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Supersymmetric $J\bar{T}$ and $T\bar{J}$ deformations

Hongliang Jiang[a] **and Gabriele Tartaglino-Mazzucchelli**[a,b]

[a] *Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland*

[b] *School of Mathematics and Physics, University of Queensland, St Lucia, Brisbane, Queensland 4072, Australia*

E-mail: jiang@itp.unibe.ch, g.tartaglino-mazzucchelli@uq.edu.au

Abstract: We explore the $J\bar{T}$ and $T\bar{J}$ deformations of two-dimensional field theories possessing $\mathcal{N} = (0,1), (1,1)$ and $(0,2)$ supersymmetry. Based on the stress-tensor and flavor current multiplets, we construct various bilinear supersymmetric primary operators that induce the $J\bar{T}/T\bar{J}$ deformation in a manifestly supersymmetric way. Moreover, their supersymmetric descendants are shown to agree with the conventional $J\bar{T}/T\bar{J}$ operator on-shell. We also present some examples of $J\bar{T}/T\bar{J}$ flows arising from the supersymmetric deformation of free theories. Finally, we observe that all the deformation operators fit into a general pattern which generalizes the Smirnov-Zamolodchikov type composite operators.

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Contents

1 Introduction

Recently, a new type of deformation of two-dimensional quantum field theories, dubbed $T\bar{T}$ deformation, received a lot of attention. This arises by deforming a quantum field theory with a bilinear composite operator built as the determinant of the stress-energy tensor [1–3], leading to an *irrelevant* deformation. Nevertheless, the $T\bar{T}$ operator is free of short-distance divergences and hence proves to be a well-defined composite local operator [1, 2]. Remarkably, the seminal works [2, 3] have shown that: for any two-dimensional quantum field theory the spectrum of the $T\bar{T}$-deformed theory can be expressed in terms of the undeformed one in a simple way; if the undeformed theory is integrable so is the $T\bar{T}$-deformed one. For these reasons, the $T\bar{T}$ deformation has recently been shown to play an important role in many different areas of research and has stirred up excitements in various subjects of high-energy theoretical physics.[1]

The $T\bar{T}$ deformation is the simplest member in a more general family of deformations [2]. Another simple member of this family is the $J\bar{T}/T\bar{J}$ deformation [5], which is constructed out of the stress-energy tensor and a U(1) current. This deformation explicitly breaks Lorentz invariance, but it still enjoys several virtues of the $T\bar{T}$ deformation.[2] For example, the $J\bar{T}/T\bar{J}$ composite operators are also well-defined at the quantum level, and they preserve the solvability enjoyed by the $T\bar{T}$ deformation. Various aspects of the $J\bar{T}/T\bar{J}$ deformations have been studied so far, including holography [7, 8], path integral formulation [9, 10], modular invariance [11], correlation functions [12], and their role in string theory [13, 14]. See also [6, 15–23] for further results.

In this paper, we are going to discuss supersymmetry in the context of $J\bar{T}$ and $T\bar{J}$ deformations. The strategy parallels with the analysis of supersymmetric $T\bar{T}$ deformations that was recently discussed in a series of papers [24–28].[3] There, for theories possessing $\mathcal{N} = (0,1)$, $(1,1)$, $(0,2)$ and $(2,2)$ supersymmetry, it was shown how to induce manifestly supersymmetric deformations in terms of primary operators, which are constructed out of bilinears of the supercurrent multiplets (the supersymmetric counterparts of the stress-energy tensor). Remarkably, the manifestly supersymmetric deformations prove to be the same as the ordinary $T\bar{T}$ deformations, up to total derivatives and equations of motion. This result then implies that the ordinary $T\bar{T}$ deformation preserves supersymmetry and indicates, for example, how to study the $T\bar{T}$-flow of a Lagrangian in a manifestly supersymmetric way.

[1]We do not aim at reviewing here the large bulk of recent research on this subject and we simply refer to [4] for a recent, though not necessarily comprehensive, overview and list of references.

[2]The $J\bar{T}/T\bar{J}$ deformation also shares some seeming pathologies with $T\bar{T}$ deformation, such as the complex spectra in some regime. A clear understanding of those unusual properties is still not settled in the literature and would shed new light on QFTs. To better understand those issues, it might be beneficial to consider the combination of $T\bar{T}$ and $J\bar{T}/T\bar{J}$ deformations, which is also integrable, see e.g. [6].

[3]See [29] for an alternative geometric method to calculate the $T\bar{T}$-deformed action of an arbitrary (supersymmetric or non-supersymmetric) theory. When applied to the pure bosonic or fermionic theory, this method is especially powerful compared to the direct method in [3, 30]. But it is interesting to see whether one can make supersymmetry manifest in this formalism.

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As we are going to show in our paper, all these results can be generalized to the $J\bar{T}$ and $T\bar{J}$ case in a similar way. As the starting point, one needs to find the supersymmetric counterparts of the stress-energy tensor and U(1) currents. Although the former has been studied extensively, the latter, especially its conservation equation, has scattered results across the literature. Here we are going to provide a systematic construction of the flavor current multiplets and their conservation equations with $\mathcal{N} = (0,1), (1,1)$ and $(0,2)$ supersymmetries. This is done by considering the vector multiplets, coupling them to flavor currents, and inspecting their gauge invariances which yield the conservation laws of the flavor current multiplets. See appendix A for details.

In the case of $\mathcal{N} = (0,1)$, $(1,1)$ and $(0,2)$ supersymmetry, by using the stress-tensor multiplets and flavor current multiplets, we extend the analysis of [24–26] and construct various supersymmetric primary operators out of their bilinears which induce the manifestly supersymmetric $J\bar{T}/T\bar{J}$ deformations. Of particular interest are the cases of chiral supersymmetries, $\mathcal{N} = (0,1)$ and $\mathcal{N} = (0,2)$, where the supersymmetric extensions of $J\bar{T}$ and $T\bar{J}$ are structurally different. Like the $T\bar{T}$ case, a fundamental result is that the descendants of the $J\bar{T}/T\bar{J}$ primary operators coincide, on-shell and up to total derivatives, with the conventional $J\bar{T}/T\bar{J}$ operators. A central aspect of our paper is to elaborate on these results and understand in detail the properties of the $J\bar{T}/T\bar{J}$ operators.

An interesting observation arising from these analyses is that all the $T\bar{T}$, $J\bar{T}$ and $T\bar{J}$ primary operators appear to fit into the following general pattern:

$$\mathcal{O} = \mathcal{A}\mathcal{B} - s\mathcal{X}\mathcal{Y} \,. \tag{1.1}$$

Here s is a constant number and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{X}$ and $\mathcal{Y}$ are superfields satisfying the following constraints

$$\mathbb{L}\mathcal{A} = \mathbb{R}\mathcal{Y}, \qquad \mathbb{L}\mathcal{X} = \mathbb{R}\mathcal{B}\,, \tag{1.2}$$

where $\mathbb{L}$, $\mathbb{R}$ are differential operators constructed out of the superspace covariant derivatives. These generalize the Smirnov-Zamolodchikov type composite operators which corresponds to $\mathbb{L} = \partial_{--}$, $\mathbb{R} = \partial_{++}$, $s = 1$ [2]. The operator (1.1) is invariant under improvement transformation with certain assumptions as we show in appendix B. Furthermore, since the original Smirnov-Zamolodchikov composite operators were shown to be well-defined at the quantum level [2], we believe that the quantum well-definedness also holds for our generalized Smirnov-Zamolodchikov type composite operators (1.1) with appropriate s. Indeed, the well-definedness of our pattern is justified in all the cases considered so far. For $\mathcal{N} = (0,1)$, $(1,1)$ and $(0,2)$ supersymmetric $T\bar{T}$ primary operator, the well-definedness was already elaborated in [24–26]. And for the $J\bar{T}/T\bar{J}$ super-primary operators in this paper, they will also be shown to be well-defined in appendix B.

Note that, exactly as in the $T\bar{T}$ case, the equivalence of the manifestly supersymmetric and the original $J\bar{T}/T\bar{J}$ deformations ensures that, as far as the analysis of the spectrum goes, nothing changes compared to the results of [5]; for this reason we avoid to reiterate the analysis of this problem here. However, the construction of explicit $J\bar{T}/T\bar{J}$-flows for actions and their supersymmetry is largely sensible to the type of deformation we use. We will show this feature by constructing some $J\bar{T}/T\bar{J}$-deformed Lagrangians explicitly. In particular, we focus on the *chiral* $J\bar{T}/T\bar{J}$ deformations with J being a chiral U(1) current, which was argued in [5] to be the condition of solvability.[4] We thus present in our paper several examples of chiral $J\bar{T}$ and $T\bar{J}$ deformations of free actions with $\mathcal{N} = (0,1)$ and $\mathcal{N} = (0,2)$ supersymmetry.

The paper is organized as follows. In section 2, we set up the notations and review the stress-tensor multiplets with $\mathcal{N} = (0,1)$, $(1,1)$ and $(0,2)$ supersymmetry. In section 3, we present the conservation equations for flavor current multiplets which are derived in appendix A. In section 4, we construct the primary operators for $J\bar{T}/T\bar{J}$ deformations and show that their descendants coincide with the conventional $J\bar{T}/T\bar{J}$ operator. In section 5, we discuss some examples of $J\bar{T}/T\bar{J}$-deformed free theories. In section 6, we conclude and discuss possible future directions. For the reader's convenience, we relegate to two appendices main technical analyses which, however, we believe represent an important part of our results. In appendix A, we derive in a systematic way the conservation equations for the flavor current multiplets with $\mathcal{N} = (0,1)$, $(1,1)$ and $(0,2)$ supersymmetry. In appendix B, we elaborate on our observation that all the $T\bar{T}$, $J\bar{T}$ and $T\bar{J}$ deformations fit into the general pattern (1.1) which goes beyond the Smirnov-Zamolodchikov type of operators. In appendix B, we also discuss the well-definedness of the $J\bar{T}$ and $T\bar{J}$ primary operators with $\mathcal{N} = (0,1)$, $(1,1)$ and $(0,2)$ supersymmetry.

2 Stress-tensor multiplets

In this and the next section, we will introduce the stress-tensor multiplets[5] and flavor current multiplets with various amount of supersymmetries. These conserved current multiplets are the building blocks to construct supersymmetric $J\bar{T}/T\bar{J}$ operators.

This section is first devoted to reviewing the stress-tensor multiplet of two-dimensional relativistic quantum field theories. After that, since $J\bar{T}/T\bar{J}$ deformations break Lorentz invariance, we will also present the non-relativistic extensions of the stress-tensor multiplets. This section is also aiming to set up the conventions for the whole paper.

2.1 $\mathcal{N} = (0,1)$

We begin with two-dimensional quantum field theories possessing $\mathcal{N} = (0,1)$ supersymmetry. The flat 2D $\mathcal{N} = (0,1)$ superspace is parametrized by

$$\zeta^M = (\sigma^{++}, \sigma^{--}, \vartheta^+)\,, \tag{2.1}$$

with $\sigma^{\pm\pm}$ being the bosonic light-cone coordinates and ϑ^+ a real Grassmann coordinate. The spinor covariant derivatives and supercharges are given by

$$\mathcal{D}_+ = \frac{\partial}{\partial\vartheta^+} - \mathrm{i}\vartheta^+\partial_{++}\,, \qquad \mathcal{Q}_+ = \mathrm{i}\,\frac{\partial}{\partial\vartheta^+} - \vartheta^+\partial_{++}\,, \tag{2.2}$$

[4] However, see also the very recent paper [10] that solves the spectrum of general JT_a deformations by using a path integral approach.

[5] They are also commonly called supercurrent multiplets. But in order to avoid confusion with flavor current multiplets, that are also supersymmetric current multiplets and that will be introduced in the next section, we will simply call the supercurrent multiplet as stress-tensor multiplet.

and obey the anti-commutation relations

$$\{\mathcal{D}_+, \mathcal{D}_+\} = -2\mathrm{i}\partial_{++}\,, \qquad \{\mathcal{Q}_+, \mathcal{Q}_+\} = -2\mathrm{i}\partial_{++}\,, \qquad \{\mathcal{Q}_+, \mathcal{D}_+\} = 0\,. \tag{2.3}$$

Given an $\mathcal{N} = (0,1)$ superfield $\mathcal{F}(\zeta) = \mathcal{F}(\sigma, \vartheta^+)$ its supersymmetry transformation is

$$\delta_Q \mathcal{F}(\zeta) := -\mathrm{i}\epsilon_- \mathcal{Q}_+ \mathcal{F}(\zeta)\,, \tag{2.4}$$

where ϵ_- is the constant supersymmetry transformation parameter. If $F(\sigma)$ is the operator defined as the $\vartheta = 0$ component of the superfield $\mathcal{F}(\zeta)$, $F(\sigma) := \mathcal{F}(\sigma, \vartheta^+)|_{\vartheta=0}$, then its supersymmetry transformation is such that

$$\delta_Q F(\sigma) = -\mathrm{i}\epsilon_- \big[Q_+, F(\sigma)\big\} = -\mathrm{i}\epsilon_- \mathcal{Q}_+ \mathcal{F}(\sigma, \vartheta^+)\Big|_{\vartheta=0} = \epsilon_- \mathcal{D}_+ \mathcal{F}(\sigma, \vartheta^+)\Big|_{\vartheta=0}\,. \tag{2.5}$$

In our paper we will indicate with Q_+ the supersymmetry generator *acting on a component operator* while $\mathcal{Q}_+$ is the linear superspace differential operator *acting on superfields*.

For 2D $\mathcal{N} = (0,1)$ supersymmetric and Lorentz invariant theories, the stress-tensor multiplet is described by three superfields $\mathcal{T}_{----}$, $\mathcal{J}_{+++}$, and $\mathcal{J}_-$ satisfying the conservation equations:

$$\mathcal{D}_+ \mathcal{T}_{----} = \mathrm{i}\partial_{--}\mathcal{J}_-\,, \tag{2.6a}$$

$$\partial_{--}\mathcal{J}_{+++} = -\partial_{++}\mathcal{J}_-\,. \tag{2.6b}$$

See [24, 25] for derivations of these conservation equations (either through the Noether procedure or by requiring the superdiffeomorphism invariance when coupling to supergravity). In the superconformal case it holds $\mathcal{J}_-(\zeta) = 0$.

To describe the stress-tensor multiplet it is convenient to also define the following two descendant superfields

$$\mathcal{T}_{++++} := \mathcal{D}_+ \mathcal{J}_{+++}\,, \qquad \mathcal{T} := \mathcal{D}_+ \mathcal{J}_-\,. \tag{2.7}$$

They satisfy

$$\mathcal{D}_+ \mathcal{T}_{++++} = -\mathrm{i}\partial_{++}\mathcal{J}_{+++}\,, \qquad \mathcal{D}_+ \mathcal{T} = -\mathrm{i}\partial_{++}\mathcal{J}_-\,, \tag{2.8}$$

and the conservation equations

$$\partial_{++}\mathcal{T}_{----} = -\partial_{--}\mathcal{T}\,, \tag{2.9a}$$

$$\partial_{--}\mathcal{T}_{++++} = -\partial_{++}\mathcal{T}\,. \tag{2.9b}$$

The lowest $\vartheta = 0$ components of $\mathcal{T}_{++++}$, $\mathcal{T}_{----}$ and $\mathcal{T}$ describe the components of the symmetric stress-energy tensor in light-cone coordinates

$$T_{----}(\sigma) = \mathcal{T}_{----}(\zeta)|_{\vartheta=0}\,, \quad T_{++++}(\sigma) = \mathcal{T}_{++++}(\zeta)|_{\vartheta=0}\,, \quad \Theta(\sigma) = \mathcal{T}(\zeta)|_{\vartheta=0}\,, \tag{2.10}$$

while the lowest components of $\mathcal{J}_{+++}(\zeta)$ and $\mathcal{J}_-(\zeta)$ define the supersymmetry currents

$$J_{+++}(\sigma) = \mathcal{J}_{+++}(\zeta)|_{\vartheta=0}\,, \qquad J_-(\sigma) = \mathcal{J}_-(\zeta)|_{\vartheta=0}\,. \tag{2.11}$$

In components, the superfields of the stress-tensor multiplet have the following expansion

$$\mathcal{J}_{+++}(\zeta) = J_{+++}(\sigma) + \vartheta^+ T_{++++}(\sigma)\,, \tag{2.12a}$$

$$\mathcal{J}_{-}(\zeta) = J_{-}(\sigma) + \vartheta^+ \Theta(\sigma)\,, \tag{2.12b}$$

$$\mathcal{T}_{----}(\zeta) = T_{----}(\sigma) + \mathrm{i}\vartheta^+ \partial_{--} J_{-}(\sigma)\,. \tag{2.12c}$$

Due to (2.6a)–(2.9b), the operators $T_{\pm\pm\pm\pm}$, Θ, J_{+++} and J_- satisfy the conservation equations

$$\{Q_+, J_{+++}\} = \mathrm{i}T_{++++}\,, \qquad \{Q_+, J_-\} = \mathrm{i}\Theta\,, \tag{2.13a}$$

$$[Q_+, T_{++++}] = \partial_{++}J_{+++}\,, \quad [Q_+, T_{----}] = -\partial_{--}J_-\,, \qquad [Q_+, \Theta] = \partial_{++}J_-\,, \tag{2.13b}$$

$$\partial_{--}J_{+++} = -\partial_{++}J_-\,, \qquad \partial_{++}T_{----} = -\partial_{--}T\,, \quad \partial_{--}T_{++++} = -\partial_{++}T\,. \tag{2.13c}$$

2.2 $\mathcal{N} = (1,1)$

Let us now turn to $\mathcal{N} = (1,1)$ supersymmetry. The $\mathcal{N} = (1,1)$ Minkowski superspace is parametrized by the coordinates $\zeta^M = (\sigma^{++}, \sigma^{--}, \vartheta^+, \vartheta^-)$. The covariant derivatives and supercharges are defined as

$$\mathcal{D}_\pm = \frac{\partial}{\partial\vartheta^\pm} - \mathrm{i}\vartheta^\pm\partial_{\pm\pm}\,, \qquad \mathcal{Q}_\pm = \mathrm{i}\,\frac{\partial}{\partial\vartheta^\pm} - \vartheta^\pm\partial_{\pm\pm}\,, \tag{2.14}$$

and the anti-commutators read

$$\{\mathcal{D}_\pm, \mathcal{D}_\pm\} = -2\mathrm{i}\partial_{\pm\pm}\,, \qquad \{\mathcal{Q}_\pm, \mathcal{Q}_\pm\} = -2\mathrm{i}\partial_{\pm\pm}\,, \tag{2.15a}$$

$$\{\mathcal{D}_+, \mathcal{D}_-\} = \{\mathcal{D}_\pm, \mathcal{Q}_\pm\} = \{\mathcal{Q}_+, \mathcal{Q}_-\} = 0\,. \tag{2.15b}$$

The definition of the $\mathcal{N} = (1,1)$ supersymmetry transformations of an $\mathcal{N} = (1,1)$ superfield and its lowest component, and accordingly the definition of the generators $Q_\pm$ acting on component operators, is a straightforward extension of the $\mathcal{N} = (0,1)$ case, see eqs. (2.4)–(2.5), where supersymmetry transformations are parametrized by $\epsilon_\pm$ in the $\mathcal{N} = (1,1)$ case.

Field theories that are $\mathcal{N} = (1,1)$ supersymmetric and Lorentz invariant possess two pairs of superfields, $(\mathcal{J}_{+++}(\zeta), \mathcal{J}_-(\zeta))$ and $(\mathcal{J}_{---}(\zeta), \mathcal{J}_+(\zeta))$, which describe the stress-tensor multiplet. The conservation equations are encoded in the following equations (see [24, 25] for recent derivations)[6]

$$\mathcal{D}_+\mathcal{J}_{---} = \mathcal{D}_-\mathcal{J}_-\,, \qquad \mathcal{D}_-\mathcal{J}_{+++} = \mathcal{D}_+\mathcal{J}_+\,, \qquad \mathcal{D}_+\mathcal{J}_- = \mathcal{D}_-\mathcal{J}_+ := \mathcal{T}\,. \tag{2.16}$$

We define the following descendant superfields

$$\mathcal{T}_{\pm\pm\pm\pm} := \mathcal{D}_\pm\mathcal{J}_{\pm\pm\pm}\,, \qquad \mathcal{Z}_{\pm\pm} := \mathcal{D}_\pm\mathcal{J}_\pm\,. \tag{2.17}$$

Such definitions, together with eq. (2.16), imply

$$\mathcal{D}_\pm\mathcal{T}_{\pm\pm\pm\pm} = -\mathrm{i}\partial_{\pm\pm}\mathcal{J}_{\pm\pm\pm}\,, \quad \mathcal{D}_\mp\mathcal{T}_{\pm\pm\pm\pm} = \mathrm{i}\partial_{\pm\pm}\mathcal{J}_\pm\,, \qquad \mathcal{D}_\pm\mathcal{T} = -\mathrm{i}\partial_{\pm\pm}\mathcal{J}_\mp\,, \tag{2.18a}$$

$$\mathcal{D}_\pm\mathcal{Z}_{\pm\pm} = -\mathrm{i}\partial_{\pm\pm}\mathcal{J}_\pm\,, \qquad \mathcal{D}_\mp\mathcal{Z}_{\pm\pm} = \mathrm{i}\partial_{\pm\pm}\mathcal{J}_\mp\,, \tag{2.18b}$$

$$\partial_{\mp\mp}\mathcal{J}_{\pm\pm\pm} = -\partial_{\pm\pm}\mathcal{J}_\mp\,, \qquad \partial_{\pm\pm}\mathcal{T}_{\mp\mp\mp\mp} = -\partial_{\mp\mp}\mathcal{T}\,, \quad \partial_{--}\mathcal{Z}_{++} = -\partial_{++}\mathcal{Z}_{--}\,. \tag{2.18c}$$

[6] In the supergravity approach [25], it holds $\mathcal{J}_\pm(\zeta) = \mp\mathrm{i}\mathcal{D}_\pm\mathcal{J}(\zeta)$.

