

Chaotic string dynamics in deformed $T^{1,1}$

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ABSTRACT: Recently, Arutyunov, Bassi and Lacroix have shown that 2D non-linear sigma model with a deformed $T^{1,1}$ background is classically integrable [arXiv:2010.05573 [hep-th]]. This background includes a Kalb-Ramond two-form with a critical value. Then the sigma model has been conjectured to be non-integrable when the two-form is off critical. We confirm this conjecture by explicitly presenting classical chaos. With a winding string ansatz, the system is reduced to a dynamical system described by a set of ordinary differential equations. Then we find classical chaos, which indicates non-integrability, by numerically computing Poincaré sections and Lyapunov spectra for some initial conditions.

KEYWORDS: Integrable Field Theories, Sigma Models, Bosonic Strings

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1 Introduction

The AdS/CFT correspondence [1–3] is one of the most significant topics in String Theory. A typical example is the duality between type IIB string theory on $\text{AdS}_5 \times \text{S}^5$ and the $\mathcal{N} = 4$ $\text{SU}(N)$ super Yang-Mills theory in large N limit. A possible generalization, preserving the conformal symmetry of the boundary theory, is to replace the internal manifold S^5 by an Einstein manifold $T^{1,1}$ [4–6]. The metric of $T^{1,1}$ is given by

$$ds^2 = \frac{1}{6} \sum_{i=1}^2 \left(d\theta_i^2 + \sin^2 \theta_i d\phi_i^2 \right) + \frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 \quad (1.1)$$

(in units that the dimension of the curvature scale is unity). The $\text{AdS}_5 \times T^{1,1}$ geometry has been elaborated in [7]. In particular, the classical dynamics of a string moving on $T^{1,1}$ is chaotic and hence non-integrable [8].¹ This non-integrability has been shown also in an analytic manner [13]. Furthermore, the chaotic dynamics has been shown in a near Penrose limit [14]. The coset construction of the $T^{1,1}$ metric based on the coset

$$\frac{\text{SU}(2)_L \times \text{SU}(2)_R \times \text{U}(1)}{\text{U}(1)_L \times \text{U}(1)_R} \quad (1.2)$$

has been discussed in [15]. Yang-Baxter deformations [16–21] of $T^{1,1}$ are discussed in [15, 22, 23] (For a short summary, see [24]).

Recently, an intriguing generalization of the $T^{1,1}$ background has been considered by Arutyunov, Bassi and Lacroix [25]. The background is given by

$$ds^2 = \sum_{i=1}^2 \lambda_i^2 \left(d\theta_i^2 + \sin^2 \theta_i d\phi_i^2 \right) + \lambda^2 (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2, \quad (1.3a)$$

$$B_2 = k (d\psi + \cos \theta_1 d\phi_1) \wedge (d\psi + \cos \theta_2 d\phi_2) \quad (1.3b)$$

¹Classical chaos and non-integrability of string dynamics have also been investigated on AdS solitons [9–11]. Recently, they were also studied on $\text{AdS}_5 \times L^{a,b,c}$ [12].

and contains four real parameters λ_1 , λ_2 , λ and k . We shall denote this background by the symbol $\mathcal{T}_k^{(\lambda_1, \lambda_2, \lambda)}$. This metric (1.3a) is not Einstein in general. Notice that this background contains a Kalb-Ramond two-form B_2 .

It should be remarked that when $k = \lambda^2$, some interesting things happen. The classical string dynamics on $R \times \mathcal{T}_{\lambda^2}^{(\lambda_1, \lambda_2, \lambda)}$, where R describes the time direction in target space, is integrable [25] in the sense of Lax pair. The special case with $\lambda_1^2 = \lambda_2^2 = \frac{3}{2}\lambda^2$ gives rise to the original $T^{1,1}$ metric (1.1). Note that even for this original metric, the integrability is still preserved due to the existence of the B -field (1.3b). Also, the case with $\lambda_1 = \lambda_2 = \lambda$ has already been known as the Guadagnini-Martellini-Mintchev (GMM) model [26] and used to construct an NS-NS supergravity background with the flux $k = \lambda^2$ [27]. For recent progress on the RG flow in this model, see [28].

On the other hand, it is conjectured that the system would be non-integrable when $k \neq \lambda^2$ [25].² Hence it should be significant to show this non-integrability by following some standard manners. In this letter, we will consider cases with $k \neq \lambda^2$ and show that the string sigma model on $R \times \mathcal{T}_k^{(\lambda_1, \lambda_2, \lambda)}$ exhibits classical chaos in string motion. To reduce the sigma model to a dynamical system, we employ a winding string ansatz in which a string is wrapped around all of the isometry directions, i.e. ϕ_1 , ϕ_2 and ψ . Then the chaotic behaviors are explicitly shown by numerically computing Poincaré sections and Lyapunov spectra for some initial values.

