$, again like in the (F, H) -orbit. Instead of the parametrisation $\rho_i^\mu = M_i^j V_j^\mu$ we might now as well take $\rho = g^{-1} dg$ with $g \in \exp \mathfrak{f}$ some group element of the Lie group associated to \mathfrak{f} .⁴⁵ Then solving for $Q_i^{jk} = 0$ gives $\beta^{ij} = \langle r_g t^i, t^j \rangle$, with $r_g = \text{Ad}_g^{-1} \circ r \circ \text{Ad}_g$, $t^i = \kappa^{ij} t_j$ and r a constant antisymmetric operator. Notice only the difference in generators compared to eq. (3.4). Furthermore r must additionally satisfy

$$r^{kl} H_{ijl} = 0. \quad (3.100)$$

The R -flux equation, on the other hand, gives

$$(\text{Ad}_g)_l^i (\text{Ad}_g)_m^j (\text{Ad}_g)_n^k R^{lmn} = 3r^{l[i} r^{j]m} F_{lm}^{[k]}, \quad (3.101)$$

where we have used that Ad_g is an automorphism of \mathfrak{f} . Notice that the matrices $(\text{Ad}_g)_i^j$ and M_i^j coincide if we parametrise $g \in \exp \mathfrak{g}$ as $g = e^{m^i t_i}$, which is most easily seen when writing both in a series expansion. Indeed in that case

$$M_i^j = (\text{Ad}_g)_i^j = \sum_{N=0}^{\infty} \frac{1}{N!} (mF^N)_i^j. \quad (3.102)$$

Additionally, since $M^{ij} = 0$ we also have from (2.42) that $M^i_l M^j_m M^k_n R^{lmn} = R^{ijk}$ where $M^i_j = (M^{-1})_j^i$. These observations combined ensure that eq. (3.101) becomes simply

$$R^{ijk} = 3r^{l[i} r^{j]m} F_{lm}^{[k]}. \quad (3.103)$$

When we consider $R^{ijk} = \alpha \kappa^{il} \kappa^{jm} F_{lm}^k$, with $\alpha \in \mathbb{R}$, eq. (3.103) becomes $\alpha \kappa^{il} \kappa^{jm} F_{lm}^k = 3r^{l[i} r^{j]m} F_{lm}^{[k]}$ which is precisely the mCYBE (3.39) on \mathfrak{f} for $c^2 = \alpha$ as in the (F, R) orbit, cf. section 3.6.

⁴⁵We ignore again here the possibility of (outer) automorphisms of which the general discussion is given around eq. (2.24).

Interestingly, using a rigid $O(d, d)$ transformation the R -flux can be traded for Q -flux when we take $b = 0$ and $\beta = r$ in (2.27), so that this representative can be equally described in the (F, H, Q) -orbit. In particular, in that case we will have

$$H'_{ijk} = H_{ijk}, \quad F'_{ij}{}^k = F_{ij}{}^k, \quad Q_i{}^{jk} = -2F_{il}{}^{[j}r^{k]l}, \quad R'^{ijk} = 0, \quad (3.104)$$

so that, as in the (F, R) -orbit of section 3.6, the Q -flux are the structure constants of \mathfrak{g}_r whose Lie bracket was defined in (3.42).

On a different note, the (F, H, R) -orbit that we are describing is a simpler orbit in disguise if we can turn off at least one of the three types of fluxes. The possibilities that we have are: (i) we can turn off the H -flux if it is of the form $H_{ijk} = -3F_{[ij}{}^l b_{k]l} - b_{il} b_{jm} b_{kn} R^{lmn}$ for some constant antisymmetric matrix b satisfying $2F_{il}{}^{[j} \beta^{k]l} - b_{il} R^{ljk} + 2b_{il} b_{mn} R^{ln[j} \beta^{k]m} = 0$ for some constant antisymmetric matrix β . The F - and R -flux will obtain in that case a shift, $F'_{ij}{}^k = F_{ij}{}^k + b_{il} b_{jm} R^{lmk}$ and $R'^{ijk} = R^{ijk} + \frac{3}{2} \beta^{l[i} R^{jk]m} b_{ml}$, which vanishes when we take $\beta = 0$ (since then we must have $b_{il} R^{ljk} = 0$); (ii) to turn off the R -flux we find from $Q_i{}^{jk} = 0$ and taking $b = 0$ for simplicity in (2.27) the condition $H_{imn} \beta^{mj} \beta^{nk} - 2F_{il}{}^{[j} \beta^{k]l} = 0$, so that the R -flux should be of the form $R^{ijk} = \beta^{l[i} \beta^{j]m} F_{lm}{}^{k]}$. In that case H stays invariant while F receives a shift $F'_{ij}{}^k = F_{ij}{}^k - H_{ijl} \beta^{lk}$. Notice that while in the previous discussion we did take the R -flux of this form with $\beta = \sqrt{3}r$, the condition (3.100) also implies $F_{il}{}^{[j} r^{k]l} = 0$ which is inconsistent with the mCYBE unless $\alpha = 0$ from the very beginning, which is actually not the case that we want to consider. Finally (iii) we do not need to discuss the possibility of turning off the F -flux since in that case we would end up in the (H, R) -orbit which, as discussed before, can not be realised on the strong constraint with the methods of section 2.4, while here we do find possible representatives.

Finally let us close the discussion of this orbit by briefly commenting on the generalised fluxes \mathcal{F}_I . We have

$$\mathcal{F}_i = F_{ij}{}^j + 2\partial_i \bar{\lambda}, \quad \mathcal{F}^i = -\beta^{jl} F_{jl}{}^i + 2\beta^{il} \partial_l \bar{\lambda}. \quad (3.105)$$

Constancy of \mathcal{F}_i implies $F_{ij}{}^k \partial_k \bar{\lambda} = 0$ while one can verify that constancy of \mathcal{F}^i is guaranteed by the Bianchi $F_{ik}{}^j \mathcal{F}^k = 0$. In addition, this condition as well as $H_{ijk} R^{klm} = 0$ implies also the Bianchi $\mathcal{F}_i \mathcal{F}^i = 0$. At last we point out that when writing down all the Bianchi identities for \mathcal{F}_I explicitly one finds immediately that they are satisfied when \mathfrak{f} is semisimple, the dilaton is isometric, and \mathfrak{r} as well as \mathfrak{f} are unimodular.

3.14 H, Q, R

In the (H, Q, R) -orbit, the Bianchi identities imply that $Q_i{}^{jk}$ are structure constants for $\tilde{\mathfrak{g}} = \text{span}(\tilde{T}^i)$ and that the R -flux is a 3 cocycle of $\tilde{\mathfrak{g}}$. Furthermore we have $H_{k[ij} Q_l]{}^{mk} = 0$ and $H_{ijk} R^{klm} = 0$. Within the ansatz of 2.4 this again implies several simplifications. First notice that the R -flux will not alter the series expansion of the one-form dmm^{-1} given in (3.79) as well as the expansion of M_{ij} . Hence, upon the Bianchi $H_{ijk} R^{klm} = 0$ we have

$$V_{\mu l} R^{ljk} = 0, \quad M_{il} R^{ljk} = 0. \quad (3.106)$$

Let us now use the parametrisation $U^{(2)}$ of (2.38) where, recall, we have the general solution of the H -flux equation in terms of the two-form ω . For the expressions of the other flux equations in terms of the yet unknown ξ^{ij} we refer to the most general orbit discussed in section 2.4 by setting $F = 0$ in the equations (2.50)–(2.53). Let us point out here that due to the presence of R -flux, $\xi = 0$ will not be a solution. Using (3.106) the expressions simplify to

$$\xi^{kl}H_{ijl} = 0, \tag{3.107}$$

$$\partial_\mu \xi^{ij} + 2V_{\mu l} \xi^{m[i} Q_m{}^{j]l} = 0, \tag{3.108}$$

$$3Q_l{}^{[ij} \xi^{k]l} = R^{ijk} \tag{3.109}$$

Similarly as in section 3.12 the Bianchi identity $H_{k[ij} Q_l]{}^{mk} = 0$ implies that $V_{\mu l} \xi^{m[i} Q_m{}^{j]l} = 0$ so that ξ^{ij} must simply be a constant matrix. However, given a rigid $O(d, d)$ transformation it is possible to turn off the R -flux for this particular constant ξ^{ij} . In order to do so one should take in (2.27) the constant matrices $b_{ij} = 0$ and $\beta^{ij} = -\xi^{ij}$ such that $H_{ijk} \beta^{kl} = 0$. This rigid $O(d, d)$ will leave the other fluxes invariant and thus we are in fact describing the (H, Q) -orbit (which, recall, is a genuine orbit modulo rigid $O(d, d)$). Concluding, within the ansatz of section 2.4 we cannot describe a particular non-trivial representative in the (H, Q, R) -orbit and, therefore, we cannot describe a rigid T -transformation from the (F, H, R) -orbit as a solution-generating technique. It would be interesting to explore other ansatze for this purpose.

3.15 F, H, Q

The (F, H, Q) -orbit describes what is known as a quasi-Manin triple. The generators \tilde{T}^i span a subalgebra $\tilde{\mathfrak{g}}$ of \mathfrak{t} with structure constants $Q_i{}^{jk}$, while the generators T_i do not. Furthermore we have the Bianchi $F_{[ij}{}^k H_{lm]k} = 0$ as well as (2.14) and (2.15). To discuss this orbit we use the ansatz of section 2.4 and the parametrisation $U^{(2)}$ of (2.38) for the twist. The general solution of the H -flux equation was given in terms of ω_{ij} in (2.43). The equations to be solved for the unknown ξ^{ij} can be found in section 2.4 by setting $R = 0$ in (2.50)–(2.53). They are

$$2\xi^{kl}H_{ijl} + \xi^{km}\Lambda_{ml}F_{ij}{}^l = 0, \tag{3.110}$$

$$\hat{\partial}_i \xi^{jk} + 2\Lambda_{il} \xi^{m[j} Q_m{}^{k]l} - \xi^{m[j} \xi^{k]n} H_{imn} = 0, \tag{3.111}$$

$$3Q_l{}^{[ij} \xi^{k]l} + 3\xi^{l[i} \xi^{j]m} F_{lm}{}^{[k} - \xi^{l[i} \xi^{j]m} \xi^{[k]n} \Lambda_{np} F_{lm}{}^p = 0 \tag{3.112}$$

In the first place, we always have the trivial solution $\xi = 0$. Recall that no genuine representatives exist in the (F, Q, R) -orbit, so that we cannot employ a rigid T -transformation as a solution-generating technique in this case. On the other hand, it is possible to consider a rigid β -transformation relating this representative and the one of the (F, H, R) -orbit. It would be interesting to find also non-trivial solutions for ξ^{ij} within this orbit in the hope of having solution-generating techniques mapping cases with different ξ 's. We leave this problem open.

4 Type II superstring and Ramond-Ramond fields

Let us now include also the RR fields in the discussion of the solution-generating techniques, which in general may be relevant for type II backgrounds. RR fields in the doubled formulation were discussed in various works, see for example [88–90]. Here we will employ the spinorial formulation of [88] following the rewriting of [20]. In this section we will review only the essential ingredients of the construction, and we refer to appendix D for more details. One uses a democratic formulation [91, 92] where all even (odd) forms from 1 to $D = 10$ are used for the RR fields strengths of type IIA (IIB). The RR potentials are encoded in a spinor $|\mathfrak{c}\rangle$ and the RR field strengths in a spinor⁴⁶ $|\mathfrak{F}\rangle$. Given Gamma matrices Γ^A in $2D$ -dimensions satisfying the Clifford algebra relations $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$ (notice the flat indices) we write $\Gamma^A = \sqrt{2}\psi^A$ so that $\{\psi_a, \psi^b\} = \delta_a^b$, $\{\psi_a, \psi_b\} = 0$, $\{\psi^a, \psi^b\} = 0$ are anticommutation relations for the fermionic oscillators ψ_a, ψ^a . Starting from the Clifford vacuum $|0\rangle$ such that $\psi_a|0\rangle = 0$ for all a , we rewrite the spinor $|\mathfrak{F}\rangle$ as

$$|\mathfrak{F}\rangle = \sum_{p=0}^D \frac{e^\phi}{p!} \hat{F}_{m_1 \dots m_p} e_{a_1}^{m_1} \dots e_{a_p}^{m_p} \psi^{a_1} \dots \psi^{a_p} |0\rangle. \quad (4.1)$$

This rewriting relies on the one-to-one map between spinors $|\mathfrak{F}\rangle$ and polyforms \hat{F} , where $\hat{F} = \sum_{p=0}^D \frac{1}{p!} \hat{F}_{m_1 \dots m_p} dx^{m_1} \dots dx^{m_p}$ on $\mathbb{R}^{1,D-1}$. Here we are using e_a^m which is the (inverse) vielbein for the metric G_{mn} . Importantly, $\hat{F}_{m_1 \dots m_p}$ are the RR field strengths that are commonly used in type II supergravity (they are the \hat{F} of [88]), and they are the ones that appear in the quadratic couplings of the fermions in the Green-Schwarz formulation of the superstring (they are the F of [93]). The two spinors $|\mathfrak{F}\rangle$ and $|\mathfrak{c}\rangle$ are related by

$$|\mathfrak{F}\rangle = \left(\psi^A \partial_A - \frac{1}{6} \mathcal{F}_{ABC} \psi^{ABC} - \frac{1}{2} \mathcal{F}_A \psi^A \right) |\mathfrak{c}\rangle = \mathcal{V} |\mathfrak{c}\rangle, \quad \mathcal{V} \equiv \not{\partial} - \mathcal{F}^{(3)} - \frac{1}{2} \mathcal{F}^{(1)}, \quad (4.2)$$

where $\mathcal{F}^{(n)}$ includes the $1/n!$ factor. Gauge transformations of RR potentials read as $\delta_\lambda |\mathfrak{c}\rangle = \mathcal{V} |\lambda\rangle$ and Bianchi identities as $\mathcal{V} |\mathfrak{F}\rangle = 0$. Notice that these Bianchi identities are a consequence of $\mathcal{V}^2 = 0$ which holds on the strong constraint. After defining $\Psi_- = (\psi_0 + \psi^0)(\psi_1 - \psi^1) \dots (\psi_9 - \psi^9)$ we also impose the self-duality condition $\Psi_- |\mathfrak{F}\rangle = |\mathfrak{F}\rangle$. Notice that this differs from [20], see appendix D. The self-duality condition translates into the duality conditions for the p -forms as⁴⁷ $\hat{F}^{(p)} = -(-1)^{\frac{1}{2}p(p+1)} * \hat{F}^{(10-p)}$.