It is clear that $\mathcal{J}_{\pm\pm\pm}$ and $\mathcal{J}_{\pm}$ belong to a stress-tensor multiplet where $\mathcal{J}_{\pm}$ play the role of the supertrace. In fact, if the matter system is superconformal then it holds $\mathcal{J}_{\pm} = 0$. The $\vartheta^{\pm} = 0$ components of $\mathcal{J}_{\pm\pm\pm}(\zeta)$ and $\mathcal{J}_{\pm}(\zeta)$ define the supersymmetry currents $J_{\pm\pm\pm}(\sigma) := \mathcal{J}_{\pm\pm\pm}(\zeta)|_{\vartheta^{\pm}=0}$ and $J_{\pm}(\sigma) := \mathcal{J}_{\pm}(\zeta)|_{\vartheta^{\pm}=0}$, respectively. The lowest component of $\mathcal{Z}_{\pm\pm}$, $Z_{\pm\pm}(\sigma) := \mathcal{Z}_{\pm\pm}(\zeta)|_{\vartheta^{\pm}=0}$, define a central charge current. The components of the symmetric stress-energy tensor in light-cone coordinates can be defined as

$$T_{\pm\pm\pm\pm}(\sigma) := \mathcal{T}_{\pm\pm\pm\pm}|_{\vartheta^{\pm}=0} = \mathcal{D}_{\pm}\mathcal{J}_{\pm\pm\pm}|_{\vartheta^{\pm}=0}\,, \tag{2.19a}$$

$$\Theta(\sigma) := \mathcal{T}|_{\vartheta^{\pm}=0} = \mathcal{D}_{+}\mathcal{J}_{-}|_{\vartheta^{\pm}=0} = \mathcal{D}_{-}\mathcal{J}_{+}|_{\vartheta^{\pm}=0}\,. \tag{2.19b}$$

The expansion in components of $\mathcal{J}_{\pm\pm\pm}$ and $\mathcal{J}_{\pm}$ read

$$\mathcal{J}_{\pm\pm\pm}(\zeta) = J_{\pm\pm\pm}(\sigma) + \vartheta^{\pm}T_{\pm\pm\pm\pm}(\sigma) + \vartheta^{\mp}Z_{\pm\pm}(\sigma) \pm \mathrm{i}\vartheta^{+}\vartheta^{-}\,\partial_{\pm\pm}J_{\pm}(\sigma)\,, \tag{2.20a}$$

$$\mathcal{J}_{\pm}(\zeta) = J_{\pm}(\sigma) + \vartheta^{\pm}Z_{\pm\pm}(\sigma) + \vartheta^{\mp}\Theta(\sigma) \pm \mathrm{i}\vartheta^{+}\vartheta^{-}\partial_{\pm\pm}J_{\mp}(\sigma)\,. \tag{2.20b}$$

It is straightforward to prove that the operators $T_{\pm\pm\pm\pm}$, Θ, $Z_{\pm\pm}$, $J_{\pm\pm\pm}$ and $J_{\pm}$ satisfy the conservation equations

$$\{Q_{\pm}, J_{\pm\pm\pm}\} = \mathrm{i}T_{\pm\pm\pm\pm}\,, \qquad \{Q_{\pm}, J_{\mp}\} = \mathrm{i}\Theta\,, \qquad [Q_{\pm}, T_{\pm\pm\pm\pm}] = \partial_{\pm\pm}J_{\pm\pm\pm}\,, \tag{2.21a}$$

$$\{Q_{\pm}, J_{\pm}\} = \{Q_{\mp}, J_{\pm\pm\pm}\} = \mathrm{i}Z_{\pm\pm}\,, \qquad [Q_{\mp}, T_{\pm\pm\pm\pm}] = -\partial_{\pm\pm}J_{\pm}\,, \tag{2.21b}$$

$$[Q_{\pm}, \Theta] = -[Q_{\mp}Z_{\pm\pm}] = \partial_{\pm\pm}J_{\mp}\,, \qquad [Q_{\pm}, Z_{\pm\pm}] = \partial_{\pm\pm}J_{\pm}\,, \tag{2.21c}$$

$$\partial_{\mp\mp}J_{\pm\pm\pm} = -\partial_{\pm\pm}J_{\mp}\,, \quad \partial_{\mp\mp}T_{\pm\pm\pm\pm} = -\partial_{\pm\pm}T\,, \qquad \partial_{--}Z_{++} = -\partial_{++}Z_{--}\,. \tag{2.21d}$$

2.3 $\mathcal{N} = (0,2)$

Finally, we discuss the case of $\mathcal{N} = (0,2)$ supersymmetry. Its Minkowski superspace is parametrized by $\zeta^M = (\sigma^{++}, \sigma^{--}, \vartheta^{+}, \bar{\vartheta}^{+})$ with ϑ^{+} now a complex Grassmann coordinate and $\bar{\vartheta}^{+} = \overline{(\vartheta^{+})}$.

The covariant derivatives and supercharges are defined as (for later convenience in deriving the flavor current multiplet, we follow the notation in [31])

$$\mathcal{D}_{+} = \frac{\partial}{\partial\vartheta^{+}} + \mathrm{i}\bar{\vartheta}^{+}\partial_{++}\,, \qquad \bar{\mathcal{D}}_{+} = \frac{\partial}{\partial\bar{\vartheta}^{+}} + \mathrm{i}\vartheta^{+}\partial_{++}\,, \tag{2.22a}$$

$$\mathcal{Q}_{+} = \mathrm{i}\frac{\partial}{\partial\vartheta^{+}} + \bar{\vartheta}^{+}\partial_{++}, \qquad \bar{\mathcal{Q}}_{+} = \mathrm{i}\frac{\partial}{\partial\bar{\vartheta}^{+}} + \vartheta^{+}\partial_{++}\,, \tag{2.22b}$$

satisfying the following (anti-)commutation relations

$$\mathcal{D}_{+}^2 = \bar{\mathcal{D}}_{+}^2 = 0\,, \qquad \{\mathcal{D}_{+}, \bar{\mathcal{D}}_{+}\} = 2\mathrm{i}\partial_{++}\,, \qquad [\mathcal{D}_{+}, \partial_{\pm\pm}] = [\bar{\mathcal{D}}_{+}, \partial_{\pm\pm}] = 0\,, \tag{2.23}$$

with equivalent relations satisfied by $\mathcal{Q}_{+}$, $\bar{\mathcal{Q}}_{+}$ and $\partial_{\pm\pm}$.

For a Lorentz and $\mathcal{N} = (0,2)$ supersymmetric theory the general stress-tensor multiplet, or supercurrent "$\mathcal{S}$-multiplet", was studied in [32]. In terms of our notation, the

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$\mathcal{S}$-multiplet is determined by the following differential constraints[7]

$$\partial_{--}\mathcal{S}_{++} = \mathcal{D}_+\mathcal{W}_- + \bar{\mathcal{D}}_+\bar{\mathcal{W}}_- \,, \tag{2.24a}$$
$$\bar{\mathcal{D}}_+\mathcal{T}_{----} = \partial_{--}\mathcal{W}_- \,, \tag{2.24b}$$
$$\mathcal{D}_+\mathcal{T}_{----} = -\partial_{--}\bar{\mathcal{W}}_- \,, \tag{2.24c}$$
$$\bar{\mathcal{D}}_+\mathcal{W}_- = \mathcal{D}_+\bar{\mathcal{W}}_- = 0 \,. \tag{2.24d}$$

In our notation, the component expansions of the superfields $\mathcal{S}_{++}$, $\mathcal{T}_{----}$ and $\mathcal{W}_-$ solving the previous constraints are given by

$$\mathcal{T}_{----}(\zeta) = T_{----}(\sigma) + \frac{1}{2}\vartheta^+\partial_{--}S_{+--}(\sigma) - \frac{1}{2}\bar{\vartheta}^+\partial_{--}\bar{S}_{+--}(\sigma) - \frac{1}{2}\vartheta^+\bar{\vartheta}^+\partial^2_{--}j_{++}(\sigma) \,, \tag{2.25a}$$
$$\mathcal{S}_{++}(\zeta) = j_{++}(\sigma) + \mathrm{i}\vartheta^+S_{+++}(\sigma) + \mathrm{i}\bar{\vartheta}^+\bar{S}_{+++}(\sigma) + 2\vartheta^+\bar{\vartheta}^+T_{++++}(\sigma) \,, \tag{2.25b}$$
$$\mathcal{W}_-(\zeta) = -\frac{1}{2}\bar{S}_{+--}(\sigma) - \mathrm{i}\vartheta^+\left[\Theta(\sigma) + \frac{\mathrm{i}}{2}\partial_{--}j_{++}(\sigma)\right] - \frac{\mathrm{i}}{2}\vartheta^+\bar{\vartheta}^+\partial_{++}\bar{S}_{+--}(\sigma) \,. \tag{2.25c}$$

The operators $T_{\pm\pm\pm\pm}$ and Θ are the light-cone components of the symmetric stress-tensor while $S_{+\pm\pm}$ and its conjugate $\bar{S}_{+\pm\pm}$ are the $\mathcal{N} = (0, 2)$ supersymmetry currents. They satisfy the conservation equation

$$\partial_{\mp\mp}T_{\pm\pm\pm\pm} = -\partial_{\pm\pm}\Theta \,, \qquad \partial_{++}S_{+--} = -\partial_{--}S_{+++} \,. \tag{2.26}$$

Note that, by using (2.24)–(2.25), as for the $\mathcal{N} = (0, 1)$ and $\mathcal{N} = (1, 1)$ cases, it is straightforward to compute the action of the Q_+ and $\bar{Q}_+$ generators on the component fields of the $\mathcal{S}$-multiplet.[8]

We also define the descendant superfields

$$\mathcal{T}_{++++} := \frac{1}{4}[\bar{\mathcal{D}}_+, \mathcal{D}_+]\mathcal{S}_{++} \,, \qquad \mathcal{T} := \frac{\mathrm{i}}{2}\Big(\mathcal{D}_+\mathcal{W}_- - \bar{\mathcal{D}}_+\bar{\mathcal{W}}_-\Big) \,, \tag{2.27}$$

whose lowest components are T_{++++} and Θ.

One can then check that it holds:

$$\mathcal{D}_+\Big(\partial_{--}\mathcal{S}_{++} - 2\mathrm{i}\mathcal{T}\Big) = 0 \,, \qquad \bar{\mathcal{D}}_+\Big(\partial_{--}\mathcal{S}_{++} + 2\mathrm{i}\mathcal{T}\Big) = 0 \,, \tag{2.28}$$

and

$$[\bar{\mathcal{D}}_+, \mathcal{D}_+]\mathcal{T} = \partial_{++}\partial_{--}\mathcal{S}_{++} \,. \tag{2.29}$$

The $\mathcal{S}$-multiplet is in general reducible [32]. For instance, for $\mathcal{N} = (0, 2)$ supersymmetric theories admitting a conserved $\mathrm{U}(1)_R$ R-symmetry, the $\mathcal{S}$-multiplet can be improved to

[7] Note that for simplicity we set to zero the $\mathcal{S}$-multiplet space-filling brane charge appearing in the constraint $\bar{\mathcal{D}}_+\mathcal{W}_- = C$ since it is linked to supersymmetry breaking [32, 33].

[8] Here for a superfield $\mathcal{F}(\zeta)$ with lowest component $F(\sigma) = \mathcal{F}(\zeta)|_{\vartheta=0}$, the supersymmetry transformations act on $F(\sigma)$ as $\delta_Q F(\sigma) = -\mathrm{i}\big[\epsilon_- Q_+ + \bar{\epsilon}_-\bar{Q}_+, F(\sigma)\big\} = \delta_Q\mathcal{F}|_{\vartheta=0}$ with $\delta_Q\mathcal{F}(\zeta) := -\mathrm{i}\big(\epsilon_-\mathcal{Q}_+ + \bar{\epsilon}_-\bar{\mathcal{Q}}_+\big)\mathcal{F}(\zeta)$.

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the so-called $\mathcal{R}$-multiplet. In this case, the superfield currents $\mathcal{W}_-(\zeta)$ and $\bar{\mathcal{W}}_-(\zeta)$ are the descendants of a real superfield $\mathcal{R}_{--}(\zeta)$

$$\mathcal{W}_- = \frac{\mathrm{i}}{2}\bar{\mathcal{D}}_+\mathcal{R}_{--}\,, \qquad \bar{\mathcal{W}}_- = \frac{\mathrm{i}}{2}\mathcal{D}_+\mathcal{R}_{--}\,. \tag{2.30}$$

Then, once we redefine $\mathcal{S}_{++}(\zeta) \equiv \mathcal{R}_{++}(\zeta)$ for the $\mathcal{R}$-multiplet, the conservation equations (2.24) turn into

$$\partial_{--}\mathcal{R}_{++} = -\partial_{++}\mathcal{R}_{--}\,, \tag{2.31a}$$

$$\bar{\mathcal{D}}_+\Big(\mathcal{T}_{----} - \frac{\mathrm{i}}{2}\partial_{--}\mathcal{R}_{--}\Big) = \mathcal{D}_+\Big(\mathcal{T}_{----} + \frac{\mathrm{i}}{2}\partial_{--}\mathcal{R}_{--}\Big) = 0\,. \tag{2.31b}$$

The conserved vector R-symmetry current is then given by the component operators $j_{++}(\sigma) = \mathcal{R}_{++}(\zeta)|_{\vartheta=0}$ and $j_{--}(\sigma) = \mathcal{R}_{--}(\zeta)|_{\vartheta=0}$ such that

$$\partial_{--}j_{++} = -\partial_{++}j_{--}\,. \tag{2.32}$$

See [32] for more detail and [26] for a recent derivation by using $\mathcal{N} = (0, 2)$ supergravity. For the $\mathcal{R}$-multiplet, note that following useful relation, which derive from (2.31), also hold

$$\mathcal{T} = \frac{1}{4}[\bar{\mathcal{D}}_+, \mathcal{D}_+]\mathcal{R}_{--}\,, \qquad [\bar{\mathcal{D}}_+, \mathcal{D}_+]\mathcal{T}_{----} = \partial_{++}\partial_{--}\mathcal{R}_{--}\,. \tag{2.33}$$

To conclude, note that if the field theory is $\mathcal{N} = (0, 2)$ superconformal then it holds $\mathcal{W}_- = \bar{\mathcal{W}}_- = 0$ and the $\mathcal{S}$-multiplet is accordingly further simplified.

2.4 Caveat on the non-Lorentz-invariant case

In the previous subsections, we have reviewed the stress-tensor multiplets of relativistic quantum field theories possessing various types of supersymmetries. However, since $T\bar{J}$ and $J\bar{T}$ deformations break Lorentz invariance, the deformed theory is not Lorentz invariant any longer. For this reason, in this subsection we are going to extend the description of the stress-tensor multiplets to non-Lorentz-invariant supersymmetric field theories.

Given a supersymmetric theory, since translational and supersymmetry invariance are always preserved, according to the Noether theorem, the stress-energy tensor and supercharges are always well-defined and conserved. However, if Lorentz invariance is missing, the stress-energy tensor is no longer symmetric — in light-cone coordinates the two off-diagonal components of the stress-energy tensor

$$\Theta(\sigma) = T_{++--}(\sigma)\,, \qquad \tilde{\Theta}(\sigma) := T_{--++}(\sigma)\,, \tag{2.34}$$

are independent $\Theta(\sigma) \neq \tilde{\Theta}(\sigma)$. Translation invariance implies the conservation equations for the pairs of currents (T_{++++}, Θ) and $(T_{----}, \tilde{\Theta})$ separately

$$\partial_{--}T_{++++} = -\partial_{++}\Theta\,, \qquad \partial_{++}T_{----} = -\partial_{--}\tilde{\Theta}\,. \tag{2.35}$$

If Θ or $\tilde{\Theta}$ are zero then the field theory possesses a chiral $\mathrm{SL}(2, \mathbb{R})$ symmetry, which enhances to a chiral Virasoro algebra. Despite these differences, it is straightforward to extend the analysis of the supersymmetric stress-tensor multiplets. In fact, as we are now going to describe, up to appropriately distinguishing T_{++--} and T_{--++}, we can effectively use the results of the relativistic field theories.

Non-Lorentz-invariant $\mathcal{N} = (0,1)$. When there are Lorentz anomalies, the supercurrents were discussed, for example, in [34]. The conservation equations corresponding to different symmetries are given by:

1. Translation invariance:

$$\mathcal{D}_+\mathcal{T}_{----} = \mathrm{i}\partial_{--}\tilde{\mathcal{J}}_- \,, \tag{2.36a}$$

$$\partial_{--}\mathcal{J}_{+++} = -\partial_{++}\mathcal{J}_- \;. \tag{2.36b}$$

2. Dilatation invariance:

$$\tilde{\mathcal{J}}_- + \mathcal{J}_- = 0 \;. \tag{2.37}$$

3. Lorentz invariance:

$$\tilde{\mathcal{J}}_- - \mathcal{J}_- = 0 \;. \tag{2.38}$$

Hence, if we require Lorentz invariance, we obtain exactly the conservation equations we discussed before for relativistic quantum field theory.

To consider the deformation of non-Lorentz invariant field theory, we can only use the first two equations (2.36a) and (2.36b) and remain with an independent set of superfield currents given by $(\mathcal{T}_{----}(\zeta),\, \mathcal{J}_{+++}(\zeta),\, \mathcal{J}_-(\zeta),\, \tilde{\mathcal{J}}_-(\zeta))$. In this case, the stress-energy tensor is not symmetric:

$$\Theta \equiv \mathcal{T}|_{\vartheta=0} \neq \tilde{\mathcal{T}}|_{\vartheta=0} \equiv \tilde{\Theta} \,, \tag{2.39}$$

with $\mathcal{T} = \mathcal{D}_+\mathcal{J}_-$, $\tilde{\mathcal{T}} = \mathcal{D}_+\tilde{\mathcal{J}}_-$.

Non-Lorentz-invariant $\mathcal{N} = (1,1)$. Similarly, in the $\mathcal{N} = (1,1)$ case, the conservation equations corresponding to different symmetries are given by:

1. Translation invariance:

$$\mathcal{D}_+\mathcal{J}_{---} = -\partial_{--}\tilde{\mathcal{J}} \,, \tag{2.40a}$$

$$\mathcal{D}_-\mathcal{J}_{+++} = -\partial_{++}\mathcal{J} \;. \tag{2.40b}$$

2. Dilatation invariance:

$$\tilde{\mathcal{J}} + \mathcal{J} = 0 \;. \tag{2.41}$$

3. Lorentz invariance:

$$\tilde{\mathcal{J}} - \mathcal{J} = 0 \;. \tag{2.42}$$

For Lorentz invariant theory, we thus have $\tilde{\mathcal{J}} - \mathcal{J} = 0$, and (2.40a), (2.40b) reduce to (2.16) with $\mathcal{J}_\pm = \mp\mathrm{i}\mathcal{D}_\pm\mathcal{J}$. While in our non-Lorentz invariant field theories, we should use (2.40a) and (2.40b) while keeping $(\tilde{\mathcal{J}}_-(\zeta) \neq \mathcal{J}_-(\zeta))$ $\tilde{\mathcal{J}} \neq \mathcal{J}$.

Non-Lorentz-invariant $\mathcal{N}=(0,2)$. To accommodate the fact that for non-Lorentz invariant theories the stress-energy tensor is not necessarily symmetric, it turns out that the $(0,2)$ $\mathcal{S}$-multiplet constraints (2.24) should be modified as follows

$$\partial_{--}\mathcal{S}_{++} = \mathcal{D}_+\mathcal{W}_- + \bar{\mathcal{D}}_+\bar{\mathcal{W}}_- \,, \tag{2.43a}$$
$$\bar{\mathcal{D}}_+\mathcal{T}_{----} = \partial_{--}\widetilde{\mathcal{W}}_- \,, \tag{2.43b}$$
$$\mathcal{D}_+\mathcal{T}_{----} = -\partial_{--}\bar{\widetilde{\mathcal{W}}}_- \,, \tag{2.43c}$$
$$\bar{\mathcal{D}}_+\mathcal{W}_- = \mathcal{D}_+\bar{\mathcal{W}}_- = \bar{\mathcal{D}}_+\widetilde{\mathcal{W}}_- = \mathcal{D}_+\bar{\widetilde{\mathcal{W}}}_- = 0\,, \tag{2.43d}$$

with

$$\mathcal{W}_- = -\mathrm{i}\vartheta^+\Big(\Theta + \frac{\mathrm{i}}{2}\partial_{--}j_{++}\Big) + \cdots\,, \qquad \bar{\mathcal{W}}_- = \mathrm{i}\bar{\vartheta}^+\Big(\Theta - \frac{\mathrm{i}}{2}\partial_{--}j_{++}\Big) + \cdots\,, \tag{2.44a}$$
$$\widetilde{\mathcal{W}}_- = -\mathrm{i}\vartheta^+\Big(\tilde{\Theta} + \frac{\mathrm{i}}{2}\partial_{--}j_{++}\Big) + \cdots\,, \qquad \bar{\widetilde{\mathcal{W}}}_- = \mathrm{i}\bar{\vartheta}^+\Big(\tilde{\Theta} - \frac{\mathrm{i}}{2}\partial_{--}j_{++}\Big) + \cdots\,. \tag{2.44b}$$

For the multiplet, the dots above, and all other superfields, are the same as the ones in (2.25). The $(0,2)$ $\mathcal{R}$-multiplet for a non-Lorentz invariant theory can similarly be derived in a straightforward way from the Lorentz invariant case and we leave to the reader the details for its derivation.

It is crucial to emphasize that these modifications for non-Lorentz-invariant theories are actually not needed for our $T\bar{J}/J\bar{T}$-deformations. In fact, as we will see later in our analysis, these composite operators always involve *only one* off-diagonal component, either Θ or $\tilde{\Theta}$.[9]

Since this difference proves to be irrelevant for our analysis, we will "pretend" to be working in relativistic theories, namely we will just use the normal conservation equations described in the last subsections without using the tildes when we refer to $\tilde{\mathcal{J}}_-, \tilde{\mathcal{J}}$ or $\widetilde{\mathcal{W}}_-$.

3 Flavour current multiplets

To construct the supersymmetric $T\bar{J}/J\bar{T}$ primary operators, we also need to derive the supercurrent multiplet for a gauge/flavour symmetry. For simplicity, we will restrict to the Abelian case with U(1) symmetry.

The flavor current multiplet can be found in a standard fashion as follows. For a given amount of supersymmetry, we first need to find the gauge multiplet as well as their gauge transformations rules, then we couple the gauge multiplet to the corresponding flavor current multiplet. To linearized order, the gauge invariance of gauge-current couplings gives rise to the conservation equations of the flavor current multiplets. We defer the details of the derivations to appendix A and here present only the final results for the conservation equations.

[9] Though it will not play a role in our discussion, note that the difference between Θ and $\tilde{\Theta}$ might instead be relevant for other types of composite operators, for example the $T\bar{T}$-operator in non-Lorentz invariant field theories discussed in [35].