This letter is organized as follows. In section 2, we introduce the classical string sigma model action on $R \times \mathcal{T}_k^{(\lambda_1, \lambda_2, \lambda)}$. Then the system is reduced by using a winding string ansatz. In section 3, Poincaré sections and Lyapunov spectra are computed numerically for some initial values and classical chaos appears when $k \neq \lambda^2$. Section 4 is devoted to conclusion and discussion.

2 Reducing string sigma model on $R \times \mathcal{T}_k^{(\lambda_1, \lambda_2, \lambda)}$

We shall consider a classical string on the background $R \times \mathcal{T}_k^{(\lambda_1, \lambda_2, \lambda)}$. As discussed in [25], the original isometries as $T^{1,1}$ are retained in the presence of B -field. In particular, one can easily see in (1.3) that constant shifts in (ϕ_1, ϕ_2, ψ) -directions are still symmetries.

Note that we suppose that the background $R \times \mathcal{T}_k^{(\lambda_1, \lambda_2, \lambda)}$ should be obtained from a supergravity embedding of the background (1.3), though it has not been done other than for special values of the parameters. So this is just analogy of $R \times S^3$ in the case of $\text{AdS}_5 \times S^5$, where R comes from AdS time.

2.1 String sigma model on $R \times \mathcal{T}_k^{(\lambda_1, \lambda_2, \lambda)}$

Let $G_{\mu\nu}$ and $B_{\mu\nu}$ ($\mu, \nu = t, \theta_1, \phi_1, \theta_2, \phi_2, \psi$) denote the background metric and B -field, h_{ab} ($a, b = \tau, \sigma$) the world-sheet metric, and X^μ the target-space coordinates. The classical action of the string σ -model is given by

$$S = -\frac{1}{2} \int d\tau d\sigma \left(\sqrt{-h} h^{ab} G_{\mu\nu} + \epsilon^{ab} B_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu, \quad (2.1)$$

²In this sense, the particular value $k = \lambda^2$ may be called critical and otherwise off critical.

where $h \equiv \det h_{ab}$ and the antisymmetric symbol is defined as $\epsilon^{\tau\sigma} = -1$. The equation of motion for X^μ is given by

$$\partial_a \partial^a X^\mu + \Gamma_{\rho\sigma}^\mu h^{ab} \partial_a X^\rho \partial_b X^\sigma - \mathcal{T}_{\rho\sigma}^\mu \varepsilon^{ab} \partial_a X^\rho \partial_b X^\sigma = 0, \quad (2.2)$$

where $\varepsilon^{ab} \equiv \epsilon^{ab}/\sqrt{-h}$ and

$$\Gamma_{\rho\sigma}^\mu = \frac{1}{2} G^{\mu\nu} (G_{\nu\sigma,\rho} + G_{\rho\nu,\sigma} - G_{\rho\sigma,\nu}), \quad \mathcal{T}_{\rho\sigma}^\mu = \frac{1}{2} G^{\mu\nu} (G_{\rho\sigma,\nu} + G_{\nu\rho,\sigma} + G_{\sigma\nu,\rho}). \quad (2.3)$$

The equation of motion for h^{ab} imposes that the stress-energy tensor T_{ab} on the world-sheet should vanish:

$$0 = T_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} h_{ab} h^{cd} G_{\mu\nu} \partial_c X^\mu \partial_d X^\nu. \quad (2.4)$$

Note here that the variation of h^{ab} is irrelevant to the antisymmetric symbol. The vanishing stress-energy tensor leads to the Virasoro constraints which restrict the string embedding specified by X^μ .

Thanks to the world-sheet diffeomorphism, one can take the conformal gauge $h_{ab} = e^{2\omega(\tau,\sigma)} \eta_{ab}$, where $\eta_{ab} = \text{diag}(-1, 1)$. In this gauge, the action takes the form:

$$S^{(c)} = \int d\tau d\sigma \mathcal{L} = -\frac{1}{2} \int d\tau d\sigma \left(\eta^{ab} G_{\mu\nu} + \epsilon^{ab} B_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu. \quad (2.5)$$

The conjugate momentum is given by

$$p_\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\tau X^\mu)} = G_{\mu\nu} \partial_\tau X^\nu + B_{\mu\nu} \partial_\sigma X^\nu. \quad (2.6)$$

Then the Hamiltonian is given by

$$\mathcal{H} = p_\mu \partial_\tau X^\mu - \mathcal{L} = \frac{1}{2} G_{\mu\nu} \partial_\tau X^\mu \partial_\tau X^\nu - \frac{1}{2} G_{\mu\nu} \partial_\sigma X^\mu \partial_\sigma X^\nu. \quad (2.7)$$

While the B -field appears in (2.6), it cancels with the B -field from \mathcal{L} in the above expression. As a result, the canonical stress-energy tensor made of the conjugate momentum gives the same result as (2.4) as it can be checked easily.

