

Two-dimensional lattice $SU(N_c)$ gauge theories with multiflavor adjoint scalar fields

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ABSTRACT: We consider two-dimensional lattice $SU(N_c)$ gauge theories with N_f real scalar fields transforming in the adjoint representation of the gauge group and with a global $O(N_f)$ invariance. Focusing on systems with $N_f \geq 3$, we study their zero-temperature limit, to understand under which conditions a continuum limit exists, and to investigate the nature of the associated quantum field theory. Extending previous analyses, we address the role that the gauge-group representation and the quartic scalar potential play in determining the nature of the continuum limit (when it exists). Our results further corroborate the conjecture that the continuum limit of two-dimensional lattice gauge models with multiflavor scalar fields, when it exists, is associated with a σ model defined on a symmetric space that has the same global symmetry as the lattice model.

KEYWORDS: Gauge Symmetry, Global Symmetries, Lattice Quantum Field Theory

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1 Introduction

Gauge theories represent a unifying theme of modern theoretical physics, being used to describe both fundamental processes in high-energy particle theories [1–3] and emerging phenomena in condensed matter physics [3–5]. In the framework of statistical field theory, one is typically interested in determining the low-energy spectrum of the theory, the phase structure (in the context of gauge theories with scalar fields the different phases are related to different realizations of the Higgs mechanism), and the nature of their critical behavior, or equivalently their continuum limit. A deep understanding of the interplay between the global and the local symmetries of the theory is of fundamental importance for all these topics. In this paper we address such a problem in two-dimensional (2D) lattice gauge theories, to identify the key features that eventually determine the nature of their continuum limit and critical behavior.

According to the Mermin-Wagner theorem [6, 7], 2D models with global continuous symmetries do not show magnetized phases characterized by the condensation of an order

parameter, and therefore they do not undergo phase transitions associated with the spontaneous breaking of the global symmetry. However, 2D systems with global nonabelian symmetries may develop a critical behavior in the zero-temperature limit. For example, in the $O(N)$ σ model with $N \geq 3$ and in the CP^{N-1} model with $N \geq 2$, correlation functions in the thermodynamic limit are characterized by a length scale ξ that diverges as $T^p e^{c/T}$ for $T \rightarrow 0$; see, e.g., refs. [3, 8]. Systems with an Abelian $O(2)$ global symmetry are peculiar in this respect, since they may undergo a finite-temperature topological Berezinskii-Kosterlitz-Thouless (BKT) transition [9–11], which separates the high- T disordered phase from the low-temperature nonmagnetized spin-wave phase characterized by correlation functions that decay algebraically.

In the case of models characterized by both global and gauge symmetries, the asymptotic critical behavior is expected to arise from the interplay between the two different symmetries. For the purpose of understanding which features are relevant and which continuum limits are effectively realized, several 2D lattice models presenting both global and gauge continuous symmetries have been investigated [12–15], such as the lattice Abelian-Higgs model characterized by a global $SU(N_f)$ ($N_f \geq 2$) and a local $U(1)$ symmetry, the lattice scalar quantum chromodynamics with a global $SU(N_f)$ and a local $SU(N_c)$ symmetry, and a lattice $SO(N_c)$ gauge model with a global $O(N_f)$ and a local $SO(N_c)$ symmetry. These studies support the following general conjecture: the universal low-temperature critical behavior, and therefore the continuum limit, of 2D lattice gauge models with scalar fields is the same as that of 2D σ models defined on symmetric spaces [3, 16], which have the same global symmetry.

In this paper we extend the above analyses in two different directions. First, we want to understand whether the above conjecture also holds when the matter fields transform under a higher (than the fundamental) representation of the nonabelian gauge group. Second, we consider general quartic potentials, that allow us to obtain different low-temperature behaviors. For this purpose we consider a 2D lattice gauge model with a matrix scalar field, which is invariant under $O(N_f)$ global transformations and $SU(N_c)$ gauge transformations, and in which the scalar field transforms according to the adjoint representation of the gauge group. It is worth mentioning that, for $N_c = 2$, this model has been recently considered as an emerging gauge theory for high- T_c superconductors [17, 18].

The above issues are investigated by scrutinizing the nature of the low-energy configurations that are relevant in the zero-temperature limit, and by performing numerical finite-size scaling (FSS) analyses of Monte Carlo (MC) results. We present results for $N_c = 2, 3$ and $N_f = 3, 4$. As we shall see, our results confirm the aforementioned conjecture. We consider first a scalar model which is maximally symmetric in the absence of the gauge fields, i.e., it is an $O(M)$ σ model with $M = N_f(N_c^2 - 1)$. In this case the lattice gauge model with N_f scalar flavors in the adjoint gauge-group representation shows an asymptotic zero-temperature critical behavior that belongs to the universality class of the 2D RP^{N_f-1} model, defined on the symmetric space $O(N_f)/O(N_f - 1)$. Then, we generalize the model introducing a scalar potential that reduces the symmetry of the ungauged model to $O(N_f) \otimes O(N_c^2 - 1)$. In this case, different behaviors are observed, depending on the sign of one of the parameters appearing in the quartic potential. For negative values

of the parameter, the RP^{N_f-1} behavior is still observed. A different behavior is observed instead for positive values. If $N_f \leq N_c^2 - 1$, no continuum limit can be defined: correlations are always short-ranged, even in the zero-temperature limit. On the other hand, for $N_f > N_c^2 - 1$, long-range correlations are observed. We conjecture that the continuum limit is associated with a σ model defined in the symmetric space $\text{O}(N_f)/\text{O}(q) \otimes \text{O}(N_f - q)$ with $q = N_c^2 - 1$. Numerical results for $N_c = 2$ and $N_f = 4$ are in full agreement with this conjecture. The different behavior for positive and negative values of the quartic potential parameter is due to the qualitative differences of the minimum-action configurations that control the zero-temperature limit.

The paper is organized as follows. In section 2 we define the lattice $\text{SU}(N_c)$ gauge model with scalar fields in the adjoint representation. In section 3 we discuss the expected low-temperature behavior. We determine the minimum-action configurations and derive the corresponding effective models. In section 4 we present Monte Carlo results that fully confirm the predictions of section 3. Finally, in section 5 we summarize our results and draw our conclusions. In appendix A we study the role that gauge fields play in determining the relevant low-temperature configurations. In appendix B we report some details of the MC simulations.

2 2D lattice $\text{SU}(N_c)$ gauge models with scalar fields in the adjoint $\text{SU}(N_c)$ representation

We consider multiflavor lattice gauge models defined on a square lattice of linear size L with periodic boundary conditions, which are invariant under local $\text{SU}(N_c)$ and global $\text{O}(N_f)$ transformations. The fundamental variables are real matrices Φ_x^{af} defined on the sites of the lattice, with $a = 1, \dots, N_c^2 - 1$ (*color* index) and $f = 1, \dots, N_f$ (*flavor* index). They transform under the adjoint representation of the $\text{SU}(N_c)$ gauge group and under the fundamental representation of the $\text{O}(N_f)$ group:

$$\Phi_x^{af} \rightarrow \sum_b \tilde{V}_x^{ab} \Phi_x^{bf} \quad \Phi_x^{af} \rightarrow \sum_b W^{fg} \Phi_x^{ag}, \quad (2.1)$$

where \tilde{V}_x is a matrix belonging to the adjoint representation of the $\text{SU}(N_c)$ gauge group and W is an orthogonal matrix. Using the Wilson approach [2], we introduce gauge variables $U_{x,\mu} \in \text{SU}(N_c)$ associated with each link (x, μ) of the lattice. The model is defined by the partition function

$$Z = \sum_{\{\Phi, U\}} e^{-\beta S}, \quad \beta = 1/T, \quad S = S_K(\Phi, U) + S_V(\Phi) + S_G(U), \quad (2.2)$$

where the action S is written as a sum of three terms: S_K is the kinetic term for the scalar field, S_V is the local scalar potential, and S_G is the gauge action.

The kinetic term S_K is given by

$$S_K(\Phi, U) = -J \frac{N_f}{2} \sum_{x,\mu} \text{Tr} \Phi_x^t \tilde{U}_{x,\mu} \Phi_{x+\hat{\mu}}, \quad (2.3)$$

where $\tilde{U}_{x,\mu}^{ab}$ is the adjoint representation of the link variable $U_{x,\mu}$. It can be written as

$$\tilde{U}^{ab} = 2 \text{Tr} U^\dagger T^a U T^b, \quad a, b = 1, \dots, N_c^2 - 1, \quad (2.4)$$

where T^a are the $(N_c^2 - 1)$ generators of the $\text{SU}(N_c)$ algebra in the fundamental representation, normalized so that $\text{Tr} T^a T^b = \frac{1}{2} \delta^{ab}$.¹ We set the lattice spacing equal to one, so that all lengths are measured in units of the lattice spacing. Using eq. (2.4) we can rewrite the kinetic term as

$$S_K = -J N_f \sum_{\mathbf{x}, \mu} \sum_f \text{Tr} U_{\mathbf{x}, \mu}^\dagger \phi_{\mathbf{x}}^f U_{\mathbf{x}, \mu} \phi_{\mathbf{x}+\hat{\mu}}^f, \quad (2.5)$$

where the trace is taken in the fundamental representation of $\text{SU}(N_c)$ and

$$\phi_{ij}^f = \sum_a \Phi^{af} T_{ij}^a, \quad \Phi^{af} = 2 \text{Tr} \phi^f T^a. \quad (2.6)$$

In the following we set $J = 1$, so that energies are measured in units of J .