The transformation rules of RR fields $\hat{F}_{m_1 \dots m_p}$ under the $O(D, D)$ solution-generating techniques are found by the observation that $|\mathfrak{F}\rangle$ is in fact invariant under these transformations. Notice that knowing that the generalised fluxes \mathcal{F} are invariant under the $O(D, D)$ solution-generating techniques, it is obvious that keeping also $|\mathfrak{F}\rangle$ invariant ensures that even when including the RR sector we still have a solution of the supergravity equations, and that we correctly satisfy the Bianchi identities and the constraints. The

⁴⁶Our $|\mathfrak{F}\rangle$ is \mathcal{G} of [20]. Our $|\mathfrak{F}\rangle$ and $|\hat{F}\rangle$ appearing later are respectively $|F\rangle$ and $|\hat{F}\rangle$ of [88].

⁴⁷The Hodge dual is defined as $(*A)_{m_1 \dots m_p} = \frac{1}{(10-p)!} G_{m_1 n_1} \dots G_{m_p n_p} \epsilon^{k_{p+1} \dots k_{10} n_1 \dots n_p} A_{k_{p+1} \dots k_{10}}$ where $\epsilon^{1 \dots 10} = 1$, $\epsilon_{1 \dots 10} = -1$, $\epsilon^{m_1 \dots m_{10}} = \frac{1}{\sqrt{-G}} \epsilon^{m_1 \dots m_{10}}$, $\epsilon_{m_1 \dots m_{10}} = \sqrt{-G} \epsilon_{m_1 \dots m_{10}}$. We also have $* * \omega^{(p)} = -(-1)^{p(10-p)} s\omega^{(p)}$ where the additional minus sign is due to the Lorentz signature.

fact that we can identify a dynamical field (in this case $|\mathfrak{F}\rangle$) that remains invariant under the transformation is again a confirmation of the usefulness of the DFT formulation that we are employing.

The fact that $|\mathfrak{F}\rangle$ is invariant does not mean that the RR fields $\hat{\mathbf{F}}_{m_1 \dots m_p}$ remain invariant. This is similar to what we saw in the NSNS sector: while the generalised fluxes are invariant, the background metric \mathbf{G}_{mn} , the field \mathbf{B}_{mn} and the dilaton ϕ do transform. To present the transformation rules it is useful to define $|\mathbf{F}\rangle \equiv e^{-d} S_{[\mathbf{E}]}^{-1} |\mathfrak{F}\rangle$ where $S_{[\mathbf{E}]}$ is a $\text{Spin}(D, D)$ element corresponding to the $O(D, D)$ element \mathbf{E} . See the appendix for more details. Another useful rewriting of the spinor for the RR field strengths is $|\hat{\mathbf{F}}\rangle \equiv S_{[\mathbf{B}]} |\mathbf{F}\rangle$, where we use the $\text{Spin}(D, D)$ element $S_{[\mathbf{B}]} \equiv \exp(-\frac{1}{2} \mathbf{B}_{mn} \psi^m \psi^n)$ with \mathbf{B}_{mn} the B-field of the supergravity background. We will then write $|\mathbf{F}\rangle = \sum_{p=1}^D \frac{1}{p!} \mathbf{F}_{m_1 \dots m_p} \psi^{m_1} \dots \psi^{m_p} |0\rangle$ and $|\hat{\mathbf{F}}\rangle = \sum_{p=1}^D \frac{1}{p!} \hat{\mathbf{F}}_{m_1 \dots m_p} \psi^{m_1} \dots \psi^{m_p} |0\rangle$, which is compatible with (4.1). From the invariance of $|\mathfrak{F}\rangle$ and using that $\mathbf{E} = \dot{\mathbf{E}}U$ and $\mathbf{d} = \dot{\mathbf{d}} + \lambda$, it follows that

$$|\dot{\mathbf{F}}\rangle \equiv e^\lambda S_{[U]} |\mathbf{F}\rangle, \quad (4.3)$$

is also invariant under the $O(D, D)$ transformations. If we use primes to denote the new background related by the $O(D, D)$ transformation, this fact can be exploited to compute the RR fields of the new solution just by identifying $|\dot{\mathbf{F}}\rangle = e^{\lambda'} S_{[U']} |\mathbf{F}'\rangle$.

From an operational point of view, starting from the polyform $\hat{\mathbf{F}} = \sum_{p=1}^D \frac{1}{p!} \hat{\mathbf{F}}_{m_1 \dots m_p} dx^{m_1} \dots dx^{m_p}$ of a supergravity solution, we can get $\mathbf{F} = \exp(\mathbf{B}^{(2)}) \wedge \hat{\mathbf{F}}$ with $\mathbf{B}^{(2)} \equiv \frac{1}{2} \mathbf{B}_{mn} dx^m \wedge dx^n$, which is a consequence of the relation between the corresponding spinors. To obtain $\dot{\mathbf{F}}$ we write the $\text{Spin}(D, D)$ element $S_{[U]} = S_{[\rho]} S_{[\beta]} S_{[\bar{b}]}^{-1}$ which follows from the rewriting of the twist U as

$$U = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} \rho^t & 0 \\ 0 & \rho^{-1} \end{pmatrix} = \begin{pmatrix} \rho^t & 0 \\ 0 & \rho^{-1} \end{pmatrix} \begin{pmatrix} 1 & \bar{\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{b} & 1 \end{pmatrix} \quad (4.4)$$

where $\bar{\beta} = \rho^{-t} \beta \rho^{-1}$ and $\bar{b} = \rho b \rho^t$ have curved indices. We prefer this rewriting because it simplifies the translation of the action on polyforms. In fact

$$|\mathbf{F}^{(\bar{b})}\rangle = S_{[\bar{b}]}^{-1} |\mathbf{F}\rangle = e^{\frac{1}{2} \bar{b}_{\mu\nu} \psi^\mu \psi^\nu} |\mathbf{F}\rangle \implies \mathbf{F}^{(\bar{b})} = e^{\bar{b} \wedge} \mathbf{F} = \mathbf{F} + \bar{b} \wedge \mathbf{F} + \frac{1}{2!} \bar{b} \wedge \bar{b} \wedge \mathbf{F} + \dots, \quad (4.5)$$

where $\bar{b} \equiv \frac{1}{2} \bar{b}_{\mu\nu} dy^\mu \wedge dy^\nu$ and

$$|\mathbf{F}^{(\bar{\beta})}\rangle = S_{[\bar{\beta}]} |\mathbf{F}\rangle = e^{\frac{1}{2} \bar{\beta}^{\mu\nu} \psi_\mu \psi_\nu} |\mathbf{F}\rangle \implies \mathbf{F}^{(\bar{\beta})} = e^{\bar{\beta} \vee} \mathbf{F} = \mathbf{F} + \bar{\beta} \vee \mathbf{F} + \frac{1}{2!} \bar{\beta} \vee \bar{\beta} \vee \mathbf{F} + \dots, \quad (4.6)$$

where $\bar{\beta} \vee \mathbf{F} = \frac{1}{2} \bar{\beta}^{\mu\nu} \iota_\mu \iota_\nu \mathbf{F}$ and $\iota_m dx^n = \delta_m^n$, $\iota_m (\omega^{(p)} \wedge \chi^{(q)}) = \iota_m \omega^{(p)} \wedge \chi^{(q)} + (-1)^p \omega^{(p)} \wedge \iota_m \chi^{(q)}$. Finally one has

$$\begin{aligned} S_{[\rho]} |\mathbf{F}\rangle &= (\det \rho)^{1/2} \exp(-\psi^m \mathbf{R}_m{}^n \psi_n) \sum_p \frac{1}{p!} \mathbf{F}_{m_1 \dots m_p} \psi^{m_1} \dots \psi^{m_p} |0\rangle \\ &= (\det \rho)^{1/2} \sum_p \frac{1}{p!} \mathbf{F}_{m_1 \dots m_p} \rho_{i_1}^{m_1} \dots \rho_{i_p}^{m_p} \psi^{i_1} \dots \psi^{i_p} |0\rangle \end{aligned} \quad (4.7)$$

where $\rho = \exp \mathbf{R}$ acts as the identity on spectator coordinates \hat{x}^μ and coincides with ρ on the y -block. In other words the only action of $S_{[\rho]}$ is to translate curved indices μ, ν into algebra-like indices i, j , and multiply by $(\det \rho)^{1/2}$. Notice that this factor cancels in $|\check{\mathbf{F}}\rangle$ once we rewrite $\lambda = \bar{\lambda} - \frac{1}{2} \log \det \rho$. With the above formulas and starting from $\check{\mathbf{F}}, U, \lambda$ one can obtain $\check{\mathbf{F}}$, and given another representative with U', λ' one can similarly compute \mathbf{F}' and $\hat{\mathbf{F}}'$ by the same formulas.

As we argued in the previous sections, the solution-generating techniques that we are studying cover not only those that leave the generalised fluxes invariant. They also cover those that relate different sets of generalised fluxes by constant $O(D, D)$ transformations, the prominent example being the rigid T -transformation implemented by the matrix in (2.28). Let us show how the RR fields transform when this T -transformation is involved. When the fluxes are related as in $\mathcal{F}'_{A'B'C'} = T_{A'}^A T_{B'}^B T_{C'}^C \mathcal{F}_{ABC}$ and $\mathcal{F}'_{A'} = T_{A'}^A \mathcal{F}_A$, from (4.2) and using $S_{[T]} \psi^A S_{[T]}^{-1} = \psi^B \mathbf{T}_B^A$ one sees that we have the relations $|\check{\mathfrak{F}}'\rangle = S_{[T]} |\check{\mathfrak{F}}\rangle$ and $|\mathbf{c}'\rangle = S_{[T]} |\mathbf{c}\rangle$, provided that $|\mathbf{c}\rangle$ is taken to be independent of the coordinates that are being dualised.⁴⁸ Notice that the chirality of the spinors remains the same if we dualise in even d dimensions, and changes in odd d . In addition notice that this transformation⁴⁹ gives $\Psi'_- = S_T \Psi_- S_T^{-1} = (-1)^d \Psi_-$, therefore in d -even dimensions it preserves the self-duality condition, while in d odd it changes the sign, see also [88].

To obtain the transformation rules of RR fields one uses the fact that the relation to the dualised model is via a twist

$$\tilde{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} \rho^t & 0 \\ 0 & \rho^{-1} \end{pmatrix} = \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \bar{\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{b} & 1 \end{pmatrix} \quad (4.8)$$

that now has an additional matrix implementing the T -transformation. Also in this case it is preferable to pull the $GL(d)$ block with ρ to the left, because this rewriting permits to implement the action of the T-duality matrix (the second block in the last equation) on polyforms in a simple way. Notice that because of the above rewriting we can think of the operation as a dualisation of d coordinates not just in the case of abelian T-dualities, but also in their generalisations. When dualising more than one coordinate we can think of it as a factorised product of single T-dualities along the m direction implemented by $S_{[\mu]} = (\psi^\mu - \psi_\mu)(-1)^{N_F}$. Under such transformation one has in polyform notation

$$|\mathbf{F}^{(\mu)}\rangle = S_{[\mu]} |\mathbf{F}\rangle \implies \mathbf{F}^{(\mu)} = \mathbf{F} \wedge dy^\mu + \mathbf{F} \vee dy^\mu, \quad (4.9)$$

where if A is a p -form $A \vee dy^\mu = (-1)^{p-1} dy^\mu \vee A = (-1)^{p-1} \iota_\mu A$. A similar point of view to obtain the transformation rules of RR fields under PL duality was used in [23].

