Published for SISSA by Springer

Received: *February 25, 2020*
Accepted: *May 4, 2020*
Published: *May 25, 2020*

Three-point functions in $\mathcal{N} = 4$ SYM at finite N_c and background independence

Ryo Suzuki

Shing-Tung Yau Center of Southeast University,
15th Floor, Yifu Architecture Building, No.2 Sipailou,
Xuanwu district, Nanjing, Jiangsu, 210096, China

E-mail: rsuzuki.mp@gmail.com

Abstract: We compute non-extremal three-point functions of scalar operators in $\mathcal{N} = 4$ super Yang-Mills at tree-level in $g_{\rm YM}$ and at finite N_c, using the operator basis of the restricted Schur characters. We make use of the diagrammatic methods called quiver calculus to simplify the three-point functions. The results involve an invariant product of the generalized Racah-Wigner tensors ($6j$ symbols). Assuming that the invariant product is written by the Littlewood-Richardson coefficients, we show that the non-extremal three-point functions satisfy the large N_c background independence; correspondence between the string excitations on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and those in the LLM geometry.

Keywords: AdS-CFT Correspondence, 1/N Expansion, Matrix Models

ArXiv ePrint: 2002.07216

https://doi.org/10.1007/JHEP05(2020)118

JHEP05(2020)118

Contents

JHEP05(2020)118

1 Introduction

Recently we have seen remarkable progress in the computation of the correlation functions of $\mathcal{N}=4$ super Yang-Mills theory (SYM) in the hope of establishing the AdS/CFT correspondence [1]. There are two complementary approaches to this problem.

The first approach is based on the integrability of $\mathcal{N}=4$ SYM in the planar limit. The planar three-point functions of single-trace operators are regarded as a pair of hexagons glued together, where each hexagon form-factor is severely constrained by the centrally-extended $\mathfrak{su}(2|2)$ symmetry [2]. The n-point functions of BPS operators can be studied by *hexagonization*. The gluing of four hexagons give us the planar four-point functions [3–5], and the gluing of $2n-4+4g$ hexagons should give the g-th non-planar corrections [6–8]. Furthermore, certain four-point functions in the large charge limit decompose into a pair of octagons [9, 10], which can be resummed [11, 12].

The integrability approach tells us how single-trace correlation functions depend on the 't Hooft coupling $\lambda = N_c\, g_{\rm YM}^2$. However, only the non-extremal correlation functions have been studied, because the non-extremality is related to the so-called bridge length (the number of Wick contractions between a pair of operators), which suppresses the complicated wrapping corrections to the asymptotic formula [13–17].

The second approach is based on the finite-group theory. In this approach, one obtains the results valid for any values of N_c, though most results are limited to tree-level or a few orders of small λ expansion. In the finite-group approach, extremal correlation functions are often studied, because they are roughly equal to the two-point functions at tree level.

Quite recently the author studied the n-point functions of multi-trace scalar operators at tree-level of $\mathcal{N}=4$ SYM with $\mathrm{U}(N_c)$ gauge group, based on the finite group methods [18]. Those results are written in terms of permutations, meaning that they are valid to any orders of $1/N_c$ expansions, but not at any values of N_c because the finite-N_c constraints are not taken into consideration. The primary purpose of this paper is to generalize the permutation-based results to finite N_c, by taking a Fourier transform of symmetric groups.

Two types of operator bases of $\mathcal{N}=4$ SYM are well-known, which carry a set of Young diagrams as the operator label, diagonalize tree-level two-point functions at finite N_c, generalizing the pioneering work of [19]. The covariant basis (also called BHR basis) introduced in [20, 21] respects the global (or flavor) symmetry of the operator. As such, one can construct $O(N_f)$ singlets for general N_f [22]. The restricted Schur basis was introduced in a series of papers [23–25] and related to multi-matrix models in [26, 27].[1] The restricted Schur basis respects the permutation symmetry of the operator, and suitable for explicit calculation. In other words, one has to specify a state inside the irreducible representation of the global (or flavor) symmetry, like the highest weight state. Here is a brief comparison of the two representation bases [28]:

Operator basis	Symmetry respected	Analogy
Covariant	Global symmetry	Spherical coordinates
Restricted Schur	Permutation of constituents	Cartesian coordinates

[1] Note that the restricted Schur basis can compute the observables of a multi-matrix model, which are not the function of the multi-matrix eigenvalues only.

In this paper, we consider general non-extremal three-point functions of the scalar operators in the restricted Schur basis. There are several important ideas in this computation. The first idea is the Schur-Weyl duality between $\mathrm{U}(N_c)$ and S_L, which converts powers of N_c into the irreducible characters of the symmetric group S_L. The second idea is the *quiver calculus* initiated by [29]. This is a set of diagrammatic rules which enormously simplify the manipulation of representation-theoretical objects. The third idea is the generalized Racah-Wigner tensor. Since the three-point function is non-extremal, we need to compute a non-trivial overlap between the states under different subgroup decompositions of S_L. The invariant products we encounter are more general than Wigner's $6j$ symbols.[2]

Let us summarize the main results. Our notation is explained in appendix A. We are particularly interested in two types of the non-extremal three-point functions (or equivalently non-extremal OPE coefficients). The first type is the super-protected three-point functions [32] in the restricted Schur basis, given by (3.70)

$$\begin{aligned}&\text{Fourier transform of } \left\langle \mathrm{tr}_{L_1}\big(\alpha_1\, Z^{\otimes L_1}\big)\ \mathrm{tr}_{L_2}\big(\alpha_2\, \tilde{Z}^{L_2}\big)\ \mathrm{tr}_{L_3}\big(\alpha_3\, \overline{Z}^{L_3}\big)\right\rangle \\ &= \left(\prod_{i=1}^{3}\frac{L_i!}{\overline{L}_i!}\right)\sum_{\hat{R}\vdash L}\frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1}d_{R_2}d_{R_3}}\sum_{Q_1\vdash\overline{L}_2}\sum_{Q_2\vdash\overline{L}_3}\sum_{Q_3\vdash\overline{L}_1}\left(\prod_{i=1}^{3}d_{Q_i}\right)\mathcal{G}_{123}\,. \end{aligned}\tag{1.1}$$