The scalar potential term S_V can be written as²

$$S_V(\Phi) = \sum_{\mathbf{x}} V(\Phi_{\mathbf{x}}), \quad V(\Phi) = \frac{r}{2} \text{Tr} \Phi^t \Phi + \frac{u}{4} \left(\text{Tr} \Phi^t \Phi \right)^2 + \frac{v}{4} \text{Tr} \Phi^t \Phi \Phi^t \Phi, \quad (2.7)$$

which is the most general quartic potential that is invariant under $\text{O}(N_f) \otimes \text{O}(N_c^2 - 1)$ transformations. Note that, for $v = 0$, the symmetry of the scalar potential enlarges to $\text{O}(M)$ with $M = N_f(N_c^2 - 1)$. Finally, we define the gauge action

$$S_G(U) = -\frac{\gamma}{N_c} \sum_{\mathbf{x}} \text{Re} \text{Tr} \Pi_{\mathbf{x}}, \quad \Pi_{\mathbf{x}} = U_{\mathbf{x},1} U_{\mathbf{x}+\hat{1},2} U_{\mathbf{x}+\hat{2},1}^\dagger U_{\mathbf{x},2}^\dagger, \quad (2.8)$$

in which the plaquette parameter γ plays the role of inverse gauge coupling.

The action S defined in eq. (2.2) is invariant under the global $\text{O}(N_f)$ transformations (2.1) and under local $\text{SU}(N_c)$ transformations (the scalar field transforms as in eq. (2.1), while $U_{x,\mu} \rightarrow V_x U_{x,\mu} V_{x+\hat{\mu}}^\dagger$; \tilde{V} corresponds to V in the adjoint representation). For $\gamma \rightarrow \infty$ the link variables $U_{x,\mu}$ become equal to the identity, modulo gauge transformations. Thus, in this limit, one recovers a matrix scalar model. For $v \neq 0$ the global symmetry group of this scalar model is $\text{O}(N_f) \otimes \text{O}(N_c^2 - 1)$. For $v = 0$ the symmetry group is $\text{O}(M)$ with $M = N_f(N_c^2 - 1)$.

For $\gamma = 0$ it is easily seen from the expression of \tilde{U}^{ab} in eq. (2.4) (or equivalently from eq. (2.5)) that each matrix $U_{x,\mu}$ can be multiplied by an arbitrary (\mathbf{x}, μ) -dependent element of the gauge group center without changing the action: for $\gamma = 0$ the gauge group is in fact $\text{SU}(N_c)/\mathbb{Z}_{N_c}$. This is responsible for the vanishing of the average value of the plaquette, $\langle \text{Tr} \Pi_{\mathbf{x}} \rangle = 0$, for $\gamma = 0$. Finally, note that for $N_c = 2$ and again $\gamma = 0$, because of the isomorphism $\text{SU}(2)/\mathbb{Z}_2 = \text{SO}(3)$, we are dealing with a theory with $\text{SO}(3)$ local symmetry.

¹Using the completeness relation $\sum_a T_{ij}^a T_{kl}^a = \frac{1}{2}(\delta_{il} \delta_{jk} - N_c^{-1} \delta_{ij} \delta_{kl})$ it is easily shown that \tilde{U}^{ab} is a representation of $\text{SU}(N_c)$. Close to the identity, if $U_{ij} \approx \delta_{ij} + i \varepsilon^a T_{ij}^a$, one obtains $\tilde{U}^{ab} \approx \delta^{ab} + i \varepsilon^c (-i f^{abc})$, where f^{abc} are the structure constants of the $\text{SU}(N_c)$ group satisfying $[T^a, T^b] = i f^{abc} T^c$. This proves that \tilde{U} belongs to the adjoint representation.

²One can easily express the potential in terms of the variable ϕ defined in eq. (2.6) using $\text{Tr} \Phi^t \Phi = 2 \sum_f \text{Tr} \phi^f \phi^f$ and $\text{Tr} \Phi^t \Phi \Phi^t \Phi = 4 \sum_{fg} (\text{Tr} \phi^f \phi^g)^2$.

In the following we consider a simplified model, which can be formally obtained by setting $r = -2u$, and taking the limit $u \rightarrow \infty$. The model has fixed-length fields and a simpler potential:

$$\text{Tr } \Phi_x^t \Phi_x = 2, \quad V(\Phi) = \frac{v}{4} \text{Tr } \Phi^t \Phi \Phi^t \Phi. \quad (2.9)$$

In terms of the variables ϕ_{ij}^f defined in eq. (2.6) we have

$$\sum_f \text{Tr } \phi_x^f \phi_x^f = 1, \quad V(\phi) = v \sum_{fg} (\text{Tr } \phi^f \phi^g)^2. \quad (2.10)$$

Therefore, we consider the action

$$S = -\frac{N_f}{2} \sum_{x,\mu} \text{Tr } \Phi_x^t \tilde{U}_{x,\mu} \Phi_{x+\hat{\mu}} + \frac{v}{4} \sum_x \text{Tr } \Phi_x^t \Phi_x \Phi_x^t \Phi_x - \frac{\gamma}{N_c} \sum_x \text{Re Tr } \Pi_x. \quad (2.11)$$

We expect this simplified model to show all universal features of the models with generic values of r and u .

The critical properties in the zero-temperature limit can be monitored by the correlation functions of the gauge-invariant bilinear operators

$$B_x^{fg} = \frac{1}{2} \sum_a \Phi_x^{af} \Phi_x^{ag}, \quad Q_x^{fg} = B_x^{fg} - \frac{1}{N_f} \delta^{fg}, \quad (2.12)$$

which satisfy $\text{Tr } B_x = 1$ and $\text{Tr } Q_x = 0$, due to the fixed-length constraint. Assuming translation invariance, holding for finite-size systems with periodic boundary conditions, we define the two-point correlation function

$$G(\mathbf{x} - \mathbf{y}) = \langle \text{Tr } Q_x Q_y \rangle, \quad (2.13)$$

the corresponding susceptibility $\chi = \sum_x G(\mathbf{x})$ and second-moment correlation length

$$\xi^2 = \frac{1}{4 \sin^2(\pi/L)} \frac{\tilde{G}(\mathbf{0}) - \tilde{G}(\mathbf{p}_m)}{\tilde{G}(\mathbf{p}_m)}, \quad (2.14)$$

where $\tilde{G}(\mathbf{p}) = \sum_x e^{i\mathbf{p}\cdot\mathbf{x}} G(\mathbf{x})$ is the Fourier transform of $G(\mathbf{x})$, and $\mathbf{p}_m = (2\pi/L, 0)$. In addition, we consider universal renormalization-group (RG) invariant quantities, such as the ratio

$$R_\xi \equiv \xi/L, \quad (2.15)$$

and the Binder parameter

$$U = \frac{\langle \mu_2^2 \rangle}{\langle \mu_2 \rangle^2}, \quad \mu_2 = \frac{1}{V^2} \sum_{x,y} \text{Tr } Q_x Q_y, \quad V = L^2. \quad (2.16)$$

3 Zero-temperature limit

Let us now discuss the expected critical behavior. We only consider systems with $N_f \geq 3$, in which the global symmetry is nonabelian. In this case, we do not expect a critical behavior for finite β , but only in the zero-temperature limit. According to the conjecture reported in the introduction, the critical behavior should be the same as that of the 2D σ models defined on the symmetric spaces with the same global symmetry, that is the models defined on [3, 16] $O(N_f)/O(p) \otimes O(N_f - p)$ for different values of p .

3.1 Zero-temperature relevant configurations

As a first step, we identify the relevant configurations for $\beta \rightarrow \infty$, which are controlled by the action terms $S_K(\Phi, U)$ and $S_V(\Phi)$. As in two dimensions there is no critical pure-gauge dynamics, we expect, and we will verify numerically, that $S_G(U)$ does not play a relevant role. Although we will be interested in systems with $N_f \geq 3$, the results for the zero-temperature configurations also hold for $N_f = 2$.