Once the gauge has been fixed, the Virasoro constraints can be expressed as follows. The (τ, τ) -component of the stress-energy tensor is precisely the Hamiltonian, i.e. $T_{\tau\tau} = \mathcal{H}$ and hence the Virasoro constraints indicate that $\mathcal{H} = 0$ (the Hamiltonian constraint). We also find that $T_{\tau\tau} = T_{\sigma\sigma}$ because $T^a_a = 0$. Substituting (2.6) into (2.4) in the conformal gauge, the other independent component $T_{\tau\sigma}$ can be rewritten as

$$T_{\tau\sigma} = p_\mu \partial_\sigma X^\mu = 0. \quad (2.8)$$

When $k = \lambda^2$, this string sigma model has a classical Lax pair and is classically integrable [25]. Our purpose here is to see the anticipated non-integrability when $k \neq \lambda^2$.

2.2 Reduction of the system with a winding string ansatz

In order to show non-integrability, it is sufficient to find out chaotic motions in a subsector. Hence it is convenient to consider the simplest subsector which contains chaotic motions. To reduce the system to a set of ordinary differential equations, let us suppose the following winding string ansatz:

$$\begin{aligned} t &= t(\tau), & \theta_1 &= \theta_1(\tau), & \theta_2 &= \theta_2(\tau), \\ \phi_1 &= \phi_1(\tau) + \alpha_1 \sigma, & \phi_2 &= \phi_2(\tau) + \alpha_2 \sigma, & \psi &= \psi(\tau) + \alpha \sigma. \end{aligned} \quad (2.9)$$

This ansatz describes a string wrapped on (ϕ_1, ϕ_2, ψ) -directions. Since the periodicity of the compact coordinates are $\phi_i \simeq \phi_i + 2\pi$ ($i = 1, 2$) and $\psi \simeq \psi + 4\pi$, and $\sigma \simeq \sigma + 2\pi$, the winding numbers α_i ($i = 1, 2$) and α should satisfy $\alpha_i, \alpha/2 \in \mathbb{Z}$.

By substituting the ansatz (2.9) into (2.6), the momentum components associated with the isometries (and hence conserved) are given by

$$p_t = -\dot{t}, \quad (2.10a)$$

$$\begin{aligned} p_{\phi_1} &= \left(\lambda_1^2 \sin^2 \theta_1 + \lambda^2 \cos^2 \theta_1 \right) \dot{\phi}_1 \\ &\quad + \lambda^2 \cos \theta_1 \cos \theta_2 \dot{\phi}_2 + \lambda^2 \cos \theta_1 \dot{\psi} + k \cos \theta_1 (\alpha_2 \cos \theta_2 + \alpha), \end{aligned} \quad (2.10b)$$

$$\begin{aligned} p_{\phi_2} &= \left(\lambda_2^2 \sin^2 \theta_2 + \lambda^2 \cos^2 \theta_2 \right) \dot{\phi}_2 \\ &\quad + \lambda^2 \cos \theta_1 \cos \theta_2 \dot{\phi}_1 + \lambda^2 \cos \theta_2 \dot{\psi} - k \cos \theta_2 (\alpha_1 \cos \theta_1 + \alpha), \end{aligned} \quad (2.10c)$$

$$p_\psi = \lambda^2 \cos \theta_1 \dot{\phi}_1 + \lambda^2 \cos \theta_2 \dot{\phi}_2 + \lambda^2 \dot{\psi} - k \alpha_1 \cos \theta_1 + k \alpha_2 \cos \theta_2. \quad (2.10d)$$

Under the ansatz (2.9), the constraint (2.8) can be rewritten as

$$T_{\tau\sigma} = p_{\phi_1} \alpha_1 + p_{\phi_2} \alpha_2 + p_\psi \alpha = 0. \quad (2.11)$$