The second type is the three-point functions of the scalar operators made of three pairs of complex scalars in $\mathcal{N}=4$ SYM, given by (3.90)

$$\begin{aligned}&\text{Fourier transform of } \Big\langle \mathrm{tr}_{L_1}\Big(\alpha_1\, \overline{X}^{\otimes(\ell_{31}-h_2)}\, \overline{Y}^{\otimes h_3}\, Z^{\otimes(\ell_{12}-h_3+h_2)}\Big)\ \times \\ &\mathrm{tr}_{L_2}\Big(\alpha_2\, \overline{X}^{\otimes h_1}\, Y^{\otimes(\ell_{23}-h_1+h_3)}\, \overline{Z}^{\otimes(\ell_{12}-h_3)}\Big)\ \mathrm{tr}_{L_3}\Big(\alpha_3\, X^{\otimes(\ell_{31}-h_2+h_1)}\, \overline{Y}^{\otimes(\ell_{23}-h_1)}\, \overline{Z}^{\otimes h_2}\Big)\Big\rangle \\ &= \left(\prod_{i=1}^{3}\frac{L_i!}{\overline{L}_i!}\right)\sum_{\hat{R}\vdash L}\frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1}d_{R_2}d_{R_3}}\left(d_{q_1}\, d_{q_2}\, d_{r_1}\, d_{r_3}\, d_{s_2}\, d_{s_3}\right)\overline{\delta}^{\,\nu_{1-}\,\nu_{2+}}\,\overline{\delta}^{\,\nu_{2-}\,\nu_{3+}}\,\overline{\delta}^{\,\nu_{3-}\,\nu_{1+}}\,\mathcal{G}'_{123}\,. \end{aligned}\tag{1.2}$$

The objects $\mathcal{G}_{123}$ and $\mathcal{G}'_{123}$ are related to the invariant products of the generalized Racah-Wigner tensors.

Mathematically, the branching coefficient of $R=\bigoplus_{r,s}(r\otimes s)$ is the building block of the restricted Schur character and the generalized Racah-Wigner tensor. In the literature, the orthonormal basis of $r\otimes s$ is called the split basis [33], and the branching coefficients are called fractional parentage coefficients [34], subduction coefficients [35, 36] or the split-standard transformation coefficients [33, 37, 38]. In general, explicit computation of the branching coefficients is a hard problem. See [39–41] for the recent results on the branching coefficients, and on the construction of the restricted Schur basis [42].

Likewise, it is difficult to compute $\mathcal{G}_{123}\,,\mathcal{G}'_{123}$ explicitly. We conjecture that they can be written by the Littlewood-Richardson coefficients, based on the fact that they satisfy certain sum rules.

[2]The $6j$ symbol is also called Racah's W coefficient or recoupling coefficient. The $6j$ symbols of symmetrical groups are called $6f$ symbols in [30], and they are related to the $6j$ symbols of unitary groups by the through the duality factor [31].

From (1.1) and (1.2), it is straightforward to show the large N_c background independence in $\mathcal{N}=4$ SYM [43]. The background independence is a conjectured correspondence between the operators with $\mathcal{O}(N_c^0)$ canonical dimensions and those with $\mathcal{O}(N_c^2)$ canonical dimensions, where the latter is constructed from the former by "attaching" a large number of background boxes. By AdS/CFT, this conjecture implies that the stringy excitations in $\mathrm{AdS}_5 \times \mathrm{S}^5$ and those in the (concentric circle configuration of) LLM geometry [44].

On the gauge theory side, the large N_c background independence has been checked for the case of two-point functions and extremal n-point functions. On the gravity side, some string spectrum of in the SL(2) sector has been studied in [45]. We find that the non-extremal OPE coefficients in the LLM background are essentially given by the rescaling of N_c in (1.1), (1.2). Our results provide strong support that the large N_c background independence can be found also in the string interactions.

2 Two-point functions in the representation basis

We review the construction of the restricted Schur basis, and introduce the diagrammatic computation methods called quiver calculus.

2.1 Set-up

We consider $\mathcal{N}=4$ SYM of $\mathrm{U}(N_c)$ gauge group at tree-level. This theory has three complex scalars (X,Y,Z), which satisfy the $\mathrm{U}(N_c)$ Wick rule,

$$\overbrace{X^b_a(x)\,\overline{X}}{}^d_c(0) = \overbrace{Y^b_a(x)\,\overline{Y}}{}^d_c(0) = \overbrace{Z^b_a(x)\,\overline{Z}}{}^d_c(0) = |x|^{-2}\,\delta^d_a\,\delta^b_c\,. \tag{2.1}$$

With $\alpha \in S_{l+m+n}$, we define a multi-trace operator in the permutation basis

$$\begin{aligned}\mathcal{O}^{(l,m,n)}_\alpha &= \mathrm{tr}_{m+n}\left(\alpha\, X^{\otimes l}\, Y^{\otimes m}\, Z^{\otimes n}\right) \\ &\equiv \sum_{i_1,i_2,\ldots,i_{l+m+n}=1}^{N_c} X^{i_1}_{i_{\alpha(1)}} \ldots X^{i_l}_{i_{\alpha(l)}}\, Y^{i_{m+1}}_{i_{\alpha(l+1)}} \ldots Y^{i_{l+m}}_{i_{\alpha(l+m)}}\, Z^{i_{l+m+1}}_{i_{\alpha(l+m+1)}} \ldots Z^{i_{l+m+n}}_{i_{\alpha(l+m+n)}}\,.\end{aligned} \tag{2.2}$$

The usual single-trace operators can be expressed in the permutation basis as

$$\mathrm{tr}\,(X^l\, Y^m\, Z^n) \;\to\; \mathrm{tr}_L(\alpha\, X^{\otimes l}\, Y^{\otimes m}\, Z^{\otimes n}), \qquad (\alpha_i \in \mathbb{Z}_{l+m+n}). \tag{2.3}$$

The correspondence between a multi-trace operator and $\alpha \in S_L$ is not one-to-one, because α is defined modulo conjugation,

$$\mathcal{O}^{(l,m,n)}_\alpha = \mathcal{O}^{(l,m,n)}_{\gamma\alpha\gamma^{-1}}, \qquad \gamma \in S_l \otimes S_m \otimes S_n \tag{2.4}$$

which we call the flavor symmetry (or global symmetry). For example,

$$\begin{aligned}\mathrm{tr}\,(XXZZ) &= \mathrm{tr}_{L=4}((1234)\, X^{\otimes 2} Z^{\otimes 2}) = \mathrm{tr}_{L=4}((2143)\, X^{\otimes 2} Z^{\otimes 2}) = \ldots \\ \mathrm{tr}\,(XZXZ) &= \mathrm{tr}_{L=4}((1324)\, X^{\otimes 2} Z^{\otimes 2}) = \mathrm{tr}_{L=4}((3142)\, X^{\otimes 2} Z^{\otimes 2}) = \ldots\end{aligned} \tag{2.5}$$

where ... represents the other permutations generated by the flavor symmetry (2.4).