Let us first consider the potential term $S_V(\Phi)$. For $\beta \rightarrow \infty$, the relevant configurations are those that minimize $V(\Phi)$ defined in eq. (2.9). To determine the minima, we use the singular value decomposition that allows us to rewrite the field Φ as

$$\Phi^{af} = \sum_{bg} C^{ab} W^{bg} F^{gf}, \quad (3.1)$$

where $C \in O(N_c^2 - 1)$ and $F \in O(N_f)$ are orthogonal matrices, and W is an $(N_c^2 - 1) \times N_f$ rectangular matrix with zero nondiagonal elements ($W_{ij} = 0$ for $i \neq j$). We set $W_{ii} = w_i$ with $i = 1, \dots, q$, where

$$q = \text{Min}[N_f, N_c^2 - 1]. \quad (3.2)$$

Without loss of generality, we assume that $w_i \geq 0$. Substitution in $V(\Phi)$ gives

$$V(\Phi) = \frac{v}{4} \sum_{i=1}^q w_i^4. \quad (3.3)$$

If we minimize $V(\Phi)$ subject to the constraint $\text{Tr} \Phi^t \Phi = \sum_{i=1}^q w_i^2 = 2$, it is easy to verify that there are two solutions that depend on the sign of v :

$$\begin{aligned} \text{(I)} \quad w_1 &= \sqrt{2}, & w_i &= 0 \quad \text{for } i \geq 2, \\ \text{(II)} \quad w_1 &= \dots = w_q = (2/q)^{1/2}. \end{aligned} \quad (3.4)$$

Solution (I) is the relevant one for $v < 0$, while solution (II) is the relevant one for $v > 0$. It is interesting to observe that this result also holds for the general potential (2.7), as long as $r < 0$. For $r > 0$, the minimum of the potential corresponds to $w_1 = \dots = w_q = 0$: no critical behavior is expected in this case.

For solutions of type (I), we can rewrite the field as

$$\Phi^{af} = \sqrt{2} s^a z^f, \quad (3.5)$$

where s and z are unit real vectors of dimension $N_c^2 - 1$ and N_f , respectively. For solutions of type (II), we have instead

$$\Phi^{af} = \sqrt{\frac{2}{q}} \sum_{k=1}^q C^{ak} F^{kf}. \quad (3.6)$$

This expression can be simplified, parametrizing Φ in terms of a single orthogonal matrix. We should distinguish two cases. If $N_f \geq N_c^2 - 1 = q$, let us define an N_f -dimensional orthogonal matrix $\hat{C} = C \oplus I_{N_f - q}$, where I_p is the p -dimensional identity matrix. We can rewrite eq. (3.6) as

$$\Phi^{af} = \sqrt{\frac{2}{q}} \sum_{g=1}^{N_f} \hat{C}^{ag} F^{gf}. \quad (3.7)$$

Since \widehat{C} is an orthogonal matrix, we can express Φ in terms of a single orthogonal matrix $F' = \widehat{C}F$, i.e., we can set $C = I$ in eq. (3.6). Of course, because of gauge invariance, see eq. (2.1), F is not uniquely defined and it is, more properly, an element of $O(N_f)/\text{SU}(N_c)_{\text{adj}}$ [$\text{SU}(N_c)_{\text{adj}}$ is the group of block-diagonal matrices $\widetilde{V} \oplus I_{N_f-q}$, where \widetilde{V} belongs to the adjoint representation of $\text{SU}(N_c)$]. For $N_c = 2$ the quotient becomes $\text{SO}(N_f)/\text{SO}(3)$. If $N_f \leq N_c^2 - 1$, we can repeat the same argument to prove that one can set $F = I$ and $\Phi^{af} = C^{af}$, without loss of generality. Note that, for $N_c = 2$, we can use the gauge transformations to further simplify the field. Indeed, the matrix \widetilde{V} appearing in eq. (2.1) is a generic orthogonal matrix. Thus, choosing $\widetilde{V} = C^t$, we obtain $\Phi^{af} = \delta^{af}$: the minimum-potential field configuration is completely determined.

In the previous calculation we have assumed that the relevant scalar-field configurations in the large- β limit are only determined by the potential term $S_V(\Phi)$. In appendix A, we discuss the role of the kinetic term $S_K(\Phi, U)$ and show that this quantity is not relevant for the determination of the low-temperature behavior of the scalar field for $v \neq 0$. The kinetic term is only relevant for $v = 0$. In this case, we can show that, for $N_c = 2$, the model with $v = 0$ behaves as for $v < 0$ (see appendix A): the relevant configurations correspond to solution (I) reported above. We do not have exact results for $N_c > 2$. However, the numerical results we will present below indicate that also for $N_c > 2$, the relevant configurations for $v = 0$ are those of type (I).

To distinguish the nature of the zero-temperature configurations, one can use the order parameter B_x defined in eq. (2.12). If the field is parametrized as in eq. (3.1), we have

$$\text{Tr } B^2 = \frac{1}{4} \sum_{i=1}^q w_i^4, \tag{3.8}$$

so that

$$\begin{aligned} \text{(I)} \quad & \text{Tr } B^2 = 1, \\ \text{(II)} \quad & \text{Tr } B^2 = \frac{1}{q}, \end{aligned} \tag{3.9}$$

for solutions of type (I) and (II), respectively [see eq. (3.4)].

It is interesting to note that in this discussion the gauge group does not play any role: the only relevant quantity is the dimension of the gauge representation. In particular, one would obtain exactly the same results for the minimum configuration and the behavior of the order parameter Q for a gauge theory in which the fields transform under the fundamental representation of the $O(N_c^2 - 1)$ group.

In the previous discussion, we focused on the minimum configurations of the scalar fields. We wish now to discuss the large- β behavior of the gauge fields. If we minimize the kinetic term (2.3), we obtain

$$\Phi_x = \widetilde{U}_{x,\mu} \Phi_{x+\hat{\mu}}. \tag{3.10}$$

Repeated applications of this relation along a plaquette give

$$\Phi_x = \widetilde{\Pi}_x \Phi_x \quad \widetilde{\Pi}_x = \widetilde{U}_{x,1} \widetilde{U}_{x+\hat{1},2} \widetilde{U}_{x+\hat{2},1}^t \widetilde{U}_{x,2}^t. \tag{3.11}$$

For minimum configurations of type (I), using eq. (3.5), we have

$$s^a = \sum_b \tilde{\Pi}^{ab} s^b, \tag{3.12}$$

i.e., $\tilde{\Pi}_x$ has necessarily a unit eigenvalue. A detailed analysis shows that $\tilde{\Pi}_x$ can be written as $\exp(i \sum_a \alpha^a \tilde{T}^a)$, where \tilde{T}^a are the generators in the adjoint representation of a smaller subgroup isomorphic to $U(1) \oplus U(N_c - 2)$. Thus, for $\beta \rightarrow \infty$ there is still a residual dynamics of the gauge fields, i.e., we end up with a $U(1) \oplus U(N_c - 2)$ pure gauge model with Hamiltonian $H_G(U)$. In two dimensions, however, this dynamics is unable to give rise to a critical behavior.

Let us now consider the case in which the relevant configurations are those of type (II), see eq. (3.4). In this case, $\tilde{\Pi}$ has q unit eigenvalues, which further reduce the dynamics of the gauge fields. In particular, for $N_f \geq N_c^2 - 1$, $\tilde{\Pi}_x = 1$. Note, however, that this still leaves open the possibility of a nontrivial dynamics for the fields U_x . Indeed, the condition $\tilde{\Pi}_x = 1$ implies that Π_x belongs to the center \mathbb{Z}_{N_c} of the group, so that, in the limit $\beta \rightarrow \infty$, we end up with a \mathbb{Z}_{N_c} pure gauge theory. Again, as we are in two dimensions, this gauge model is not expected to become critical and therefore it should not be relevant for the critical dynamics of the model.

3.2 Effective models for the low-temperature behavior

Let us now analyze the effective behavior in the zero-temperature limit. We first assume that v is negative or vanishes, so that the relevant minimum configurations are those of type (I). Then, we assume that, for $\beta \rightarrow \infty$, the relevant fluctuations are those that locally satisfy the minimum potential conditions, i.e., that we can parametrize the field as in eq. (3.5) with site dependent vectors z_x and s_x . The field (3.5) satisfies the minimum condition exactly. Fluctuations are possible as we do not assume translation invariance, so that z_x and s_x are site dependent. For this type of field configurations the kinetic term becomes

$$S_K = -JN_f \sum_{x,\mu} j_{x,\mu} z_x \cdot z_{x+\hat{\mu}}, \quad j_{x,\mu} = \sum_{ab} s_x^a \tilde{U}_{x,\mu}^{ab} s_{x+\hat{\mu}}^b. \tag{3.13}$$

It is trivial to see that the coupling $j_{x,\mu}$ satisfies $|j_{x,\mu}| \leq 1$. For large values of β , S_K should be minimized, which requires either $z_x = z_{x+\hat{\mu}}$ and $j_{x,\mu} = 1$ or $z_x = -z_{x+\hat{\mu}}$ and $j_{x,\mu} = -1$. As these two possibilities occur with the same probability, the effective model for the fluctuations is a gauge RP^{N_f-1} model, in which $j_{x,\mu}$ is a gauge field that takes the values ± 1 with equal probability.