5 Conclusions and outlook

We have discussed an ansatz for the generalised vielbein of DFT by demanding that it takes a “twisted” form, as in generalised Scherk-Schwarz reductions of D -dimensional backgrounds on d -dimensional spaces, and that the twist U gives rise to constant generalised

⁴⁸Notice that the ansatz for the spinor taken in [94] is equivalent to this condition, and a similar condition is taken also in [22, 23].

⁴⁹In the presence of RR fields we assume that we do not dualise the time direction.

fluxes in the d -dimensional space when imposing the strong constraint of DFT. The results are organised as in figure 1 into orbits depending on which of the different fluxes F, H, Q and R are turned on, and in general they can be related by rigid T -transformations by $F \leftrightarrow Q$ and $H \leftrightarrow R$, or more generic $O(d, d)$ transformations. Our classification of the representatives is complete when the H -flux is vanishing, while we have employed particular methods for the cases with non-trivial H -flux which may not cover all possible representatives.

When an orbit contains more than one representative, or when by means of a rigid $O(d, d)$ transformation (including a rigid T -transformation) we can relate it to another orbit that admits representatives, we can view the maps relating two of them as a solution-generating technique in (super)gravity, as well as a canonical transformation at the level of the σ -models. Therefore when one of the two σ -models is classically integrable it follows that the other enjoys the same property. Our results add new possibilities to the known zoo of generalised T-dualities and Yang-Baxter deformations. Generic $O(d, d)$ transformations relating possibly different orbits may be seen as generalisations of the so called PL-*plurality*, which is traditionally defined as the possibility of decomposing in different ways the same Drinfel'd double. Let us remark that in this paper we have only imposed the $O(d, d)$ symmetry of the background (i.e. the fact that it takes the form of a generalised Scherk-Schwarz ansatz and that it gives rise to constant $\mathcal{F}_{IJK}, \mathcal{F}_I$). At no point we have imposed the DFT (or supergravity) equations of motion, and this is actually not relevant if one is only interested in canonical transformations of σ -models. In general, one may have to choose appropriate RR fields and/or spectator background fields in order to solve the DFT/supergravity equations. Similar comments apply to the integrability of the σ -models giving rise to the backgrounds that we describe. At no point we imposed the existence of a Lax connection, and in general this may introduce additional conditions.

Let us now summarise our classification. In the (F) -orbit one can independently switch on a twist β satisfying the CYBE, which corresponds to possible (homogeneous) YB deformations [33–35] of isometric backgrounds. This orbit is related by rigid T -transformations to the (Q) -orbit, which contains backgrounds that arise from applying NATD [4] to the ones in the (F) -orbit, and more generally include the DTD models of [36, 63]. The (F, Q) orbit contains representatives that are PL-symmetric [11, 12], and it allows also for novel deformations that can be understood as the natural generalisation of the homogeneous YB-deformations, now without the need of having isometries in the initial background. Because the fluxes remain invariant, these deformations preserve the integrability of the σ -model. When we perform this generalised deformation combined with a rigid T -transformation, the resulting map may be understood as a generalised notion of T-duality. The (R) -orbit is empty, it contains no non-trivial representative when demanding the strong constraint. The (F, R) orbit contains the so-called inhomogeneous YB-model (or η -model) [34, 35], slightly generalised in our treatment by the presence of spectators. The (Q, R) and (F, Q, R) orbits are trivial in the sense that one can turn off the R -flux by a rigid $O(d, d)$ transformation.

When the H -flux is non-zero our classification is not exhaustive, because the methods of section 2.4 that we employ in these cases do not guarantee that we are covering all possible representatives. Nevertheless we identify interesting possibilities for the $(H), (F, H), (H, Q)$ and (F, H, R) orbits. The (H, R) and (H, Q, R) orbits seem trivial within these methods

since the R -flux can again be turned off by a rigid $O(d, d)$. The (F, H) -orbit contains the PCM+WZ model as well as a novel generalisation of the homogeneous YB-deformation in which the YB-operator is constrained to be compatible with H . When applied to the integrable PCM+WZ model, this deformation preserves integrability. The (H, Q) orbit contains the integrable (asymmetrical) λ -model [82, 83] as well as additional novel deformations by 2-cocycles compatible with the H -flux, which are reminiscent of the deformations of NATD in the (Q) -orbit that were called DTD. The map between an (F, R) representative and the generalised DTD representative of (H, Q) by means of a rigid T -transformation can be understood as another generalised T-duality. In the (F, H, R) -orbit we can describe a generalisation of the inhomogeneous YB-model⁵⁰ as well as the fact that a rigid $O(d, d)$ relates it to the (F, H, Q) -orbit. Within the methods that we use, the (F, H, Q) -orbit admits at least the solutions of [22]. The question of having more general representatives in this orbit, as well as representatives in the most general (F, H, Q, R) -orbit, remains open. Finally in the (H) -orbit we identified possible deformations of the torus with H -flux without relying on the methods of section 2.4. This confirms that it should be possible to go beyond these methods, and it would certainly be interesting to do so in a systematic way in the other orbits with non-vanishing H -flux. It would also be interesting to look at this classification from the point of view of para-Hermitian geometry [95]. Let us stress that all the maps connecting different representatives of the same orbit (up to rigid $O(d, d)$) can be used to generate integrable σ -models when starting from another known one.

A general observation that we made, which generalises previously known results, is that a constant automorphism W of the pre-Roytenberg algebra \mathfrak{r} (modulo gauge transformations) may generate a new representative from a known representative for the twist. They must not be mistaken with the rigid $O(d, d)$ transformations of (2.23), since these automorphisms do not involve a compensating transformation of the spectator background. For instance, in the (F) -orbit the automorphism reduced to the subalgebra generated by F_{ij}^k must be an outer automorphism. An interesting possibility is that for non-semisimple algebras outer automorphisms may involve continuous parameters and can therefore be seen as deforming the background.

While we have focused on the NSNS sector in most of the paper, in section 4 we have explained how to obtain the transformation rules of RR fluxes by demanding that the maps under consideration are in fact mapping type II solutions to type II solutions.

Let us now comment on several interesting open questions.

While we have discussed in our classification of representatives various known integrable 2-dimensional σ -models, it would be interesting to rewrite in this language also others such as the bi-YB-deformation of [80, 81, 96] and the deformations of [97].

In this paper we have not analysed the special case of solution-generating techniques involving (super)cosets, and this would be a very interesting future direction. In the (super)coset case the spectator background fields are expected to project on the coset part of the algebra, and can therefore give rise to solution-generating techniques even when

⁵⁰We stress that we cannot view it as a solution-generating technique from the PCM+WZ model since we are going from the (F, H) to the (F, H, R) orbit by turning on R .

demanding conditions that are weaker than the invariance of the $O(d, d)$ generalised fluxes $\mathcal{F}_{IJK}, \mathcal{F}_I$, see [58, 98] for an observation along these lines. In this paper we cannot view the inhomogeneous YB-deformations (the η -deformation) as a solution-generating technique from $\eta = 0$ to $\eta \neq 0$, because in general it entails going from the (F) -orbit to the (F, R) -orbit, but it is possible that the (unimodular) η -deformation of the superstring [99] turns out to be a solution-generating technique because of these additional features of the supercoset. The transformation rules of the NSNS and RR fields [60] are in fact strongly suggesting the underlying $O(D, D)$ structure also in this case. Similar comments apply to the λ -deformation of the superstring [86], as well as to the construction of [87].

Another possible future direction, which is also necessary for the point above, is to discuss solution-generating techniques that involve *superalgebras*. It is likely that for this purpose the formulation of [90] for type II superstrings in DFT language will be more useful than the formulation of [88].

It would be interesting to relax some assumptions that we have made. In particular, the solution-generating techniques discussed in this paper arise by demanding that the generalised fluxes (and their flat derivatives) remain invariant under the map. To find more general solution-generating techniques in supergravity, one may try to look for more complicated symmetries of the DFT equations of motion that do not necessarily leave the generalised fluxes invariant, see [58] for a step in this direction. Additionally we were interested in backgrounds that satisfy the strong constraint of DFT, namely those whose fields depend only on the physical coordinates y and not on the dual \tilde{y} . In the context of gauged DFT it is possible to relax the strong constraint as done in [54], and it would be interesting to look at this generalisation of the classification as well. One may also relax our analysis by looking for solution-generating techniques in the context of the generalised supergravity equations of [100, 101].

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A Notation

We follow the convention of using a boldface notation for the objects (and their indices) of the full D -dimensional space, both before and after the doubling (e.g. we have the metric \mathbf{G}_{mn} and the generalised fluxes \mathcal{F}_{ABC}), and the same notation but without boldface for the corresponding quantities in d dimensions, (e.g. G_{mn} and \mathcal{F}_{ABC}). A recap on our indices

conventions is as follows:

$\mathbf{m}, \mathbf{n}, \dots$	Curved D -dimensional indices of coordinates \mathbf{x}^m
$\mu, \nu, \dots, \alpha, \beta, \dots$	Curved d -dimensional indices of y^μ
$\dot{\mu}, \dot{\nu}, \dots$	Curved $(D - d)$ -dimensional indices of spectator coordinates $\dot{x}^{\dot{\mu}}$
M, N, \dots	Curved $O(D, D)$ indices
M, N, \dots	Curved $O(d, d)$ indices
$\mathbf{A}, \mathbf{B}, \dots$	Flat $O(1, D - 1) \times O(D - 1, 1)$ -dimensional indices
I, J, \dots	Algebra indices in $2d$ dimensions
i, j, \dots	Algebra indices in d dimensions

B Brief recap on DFT and gDFT

The generalised vielbein may be parametrised as

$$\mathbf{E}_A{}^M = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{(+)\mathbf{a}n}(\mathbf{G} - \mathbf{B})_{nm} & e^{(+)\mathbf{a}m} \\ -e_a^{(-)n}(\mathbf{G} + \mathbf{B})_{nm} & e_a^{(-)m} \end{pmatrix}. \quad (\text{B.1})$$

Here $e^{(\pm)}$ are two vielbeins for the metric \mathbf{G}_{mn} and \mathbf{B}_{mn} is the Kalb-Ramond field. The generalised vielbein satisfies the following relations with the $O(D, D)$ metric (1.1) and the generalised metric (1.2)

$$\eta^{MN} = \mathbf{E}_A{}^M \eta^{AB} \mathbf{E}_B{}^N, \quad \mathcal{H}^{MN} = \mathbf{E}_A{}^M \mathcal{H}^{AB} \mathbf{E}_B{}^N, \quad (\text{B.2})$$

where

$$\eta_{AB} = \begin{pmatrix} \bar{\eta}^{ab} & \mathbf{0} \\ \mathbf{0} & -\bar{\eta}_{ab} \end{pmatrix}, \quad \mathcal{H}^{AB} = \begin{pmatrix} \bar{\eta}_{ab} & \mathbf{0} \\ \mathbf{0} & \bar{\eta}^{ab} \end{pmatrix}. \quad (\text{B.3})$$

Curved indices M, N, \dots are raised and lowered with η^{MN} and η_{MN} , while flat indices $\mathbf{A}, \mathbf{B}, \dots$ with η^{AB} and η_{AB} . The generalised dilaton is $\mathbf{d} = \phi - \frac{1}{4} \log(-\det \mathbf{G})$ with ϕ the usual dilaton. The generalised fluxes are defined as

$$\mathcal{F}_{ABC} = 3\Omega_{[ABC]}, \quad \mathcal{F}_A = \Omega^B{}_{BA} + 2\mathbf{E}_A{}^M \partial_M \mathbf{d} \quad (\text{B.4})$$

in terms of the generalized Weitzenböck connection

$$\Omega_{ABC} = \mathbf{E}_A{}^M \partial_M \mathbf{E}_B{}^N \mathbf{E}_{CN}, \quad (\text{B.5})$$

and they satisfy the following Bianchi identities

$$\begin{aligned} \partial_{[A} \mathcal{F}_{BCD]} - \frac{3}{4} \mathcal{F}_{[AB}{}^E \mathcal{F}_{CD]E} &= \mathcal{Z}_{ABCD}, \\ 2\partial_{[A} \mathcal{F}_{B]} + (\partial^C - \mathcal{F}^C) \mathcal{F}_{ABC} &= \mathcal{Z}_{AB}, \\ \partial^A \mathcal{F}_A - \frac{1}{2} \mathcal{F}^A \mathcal{F}_A + \frac{1}{12} \mathcal{F}^{ABC} \mathcal{F}_{ABC} &= \mathcal{Z} \end{aligned} \quad (\text{B.6})$$

where

$$\begin{aligned}\mathcal{Z}_{ABCD} &\equiv -\frac{3}{4}\Omega_{E[AB}\Omega^E{}_{CD]}, \\ \mathcal{Z}_{AB} &\equiv (\partial^M\partial_M E_{[A}{}^N)E_{B]N} - 2\Omega^C{}_{AB}\partial_C d, \\ \mathcal{Z} &\equiv -2\partial^A d\partial_A d + 2\partial^A\partial_A d + \frac{1}{4}\Omega^{ABC}\Omega_{ABC}.\end{aligned}\tag{B.7}$$

On the strong constraint one has $\mathcal{Z}_{ABCD} = 0$, $\mathcal{Z}_{AB} = 0$ and $\mathcal{Z} = 0$.