We define the complex conjugate operator by

$$\overline{\mathcal{O}}_{\alpha}^{(l,m,n)} = \mathrm{tr}_{m+n}\left(\alpha\, \overline{X}^{\otimes l}\, \overline{Y}^{\otimes m}\, \overline{Z}^{\otimes n}\right) \tag{2.6}$$

The two-point function between $\mathcal{O}_{\alpha_1}^{(l,m,n)}$ and $\overline{\mathcal{O}}_{\alpha_2}^{(l,m,n)}$ at tree-level is given by

$$\langle \mathcal{O}_{\alpha_1}^{(l,m,n)}(x)\, \overline{\mathcal{O}}_{\alpha_2}^{(l,m,n)}(0)\rangle = |x|^{-2(l+m+n)} \sum_{\gamma \in S_l \otimes S_m \otimes S_n} N_c^{C(\alpha_1 \gamma \alpha_2 \gamma^{-1})} \tag{2.7}$$

where $C(\omega)$ counts the number of cycles in $\omega \in S_{l+m+n}$. We write $\langle \mathcal{O}_1 \overline{\mathcal{O}}_2 \rangle \equiv \langle \mathcal{O}_1(1) \overline{\mathcal{O}}_2(0)\rangle$.

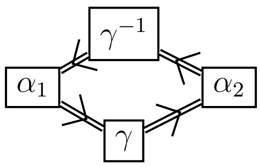
2.2 Diagonalizing the tree-level two-point

Following [29], we show how to "derive" the representation basis of operators starting from the two-point functions on the permutation basis (2.7). The resulting tree-level two-point functions are diagonal at any N_c. The readers familiar with the restricted Schur basis can skip this subsection. The basic formulae are summarized in appendix A.3.

First, we rewrite the equation (2.7) by using (A.41) as

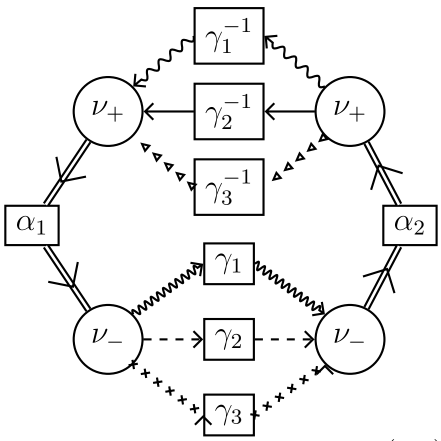
$$\langle \mathcal{O}_{\alpha_1}^{(l,m,n)}\, \overline{\mathcal{O}}_{\alpha_2}^{(l,m,n)}\rangle = \sum_{\gamma \in S_l \otimes S_m \otimes S_n} \sum_{R \vdash (l+m+n)} \mathrm{Dim}_{N_c}(R) \chi^R(\alpha_1 \gamma \alpha_2 \gamma^{-1})$$

$$= \sum_{R \vdash (l+m+n)} \mathrm{Dim}_{N_c}(R) \sum_{\gamma \in S_l \otimes S_m \otimes S_n} \tag{2.8}$$



where we used the quiver calculus notation of appendix B in the second line. We introduce $\gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3 \in S_l \otimes S_m \otimes S_n$ and the branching coefficients for $S_{l+m+n} \downarrow (S_l \otimes S_m \otimes S_n)$ to make use of the identity (A.24) for $\ell = 3$. The equation (2.8) becomes

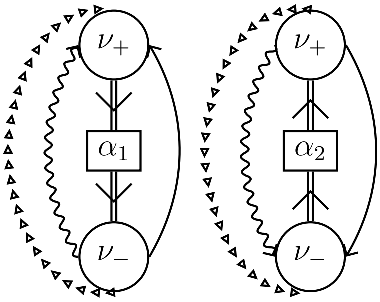
$$\langle \mathcal{O}_{\alpha_1}^{(l,m,n)}\, \overline{\mathcal{O}}_{\alpha_2}^{(l,m,n)}\rangle = \sum_{R \vdash (l+m+n)} \mathrm{Dim}_{N_c}(R) \sum_{\substack{\gamma_1 \in S_l \\ \gamma_2 \in S_m \\ \gamma_3 \in S_n}} \sum_{\substack{r_1, r_2, r_3, \nu_- \\ s_1, s_2, s_3, \nu_+}}$$



(2.9)

We apply the grand orthogonality (B.4) to the matrix elements of γ_1 , γ_2 and γ_3 to obtain

$$\langle \mathcal{O}^{(l,m,n)}_{\alpha_1}\, \overline{\mathcal{O}}^{(l,m,n)}_{\alpha_2} \rangle = \sum_{R\vdash(l+m+n)} \mathrm{Dim}_{N_c}(R) \sum_{r_1,r_2,r_3,\nu_-,\nu_+} \frac{l!\, m!\, n!}{d_{r_1} d_{r_2} d_{r_3}}$$



$$= \sum_{R,r_1,r_2,r_3,\nu_-,\nu_+} \mathrm{Dim}_{N_c}(R) \frac{l!\, m!\, n!}{d_{r_1} d_{r_2} d_{r_3}} \chi^{R,(r_1,r_2,r_3),(\nu_+,\nu_-)}(\alpha_1)\, \chi^{R,(r_1,r_2,r_3),(\nu_-,\nu_+)}(\alpha_2)$$

where $\chi^{R,(r_1,r_2,r_3),(\nu_+,\nu_-)}(\alpha)$ is the restricted characters defined through branching coefficients,

$$\chi^{R,(r_1,r_2,r_3),\nu_+,\nu_-}(\sigma) \equiv \sum_{I,J} \sum_{i,j} B^{R\to(r_1,r_2,r_3)\nu_+}_{I\to(i,j,k)} \, (B^T)^{R\to(r_1,r_2,r_3)\nu_-}_{J\to(i,j,k)} \, D^R_{IJ}(\sigma). \tag{2.10}$$

The restricted characters satisfy the orthogonality relations (A.52). It is straightforward to find a linear combination of operators which diagonalizes the two-point function;

$$\begin{aligned} \mathcal{O}^{S,(s_1,s_2,s_3),\mu_+,\mu_-}(x) &= \frac{1}{l!\, m!\, n!} \sum_{\alpha\in S_{l+m+n}} \chi^{S,(s_1,s_2,s_3),\mu_+,\mu_-}(\alpha)\, \mathcal{O}^{(l,m,n)}_{\alpha}(x) \\ \overline{\mathcal{O}}^{T,(t_1,t_2,t_3),\eta_+,\eta_-}(y) &= \frac{1}{l!\, m!\, n!} \sum_{\alpha\in S_{l+m+n}} \chi^{T,(t_1,t_2,t_3),\eta_+,\eta_-}(\alpha)\, \overline{\mathcal{O}}^{(l,m,n)}_{\alpha}(y). \end{aligned} \tag{2.11}$$