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3.1 $\mathcal{N} = (0,1)$

For quantum field theories with $\mathcal{N} = (0,1)$ supersymmetry, as derived in appendix A.1, the flavor current multiplet of an Abelian symmetry consists of two superfields $\mathcal{G}_{--}(\zeta)$ and $\mathcal{G}_{+}(\zeta)$ satisfying the following constraint:

$$\mathcal{D}_+\mathcal{G}_{--} = \mathrm{i}\partial_{--}\mathcal{G}_+ \ . \tag{3.1}$$

If we define

$$\mathcal{G}_{++} := \mathcal{D}_+\mathcal{G}_+ \, , \tag{3.2}$$

then we have

$$\mathcal{D}_+\mathcal{G}_{++} = -\mathrm{i}\partial_{++}\mathcal{G}_+ \, , \tag{3.3}$$

and

$$\partial_{++}\mathcal{G}_{--} = -\partial_{--}\mathcal{G}_{++} \ . \tag{3.4}$$

In components, the current multiplet is given by

$$\mathcal{G}_{--}(\zeta) = G_{--}(\sigma) + \mathrm{i}\vartheta^+\partial_{--}g_+(\sigma) \, , \qquad \mathcal{G}_+(\zeta) = g_+(\sigma) + \vartheta^+ G_{++}(\sigma) \, , \tag{3.5}$$

whose components $G_{\pm\pm}(\sigma)$ satisfy the conservation equation for a vector current

$$\partial_{++}G_{--} + \partial_{--}G_{++} = 0 \ . \tag{3.6}$$

3.2 $\mathcal{N} = (1,1)$

As shown in appendix A.2, the flavor current multiplet consists of two superfields $\mathcal{G}_-(\zeta)$ and $\mathcal{G}_+(\zeta)$ satisfying the following constraint:

$$\mathcal{D}_+\mathcal{G}_- - \mathcal{D}_-\mathcal{G}_+ = 0 \ . \tag{3.7}$$

Acting with $\mathcal{D}_+\mathcal{D}_-$ on both sides of the previous equation gives

$$\partial_{++}\mathcal{G}_{--} + \partial_{--}\mathcal{G}_{++} = 0 \, , \tag{3.8}$$

where we have defined the following descendant superfields

$$\mathcal{G}_{++} = \mathcal{D}_+\mathcal{G}_+ \, , \qquad \mathcal{G}_{--} = \mathcal{D}_-\mathcal{G}_- \ . \tag{3.9}$$

The flavor current multiplet can be expressed in terms of component fields as:

$$\mathcal{G}_+(\zeta) = g_+(\sigma) + \vartheta^+ G_{++}(\sigma) + \vartheta^- p(\sigma) + \mathrm{i}\vartheta^+\vartheta^-\partial_{++}g_-(\sigma) \, , \tag{3.10a}$$

$$\mathcal{G}_-(\zeta) = g_-(\sigma) + \vartheta^- G_{--}(\sigma) + \vartheta^+ p(\sigma) + \mathrm{i}\vartheta^-\vartheta^+\partial_{--}g_+(\sigma) \, , \tag{3.10b}$$

where $G_{\pm\pm}(\sigma)$ are the components of a vector current field

$$\partial_{++}G_{--} + \partial_{--}G_{++} = 0 \ . \tag{3.11}$$

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3.3 $\mathcal{N} = (0,2)$

Finally, the flavor current multiplet for $\mathcal{N} = (0,2)$ supersymmetric theories is derived in the appendix A.3. It contains two real superfields $\mathcal{G}(\zeta)$ and $\mathcal{G}_{--}(\zeta)$ satisfying the constraints:

$$\bar{\mathcal{D}}_+(\mathcal{G}_{--} - \mathrm{i}\partial_{--}\mathcal{G}) = 0\,, \qquad \mathcal{D}_+(\mathcal{G}_{--} + \mathrm{i}\partial_{--}\mathcal{G}) = 0\ . \tag{3.12}$$

These two equations are conjugate to each other.

If we define the following descendant superfield

$$\mathcal{G}_{++} = -\frac{1}{2}[\mathcal{D}_+, \bar{\mathcal{D}}_+]\mathcal{G}\,, \tag{3.13}$$

then we have the conservation equation

$$\partial_{++}\mathcal{G}_{--} + \partial_{--}\mathcal{G}_{++} = 0\,, \tag{3.14}$$

together with

$$[\mathcal{D}_+, \bar{\mathcal{D}}_+]\mathcal{G}_{--} = -2\partial_{++}\partial_{--}\mathcal{G}\ . \tag{3.15}$$

The flavor current multiplet is then described by the following decomposition in component fields

$$\mathcal{G}(\zeta) = g(\sigma) + \mathrm{i}\vartheta^+ p_+(\sigma) + \mathrm{i}\bar{\vartheta}^+ \bar{p}_+(\sigma) + \vartheta^+\bar{\vartheta}^+ G_{++}(\sigma)\,, \tag{3.16a}$$

$$\mathcal{G}_{--}(\zeta) = G_{--}(\sigma) + \vartheta^+\partial_{--}p_+(\sigma) - \bar{\vartheta}^+\partial_{--}\bar{p}_+(\sigma) + \vartheta^+\bar{\vartheta}^+\partial_{--}\partial_{++}g(\sigma)\,, \tag{3.16b}$$

where

$$\partial_{++}G_{--} + \partial_{--}G_{++} = 0\,, \tag{3.17}$$

which is just the lowest component projection of (3.14), and indicates, once more, that $G_{\pm\pm}(\sigma)$ are the components of a vector current field.

4 Supersymmetric $J\bar{T}$ and $T\bar{J}$ primary operators

Let us first recall that, in light-cone notation, the standard $T\bar{J}$ and $J\bar{T}$ composite operators are defined as [5]

$$O^{T\bar{J}}_{--}(\sigma) := T_{----}(\sigma)G_{++}(\sigma) - \Theta(\sigma)G_{--}(\sigma)\,, \tag{4.1a}$$

$$O^{J\bar{T}}_{++}(\sigma) := T_{++++}(\sigma)G_{--}(\sigma) - \Theta(\sigma)G_{++}(\sigma)\ . \tag{4.1b}$$

These two operators may be quite different in theories that are not parity invariant. This will indeed be the case for theories with chiral supersymmetry, such as $\mathcal{N} = (0,1)$ and $\mathcal{N} = (0,2)$, that we are going to consider in our paper.

As already emphasized, $T\bar{J}$ and $J\bar{T}$ deformations break Lorentz invariance. This implies that the stress-energy tensor is not symmetric anymore, $T_{++--} \neq T_{--++}$. Hence, the component Θ in the above two equations has two different meanings: in (4.1a), $\Theta = T_{++--}$, while in (4.1b), $\Theta = T_{--++}$ where the latter was defined as $\tilde{\Theta}$ in (2.34). As already mentioned before, since T_{++--}, T_{--++} never appear simultaneously, in the following analysis we can forget about tildes. We only need to make sure that the correct Θ is used and satisfies the appropriate conservation equations.

In this section, we will show that the $O^{T\bar{J}}_{--}$ and $O^{J\bar{T}}_{++}$ operators preserve supersymmetry in complete analogy with the $T\bar{T}$ case of [24–26]. More precisely, we will make use of the stress-tensor multiplets and flavor current multiplets introduced in the previous section to construct supersymmetric primary $J\bar{T}$ and $T\bar{J}$ operators and show that the $O^{T\bar{J}}_{--}$ and $O^{J\bar{T}}_{++}$ operators are supersymmetric descendants of the primary ones (up to total derivatives and equations of motion). Note that in the appendix B.2 we discuss well-definedness properties of all the supersymmetric primary operators given in our paper. Interestingly, as already indicated in the introduction and elaborated in more detail in appendix B, it turns out that all the primary operators fit into a general pattern which extends the original analysis of [2] and the supersymmetric extensions of [24–26].

4.1 $\mathcal{N} = (0, 1)$

In section 2.1 and section 3.1, we have presented the structure of the $\mathcal{N} = (0, 1)$ stress-tensor multiplet and the $\mathcal{N} = (0, 1)$ flavor current multiplet as well as their conservation equations. With these ingredients, we can immediately construct two different bilinear superfields that work as supersymmetric primary operators for $T\bar{J}$ and $J\bar{T}$ in (4.1a). They are

$$\mathcal{O}^{J\bar{T}}_{+}(\zeta) := \mathcal{J}_{+++}(\zeta)\mathcal{G}_{--}(\zeta) - \mathcal{J}_{-}(\zeta)\mathcal{G}_{++}(\zeta)\,, \tag{4.2a}$$
$$\mathcal{O}^{T\bar{J}}_{---}(\zeta) := \mathcal{T}_{----}(\zeta)\mathcal{G}_{+}(\zeta) - \mathcal{J}_{-}(\zeta)\mathcal{G}_{--}(\zeta)\;. \tag{4.2b}$$

From these, it is in fact possible to construct the manifestly supersymmetric operators described by the following descendants

$$\mathsf{O}^{J\bar{T}}_{++}(\sigma) = \int \mathrm{d}\vartheta^{+}\,\mathcal{O}^{J\bar{T}}_{+}(\zeta)\,, \tag{4.3a}$$

and

$$\mathsf{O}^{T\bar{J}}_{--}(\sigma) = \int \mathrm{d}\vartheta^{+}\,\mathcal{O}^{T\bar{J}}_{---}(\zeta)\;. \tag{4.3b}$$

These, up to conservation equations and total derivatives, prove to be equivalent to the $O^{T\bar{J}}_{--}$ and $O^{J\bar{T}}_{++}$ operators. As explained in more details for the $T\bar{T}$ case in [25], this equivalence defines precisely how $T\bar{J}$ and $J\bar{T}$ deformations preserve $\mathcal{N} = (0, 1)$ supersymmetry. Similar results will hold for the $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (0, 2)$ cases.

Let us start with the $J\bar{T}$ primary operator (4.2a). One can straightforwardly compute its descendant and obtain the following result

$$\begin{aligned}\mathcal{D}_{+}\mathcal{O}^{J\bar{T}}_{+} &= \mathcal{T}_{++++}\mathcal{G}_{--} - \mathcal{T}\mathcal{G}_{++} + \mathrm{i}\big(\partial_{--}\mathcal{J}_{+++} + \partial_{++}\mathcal{J}_{-}\big)\mathcal{G}_{+} + \mathcal{J}_{-}\big(\mathcal{D}_{+}\mathcal{G}_{++} + \mathrm{i}\partial_{++}\mathcal{G}_{+}\big)\\ &\quad -\mathcal{J}_{+++}\big(\mathcal{D}_{+}\mathcal{G}_{--} - \mathrm{i}\partial_{--}\mathcal{G}_{+}\big) - \mathrm{i}\partial_{--}\big(\mathcal{J}_{+++}\mathcal{G}_{+}\big) - \mathrm{i}\partial_{++}\big(\mathcal{J}_{-}\mathcal{G}_{+}\big)\;. \end{aligned}\tag{4.4}$$

It is easy to recognize that the quantities in the first three brackets are exactly the conservation equations of the stress-tensor and flavor current multiplets while the last two terms are just total derivatives that do not contribute once one integrates over the $\sigma^{\pm\pm}$ bosonic coordinates. On the other hand, the lowest $\vartheta^{+} = 0$ component of the first two terms in (4.4) are precisely the $J\bar{T}$ operator, $O^{J\bar{T}}_{++}(\sigma)$. Therefore, up to total derivatives and equations of motion, the descendant of the primary operator (4.2a) is exactly the standard $T\bar{J}$ operator in (4.1a):

$$\mathsf{O}^{J\bar{T}}_{++}(\sigma) = \mathcal{D}_+\mathcal{O}^{J\bar{T}}_{+}(\zeta)|_{\vartheta^+=0} + \text{total derivatives} = O^{J\bar{T}}_{++}(\sigma) + \text{EoMs} + \text{total derivatives}\,. \quad (4.5)$$

Here "EoMs" means those quantities which vanish once the equations of motion, or more precisely the conservation equations, are used.

For the $T\bar{J}$ case one can similarly compute

$$\begin{aligned}\mathcal{D}_+\mathcal{O}^{T\bar{J}}_{---} &= \mathcal{T}_{----}\mathcal{G}_{++} - \mathcal{T}\mathcal{G}_{--} + \left(\mathcal{D}_+\mathcal{T}_{----} - \mathrm{i}\partial_{--}\mathcal{J}_-\right)\mathcal{G}_+ + \mathcal{T}_{----}\left(\mathcal{D}_+\mathcal{G}_+ - \mathcal{G}_{++}\right)\\ &\quad + \mathcal{J}_-\left(\mathcal{D}_+\mathcal{G}_{--} - \mathrm{i}\partial_{--}\mathcal{G}_+\right) + \mathrm{i}\partial_{--}\left(\mathcal{J}_-\mathcal{G}_+\right)\,. \end{aligned} \quad (4.6)$$

Again, the quantities in the first three brackets are exactly the conservation equations, thus vanish on-shell. Therefore, exactly as in the $J\bar{T}$ case, it holds

$$\mathsf{O}^{T\bar{J}}_{--}(\sigma) = O^{T\bar{J}}_{--}(\sigma) + \text{EoMs} + \text{total derivatives}\,. \quad (4.7)$$

Remember that Smirnov and Zamolodchikov, by extending the analysis by Zamolodchikov for $T\bar{T}$ deformations [1], have proven that, given any pairs of currents $(A_s,\, B_{s+2})$ and $(A'_{s'},\, B'_{s'-2})$ satisfying the conservation equations

$$\partial_{++}A_s = -\partial_{--}B_{s+2}\,, \qquad \partial_{--}A'_{s'} = -\partial_{++}B'_{s'-2}\,, \quad (4.8)$$

where s and s' label the spins of the operators, then the following bilinear operators

$$O^{\mathrm{SZ}}_{s+s'}(\sigma) := A_s(\sigma)\,A'_{s'}(\sigma) - B_{s+2}(\sigma)\,B_{s'-2}(\sigma)\,, \quad (4.9)$$

can be proven to be free of short distance singularities and well defined by a point splitting procedure [2]. Both $O^{J\bar{T}}_{++}(\sigma)$ and $O^{T\bar{J}}_{--}(\sigma)$ are Smirnov-Zamolodchikov operators. Note that the structure of $\mathcal{O}^{\bar{T}J}_{+}(\zeta)$ (4.2a) is the one of a Smirnov-Zamolodchikov type of operator given in (4.9). This implies that, exactly as the $\mathcal{N} = (0,1)$ $T\bar{T}$ primary operator introduced in [24, 25], $\mathcal{O}^{\bar{T}J}_{+}(\zeta)$, despite being a composite irrelevant operator, is free of short distance singularities and well defined by a point splitting procedure as for the analysis in [2]. Interestingly, the $\mathcal{O}^{T\bar{J}}_{---}(\zeta)$ is not of Smirnov-Zamolodchikov type. Despite that, as described in appendix B.2, one can show that $\mathcal{O}^{T\bar{J}}_{---}(\zeta)$ is also well defined, in complete analogy to the analysis of [26] where the $\mathcal{N} = (0,2)$ $T\bar{T}$ operator was shown to be well defined even though not being of Smirnov-Zamolodchikov type.[10]

4.2 $\mathcal{N} = (1,1)$

From the stress-tensor multiplet and flavor current multiplet in subsections 2.2 and 3.2, we can construct the following primary operator

$$\mathcal{O}^{J\bar{T}}_{++}(\zeta) = \mathcal{J}_{+++}(\zeta)\mathcal{G}_-(\zeta) + \mathcal{J}_+(\zeta)\mathcal{G}_+(\zeta)\,. \quad (4.10)$$

[10] See also [27] for the $\mathcal{N} = (2,2)$ case which is also described by a $T\bar{T}$ primary operator that is not of Smirnov-Zamolodchikov type.

By using conservation equations, a straightforward calculation gives

$$\mathcal{D}_-\mathcal{D}_+\mathcal{O}_{++} = \mathcal{T}_{++++}\mathcal{G}_{--} - \mathcal{T}\mathcal{G}_{++} + \text{total derivatives } + \text{EoMs} \,. \tag{4.11}$$

In complete analogy to the $\mathcal{N} = (0,1)$ case of the previous subsection, this result implies that the $J\bar{T}$ operator is equivalent to the descendant of the operator $\mathcal{O}^{J\bar{T}}_{++}$, up to conservation equations and total derivatives:

$$O^{J\bar{T}}_{++}(\sigma) = \int \mathrm{d}\vartheta^-\mathrm{d}\vartheta^+\, \mathcal{O}^{J\bar{T}}_{++}(\zeta) + \text{EoMs} + \text{total derivatives} \,. \tag{4.12}$$

For the $T\bar{J}$ case we can construct the following primary operator

$$\mathcal{O}^{T\bar{J}}_{--}(\zeta) = \mathcal{J}_{---}(\zeta)\mathcal{G}_+(\zeta) + \mathcal{J}_-(\zeta)\mathcal{G}_-(\zeta) \,. \tag{4.13}$$

In complete analogy to all the other cases considered so far, one can prove the equivalence of its descendant operator with the $O^{T\bar{J}}_{--}(\sigma)$:

$$O^{T\bar{J}}_{--}(\sigma) = \int \mathrm{d}\vartheta^-\mathrm{d}\vartheta^+\, \mathcal{O}^{T\bar{J}}_{--}(\zeta) + \text{EoMs} + \text{total derivatives} \,. \tag{4.14}$$

Note that, since $\mathcal{N} = (1,1)$ supersymmetry is left-right symmetric, the two $T\bar{J}$ and $J\bar{T}$ primary operators above are simply related through a parity transformation which exchanges the left and right moving sectors.

To conclude this subsection, note also that in the $(1,1)$ case both these operators are not of Smirnov-Zamolodchikov type. Despite that, as described in appendix B.2, it is once more possible to use the arguments originally presented in [26] and show that $T\bar{J}$ and $J\bar{T}$ primary operators are both well defined.

4.3 $\mathcal{N} = (0,2)$

Finally we can turn to discuss the $T\bar{J}/J\bar{T}$ supersymmetric primary operators in $\mathcal{N} = (0,2)$ theories that are constructed as bilinears of the stress-tensor multiplet and flavor current multiplet given in subsection 2.3 and 3.3. The well-definedness of various operators is analyzed in appendix B.2.

- **$\mathcal{N} = (0,2)$ $J\bar{T}$.** For the $J\bar{T}$ case, we can naturally construct the following primary operator

$$\mathcal{O}^{J\bar{T}}(\zeta) = \mathcal{S}_{++}(\zeta)\mathcal{G}_{--}(\zeta) - 2\mathcal{G}(\zeta)\mathcal{T}(\zeta) \,. \tag{4.15}$$

It is easy to check that it holds

$$\frac{1}{4}[\bar{\mathcal{D}}_+, \mathcal{D}_+]\mathcal{O}^{J\bar{T}}(\zeta) = \mathcal{T}_{++++}(\zeta)\mathcal{G}_{--}(\zeta) - \mathcal{T}(\zeta)\mathcal{G}_{++}(\zeta) + \text{EoMs} + \text{total derivatives} \,, \tag{4.16}$$

which implies

$$O^{J\bar{T}}_{++}(\sigma) = \frac{1}{2}\int \mathrm{d}\bar{\vartheta}^+\mathrm{d}\vartheta^+\, \mathcal{O}^{J\bar{T}}(\zeta) + \text{EoMs} + \text{total derivatives} \,, \tag{4.17}$$

as expected.

By remembering from eq. (2.27) that $\mathcal{T} := \frac{\mathrm{i}}{2}\big(\mathcal{D}_+ W_- - \bar{\mathcal{D}}_+ \bar{W}_-\big)$, it is clear that, up to terms that are $\mathcal{D}_+$ and/or $\bar{\mathcal{D}}_+$ acting on a superfield, the second term in (4.15) can be written in different equivalent ways while preserving the main result (4.17). In fact, for $\mathcal{N} = (0,2)$ theories with an R-symmetry there is a very natural variant definition of the supersymmetric primary operator in terms of the $\mathcal{R}$-multiplet superfields $\mathcal{R}_{\pm\pm}$. This is given by the following operator

$$\mathcal{O}^{J\bar{T}}_{\mathcal{R}}(\zeta) = \mathcal{R}_{++}(\zeta)\mathcal{G}_{--}(\zeta) - \mathcal{R}_{--}(\zeta)\mathcal{G}_{++}(\zeta)\,, \tag{4.18}$$

which is such that $\mathcal{O}^{J\bar{T}}_{\mathcal{R}}(\zeta) = \mathcal{O}^{J\bar{T}}(\zeta) + \mathcal{D}_+(\cdots) + \bar{\mathcal{D}}_+(\cdots)$ and clearly also satisfies eq. (4.17). Note that $\mathcal{O}^{J\bar{T}}_{\mathcal{R}}(\zeta)$ in (4.18) is of Smirnov-Zamolodchikov type.

- $\boldsymbol{\mathcal{N} = (0,2)\ T\bar{J}}$**.** For simplicity, in the $T\bar{J}$ case let us start directly from an $\mathcal{N} = (0,2)$ supersymmetric theory possessing an R-symmetry. After that, we will extend the $\mathcal{R}$-multiplet results to the general case in which the stress-tensor multiplet is an $\mathcal{S}$-multiplet.

Assuming the existence of an $\mathcal{R}$-multiplet, we can construct the following supersymmetric primary operator

$$\mathcal{O}^{T\bar{J}}_{----}(\zeta) = \mathcal{T}_{----}(\zeta)\mathcal{G}(\zeta) - \frac{1}{2}\mathcal{R}_{--}(\zeta)\mathcal{G}_{--}(\zeta)\ . \tag{4.19}$$

In analogy to the other supersymmetric primary operators considered so far, a straightforward calculation leads to the following result

$$\frac{1}{2}[\bar{\mathcal{D}}_+, \mathcal{D}_+]\mathcal{O}_{----} = \mathcal{T}_{----}\mathcal{G}_{++} - \mathcal{T}\mathcal{G}_{--} + \text{EoM} + \text{total derivatives}\,, \tag{4.20}$$

which implies

$$O^{T\bar{J}}_{--}(\sigma) = \int \mathrm{d}\bar{\vartheta}^+ \mathrm{d}\vartheta^+\, \mathcal{O}^{T\bar{J}}_{----}(\zeta) + \text{EoMs} + \text{total derivatives}\,, \tag{4.21}$$

as expected.