A generalised diffeomorphism is implemented on tensors $V_M{}^N$ and on the generalised dilaton (transforming as a density) as

$$\begin{aligned}\delta_\xi V_M{}^N &= \hat{\mathcal{L}}_\xi V_M{}^N = \xi^P\partial_P V_M{}^N + (\partial^N\xi_P - \partial_P\xi^N)V_M{}^P + (\partial_M\xi^P - \partial^P\xi_M)V_P{}^N, \\ \delta_\xi e^{-2d} &= \hat{\mathcal{L}}_\xi e^{-2d} = \partial_M(\xi^M e^{-2d}), \quad \iff \quad \delta_\xi d = \hat{\mathcal{L}}_\xi d = \xi^M\partial_M d - \frac{1}{2}\partial_M\xi^M,\end{aligned}\tag{B.8}$$

where $\hat{\mathcal{L}}_\xi$ is the generalised Lie derivative and ξ^M the parameter of the transformation. On the strong constraint the generalised fluxes transform as scalars under generalised diffeomorphisms. In terms of the generalised Lie derivative they may be written also as

$$\mathcal{F}_{ABC} = \hat{\mathcal{L}}_{E_A} E_B{}^M E_{CM}, \quad \mathcal{F}_A = 2\hat{\mathcal{L}}_{E_A} d.\tag{B.9}$$

The strong constraint is again a sufficient condition also for the closure of the algebra of generalised diffeomorphisms, so that $[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}] = \hat{\mathcal{L}}_{\xi_{12}}$ with $\xi_{12} = [\xi_1, \xi_2]_{(C)}^M = \frac{1}{2}(\delta_{\xi_1}\xi_2^M - \delta_{\xi_2}\xi_1^M) = 2\xi_{[1}^P\partial_P\xi_2^M + \partial^M\xi_{[1P}\xi_2^P]$ given by the C-bracket.

The point of view of this paper is similar to the setup of gauged DFT [46, 102]. Rather than the original interpretation of [102], where the $O(D, D)$ theory is gauged by shifting the generalised fluxes by some constant gaugings, we are closer to the interpretation of [46] where an appropriate generalised Scherk-Schwarz (gSS) reduction from D to $(D - d)$ dimensions essentially gives the same construction, i.e. generalised fluxes for the $(D - d)$ -dimensional theory that are gauged by the fluxes of the d -dimensional space. If tensors are decomposed as in the gSS ansatz $V_M{}^N(x) = \dot{V}_I{}^J(\dot{x})U^I{}_M(y)U_J{}^N(y)$ then generalised diffeomorphisms with parameter $\xi^M = \dot{\xi}^I U_I{}^M$ respect the ansatz in the sense that $\delta_\xi V_M{}^N = \delta_{\dot{\xi}} \dot{V}_I{}^J U^I{}_M U_J{}^N$ where in the case of a vector $\delta_{\dot{\xi}} \dot{V}^I = \delta_{\dot{\xi}} \dot{V}^I + \mathcal{F}^I{}_{JK} \dot{\xi}^J \dot{V}^K$. Imposing the strong constraint both on the “external space” (with coordinates \dot{x}) and “internal space” (with coordinates y) is a sufficient condition for closure of the algebra of the $\delta_{\dot{\xi}}$ transformations, since $[\delta_{\dot{\xi}_1}, \delta_{\dot{\xi}_2}] = \delta_{\dot{\xi}_{12}}$ where $\dot{\xi}_{12}^I = [\dot{\xi}_1, \dot{\xi}_2]_{(\mathcal{F})}^I \equiv [\dot{\xi}_1, \dot{\xi}_2]_{(C)}^I + \mathcal{F}^I{}_{JK} \dot{\xi}_1^J \dot{\xi}_2^K$. Notice the relation to the C-bracket $[\xi_1, \xi_2]_{(C)}^M = [\dot{\xi}_1, \dot{\xi}_2]_{(\mathcal{F})}^I U_I{}^M$. While the strong constraint on the internal space is a sufficient condition to define consistently gauged DFT, it may be relaxed, as long as $\mathcal{Z}_{IJKL} = \mathcal{Z}_{IJ} = \mathcal{Z} = 0$.

B.1 Geometric interpretation

The fact that the solution-generating techniques that we discuss do not modify the generalised fluxes when written in flat indices, still allows for the possibility of finding maps between geometric and (globally) non-geometric backgrounds. (Non)geometry is in fact captured by the fluxes written in *curved* indices — otherwise their definition in flat indices is dependent on the chosen double-Lorentz gauge, see [49] for a discussion. We refer

to [61] for a comprehensive review on non-geometric backgrounds in string theory. After going from flat to curved indices, the independent components of \mathcal{F}_{MNP} are usually divided into geometric fluxes $F_{mn}{}^p, H_{mnp}$ and non-geometric fluxes $Q_m{}^{np}, R^{mnp}$ [49], see also [20, 103]. These fluxes are often used to distinguish between

- *geometric backgrounds*, i.e. backgrounds that are well defined globally when using diffeomorphisms and B-field gauge transformations as transition functions in different patches. They have $Q_m{}^{np} = R^{mnp} = 0$.
- *globally non-geometric but locally geometric backgrounds*, e.g. T-folds, that require T-duality transformations to glue different patches. They have $Q_m{}^{np} \neq 0$ and $R^{mnp} = 0$.
- *locally non-geometric backgrounds*, i.e. depending on dual coordinates. They have $Q_m{}^{np} \neq 0$ and $R^{mnp} \neq 0$.

We do not consider locally non-geometric backgrounds because we always impose the strong constraint. If $\mathcal{F}'_{ABC} = \mathcal{F}_{ABC}$ then the generalised fluxes in curved indices are related as

$$\mathcal{F}'_{MNP} = h_M{}^Q h_N{}^R h_P{}^S \mathcal{F}_{QRS}, \quad \text{where} \quad h_M{}^Q = E'_M{}^A E_A{}^Q = U'_M{}^I U_I{}^Q. \quad (\text{B.10})$$

Notice that because of the assumptions we made on the twists U, U' , the matrix $h_M{}^Q$ is of block diagonal form as U in (2.2), i.e. it acts as the identity in the block for the (doubled) \dot{x} coordinates, while it is

$$h_M{}^Q = U'_M{}^I U_I{}^Q, \quad (\text{B.11})$$

in the block for the y -coordinates. If $h_M{}^Q$ has off-diagonal components $h_{\mu\rho}, h^{\mu\rho}$ then it is possible that the type of fluxes (in curved indices) change after the transformation. To show examples we say how they change in solution-generating techniques involving the (F) and (Q)-orbits. Notice that

$$\mathcal{F}_{MNP} = U_M{}^I U_N{}^J U_P{}^K (\dot{\mathcal{F}}_{IJK} + \mathcal{F}_{IJK}), \quad (\text{B.12})$$

where $\dot{\mathcal{F}}_{IJK} = \dot{E}_I{}^A \dot{E}_J{}^B \dot{E}_K{}^C \dot{\mathcal{F}}_{ABC}$ and $\dot{\mathcal{F}}_{IJK} = 3\delta_{[I}{}^\mu \dot{E}_{J]}{}^B \partial_\mu \dot{E}_{B|K]}$. Because of the derivative at least one leg is dotted in $\dot{\mathcal{F}}_{IJK}$, and the corresponding U and h act as the identity on that leg. The other two legs are not necessarily in the dotted directions, so that U and h can act non-trivially on them. Therefore let us stress that in general it is important to include the contribution of $\dot{\mathcal{F}}_{IJK}$ to understand how the fluxes change under the transformation.⁵¹ However, in order to have a more concrete discussion, we will look in the following only at the piece coming from \mathcal{F}_{IJK} , and we remind that this is non-zero only for $IJK = IJK$. We are therefore looking at flux components with all legs along y -directions, and these do not mix with the contributions from $\dot{\mathcal{F}}$ that have at least one leg along dotted coordinates. Hence we will analyse only these contributions to the fluxes

$$\mathcal{F}_{MNP} = U_M{}^I U_N{}^J U_P{}^K \mathcal{F}_{IJK}, \quad \mathcal{F}'_{MNP} = U'_M{}^I U'_N{}^J U'_P{}^K \mathcal{F}_{IJK}, \quad (\text{B.13})$$

⁵¹In the standard example of the T-duality chain on the 3-torus with H -flux done for example in [76] one dualises one leg at a time, and therefore the contribution to the flux that is discussed in each step is indeed coming from $\dot{\mathcal{F}}_{IJK}$, with two legs undotted.

that are related as

$$\mathcal{F}'_{MNP} = h_M^Q h_N^R h_P^S \mathcal{F}_{QRS}. \quad (\text{B.14})$$

Separating the components we have

$$\begin{aligned} \mathcal{F}'_{\mu\nu\rho} &= h_\mu^\alpha h_\nu^\beta h_\rho^\gamma \mathcal{F}_{\alpha\beta\gamma} + 3h_{[\mu}^\alpha h_\nu^\beta h_{\rho]\gamma} \mathcal{F}_{\alpha\beta}{}^\gamma + 3h_{[\mu}^\alpha h_{\nu|\beta} h_{|\rho]\gamma} \mathcal{F}_\alpha{}^{\beta\gamma} + h_{\mu\alpha} h_{\nu\beta} h_{\rho\gamma} \mathcal{F}^{\alpha\beta\gamma}, \\ \mathcal{F}'_{\mu\nu}{}^\rho &= h_\mu^\alpha h_\nu^\beta h^{\rho\gamma} \mathcal{F}_{\alpha\beta\gamma} + (h_\mu^\alpha h_\nu^\beta h^\rho{}_\gamma + 2h_{[\mu}^\beta h_{\nu]\gamma} h^{\rho\alpha}) \mathcal{F}_{\alpha\beta}{}^\gamma \\ &\quad + (h_{\mu\beta} h_{\nu\gamma} h^{\rho\alpha} + 2h_{[\mu}^\alpha h_{\nu]\beta} h^\rho{}_\gamma) \mathcal{F}_\alpha{}^{\beta\gamma} + h_{\mu\alpha} h_{\nu\beta} h^\rho{}_\gamma \mathcal{F}^{\alpha\beta\gamma}, \\ \mathcal{F}'_{\mu}{}^{\nu\rho} &= h_\mu^\alpha h^{\nu\beta} h^{\rho\gamma} \mathcal{F}_{\alpha\beta\gamma} + (h_{\mu\gamma} h^{\nu\alpha} h^{\rho\beta} + 2h_\mu^\alpha h^{[\nu|\beta} h^{|\rho]\gamma}) \mathcal{F}_{\alpha\beta}{}^\gamma \\ &\quad + (h_\mu^\alpha h^\nu{}_\beta h^\rho{}_\gamma + 2h_{\mu\beta} h^{[\nu}{}_\gamma h^{\rho]\alpha}) \mathcal{F}_\alpha{}^{\beta\gamma} + h_{\mu\alpha} h^\nu{}_\beta h^\rho{}_\gamma \mathcal{F}^{\alpha\beta\gamma}, \\ \mathcal{F}'^{\mu\nu\rho} &= h^{\mu\alpha} h^{\nu\beta} h^{\rho\gamma} \mathcal{F}_{\alpha\beta\gamma} + 3h^{[\mu|\alpha} h^{|\nu|\beta} h^{|\rho]\gamma} \mathcal{F}_{\alpha\beta}{}^\gamma + 3h^{[\mu|\alpha} h^{|\nu}{}_\beta h^{|\rho]\gamma} \mathcal{F}_\alpha{}^{\beta\gamma} + h^\mu{}_\alpha h^\nu{}_\beta h^\rho{}_\gamma \mathcal{F}^{\alpha\beta\gamma} \end{aligned} \quad (\text{B.15})$$

C On the $O(d, d)$ parametrisation of the twist

Here we explain why we take (2.20) as a parametrisation of the twist U . The $O(d, d)$ group is generated by the matrices [103]

$$R_M{}^N = \begin{pmatrix} (\rho^t)^\mu{}_\nu & 0 \\ 0 & (\rho^{-1})_\mu{}^\nu \end{pmatrix}, \quad J_M{}^N = \begin{pmatrix} \delta^\mu{}_\nu & 0 \\ b_{\mu\nu} & \delta_\mu{}^\nu \end{pmatrix}, \quad T_M^{[\sigma]N} = \begin{pmatrix} \delta^\mu{}_\nu - u_{(\sigma)}^\mu{}_\nu & u_{(\sigma)}^{\mu\nu} \\ u_{\mu\nu}^{(\sigma)} & \delta_\mu{}^\nu - u_{\mu}^{(\sigma)\nu} \end{pmatrix}, \quad (\text{C.1})$$

where $\rho \in \text{GL}(d)$, $b^t = -b$ and $u_{(\sigma)} = u^{(\sigma)} = \text{diag}(0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is at the position $\sigma \in \{1, \dots, d\}$. The matrices R and J generate two separate subgroups, which together generate a larger subgroup of $O(d, d)$ that we call G_{geom} . Notice that $RJ = J'R$, with $b' = \rho^{-1} b \rho^{-t}$. Hence we can always use this move to write any element of G_{geom} e.g. in the order JR . Each $T^{[\sigma]}$ can be understood as a T-duality along the direction σ . When we involve the matrices $T^{[\sigma]}$ we generate also different types of matrices. A distinguished one is given by the multiplication of all the $T^{[\sigma]}$

$$T_M{}^N = (T^{[1]} T^{[2]} \dots T^{[d]})_M{}^N = \begin{pmatrix} 0 & \delta^{\mu\nu} \\ \delta_{\mu\nu} & 0 \end{pmatrix}, \quad (\text{C.2})$$

which we will call a rigid T -transformation since it is constant. Notice that $T^2 = 1$ and that T is not equivalent to the $O(d, d)$ metric η , because of the position of the indices in the definitions. We also have matrices $S \equiv TR$

$$S_M{}^N = \begin{pmatrix} 0 & (\sigma^{-1})^{\mu\nu} \\ (\sigma^t)_{\mu\nu} & 0 \end{pmatrix}, \quad (\text{C.3})$$

with $\sigma \in \text{GL}(d)$, which form a subgroup on their own. Notice that $TRT = R'$ where $\rho' = \rho^{-t}$. Finally, we also have matrices $K \equiv TJT$

$$K_M{}^N = \begin{pmatrix} \delta^\mu{}_\nu & \beta^{\mu\nu} \\ 0 & \delta_\mu{}^\nu \end{pmatrix}, \quad (\text{C.4})$$

with $\beta^t = -\beta$, which again form a subgroup on their own. More complicated elements occur if instead of T we multiply by single $T^{[\sigma]}$ matrices. However, we will not consider this case here since we will assume that we have already reduced from D to d dimensions, so that T-duality is implemented on all the d coordinates.