It follows that

$$\begin{aligned} &\left\langle \mathcal{O}^{S,(s_1,s_2,s_3),\mu_+,\mu_-}\, \overline{\mathcal{O}}^{T,(t_1,t_2,t_3),\eta_+,\eta_-} \right\rangle = \left(\frac{1}{l!\, m!\, n!}\right)^2 \sum_{R,r_1,r_2,r_3,\nu_-,\nu_+} \mathrm{Dim}_{N_c}(R) \frac{l!m!n!}{d_{r_1} d_{r_2} d_{r_3}} \times \\ &\sum_{\alpha_1\,,\alpha_2\in S_{l+m+n}} \chi^{S,(s_1,s_2,s_3),\mu_+,\mu_-}(\alpha_1)\, \chi^{T,(t_1,t_2,t_3),\eta_+,\eta_-}(\alpha_2) \chi^{R,(r_1,r_2,r_3),(\nu_+,\nu_-)}(\alpha_1) \chi^{R,(r_1,r_2,r_3),(\nu_-,\nu_+)}(\alpha_2) \\ &= \mathrm{Dim}_{N_c}(S) \frac{(l+m+n)!^2}{l!\, m!\, n!} \frac{d_{s_1} d_{s_2} d_{s_3}}{d_S^2} \delta^{ST} \delta^{s_1 t_1} \delta^{s_2 t_2} \delta^{s_3 t_3} \delta^{\mu_+ \eta_-} \delta^{\mu_- \eta_+} \\ &= \mathrm{Wt}_{N_c}(S) \frac{\mathrm{hook}_S}{\mathrm{hook}_{s_1} \mathrm{hook}_{s_2} \mathrm{hook}_{s_3}} \delta^{ST} \delta^{s_1 t_1} \delta^{s_2 t_2} \delta^{s_3 t_3} \delta^{\mu_+ \eta_-} \delta^{\mu_- \eta_+} \end{aligned} \tag{2.12}$$

where we used (A.5).

Recall that $\mathcal{O}^{(l,m,n)}_{\alpha}$ in (2.2) becomes half-BPS when $l = m = 0$, and the restricted character (2.10) reduces to the usual irreducible characters of S_n . The two-point function (2.12) becomes

$$\left\langle \mathcal{O}^S\, \overline{\mathcal{O}}^T \right\rangle = \mathrm{Wt}_{N_c}(S)\, \delta^{ST} \tag{2.13}$$

which gives the same normalization of half-BPS operators as in [19].

3 Three-point functions in the representation basis

In [18], tree-level formulae of the n-point functions of general scalar operators in the permutation basis have been derived. We consider three-point functions of scalar operators in the restricted Schur basis below. The three-point functions of $\mathcal{N} = 4$ SYM are related to the OPE coefficients by

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}\, |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}\, |x_3 - x_1|^{\Delta_3+\Delta_1-\Delta_2}} \tag{3.1}$$

thanks to the conformal symmetry. By abuse of notation, we write (3.1) as

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = C_{123} \, . \tag{3.2}$$

3.1 Set-up

Let us recall the tree-level permutation formula for three-point functions in [18]. That formula has been derived based on the following idea. Consider a non-extremal three-point function of the operators labeled by $\alpha_i \in S_{L_i}$ for $i = 1, 2, 3$. We expect that the tree-level Wick contractions give the quantity like $N_c^{C(\alpha_1\alpha_2\alpha_3)}$. However, we cannot define the multiplication of elements in S_{L_1} and S_{L_2} if $L_1 \neq L_2$. This problem can be solved by extending α_i to $\hat\alpha_i \in S_L$ for some L , which makes the quantity $N_c^{C(\hat\alpha_1\hat\alpha_2\hat\alpha_3)}$ well-defined.

Let us explain how this idea works. First, we extend the operator $\mathcal{O}_i$ by adding identity fields,

$$\hat{\mathcal{O}}_i \equiv \mathcal{O}_{\alpha_i} \times \mathrm{tr}\,(\mathbf{1})^{\overline{L}_i} \equiv \prod_{p=1}^{L} (\Phi^{\hat A_p^{(i)}})^{a_p}_{a_{\hat\alpha_i(p)}} \, , \qquad \hat\alpha_i = \alpha_i \circ \mathbf{1}_{\overline{L}_i} \in S_{L_i} \times S_{\overline{L}_i} \subset S_L \tag{3.3}$$

where

$$L = \frac{L_1 + L_2 + L_3}{2} \, , \qquad \overline{L}_i = L - L_i \, , \qquad \Phi^{\hat A_p^{(i)}} \in (X, \overline{X}, Y, \overline{Y}, Z, \overline{Z}, \mathbf{1}). \tag{3.4}$$

The permutation $\hat\alpha_i$ acts as the identity at the position p at which $\Phi^{\hat A_p^{(i)}} = \mathbf{1}$. The (edge-type) permutation formula reads

$$C_{123} = \frac{1}{\prod_{i=1}^{3} \overline{L}_i!} \frac{1}{L!} \sum_{\{U_i\} \in S_L^{\otimes 3}} \left(\prod_{p=1}^{L} h^{\check A_p^{(1)} \check A_p^{(2)} \check A_p^{(3)}} \right) N_c^{C(\check\alpha_1\, \check\alpha_2\, \check\alpha_3)} \tag{3.5}$$

where $\check A_p^{(i)} \equiv \hat A_{U_i(p)}^{(i)}$, $\check\alpha_i \equiv U_i^{-1}\, \hat\alpha_i\, U_i$ and

$$h^{ABC} = h^{AB}\, \delta_{\mathbf{1}}^{C} + h^{BC}\, \delta_{\mathbf{1}}^{A} + h^{CA}\, \delta_{\mathbf{1}}^{B} \, , \quad h^{AB} = \begin{cases} g^{AB} \equiv \langle \Phi^A(1)\Phi^B(0)\rangle & (\text{both } \Phi^A, \Phi^B \neq \mathbf{1}) \\ 0 & (\text{otherwise}). \end{cases} \tag{3.6}$$

We call h^{ABC} a triple Wick contraction.