From this expression, one finds that this constraint can be satisfied by simply taking $p_{\phi_1} = p_{\phi_2} = p_\psi = 0$. It is also helpful to introduce the energy E defined as $E \equiv -p_t$. Then, from (2.10), we obtain

$$\dot{t} = E, \quad (2.12a)$$

$$\dot{\phi}_1 = -\frac{k \cos \theta_1 (\alpha + \alpha_1 \cos \theta_1)}{\lambda_1^2 \sin^2 \theta_1}, \quad (2.12b)$$

$$\dot{\phi}_2 = \frac{k \cos \theta_2 (\alpha + \alpha_2 \cos \theta_2)}{\lambda_2^2 \sin^2 \theta_2}, \quad (2.12c)$$

$$\dot{\psi} = k \left(\frac{\alpha_1 \cos \theta_1 - \alpha_2 \cos \theta_2}{\lambda^2} + \frac{\cos^2 \theta_1 (\alpha + \alpha_1 \cos \theta_1)}{\lambda_1^2 \sin^2 \theta_1} - \frac{\cos^2 \theta_2 (\alpha + \alpha_2 \cos \theta_2)}{\lambda_2^2 \sin^2 \theta_2} \right). \quad (2.12d)$$

The other components are given by

$$p_{\theta_i} = \lambda_i^2 \dot{\theta}_i. \quad (2.13)$$

The string motion is described by coupled ordinary differential equations for θ_i . Substituting (2.9) into (2.2) and using (2.12), we obtain the evolution equations for θ_i as

$$\begin{aligned} \frac{\ddot{\theta}_1}{\sin \theta_1} = & \alpha_1^2 \left(\frac{\lambda^2}{\lambda_1^2} - 1 + \frac{k^2}{\lambda^2 \lambda_1^2} \left(1 - \frac{\lambda^2}{\lambda_1^2} + \frac{\lambda^2}{\lambda_1^2 \sin^4 \theta_1} \right) \right) \cos \theta_1 - \alpha_1 \alpha_2 \frac{k^2 - \lambda^4}{\lambda^2 \lambda_1^2} \cos \theta_2 \\ & + \frac{\alpha \alpha_1}{\lambda_1^4} \left(\lambda^2 \lambda_1^2 - k^2 + \frac{k^2 (\cos^2 \theta_1 + 1)}{\sin^4 \theta_1} \right) + \frac{\alpha^2 k^2 \cos \theta_1}{\lambda_1^4 \sin^4 \theta_1}, \end{aligned} \quad (2.14a)$$

$$\begin{aligned} \frac{\ddot{\theta}_2}{\sin \theta_2} = & \alpha_2^2 \left(\frac{\lambda^2}{\lambda_2^2} - 1 + \frac{k^2}{\lambda^2 \lambda_2^2} \left(1 - \frac{\lambda^2}{\lambda_2^2} + \frac{\lambda^2}{\lambda_2^2 \sin^4 \theta_2} \right) \right) \cos \theta_2 - \alpha_1 \alpha_2 \frac{k^2 - \lambda^4}{\lambda^2 \lambda_2^2} \cos \theta_1 \\ & + \frac{\alpha \alpha_2}{\lambda_2^4} \left(\lambda^2 \lambda_2^2 - k^2 + \frac{k^2 (\cos^2 \theta_2 + 1)}{\sin^4 \theta_2} \right) + \frac{\alpha^2 k^2 \cos \theta_2}{\lambda_2^4 \sin^4 \theta_2}. \end{aligned} \quad (2.14b)$$

Note that the two equations are decoupled when $k = \lambda^2$ and no chaos appears obviously.

The Hamiltonian becomes

$$\mathcal{H} = -\frac{1}{2}E^2 + \frac{1}{2\lambda_1^2}p_{\theta_1}^2 + \frac{1}{2\lambda_2^2}p_{\theta_2}^2 + V(\theta_1, \theta_2), \quad (2.15)$$

where the potential $V(\theta_1, \theta_2)$ is defined as

$$\begin{aligned} V(\theta_1, \theta_2) \equiv & \frac{\alpha_1^2}{2\lambda^2 \lambda_1^2} \frac{(\lambda^2 \cos^2 \theta_1 + \lambda_1^2 \sin^2 \theta_1) (k^2 \cos^2 \theta_1 + \lambda^2 \lambda_1^2 \sin^2 \theta_1)}{\sin^2 \theta_1} \\ & + \frac{\alpha_2^2}{2\lambda^2 \lambda_2^2} \frac{(\lambda^2 \cos^2 \theta_2 + \lambda_2^2 \sin^2 \theta_2) (k^2 \cos^2 \theta_2 + \lambda^2 \lambda_2^2 \sin^2 \theta_2)}{\sin^2 \theta_2} \\ & - \alpha_1 \alpha_2 \frac{k^2 - \lambda^4}{\lambda^2} \cos \theta_1 \cos \theta_2 \\ & + \frac{\alpha^2}{2} \left(\lambda^2 + \frac{k^2 \cot^2 \theta_1}{\lambda_1^2} + \frac{k^2 \cot^2 \theta_2}{\lambda_2^2} \right) \\ & + \frac{\alpha \alpha_1}{\lambda_1^2} (k^2 \cot^2 \theta_1 + \lambda^2 \lambda_1^2) \cos \theta_1 + \frac{\alpha \alpha_2}{\lambda_2^2} (k^2 \cot^2 \theta_2 + \lambda^2 \lambda_2^2) \cos \theta_2. \end{aligned} \quad (2.16)$$

Recall that the Hamiltonian has to be zero, i.e. $\mathcal{H} = 0$, due to the Virasoro constraints. Since the Hamiltonian is conserved, this constraint is always satisfied along time evolution once the initial data is prepared so as to satisfy $\mathcal{H} = 0$.