At a more intuitive level, the correspondence between the critical behavior of the gauge model and of the RP^{N_f-1} model can be established by noting that the order parameter Q , or equivalently B , defined in eq. (2.12), can be written as

$$B_x^{fg} = z_x^f z_x^g = P_x^{fg}, \tag{3.14}$$

which shows that B_x is a local projector P^{fg} onto a one-dimensional space. If we assume that the dynamics in the gauge model is completely determined by the fluctuations of the

order parameter B_x , we immediately identify the effective scalar model as the RP^{N_f-1} model. Indeed, the standard nearest-neighbor RP^{N-1} action is obtained by taking the simplest action for a local projector P_x^{fg} :

$$S_{\text{RP}} = -J \sum_{x,\mu} \text{Tr} P_x P_{x+\hat{\mu}}, \quad P_x^{fg} = \varphi_x^f \varphi_x^g, \quad (3.15)$$

where φ_x^a is a unit vector and $P_x^{ab} = \varphi_x^a \varphi_x^b$ is a local projector onto a one-dimensional space, i.e., it satisfies $P_x = P_x^2$ and $\text{Tr} P_x = 1$. The zero-temperature critical behavior of 2D RP^{N-1} models is still debated; see, e.g., refs. [19–25]. Although 2D RP^{N-1} and $O(N)$ σ models have the same perturbative behavior [21], there is numerical evidence that their nonperturbative behavior differs. This is due to topological \mathbb{Z}_2 defects that are present in the RP^{N-1} model, which are apparently relevant perturbations of the zero-temperature 2D $O(N)$ fixed point, leading to a different universal asymptotic behavior in the nonperturbative regime [25].

Let us now assume $v > 0$. The relevant solutions are those of type (II). In this case, we must distinguish two cases. If $N_f \leq N_c^2 - 1$ we find

$$B = \frac{1}{2} F^t W^t W F = \frac{1}{N_f} I_{N_f}, \quad (3.16)$$

where I_{N_f} is the N_f -dimensional unit matrix. Correspondingly, the order parameter Q vanishes in the limit $\beta \rightarrow \infty$. Therefore, correlations of Q_x , and also the Binder parameter, depend on the fluctuations of the field Φ_x^{af} around the minimum configurations. We do not have predictions for their behavior. However, we will show numerically below that these fluctuations do not show long-range correlations. Indeed, for $T \rightarrow 0$, the Binder parameter takes the high-temperature value appropriate for disordered configurations:

$$\lim_{\beta \rightarrow 0} U = 1 + \frac{4}{(N_f - 1)(N_f + 2)}. \quad (3.17)$$

Note that this result is consistent with what we assumed for $v < 0$: fluctuations around the minimum configurations are irrelevant and the critical behavior is only due to the fluctuations of the fields that locally minimize the quartic potential. In the case we are discussing now, once the fields minimize the quartic potential, the order parameter Q is fixed — it vanishes — and therefore no long-range fluctuations of Q_x are possible.

Let us finally suppose that $N_f > N_c^2 - 1$. In this case, the order parameter is non-trivial and the system orders. To identify the effective model, note that $\tilde{\Pi} = 1$, so that $\tilde{U}_{x,\mu} = \tilde{V}_x \tilde{V}_{x+\hat{\mu}}$. As in the discussion for $v < 0$, we assume that the fields locally minimize the potential, so that Φ_x^{af} can be parametrized as in eq. (3.6) with $C = I$ and a site-dependent orthogonal matrix F_x . Substituting this parametrization in the kinetic term of the action we obtain

$$S_K = -\frac{N_f}{q} \sum_{x,\mu} \text{Tr} (F_x^t \hat{V}_x Y_{N_f}^q \hat{V}_{x+\hat{\mu}}^t F_{x+\hat{\mu}}), \quad (3.18)$$

where³ $Y_{N_f}^q = I_q \oplus 0$ is an $N_f \times N_f$ diagonal matrix in which the first q elements are 1 and the other $(N_f - q)$ elements are 0 and $\hat{V} = \tilde{V} \oplus I_{N_f-q}$. Note that action is invariant

³We indicate with $A \oplus B$ a block-diagonal matrix, where A and B are square matrices of dimension q and $N_f - q$, respectively.

N_c	N_f	$\langle \text{Tr } B^2 \rangle_{v=0}$	$\langle \text{Tr } B^2 \rangle_{v=1}$	$U_{v=1}$
2	2	0.9998(4)	0.49981(13)	2.000(2)
2	3	0.9999(5)	0.3331(2)	1.4014(7)
2	4	1.0000(4)	0.3332(2)	0.9997(2)
2	5		0.3333(5)	0.9999(2)
2	6		0.3335(4)	1.0000(1)
3	2	1.0006(12)	0.4999(5)	2.012(5)
3	3	0.9999(12)	0.3327(12)	1.401(2)
3	4	1.0002(14)	0.251(2)	
4	2	0.99(2)	0.500(3)	1.991(6)
4	3	1.000(2)	0.330(5)	1.407(3)
4	4	0.999(3)	0.236(8)	

Table 1. Results for $\langle \text{Tr } B^2 \rangle$ and the Binder parameter U in the large- β limit. We consider a square lattice of size $L = 4$, two values of v , $v = 0$ and $v = 1$, and $\gamma = 0$. The values of $U_{v=1}$ that differ from 1 are consistent with eq. (3.17), which predicts $U = 2$ and $7/5$ for $N_f = 2$ and 3, respectively.

under $SU(N_c)_{\text{adj}} \otimes O(N_f - q)$ transformations defined by $F \rightarrow W_F F$, $\hat{V} \rightarrow W_V \hat{V}$, where $W_F = W_1 \oplus W_2$, $W_V = W_1 \oplus I$, with $W_1 \in SU(N_c)_{\text{adj}}$ and $W_2 \in O(N_f - q)$. As we already mentioned in section 3.1, we expect the same critical behavior if we consider $O(q)$ gauge fields, leading to an effective enlargement of the symmetry to $O(q) \otimes O(N_f - q)$. The resulting effective model is therefore a lattice σ model defined on the symmetric space $O(N_f)/O(q) \otimes O(N_f - q)$ [3, 16]. For $N_f = q + 1$ the symmetric space is isomorphic to the sphere in N_f dimensions, and thus the effective model is simply the $O(N_f)$ -invariant vector σ model with Hamiltonian

$$S_{O(N)} = -J \sum_{\mathbf{x}, \mu} \varphi_{\mathbf{x}} \cdot \varphi_{\mathbf{x}+\hat{\mu}}, \quad \varphi_{\mathbf{x}} \cdot \varphi_{\mathbf{x}} = 1. \tag{3.19}$$

3.3 Numerical results

To verify the above-reported predictions, we have performed MC simulations (see appendix B for details) for large values of β , $\gamma = 0$ (the gauge action is not expected to play an important role), and relatively small systems. The extrapolations of the results provide information on the nature of the relevant low-temperature configurations. In table 1 we report the large- β extrapolations of $\langle \text{Tr } B^2 \rangle$ for $v = 0$ and $v = 1$. The results should be compared with the prediction (3.9). For $v = 0$, the average is always consistent with 1, confirming that the relevant configurations correspond to solution (I). Apparently, for any N_c , for $v = 0$ the model behaves as for $v < 0$, a result that we have only proved for $N_c = 2$. For $v = 1$, results are instead consistent with $1/q$, confirming that the relevant configurations are those of type (II). Note that this result also applies when $N_f = 2$. In this case, the flavor symmetry is abelian and therefore a BKT finite-temperature transition is possible.

We also computed the Binder parameter U . For $v = 0$, it always converges to 1 as $\beta \rightarrow \infty$ (data not shown), indicating that long-range correlations set in the limit. Results for $v = 1$ are reported in table 1. For $N_f \leq N_c^2 - 1$ (all results we report for $N_c = 3, 4$ satisfy this condition) we observe that U is always approximately equal to the high-temperature value (3.17), $U = 2$ and $7/5$ for $N_f = 2$ and 3 , respectively. This indicates the absence of long-range correlations. In the opposite case $N_f > N_c^2 - 1$, we find instead $U = 1$, consistent with the presence of critical fluctuations.

3.4 Summary

To conclude the section, let us summarize the expected low-temperature behavior of the model for $N_f \geq 3$:

- (i) For $v \leq 0$, the model has a zero-temperature critical (continuum) limit independent of N_c , analogous to that of the RP^{N_f-1} σ model.
- (ii) For $v > 0$ and $N_f \leq N_c^2 - 1$, Q correlations are always short-ranged, even in the limit $T \rightarrow 0$.
- (iii) For $v > 0$ and $N_f > N_c^2 - 1$ the model has a zero-temperature critical (continuum) limit that depends both on N_f and N_c . It is the same as that of the σ model defined on the symmetric space [3, 16] $\text{O}(N_f)/\text{O}(q) \otimes \text{O}(N_f - q)$: correlations of the order parameter Q have the same critical behavior, i.e., continuum limit, in the two models. For $N_f = N_c^2$, we obtain the same behavior as that of the $\text{O}(N_f)$ vector σ model.