We will consider two types of three-point functions. The first type is the three-point functions of half-BPS multi-trace operators,

$$C_{\circ\circ\circ} = \Big\langle \mathrm{tr}_{L_1}\big(\alpha_1\, Z^{\otimes L_1}\big)\ \mathrm{tr}_{L_2}\big(\alpha_2\, \tilde{Z}^{\otimes L_2}\big)\ \mathrm{tr}_{L_3}\big(\alpha_3\, \overline{Z}^{\otimes L_3}\big)\Big\rangle, \qquad \tilde{Z} = (Z + \overline{Z} + Y - \overline{Y}). \tag{3.7}$$

The field $\tilde{Z}$ belongs to the one-parameter family of operators used in [2, 32],

$$\mathfrak{Z}_i(a) = (Z + a_i\,(Y - \overline{Y}) + a_i^2\,\overline{Z})(x_i), \qquad x_i = (0, a_i, 0, 0). \tag{3.8}$$

The second type is general three-point functions of the scalar multi-trace operator (2.2),

$$\begin{aligned} C^{XYZ}_{\vec{h}} = \Big\langle \mathrm{tr}_{L_1}\Big(\alpha_1\, \overline{X}^{\otimes(\ell_{31}-h_2)}\, \overline{Y}^{\otimes h_3}\, Z^{\otimes(\ell_{12}-h_3+h_2)}\Big) \times \\ \mathrm{tr}_{L_2}\Big(\alpha_2\, \overline{X}^{\otimes h_1} Y^{\otimes(\ell_{23}-h_1+h_3)} \overline{Z}^{\otimes(\ell_{12}-h_3)}\Big)\, \mathrm{tr}_{L_3}\Big(\alpha_3\, X^{\otimes(\ell_{31}-h_2+h_1)} \overline{Y}^{\otimes(\ell_{23}-h_1)} \overline{Z}^{\otimes h_2}\Big)\Big\rangle \end{aligned} \tag{3.9}$$

where ℓ_{ij} is the number of tree-level Wick contractions between $\mathcal{O}_i$ and $\mathcal{O}_j$ (called the bridge length), given by

$$\ell_{12} = \frac{L_1 + L_2 - L_3}{2}\,, \qquad \ell_{23} = \frac{L_2 + L_3 - L_1}{2}\,, \qquad \ell_{31} = \frac{L_3 + L_1 - L_2}{2} \tag{3.10}$$

and h_i is an integer inside the range

$$0 \le h_1 \le \ell_{23}\,, \qquad 0 \le h_2 \le \ell_{31}\,, \qquad 0 \le h_3 \le \ell_{12}\,. \tag{3.11}$$

3.2 Partial Fourier transform

We construct the three-point functions in the restricted Schur basis by taking the Fourier transform of $C_{\circ\circ\circ}$ in (3.7) and $C^{XYZ}_{\vec{h}}$ (3.9). Recall that the usual Fourier transform of the delta function is a constant. In the Fourier transform over a finite group, the Fourier transform of the identity permutation should be a sum over all representations. In other words, if we write

$$R_i \vdash L_i \quad \leftrightarrow \quad \text{FT of } \alpha_i \in S_{L_i}\,, \qquad t_i \vdash \overline{L}_i \quad \leftrightarrow \quad \text{FT of } \mathbf{1}^{\overline{L}_i} \in S_{\overline{L}_i} \tag{3.12}$$

then we should sum t_i over all possible partitions of $\overline{L}_i$. In fact, t_i is an unphysical parameter, and we can perform a calculation without using t_i . Thus we call the procedure (3.12) a partial Fourier transform.

In order to treat $C_{\circ\circ\circ}$ and $C^{XYZ}_{\vec{h}}$ simultaneously, we extend the multi-trace operator (2.2) as in (3.3),

$$\begin{gathered} \mathcal{O}^{(l_i, m_i, n_i, \overline{L}_i)}_{\hat{\alpha}_i}[X, Y, Z, \mathbf{1}] = \mathrm{tr}_{L_i}\Big(\alpha_i\, X^{\otimes l_i}\, Y^{\otimes m_i}\, Z^{\otimes n_i}\Big) \times \mathrm{tr}\,(\mathbf{1})^{\overline{L}_i} \\ l_i + m_i + n_i = L_i\,, \quad L_i + \overline{L}_i = L\,, \quad \hat{\alpha}_i = \alpha_i \circ \mathbf{1}_{\overline{L}_i} \in S_L \end{gathered} \tag{3.13}$$

and define the partial Fourier transform by

$$\begin{gathered} \hat{\mathcal{O}}^{\boldsymbol{R}_i\,(\overline{L}_i)}[X, Y, Z, \mathbf{1}] = \frac{1}{l_i!\, m_i!\, n_i!} \sum_{\alpha_i \in S_{L_i}} \chi^{\boldsymbol{R}_i}(\alpha_i)\, \mathcal{O}^{(l_i, m_i, n_i, \overline{L}_i)}_{\hat{\alpha}_i}[X, Y, Z, \mathbf{1}] \\ \boldsymbol{R}_i = \{R_i, (q_i, r_i, s_i), \nu_{i-}, \nu_{i+}\}\,, \quad (R_i \vdash L_i,\ q_i \vdash l_i,\ r_i \vdash m_i,\ s_i \vdash n_i)\,. \end{gathered} \tag{3.14}$$

The partial Fourier transform can be rewritten as a linear combination of the complete Fourier transform. To see this, we recall (A.34) and

$$\chi^{\boldsymbol{R}_i\otimes t_i}(\alpha_i\circ \mathbf{1}_{\overline{L}_i}) = \chi^{\boldsymbol{R}_i}(\alpha_i)\, d_{t_i}\,, \qquad \sum_{t_i\vdash \overline{L}_i} d_{t_i}^2 = \overline{L}_i \tag{3.15}$$

giving us a dummy representation t_i to be summed over the partitions of $\overline{L}_i$. It follows that

$$\hat{\mathcal{O}}^{\boldsymbol{R}_i\,(\overline{L}_i)}[X,Y,Z,\mathbf{1}] = \frac{1}{l_i!\,m_i!\,n_i!\,\overline{L}_i!}\sum_{t_i\vdash\overline{L}_i}\ \sum_{\hat\alpha_i\in S_{L_i}\times\mathbf{1}_{\overline{L}_i}} d_{t_i}\,\chi^{\boldsymbol{R}_i\otimes t_i}(\hat\alpha_i)\,\mathcal{O}_{\hat\alpha_i}^{(l_i,m_i,n_i,\overline{L}_i)}[X,Y,Z,\mathbf{1}]. \tag{3.16}$$

As for $C_{\circ\circ\circ}$, we introduce the Fourier transform of the half-BPS operators as

$$\widetilde{\mathcal{O}}_1 = \hat{\mathcal{O}}_1^{R_1(\overline{L}_1)}[Z,\mathbf{1}],\quad \widetilde{\mathcal{O}}_2 = \hat{\mathcal{O}}_2^{R_2(\overline{L}_2)}[\tilde{Z},\mathbf{1}],\quad \widetilde{\mathcal{O}}_3 = \hat{\mathcal{O}}_3^{R_3(\overline{L}_3)}[\overline{Z},\mathbf{1}],\quad \boldsymbol{R}_i = R_i\vdash L_i \tag{3.17}$$

and define

$$\tilde{C}_{\circ\circ\circ} = \left\langle \hat{\mathcal{O}}_1^{R_1(\overline{L}_1)}[Z,\mathbf{1}]\,\hat{\mathcal{O}}_2^{R_2(\overline{L}_2)}[\tilde{Z},\mathbf{1}]\,\hat{\mathcal{O}}_3^{R_3(\overline{L}_3)}[\overline{Z},\mathbf{1}]\right\rangle. \tag{3.18}$$