In the absence of a conserved R-symmetry in the stress-tensor multiplet, one can not construct the $T\bar{J}$ supersymmetric primary operator as a unique D-term whose full superspace integral leads to $O^{T\bar{J}}_{--}$. The reason is simply that there might not exist in the $\mathcal{S}$-multiplet the $\mathcal{R}_{--}$ operator such that $\mathcal{W}_- = \frac{\mathrm{i}}{2}\bar{\mathcal{D}}_+\mathcal{R}_{--}$ and $\mathcal{W}_- = \frac{\mathrm{i}}{2}\bar{\mathcal{D}}_+\mathcal{R}_{--}$, eq. (2.30). In such a case, $\mathcal{O}^{T\bar{J}}_{----}$ of eq. (4.19) will not exist and consequently eq. (4.21) will not hold. Nevertheless, for a general $\mathcal{N} = (0,2)$ supersymmetric theory it is still possible to show that $O^{T\bar{J}}_{--}(\sigma)$ arises as a linear combination of a full superspace integral and a chiral half superspace integral.[11] This extends eq. (4.21) and proves again that $T\bar{J}$ deformations preserve supersymmetry. Let us turn to the precise description of this case.

First, note that the constraints defining the $\mathcal{N} = (0,2)$ flavor current multiplet, eq. (3.12), tell us that the superfields

$$\mathcal{H}_{--} := \mathcal{G}_{--} - \mathrm{i}\partial_{--}\mathcal{G}\,, \qquad \bar{\mathcal{H}}_{--} = \mathcal{G}_{--} + \mathrm{i}\partial_{--}\mathcal{G}\,, \tag{4.22}$$

[11] Note that the same happens with $T\bar{T}$ deformations for general $\mathcal{N} = (2,2)$ supersymmetric models described by an $\mathcal{S}$-multiplet [27] where the $T\bar{T}$ operator is related to a linear combination of full and chiral superspace integrals.

are chiral and anti-chiral, respectively: $\bar{\mathcal{D}}_+\mathcal{H}_{--} = 0$, $\mathcal{D}_+\bar{\mathcal{H}}_{--} = 0$. Moreover, they satisfy the following relations

$$\mathcal{D}_+\mathcal{H}_{--} = -2\mathrm{i}\partial_{--}\mathcal{D}_+\mathcal{G}\,, \qquad \bar{\mathcal{D}}_+\bar{\mathcal{H}}_{--} = 2\mathrm{i}\partial_{--}\bar{\mathcal{D}}_+\mathcal{G}\ . \tag{4.23}$$

By using $\mathcal{H}_{--}$ and $\bar{\mathcal{H}}_{--}$ together with the superfields of the $\mathcal{S}$-multiplet, we can define the following $T\bar{J}$ superfield

$$\mathcal{O}^{T\bar{J}}_{--} = \frac{1}{2}[\bar{\mathcal{D}}_+, \mathcal{D}_+](\mathcal{T}_{----}\mathcal{G}) - \frac{\mathrm{i}}{2}\mathcal{D}_+(\mathcal{W}_-\mathcal{H}_{--}) + \frac{\mathrm{i}}{2}\bar{\mathcal{D}}_+(\bar{\mathcal{W}}_-\bar{\mathcal{H}}_{--})\ . \tag{4.24}$$

This can be easily shown to be

$$\mathcal{O}^{T\bar{J}}_{--} = \mathcal{T}_{----}\mathcal{G}_{++} - \mathcal{T}\mathcal{G}_{--} + \mathrm{EoM} + \text{total derivatives}\,, \tag{4.25}$$

whose lowest $\vartheta = 0$ component is just the standard $T\bar{J}$-operator

$$O^{T\bar{J}}_{--} = \int \mathrm{d}\bar{\vartheta}^+\mathrm{d}\vartheta^+\mathcal{T}_{----}\mathcal{G} - \frac{\mathrm{i}}{2}\Big(\int \mathrm{d}\vartheta^+\mathcal{W}_-\mathcal{H}_{--} - \int \mathrm{d}\bar{\vartheta}^+\bar{\mathcal{W}}_-\bar{\mathcal{H}}_{--}\Big)$$
$$+\mathrm{EoM} + \text{total derivatives}\,, \tag{4.26}$$

as expected.

5 Examples of supersymmetric $J\bar{T}/T\bar{J}$ deformations

In this section, we will present some explicit examples of supersymmetric $J\bar{T}/T\bar{J}$ deformations. As argued in [5], $J\bar{T}/T\bar{J}$ is solvable when the U(1) current is chirally conserved. With the aim of extending the results of [5], in this section we will only focus on supersymmetric examples arising from *chiral* $J\bar{T}/T\bar{J}$ deformations. Our analysis will be purely classical here but we will manage to construct explicit $J\bar{T}/T\bar{J}$ flows for some simple supersymmetric example.

In both the $\mathcal{N} = (0,1)$ and $\mathcal{N} = (0,2)$ cases, we will present two models induced by $J\bar{T}$ and $T\bar{J}$ deformations, respectively. The chiral $J\bar{T}/T\bar{J}$ deformations in $\mathcal{N} = (1,1)$ theories seem to resist illustrations in simple examples. We will comment more on these cases in the conclusion.

5.1 $\mathcal{N} = (0,1)$ $J\bar{T}$

Here we are going to present the simplest example of $J\bar{T}$ deformation with $\mathcal{N} = (0,1)$ supersymmetry. It consists of a left-moving complex fermion which has the U(1) symmetry, and a right-moving supersymmetric sector which consists of a real scalar and a real fermion.

5.1.1 Component form

Inspired by the non-supersymmetric example in [5], we propose that the following action satisfies a $J\bar{T}$ flow.

$$S_\alpha = \int d^2\sigma \mathcal{L}_\alpha = \int d^2\sigma \Big(\mathcal{L}_L + \mathcal{L}_R + \alpha\mathcal{L}_{\text{def}}\Big) , \tag{5.1a}$$

$$\mathcal{L}_L = \frac{\mathrm{i}}{2}\bar{\chi}_-\partial_{++}\chi_- , \tag{5.1b}$$

$$\mathcal{L}_R = \frac{1}{2}\partial_{++}\phi\partial_{--}\phi + \frac{\mathrm{i}}{2}\psi_+\partial_{--}\psi_+ , \tag{5.1c}$$

$$\mathcal{L}_{\text{def}} = -\bar{\chi}_-\chi_-\Big(\partial_{++}\phi\partial_{++}\phi + \mathrm{i}\psi_+\partial_{++}\psi_+\Big) . \tag{5.1d}$$

where χ_- is a complex fermion, while $\psi_+ = \bar{\psi}_+$ and $\phi = \bar{\phi}$ are a real fermion and a real scalar, respectively.

The left-moving complex fermion possess the following U(1) symmetry

$$\chi_- \to e^{\mathrm{i}\rho}\chi_- , \qquad \bar{\chi}_- \to e^{-\mathrm{i}\rho}\bar{\chi}_- . \tag{5.2}$$

According to Noether's theorem, this gives rise to the following U(1) current

$$G_{--} = \bar{\chi}_-\chi_- , \tag{5.3}$$

which we will shortly show to be chiral.

The T_{++++} component of the stress-energy tensor for the action (5.1a) can be easily computed and turns out to be[12]

$$T_{++++} = -\partial_{++}\phi\partial_{++}\phi - \mathrm{i}\psi_+\partial_{++}\psi_+ . \tag{5.4}$$

In particular, note that $\mathcal{L}_{\text{def}}$ does not contribute to this component.

Note that the equation of motion of the complex fermion χ_- is

$$\partial_{++}\chi_- = -2\alpha\mathrm{i}\chi_-\Big(\partial_{++}\phi\partial_{++}\phi + \mathrm{i}\psi_+\partial_{++}\psi_+\Big) = 2\alpha\mathrm{i}\chi_- T_{++++} . \tag{5.5}$$

Together with its complex conjugate, a short but instructive calculation which uses (5.5) shows that it holds

$$\partial_{++}G_{--} = 0 . \tag{5.6}$$

Therefore, the U(1) current is chirally conserved and $G_{++} = 0$, which is expected from the symmetry (5.2) where there is even no notion of G_{++}.

Finally, we easily notice that the deformation part of the Lagrangian (5.1a) satisfies (remember that $G_{++} = 0$)

$$\frac{\partial\mathcal{L}_\alpha}{\partial\alpha} = \mathcal{L}_{\text{def}} = T_{++++}G_{--} = O^{J\bar{T}} . \tag{5.7}$$

[12]Our conventions for the stress-energy tensor is $T_{ab} = \eta_{ac}\frac{\partial\mathcal{L}}{\partial\partial_c\varphi}\partial_b\varphi - \eta_{ab}\mathcal{L}$ where in light-cone notations the Minkowski metric is $\eta_{\pm\pm,\pm\pm} = -2, \eta^{\pm\pm,\pm\pm} = -\frac{1}{2}, \eta_{\pm\pm,\mp\mp} = \eta^{\pm\pm,\mp\mp} = 0$.

This shows that the action we proposed in eq. (5.1a) arises from a $J\bar{T}$ deformation as expected. Since the U(1) current is chirally conserved, the deformation is thus a *chiral* $J\bar{T}$ deformation.

So far we have not discussed whether the model described by (5.1a) is supersymmetric, though from our general discussion we expect this to be the case. To prove this statement explicitly we turn to describing the same model in $\mathcal{N}=(0,1)$ superspace.

5.1.2 Superfield form

We start by introducing the following superfields

$$\Phi = \phi - \mathrm{i}\vartheta^+\psi_+\,, \tag{5.8}$$

and

$$\Upsilon_- = \chi_- - \mathrm{i}\vartheta^+ B\,, \qquad \bar{\Upsilon}_- = \bar{\chi}_- + \mathrm{i}\vartheta^+\bar{B}\,, \tag{5.9}$$

that embed the component fields ϕ, ψ_+, χ_- and $\bar{\chi}_-$ into appropriate supermultiplets.

A natural manifestly supersymmetric extension of the action (5.1a) is

$$S_\alpha = \int \mathrm{d}^2\sigma \mathrm{d}\vartheta^+\left(\frac{\mathrm{i}}{2}\mathcal{D}_+\Phi\partial_{--}\Phi + \frac{1}{2}\bar{\Upsilon}_-\mathcal{D}_+\Upsilon_- - \mathrm{i}\alpha\bar{\Upsilon}_-\Upsilon_-\mathcal{D}_+\Phi\partial_{++}\Phi\right)\,. \tag{5.10}$$

To show the equivalence with (5.1a), we reduce (5.10) to components and obtain

$$\begin{aligned} S_\alpha = \int \mathrm{d}^2\sigma \Bigg[& \frac{1}{2}\partial_{++}\phi\partial_{--}\phi + \frac{\mathrm{i}}{2}\psi_+\partial_{--}\psi_+ + \frac{\mathrm{i}}{2}\bar{\chi}_-\partial_{++}\chi_- + \frac{1}{2}B\bar{B} \\ & -\mathrm{i}\alpha(\chi_-\bar{B} + \bar{\chi}_- B)\psi_+\partial_{++}\phi - \alpha\bar{\chi}_-\chi_-\Big(\partial_{++}\phi\partial_{++}\phi + \mathrm{i}\psi_+\partial_{++}\psi_+\Big)\Bigg]\,. \end{aligned} \tag{5.11}$$

Note that the previous action is identical to (5.1a) except for all the terms involving the complex auxiliary fields B and $\bar{B}$. It is simple to show that these terms are identically zero once we integrate out B and $\bar{B}$. In fact, these can be solved in terms of the physical fields by using their algebraic equation of motion:

$$B = 2\mathrm{i}\alpha\chi_-\psi_+\partial_{++}\phi\,, \qquad \bar{B} = 2\mathrm{i}\alpha\bar{\chi}_-\psi_+\partial_{++}\phi\,. \tag{5.12}$$

By substituting this result back into (5.11), one can see that the auxiliary fields B and $\bar{B}$ have no contribution due to the fermionic property $\psi_+^2 = 0$. Thus the manifestly supersymmetric action (5.10)–(5.11) is equivalent to the $J\bar{T}$ deformed action (5.1a). The above construction also tells us that the action (5.1a) is supersymmetric.

To see the supersymmetry more explicitly, we can work out the supersymmetry transformation rules. The off-shell $\mathcal{N}=(0,1)$ supersymmetry transformation of an arbitrary superfield $\mathcal{F}$ was given in (2.4) and we repeat them here for the reader's convenience:

$$\delta\mathcal{F} = -\mathrm{i}\epsilon_-\mathcal{Q}_+\mathcal{F} = -\mathrm{i}\epsilon_-\left(\mathrm{i}\frac{\partial}{\partial\vartheta^+} - \vartheta^+\partial_{++}\right)\mathcal{F}\,. \tag{5.13}$$

By using this rule for Φ, eq. (5.8), and Υ_-, eq. (5.9), one can derive the off-shell supersymmetry transformations of their component fields

$$\delta\chi_- = -\mathrm{i}\epsilon_- B\,, \qquad \delta\bar\chi_- = \mathrm{i}\epsilon_-\bar B\,, \qquad \delta\phi = -\mathrm{i}\epsilon_-\psi_+\,, \qquad \delta\psi_+ = \epsilon_-\partial_{++}\phi\,. \tag{5.14}$$

One can check explicitly that (5.11) is invariant under the previous transformations. Note also that the equations of motion for B and $\bar B$ given by (5.12) are also consistent with these supersymmetry transformations. In fact, one can also verify that (5.1a) is invariant under (5.14) *on-shell*, meaning when (5.12) are satisfied. In particular, one can check that in this case $\delta(\bar\chi_-\chi_-) = 0$ which guarantees that no higher-order terms in α would be generated in the on-shell transformation rules. We can then conclude that the model described by the action (5.1a), which we have previously shown to be a standard $J\bar T$ deformation, is supersymmetric as expected.

Let us look back at the manifestly off-shell supersymmetric action (5.10) and show that it is a manifestly supersymmetric deformation associated to the operator (4.2a). First, we rewrite the action (5.10) as

$$S_\alpha = \int \mathrm{d}^2\sigma\mathrm{d}\vartheta^+\mathcal{A}_\alpha\,, \quad \mathcal{A}_\alpha = \left(\frac{\mathrm{i}}{2}\mathcal{D}_+\Phi\partial_{--}\Phi + \frac{1}{2}\bar\Upsilon_-\mathcal{D}_+\Upsilon_- - \mathrm{i}\alpha\bar\Upsilon_-\Upsilon_-\mathcal{D}_+\Phi\partial_{++}\Phi\right)\,. \tag{5.15}$$

We can derive the stress-tensor multiplet for example by using the Noether procedure of [24]. We obtain

$$\mathcal{J}_{+++} = -\mathrm{i}\frac{\delta\mathcal{A}_\alpha}{\delta\partial_{--}\Phi}\partial_{++}\Phi = -\mathrm{i}\mathcal{D}_+\Phi\partial_{++}\Phi\,. \tag{5.16}$$

Then it is easy to see that the superspace Lagrangian $\mathcal{A}_\alpha$ satisfies the supersymmetric $J\bar T$ flow equation

$$\frac{\partial\mathcal{A}_\alpha}{\partial\alpha} = \mathcal{J}_{+++}\mathcal{G}_{--} = \mathcal{O}_+^{J\bar T}\,, \tag{5.17}$$

where

$$\mathcal{G}_{--} = \bar\Upsilon_-\Upsilon_- = \bar\chi_-\chi_- + \mathrm{i}\vartheta^+(\bar\chi_- B + \chi_-\bar B)\,, \tag{5.18}$$

and $\mathcal{G}_{++} = 0$ in the supersymmetric primary operator $\mathcal{O}_+^{J\bar T}$ of eq. (4.2a). Let us in fact verify at the superspace level that $\mathcal{G}_{--}$ is chirally conserved. By using the superspace equations of motion for the superfields Υ_- and $\bar\Upsilon_-$ which read

$$\mathcal{D}_+\Upsilon_- = 2\mathrm{i}\alpha\Upsilon_-\mathcal{D}_+\Phi\partial_{++}\Phi\,, \qquad \mathcal{D}_+\bar\Upsilon_- = -2\mathrm{i}\alpha\bar\Upsilon_-\mathcal{D}_+\Phi\partial_{++}\Phi\,, \tag{5.19}$$

it is a straightforward calculation to prove the following result

$$\mathcal{D}_+\mathcal{G}_{--} = \mathcal{D}_+\bar\Upsilon_-\cdot\Upsilon_- - \mathcal{D}_+\Upsilon_-\cdot\bar\Upsilon_- = -2\mathrm{i}\alpha(\bar\Upsilon\Upsilon_- + \Upsilon\bar\Upsilon_-)\mathcal{D}_+\Phi\partial_{++}\Phi = 0\,. \tag{5.20}$$

Note that the conservation equation (5.20) is expected considering that the action (5.15) is invariant under the following symmetry

$$\Upsilon_- \to e^{\mathrm{i}\rho}\Upsilon_-\,, \qquad \bar\Upsilon_- \to e^{-\mathrm{i}\rho}\bar\Upsilon_-\,. \tag{5.21}$$

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Actually, the super flavor current (5.18) can also be constructed directly by promoting (5.21) to a gauge symmetry and then covariantizing[13] the action (5.15). Comparing the linearized action with (A.12) gives (5.18) and $\mathcal{G}_+ = 0$.

To summarize, we have shown that the superspace action (5.15) arises from a $J\bar{T}$ deformation with $\mathcal{N} = (0,1)$ supersymmetry, and satisfies the manifestly supersymmetric $J\bar{T}$ flow equation (5.17) driven by the supersymmetric primary operator $\mathcal{O}_+^{J\bar{T}}$ of eq. (4.2a).

5.2 $\mathcal{N} = (0,1)$ $T\bar{J}$

In this subsection, we are going to present an $\mathcal{N} = (0,1)$ supersymmetric model which arises from the chiral $T\bar{J}$ deformation. This is going to be a supersymmetric generalization of the model first presented by Guica in [5]. We start by reconstructing this model in the case without supersymmetry and then turn to its $\mathcal{N} = (0,1)$ supersymmetric extension.

5.2.1 A bosonic $T\bar{J}$ model from a new perspective

In [5], Guica worked out the $T\bar{J}$-deformation of a free scalar field action. The U(1) current is associated with the shift symmetry of the real free massless scalar field. Here we would like to rederive this model from a slightly different point of view which will be used in constructing the supersymmetric extension.

We can make the following educated guess for the action of the $T\bar{J}$-deformed real free massless scalar field

$$S_\lambda = \int d^2\sigma \, \mathcal{L}_\lambda = \int d^2\sigma \, \partial_{++}\phi\partial_{--}\phi \, F(\lambda\partial_{--}\phi) \ . \tag{5.22}$$

Here $F(x)$ is an arbitrary analytic function such that $F(0) = 1$, which ensures that the undeformed action, S_0, is the one of a free massless scalar field. The T_{----} component of the stress-energy tensor of the action (5.22) proves to be

$$T_{----} = -2(\partial_{--}\phi)^2 F \ . \tag{5.23}$$

We want to impose the action (5.22) to have a conserved chiral current G_{++}, which is $\partial_{--}G_{++} = 0$, and to be a $T\bar{J}$ flow, namely:

$$\partial_\lambda \mathcal{L}_\lambda = G_{++}T_{----} \ . \tag{5.24}$$

As already mentioned, the reason to consider a chiral $T\bar{J}$-deformation is that this is the case for which the quantum spectrum of the model is still solvable [5]. The $T\bar{J}$ flow equation (5.24) together with (5.23) can be used to determine the U(1) current

$$G_{++} = -\frac{F'}{2F}\partial_{++}\phi \ . \tag{5.25}$$

The previous result is consistent only when we assume G_{++} to be chirally conserved on-shell, which turns into the following constraint

$$\partial_{--}G_{++} = \partial_{--}\Big(-\frac{F'}{2F}\partial_{++}\phi\Big) = 0 \ . \tag{5.26}$$

[13]More specifically, by using the covariant derivative ∇_A in (A.1) in place of $\mathcal{D}_A$.

Using the equation of motion for the action (5.22), the above conservation equation leads to the following differential equation for the function $F(x)$:

$$\frac{F' + \frac{1}{2}xF''}{F + xF'} = \frac{F''}{F'} - \frac{F'}{F}, \qquad x = \lambda\partial_{--}\phi\,. \tag{5.27}$$

Solving this equation, one gets

$$F = \frac{c_2}{x + c_1}, \qquad \text{or} \quad F = \frac{c}{x^2}\,. \tag{5.28}$$

Once we impose the boundary condition $F(0) = 1$, the second solution is discarded and the most general solution turns out to be:

$$F(x) = \frac{c}{x + c}\,. \tag{5.29}$$

By plugging this result into (5.25), the chiral current is then given by

$$G_{++} = -\frac{F'}{2F}\partial_{++}\phi = \frac{1}{2}\frac{1}{c + \lambda\partial_{--}\phi}\partial_{++}\phi\,. \tag{5.30}$$

In the undeformed limit $\lambda = 0$, it holds $G_{++} = \frac{1}{2c}\partial_{++}\phi$. Therefore, the seemingly extra parameter c just corresponds to the normalization of the current which we have not specified yet and has no physical meaning. To be consistent with [5], we choose the normalization $c = -4$, hence the function $F(x)$ is

$$F(x) = \frac{1}{1 - \frac{1}{4}x}\,. \tag{5.31}$$

To conclude, we have shown that the action (5.22) with F given by (5.31) describes a chiral $T\bar{J}$ flow, eq. (5.24), with chiral current given by (5.25).

5.2.2 A $T\bar{J}$ deformed model with $\mathcal{N} = (0, 1)$ supersymmetry

Now we would like to extend the analysis of the previous subsection to the supersymmetric case and find the $T\bar{J}$-deformation of a free $\mathcal{N} = (0, 1)$ scalar multiplet action. The natural manifestly off-shell supersymmetric extension of (5.22) is given by the following ansatz

$$S_\lambda = \mathrm{i}\int \mathrm{d}^2\sigma\mathrm{d}\vartheta^+\, \mathcal{D}_+\Phi\partial_{--}\Phi F(\lambda\partial_{--}\Phi)\,, \tag{5.32}$$

where the real scalar superfield $\Phi(\zeta)$ is the same as (5.8) and the analytic function $F(x)$ is such that $F(0) = 1$ but otherwise arbitrary. Similarly to the $\mathcal{N} = (0, 1)$ $J\bar{T}$ deformation, we will first analyze the previous ansatz in components and then directly in superspace.