The presence of the background B -field restricts the domain of the string motion. When $k \neq 0$, the string cannot reach the poles of S^2 ($\theta_i = 0$ and π), because $V(\theta_1, \theta_2)$ diverges there. Therefore, the string can move in the range $0 < \theta_i < \pi$. When $k = 0$, the divergent terms are absent in the potential, and the string can shrink and pass through the poles.

Note also that the potential is tilted because of the winding in the ψ -direction (i.e., $\alpha \neq 0$). If $\alpha = 0$, the potential is symmetric under $(\theta_1, \theta_2) \rightarrow (\pi - \theta_1, \pi - \theta_2)$, while it is not otherwise. Hence, there is one global minimum when $\alpha \neq 0$. In practice, the minimum can be found numerically with specified parameters.

3 Chaotic string dynamics

In this section, we present classical chaos in the reduced system (2.14) by computing Poincaré sections and Lyapunov spectra for some values when $k \neq \lambda^2$.

For numerical computations, we set the parameters as

$$\alpha_1 = \alpha_2 = 1, \quad \alpha = 2, \quad \lambda_1 = \lambda_2 = \lambda = 1 \quad (3.1)$$

for simplicity. We have examined various values of k , but here we will show results for the cases that $k = 0, 0.5$ and 10 . For other values of k as far as we have examined, classical chaos has been observed (except the special case with $k = \lambda^2$). Poincaré sections are computed for

$$\theta_2 = \pi \ (k = 0) \quad \text{or} \quad \theta_2 = \pi/2 \ (k = 0.5, 10) \quad \text{and} \quad p_{\theta_2} \geq 0. \quad (3.2)$$

In each figure, the energy E is fixed and the initial conditions for the differential equations are changed. Different colors correspond to different initial conditions.

In addition, to measure the strength of chaos, we have examined the Lyapunov spectrum as well (For the detail of the computation, for example, see [29]). In the system (2.14), the Lyapunov spectrum contains four exponents and the largest one is the most significant for the growth of chaos.

As a remark on α , for $\alpha = 0$ and $\lambda_1 = \lambda_2 = \lambda$, we could not find any chaos in string motion even for $k \neq \lambda^2$. This would be simply because this case corresponds to an integrable subsector.

In the following, let us present Poincaré sections and Lyapunov spectra for $k = 0, 0.5$ and 10 , respectively.

(i) $k = 0$ case. Figure 1 shows Poincaré sections for $k = 0$. The potential minimum is $(\theta_1, \theta_2) = (\pi, \pi)$. In the low-energy region ($E = 1$), there are only Kolmogorov-Arnold-Moser (KAM) tori [30–32] and is no chaos [figure 1(a)]. One can see a separatrix structure around $\theta_1 = 2.44$ and $p_{\theta_1} = 0$. When $E = 1.5$, the separatrix begins to collapse and classical chaos appears [figure 1(b)]. When $E = 5$, a lot of KAM tori are broken and an island of KAM tori survives on the right-hand side [figure 1(c)]. When $E = 10$, all of the trajectories become KAM tori again and no chaos is found [figure 1(d)]. This is a typical behavior of classical chaos in the high-energy region.

Figure 1(e) is a Lyapunov spectrum for $E = 5$. The initial values of $\theta_1(t)$, $\theta_2(t)$, $p_{\theta_1}(t)$ and $p_{\theta_2}(t)$ are taken as $\theta_1(0) = 1$, $\theta_2(0) = \pi$, $p_{\theta_1}(0) = 1$, and $p_{\theta_2}(0) \sim 4.574$. The maximum Lyapunov exponent is evaluated as the average for $150 \leq t \leq 200$. Then it is 0.3990 ± 0.0116 , where the errors are given by the standard deviation.

(ii) $k = 0.5$ case. Figure 2 shows Poincaré sections for $k = 0.5$. The potential minimum is $(\theta_1, \theta_2) \sim (2.438, 2.438)$. In the low-energy region ($E = 3$), there are only KAM tori only and is no chaos [figure 2(a)]. As the energy is a bit increased to $E = 3.3$, a separatrix structure appears around $\theta_1 = 1.51$ and $p_{\theta_1} = 1.6$ as shown in figure 2(b). When $E = 4$, one can see that the separatrix begins to collapse and classical chaos appears [figure 2(c)].