As for $C_{\vec{h}}^{XYZ}$, we take the Fourier transform of the operators in (3.9) as

$$\begin{aligned}
\widetilde{\mathcal{O}}_1 &= \hat{\mathcal{O}}_1^{\boldsymbol{R}_1(\overline{L}_1)}[\overline{X},\overline{Y},Z,\mathbf{1}] &\quad (l_1,m_1,n_1) &= (\ell_{31}-h_2,\ h_3,\ \ell_{12}-h_3+h_2)\\
\widetilde{\mathcal{O}}_2 &= \hat{\mathcal{O}}_2^{\boldsymbol{R}_2(\overline{L}_2)}[\overline{X},Y,\overline{Z},\mathbf{1}] &\quad (l_2,m_2,n_2) &= (h_1,\ \ell_{23}-h_1+h_3,\ \ell_{12}-h_3)\\
\widetilde{\mathcal{O}}_3 &= \hat{\mathcal{O}}_3^{\boldsymbol{R}_3(\overline{L}_3)}[X,\overline{Y},\overline{Z},\mathbf{1}] &\quad (l_3,m_3,n_3) &= (\ell_{31}-h_2+h_1,\ \ell_{23}-h_1,\ h_2)
\end{aligned} \tag{3.19}$$

and define

$$\tilde{C}_{\vec{h}}^{XYZ} = \left\langle \hat{\mathcal{O}}_1^{\boldsymbol{R}_1(\overline{L}_1)}[\overline{X},\overline{Y},Z,\mathbf{1}]\,\hat{\mathcal{O}}_2^{\boldsymbol{R}_2(\overline{L}_2)}[\overline{X},Y,\overline{Z},\mathbf{1}]\,\hat{\mathcal{O}}_3^{\boldsymbol{R}_3(\overline{L}_3)}[X,\overline{Y},\overline{Z},\mathbf{1}]\right\rangle. \tag{3.20}$$

We collectively denote the three-point functions of the operators in the representation basis by

$$\tilde{C}_{123} \equiv \left\langle \widetilde{\mathcal{O}}_1\,\widetilde{\mathcal{O}}_2\,\widetilde{\mathcal{O}}_3\right\rangle. \tag{3.21}$$

From (3.5) we get

$$\begin{aligned}
\tilde{C}_{123} = {}& \frac{1}{\prod_{i=1}^3 l_i!\,m_i!\,n_i!\,(\overline{L}_i!)^2}\,\frac{1}{L!}\sum_{\{U_i\}\in S_L^{\otimes 3}}\left(\prod_{p=1}^{L} h^{\hat{A}^{(1)}_{U_1(p)}\hat{A}^{(2)}_{U_2(p)}\hat{A}^{(3)}_{U_3(p)}}\right)\sum_{\{t_i\vdash\overline{L}_i\}}\left(\prod_{i=1}^3 d_{t_i}\right)\times\\
&\sum_{\{\hat\alpha_i\in S_{L_i}\times\mathbf{1}_{\overline{L}_i}\}}\left(\prod_{i=1}^3\chi^{\boldsymbol{R}_i\otimes t_i}(\hat\alpha_i)\right)N_c^{C(U_1^{-1}\hat\alpha_1 U_1U_2^{-1}\hat\alpha_2 U_2U_3^{-1}\hat\alpha_3 U_3)}.
\end{aligned} \tag{3.22}$$

Consider the second line of (3.22). We use the identity (A.41) and (A.9) to obtain

$$\begin{aligned}
&\sum_{\{\hat\alpha_i\in S_{L_i}\times\mathbf{1}_{\overline{L}_i}\}}\left(\prod_{i=1}^3\chi^{\boldsymbol{R}_i\otimes t_i}(\hat\alpha_i)\right)N_c^{C(U_1^{-1}\hat\alpha_1 U_1U_2^{-1}\hat\alpha_2 U_2U_3^{-1}\hat\alpha_3 U_3)} \qquad (3.23)\\
&= \sum_{\{\hat\alpha_i\in S_{L_i}\times\mathbf{1}_{\overline{L}_i}\}}\ \sum_{\hat{R}\vdash L}\mathrm{Dim}_{N_c}(\hat{R})\left(\prod_{i=1}^3\chi^{\boldsymbol{R}_i\otimes t_i}(\hat\alpha_i)D^{\hat{R}}_{\hat{I}_i\hat{J}_i}(\hat\alpha_i)\right)\\
&\quad\times D^{\hat{R}}_{\hat{J}_1\hat{I}_2}(U_1U_2^{-1})D^{\hat{R}}_{\hat{J}_2\hat{I}_3}(U_2U_3^{-1})D^{\hat{R}}_{\hat{J}_3\hat{I}_1}(U_3U_1^{-1}).
\end{aligned}$$

JHEP05(2020)118

We simplify the sum over $\{\hat{\alpha}_i\}$ in the last line. The character is given by (3.15). We decompose the matrix elements $D^{\hat{R}}_{\hat{I}_i\hat{J}_i}(\hat{\alpha}_i)$ according to the restriction

$$S_L \downarrow (S_{L_i} \otimes S_{\overline{L}_i}), \qquad \hat{R} = \bigoplus_{R'_i \vdash L_i} \bigoplus_{T_i \vdash \overline{L}_i} \bigoplus_{\mu_i=1}^{g(R'_i,t'_i;\hat{R})} (R'_i \otimes T_i)_{\mu_i} \,. \tag{3.24}$$

When $\tilde{C}_{123} = \tilde{C}_{\circ\circ\circ}$, we have $\boldsymbol{R}_i = R_i$. From (3.24) we get