- ***Component approach.*** Once the superfield $\Phi(\zeta)$ is reduced to its real component fields $\phi(\sigma)$ and $\psi_+(\sigma)$, see eq. (5.8), and the Grassmann integral is performed, the action (5.32) takes the form

$$S_\lambda = \int \mathrm{d}^2\sigma\, \mathcal{L}_\lambda = \int \mathrm{d}^2\sigma\, \Big\{\partial_{++}\phi\partial_{--}\phi F + \mathrm{i}\psi_+\partial_{--}\psi_+\big(F + \lambda\partial_{--}\phi F'\big)\Big\}\,. \tag{5.33}$$

The T_{----} component of the stress-energy tensor for the previous model proves to be

$$T_{----} = -2(\partial_{--}\phi)^2 F \ . \tag{5.34}$$

Interestingly, this is exactly the same as the bosonic case (5.23).

As in the pure bosonic case, we want to interpret the action (5.33) as a chiral $T\bar{J}$-deformation satisfying

$$\partial_\lambda \mathcal{L}_\lambda = G_{++} T_{----} \ . \tag{5.35}$$

This flow equation enables us to determine the U(1) current to be

$$G_{++} = -\frac{1}{2F}\Big(F'\partial_{++}\phi + \mathrm{i}\lambda F''\psi_+\partial_{--}\psi_+\Big) - \mathrm{i}\frac{F'}{F}\frac{\psi_+\partial_{--}\psi_+}{\partial_{--}\phi} \ . \tag{5.36}$$

Consistency of (5.35) requires that G_{++} is chiral, $\partial_{--}G_{++} = 0$, which we are going to study next.

The equation of motion for the fermion of the action (5.33) is given by:

$$0 = 2\mathrm{i}\partial_{--}\psi_+ \cdot (F + \lambda F'\partial_{--}\phi) + \mathrm{i}\psi_+\partial_{--}(F + \lambda F'\partial_{--}\phi) \, , \tag{5.37}$$

which yields

$$\partial_{--}\psi_+ = -\psi_+ \frac{\partial_{--}(F + xF')}{2(F + xF')} \ . \tag{5.38}$$

Multiplying by ψ_+, we get the following non-trivial simplification

$$\psi_+\partial_{--}\psi_+ = 0 \ . \tag{5.39}$$

Interestingly, the previous result implies that G_{++}, eq. (5.36), has no contribution from the fermion ψ_+ once its equation of motion is used. In this case, (5.36) simplifies to

$$G_{++} = -\frac{F'}{2F}\partial_{++}\phi \, , \tag{5.40}$$

which is precisely the same as the purely bosonic case, eq. (5.25). Note also that by using (5.39) the fermion terms disappear from the action (5.33). This implies that the dynamics of the boson ϕ can be treated independently from the fermion ψ_+, once (5.39) holds. Therefore, for the purpose of imposing that G_{++} is chiral on-shell, effectively one can use eq. (5.40) and note that ϕ has the same equation of motion as for the purely bosonic action (5.22). This immediately implies that the condition for G_{++} to be a chiral current is solved by the same function F as in the non-supersymmetric case, namely (5.29). This concludes the proof that the supersymmetric action (5.33), and equivalently (5.32), satisfies the $T\bar{J}$ flow (5.35) with a chirally conserved current G_{++} given by (5.36).

- ***Superfield approach.*** In the discussions above, we have worked out the $T\bar{J}$ deformation in terms of component fields. It is natural to expect that (5.32) satisfies a $T\bar{J}$ flow equation driven by the superfield operator (4.2b). We show this to be true in the following.

The action (5.32) is given by the superspace integral of the superfield Lagrangian $\mathcal{A}_\lambda$:

$$S_\lambda = \int \mathrm{d}^2\sigma \mathrm{d}\vartheta^+ \, \mathcal{A}_\lambda \, , \qquad \mathcal{A}_\lambda = \mathrm{i}\mathcal{D}_+\Phi\partial_{--}\Phi F(\lambda\partial_{--}\Phi) \ . \tag{5.41}$$

By using for example the Noether techniques for $\mathcal{N} = (0,1)$ superspace described in [24], one can compute the $\mathcal{T}_{----}$ superfield component of the stress-tensor multiplet:

$$\mathcal{T}_{----} = 2\mathrm{i}\left[\frac{\delta\mathcal{A}}{\delta\mathcal{D}_+\Phi}\partial_{--}\Phi - \mathrm{i}\mathcal{D}_+\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\partial_{--}\Phi\right)\right] = -2(\partial_{--}\Phi)^2F(\lambda\partial_{--}\Phi)\ . \tag{5.42}$$

Note that its lowest $\vartheta = 0$ component gives the corresponding component of the stress-energy tensor, $\mathcal{T}_{----}|_{\vartheta=0} = T_{----}$, as expected.

If the action (5.41) arises from a chiral $T\bar{J}$ deformation, it should satisfy the following flow equation

$$\partial_\lambda\mathcal{A}_\lambda = \mathcal{O}^{T\bar{J}}_{---} = \mathcal{T}_{----}\mathcal{G}_+\,, \tag{5.43}$$

with $\mathcal{G}_{--} = 0$. Thus, by imposing the previous flow equation for the Lagrangian $\mathcal{A}_\lambda$ in (5.41) and the expression for $\mathcal{T}_{----}$ given by (5.42), the superfield $\mathcal{G}_+$ can be solved as

$$\mathcal{G}_+ = -\mathrm{i}\mathcal{D}_+\Phi\frac{F'}{2F}\ . \tag{5.44}$$

For consistency, with $\mathcal{G}_{--} = 0$, $\mathcal{G}_+$ should describe a chiral current multiplet satisfying (3.1)

$$\partial_{--}\mathcal{G}_+ = 0\ . \tag{5.45}$$

By imposing this constraint on (5.44) one obtains

$$\partial_{--}\mathcal{D}_+\Phi FF' + \lambda\mathcal{D}_+\Phi\partial^2_{--}\Phi(FF'' - F'^2) = 0\,, \tag{5.46}$$

which should hold on-shell. The superspace equation of motion for the real scalar superfield Φ can be easily computed by varying the action (5.41) and is given by

$$2G'\partial_{--}\mathcal{D}_+\Phi + \lambda\mathcal{D}_+\Phi\partial^2_{--}\Phi G'' = 0\,, \qquad G(x) = xF(x)\,, \qquad x = \lambda\partial_{--}\Phi\ . \tag{5.47}$$

By using this result in (5.46) we can obtain the following equation

$$-\frac{\partial_{--}\mathcal{D}_+\Phi}{\lambda\mathcal{D}_+\Phi\partial^2_{--}\Phi} = \frac{G''}{2G'} = \frac{FF'' - F'^2}{FF'}\ . \tag{5.48}$$

Using $G(x) = xF(x)$, we get

$$\frac{(2F + xF')(-2F'^2 + FF'')}{2FF'(F + xF')} = 0\ . \tag{5.49}$$

One can easily check that this differential equation is equivalent to the one we obtained in the bosonic case, eq. (5.27). Thus the solution, of the above differential equation is also given by the bosonic one (5.31). This is consistent with our previous component approach.

To make more clear the connection with the components results given above, we can further calculate[14]

$$\mathcal{G}_{++} = \mathcal{D}_+\mathcal{G}_+ = -\frac{1}{8}\partial_{++}\Phi F - \frac{\mathrm{i}}{32}\lambda F^2\mathcal{D}_+\Phi\partial_{--}\mathcal{D}_+\Phi\,, \tag{5.50a}$$

$$= -\frac{1}{8}\partial_{++}\Phi F\,, \tag{5.50b}$$

[14] Here we used the relation $F' = \frac{1}{4}F^2$ which holds for (5.31).

where in the last equality we used the relation $\mathcal{D}_+\Phi\partial_{--}\mathcal{D}_+\Phi = 0$ which can be obtained by multiplying the equation of motion (5.47) with $\mathcal{D}_+\Phi$. These can be seen to be in agreement with the components results for G_{++} given above. In particular, the $\vartheta = 0$ component projection of $\mathcal{G}_{++}$ on-shell is given by

$$G_{++} = \mathcal{G}_{++}|_{\vartheta=0} = -\frac{1}{8}\partial_{++}\phi F\,, \tag{5.51}$$

which is in agreement with (5.40). In particular, it follows that $\partial_{--}G_{++} = 0$ on-shell.

5.3 $\mathcal{N} = (0,2)$ $J\bar{T}$

In this subsection, we are going to generalize the model constructed for the $\mathcal{N} = (0,1)$ case by complexifying its right-moving sector. The resulting model will possess the left-moving complex fermion χ_- and $\bar{\chi}_-$, which generates the U(1) symmetry, and a complex $\mathcal{N} = (0,2)$ supersymmetric sector which consists of a complex scalar ϕ and $\bar{\phi}$ together with a complex right-chirality fermion ψ_+ and $\bar{\psi}_+$. A natural generalization of the action (5.1a) is the following

$$S_\alpha = \int \mathrm{d}^2\sigma \Big(\mathcal{L}_L + \mathcal{L}_R + \alpha\mathcal{L}_{\text{def}}\Big)\,, \tag{5.52a}$$

$$\mathcal{L}_L = \frac{\mathrm{i}}{2}\bar{\chi}_-\partial_{++}\chi_-\,, \tag{5.52b}$$

$$\mathcal{L}_R = \frac{1}{2}\partial_{++}\bar{\phi}\partial_{--}\phi - \frac{\mathrm{i}}{2}\bar{\psi}_+\partial_{--}\psi_+\,, \tag{5.52c}$$

$$\mathcal{L}_{\text{def}} = -\bar{\chi}_-\chi_-\Big(\partial_{++}\bar{\phi}\partial_{++}\phi - \mathrm{i}\bar{\psi}_+\partial_{++}\psi_+\Big)\,. \tag{5.52d}$$

Let us check that this action describes a $J\bar{T}$ flow.

As for the $\mathcal{N} = (0,1)$ case, the left-moving complex fermion has U(1) symmetry and the associated U(1) current is given by

$$G_{--} = \bar{\chi}_-\chi_-\,. \tag{5.53}$$

The T_{++++} component of the stress-energy tensor proves to be

$$T_{++++} = -\partial_{++}\bar{\phi}\partial_{++}\phi + \frac{\mathrm{i}}{2}\bar{\psi}_+\partial_{++}\psi_+ + \frac{\mathrm{i}}{2}\psi_+\partial_{++}\bar{\psi}_+\,, \tag{5.54}$$

while the equation of motion for the fermion χ_- is

$$\partial_{++}\chi_- = 2\alpha\mathrm{i}\chi_- T_{++++}\,. \tag{5.55}$$

Together with its complex conjugate, this implies that the U(1) flavor current G_{--} is chiral

$$\partial_{++}G_{--} = 0\,, \tag{5.56}$$

and that our action (5.52) arises from a chiral $J\bar{T}$ deformation:

$$\frac{\partial\mathcal{L}_\alpha}{\partial\alpha} = \mathcal{L}_{\text{def}} = T_{++++}G_{--}\,. \tag{5.57}$$

However, it remains to show that the action is $\mathcal{N}=(0,2)$ supersymmetric. For this we turn to superspace.

The fields ϕ and ψ_+ are going to describe a chiral $\mathcal{N}=(0,2)$ multiplet while their complex conjugate fits in an anti-chiral one. We can introduce the chiral and anti-chiral $\mathcal{N}=(0,2)$ complex scalar superfields

$$\Phi=\phi+\mathrm{i}\sqrt{2}\vartheta^+\psi_+ +\mathrm{i}\vartheta^+\bar{\vartheta}^+\partial_{++}\phi\,,\qquad \bar{\Phi}=\bar{\phi}+\mathrm{i}\sqrt{2}\bar{\vartheta}^+\bar{\psi}_+ -\mathrm{i}\vartheta^+\bar{\vartheta}^+\partial_{++}\bar{\phi}\,, \tag{5.58}$$

satisfying the constraints

$$\bar{\mathcal{D}}_+\Phi=\mathcal{D}_+\bar{\Phi}=0\ . \tag{5.59}$$

We also introduce the $\mathcal{N}=(0,2)$ complex Fermi-multiplet through the following superfields

$$\Upsilon_-=\chi_-+\vartheta^+\mathsf{F}+\mathrm{i}\vartheta^+\bar{\vartheta}^+\partial_{++}\chi_-\,,\qquad \bar{\Upsilon}_-=\bar{\chi}_-+\bar{\vartheta}^+\bar{\mathsf{F}}-\mathrm{i}\vartheta^+\bar{\vartheta}^+\partial_{++}\bar{\chi}_-\ . \tag{5.60}$$

These are also chiral and anti-chiral respectively

$$\bar{\mathcal{D}}_+\Upsilon_-=\mathcal{D}_+\bar{\Upsilon}_-=0\ . \tag{5.61}$$

Note that the extra complex fields F and $\bar{\mathsf{F}}$ are necessary to close $\mathcal{N}=(0,2)$ supersymmetry off-shell and will play the role of auxiliary fields, analogously to the $\mathcal{N}=(0,1)$ case. The natural ansatz for the $J\bar{T}$-deformed action in superspace is then given by

$$\mathcal{L}_\alpha=\frac{1}{4}\int\mathrm{d}\bar{\vartheta}^+\mathrm{d}\vartheta^+\bar{\Upsilon}_-\Upsilon_- +\frac{\mathrm{i}}{4}\int\mathrm{d}\bar{\vartheta}^+\mathrm{d}\vartheta^+\bar{\Phi}\partial_{--}\Phi-\frac{\alpha}{4}\int\mathrm{d}\bar{\vartheta}^+\mathrm{d}\vartheta^+\bar{\Upsilon}_-\Upsilon_-\mathcal{D}_+\Phi\bar{\mathcal{D}}_+\bar{\Phi}\ . \tag{5.62}$$

We can compute the equation of motion of the chiral Fermi superfields that gives

$$\mathcal{D}_+\Upsilon_-=2\mathrm{i}\alpha\Upsilon_-\mathcal{D}_+\Phi\partial_{++}\bar{\Phi}\,,\quad \bar{\mathcal{D}}_+\bar{\Upsilon}_-=-2\mathrm{i}\alpha\bar{\Upsilon}_-\bar{\mathcal{D}}_+\bar{\Phi}\partial_{++}\Phi\ . \tag{5.63}$$

The action (5.62) is invariant under the following symmetry:

$$\Upsilon_-\to e^{\mathrm{i}\Lambda}\Upsilon_-,\qquad \bar{\Upsilon}_-\to e^{-\mathrm{i}\bar{\Lambda}}\bar{\Upsilon}_-\,, \tag{5.64}$$

with $\mathcal{D}_+\bar{\Lambda}=\bar{\mathcal{D}}_+\Lambda=0$. Promoting this symmetry to a gauge symmetry, we can couple the Fermi multiplet to a real gauge prepotential superfield V exactly in the same way as the well known 4D case: $\bar{\Upsilon}_-e^V\Upsilon_-$. By expanding the resulting gauged action to leading order in V and comparing to (A.50), one can get the flavor current superfields

$$\mathcal{G}_{--}=\bar{\Upsilon}_-\Upsilon_-\Big(1-\alpha\mathcal{D}_+\Phi\bar{\mathcal{D}}_+\bar{\Phi}\Big)\,,\qquad \mathcal{G}=0\ . \tag{5.65}$$

Noether theorem guarantees that $\mathcal{G}_{--}$ is a conserved chiral current. Indeed, using the equation of motion (5.63), one can verify that it holds

$$\mathcal{D}_+\mathcal{G}_{--}=-\bar{\Upsilon}_-\mathcal{D}_+\Upsilon_-\Big(1-\alpha\mathcal{D}_+\Phi\bar{\mathcal{D}}_+\bar{\Phi}\Big)+2\mathrm{i}\alpha\bar{\Upsilon}_-\Upsilon_-\mathcal{D}_+\Phi\partial_{++}\bar{\Phi}=0\ . \tag{5.66}$$

The stress-tensor multiplet can also be straightforwardly computed. In particular, the $\mathcal{S}_{++}$ superfield can be shown to be

$$\mathcal{S}_{++}=\frac{1}{2}\bar{\mathcal{D}}_+\bar{\Phi}\mathcal{D}_+\Phi\ . \tag{5.67}$$

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With this, one can compute $\mathcal{T}_{++++} = \frac{1}{4}[\bar{\mathcal{D}}_+, \mathcal{D}_+]\mathcal{S}_{++}$ and find that its lowest component gives (5.54).

The supersymmetric chiral $J\bar{T}$ primary operator, see eq. (4.15) with $\mathcal{G} \equiv 0$, is thus given by

$$\mathcal{O}^{J\bar{T}} = \mathcal{S}_{++}\mathcal{G}_{--} = \frac{1}{2}\bar{\mathcal{D}}_+\bar{\Phi}\mathcal{D}_+\Phi \cdot \bar{\Upsilon}_-\Upsilon_-\Big(1 + \alpha\mathcal{D}_+\Phi\bar{\mathcal{D}}_+\bar{\Phi}\Big) = \frac{1}{2}\bar{\mathcal{D}}_+\bar{\Phi}\mathcal{D}_+\Phi\bar{\Upsilon}_-\Upsilon_- \,, \tag{5.68}$$

which is independent of deformation parameter α. It is then obvious that the action (5.62) satisfies the following chiral supersymmetric $J\bar{T}$ flow equation

$$\frac{\partial\mathcal{A}_\alpha}{\partial\alpha} = -\frac{1}{4}\bar{\Upsilon}_-\Upsilon_-\mathcal{D}_+\Phi\bar{\mathcal{D}}_+\bar{\Phi} = \frac{1}{2}\mathcal{S}_{++}\mathcal{G}_{--} = \frac{1}{2}\mathcal{O}^{J\bar{T}} \,. \tag{5.69}$$

Therefore, (5.62) is indeed a manifestly off-shell $\mathcal{N} = (0, 2)$ supersymmetric $J\bar{T}$ deformed action.

To analyze in more detail the action (5.62) and check its relation with (5.52), we would like to expand the superspace action in components. We find that (5.62) after integrating the Grassmann variables gives

$$\begin{aligned}\mathcal{L}_\alpha = &\frac{1}{2}\partial_{++}\bar{\phi}\partial_{--}\phi - \frac{\mathrm{i}}{2}\bar{\psi}_+\partial_{--}\psi_+ + \frac{\mathrm{i}}{4}\bar{\chi}_-\partial_{++}\chi_- + \frac{\mathrm{i}}{4}\chi_-\partial_{++}\bar{\chi}_- - \frac{1}{4}\mathsf{F}\bar{\mathsf{F}}\\ &-\frac{\alpha}{4}\Bigg[4\chi_-\bar{\chi}_-\Big(- \partial_{++}\bar{\phi}\partial_{++}\phi + \frac{\mathrm{i}}{2}\bar{\psi}_+\partial_{++}\psi_+ + \frac{\mathrm{i}}{2}\psi_+\partial_{++}\bar{\psi}_+\Big)\\ &-2\bar{\psi}_+\psi_+\Big(- \mathrm{i}\chi_-\partial_{++}\bar{\chi}_- - \mathrm{i}\bar{\chi}_-\partial_{++}\chi_- + \mathsf{F}\bar{\mathsf{F}}\Big)\\ &-2\sqrt{2}\bar{\chi}_-\bar{\psi}_+\partial_{++}\phi\mathsf{F} - 2\sqrt{2}\psi_+\chi_-\partial_{++}\bar{\phi}\bar{\mathsf{F}}\Bigg] \,.\end{aligned} \tag{5.70}$$

We stress that the previous Lagrangian leads to an action which is $\mathcal{N} = (0, 2)$ supersymmetric off-shell. Solving the auxiliary field equations of motion gives

$$\bar{\mathsf{F}} = 2\sqrt{2}\alpha\bar{\chi}_-\bar{\psi}_+\partial_{++}\phi \,, \qquad \mathsf{F} = 2\sqrt{2}\alpha\psi_+\chi_-\partial_{++}\bar{\phi} \,. \tag{5.71}$$

By using this result the action turns into

$$\begin{aligned}S_\alpha = \int \mathrm{d}^2\sigma \Bigg[&\frac{1}{2}\partial_{++}\bar{\phi}\partial_{--}\phi - \frac{\mathrm{i}}{2}\bar{\psi}_+\partial_{--}\psi_+ + \frac{\mathrm{i}}{4}\Big(\bar{\chi}_-\partial_{++}\chi_- + \chi_-\partial_{++}\bar{\chi}_-\Big)\Big(1 - 2\alpha\bar{\psi}_+\psi_+\Big)\\ &-\alpha\bar{\chi}_-\chi_-\Big(\partial_{++}\bar{\phi}\partial_{++}\phi - \frac{\mathrm{i}}{2}\bar{\psi}_+\partial_{++}\psi_+ - \frac{\mathrm{i}}{2}\psi_+\partial_{++}\bar{\psi}_+\Big)\\ &-2\alpha^2\chi_-\bar{\chi}_-\bar{\psi}_+\psi_+\partial_{++}\bar{\phi}\partial_{++}\phi\Bigg] \,.\end{aligned} \tag{5.72}$$

Compared to the original component action (5.52), we see that there are two extra pieces: one is the α^2 term, and the other one multiplies the kinetic term of the left fermions. As we will see, these extra terms can be redefined away.

Note that the flavor supercurrent is given by (5.65) and its lowest component gives the conventional flavour current

$$G_{--} = \bar{\chi}_- \chi_- (1 - 2\alpha \bar{\psi}_+ \psi_+) \; . \tag{5.73}$$

Following our previous superfield approach, this current is a chiral conserved current $\partial_{++} G_{--} = 0$.

To see the role of this current, we can rewrite the action (5.72) in the following form:

$$\begin{aligned} S_\alpha = \int \mathrm{d}^2\sigma \bigg[& \frac{1}{2}\partial_{++}\bar{\phi}\partial_{--}\phi - \frac{\mathrm{i}}{2}\bar{\psi}_+\partial_{--}\psi_+ + \frac{\mathrm{i}}{4}\Big(\bar{\chi}_-\partial_{++}\chi_- + \chi_-\partial_{++}\bar{\chi}_-\Big)\Big(1 - 2\alpha\bar{\psi}_+\psi_+\Big) \\ & -\alpha\bar{\chi}_-\chi_-\Big(1 - 2\alpha\bar{\psi}_+\psi_+\Big)\Big(\partial_{++}\bar{\phi}\partial_{++}\phi - \frac{\mathrm{i}}{2}\bar{\psi}_+\partial_{++}\psi_+ - \frac{\mathrm{i}}{2}\psi_+\partial_{++}\bar{\psi}_+\Big)\bigg] \; . \end{aligned} \tag{5.74}$$

The first line can be thought as the undeformed action where the left-moving fermion still has a U(1) symmetry. The associated current is exactly given by (5.73). And the second line is then just the $J\bar{T}$ deformation with a modified current (5.73).