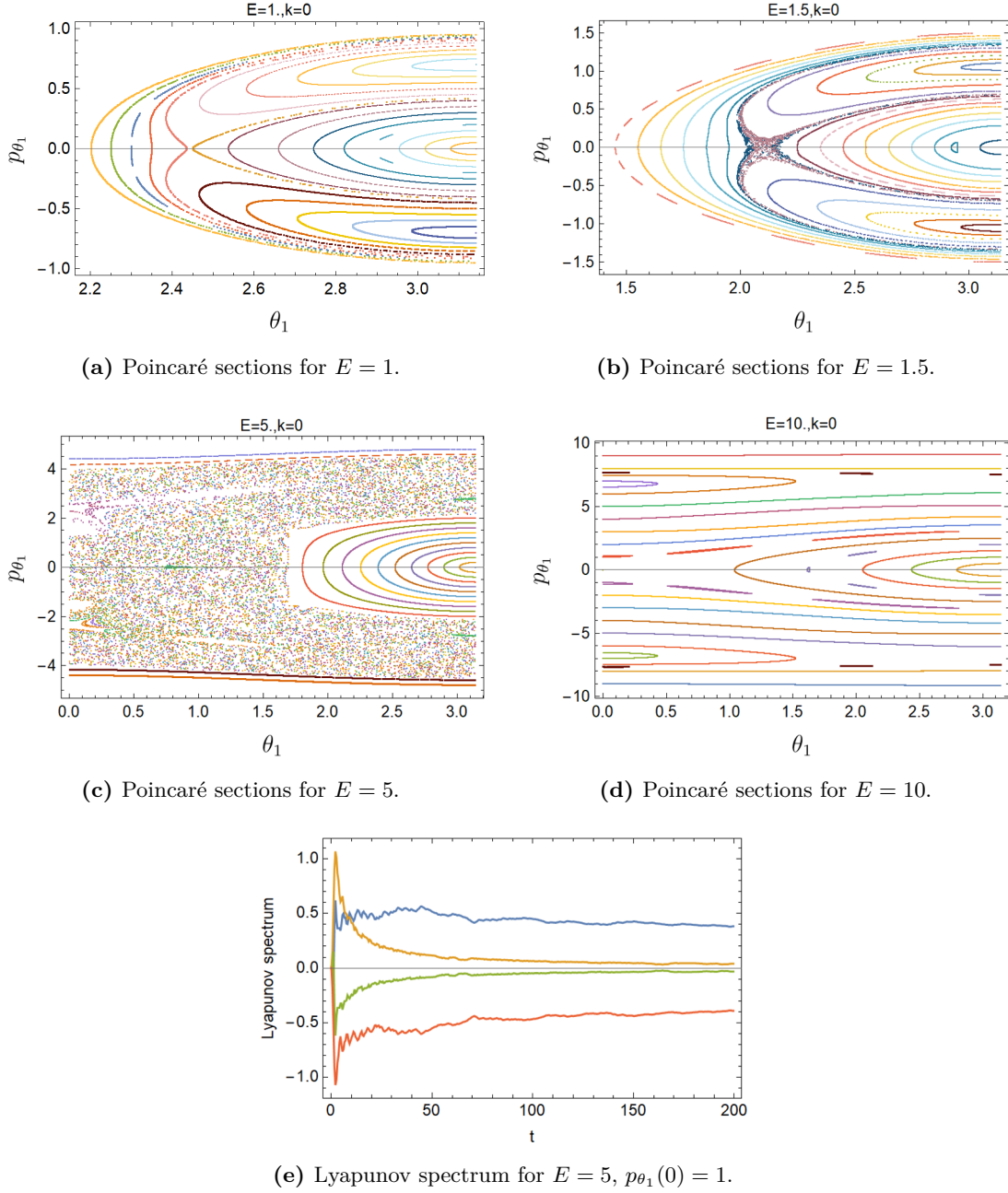


Figure 1. Poincaré sections and Lyapunov spectrum for $k = 0$.

When $E = 5$, chaos is clearly observed while some islands of KAM tori survive [figure 2(d)]. In the higher energy region ($E = 15$), there are only KAM tori and the chaos disappears again [figure 2(e)].

Figure 2(f) is a Lyapunov spectrum for $E = 5$. The initial values of $\theta_1(t)$, $\theta_2(t)$, $p_{\theta_1}(t)$, and $p_{\theta_2}(t)$ are taken as $\theta_1(0) = \theta_2(0) = \pi/2$, $p_{\theta_1}(0) = 3$, and $p_{\theta_2}(0) = \sqrt{10}$. Then the maximum Lyapunov exponent is 0.1999 ± 0.0075 .