One way to classify the possibilities in the parametrisation of the most general $O(d, d)$ element is to observe whether they involve an even or an odd number of T 's, which we consequently call “even” and “odd” elements respectively. Let us first analyse the most generic parametrisation for an even twist and argue that they are generated by the matrices R, J, K . As we have remarked $TRT = R'$ and we can pull any R to the right of any J . Notice that the same considerations apply for K . Therefore, any even twist can be written as a generic product of J 's and K 's times a single R . Finally notice that we can always write $JK = K'J'R'$ where

$$\rho' = (1 + b\beta)^{-1}, \quad b' = b(1 + \beta b), \quad \beta' = \beta(1 + b\beta)^{-1}, \quad (C.5)$$

if we assume that $1 + b\beta$ is invertible. Therefore, under this assumption we can always re-arrange the order of the matrices J and K in the product, at the cost of introducing a matrix of type R . We can therefore conclude that any twist U of even type can be written as the product

$$U_{\text{even}} = KJR, \quad (C.6)$$

for some K, J, R . Notice, however, because of the previous assumption we may be excluding twists involving products of the type JK with $1 + b\beta$ not invertible. Let us finally discuss the most generic parametrisation for odd twists. As they must be generated by an odd number of T 's we can write them as a generic product of U_{even} 's and T 's (odd). By noticing that $TKJR = (TKT)(TJT)(TRT)T = J'K'R'T = K''J''R''T$ where we used $T^2 = 1$, we conclude that we can always pull T to the left, for example. Then the most generic parametrisation for an odd twist is

$$U_{\text{odd}} = TKJR. \quad (C.7)$$

The only effect of the rigid T matrix in the odd twist is to map an orbit to its dual.

D Details on RR fields and type II

In this appendix we collect some additional details that are useful for the DFT formulation of type II supergravity. We closely follow [88] and [20] and highlight the differences when present. In this appendix we will always assume that $D = 10$. We will also remove the boldface from all objects and indices, since the reduction from D to d dimensions is not relevant here, and we prefer to have a simpler notation.

Given the Clifford algebra $C(D, D)$ generated by the Gamma matrices Γ_M satisfying $\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}$ (notice the *curved* indices as in [88]) we can define $\psi_M = \Gamma_M/\sqrt{2}$ so that we can have a representation in terms of fermionic oscillators ψ^m, ψ_m satisfying $\{\psi_m, \psi^n\} = \delta_m^n$, $\{\psi_m, \psi_n\} = 0$, $\{\psi^m, \psi^n\} = 0$. The Clifford vacuum $|0\rangle$ is defined such that $\psi_m|0\rangle = 0$ for all m . A generic spinor can be rewritten as $|\chi\rangle = \sum_{p=0}^D \frac{1}{p!} C_{m_1 \dots m_p} \psi^{m_1} \dots \psi^{m_p} |0\rangle$ which is consequence of the one-to-one map between spinors χ and polyforms $C = \sum_{p=0}^D \frac{1}{p!} C_{m_1 \dots m_p} dx^{m_1} \dots dx^{m_p}$ on $\mathbb{R}^{1, D-1}$. One fixes the normalisation $\langle 0|0\rangle = 1$ and defines conjugation such as $(\psi^{m_1} \dots \psi^{m_p} |0\rangle)^\dagger = \langle 0| \psi_{m_p} \dots \psi_{m_1}$.

$\text{Pin}(D, D)$ is a subgroup of $C(D, D)$, see [88] for its definition. The algebras of $O(D, D)$ and $\text{Pin}(D, D)$ are isomorphic and if we call J_{MN} the generators of the algebra of $O(D, D)$ satisfying $[J_{MN}, J_{PQ}] = \eta_{MP}J_{QN} - \eta_{NP}J_{QM} - \eta_{MQ}J_{PN} + \eta_{NQ}J_{PM}$ then in the spinor representation we can identify $J^{MN} \equiv \frac{1}{2}\Gamma^{MN}$, where $\Gamma^{M_1 \dots M_p}$ is the totally antisymmetric product of p Gamma matrices, with the factor $1/p!$ included, e.g. $\Gamma^{MN} = \frac{1}{2}[\Gamma^M, \Gamma^N]$. We denote by ρ the group homomorphisms $\rho : \text{Pin}(D, D) \rightarrow O(D, D)$. $\text{Pin}(D, D)$ is in fact the double cover of $O(D, D)$ because $\rho(S) = \rho(-S)$ for $S \in \text{Pin}(D, D)$. We have

$$S\Gamma^M S^{-1} = \Gamma^N h_N^M, \quad \text{where} \quad \rho(S) = h_{\bullet}^{\bullet}. \quad (\text{D.1})$$

This is a change of conventions compared to [88], where they define ρ such that $\rho(S) = h_{\bullet}^{\bullet}$. Notice that this amounts to swapping the two diagonal blocks of h , and the two off-diagonal blocks. $\text{Spin}(D, D)$ is the subgroup generated by the S made out of only an even number of Gamma matrices and under the group homomorphism ρ it is mapped to $\text{SO}(D, D)$.

There are some distinguished elements of $\text{Spin}^+(D, D)$ that are important for the rest of the section. Their definition and the corresponding $\text{SO}^+(D, D)$ elements are

$$S_{[b]} = \exp(-\frac{1}{2}b_{mn}\psi^m\psi^n), \quad S_{[\beta]} = \exp(\frac{1}{2}\beta^{mn}\psi_m\psi_n), \quad S_{[r]} = (\det r)^{-1/2} \exp(\psi^m R_m^n \psi_n),$$

$$h_{[b]} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}, \quad h_{[\beta]} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad h_{[r]} = \begin{pmatrix} r^{-T} & 0 \\ 0 & r \end{pmatrix}.$$

Above we have $r_m^n = (\exp R)_m^n \in \text{GL}^+(D)$, where the plus stands for elements with a positive determinant. Other useful elements of $\text{Pin}(D, D)$ that are not in $\text{Spin}(D, D)$ are $S_m^\pm = \psi^m \pm \psi_m$. They satisfy $(S_m^\pm)^2 = \pm 1$ and

$$\rho(S_m^+) = - \begin{pmatrix} 1 - u_m & -u_m \\ -u_m & 1 - u_m \end{pmatrix}, \quad \rho(S_m^-) = - \begin{pmatrix} 1 - u_m & u_m \\ u_m & 1 - u_m \end{pmatrix}, \quad (\text{D.2})$$

where $u_m = \text{diag}(0, \dots, 0, 1, 0, \dots, 0)$ has 1 only at position m . It is nevertheless preferable to implement one factorised T-duality along the direction m as in [23] by $S_{[m]} = (\psi^m - \psi_m)(-1)^{N_F}$ with $(-1)^{N_F}$ to be defined below. In fact this corresponds to $S_{[m]}\psi^M S_{[m]}^{-1} = \psi^N T_N^{[m]M}$ where $T^{[m]}$ implements T-duality along direction m , and acts as the identity on other coordinates, without other unwanted signs.

The charge conjugation matrix is defined as $C = (\psi^0 - \psi_0)(\psi^1 - \psi_1) \dots (\psi^9 - \psi_9)$ and it satisfies $C^{-1} = -C$. One has the relations $C\psi_m C^{-1} = \psi^m$, $C\psi^m C^{-1} = \psi_m$ and $C\Gamma^M C^{-1} = (\Gamma^M)^\dagger$, $C\Gamma_M C^{-1} = (\Gamma_M)^\dagger$. Defining the T-duality matrix as in (2.28) one sees that $C\Gamma_M C^{-1} = T_M^N \Gamma_N$ and therefore $\rho(C) = T$. Under conjugation $C^\dagger = C^{-1}$, and we have that $S^\dagger = CS^{-1}C^{-1}$ if $S \in \text{Spin}^+(D, D)$, while $S^\dagger = -CS^{-1}C^{-1}$ if $S \in \text{Spin}^-(D, D)$. Moreover $(S_m^\pm)^\dagger = CS_m^\pm C^{-1}$.

One can define chiral spinors from the number operator $N_F = \sum_m \psi^m \psi_m$ that gives $N_F |\chi\rangle_p = p |\chi\rangle_p$ and $(-1)^{N_F} |\chi\rangle = \sum_{p=0}^D (-1)^p \frac{1}{p!} C_{m_1 \dots m_p} \psi^{m_1} \dots \psi^{m_p} |0\rangle$. Chiral projections of spinors are therefore defined as $|\chi_\pm\rangle = \frac{1}{2}(1 \pm (-1)^{N_F}) |\chi\rangle$ so that $(-1)^{N_F} |\chi_\pm\rangle = \pm |\chi_\pm\rangle$. We have that χ_+ is mapped to even forms and χ_- to odd forms. Notice that the action of $\text{Spin}(D, D)$ preserves the chirality.

The generalised metric \mathcal{H} is an element of $\text{SO}^-(D, D)$ because it has determinant 1 and we are in Lorentz signature. One therefore takes an $\mathbb{S} \in \text{Spin}^-(D, D)$ such that $\rho(\mathbb{S}) = \mathcal{H}^{\bullet\bullet}$. Notice again the change in conventions, in [88] one has $\rho(\mathbb{S}) = \mathcal{H}_{\bullet\bullet}$. Here it is \mathbb{S} rather than \mathcal{H} that is considered the fundamental field. We have $\mathbb{S} = \mathbb{S}^\dagger = -C\mathbb{S}^{-1}C^{-1}$. Let us further define $S_{[\bar{\eta}]} = \psi^0\psi_0 - \psi_0\psi^0$, then $S_{[\bar{\eta}]} = S_{[\bar{\eta}]}^\dagger = S_{[\bar{\eta}]}^{-1} \in \text{Spin}^-(D, D)$. If G_{mn} is the NSNS metric of the background and $G = e\bar{\eta}e^T$ in terms of a standard vielbein e and the Minkowski metric $\bar{\eta}$, then we can define $S_{[G]} \equiv S_{[e]}S_{[\bar{\eta}]}S_{[e]}^\dagger$ where $S_{[e]} = S_{[r=e]}$, and where we used the definition of $S_{[r]}$ above. We then have $\rho(S_{[G]}^{-1}) = \text{diag}(G, G^{-1})$. Given the NSNS field B_{mn} of the background we can further define $S_{[B]} \equiv S_{[b=B]}$ using the definition of $S_{[b]}$ above. Then if we define $S_{\mathcal{H}} \equiv S_{[B]}^\dagger S_{[G]}^{-1} S_{[B]}$ we have $\rho(S_{[\mathcal{H}]}) = \mathcal{H}^{\bullet\bullet}$. Notice that because of our identification $\rho(S) = h_{\bullet\bullet}$, if we consider an $O(D, D)$ transformation $\mathcal{H}' = h^T \mathcal{H} h$ on the generalised metric, then it corresponds to $\mathbb{S}' = S^\dagger \mathbb{S} S$ which again differs from [88]. For later convenience it will be useful to define also $\mathcal{K} = C^{-1}\mathbb{S}$, so that $\rho(\mathcal{K}) = \rho(C^{-1})\rho(\mathbb{S}) = T\mathcal{H}^{\bullet\bullet} = \mathcal{H}_{\bullet\bullet}$.