The action (5.74) can also be obtained from (5.52) through a field redefinition:

$$\chi_- \to \chi_-\Big(1 - \alpha\bar{\psi}_+\psi_+\Big) \, , \qquad \bar{\chi}_- \to \bar{\chi}_-\Big(1 - \alpha\bar{\psi}_+\psi_+\Big) \; . \tag{5.75}$$

To conclude, the supersymmetric $J\bar{T}$ deformation in (5.62), (5.74) and the conventional $J\bar{T}$ deformation in (5.52) coincide up to field redefinitions. This implies that these actions are the same on-shell, as expected from the general equivalence of the manifestly supersymmetric $J\bar{T}$ deformation and the one given by the operator $O^{J\bar{T}}_{++}$, eq. (4.1b).

5.4 $\mathcal{N} = (0, 2)$ $T\bar{J}$

In this subsection we shortly present a model for an $\mathcal{N} = (0, 2)$ $T\bar{J}$ deformation which extends the bosonic and $\mathcal{N} = (0, 1)$ cases presented in section 5.2. To make the presentation more concise and manifestly supersymmetric, we will work directly in superspace.

We are going to show that the following model

$$S_\lambda = \int \mathrm{d}^2\sigma \mathrm{d}\bar{\vartheta}^+ \mathrm{d}\vartheta^+ \mathcal{A}_\lambda = \frac{\mathrm{i}}{4}\int \mathrm{d}^2\sigma \mathrm{d}\bar{\vartheta}^+ \mathrm{d}\vartheta^+ \; \bar{\Phi}\partial_{--}\Phi F(\lambda\partial_{--}\bar{\Phi}) \, , \tag{5.76}$$

is a $T\bar{J}$ flow and, in particular, a $\mathcal{N} = (0, 2)$ extension of the action (5.22).

By considering the variation with respect to the $\mathcal{N} = (0, 2)$ chiral superfield Φ, we get the following equation of motion

$$\partial_{--}\bar{\mathcal{D}}_+(\bar{\Phi}F) = 0 \; . \tag{5.77}$$

Compared with (3.12), we can naturally identify the following chiral U(1) current[15]

$$\mathcal{G} = -\mathrm{i}\gamma\bar{\Phi}F \, , \qquad \mathcal{G}_{--} = 0 \, , \tag{5.78}$$

[15]Note that this current is now complex, so only the first equation in (3.12) is satisfied. We will comment on it later.

where γ is an arbitrary normalization constant introduced for convenience. In the undeformed limit $F = 1$, this is indeed the U(1) current associated with the shift symmetry of the superfield Φ.

It is also straightforward to compute the $\mathcal{T}_{----}$ component of the stress-tensor multiplet which can be shown to be

$$\mathcal{T}_{----} = -\partial_{--}\bar{\Phi}\partial_{--}\Phi F \, . \tag{5.79}$$

By requiring that the $\mathcal{A}_\lambda$ superfield Lagrangian satisfies the chiral $T\bar{J}$ flow equation (4.19)

$$\frac{\partial \mathcal{A}_\lambda}{\partial \lambda} = \mathcal{O}^{T\bar{J}}_{----} = \mathcal{T}_{----}\mathcal{G} \, , \tag{5.80}$$

we obtain the following condition for the function F

$$\frac{\mathrm{i}}{4}\bar{\Phi}\partial_{--}\Phi\partial_{--}\bar{\Phi}F' = \mathrm{i}\gamma\bar{\Phi}\partial_{--}\bar{\Phi}\partial_{--}\Phi F^2 \qquad \Longrightarrow \qquad F' = 4\gamma F^2 \, . \tag{5.81}$$

By imposing $F(0) = 1$, the solution is[16]

$$F(x) = \frac{1}{1 - 4\gamma x} \, . \tag{5.82}$$

Therefore, we have shown that the action S_λ in (5.76) satisfies a chiral $T\bar{J}$ flow. However, it is clear that the action is actually a little pathological because it is not real due the complex chiral current $\mathcal{G}(\zeta)$ and its descendant $G_{++}(\sigma) := -\frac{1}{2}[\mathcal{D}_+, \bar{\mathcal{D}}_+]\mathcal{G}(\zeta)|_{\vartheta=0}$. This does not change or spoil the basic properties of the $T\bar{J}$ deformation. However, it would be interesting and important to see whether this is an intrinsic pathology on $T\bar{J}$ deformation in this case or one could modify the action to get a theory arising from a chiral $T\bar{J}$ deformation with *real* U(1) current. We leave this for future analysis.

6 Conclusion and outlook

In this paper we have analyzed $J\bar{T}/T\bar{J}$-deformations for theories possessing $\mathcal{N} = (0,1)$, $(1,1)$ and $(0,2)$ supersymmetry. We have first discussed the conservation equations of the stress-tensor multiplets and flavor current multiplets. Based on those multiplets, we have then constructed the $J\bar{T}/T\bar{J}$ supersymmetric primary operators. We have further shown that their descendants are equivalent to the conventional $J\bar{T}/T\bar{J}$ operators up to conservation equations and total derivatives. Several examples of Lagrangians arising from the chiral $J\bar{T}/T\bar{J}$ deformation of free supersymmetric theories were also presented.

To construct the $J\bar{T}/T\bar{J}$ operator, a conserved U(1) current is needed. In this paper we have been focusing exclusively on a flavor U(1) current that does not belong to the stress-tensor multiplet. However, in some supersymmetric theories, there is also an R-symmetry which can give rise to a U(1) R-current. A natural question is: can we construct the $J\bar{T}/T\bar{J}$ operators out of the stress-energy tensor and the U(1) R-current? In our $\mathcal{N} = (0,2)$ case,

[16] One can then choose the normalization $\gamma = \frac{1}{16}$ such that the solution agrees with (5.31).

the $\mathcal{R}$-multiplet is given in (2.31) and it contains both the stress-energy tensor $T_{++\pm\pm}$ and R-current $j_{\pm\pm}$ which enables one to construct the conventional $J\bar{T}/T\bar{J}$ operator. However, a supersymmetric primary built out of the $\mathcal{R}$-multiplet seems to evade the constructions in this paper. It would be interesting to investigate in detail the underlying reasons of the failure/success of these R-symmetry deformations and analyze them also in theories with more supersymmetries, say $\mathcal{N} = (2, 2)$. We leave this problem for the future.

In our paper, we have only considered Lagrangians arising from the chiral $J\bar{T}/T\bar{J}$ deformations of free theories because these deformed models are simple and argued to be solvable. Starting from the relativistic free theory, we indeed find several simple theories arising from chiral $J\bar{T}$ and $T\bar{J}$ deformations with $\mathcal{N} = (0, 1)$ supersymmetry and a chiral $J\bar{T}$ deformation with $\mathcal{N} = (0, 2)$ supersymmetry. However, in the $\mathcal{N} = (1, 1)$ case, we did not find a simple realization of chiral $J\bar{T}/T\bar{J}$ deformations.[17] It would be interesting to see whether this type of $\mathcal{N} = (1, 1)$ chiral $J\bar{T}/T\bar{J}$ deformations can be realized in a more complicated or broad class of theories. For example, since $J\bar{T}/T\bar{J}$ deformations break Lorentz invariance, one can naturally start with a non-Lorentz-invariant but supersymmetric theory and see whether it admits a chiral $J\bar{T}/T\bar{J}$ deformation with some amount of supersymmetry. In the $\mathcal{N} = (0, 2)$ case, as an example, we have presented a chiral $T\bar{J}$ deformed Lagrangian with *complex* current. It remains to see how to construct a *real* chiral $T\bar{J}$ deformed theory with $\mathcal{N} = (0, 2)$ supersymmetry.

Another question is the symmetry enhancement. As argued in [5, 7], the $J\bar{T}/T\bar{J}$ deformation breaks the original two-dimensional conformal group $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ down to the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ subgroup as the global symmetry of the deformed theory, but these symmetries would be enhanced to the infinite-dimensional Virasoro $\times$ Virasoro. Now with supersymmetries, it is natural to expect that the enhancement is given by a super-Virasoro $\times$ super-Virasoro symmetry.[18]

Last but not least, as shown in appendix B, all the operators associated to $T\bar{T}$ and $J\bar{T}/T\bar{J}$ deformations fit into a general pattern which generalizes the Smirnov-Zamolodchikov type of composite operators. In appendix B, we have also shown that under certain assumptions, the generalized composite operator is invariant under improvement transformations. The original Smirnov-Zamolodchikov type composite operators are proved to be well-defined at the quantum level. For our generalization, this quantum definedness has also been shown to hold in several examples explicitly. It is thus reasonable to speculate that our generalized Smirnov-Zamolodchikov composite operators are also well-defined at the quantum level in general. The proof of this statement and its implications will be an interesting and important future research problem.

[17] For example, the $J\bar{T}$ construction in subsection 5.1 is not obvious because the $\mathcal{N} = (1, 1)$ supersymmetric generalization of the left moving sector in (5.1b) requires the embedding of the *complex* fermion χ_- into a superfield which necessarily introduces also many other fields. For the naive $\mathcal{N} = (1, 1)$ supersymmetric generalization of $T\bar{J}$ construction in (5.22), the EoMs contain many types of derivatives $\mathcal{D}_\pm, \partial_{\pm\pm}$ and thus fails to guarantee the chiral conservation of the U(1) current $\mathcal{D}_+\mathcal{G}_- = 0$ or $\mathcal{D}_-\mathcal{G}_+ = 0$ in a simple way.

[18] Besides, there is also a chiral $\mathrm{U}(1)_J$ symmetry generated by the current; this symmetry is now expected to enhance to super-Kac-Moody.

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A Deriving the conservation laws of flavor current multiplets

In this appendix we derive the various flavor current multiplets described in section 3. The derivation is conceptually the same for all the types of supersymmetries. As a first step we describe a supersymmetric abelian vector multiplet and its gauge transformation rules. Then we couple the gauge multiplet to a corresponding flavor current multiplet and impose the gauge invariance of such coupling. As a result we obtain the conservation equations of the supersymmetric flavor current multiplets.

A.1 $\mathcal{N} = (0, 1)$

By looking for example at [36], pages 5-6, we see that an $\mathcal{N} = (1, 0)$ abelian vector multiplet is described by a gauge connection Γ_A and gauge covariant derivatives

$$\nabla_A = \mathcal{D}_A - \mathrm{i}\Gamma_A \,, \tag{A.1}$$

satisfying the following algebra

$$\{\nabla_+, \nabla_+\} = -2\mathrm{i}\nabla_{++} \,, \tag{A.2a}$$

$$[\nabla_+, \nabla_{--}] = \mathrm{i}\mathcal{W}_- \,, \qquad [\nabla_+, \nabla_{++}] = 0 \,, \tag{A.2b}$$

$$[\nabla_{++}, \nabla_{--}] = -\nabla_+\mathcal{W}_- \,. \tag{A.2c}$$

Here the superfield $\mathcal{W}_-(\zeta)$ is an unconstrained real spinorial field strength. The previous algebra correctly satisfies the super-Jacobi identities and in fact it is interesting to note that the form of the commutator $[\nabla_{++}, \nabla_{--}]$ is fixed by the Bianchi identities

$$[\nabla_{++}, \nabla_{--}] = \frac{\mathrm{i}}{2}[\{\nabla_+, \nabla_+\}, \nabla_{--}] = \mathrm{i}\{\nabla_+, [\nabla_+, \nabla_{--}]\} = -\{\nabla_+, \mathcal{W}_-\} = -\nabla_+\mathcal{W}_- \,. \tag{A.3}$$

The anti-commutator (A.2a) implies that Γ_{++} can be solved in terms of Γ_+,[19]

$$\Gamma_{++} = \mathrm{i}\mathcal{D}_+\Gamma_+ \,, \tag{A.4}$$

[19]Note that we are considering an Abelian gauge symmetry, so that the connections (anti-)commute.

while Γ_+ and Γ_{--} remain independent and unconstrained gauge connections. The $\mathcal{N} = (0,1)$ superfields $(\Gamma_+(\zeta), \Gamma_{--}(\zeta))$ then play exactly the same role of unconstrained component gauge connection fields, $(A_{++}(\sigma), A_{--}(\sigma))$, gauging an Abelian symmetry in the standard two-dimensional Minkowski space-time. The first equation in (A.2b) can be used to express $\mathcal{W}_-(\zeta)$ in terms of the unconstrained connections

$$\mathcal{W}_- = -\mathcal{D}_+\Gamma_{--} + \partial_{--}\Gamma_+ \,. \tag{A.5}$$

All the other constraints associated with the algebra (A.2) are then identically satisfied once (A.4) and (A.5) are imposed.

Note that the gauge transformations of Γ_{--} and Γ_+ are

$$\delta_G\Gamma_{--} = \mathrm{i}\partial_{--}\tau\,, \qquad \delta_G\Gamma_+ = \mathrm{i}\mathcal{D}_+\tau\,, \tag{A.6}$$

with $\tau(\zeta)$ an unconstrained real gauge superfield parameter. It is easy to see that

$$\delta_G\mathcal{W}_- = 0\,, \tag{A.7}$$

so the field strength is gauge invariant as expected.

In components, the multiplet of connections reads

$$\Gamma_+(\zeta) = \chi_+(\sigma) + \vartheta^+A_{++}(\sigma)\,, \qquad \Gamma_{--}(\zeta) = \mathrm{i}A_{--}(\sigma) - \vartheta^+\lambda_-(\sigma)\,, \tag{A.8}$$

and the field strength is

$$\mathcal{W}_-(\zeta) = \lambda_-(\sigma) + \partial_{--}\chi_+(\sigma) + \vartheta^+\Big(\partial_{--}A_{++}(\sigma) - \partial_{++}A_{--}(\sigma)\Big)\,. \tag{A.9}$$

Then under the gauge transformation (A.6) with $\tau(\zeta) = \phi(\sigma) + \vartheta^+\psi_+(\sigma)$, the component fields transform as

$$\delta_G\chi_+ = \mathrm{i}\psi_+\,, \qquad \delta_G\lambda_- = -\mathrm{i}\partial_{--}\psi_+\,, \qquad \delta_GA_{++} = \partial_{--}\phi\,, \qquad \delta_GA_{--} = \partial_{++}\phi\,. \tag{A.10}$$

These transformations obviously leave the components of the field strength (A.9) invariant. Furthermore, they imply that χ_+ is pure gauge and can be set to zero. Then the two independent components of the field strength multiplet are the gaugino $\lambda(\sigma)$ and the field strength $F(\sigma)$:

$$\lambda_-(\sigma) = \mathcal{W}_-(\zeta)|_{\vartheta=0}\,, \qquad F(\sigma) = \nabla_+\mathcal{W}_-(\zeta)|_{\vartheta=0} = \partial_{--}A_{++}(\sigma) - \partial_{++}A_{--}(\sigma)\,. \tag{A.11}$$

Note that F is a pseudo-scalar field that arises from the Hodge dual of the field strength $F_{ab} = \partial_{[a}A_{b]}$.

Now that we have reviewed the structure of an $\mathcal{N} = (0,1)$ vector multiplet, we can derive the multiplet of currents for an Abelian symmetry. Consider a U(1) invariant action S for a matter system. If we couple it to a background U(1) gauge multiplet described by the independent superfields (Γ_+, Γ_{--}), at first order in the gauge connections it holds

$$S = -\mathrm{i}\int \mathrm{d}^2\sigma\,\mathrm{d}\vartheta^+\Big[\mathcal{G}_+\Gamma_{--} + \mathrm{i}\mathcal{G}_{--}\Gamma_+\Big]\,. \tag{A.12}$$

Assuming that the equations of motion for the matter multiplets are satisfied, the variation of the action under arbitrary local U(1) transformations (A.6), after some integrations by parts, takes the form

$$\delta_G S = \mathrm{i}\int \mathrm{d}^2\sigma\,\mathrm{d}\vartheta^+\,\tau\big(\partial_{--}\mathcal{G}_+ + \mathrm{i}\mathcal{D}_+\mathcal{G}_{--}\big)\ . \tag{A.13}$$

Imposing that the action is invariant $\delta_G S = 0$ then leads to the following supercurrent conservation equations for a U(1) symmetry:

$$\mathcal{D}_+\mathcal{G}_{--} = \mathrm{i}\partial_{--}\mathcal{G}_+\ . \tag{A.14}$$

It is simple to see that the previous conservation equation implies

$$\mathrm{i}\partial_{--}\mathcal{D}_+\mathcal{G}_+ = \mathcal{D}_+\mathcal{D}_+\mathcal{G}_{--} = -\mathrm{i}\partial_{++}\mathcal{G}_{--}\ . \tag{A.15}$$

Thus by defining

$$\mathcal{G}_{++} := \mathcal{D}_+\mathcal{G}_+\,, \tag{A.16}$$

one gets the conservation equation for a U(1) flavor current

$$\partial_{++}\mathcal{G}_{--} = -\partial_{--}\mathcal{G}_{++}\ . \tag{A.17}$$

Note that by construction, due to (A.16), it also holds

$$\mathcal{D}_+\mathcal{G}_{++} = -\mathrm{i}\partial_{++}\mathcal{G}_+\ . \tag{A.18}$$

In components, the superfields of the U(1) flavor current multiplet are given by

$$\mathcal{G}_+(\zeta) = g_+(\sigma) + \vartheta^+ G_{++}(\sigma)\,, \qquad \mathcal{G}_{--}(\zeta) = G_{--}(\sigma) + \mathrm{i}\vartheta^+\partial_{--}g_+(\sigma)\ . \tag{A.19}$$

Due to eq. (A.17), $G_{\pm\pm}$ satisfy the ordinary vector conservation equation

$$\partial_{--}G_{++} + \partial_{++}G_{--} = 0\ . \tag{A.20}$$

A.2 $\mathcal{N} = (1,1)$

The Abelian current multiplet with $\mathcal{N} = (1,1)$ supersymmetry can be derived in a similar fashion as that in the $\mathcal{N} = (0,1)$ case. In practice, we can appropriately combine the two copies of $\mathcal{N} = (0,1)$ and $\mathcal{N} = (1,0)$ currents that arise from parity transformations of one to the other. A description of the off-shell vector multiplet for $\mathcal{N} = (1,1)$ can be found in [37].

The superspace Abelian gauge covariant derivatives are given in terms of connections $\Gamma_A(\zeta)$ by

$$\nabla_A = \mathcal{D}_A - \mathrm{i}\Gamma_A\,, \tag{A.21}$$

where the flat spinor derivatives are given in (2.14). To describe an irreducible vector multiplet, the covariant derivatives are constrained to satisfy the following algebra

$$\{\nabla_+,\nabla_+\} = -2\mathrm{i}\nabla_{++}\,,\quad \{\nabla_-,\nabla_-\} = -2\mathrm{i}\nabla_{--}\,,\quad \{\nabla_+,\nabla_-\} = -\mathrm{i}\mathcal{W}\,, \tag{A.22a}$$

$$[\nabla_+,\nabla_{--}] = -\nabla_-\mathcal{W}\,,\quad [\nabla_-,\nabla_{++}] = -\nabla_+\mathcal{W}\,,\quad [\nabla_-,\nabla_{--}] = [\nabla_+,\nabla_{++}] = 0\,, \tag{A.22b}$$

$$[\nabla_{++},\nabla_{--}] = -\mathrm{i}\nabla_+\nabla_-\mathcal{W}\ . \tag{A.22c}$$

We would like to describe the previous algebra completely in terms of independent connections. By analyzing the first two anti-commutators in (A.22a) we can express the vector connections $\Gamma_{\pm\pm}$ in terms of the spinor ones $\Gamma_\pm$ as

$$\Gamma_{++} = \mathrm{i}\mathcal{D}_+\Gamma_+ \,, \qquad \Gamma_{--} = \mathrm{i}\mathcal{D}_-\Gamma_- \,. \tag{A.23}$$

Moreover, from the third anti-commutator in (A.22a) we obtain the expression of the scalar superfield strength $\mathcal{W}(\zeta)$ in terms of the independent connections $\Gamma_\pm(\zeta)$

$$\mathcal{W} = \mathcal{D}_+\Gamma_- + \mathcal{D}_-\Gamma_+ \,. \tag{A.24}$$

With these relations holding, it is easy to verify that the rest of the algebra is completely determined in terms of the unconstrained connection superfields Γ_+ and Γ_-.

The gauge transformation is given by

$$\delta_G\Gamma_+ = \mathrm{i}\mathcal{D}_+\tau \,, \qquad \delta_G\Gamma_- = \mathrm{i}\mathcal{D}_-\tau \,. \tag{A.25}$$

It leaves the field strength invariant $\delta_G\mathcal{W} = 0$.

Note that in components, the previous Abelian vector multiplet is reduced in the following way. The connections are

$$\Gamma_+(\zeta) = \chi_+(\sigma) + \vartheta^+ A_{++}(\sigma) + \vartheta^- B_{-+}(\sigma) + \mathrm{i}\vartheta^+\vartheta^-\eta_+(\sigma) \,, \tag{A.26a}$$

$$\Gamma_-(\zeta) = \chi_-(\sigma) + \vartheta^- A_{--}(\sigma) + \vartheta^+ B_{+-}(\sigma) + \mathrm{i}\theta^-\theta^+\eta_-(\sigma) \,. \tag{A.26b}$$

The field strength $\mathcal{W}$ is consequently given by

$$\begin{aligned}\mathcal{W}(\zeta) = {} & B_{-+}(\sigma) + B_{+-}(\sigma) - \mathrm{i}\vartheta^+\big(\eta_+(\sigma) + \partial_{++}\chi_-(\sigma)\big) - \mathrm{i}\vartheta^-\big(\eta_-(\sigma) + \partial_{--}\chi_+(\sigma)\big) \\ & -\mathrm{i}\vartheta^+\vartheta^-\big(\partial_{++}A_{--}(\sigma) - \partial_{--}A_{++}(\sigma)\big) \,.\end{aligned} \tag{A.27}$$

Under the gauge transformation (A.25) with gauge parameter

$$\tau(\zeta) = \phi(\sigma) + \mathrm{i}\vartheta^+\psi_+(\sigma) + \mathrm{i}\vartheta^-\psi_-(\sigma) + \mathrm{i}\vartheta^+\vartheta^- C(\sigma) \,. \tag{A.28}$$

the connections (A.26) transform as

$$\delta_G\Gamma_+ = \mathrm{i}\mathcal{D}_+\tau = -\psi_+ + \vartheta^+\partial_{++}\phi - \vartheta^- C + \mathrm{i}\vartheta^+\vartheta^-\partial_{++}\psi_- \,, \tag{A.29a}$$

$$\delta_G\Gamma_- = \mathrm{i}\mathcal{D}_-\tau = -\psi_- + \vartheta^-\partial_{--}\phi + \vartheta^+ C - \mathrm{i}\vartheta^+\vartheta^-\partial_{--}\psi_+ \,. \tag{A.29b}$$

One can check that under this gauge transformation, the components of $\mathcal{W}$ are indeed invariant. We can choose a WZ gauge such that $\chi_+ = \chi_- = 0$, then

$$\mathcal{W} = B - \mathrm{i}\vartheta^+\eta_+ - \mathrm{i}\vartheta^-\eta_- - \mathrm{i}\vartheta^+\vartheta^- F \,, \tag{A.30}$$

where

$$B = B_{-+} + B_{+-} \,, \qquad F = \partial_{++}A_{--} - \partial_{--}A_{++} \,. \tag{A.31}$$

Then the physical degrees of freedoms include two real gaugni $\eta_\pm$ and one real scalar B as well as one pseudo-real scalar F [37].