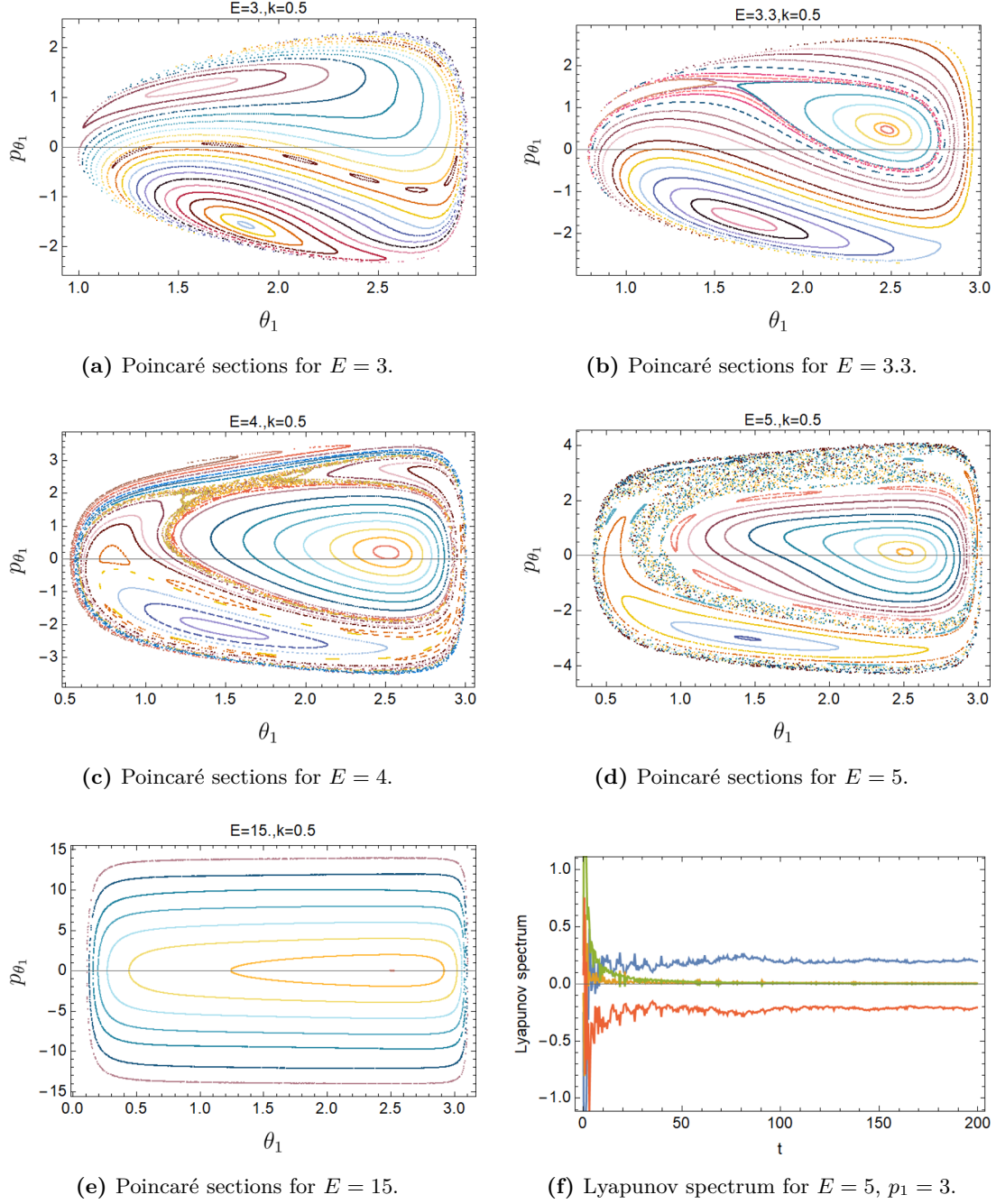


Figure 2. Poincaré sections and Lyapunov spectrum for $k = 0.5$.