$$\begin{aligned}
&\sum_{\hat{\alpha}_i} \chi^{\boldsymbol{R}_i \otimes t_i}(\hat{\alpha}_i) D^{\hat{R}}_{\hat{I}_i\hat{J}_i}(\hat{\alpha}_i) \\
&= \sum_{\alpha_i \in S_{L_i}} \sum_{R'_i \vdash L_i} \sum_{T_i \vdash \overline{L}_i} \sum_{\mu_i=1}^{g(R'_i,T_i;\hat{R})} \chi^{R_i}(\alpha_i)\, d_{t_i}\, B^{\hat{R}\to(R'_i,T_i),\mu_i}_{\hat{I}_i\to(I_i,c_i)}\, (B^T)^{\hat{R}\to(R'_i,T_i),\mu_i}_{\hat{J}_i\to(J_i,c_i)}\, D^{R'_i}_{I_iJ_i}(\alpha_i) \\
&= \sum_{R'_i,T_i,\mu_i} \Big\{ \sum_{\alpha_i \in S_{L_i}} \chi^{R_i}(\alpha_i) D^{R'_i}_{I_iJ_i}(\alpha_i) \Big\} d_{t_i}\, B^{\hat{R}\to(R'_i,T_i),\mu_i}_{\hat{I}_i\to(I_i,c_i)}\, (B^T)^{\hat{R}\to(R'_i,T_i),\mu_i}_{\hat{J}_i\to(J_i,c_i)} \\
&= \sum_{T_i,\mu_i} \frac{L_i!\, d_{t_i}}{d_{R_i}}\, B^{\hat{R}\to(R_i,T_i),\mu_i}_{\hat{I}_i\to(I_i,c_i)}\, (B^T)^{\hat{R}\to(R_i,T_i),\mu_i}_{\hat{J}_i\to(I_i,c_i)} \\
&= \sum_{T_i \vdash \overline{L}_i} \sum_{\mu_i=1}^{g(R_i,T_i\,;\hat{R})} \frac{L_i!\, d_{t_i}}{d_{R_i}}\, \mathscr{P}^{\hat{R}\to(R_i,T_i),\mu_i,\mu_i}_{\hat{I}_i\hat{J}_i}
\end{aligned} \tag{3.25}$$

where we used (3.15), (A.20), (A.30) and (A.47). When $\tilde{C}_{123} = \tilde{C}^{XYZ}_{\vec{h}}$, by using the definition of the restricted character (A.25) we find

$$\begin{aligned}
&\sum_{\hat{\alpha}_i \in S_{L_i} \times \mathbf{1}_{\overline{L}_i}} \chi^{\boldsymbol{R}_i \otimes t_i}(\hat{\alpha}_i) D^{\hat{R}}_{\hat{I}_i\hat{J}_i}(\hat{\alpha}_i) \\
&= \sum_{R'_i,T_i,\mu_i} \Big\{ \sum_{\alpha_i \in S_{L_i}} D^{R_i}_{I'J'}(\alpha_i) D^{R'_i}_{I_iJ_i}(\alpha_i) \Big\} d_{t_i} \\
&\quad \times B^{R_i\to(q_i,r_i,s_i)\nu_{i-}}_{I'\to(j',k',l')}\, (B^T)^{R_i\to(q_i,r_i,s_i)\nu_{i+}}_{J'\to(j',k',l')}\, B^{\hat{R}\to(R'_i,T_i),\mu_i}_{\hat{I}_i\to(I_i,c_i)}\, (B^T)^{\hat{R}\to(R'_i,T_i),\mu_i}_{\hat{J}_i\to(J_i,c_i)} \\
&= \sum_{T_i,\mu_i} \frac{L_i!\, d_{t_i}}{d_{R_i}}\, B^{\hat{R}\to(R_i,T_i),\mu_i}_{\hat{I}_i\to(I_i,c_i)}\, B^{R_i\to(q_i,r_i,s_i)\nu_{i-}}_{I_i\to(j',k',l')}\, (B^T)^{\hat{R}\to(R_i,T_i),\mu_i}_{\hat{J}_i\to(J_i,c_i)}\, (B^T)^{R_i\to(q_i,r_i,s_i)\nu_{i+}}_{J_i\to(j',k',l')} \\
&\equiv \sum_{T_i,\mu_i} \frac{L_i!\, d_{t_i}}{d_{R_i}}\, \mathscr{P}^{\hat{R}\to\boldsymbol{RT}_{i-,i+}}_{\hat{I}\hat{J}}
\end{aligned} \tag{3.26}$$

where we introduced the double projector

$$\mathscr{P}^{\hat{R}\to\boldsymbol{RT}_{i-,i+}}_{\hat{I}\hat{J}} = \sum_{j,k,l,c} \mathcal{B}^{\hat{R}\to\boldsymbol{RT}_{i-}}_{\hat{I}\longrightarrow(j,k,l,c)}\, (\mathcal{B}^T)^{\hat{R}\to\boldsymbol{RT}_{i+}}_{\hat{J}\longrightarrow(j,k,l,c)} \tag{3.27}$$

$$\mathcal{B}^{\hat{R}\to\boldsymbol{RT}_{i\mp}}_{\hat{I}\longrightarrow(j,k,l,c)} \equiv \sum_{I=1}^{d_{R_i}} B^{\hat{R}\to(R_i,T_i),\mu_i}_{\hat{I}\to(I,c)}\, B^{R_i\to(q_i,r_i,s_i),\nu_{i\mp}}_{I\to(j,k,l)} \,. \tag{3.28}$$

$$\Big\{ \hat{R} \to \boldsymbol{RT}_{i\mp} \Big\} = \Big\{ \hat{R} \to (R_i,T_i), \mu_i \to (q_i,r_i,s_i,T_i), (\mu_i,\nu_{i\mp}) \Big\} \tag{3.29}$$

which come from the double restriction $S_L \downarrow (S_{L_i} \otimes S_{\overline{L}_i}) \downarrow (S_{l_i} \otimes S_{m_i} \otimes S_{n_i} \otimes S_{\overline{L}_i})$. Here we should keep in mind that the restriction to the subgroup of S_L is different for each $i = 1, 2, 3$. We will revisit this issue in section 3.4.

Now the equation (3.23) is simplified as

$$\begin{aligned} &\sum_{\{\hat{\alpha}_i \in S_{L_i} \times \mathbf{1}_{\overline{L}_i}\}} \left(\prod_{i=1}^{3} \chi^{\boldsymbol{R}_i \otimes t_i}(\hat{\alpha}_i) \right) N_c^{C(U_1^{-1}\,\hat{\alpha}_1\,U_1 U_2^{-1}\,\hat{\alpha}_2\,U_2 U_3^{-1}\,\hat{\alpha}_3\,U_3)} \\ &= \sum_{\{T_i, \mu_i\}} \left(\prod_{i=1}^{3} \mathscr{P}_{\hat{I}_i \hat{J}_i}^{\hat{R} \to \mathrm{sub}} \right) D^{\hat{R}}_{\hat{J}_1 \hat{I}_2}(U_1 U_2^{-1})\, D^{\hat{R}}_{\hat{J}_2 \hat{I}_3}(U_2 U_3^{-1})\, D^{\hat{R}}_{\hat{J}_3 \hat{I}_1}(U_3 U_1^{-1}) \end{aligned} \tag{3.30}$$

where the projector $\mathscr{P}_{\hat{I}_i \hat{J}_i}^{\hat{R} \to \mathrm{sub}}$ is given by