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As before for the $\mathcal{N}=(0,1)$ case, we can couple the vector multiplet to the Abelian current superfields $\mathcal{G}_\pm(\zeta)$:

$$S=\int \mathrm{d}^2\sigma \mathrm{d}\vartheta^+\mathrm{d}\vartheta^-\Big(\Gamma_-\mathcal{G}_+-\Gamma_+\mathcal{G}_-\Big)\ . \tag{A.32}$$

Under the gauge transformation (A.25), the action transforms as

$$\delta_G S=-\mathrm{i}\int \mathrm{d}^2\sigma \mathrm{d}\vartheta^+\mathrm{d}\vartheta^-\ \tau\Big(\mathcal{D}_+\mathcal{G}_--\mathcal{D}_-\mathcal{G}_+\Big)\ . \tag{A.33}$$

By imposing gauge invariance, we obtain the conservation equation for the U(1) current

$$\mathcal{D}_+\mathcal{G}_--\mathcal{D}_-\mathcal{G}_+=0\ . \tag{A.34}$$

We can define the descendant superfields

$$\mathcal{G}_{++}=\mathcal{D}_+\mathcal{G}_+\,,\qquad \mathcal{G}_{--}=\mathcal{D}_-\mathcal{G}_-\ . \tag{A.35}$$

Then acting with $\mathcal{D}_+\mathcal{D}_-$ on both sides of equation (A.34) yields

$$\partial_{--}\mathcal{G}_{++}+\partial_{++}\mathcal{G}_{--}=0\ . \tag{A.36}$$

In components, the U(1) current multiplet reads

$$\mathcal{G}_+=g_++\vartheta^+G_{++}+\vartheta^-p+\mathrm{i}\vartheta^+\vartheta^-\partial_{++}g_-\ , \tag{A.37a}$$

$$\mathcal{G}_-=g_-+\vartheta^-G_{--}+\vartheta^+p-\mathrm{i}\vartheta^+\vartheta^-\partial_{--}g_+\ . \tag{A.37b}$$

The lowest component of (A.36) is just the conventional U(1) vector current conservation equation

$$\partial_{--}G_{++}+\partial_{++}G_{--}=0\ . \tag{A.38}$$

A.3 $\mathcal{N}=(0,2)$

In this section, we will first review the gauge multiplet with $\mathcal{N}=(0,2)$ supersymmetry following [31]. After that, by following the same standard approach used above for the $\mathcal{N}=(0,1)$ and $\mathcal{N}=(1,1)$ cases, we will derive the current multiplet for $\mathcal{N}=(0,2)$ supersymmetric theories.

The Abelian vector multiplet can be constructed by introducing the gauge covariant derivatives:

$$\nabla_+=\mathcal{D}_+-\mathrm{i}\Gamma_+\,,\quad \bar\nabla_+=\bar{\mathcal{D}}_+-\mathrm{i}\bar\Gamma_+\,,\quad \nabla_{\pm\pm}=\partial_{\pm\pm}-\mathrm{i}\Gamma_{\pm\pm}\,, \tag{A.39}$$

where the spinor covariant derivatives were introduced in (2.22). Note also the conjugation properties $\bar{\mathcal{D}}_+=-(\mathcal{D}_+)^\dagger, \bar\Gamma_+=-(\Gamma_+)^\dagger, \bar\nabla_+=-(\nabla_+)^\dagger$.

An irreducible vector multiplet is obtained by imposing the following constraints on the algebra:[20]

$$\{\nabla_+,\nabla_+\}=\{\bar\nabla_+,\bar\nabla_+\}=[\nabla_+,\nabla_{++}]=[\bar\nabla_+,\nabla_{++}]=0\,,\quad \{\nabla_+,\bar\nabla_+\}=2\mathrm{i}\nabla_{++}\,, \tag{A.40a}$$

$$[\nabla_+,\nabla_{--}]=-\mathrm{i}\bar{\mathcal{W}}_-\,,\qquad [\bar\nabla_+,\nabla_{--}]=-\mathrm{i}\mathcal{W}_-\,,\qquad [\nabla_{++},\nabla_{--}]=-\mathrm{i}\mathcal{F}\,, \tag{A.40b}$$

[20] Note the conjugation properties: $(\mathcal{F})^\dagger=\mathcal{F}, (\mathcal{W}_-)^\dagger=-\bar{\mathcal{W}}_-$. Note also that the field strengths $\mathcal{W}_-$ and $\bar{\mathcal{W}}_-$ should not be confused with the trace currents of the $\mathcal{N}=(0,2)$ stress-tensor multiplet used in section (2.3).

where the superfield strengths satisfy the following Bianchi identities

$$\nabla_+\bar{\mathcal{W}}_- = \bar{\nabla}_+\mathcal{W}_- = 0\,, \qquad \nabla_+\mathcal{W}_- + \bar{\nabla}_+\bar{\mathcal{W}}_- = 2\mathrm{i}\mathcal{F}\,, \tag{A.41a}$$

$$\nabla_+\mathcal{F} = \nabla_{++}\bar{\mathcal{W}}_-\,, \qquad \bar{\nabla}_+\mathcal{F} = \nabla_{++}\mathcal{W}_-\,. \tag{A.41b}$$

These imply

$$\nabla_+\mathcal{W}_- = \mathcal{R} + \mathrm{i}\mathcal{F}\,, \qquad \bar{\nabla}_+\bar{\mathcal{W}}_- = -\mathcal{R} + \mathrm{i}\mathcal{F}\,, \qquad (\mathcal{R})^\dagger = \mathcal{R}\,. \tag{A.42}$$

We are interested in the Abelian gauge theory. It is easy to show that the vanishing of the first two anti-commutators in (A.40a) gives $\mathcal{D}_+\Gamma_+ = \bar{\mathcal{D}}_+\bar{\Gamma}_+ = 0$. Since it holds $\mathcal{D}_+^2 = \bar{\mathcal{D}}_+^2 = 0$, we can rewrite the spinor connections in terms of the real unconstrained prepotential V as

$$\Gamma_+ = \mathrm{i}e^{-V}\mathcal{D}_+e^V = \mathrm{i}\mathcal{D}_+V\,, \tag{A.43a}$$

$$\bar{\Gamma}_+ = \mathrm{i}e^{V}\bar{\mathcal{D}}_+e^{-V} = -\mathrm{i}\bar{\mathcal{D}}_+V\,. \tag{A.43b}$$

Moreover, the last anti-commutator in (A.40a) expresses the vector connection Γ_{++} in terms of the spinor ones:

$$\Gamma_{++} = -\frac{\mathrm{i}}{2}(\mathcal{D}_+\bar{\Gamma}_+ + \bar{\mathcal{D}}_+\Gamma_+)\,. \tag{A.44}$$

From (A.40b), we can obtain the following expressions for the superfield strengths

$$\bar{\mathcal{W}}_- = -\partial_{--}\Gamma_+ + \mathcal{D}_+\Gamma_{--}\,, \tag{A.45a}$$

$$\mathcal{W}_- = -\partial_{--}\bar{\Gamma}_+ + \bar{\mathcal{D}}_+\Gamma_{--}\,, \tag{A.45b}$$

$$\mathcal{F} = \partial_{++}\Gamma_{--} - \partial_{--}\Gamma_{++} = \partial_{++}\Gamma_{--} + \frac{\mathrm{i}}{2}\partial_{--}(\mathcal{D}_+\bar{\Gamma}_+ + \bar{\mathcal{D}}_+\Gamma_+)\,, \tag{A.45c}$$

which satisfy the Bianchi identities (A.41). As a result, the unconstrained gauge fields for the $\mathcal{N} = (0, 2)$ vector multiplet are the real prepotential V and the connection Γ_{--}.

The gauge transformation of the prepotential V is given by

$$\delta_G V = \mathrm{i}(\Lambda - \bar{\Lambda})\,, \tag{A.46}$$

where Λ and $\bar{\Lambda}$ are chiral and anti-chiral, respectively:

$$\mathcal{D}_+\bar{\Lambda} = \bar{\mathcal{D}}_+\Lambda = 0\,. \tag{A.47}$$

As a consequence, the connections transform as

$$\delta_G\Gamma_+ = -\mathcal{D}_+\Lambda\,, \tag{A.48a}$$

$$\delta_G\bar{\Gamma}_+ = -\bar{\mathcal{D}}_+\bar{\Lambda}\,, \tag{A.48b}$$

$$\delta_G\Gamma_{--} = -\partial_{--}(\Lambda + \bar{\Lambda})\,, \tag{A.48c}$$

$$\delta_G\Gamma_{++} = -\partial_{++}(\Lambda + \bar{\Lambda})\,. \tag{A.48d}$$

It is easy to verify that these gauge transformations leave the field strengths invariant:

$$\delta_G\mathcal{W}_- = \delta_G\bar{\mathcal{W}}_- = \delta_G\mathcal{F} = 0\,. \tag{A.49}$$

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By using the $\mathcal{N} = (0,2)$ Abelian vector multiplet described above, we can now derive the U(1) current multiplet. We proceed by coupling the unconstrained gauge potentials Γ_{--} and V to an Abelian current multiplet in the following way

$$S = \int \mathrm{d}^2\sigma \mathrm{d}\bar{\vartheta}^+ \mathrm{d}\vartheta^+ \Big(\Gamma_{--}\mathcal{G} + V\mathcal{G}_{--}\Big)\,, \tag{A.50}$$

where $\mathcal{G}_{--}$ and $\mathcal{G}$ are real superfields. Under a gauge transformation, the previous action transforms as

$$\delta_G S = \int \mathrm{d}^2\sigma \mathrm{d}\bar{\vartheta}^+ \mathrm{d}\vartheta^+ \Big(\Lambda(\partial_{--}\mathcal{G} + \mathrm{i}\mathcal{G}_{--}) + \bar{\Lambda}(\partial_{--}\mathcal{G} - \mathrm{i}\mathcal{G}_{--})\Big)\,. \tag{A.51}$$

Note that Λ is a chiral superfield while $\bar{\Lambda}$ is an anti-chiral superfield. Hence, the gauge invariance leads to the following two conservation equations

$$\bar{\mathcal{D}}_+(\mathcal{G}_{--} - \mathrm{i}\partial_{--}\mathcal{G}) = 0\,, \qquad \mathcal{D}_+(\mathcal{G}_{--} + \mathrm{i}\partial_{--}\mathcal{G}) = 0\,, \tag{A.52}$$

that are conjugate to each other. If we define the descendant superfield

$$\mathcal{G}_{++} = -\frac{1}{2}[\mathcal{D}_+, \bar{\mathcal{D}}_+]\mathcal{G}\,, \tag{A.53}$$

then it is straightforward to prove that the following vector conservation eqution

$$\partial_{++}\mathcal{G}_{--} + \partial_{--}\mathcal{G}_{++} = 0 \tag{A.54}$$

holds.

In components, the current multiplet is given by

$$\mathcal{G}(\zeta) = g(\sigma) + \mathrm{i}\vartheta^+ p_+(\sigma) + \mathrm{i}\bar{\vartheta}^+ \bar{p}_+(\sigma) + \vartheta^+\bar{\vartheta}^+ G_{++}(\sigma)\,, \tag{A.55a}$$
$$\mathcal{G}_{--}(\zeta) = G_{--}(\sigma) + \vartheta^+\partial_{--}p_+(\sigma) - \bar{\vartheta}^+\partial_{--}\bar{p}_+(\sigma) + \vartheta^+\bar{\vartheta}^+\partial_{--}\partial_{++}g(\sigma)\,, \tag{A.55b}$$

where, thanks to (A.54), it holds

$$\partial_{++}G_{--} + \partial_{--}G_{++} = 0\,. \tag{A.56}$$

B Generalized Smirnov-Zamolodchikov type composite operators

As already stressed in the main body of the paper, one of the important properties of the operators inducing the bosonic $T\bar{J}$ and $J\bar{T}$ deformations [5] is to be of Smirnov-Zamolodchikov type [2], see $O^{\mathrm{SZ}}_{s+s'}(\sigma)$ defined in equation (4.9). As such, despite being composite irrelevant operators, they prove to be free of short distance singularities and well-defined by a point splitting procedure, as for the analysis in [2]. In the supersymmetric cases that we have studied in this paper, the $T\bar{J}$ and $J\bar{T}$ operators prove to be supersymmetric descendants of other operators. In particular, in this section we will restrict to the $T\bar{J}$ and $J\bar{T}$ operators that arise as full superspace integrals of some primary operators. In this case, the deformation operators sit at the bottom of a long supersymmetric multiplet. If supersymmetry is not broken by quantum effects, the entire multiplet should be well defined by a point splitting regularization, not only its bottom component. This is for instance the case for the supersymmetric $T\bar{T}$ deformations studied in [24–26]. Another remarkable feature of the deformation operators is that they are all invariant under improvement transformations of the (supersymmetric) currents. As we will see, these are features that hold also for the supersymmetric $T\bar{J}$ and $J\bar{T}$ operators that we have introduced in section 4. The way we will show this here, is to actually notice that all the supersymmetric $T\bar{T}$, $T\bar{J}$ and $J\bar{T}$ operators belong to a class of composite operators that generalizes the Smirnov-Zamolodchikov one. After describing such a general pattern, we will discuss the well-definedness properties of the supersymmetric primary operators introduced in this paper which we believe extend to the general case of the operators defined below by eq. (B.1).

B.1 Generalized Smirnov-Zamolodchikov operators

It turns out that all the supersymmetric $T\bar{T}$, $T\bar{J}$ and $J\bar{T}$ primary operators studied so far in the literature fit into the following general pattern:

$$\mathcal{O}(\zeta) = \mathcal{A}(\zeta)\mathcal{B}(\zeta) - s\mathcal{X}(\zeta)\mathcal{Y}(\zeta) \ . \tag{B.1}$$

Here $\mathbb{L}, \mathbb{R}$ are superspace differential operators $\mathbb{L}, \mathbb{R} \in \{\mathcal{D}_+, \mathcal{D}_-, \partial_{++}, \partial_{--}, \partial_{++}\mathcal{D}_+, \cdots\}$ and $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}$ are superfields satisfying conservation equations of the following type

$$\mathbb{L}\mathcal{A} = \mathbb{R}\mathcal{Y} \,, \qquad \mathbb{L}\mathcal{X} = \mathbb{R}\mathcal{B} \ . \tag{B.2}$$

This generalizes the Smirnov-Zamolodchikov type of composite operators which corresponds to the case $\mathbb{L} = \partial_{--}, \mathbb{R} = \partial_{++}$ and $s = 1$.

To study some of the properties of these operators, we introduce $|A|$ to denote twice of the spin of A which can be either a superfield or a differential operator. Essentially it is given by the sum of $+$ and $-$ indices. For example

$$|\mathcal{D}_+| = |\mathcal{J}_+| = 1 \,, \qquad |\partial_{--}| = |\mathcal{G}_{--}| = -2 \,, \qquad \cdots \qquad etc. \tag{B.3}$$

This satisfies

$$|AB| = |A| + |B| \,, \qquad (-)^{|A|} = (-)^{-|A|} \ . \tag{B.4}$$

We would first like to understand the behavior of $\mathcal{O}$ under improvement transformations.[21] Suppose $\mathbb{L}, \mathbb{R}$ are either commuting or anti-commuting[22]

$$\mathbb{L}\mathbb{R} = r\mathbb{R}\mathbb{L} \,, \qquad r = \pm 1 \ . \tag{B.5}$$

Then we can have the following improvement transformations which leave the constraints (B.2) invariant:

$$\mathcal{A} \to \mathcal{A}' = \mathcal{A} + \mathbb{R}\mathcal{U} \,, \qquad \mathcal{Y} \to \mathcal{Y}' = \mathcal{Y} + r\mathbb{L}\mathcal{U} \,, \tag{B.6a}$$

$$\mathcal{X} \to \mathcal{X}' = \mathcal{X} + \mathbb{R}\mathcal{V} \,, \qquad \mathcal{B} \to \mathcal{B}' = \mathcal{B} + r\mathbb{L}\mathcal{V} \ . \tag{B.6b}$$

[21] We refer the reader to [24–26] for the improvement transformations of the various stress-tensor multiplets. The flavor current multiplets satisfy similar improvement transformations which we have not analyzed in detail in our paper. For our scopes here, it will suffice to use the abstract description given in this appendix.

[22] It should be noted that (B.5) may not be satisfied, for example in the $\mathcal{N} = (0, 2)$ $J\bar{T}$ deformation.

An explicit calculation shows that under (B.6a), $\mathcal{O}$ transforms as

$$\mathcal{O} \to \mathcal{O}' = \mathcal{A}'\mathcal{B} - s\mathcal{X}\mathcal{Y}' = \mathcal{O} - (-)^{|\mathbb{R}|\cdot|\mathcal{U}|}\mathcal{U}\Big(\mathbb{R}\mathcal{B} - srt\mathbb{L}\mathcal{X}\Big) + \mathbb{L}(\cdots) + \mathbb{R}(\cdots)\,, \tag{B.7}$$

where

$$t = (-)^{|\mathbb{L}|^2 - |\mathbb{L}|\cdot|\mathbb{R}| + |\mathcal{B}|\cdot|\mathcal{Y}|}\,. \tag{B.8}$$

If $srt = 1$, then using (B.2) gives

$$\mathcal{O}' = \mathcal{O} + \mathbb{L}(\cdots) + \mathbb{R}(\cdots)\,, \tag{B.9}$$

where $\mathbb{L}(\cdots), \mathbb{R}(\cdots)$ are superspace total derivatives and have no effect after performing the superspace integral. Then, the deformation operator is invariant under improvement transformation. One can similarly check that $srt = 1$ also ensures the improvement invariance under (B.6b).

For the reader's convenience, let us now list all the supersymmetric primary operators, together with the defining current multiplets with $\mathcal{N} = (0,1)$, $\mathcal{N} = (1,1)$ and $\mathcal{N} = (0,2)$ supersymmetry, that we have either constructed in this paper or that first appeared in the following references [24–26].[23] All the following operators are of the form given by eq. (B.1):

- $(0,1)$ $T\bar{T}$:

$$\mathcal{O}_-^{T\bar{T}} = \mathcal{T}_{----}\mathcal{J}_{+++} - \mathcal{T}\mathcal{J}_-\,, \tag{B.10a}$$

$$\mathcal{D}_+\mathcal{T}_{----} = \mathrm{i}\partial_{--}\mathcal{J}_-\,, \tag{B.10b}$$

$$\mathcal{D}_+\mathcal{T} = \mathrm{i}\partial_{--}\mathcal{J}_{+++}\,; \tag{B.10c}$$

- $(1,1)$ $T\bar{T}$:

$$\mathcal{O}^{T\bar{T}} = \mathcal{J}_{---}\mathcal{J}_{+++} - \mathcal{J}_+\mathcal{J}_-\,, \tag{B.11a}$$

$$\mathcal{D}_+\mathcal{J}_{---} = \mathcal{D}_-\mathcal{J}_-\,, \tag{B.11b}$$

$$\mathcal{D}_+\mathcal{J}_+ = \mathcal{D}_-\mathcal{J}_{+++}\,; \tag{B.11c}$$

- $(0,2)$ $T\bar{T}$:

$$\begin{aligned}\mathcal{O}_{--}^{T\bar{T}} &= \mathcal{T}_{----}\mathcal{S}_{++} - \bar{\mathcal{W}}_-\mathcal{W}_- \\ &= \mathcal{T}_{----}\mathcal{S}_{++} + \frac{1}{2}(\bar{\mathcal{W}}_- - \mathcal{W}_-)(\bar{\mathcal{W}}_- - \mathcal{W}_-)\,,\end{aligned} \tag{B.12a}$$

$$(\mathcal{D}_+ - \bar{\mathcal{D}}_+)\mathcal{T}_{----} = \partial_{--}\left(\frac{1}{2}(\bar{\mathcal{W}}_- - \mathcal{W}_-)\right), \tag{B.12b}$$

$$(\mathcal{D}_+ - \bar{\mathcal{D}}_+)(\bar{\mathcal{W}}_- + \mathcal{W}_-) = \partial_{--}\mathcal{S}_{++}\,; \tag{B.12c}$$

[23] We refer the reader to [27] for $\mathcal{N} = (2,2)$ $T\bar{T}$ deformations that share similar properties.