To write the contribution of RR fields to the DFT action according to [88] one uses a spinor $|\chi\rangle$ that encodes the RR potentials. The contribution of RR fields to the Lagrangian is

$$\mathcal{L}_{RR} = \frac{1}{4}(\not{\partial}\chi)^\dagger \mathbb{S} \not{\partial}\chi = \frac{1}{8}\partial_M \bar{\chi} \Gamma^M \mathcal{K} \Gamma^N \partial_N \chi = \frac{1}{4} \overline{(\not{\partial}\chi)} \mathcal{K} \not{\partial}\chi, \tag{D.3}$$

where we dropped the bra/ket notation and we defined $\bar{\chi} = \chi^\dagger C$ and $\not{\partial} = \frac{1}{\sqrt{2}}\Gamma^M \partial_M = \psi^m \partial_m + \psi_m \tilde{\partial}^m$. Using the $C(D, D)$ relations and the weak constraint it is easy to see that $\not{\partial}^2 = 0$. Reality of the action follows from $\mathbb{S}^\dagger = \mathbb{S}$ and the fact that χ is Grassmann even. Under the constant $\text{Spin}(D, D)$ transformation $\chi \rightarrow S^{-1}\chi$ and $X^M \rightarrow X^N h_N^M$ (where $\rho(S) = h$ so that $\partial_M \rightarrow (h^{-1})_M^N \partial_N$) one has $\not{\partial}\chi \rightarrow S^{-1}\not{\partial}\chi$, and the $\text{Spin}(D, D)$ invariance of the Lagrangian is manifest. It is not invariant under the full $\text{Pin}(D, D)$ because this would break the chirality of χ . In addition to the above Lagrangian, and after deriving the equations of motion, one imposes the duality relation

$$\not{\partial}\chi = -\mathcal{K}\not{\partial}\chi, \tag{D.4}$$

which are invariant under $\chi \rightarrow S^{-1}\chi, \mathbb{S} \rightarrow S^\dagger \mathbb{S} S$ for $S \in \text{Spin}^+(D, D)$. The restriction to only $\text{Spin}^+(D, D)$ (and not $\text{Spin}(D, D)$) invariance has to do with the Lorentz signature, so that timelike T-dualities would spoil the relations. Notice that the duality relation is consistent with $\mathcal{K}^2 = 1$. The equations of motion for χ that one obtains from the above Lagrangian are $\not{\partial}(\mathcal{K}\not{\partial}\chi) = 0$. They are consistent with the duality relation and the identity $\not{\partial}^2 = 0$ valid on the weak constraint. The abelian gauge symmetries acting on RR potentials are here implemented as $\delta_\lambda \chi = \not{\partial}\lambda$, and because $\delta_\lambda \not{\partial}\chi = \not{\partial}^2 \lambda = 0$, the invariance of the Lagrangian is obvious.

The above Lagrangian can be rewritten as a Lagrangian for the RR fields in the usual way.⁵² We recall that we are taking the spinor χ rewritten as $|\chi\rangle = \sum_{p=0}^D \frac{1}{p!} C_{m_1 \dots m_p} \psi^{m_1} \dots \psi^{m_p} |0\rangle$ which we associate to a polyform $C = \sum_{p=0}^D \frac{1}{p!} C_{m_1 \dots m_p} dx^{m_1} \dots dx^{m_p}$. Let us now define $F_{m_1 \dots m_p} \equiv p \partial_{[m_1} C_{m_2 \dots m_p]}$ so that on the strong constraint

⁵²To do so it is useful to notice that $\exp(\psi^m R_m^n \psi_n) \psi^p |0\rangle = (\exp R)_m^p \psi^m |0\rangle$ and $S_{[\bar{\eta}]} \psi^m |0\rangle =$

(i.e. taking $\tilde{\delta}^m = 0$) we have $|F\rangle = |\not\partial\chi\rangle = \sum_{p=1}^D \frac{1}{(p-1)!} \partial_{m_1} C_{m_2 \dots m_p} \psi^{m_1} \dots \psi^{m_p} |0\rangle = \sum_{p=1}^D \frac{1}{p!} F_{m_1 \dots m_p} \psi^{m_1} \dots \psi^{m_p} |0\rangle$ and in polyform notation $F = dC$. Using the NSNS B-field B_{mn} we also define $|\hat{F}\rangle \equiv \exp(-\frac{1}{2} B_{mn} \psi^m \psi^n) |F\rangle$ so that $\hat{F} = \exp(-B^{(2)}) \wedge F$ where $B^{(2)} = \frac{1}{2} B_{mn} dx^m \wedge dx^n$. The conjugate is obtained by $\langle \hat{F} | = \sum_{p=1}^D \frac{1}{p!} \langle 0 | \psi_{m_p} \dots \psi_{m_1} \hat{F}_{m_1 \dots m_p}$. With this rewriting the RR Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{RR} &= \frac{1}{4} (\not\partial\chi)^\dagger \mathbb{S} \not\partial\chi = \frac{1}{4} \langle F | S_{[\mathcal{H}]} | F \rangle = \frac{1}{4} \langle \hat{F} | S_{[G]}^{-1} | \hat{F} \rangle \\ &= \frac{1}{4} \sum_{p=1}^D \sum_{q=1}^D \frac{1}{p!q!} \langle 0 | \psi_{m_p} \dots \psi_{m_1} S_{[G]}^{-1} \psi^{n_1} \dots \psi^{n_q} | 0 \rangle \hat{F}_{m_1 \dots m_p} \hat{F}_{n_1 \dots n_q} \\ &= -\frac{\sqrt{-G}}{4} \sum_{p=1}^D |\hat{F}_p|^2, \quad |\hat{F}_p|^2 \equiv \frac{1}{p!} \hat{F}_{m_1 \dots m_p} G^{m_1 n_1} \dots G^{m_p n_p} \hat{F}_{n_1 \dots n_p}, \end{aligned} \quad (\text{D.5})$$

where we used $\langle 0 | \psi_{m_p} \dots \psi_{m_1} \psi^{n_1} \dots \psi^{n_q} | 0 \rangle = p! \delta_{p,q} \delta_{m_1}^{n_1} \dots \delta_{m_p}^{n_p}$. This is the Lagrangian in the democratic formulation, see [88]. In fact $\hat{F}_{m_1 \dots m_p}$ are the RR field strengths that are commonly used in type II supergravity (they are the \hat{F} of [88]), and they are the ones that appear in the quadratic couplings of the fermions in the Green-Schwarz formulation of the superstring (they are the F of [93]).

In terms of $|\hat{F}\rangle$ the self duality relation reads as $|\hat{F}\rangle = -S_{[G]} C |\hat{F}\rangle$ and in terms of p -forms $\hat{F}^{(p)} = -(-1)^{\frac{1}{2}p(p+1)} * \hat{F}^{(10-p)}$. The Hodge dual is defined as

$$(*A)_{m_1 \dots m_p} = \frac{1}{(10-p)!} G_{m_1 n_1} \dots G_{m_p n_p} \epsilon^{k_{p+1} \dots k_{10} n_1 \dots n_p} A_{k_{p+1} \dots k_{10}}, \quad (\text{D.6})$$

where $\epsilon^{1 \dots 10} = 1$, $\epsilon_{1 \dots 10} = -1$, $\epsilon^{m_1 \dots m_{10}} = \frac{1}{\sqrt{-G}} \epsilon^{m_1 \dots m_{10}}$, $\epsilon_{m_1 \dots m_{10}} = \sqrt{-G} \epsilon_{m_1 \dots m_{10}}$. We also have $* * \omega^{(p)} = -(-1)^{p(10-p)} \omega^{(p)}$ where the additional minus sign is due to the Lorentz signature.

Under the abelian RR gauge transformations the spinor transforms as $\delta_\lambda |\chi\rangle = \not\partial |\lambda\rangle$ which implies $\delta_\lambda C = d\lambda$. Under D -dimensional diffeomorphisms $\delta_\xi C_{m_1 \dots m_p} = \mathcal{L}_\xi C_{m_1 \dots m_p}$, i.e. it transforms with the standard Lie derivative as expected. Under the gauge transformation of the B-field with gauge parameter $\tilde{\xi}$ the transformation is $\delta_{\tilde{\xi}} C_{m_1 \dots m_p} = p(p-1) \partial_{[m_1} \tilde{\xi}_{m_2} C_{m_3 \dots m_p]}$, meaning $\delta_{\tilde{\xi}} C = d\tilde{\xi} \wedge C$ where $\delta_{\tilde{\xi}} B^{(2)} = d\tilde{\xi}$. Therefore F is not invariant under gauge transformations of the B-field, but \hat{F} is invariant. Defining $\hat{A} = e^{-B^{(2)}} \wedge C$ (i.e. $C = e^{B^{(2)}} \wedge \hat{A}$) then $\delta_{\tilde{\xi}} \hat{A} = 0$. Notice that in terms of these new potentials $\hat{F} = d\hat{A} + H \wedge \hat{A}$, with $H = dB$. If we further define $A^{(p)} = \hat{A}^{(p)}$ for $p \neq 4$ and $A^{(4)} = \hat{A}^{(4)} + \frac{1}{2} B^{(2)} \wedge \hat{A}^{(2)}$ then one finds

$$\begin{aligned} \hat{F}^{(1)} &= dA^{(0)}, & \hat{F}^{(2)} &= dA^{(1)}, \\ \hat{F}^{(3)} &= dA^{(2)} + H \wedge A^{(0)}, & \hat{F}^{(4)} &= dA^{(3)} + H \wedge A^{(1)}, \\ \hat{F}^{(5)} &= dA^{(4)} + \frac{1}{2} H \wedge A^{(2)} - \frac{1}{2} B^{(2)} \wedge dA^{(2)}, \end{aligned} \quad (\text{D.7})$$

$-\delta^{mn} \bar{\eta}_{np} \psi^p |0\rangle$. It follows that

$$\begin{aligned} S_{[G]} \psi^{m_1} \dots \psi^{m_p} |0\rangle &= -\frac{1}{\sqrt{-G}} \delta^{m_1 n_1} \dots \delta^{m_p n_p} G_{n_1 q_1} \dots G_{n_p q_p} \psi^{q_1} \dots \psi^{q_p} |0\rangle, \\ S_{[G]}^{-1} \psi^{m_1} \dots \psi^{m_p} |0\rangle &= -\sqrt{-G} G^{m_1 n_1} \dots G^{m_p n_p} \delta_{n_1 q_1} \dots \delta_{n_p q_p} \psi^{q_1} \dots \psi^{q_p} |0\rangle, \end{aligned}$$

where G is the determinant of the metric G_{mn} .

where on the left we have IIB and on the right IIA. These are the more familiar parametrisations of the RR field strengths in terms of potentials [88].

To rewrite the above results as in [20] one has to pass from a basis of Gamma matrices Γ^M in terms of curved indices to a basis Γ^A in terms of flat indices. This can be done by using $S_{[E]}$, the $\text{Pin}(D, D)$ representative of the generalised vielbein, as $S_{[E]}\Gamma^A S_{[E]}^{-1} = \Gamma^M E_M^A$. It is also necessary to change vacuum, since we have to go from a vacuum defined as $\Gamma_m |0\rangle = 0$ to a new one such that $\Gamma_a |0'\rangle = 0$. In the following we will omit the prime but always think in terms of the new vacuum. Notice also that we can still take the charge conjugation matrix C and $S_{[\bar{\eta}]}$ to be defined as before, while we interpret the ψ as being defined in terms of flat indices. We first define

$$|\mathfrak{F}\rangle = e^d S_{[E]} |F\rangle. \tag{D.8}$$

Because we can decompose $S_{[E]} = S_{[e]}^{-1} S_{[B]}$ and we get

$$|\mathfrak{F}\rangle = \sum_n \frac{e^\phi}{n!} \hat{F}_{m_1 \dots m_n} e_{a_1}^{m_1} \dots e_{a_n}^{m_n} \psi^{a_1} \dots \psi^{a_n} |0\rangle. \tag{D.9}$$

From the definition it follows that $|\mathfrak{F}\rangle = \not{\partial} |\mathfrak{c}\rangle + S_E \psi^M \partial_M S_E^{-1} |\mathfrak{c}\rangle + e^d \partial_M e^{-d} \psi^M |\mathfrak{c}\rangle$ in terms of $|\chi\rangle = e^{-d} S_{[E]}^{-1} |\mathfrak{c}\rangle$. This may be rewritten (see also [94]) using⁵³ $E_A^M \partial_M S_{[E]} S_{[E]}^{-1} = \frac{1}{2} \Omega_{ABC} \psi^{BC}$ as well as $\Omega_{ABC} \psi^A \psi^B \psi^C = \frac{1}{3} \mathcal{F}_{ABC} \psi^{ABC} + \Omega^B_{BA} \psi^A$, which is a consequence of the $C(D, D)$ relations. To conclude

$$|\mathfrak{F}\rangle = \left(\psi^A \partial_A - \frac{1}{6} \mathcal{F}_{ABC} \psi^{ABC} - \frac{1}{2} \mathcal{F}_A \psi^A \right) |\mathfrak{c}\rangle = \not{\nabla} |\mathfrak{c}\rangle, \quad \not{\nabla} \equiv \not{\partial} - \mathcal{F}^{(3)} - \frac{1}{2} \mathcal{F}^{(1)}, \tag{D.10}$$

where $\mathcal{F}^{(n)}$ includes the $1/n!$ factor. Notice that $\not{\nabla}^2 = 0$ on the strong constraint. Now gauge transformations read as $\delta_\lambda |\mathfrak{c}\rangle = \not{\nabla} |\lambda\rangle$, and the Bianchi identities as $\not{\nabla} |\mathfrak{F}\rangle = 0$. The Lagrangian of [88] is readily rewritten as

$$\mathcal{L}_{RR} = \frac{1}{4} e^{-2d} \langle \mathfrak{F} | S_{\bar{\eta}}^{-1} | \mathfrak{F} \rangle = -\frac{1}{4} e^{-2d} \langle \mathfrak{F} | C S_{\bar{\eta}}^{-1} | \mathfrak{F} \rangle. \tag{D.11}$$