Note that these effective behaviors have been obtained by making very simple assumptions. Essentially, we have assumed that the relevant configurations correspond to scalar fields $\{\Phi^{\text{min}}\}$ that locally, —i.e., at each site— minimize the potential $S_V(\Phi)$. Gauge fields are only relevant for the identification of the dynamic degrees of freedom and for restricting the focus on gauge-invariant observables. As a consequence, we expect that the model presented here has the same continuum limits of the model with fields that transform under the fundamental representation of the $\text{O}(N_c^2 - 1)$ gauge group.

4 Numerical results

In this section we present some numerical results that confirm the predictions of section 3. Results for $v = 0$ and $N_c = 2$ have been already discussed in refs. [14] ($N_f = 3, 4$) and [15] (where $N_f = 2$, so that a finite-temperature BKT transition was observed). Indeed, for $N_c = 2$ the model can be rewritten as an $\text{SO}(3)$ gauge theory with fields in the fundamental representation of the gauge group. The results presented in ref. [14] are in full agreement with the analysis presented here. In the following, we will present results for $N_c = 3$, $v = 0$ and for $N_c = 2, 3$, $v > 0$. They fully confirm the conclusions of section 3.

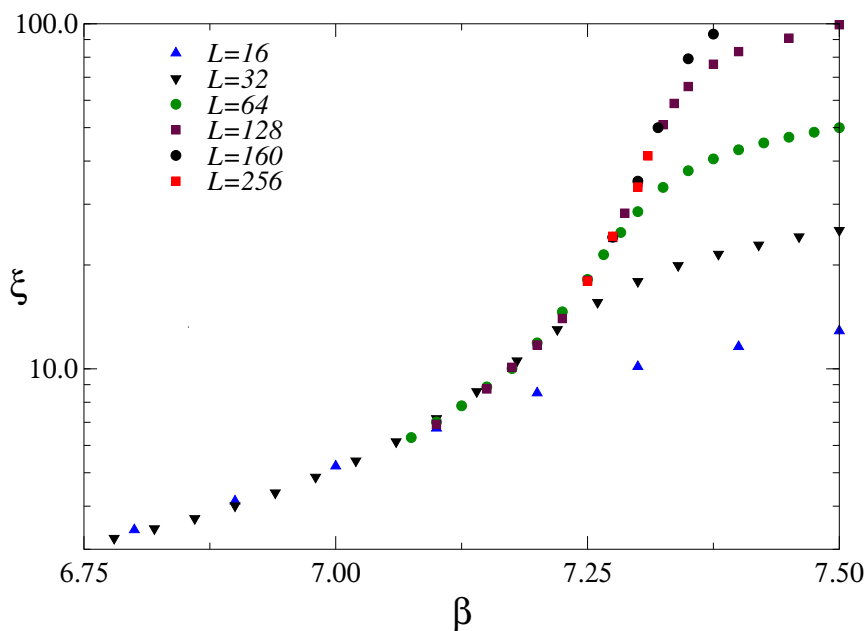


Figure 1. Estimates of ξ versus β for $N_c = 3$, $N_f = 3$, $v = 0$, and $\gamma = 0$. Results for several values of L up to 256.

4.1 Finite-size scaling analysis

To identify the nature of the zero-temperature critical behavior, we perform a FSS analysis of the MC data. We follow the strategy already employed in refs. [12–15]. In the FSS limit $L \rightarrow \infty$ at L/ξ fixed, the RG invariant quantities R_ξ and U , defined in eqs. (2.15) and (2.16), respectively, are expected to scale as

$$U(\beta, L) \approx F_U(R_\xi), \tag{4.1}$$

where $F_U(x)$ is a universal function that completely characterizes the universality class of the transition. Because of the universality of relation (4.1), one may use plots of U versus R_ξ to identify the models that have the same universal behavior. If the estimates of U for two different systems approach the same curve as $L \rightarrow \infty$ when plotted versus R_ξ , the transitions in the two models belong to the same universality class. We will apply this approach below to several different models.

4.2 Universal RP^{N_f-1} behavior for $v = 0$

Let us start by considering the model for $v = 0$. The model should have an asymptotic zero-temperature behavior analogous to that of the RP^{N_f-1} model. In ref. [14], this was verified for $N_c = 2$. We consider here $N_c = 3$ and $N_f = 3$. Numerical results for $\gamma = 0$ are reported in figures 1 and 2. The correlation length ξ rapidly increases with increasing β , see figure 1, consistently with an asymptotic exponential behavior $\xi \sim \exp(c\beta)$. The plot of U versus R_ξ reported in figure 2 shows that the data approach the universal curve of RP^2 model with increasing L , as expected. Scaling corrections are visible in figure 2, but they rapidly decay to zero (apparently as $1/L$, in the range of values of L we consider).

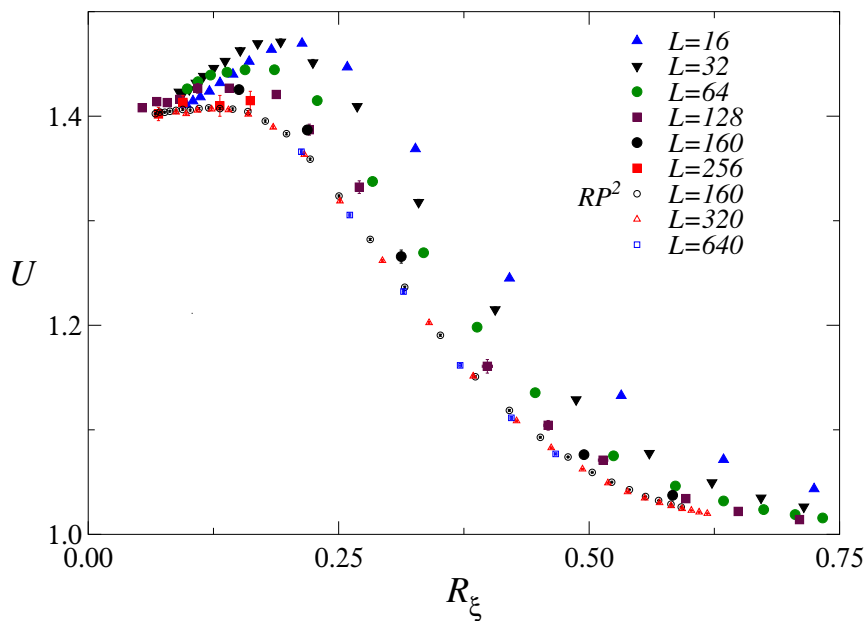


Figure 2. Plot of U versus R_ξ for $N_c = 3$, $N_f = 3$, $v = 0$, and $\gamma = 0$. Data are compared with analogous data on large lattices computed in the RP^2 model [25].

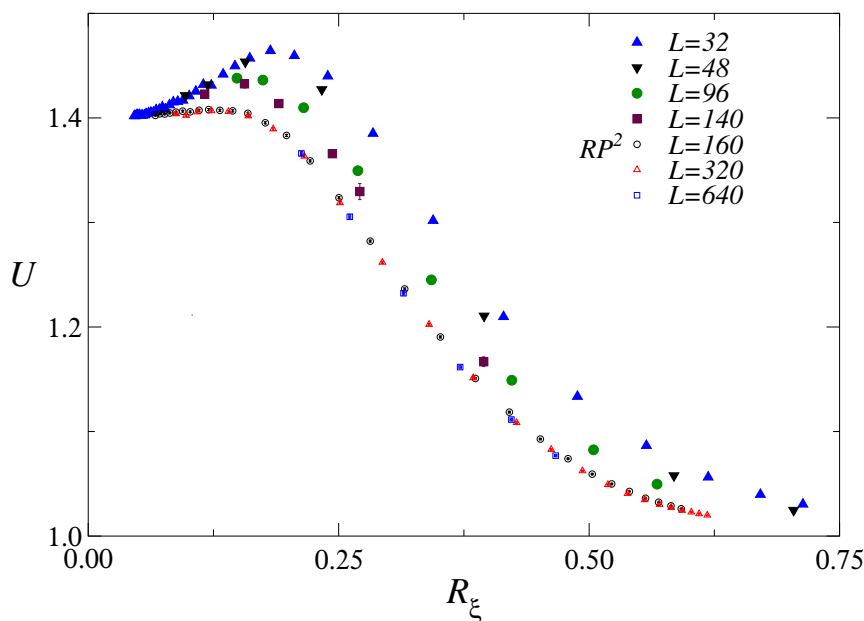


Figure 3. Plot of U versus R_ξ for $N_c = 3$, $N_f = 3$, $v = 0$, and $\gamma = 1$. Data are compared with analogous data on large lattices computed in the RP^2 model [25].

As we mentioned in section 3, the inclusion of the plaquette action should not change the asymptotic behavior. To verify this point, we performed simulations with $\gamma = 1$. Results are shown in figure 3. Also in this case, the estimates of U versus R_ξ converge towards the RP^2 universal curve.