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- $(0,1)$ $J\bar{T}$:

$$\mathcal{O}_{+}^{J\bar{T}} = \mathcal{J}_{+++}\mathcal{G}_{--} - \mathcal{G}_{++}\mathcal{J}_{-}\,, \tag{B.13a}$$

$$\partial_{--}\mathcal{J}_{+++} = -\partial_{++}\mathcal{J}_{-}\,, \tag{B.13b}$$
$$\partial_{--}\mathcal{G}_{++} = -\partial_{++}\mathcal{G}_{--}\;; \tag{B.13c}$$

- $(0,1)$ $T\bar{J}$:

$$\mathcal{O}_{---}^{T\bar{J}} = \mathcal{T}_{----}\mathcal{G}_{+} - \mathcal{G}_{--}\mathcal{J}_{-}\,, \tag{B.14a}$$

$$\mathcal{D}_{+}\mathcal{T}_{----} = \mathrm{i}\partial_{--}\mathcal{J}_{-}\,, \tag{B.14b}$$
$$\mathcal{D}_{+}\mathcal{G}_{--} = \mathrm{i}\partial_{--}\mathcal{G}_{+}\;; \tag{B.14c}$$

- $(1,1)$ $J\bar{T}$:

$$\mathcal{O}_{++}^{J\bar{T}} = \mathcal{J}_{+++}\mathcal{G}_{-} - \mathcal{G}_{+}\mathcal{J}_{+}\,, \tag{B.15a}$$

$$\mathcal{D}_{-}\mathcal{J}_{+++} = \mathcal{D}_{+}\mathcal{J}_{+}\,, \tag{B.15b}$$
$$\mathcal{D}_{-}\mathcal{G}_{+} = \mathcal{D}_{+}\mathcal{G}_{-}\;; \tag{B.15c}$$

- $(1,1)$ $T\bar{J}$:

$$\mathcal{O}_{--}^{T\bar{J}} = \mathcal{J}_{---}\mathcal{G}_{+} - \mathcal{G}_{-}\mathcal{J}_{-}\,, \tag{B.16a}$$

$$\mathcal{D}_{+}\mathcal{J}_{---} = \mathcal{D}_{-}\mathcal{J}_{-}\,, \tag{B.16b}$$
$$\mathcal{D}_{+}\mathcal{G}_{-} = \mathcal{D}_{-}\mathcal{G}_{+}\;; \tag{B.16c}$$

- $(0,2)$ $J\bar{T}$:[24]

$$\mathcal{O}^{J\bar{T}} = \mathcal{G}_{--}\mathcal{S}_{++} - 2\mathcal{T}\mathcal{G}\,, \tag{B.18a}$$

$$\mathcal{D}_{+}\mathcal{G}_{--} = -\mathrm{i}\partial_{--}\mathcal{D}_{+}\mathcal{G}\,, \tag{B.18b}$$
$$\mathcal{D}_{+}(2\mathcal{T}) = -\mathrm{i}\partial_{--}\mathcal{D}_{+}\mathcal{S}_{++}\;; \tag{B.18c}$$

- $(0,2)$ $T\bar{J}$ (in term of the $\mathcal{R}$-multiplet):

$$\mathcal{O}_{----}^{T\bar{J}} = \mathcal{T}_{----}\mathcal{G} - \mathcal{G}_{--}\cdot\frac{1}{2}\mathcal{R}_{--}\,, \tag{B.19a}$$

$$\mathcal{D}_{+}\mathcal{T}_{----} = -\mathrm{i}\partial_{--}\mathcal{D}_{+}\left(\frac{1}{2}\mathcal{R}_{--}\right), \tag{B.19b}$$
$$\mathcal{D}_{+}\mathcal{G}_{--} = -\mathrm{i}\partial_{--}\mathcal{D}_{+}\mathcal{G}\;. \tag{B.19c}$$

[24] Remember also that in the case of an $\mathcal{R}$-multiplet, the $\mathcal{N}=(0,2)$ $J\bar{T}$ operator is equivalent to

$$\mathcal{O}_{\mathcal{R}}^{J\bar{T}}(\zeta) = \mathcal{R}_{++}(\zeta)\mathcal{G}_{--}(\zeta) - \mathcal{R}_{--}(\zeta)\mathcal{G}_{++}(\zeta)\,, \tag{B.17}$$

which is of Smirnov-Zamolodchikov type.

B.2 Well-definedness of the composite operators

Of the nine operators listed above, we already know that three of them, specifically the operators in eq. (B.10a), (B.11a) and (B.13a) are well-defined (meaning free of short distance singularities in a point-splitting regularization scheme) since they are of Smirnov-Zamolodchikov type. Moreover, for the $\mathcal{N} = (0,2)$ $T\bar{T}$ operator, eq. (B.12a), we have shown in [26] that well-definedness can be proven by using supersymmetry and point splitting arguments completely analogues of the ones used in [1, 2]. It turns out that the same arguments apply also to the other operators listed in (B.10a)–(B.19a) that are not of Smirnov-Zamolodchikov's type. For this reason, we will refer the reader to [1, 2] and [26] for details and simply indicate what are the sufficient conditions required to infer well-definedness of the composite operators. We also believe these arguments might work to prove in general that operators of the form (B.1) satisfying (B.2) are well-defined.

The heart of the arguments given in [26] generalizing [1, 2] was based on the following steps:

i) Define an appropriate bilocal point-splitted version of the composite $\mathcal{O}(\zeta) = \mathcal{O}(\sigma,\vartheta)$ operator whose $\vartheta = 0$ component, $\mathcal{O}(\sigma) = \mathcal{O}(\zeta)|_{\vartheta=0}$, defines the supersymmetric primary operator. Specifically, for the operators of the type (B.1) listed above within eqs. (B.10a)–(B.19a) it suffices to consider the bilocal superspace operator given by

$$\mathcal{O}(\zeta,\zeta') = \mathcal{A}(\zeta)\mathcal{B}(\zeta') - s\mathcal{X}(\zeta')\mathcal{Y}(\zeta)\,, \tag{B.20}$$

and its $\vartheta = \vartheta'$ limit

$$\mathcal{O}(\sigma,\sigma';\vartheta) = \Big[\mathcal{A}(\sigma,\vartheta)\mathcal{B}(\sigma',\vartheta') - s\mathcal{X}(\sigma',\vartheta')\mathcal{Y}(\sigma,\vartheta)\Big]\Big|_{\vartheta=\vartheta'}\,. \tag{B.21}$$

Since divergencies cannot occur in the expansions of the Grassmann ϑ and ϑ' coordinates, the operator $\mathcal{O}(\sigma,\sigma';\vartheta)$ is the appropriate point-splitted regulated version of the composite superspace operator $\mathcal{O}(\zeta)$.

ii) Prove, by using the superspace covariant derivatives algebra, the conservation equations (B.2) (and their implications) and "integrations by parts", that the bilocal operator satisfies a relation of the following type

$$\partial_{\pm\pm}\mathcal{O}(\zeta,\zeta') = 0 + \text{EoMs} + (\partial+\partial')[\cdots] + (\mathcal{D}+\mathcal{D}')[\cdots]\,. \tag{B.22}$$

Here with "EoMs" we again refer to terms that are identically zero once the conservation equations for the current multiplets are used while with the last two terms in (B.22) we indicate terms that are superspace total derivatives, such as for example the vector derivatives $(\partial_{\pm\pm}+\partial'_{\pm\pm})$ or, for example, the spinor derivatives $(\mathcal{D}_+ + \mathcal{D}'_+)$, $(\mathcal{D}_- + \mathcal{D}'_-)$, *etc*, acting on bilocal operators.

iii) When we consider the coincident limit $\vartheta = \vartheta'$ in the Grassmann coordinates, equation (B.22) implies

$$\partial_{\pm\pm}\mathcal{O}(\sigma,\sigma';\vartheta) = 0 + \text{EoMs} + [P,\cdots] + [Q,\cdots]\,, \tag{B.23}$$

where $[P,\cdots]$ and $[Q,\cdots]$ schematically indicate a translation and supersymmetry transformation of some bilocal superfield operator. Assuming that the model under consideration has preserved translation invariance and supersymmetry, by using an extension of the OPE arguments of [1, 2], one can show that eq. (B.23) implies [26]

$$\mathcal{O}(\sigma,\sigma';\theta) = \mathcal{O}(\zeta) + \text{derivative terms} . \tag{B.24}$$

Here "derivative terms" indicate superspace covariant derivatives acting on local superfield operators while $\mathcal{O}(\zeta)$ arises from the regular, non-derivative part of the OPE of $\mathcal{O}(\sigma,\sigma';\vartheta)$. For this reason, up to total derivatives which for instance do not contribute when the operator is integrated over the full superspace, $\mathcal{O}(\sigma,\sigma';\theta)$ is free of short distance singularities in $\sigma\to\sigma'$. This concludes the arguments of well-definedness of [1, 2, 26].

Let us give an example of the calculation that leads to eq. (B.22). The simplest case is the $\mathcal{N}=(0,1)$ $T\bar{J}$ for which we define the bilocal operator

$$\mathcal{O}^{T\bar{J}}_{---}(\zeta,\zeta') = \mathcal{T}_{----}(\zeta)\mathcal{G}_{+}(\zeta') - \mathcal{G}_{--}(\zeta')\mathcal{J}_{-}(\zeta) . \tag{B.25}$$

We compute

$$\begin{aligned}\partial_{++}\mathcal{O}^{T\bar{J}}_{---}(\zeta,\zeta') &= \partial_{++}\mathcal{T}_{----}(\zeta)\mathcal{G}_{+}(\zeta') + \mathcal{J}_{-}(\zeta)\partial'_{++}\mathcal{G}_{--}(\zeta')\\ &\quad -(\partial_{++}+\partial'_{++})\big(\mathcal{J}_{-}(\zeta)\mathcal{G}_{--}(\zeta')\big)\\ &= \mathrm{i}\mathcal{D}_{+}\mathcal{D}_{+}\mathcal{T}_{----}(\zeta)\mathcal{G}_{+}(\zeta') + \mathrm{i}\mathcal{J}_{-}(\zeta)\mathcal{D}'_{+}\mathcal{D}'_{+}\mathcal{G}_{--}(\zeta')\\ &\quad -(\partial_{++}+\partial'_{++})\big(\mathcal{J}_{-}(\zeta)\mathcal{G}_{--}(\zeta')\big)\\ &= -\mathcal{D}_{+}\partial_{--}\mathcal{J}_{-}(\zeta)\mathcal{G}_{+}(\zeta') - \mathcal{J}_{-}(\zeta)\mathcal{D}'_{+}\partial'_{--}\mathcal{G}_{+}(\zeta')\\ &\quad +\mathrm{i}\mathcal{D}_{+}\big(\mathcal{D}_{+}\mathcal{T}_{----}(\zeta) - \mathrm{i}\partial_{--}\mathcal{J}_{-}(\zeta)\big)\mathcal{G}_{+}(\zeta')\\ &\quad +\mathrm{i}\mathcal{J}_{-}(\zeta)\mathcal{D}'_{+}\big(\mathcal{D}'_{+}\mathcal{G}_{--}(\zeta') - \mathrm{i}\partial_{--}\mathcal{G}_{+}(\zeta')\big)\\ &\quad -(\partial_{++}+\partial'_{++})\big(\mathcal{J}_{-}(\zeta)\mathcal{G}_{--}(\zeta')\big) ,\end{aligned} \tag{B.26}$$

where we used $\partial_{++} = \mathrm{i}\mathcal{D}_{+}\mathcal{D}_{+}$, made some "integration by parts", and completed terms that are zero once the conservation equations for the current multiplets are used. If we "integrate by parts" both the $\mathcal{D}_{+}$ and ∂_{--} derivatives in the first line of the last equivalence we obtain

$$\begin{aligned}\partial_{++}\mathcal{O}^{T\bar{J}}_{---}(\zeta,\zeta') &= \mathcal{J}_{-}(\zeta)\partial'_{--}\mathcal{D}'_{+}\mathcal{G}_{+}(\zeta') - \mathcal{J}_{-}(\zeta)\partial'_{--}\mathcal{D}'_{+}\mathcal{G}_{+}(\zeta')\\ &\quad +\mathrm{i}\mathcal{D}_{+}\big(\mathcal{D}_{+}\mathcal{T}_{----}(\zeta) - \mathrm{i}\partial_{--}\mathcal{J}_{-}(\zeta)\big)\mathcal{G}_{+}(\zeta')\\ &\quad +\mathrm{i}\mathcal{J}_{-}(\zeta)\mathcal{D}'_{+}\big(\mathcal{D}'_{+}\mathcal{G}_{--}(\zeta') - \mathrm{i}\partial_{--}\mathcal{G}_{+}(\zeta')\big)\\ &\quad -(\partial_{++}+\partial'_{++})\big(\mathcal{J}_{-}(\zeta)\mathcal{G}_{--}(\zeta')\big)\\ &\quad -(\partial_{--}+\partial'_{--})\big(\mathcal{D}_{+}\mathcal{J}_{-}(\zeta)\mathcal{G}_{+}(\zeta')\big)\\ &\quad +(\mathcal{D}_{+}+\mathcal{D}'_{+})\big(\mathcal{J}_{-}(\zeta)\partial'_{--}\mathcal{G}_{+}(\zeta')\big) ,\end{aligned} \tag{B.27}$$

where the first term is identically zero, the second and third line are zero once used the conservation equations, while the last three lines are total derivatives. A very similar calculation shows that the following result holds

$$\begin{aligned}\partial_{--}\mathcal{O}^{T\bar{J}}_{---}(\zeta,\zeta') = &-\mathrm{i}\mathcal{T}_{----}(\zeta)\big(\mathcal{D}'_+\mathcal{G}_{--}(\zeta') - \mathrm{i}\partial'_{--}\mathcal{G}_+(\zeta')\big)\\ &-\mathrm{i}\big(\mathcal{D}_+\mathcal{T}_{----}(\zeta) - \mathrm{i}\partial_{--}\mathcal{J}_-(\zeta)\big)\mathcal{G}_{--}(\zeta')\\ &+(\partial_{--}+\partial'_{--})\big(\mathcal{T}_{----}(\zeta)\mathcal{G}_+(\zeta')\big)\\ &+\mathrm{i}(\mathcal{D}_+ + \mathcal{D}'_+)\big(\mathcal{T}_{----}(\zeta)\mathcal{G}_{--}(\zeta')\big)\,,\end{aligned} \tag{B.28}$$

which, again, is zero up to total derivatives and terms that cancel once the conservation equations are used. These show that the composite bilocal operator $\mathcal{O}^{T\bar{J}}_{---}(\zeta,\zeta')$ satisfies eq. (B.22)

$$\partial_{\pm\pm}\mathcal{O}^{T\bar{J}}_{---}(\zeta,\zeta') = 0 + \text{ EoMs} + (\partial+\partial')[\cdots] + (\mathcal{D}+\mathcal{D}')[\cdots]\,, \tag{B.29}$$

and then $\mathcal{O}^{T\bar{J}}_{---}(\zeta)$ is well-defined.

Similar calculations hold for the operators defined by the equations (B.15a) and (B.16a) in the $\mathcal{N}=(1,1)$ case, while the $\mathcal{N}=(0,2)$ $J\bar{T}$ operator of eq. (B.18a), in the case of an $\mathcal{R}$-multiplet does not need any significant analysis since it is equivalent to a Smirnov-Zamolodchikov type operator, see equation (B.17) (the same is true for the bilocal forms of the $\mathcal{N}=(0,2)$ $J\bar{T}$ operators). We leave as an exercise to the reader to prove that (B.22) holds for (B.15a) and (B.16a).

We are left with the $\mathcal{N}=(0,2)$ $T\bar{J}$ operator, eq. (B.19a), which assume the existance of an $\mathcal{R}$-multiplet, and the $\mathcal{N}=(0,2)$ $J\bar{T}$ operator of eq. (B.18a) for a general $\mathcal{S}$-multiplet. Let's focus on the $T\bar{J}$ case, the general $\mathcal{N}=(0,2)$ $J\bar{T}$ analysis goes along the same lines. By doing some straightforward manipulations similar to the ones used above one can prove the following relation

$$\begin{aligned}\partial_{++}\mathcal{O}^{T\bar{J}}_{----}(\zeta,\zeta') = &-\frac{\mathrm{i}}{2}\mathcal{D}_+\Big(\bar{\mathcal{D}}_+\mathcal{T}_{----}(\zeta) - \frac{\mathrm{i}}{2}\bar{\mathcal{D}}_+\partial_{--}\mathcal{R}_{--}(\zeta)\Big)\mathcal{G}(\zeta')\\ &-\frac{\mathrm{i}}{2}\bar{\mathcal{D}}_+\Big(\mathcal{D}_+\mathcal{T}_{----}(\zeta) + \frac{\mathrm{i}}{2}\mathcal{D}_+\partial_{--}\mathcal{R}_{--}(\zeta)\Big)\mathcal{G}(\zeta')\\ &-\frac{\mathrm{i}}{4}\mathcal{R}_{--}(\zeta)\mathcal{D}'_+\Big(\bar{\mathcal{D}}'_+\mathcal{G}_{--}(\zeta') - \mathrm{i}\bar{\mathcal{D}}'_+\partial'_{--}\mathcal{G}(\zeta')\Big)\\ &-\frac{\mathrm{i}}{4}\mathcal{R}_{--}(\zeta)\bar{\mathcal{D}}'_+\Big(\mathcal{D}'_+\mathcal{G}_{--}(\zeta') + \mathrm{i}\mathcal{D}'_+\partial'_{--}\mathcal{G}(\zeta')\Big)\\ &-\frac{1}{2}(\partial_{++}+\partial'_{++})\mathcal{R}_{--}(\zeta)\mathcal{G}_{--}(\zeta')\\ &-\frac{1}{4}(\partial_{--}+\partial'_{--})\Big(\mathcal{D}_+\mathcal{R}_{--}\bar{\mathcal{D}}'_+\mathcal{G}(\zeta') - \bar{\mathcal{D}}_+\mathcal{R}_{--}\mathcal{D}'_+\mathcal{G}(\zeta')\Big)\\ &+\frac{1}{4}(\mathcal{D}_+ + \mathcal{D}'_+)\Big(\partial_{--}\bar{\mathcal{D}}_+\mathcal{R}_{--}(\zeta)\mathcal{G}(\zeta') + \mathcal{R}_{--}(\zeta)\partial'_{--}\bar{\mathcal{D}}'_+\mathcal{G}(\zeta')\Big)\\ &-\frac{1}{4}(\bar{\mathcal{D}}_+ + \bar{\mathcal{D}}'_+)\Big(\mathcal{R}_{--}(\zeta)\partial'_{--}\mathcal{D}'_+\mathcal{G}(\zeta') + \partial_{--}\mathcal{D}_+\mathcal{R}_{--}(\zeta)\mathcal{G}(\zeta')\Big)\,,\end{aligned} \tag{B.30}$$

which, as expected, is of the form

$$\partial_{++}\mathcal{O}^{T\bar{J}}_{----}(\zeta,\zeta') = 0 + \text{ EoMs} + (\partial+\partial')[\cdots] + (\mathcal{D}+\mathcal{D}')[\cdots]\,. \tag{B.31}$$

The analysis of $\partial_{--}\mathcal{O}^{T\bar{J}}_{-----}(\zeta,\zeta')$ is more intricate since it is clear that ∂_{--} acting on any superfields in the current multiplets can not be directly simplified by using the conservation equations (B.19b). As a way around, we assume that the anti-chirality constraints (and their complex conjugates) in (B.19b) can be solved on-shell in terms of two local composite complex superfields $\mathcal{P}_{-----}(\zeta)$ and $\mathcal{P}_{---}(\zeta)$ as

$$\Big(\mathcal{T}_{----} + \frac{\mathrm{i}}{2}\partial_{--}\mathcal{R}_{--}\Big) = \mathcal{D}_+\mathcal{P}_{-----}\,, \tag{B.32a}$$

$$\Big(\mathcal{G}_{--} + \mathrm{i}\partial_{--}\mathcal{G}\Big) = \mathcal{D}_+\mathcal{P}_{---}\,, \tag{B.32b}$$

which imply

$$\mathcal{T}_{----} = \frac{1}{2}\big(\mathcal{D}_+\mathcal{P}_{-----} + \bar{\mathcal{D}}_+\bar{\mathcal{P}}_{-----}\big)\,, \tag{B.33a}$$

$$\partial_{--}\mathcal{R}_{--} = -\mathrm{i}\big(\mathcal{D}_+\mathcal{P}_{-----} - \bar{\mathcal{D}}_+\bar{\mathcal{P}}_{-----}\big)\,, \tag{B.33b}$$

$$\mathcal{G}_{--} = \frac{1}{2}\Big(\mathcal{D}_+\mathcal{P}_{---} + \bar{\mathcal{D}}_+\bar{\mathcal{P}}_{---}\Big)\,, \tag{B.33c}$$

$$\partial_{--}\mathcal{G} = -\frac{\mathrm{i}}{2}\big(\mathcal{D}_+\mathcal{P}_{---} - \bar{\mathcal{D}}_+\bar{\mathcal{P}}_{---}\big)\,, \tag{B.33d}$$

where $\bar{\mathcal{P}}_{-----} = \overline{(\mathcal{P}_{-----})}$ and $\bar{\mathcal{P}}_{---} = \overline{(\mathcal{P}_{---})}$. By using the decomposition in terms of the prepotential superfields $\mathcal{P}_{-----}(\zeta)$ and $\mathcal{P}_{---}(\zeta)$ we can analyse $\partial_{--}\mathcal{O}^{T\bar{J}}_{-----}(\zeta,\zeta')$. A straightforward calculation similar to the previous cases shows that it holds

$$\begin{aligned}\partial_{--}\mathcal{O}^{T\bar{J}}_{-----}(\zeta,\zeta') = {}& (\partial_{--} + \partial'_{--})\big(\mathcal{T}_{----}(\zeta)\mathcal{G}(\zeta')\big)\\ &+\frac{\mathrm{i}}{2}(\mathcal{D}_+ + \mathcal{D}'_+)\big(\mathcal{P}_{-----}(\zeta)\mathcal{D}'_+\mathcal{P}_{---}(\zeta')\big)\\ &-\frac{\mathrm{i}}{2}(\bar{\mathcal{D}}_+ + \bar{\mathcal{D}}'_+)\big(\bar{\mathcal{P}}_{-----}(\zeta)\bar{\mathcal{D}}'_+\bar{\mathcal{P}}_{---}(\zeta')\big)\,,\end{aligned} \tag{B.34}$$

which is an equation of the form

$$\partial_{--}\mathcal{O}^{T\bar{J}}_{-----}(\zeta,\zeta') = 0 + \mathrm{EoMs} + (\partial + \partial')[\cdots] + (\mathcal{D} + \mathcal{D}')[\cdots]\,, \tag{B.35}$$

as expected. This finalizes the analysis of the well-definedness for the $\mathcal{N} = (0,2)$ $T\bar{J}$ operator. The reader can use the same on-shell resolution of the chirality constraints to show that the same analysis can be performed with the $\mathcal{N} = (0,2)$ $J\bar{T}$ operator of eq. (B.18a) for a general $\mathcal{S}$-multiplet. In fact, the arguments are almost identical considering the same structures of (B.18) and (B.19).

To conclude this section we stress, once more, that despite we have not yet attempted to prove that the generalized Smirnov-Zamolodchikov operators defined in eq. (B.1) are well-defined in general, we expect that a proof will develop along the lines of the cases analyzed so far.

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