Now we have

$$\begin{aligned} C S_{\bar{\eta}}^{-1} &= (\psi^0 - \psi_0)(\psi^1 - \psi_1) \dots (\psi^9 - \psi_9)(\psi^0 \psi_0 - \psi_0 \psi^0) \\ &= (\psi^0 - \psi_0)(\psi^0 \psi_0 - \psi_0 \psi^0)(\psi^1 - \psi_1) \dots (\psi^9 - \psi_9) \\ &= -(\psi^0 + \psi_0)(\psi^1 - \psi_1) \dots (\psi^9 - \psi_9) = \Psi_-, \end{aligned} \tag{D.12}$$

and therefore we conclude

$$\mathcal{L}_{RR} = -\frac{1}{4} e^{-2d} \langle \mathfrak{F} | \Psi_- | \mathfrak{F} \rangle. \tag{D.13}$$

⁵³Given the Lie algebra isomorphism between $O(D, D)$ and $\text{Pin}(D, D)$ one can identify the generators $J^{MN} \rightarrow \psi^{MN}$ and use them to construct group elements $E_A^M = \exp(\Lambda_{PQ} J^{PQ})_A^M$ and $S_{[E]} = \exp(\Lambda_{PQ} \psi^{PQ})$ with the same Λ_{PQ} . Therefore one has $(\partial_M E E^{-1})_{BC} = \Xi_{MPQ} (J^{PQ})_{BC}$ and $\partial_M S_{[E]} S_{[E]}^{-1} = \Xi_{MPQ} \psi^{PQ}$ with the same Ξ_{MPQ} . It follows that $E_A^M \partial_M S_{[E]} S_{[E]}^{-1} = E_A^M \Xi_{MPQ} \psi^{PQ} = \frac{1}{2} E_A^M \Xi_{MPQ} (2\delta_B^{[P} \delta_C^{Q]}) \psi^{BC} = \frac{1}{2} E_A^M \Xi_{MPQ} (J^{PQ})_{BC} \psi^{BC} = \frac{1}{2} E_A^M (\partial_M E E^{-1})_{BC} \psi^{BC} = \frac{1}{2} \Omega_{ABC} \psi^{BC}$, where we used the matrix realisation $(J^{MN})_{P^Q} = 2\delta_P^{[M} \eta^{N]Q}$. In (4.32) of [94] only the leading term in the expansion is taken into account.

Varying the Lagrangian with respect to $|\mathfrak{c}\rangle$ one finds the equations of motion $\nabla\Psi_-|\mathfrak{F}\rangle = 0$.

The above Lagrangian is different when comparing to [20], since they have Ψ_+ instead of Ψ_- , where $\Psi_{\pm} = (\psi_0 \mp \psi^0)(\psi_1 \pm \psi^1) \cdots (\psi_9 \pm \psi^9)$. To further test the mismatch we first move to prove (A.9) of [20]. Using $\Psi_{\pm}\psi^A = \mp\mathcal{H}^{AB}\psi_B\Psi_{\pm}$ one finds

$$\begin{aligned}
 \Psi_+|\omega\rangle &= \sum_n \frac{1}{n!} \omega_{a_1 \cdots a_n} \Psi_+ \psi^{a_1} \cdots \psi^{a_n} |0\rangle \\
 &= \sum_n \frac{(-1)^n}{n!} \omega_{a_1 \cdots a_n} \mathcal{H}^{a_1 B_1} \cdots \mathcal{H}^{a_n B_n} \psi_{B_1} \cdots \psi_{B_n} \Psi_+ |0\rangle \\
 &= - \sum_n \frac{(-1)^n}{n!} \omega_{a_1 \cdots a_n} \bar{\eta}^{a_1 b_1} \cdots \bar{\eta}^{a_n b_n} \psi_{b_1} \cdots \psi_{b_n} \psi^0 \psi^1 \cdots \psi^9 |0\rangle \\
 &= \sum_n \frac{(-1)^n (-1)^{\frac{n(n-1)}{2}}}{n!(10-n)!} \omega_{a_1 \cdots a_n} \bar{\eta}^{a_1 b_1} \cdots \bar{\eta}^{a_n b_n} \epsilon_{b_1 \cdots b_n b_{n+1} \cdots b_{10}} \psi^{b_{n+1}} \cdots \psi^{b_{10}} |0\rangle \\
 &= \sum_n \frac{1}{n!(10-n)!} \epsilon_{b_{n+1} \cdots b_{10}} \eta^{a_1 \cdots a_n} \omega_{a_n \cdots a_1} \psi^{b_{n+1}} \cdots \psi^{b_{10}} |0\rangle.
 \end{aligned} \tag{D.14}$$

This agrees with (A.9) of [20]. In the first step we used the commutation relations between Ψ_+ and the ψ 's. In the second step we used $\Psi_+|0\rangle = -\psi^0\psi^1 \cdots \psi^9|0\rangle$, in the third one we used (5.39) of [88] (notice that that equation is written with an ϵ with upper indices, so there is an overall minus sign when written for an epsilon with lower indices). In the last step we got a factor of $(-1)^{\frac{n(n-1)}{2}}$ by rearranging the indices of ω and a factor of $(-1)^{n(10-n)}$ by swapping the position of the $b_1 \cdots b_n$ and $b_{n+1} \cdots b_D$ indices in ϵ . In the last line, indices in the epsilon tensor are raised with the Minkowski $\bar{\eta}$.

The computation for Ψ_- is identical, since also $\Psi_-|0\rangle = -\psi^0\psi^1 \cdots \psi^9|0\rangle$, but because of the different sign in the commutation relations with the ψ 's one gets an additional factor of $(-1)^n$

$$\Psi_-|\omega\rangle = \sum_n \frac{(-1)^n}{n!(10-n)!} \epsilon_{b_{n+1} \cdots b_{10}} \eta^{a_1 \cdots a_n} \omega_{a_n \cdots a_1} \psi^{b_{n+1}} \cdots \psi^{b_{10}} |0\rangle. \tag{D.15}$$

Let us now compare to (A.11) of [20]. First we compute

$$\begin{aligned}
 C|\chi\rangle &= \sum_n \frac{1}{n!} \chi_{a_1 \cdots a_n} C \psi^{a_1} \cdots \psi^{a_n} |0\rangle \\
 &= \sum_n \frac{1}{n!} \chi_{a_1 \cdots a_n} \psi_{a_1} \cdots \psi_{a_n} C |0\rangle \\
 &= \sum_n \frac{1}{n!} \chi_{a_1 \cdots a_n} \psi_{a_1} \cdots \psi_{a_n} \psi^0 \cdots \psi^9 |0\rangle \\
 &= \sum_n \frac{(-1)^{\frac{n(n-1)}{2}}}{n!(10-n)!} \chi_{a_1 \cdots a_n} \epsilon^{a_1 \cdots a_n b_{n+1} \cdots b_{10}} \psi^{b_{n+1}} \cdots \psi^{b_{10}} |0\rangle,
 \end{aligned} \tag{D.16}$$

where in the last step we used again (5.39) of [88]. Now using $(C|\chi\rangle)^\dagger = \langle\chi|C^\dagger = -\langle\chi|C$

we find

$$\begin{aligned}
 \langle \chi | C\Psi_+ | \omega \rangle &= - \sum_{n,m} \frac{(-1)^{\frac{n(n-1)}{2}}}{n!(10-n)! m!(10-m)!} \frac{1}{\chi_{a_1 \dots a_n} \epsilon^{a_1 \dots a_n b_{n+1} \dots b_{10}} \epsilon_{d_{m+1} \dots d_{10}}^{c_1 \dots c_m} \omega_{c_m \dots c_1}} \\
 &\quad \times \langle 0 | \psi_{b_{10}} \dots \psi_{b_{n+1}} \psi^{d_{m+1}} \dots \psi^{d_{10}} | 0 \rangle \\
 &= - \sum_n \frac{(-1)^{\frac{n(n-1)}{2}}}{n! n! (10-n)!} \chi_{a_1 \dots a_n} \epsilon^{a_1 \dots a_n b_{n+1} \dots b_{10}} \epsilon_{b_{n+1} \dots b_{10}}^{c_1 \dots c_n} \omega_{c_n \dots c_1} \\
 &= - \sum_n \frac{(-1)^{\frac{n(n-1)}{2}} (-1)^{n(10-n)}}{n! n! (10-n)!} \chi_{a_1 \dots a_n} \epsilon^{b_{n+1} \dots b_{10} a_1 \dots a_n} \epsilon_{b_{n+1} \dots b_{10}}^{c_1 \dots c_n} \omega_{c_n \dots c_1} \\
 &= \sum_n \frac{(-1)^{\frac{n(n-1)}{2}} (-1)^{n(10-n)}}{n!} \chi_{a_1 \dots a_n} \bar{\eta}^{a_1 b_1} \dots \bar{\eta}^{a_n b_n} \omega_{b_n \dots b_1} \\
 &= \sum_n \frac{(-1)^{\frac{n(n-1)}{2}} (-1)^{n(10-n)} (-1)^{\frac{n(n-1)}{2}}}{n!} \chi_{a_1 \dots a_n} \bar{\eta}^{a_1 b_1} \dots \bar{\eta}^{a_n b_n} \omega_{b_1 \dots b_n} \\
 &= \sum_n \frac{(-1)^n}{n!} \chi_{a_1 \dots a_n} \bar{\eta}^{a_1 b_1} \dots \bar{\eta}^{a_n b_n} \omega_{b_1 \dots b_n}.
 \end{aligned} \tag{D.17}$$

Compared to (A.11) of [20] we have an additional factor of $(-1)^n$. In the computation above we used also $\epsilon^{b_{n+1} \dots b_{10} a_1 \dots a_n} \epsilon_{b_{n+1} \dots b_{10}}^{c_1 \dots c_n} = -n!(10-n)! \bar{\eta}^{[a_1 c_1} \dots \bar{\eta}^{a_n] c_n}$. The computation for Ψ_- works in the same way, and because of the previous factor we have in fact

$$\langle \chi | C\Psi_- | \omega \rangle = \sum_n \frac{1}{n!} \chi_{a_1 \dots a_n} \bar{\eta}^{a_1 b_1} \dots \bar{\eta}^{a_n b_n} \omega_{b_1 \dots b_n}. \tag{D.18}$$

This computation is independent from the one that was giving us Ψ_- in the Lagrangian, and it is a further confirmation that Ψ_- rather than Ψ_+ should be used there, since with Ψ_+ there would be an additional factor of $(-1)^n$ when matching the DFT action to the one of standard supergravity. In the IIB case this would affect the sign of the RR Lagrangian (because for n odd $(-1)^n = -1$) while for IIA using Ψ_- or Ψ_+ is inconsequential.

Let us also remark that the self-duality condition $\Psi_- \mathfrak{F} = \mathfrak{F}$ (rather than $\Psi_+ \mathfrak{F} = \mathfrak{F}$) is in fact consistent with the conventions of [88]. The left-hand-side is

$$\begin{aligned}
 &\sum_n \frac{(-1)^n}{n!(10-n)!} \epsilon_{b_{n+1} \dots b_{10}}^{a_1 \dots a_n} \hat{F}_{a_n \dots a_1} \psi^{b_{n+1}} \dots \psi^{b_{10}} | 0 \rangle \\
 &= \sum_n \frac{(-1)^{(10-n)}}{n!(10-n)!} \epsilon_{b_1 \dots b_n}^{a_{n+1} \dots a_{10}} \hat{F}_{a_{10} \dots a_{n+1}} \psi^{b_1} \dots \psi^{b_n} | 0 \rangle \quad (n \rightarrow 10-n) \\
 &= \sum_n \frac{(-1)^{(10-n)} (-1)^{n(10-n)}}{n!(10-n)!} \epsilon^{a_{n+1} \dots a_{10}} \epsilon_{b_1 \dots b_n} \hat{F}_{a_{10} \dots a_{n+1}} \psi^{b_1} \dots \psi^{b_n} | 0 \rangle \quad (\epsilon \dots \dots \rightarrow \epsilon \dots \dots) \\
 &= \sum_n \frac{(-1)^{(10-n)} (-1)^{n(10-n)} (-1)^{(10-n)(10-n-1)/2}}{n!(10-n)!} \epsilon^{a_{n+1} \dots a_{10}} \epsilon_{b_1 \dots b_n} \hat{F}_{a_{n+1} \dots a_{10}} \psi^{b_1} \dots \psi^{b_n} | 0 \rangle \\
 &\quad (\hat{F}_{a_{10} \dots a_{n+1}} \rightarrow \hat{F}_{a_{n+1} \dots a_{10}}) \\
 &= - \sum_n \frac{(-1)^{\frac{n(n+1)}{2}}}{n!(10-n)!} \epsilon^{a_{n+1} \dots a_{10}} \epsilon_{b_1 \dots b_n} \hat{F}_{a_{n+1} \dots a_{10}} \psi^{b_1} \dots \psi^{b_n} | 0 \rangle,
 \end{aligned} \tag{D.19}$$

which implies the self-duality relation

$$-\frac{(-1)^{\frac{n(n+1)}{2}}}{(10-n)!} \epsilon^{a_{n+1} \dots a_{10}} b_1 \dots b_n \hat{F}_{a_{n+1} \dots a_{10}} = \hat{F}_{b_1 \dots b_n}, \quad (\text{D.20})$$

in agreement with the conventions of [88].