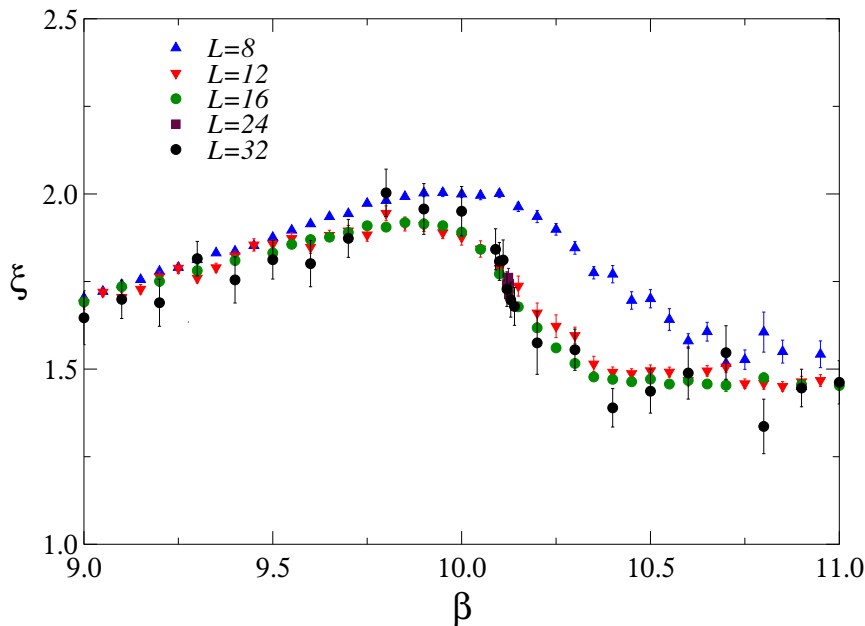


Figure 4. Plot of ξ versus β for $N_c = 3$, $N_f = 3$, $\gamma = 0$, and $v = 1$.

4.3 Behavior for $v > 0$ and $N_f \leq N_c^2 - 1$

As discussed in section 3, for $v > 0$ and $N_f \leq N_c^2 - 1$, we do not expect the correlations of the order parameter Q to become critical for $\beta \rightarrow \infty$. Therefore, the correlation length should be bounded in the limit. To verify this prediction, we performed simulations for $N_f = 3$, $N_c = 3$, $\gamma = 0$, and $v = 1$. In figure 4 we report the correlation length as a function of β . It does not increase with β and apparently $\xi \approx 1.5$ in the asymptotic regime. The data confirm that the modes associated with the scalar fields are disordered. This is also confirmed by the data of U , which is close to the high-temperature value $7/5$, see eq. (3.17).

4.4 Behavior for $v > 0$ and $N_f > N_c^2 - 1$

We finally consider the model for $v > 0$ and $N_f > N_c^2 - 1$. The analysis reported in section 3 predicts that the asymptotic zero-temperature behavior is the same as that of the σ model defined on the symmetric space $O(N_f)/O(q) \otimes O(N_f - q)$ with $q = N_c^2 - 1$. For $N_f = N_c^2$ this is equivalent to the standard $O(N_f)$ σ model.

To verify these predictions, we performed simulations for $N_f = 4$, $N_c = 2$, $v = 10$, and $\gamma = 0$. In figure 5 we show the correlation length ξ versus β . It increases with increasing β , exponentially in the region in which $\xi \ll L$. To identify the universality class, we again consider U versus R_ξ . The data are reported in figure 6. They appear to approach the corresponding ones computed in the $O(4)$ vector σ -model. Note that in the $O(4)$ model one should consider the same operator as in the gauge theory. Thus, the correlation length and the Binder parameter were computed considering the correlation functions of the spin-2 operator $\varphi_x^f \varphi_x^g - \delta^{fg}/N_f$ with periodic boundary conditions.

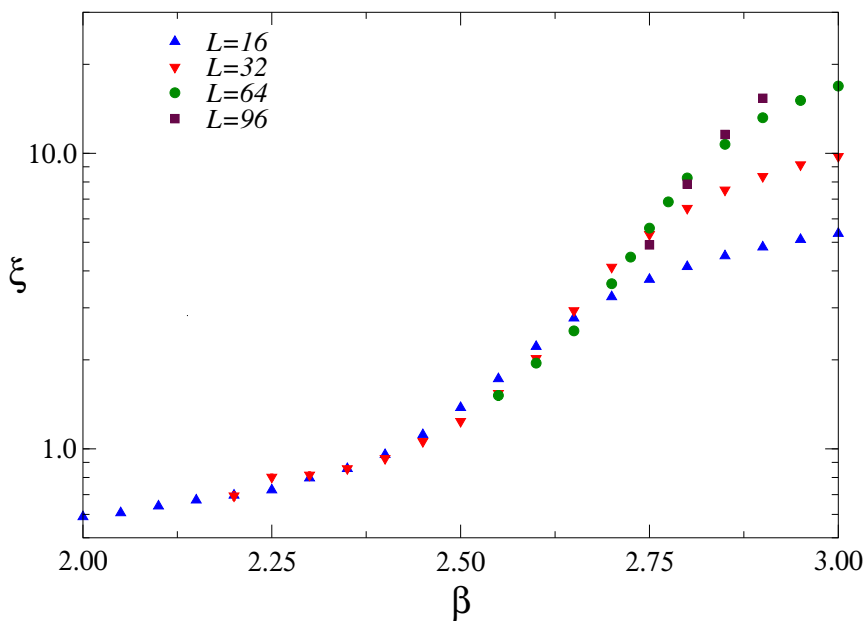


Figure 5. Correlation length ξ versus β for $N_c = 2$, $N_f = 4$, $\gamma = 0$, and $v = 10$.

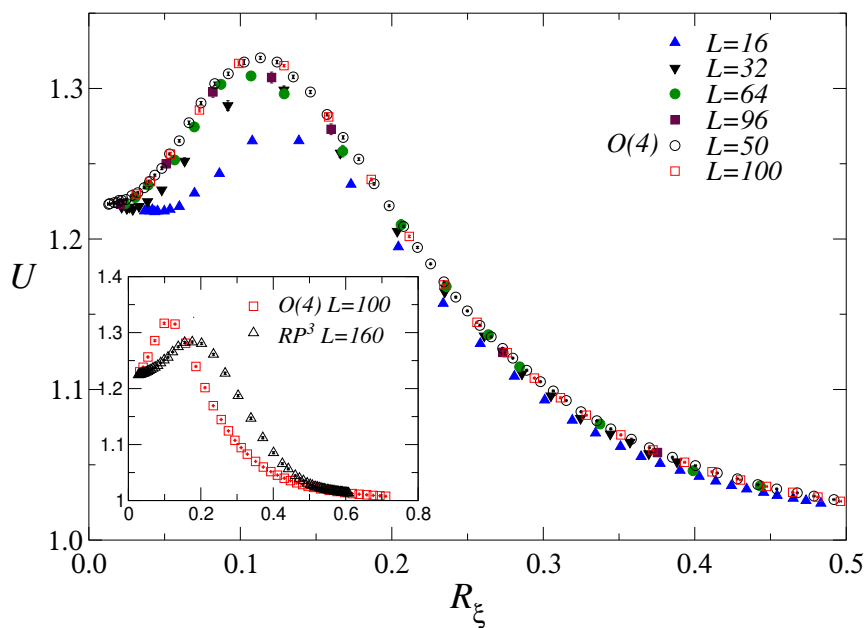


Figure 6. Plot of U versus R_ξ for $N_c = 2$, $N_f = 4$, $\gamma = 0$, and $v = 10$. Data are compared with analogous results for the spin-2 operator computed in the lattice $O(4)$ σ model with action (3.19). For comparison, we also report data for the 2D RP^3 model [25].

4.5 Crossover from the $v = 0$ to large- v regimes

In section 3 we have shown that, at $T = 0$, the relevant configurations depend on the sign of v , so that $v = 0$ is a singular point for the large-scale behavior. The presence of this singularity at $\beta = \infty$ may, in principle, give rise to singularities and discontinuities also at finite β . For instance, there might be a first-order transition line $\beta = f_{FO}(v)$ in the (β, v) plane that starts at $\beta = \infty, v = 0$ and ends at a finite value of β , with a critical endpoint where an Ising critical behavior is realized. Alternatively, it is possible that, for finite β only crossover phenomena occur without transitions. We do not have performed a thorough analysis of this issue. However, in the few cases we have considered and that we discuss below, we have no evidence of finite- β transition lines but only of crossover effects.

In the simulations at fixed $v > 0$ we have observed that the specific heat, defined as $C = \frac{1}{V}(\langle S^2 \rangle - \langle S \rangle^2)$, has a strongly nonmonotonic behavior as β increases, with a maximum at a finite value β_{\max} of the inverse temperature. Two examples, corresponding to $N_c = 2, N_f = 4$ and $N_c = N_f = 3$, are reported in figure 7. At a first-order transition, the maximum of the specific heat should increase with the lattice size, as L^2 in two dimensions. In both cases shown in figure 7, this does not occur: the maximum of C approaches a constant as L increases, indicating that only a crossover with no infinite-volume singularity occurs. The crossover is obviously present in all quantities. In figure 5 we have already reported the correlation length for $N_c = N_f = 3$. It varies quite abruptly for $\beta \approx \beta_{\max}$. The crossover is more evident in the behavior of ξ for $N_c = 2, N_f = 4$, see figure 8. At $\beta \approx \beta_{\max} \approx 3.8$, the behavior of ξ changes abruptly, providing a clear indication of a sudden change of the nature of the relevant configurations. The presence of this crossover region for $v \lesssim 1$, makes it difficult to determine the asymptotic behavior of the model for these values of v . This is the reason why, in section 4.4, we considered the model at $v = 10$, a value of v that is very far from the crossover region.