(iii) $k = 10$ case. Figure 3 shows Poincaré sections for $k = 10$. The potential minimum is $(\theta_1, \theta_2) \sim (\pi/2, \pi/2)$. When $E = 5$, there are only KAM tori and no chaos is found [figure 3(a)]. When $E = 40$, some of KAM tori collide with each other and chaos appears [figure 3(b)]. As the energy becomes much higher ($E = 300$), there are only KAM tori and chaos disappears again [figure 3(c)]. Notice here that higher energy is necessary to see chaos in this case in comparison to the previous two cases. This is simply because the

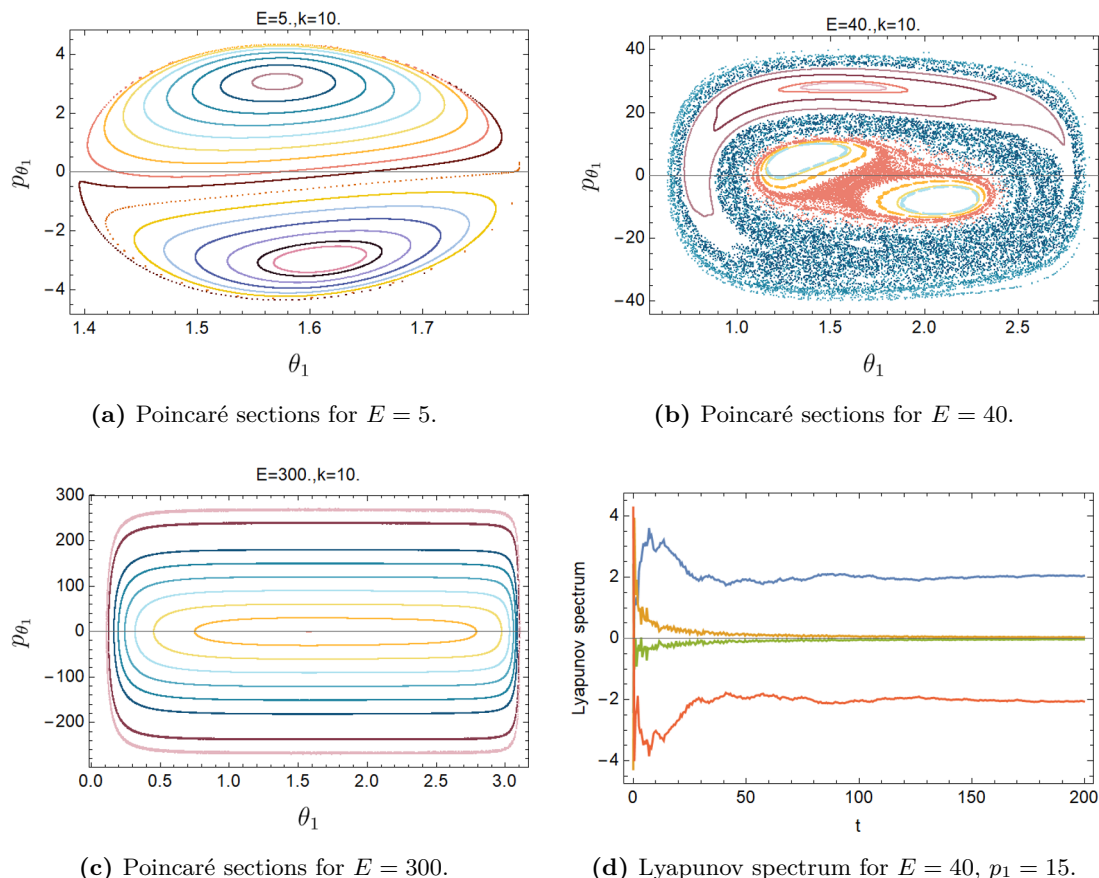


Figure 3. Poincaré sections and Lyapunov spectrum for $k = 10$.

large value of k means a strong magnetic flux. The magnetic flux acts to force the particles to move in a circle and hence the KAM tori can survive even at relatively high energy.

Figure 3(d) is a Lyapunov spectrum with $E = 40$. The initial values of $\theta_1(t)$, $\theta_2(t)$, $p_{\theta_1}(t)$, and $p_{\theta_2}(t)$ are taken as $\theta_1(0) = \theta_2(0) = \pi/2$, $p_{\theta_1}(0) = 15$, and $p_{\theta_2}(0) = 37$. Then the maximum Lyapunov exponent is 2.0341 ± 0.0182 .

4 Conclusion and discussion

In this letter, we have studied a string sigma model on $R \times \mathcal{T}_k^{(\lambda_1, \lambda_2, \lambda)}$ by using a winding string ansatz. We have shown the presence of classical chaos for $k \neq \lambda^2$ by looking at Poincaré sections. We have also calculated the Lyapunov spectra and showed that the largest Lyapunov exponent is positive for chaotic string motions. Thus we conclude that the string sigma model is not classically integrable when $k \neq \lambda^2$ as conjectured in [25].

In our analysis, we have supposed a supergravity embedding of the background (1.3). It is a significant future problem to find out this supergravity embedding for general values of the parameters. Notably, in [25], the background (1.3) has been derived as a certain limit of a more general background. It is a challenging problem to try to find out a supergravity

embedding of it. It is also a nice practice to study non-integrable parameter region of it by following the standard manner as performed in this letter.

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