E DFT equations of motion

In this appendix we continue to omit the boldface notation, since the dimensional reduction plays no role. The action in the NSNS sector is $S_{\text{NSNS}} = \int dX e^{-2d} \mathcal{R}$ where we prefer to use the rewriting of [58]

$$\mathcal{R} = -4\partial^A \mathcal{F}_A^{(-)} + 2\mathcal{F}^A \mathcal{F}_A^{(-)} - \mathcal{F}_{ABC}^{(-)} \mathcal{F}^{(-)ABC} - \frac{1}{3} \mathcal{F}_{ABC}^{(-)} \mathcal{F}^{(-)ABC}. \quad (\text{E.1})$$

Here $P_{\pm} = \frac{1}{2}(\eta \pm \mathcal{H})$ and $\mathcal{F}_A^{(\pm)} = (P_{\pm})_A^B \mathcal{F}_B$, $\mathcal{F}_{ABC}^{(\pm)} = (P_{\mp})_A^D (P_{\pm})_B^E (P_{\pm})_C^F \mathcal{F}_{DEF}$, $\mathcal{F}_{ABC}^{(\pm\pm)} = (P_{\pm})_A^D (P_{\pm})_B^E (P_{\pm})_C^F \mathcal{F}_{DEF}$. To compute the equations of motion one finds that $\delta_E S_{\text{NSNS}} = \int dX e^{-2d} \Xi_{\text{NSNS}}^{AB} \Delta_{AB}$ where $\Delta_{AB} = -\Delta_{BA} = \delta E_A^M E_{BM}$ and $\delta_d S_{\text{NSNS}} = \int dX e^{-2d} \Xi_{\text{NSNS}} \delta d$ where [20]

$$\begin{aligned} \Xi_{[AB]}^{\text{NSNS}} &= 4\partial_{[A} \mathcal{F}_{B]}^{(-)} + (\mathcal{F}^C - \partial^C) \check{\mathcal{F}}_{C[AB]} + \check{\mathcal{F}}_{CD[A} \mathcal{F}_{B]}^{CD}, \\ \Xi_{\text{NSNS}} &= -2\mathcal{R}, \end{aligned} \quad (\text{E.2})$$

where we defined $\check{\mathcal{F}}_{ABC} = \check{S}_{ABC}{}^{A'B'C'} \mathcal{F}_{A'B'C'}$ and

$$\begin{aligned} \check{S}_{ABC A'B'C'} &= \frac{1}{2} \mathcal{H}_{AA'} \eta_{BB'} \eta_{CC'} + \frac{1}{2} \eta_{AA'} \mathcal{H}_{BB'} \eta_{CC'} + \frac{1}{2} \eta_{AA'} \eta_{BB'} \mathcal{H}_{CC'} - \frac{1}{2} \mathcal{H}_{AA'} \mathcal{H}_{BB'} \mathcal{H}_{CC'} \\ &\quad - \eta_{AA'} \eta_{BB'} \eta_{CC'} \\ &= -2 \left(P_{AA'}^{(-)} P_{BB'}^{(-)} P_{CC'}^{(-)} + P_{AA'}^{(+)} P_{BB'}^{(-)} P_{CC'}^{(-)} + P_{AA'}^{(-)} P_{BB'}^{(+)} P_{CC'}^{(-)} + P_{AA'}^{(-)} P_{BB'}^{(-)} P_{CC'}^{(+)} \right). \end{aligned} \quad (\text{E.3})$$

It is convenient to write \check{S} in the second way, which is the unique way to write it as a linear combination of products of projectors. In fact it is then easy to check that

$$P_{AA'}^{(+)} P_{BB'}^{(-)} \Xi_{\text{NSNS}}^{[A'B']} = 2\Xi_{AB}^{\prime \text{NSNS}}, \quad \Xi_{AB}^{\prime \text{NSNS}} \equiv 2\partial_A^{(+)} \mathcal{F}_B^{(-)} + (\partial^C - \mathcal{F}^C) \mathcal{F}_{ABC}^{(-)} - \mathcal{F}_{CDA}^{(+)} \mathcal{F}^{(-)DC}{}_B. \quad (\text{E.4})$$

Notice the transposition of CD indices in the last term. One also has $P_{AA'}^{(+)} P_{BB'}^{(+)} \Xi_{\text{NSNS}}^{[A'B']} = 0$ and $P_{AA'}^{(-)} P_{BB'}^{(-)} \Xi_{\text{NSNS}}^{[A'B']} = 2P_{AA'}^{(-)} P_{BB'}^{(-)} \mathcal{Z}^{A'B'} \sim 0$ where the last is a constraint of DFT. Finally, using $P_{AA'}^{(-)} P_{BB'}^{(+)} \Xi_{\text{NSNS}}^{[A'B']} = -2\Xi_{BA}^{\prime \text{NSNS}}$ we can conclude that $\Xi_{[AB]}^{\text{NSNS}} \sim 4\Xi_{[AB]}^{\prime \text{NSNS}}$ (i.e. on the constraint of DFT). Hence because we can project with $P^{(\pm)}$, $\Xi_{[AB]}^{\text{NSNS}} = 0$ is equivalent to $\Xi_{AB}^{\prime \text{NSNS}} = 0$.

When considering the RR contribution to the action one has [20] $\delta_E S_{\text{RR}} = \int dX e^{-2d} \Xi_{\text{RR}}^{AB} \Delta_{AB}$ and $\delta_d S_{\text{RR}} = \int dX e^{-2d} \Xi_{\text{RR}} \delta d$ where $\Xi_{\text{RR}}^{[AB]} = -\frac{1}{4} \tilde{\mathfrak{F}} \psi^{AB} \mathfrak{F}$ and $\Xi_{\text{RR}} = 0$ after imposing the self-duality condition. The full equations of motion in the type II case are then $\Xi_{[AB]} = \Xi_{[AB]}^{\text{NSNS}} + \Xi_{[AB]}^{\text{RR}} = 0$ and $\Xi = \Xi^{\text{NSNS}} = 0$.

F Other ansatze used for orbits with H -flux

In this appendix we will report on other attempts to deal with orbits with non-vanishing H -flux in the standard parametrisation of (2.20) rather than appealing to the methodology of section 2.4. In particular, by preferring an ordinary Lie group G action on the coordinates y , we will assume here that ρ_μ^i of (2.20) are the components of a (left-invariant) Maurer-Cartan form $\rho = g^{-1}dg = dy^\mu \rho_\mu^i t_i$ where $t_i \in \mathfrak{g} = \text{Lie}(G)$ are the generators of the Lie algebra and $g \in G$ is a Lie group element. In this case the components w_{ij}^k defined in (2.33) are precisely the structure constants of \mathfrak{g} so that we can write them in the more standard way $w_{ij}^k = f_{ij}^k$. Furthermore, we can use the well-known expansion of the MC form

$$\rho = g^{-1}dg = \sum_{N=0}^{\infty} \frac{(-)^N}{(N+1)!} \text{ad}_x^N dx, \quad x \in \mathfrak{g}. \tag{F.1}$$

The main difficulty with pursuing in this way is finding the most general solution to the differential equation obtained from the Q -flux equation of (2.32). Focusing here on the simplest homogeneous equation ($Q = 0$) with $F = 0$ we must solve

$$\partial_\mu \beta^{ij} + \rho_\mu^n \beta^{l[i} \beta^{j]m} H_{nlm} = 0, \tag{F.2}$$

or equivalently upon the F -flux equation

$$\partial_\mu \beta^{ij} - \rho_\mu^n \beta^{l[i} f_{nl}^{j]} = 0. \tag{F.3}$$

Let us remark here that by lowering all indices in eq. (F.3) with the Killing form and antisymmetrising in i, j, k we find that a *necessary* (but not sufficient) condition is that as a two-form β is closed, and possibly exact. For semisimple algebras \mathfrak{h} it is known that there exists no non-trivial closed two-forms which are not exact. An exact form such as $\beta^{ij} = \rho_\mu^i \rho_\nu^j \partial^{[\mu} \lambda^{\nu]}$ for some one-form λ does not, however, solve eq. (F.3). We can therefore conclude that for semisimple algebras no non-trivial solutions to eq. (F.3) exist.

To find a non-trivial solution of (F.3) we have considered for β^{ij} the following ansatze.⁵⁴

(i) In the (H)-orbit of section 3.9 we found a non-trivial representative, where ρ is of MC form, by considering β^{ij} to be a constant matrix.

(ii) Take now a non-constant β

$$\begin{aligned} \beta^{ij} &= \langle O^t R O t^i, t^j \rangle, \quad O = \sum_{N=0}^{\infty} a_N \text{ad}_x^N, \\ &= \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} a_N a_M (\text{ad}_x^N)^i_l R^{lk} (\text{ad}_x^M)^j_k, \end{aligned} \tag{F.4}$$

⁵⁴In the calculations that follow it is useful to use the identity

$$[\text{ad}_x^N dx, y] = \sum_{K=0}^N (-)^K \frac{N!}{K!(N-K)!} \text{ad}_x^{N-K} \text{ad}_{dx} \text{ad}_x^K y,$$

which holds for any $x, y \in \text{Lie}(G)$ and which can be proved by induction and the Jacobi identity.

where R is an antisymmetric matrix and where $t^i = \kappa^{ij}t_j$ and $\kappa_{ij} = \langle t_i, t_j \rangle$ is the Killing form of \mathfrak{g} . In this case one can observe that fixing the unknown coefficients a_N becomes inconsistent at order $\mathcal{O}(\partial_\mu x, x^1)$. When generalising (F.4) as

$$\beta^{ij} = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} a_{NM} (\text{ad}_x^N)^{[i} E^{lk} (\text{ad}_x^M)^{j]}_k, \tag{F.5}$$

where $E^{ij} = S^{ij} + R^{ij}$ with S a symmetric and R an antisymmetric matrix, and a_{NM} unknown coefficients, one will find that the symmetrical part does not contribute and this is effectively equivalent to the ansatz (F.4).

- (iii) Instead take $\beta = O - O^t$ with O a series in ad_x , as defined above, which is important to resonate with the MC form. Hence

$$\beta^{ij} = 2 \sum_{N=0}^{\infty} a_N (\text{ad}_x^N)^{[i} \kappa^k{}^j]_k. \tag{F.6}$$

This ansatz turns out to be empty: for every $N \in \mathbb{N}$ we find $a_N = 0$.

We have also considered the possibility that ρ is not of MC form and therefore tried to solve the PDE (F.2) instead. Assuming nevertheless an underlying Lie algebra \mathfrak{g} we take $H_{ijk} = \alpha_1 f_{ij}{}^l \kappa_{lk}$ for some real constant α_1 , and we expand ρ as

$$\rho_\mu{}^i = \sum_{N=0}^{\infty} b_N (\text{ad}_x^N)^l{}_m \partial_\mu x_l \kappa^{mi}, \quad x \in \mathfrak{g}, \tag{F.7}$$

for some undetermined coefficients b_N . Then,

- (iv) Taking β as in (F.6) we find that the resulting recursive relations will be empty, $a_N = 0$ for all N , and independent of the coefficients b_N .
- (v) Adding the inhomogeneous term $Q_i{}^{jk} = \alpha_2 f_{im}{}^n \kappa^{jm} \kappa_{kn}$ (with α_2 a second real constant) in the PDE as $Q_i{}^{jk} = \partial_i \beta^{jk} + \beta^{lj} \beta^k{}^m H_{ilm}$ we find for the ansatz (F.6) the following solution that we verified until $N = 10$,

$$a_{2N} = 0, \quad a_{2N-1} = \frac{4^N(4^N - 1)}{2} \frac{B_{2N}}{(2N)!} \alpha_1^{N-1} \alpha_2^N b_0^{2N-1}, \tag{F.8}$$

and

$$b_{2N} = 4^{N+1}(4^{N+1} - 1) \frac{B_{2N+2}}{(2N+2)!} \alpha_1^N \alpha_2^N b_0^{2N+1}, \quad b_{2N-1} = 0, \tag{F.9}$$

where B_N is the Bernoulli number. Given the form of H and Q this should coincide with the λ -model solution of section 3.12. However, the above is clearly not workable and thus the methods of section 2.4 are preferred.

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