5 Conclusions

We have considered a class of 2D lattice non-Abelian gauge models with N_f scalar fields in the adjoint representation. They are defined by the action (2.2) and are invariant under global $O(N_f)$ and local $SU(N_c)$ transformations. For $N_f \geq 3$, the global symmetry is nonabelian and thus, a critical (continuum) limit is only possible in the limit $T \rightarrow 0$. We have therefore investigated the zero-temperature behavior, to understand whether a continuum limit exists and, if it does, to identify the corresponding 2D quantum field theory. This work extends previous results [12–15], discussing the role played by the gauge representations and by the quartic scalar potential. For this purpose, we have identified the low-energy configurations, that are relevant in the zero-temperature limit, and we have derived effective models that are expected to describe the large-scale behavior of the system. The predictions have then been checked numerically. We have performed MC simulations and determined the universal features of the low-temperature behavior using FSS methods.

We find that the continuum limit depends on the sign of the parameter v appearing in the scalar potential, see eq. (2.7). For $v \leq 0$ the lattice gauge model has the same continuum limit as the RP^{N_f-1} model, for any value of N_c . For positive v instead, the critical behavior depends on both N_c and N_f . For $N_f \leq N_c^2 - 1$, there is no continuum

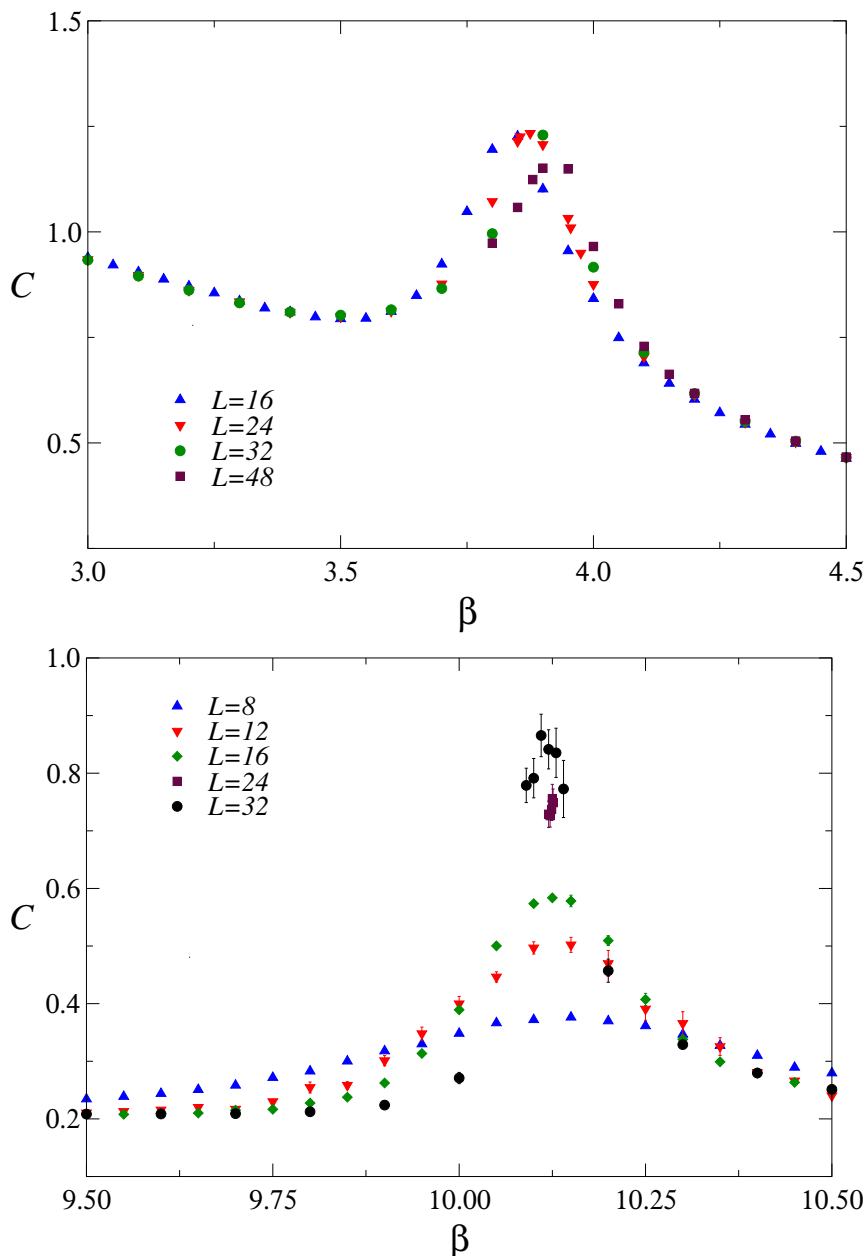


Figure 7. Plot of the specific heat C versus β for $N_c = 2$, $N_f = 4$ (top) and $N_c = 3$, $N_f = 3$ (bottom). In both cases $\gamma = 0$ and $v = 1$.

limit: correlation functions are always short ranged. On the other hand, for $N_f > N_c^2 - 1$ there are long-range correlations for $T \rightarrow 0$. The corresponding continuum limit is the same as that of the σ model defined on the symmetric space $O(N_f)/O(q) \otimes O(N_f - q)$ with $q = N_c^2 - 1$. In particular, for $N_f = N_c^2$, the gauge model is equivalent to the $O(N_f)$ vector σ model. Numerical data support these predictions. In particular, a FSS analysis of the MC data for $N_f = 4$ and $N_c = 2$ at $v = 10$ clearly supports the prediction that the critical behavior belongs to the universality class of the $O(4)$ σ model.

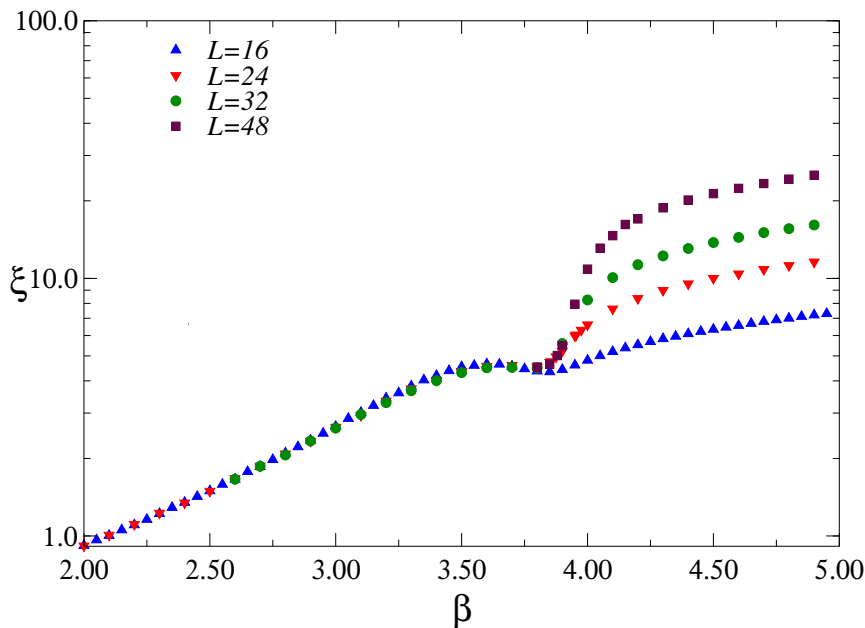


Figure 8. Plot of ξ versus β for $N_c = 2$, $N_f = 4$, $\gamma = 0$, and $v = 1$.

The results of this work provide additional support to the conjecture that the critical behavior of any 2D lattice gauge model, defined using the Wilson approach [2], belongs to the universality class of a field theory associated with one of the symmetric spaces that have the same global symmetry.

We finally mention that it is worth extending this study to analogous three-dimensional systems, whose phase diagram is expected to be more complicated, presenting various phases associated with the different Higgs mechanisms that can be realized [17, 18]. The nature of transition lines separating the various phases are expected to be crucially related to the interplay between global and local gauge symmetries [33, 34].

A Role of the gauge fields for the minima

In section 3 we showed that, in the absence of gauge fields, there are two minima whose relevance depends on the sign of the coupling v . We wish now to include the effects of the gauge fields. As suggested in refs. [17, 18], in the absence of the plaquette term, i.e., for $\gamma = 0$, we can integrate out the gauge fields, defining a local effective potential:

$$e^{-\beta\tilde{V}(D_{\mathbf{x},\mu})} = \int d\tilde{U} \exp[\text{Tr}(\tilde{U}D_{\mathbf{x},\mu})] \quad D_{\mathbf{x},\mu}^{ab} = \frac{\beta N_f}{2} \sum_f \Phi_{\mathbf{x}+\hat{\mu}}^{af} \Phi_{\mathbf{x}}^{bf}. \quad (\text{A.1})$$

If we assume translation invariance (and therefore drop the link dependence) and parametrize the field Φ^{af} as in eq. (3.1), the matrix D is given by

$$D = \frac{\beta N_f}{2} C W W^t C^t. \quad (\text{A.2})$$

We are now interested in computing the integral in the limit $\beta \rightarrow \infty$ with the purpose of understanding whether the effective term \tilde{V} changes the conclusions obtained considering only $V(\Phi)$. Note that, for $N_c = 2$, \tilde{U} is a generic orthogonal matrix and therefore we can easily check that the effective potential is independent of C (it is enough to perform the change of variable $\tilde{U}' = C^t U C$). Such an independence is not a priori expected for $N_c > 2$ and thus the integral might depend both on C and WW^t .

The integral reported here has been the object of several investigations, but there is at present no exact general result, except for $N_c = 2$, where one can take advantage of the results for the $O(N)$ link integrals [26, 27] (some results for specific matrices D are reported in refs. [28, 29]). For $N_c = 2$ the result only depends on the eigenvalues of the matrix WW^t . If $N_f \geq 3$, these eigenvalues coincide with w_1^2 , w_2^2 and w_3^2 , while for $N_f = 2$ one should consider w_1^2 , w_2^2 and 0. The results reported in ref. [26] allow us to obtain

$$\tilde{V} = -\frac{N_f}{2}(w_1^2 + w_2^2 + w_3^2) + \frac{3}{2\beta} \ln \frac{\beta N_f}{2} + \frac{1}{2\beta} \ln \left[(w_1^2 + w_2^2)(w_1^2 + w_3^2)(w_2^2 + w_3^2) \right] + O(\beta^{-2}), \tag{A.3}$$

provided that at least two eigenvalues are not zero. If only one eigenvalue is different from zero, one obtains [26]

$$\tilde{V} = -\frac{N_f}{2} w_1^2 + \frac{1}{\beta} \ln \frac{\beta N_f}{2} + \frac{1}{\beta} \ln w_1^2 + O(\beta^{-2}). \tag{A.4}$$

Since $w_1^2 + w_2^2 + w_3^2 = 2$ as a consequence of the constraint $\text{Tr} \Phi^t \Phi = 2$, the leading contribution for $\beta \rightarrow \infty$ is independent of the field configuration. Thus, the gauge fields do not change the conclusions on the relevant minimum configurations for $v > 0$ and $v < 0$. The calculation, however, allows us to determine the expected behavior for $v = 0$. Indeed, for the solution of type (I) (see eq. (3.4)), the subleading correction in \tilde{V} is $\ln \beta / \beta$, which is smaller than the subleading correction, $\frac{3}{2} \ln \beta / \beta$, that appears in \tilde{V} for configurations of type (II), for which $w_1^2 = w_2^2 = w_3^2 = 2/3$ (this is the relevant case for $N_f \geq 3$) or $w_1^2 = w_2^2 = 1$, $w_3^2 = 0$ (this is the relevant case for $N_f = 2$). This implies that, for $v = 0$, the asymptotic behavior is the same as for $v < 0$, i.e., in the RP^{N_f} universality class.

Let us now consider the case $N_c > 2$. In this case there is no general formula for the integral. We will therefore assume that the relevant minima are those that we have determined in section 3 and, for each of them, we will determine the asymptotic behavior of the one-link integral.

We start by considering $(WW^t)^{ab} = 2\delta_{a1}\delta_{b1}$, i.e. type (I) configurations. If we set $v^a = C^{a1}$ (since C is orthogonal, v^a is a unit vector), we have

$$D^{ab} = \beta N_f v^a v^b \tag{A.5}$$

The integral can then be written as

$$e^{-\beta \tilde{V}} = \int dU \exp[\text{Tr}(U^\dagger M U M)] \quad M = (2\beta N_f)^{1/2} \sum_a v_a T^a. \tag{A.6}$$

The matrix M is hermitean and traceless. If λ_a are its eigenvalues, we have

$$\sum_a \lambda_a^2 = \text{Tr} M^2 = \beta N_f \sum_a v_a^2 = \beta N_f. \tag{A.7}$$

Integral (A.6) has been computed in ref. [29]. The leading term can be rewritten in terms of the determinant of the matrix Λ , whose elements are $\Lambda_{ab} = e^{\lambda_a \lambda_b}$:

$$\tilde{V} = -\frac{\ln \det \Lambda}{\beta} + O(\ln \beta/\beta). \tag{A.8}$$

Now, $\det \Lambda$ is a sum of terms of the form

$$\exp[\lambda_1 \lambda_{i_1} + \lambda_2 \lambda_{i_2} + \dots + \lambda_N \lambda_{i_N}], \tag{A.9}$$

where (i_1, \dots, i_N) is a permutation of $(1, \dots, N = N_c^2 - 1)$. As a consequence of the Schwartz inequality

$$\lambda_1 \lambda_{i_1} + \lambda_2 \lambda_{i_2} + \dots + \lambda_N \lambda_{i_N} \leq \sum_a \lambda_a^2 = \beta N_f, \tag{A.10}$$

the equality being obtained for $i_1 = 1, i_2 = 2, \dots, i_N = N$. This implies that $\det \Lambda \sim e^{\beta N_f}$ for $\beta \rightarrow \infty$. We thus obtain for $N_c > 2$

$$\tilde{V} = -N_f + O(\ln \beta/\beta). \tag{A.11}$$

Let us now consider the configurations of type (II). We have not been able to obtain results for $N_f < N_c^2 - 1$, in which WW^t is a diagonal matrix that has both zero and unit eigenvalues. The case $N_f \geq N_c^2 - 1$, is instead easily discussed. The matrix WW^t is proportional to $\frac{1}{q}I$ and thus we obtain

$$e^{-\beta \tilde{V}} = \int d\tilde{U} \exp\left(\frac{\beta N_f}{q} \text{Tr } \tilde{U}\right) = e^{-\beta N_f/q} \int dU \exp\left(\frac{\beta N_f}{q} |\text{Tr } U|^2\right). \tag{A.12}$$

The large β behavior of the integral is obtained by expanding around $U = I$ (a few terms of the expansion are obtained in ref. [28]). The leading term for the integral is $\exp(\beta N_f N_c^2/q)$, which gives again eq. (A.11). As it happens for $N_c = 2$, the gauge contribution to the potential is the same for both types of minimum configurations.

To conclude the appendix, let us note that the gauge effective potential \tilde{V} would play a different role if one considers a different approach to $T = 0$. Indeed, let us define (again for $\gamma = 0$)

$$Z = \sum_{\{\Phi, U\}} \exp[-\beta S_K(U) + f(\beta) S_V(\Phi)] \tag{A.13}$$

where $f(\beta)$ is a function of β . In our work we have considered $f(\beta) = \beta$, but one can also consider functions with a different large- β behavior. One possibility consists in taking $f(\beta)$ finite for $\beta \rightarrow 0$. In this case the potential would play no role and the dominant term would be the gauge potential \tilde{V} . Therefore, the critical behavior would be independent of v , the same as that we observe for $v = 0$. A nontrivial behavior would only be obtained by selecting a function $f(\beta)$ that behaves as $\ln \beta$ as $\beta \rightarrow \infty$. In this case both the gauge contribution and the scalar potential would play a role. One would expect a critical v_c , such that different phases are realized for $v > v_c$ and $v < v_c$.

B Monte Carlo simulations: technical details

We performed MC simulations on square lattices with periodic boundary conditions. The gauge link variables U were updated using a standard Metropolis algorithm [30]. The new link variable was chosen close to the old one, in order to guarantee an acceptance rate of approximately 30%. The scalar fields were updated using two different Metropolis algorithms, again tuning the proposal to obtain an acceptance rate of 30%. The first update performs a rotation in flavor space

$$\phi^f \mapsto (O\phi)^f, \quad O \in \text{SO}(N_f), \quad (\text{B.1})$$

while the second one rotates the colors of a single flavor

$$\phi^f \mapsto (H\phi H^\dagger)^f, \quad H \in \text{SU}(N_c). \quad (\text{B.2})$$

In the simulations with $v = 0$, since the action is linear in the scalar fields, we also considered microcanonical steps [31] implemented *à la* Cabibbo-Marinari [32] (the relative frequency of Metropolis and microcanonical updates was chosen equal to 3/7). Microcanonical updates could not be used for $v \neq 0$, since the action is not linear in the scalar fields. Typical statistics of our runs, for a given value of the parameters and of the size of the lattice, were of order of 10^7 - 10^8 lattice sweeps (in a sweep we update all lattice variables once), with the largest number associated to runs performed without the microcanonical update. Errors were estimated using a standard blocking and jackknife procedure, with a maximum blocking size of the order of 10^5 updates.

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Numerical simulations have been performed on the CSN4 cluster of the Scientific Computing Center at INFN-PISA.

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