$$\mathscr{P}_{\hat{I}_i \hat{J}_i}^{\hat{R} \to \mathrm{sub}} \equiv \begin{cases} \mathscr{P}_{\hat{I}_i \hat{J}_i}^{\hat{R} \to (R_i, T_i)\mu_i, \mu_i} = B^{\hat{R} \to (R_i, T_i), \mu_i}_{\hat{I}_i \to (I_i, c_i)}\, (B^T)^{\hat{R} \to (R_i, T_i), \mu_i}_{\hat{J}_i \to (I_i, c_i)} & \left(\text{for } \ \tilde{C}_{\circ\circ\circ}\right) \\ \mathscr{P}_{\hat{I}_i \hat{J}_i}^{\hat{R} \to \boldsymbol{RT}_{i-,i+}} = \mathcal{B}^{\hat{R} \to \boldsymbol{RT}_{i-}}_{\hat{I} \longrightarrow (j,k,l,c)}\, (\mathcal{B}^T)^{\hat{R} \to \boldsymbol{RT}_{i+}}_{\hat{J} \longrightarrow (j,k,l,c)} & \left(\text{for } \ \tilde{C}^{XYZ}_{\vec{h}}\right). \end{cases} \tag{3.31}$$

The three-point function (3.22) becomes

$$\begin{aligned} \tilde{C}_{123} = &\left(\prod_{i=1}^{3} \frac{L_i!}{l_i!\, m_i!\, n_i!\, \overline{L}_i!} \right) \frac{1}{L!} \sum_{\{U_i\} \in S_L^{\otimes 3}} \left(\prod_{p=1}^{L} h^{\hat{A}^{(1)}_{U_1(p)} \hat{A}^{(2)}_{U_2(p)} \hat{A}^{(3)}_{U_3(p)}} \right) \sum_{\hat{R} \vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \times \\ &\sum_{\{T_i, \mu_i\}} \left(\prod_{i=1}^{3} \mathscr{P}_{\hat{I}_i \hat{J}_i}^{\hat{R} \to \mathrm{sub}} \right) D^{\hat{R}}_{\hat{J}_1 \hat{I}_2}(U_1 U_2^{-1})\, D^{\hat{R}}_{\hat{J}_2 \hat{I}_3}(U_2 U_3^{-1})\, D^{\hat{R}}_{\hat{J}_3 \hat{I}_1}(U_3 U_1^{-1}) \end{aligned} \tag{3.32}$$

where (3.15) is used to sum over t_i .

3.3 Sum over Wick contractions

We simplify the sum over the Wick contractions, denoted by $\{U_i\} \in S_L^{\otimes 3}$ in (3.32).

3.3.1 Symmetry of the permutation formula

To begin with, let us review the symmetry in the permutation formula (3.5) for a fixed $\{U_i\}$,

$$C_{123}(\{U_i\}) = \frac{1}{\overline{L}_1!\, \overline{L}_2!\, \overline{L}_3!\, L!} \left(\prod_{p=1}^{L} h^{\hat{A}^{(1)}_{U_1(p)} \hat{A}^{(2)}_{U_2(p)} \hat{A}^{(3)}_{U_3(p)}} \right) N_c^{C(U_1^{-1}\,\hat{\alpha}_1\,U_1 U_2^{-1}\,\hat{\alpha}_2\,U_2 U_3^{-1}\,\hat{\alpha}_3\,U_3)} . \tag{3.33}$$

Since $\tilde{C}_{123}$ is a linear combination of C_{123} , the equation (3.32) should inherit the same symmetry.

First, $C_{123}(\{U_i\})$ is invariant under the simultaneous transformation

$$(U_1\,, U_2\,, U_3) \;\mapsto\; (U_1 V_0\,,\, U_2 V_0\,,\, U_3 V_0)\,, \qquad \forall V_0 \in S_L \tag{3.34}$$

which corresponds to the relabeling $p \mapsto V_0(p)$ in (3.33). Second, $C_{123}(\{U_i\})$ is invariant under the permutation of identity fields

$$\begin{aligned} &(U_1\,,U_2\,,U_3) \;\mapsto\; (V_1U_1,\; V_2U_2,\; V_3U_3) \\ &(V_1\,,V_2\,,V_3) \;\in\; \Big(\mathbf{1}_{L_1}\otimes S_{\overline{L}_1}\,,\mathbf{1}_{L_2}\otimes S_{\overline{L}_2}\,,\mathbf{1}_{L_3}\otimes S_{\overline{L}_3}\Big) \;\subset\; S_L^{\otimes 3} \end{aligned} \tag{3.35}$$

which follows from the definition $\hat{\alpha}_i = \alpha_i \circ \mathbf{1}_{\overline{L}_i}$. Third, $C_{123}(\{U_i\})$ is invariant under the flavor symmetry (2.4),

$$\begin{aligned} &(U_1\,,U_2\,,U_3) \;\mapsto (W_1U_1,\; W_2U_2,\; W_3U_3)\,, \\ &(W_1\,,W_2\,,W_3) \in \Big(S_{l_1}\otimes S_{m_1}\otimes S_{n_1}\otimes \mathbf{1}_{\overline{L}_1}\,,S_{l_2}\otimes S_{m_2}\otimes S_{n_2}\otimes \mathbf{1}_{\overline{L}_2}\,,S_{l_3}\otimes S_{m_3}\otimes S_{n_3}\otimes \mathbf{1}_{\overline{L}_3}\Big) \end{aligned} \tag{3.36}$$

The redundancy (3.34) and (3.35) are unphysical, which should be canceled by the numerical factors $\overline{L}!$ and $\prod_i \overline{L}_i!$ in (3.33). The last operation (3.36) is the symmetry of the external operators, and interchanges different Wick contractions.

3.3.2 Fixing redundancy

Let us rewrite the flavor factor $\prod_p h^{ABC}$ in (3.33) as

$$\mathfrak{H}\left[\hat{A}^{(i)}_{U_i(p)}\right] \equiv \prod_{p=1}^{L} h^{\hat{A}^{(1)}_{U_1(p)}\hat{A}^{(2)}_{U_2(p)}\hat{A}^{(3)}_{U_3(p)}} \tag{3.37}$$

where $\left[\hat{A}^{(i)}_{U_i(p)}\right]$ is the $3\times L$ Wick-contraction matrix,[3]

$$\left[\hat{A}^{(i)}_{U_i(p)}\right] = \begin{bmatrix} \hat{A}^{(1)}_{U_1(1)} & \hat{A}^{(1)}_{U_1(2)} & \cdots & \hat{A}^{(1)}_{U_1(L)} \\ \hat{A}^{(2)}_{U_2(1)} & \hat{A}^{(2)}_{U_2(2)} & \cdots & \hat{A}^{(2)}_{U_2(L)} \\ \hat{A}^{(3)}_{U_3(1)} & \hat{A}^{(3)}_{U_3(2)} & \cdots & \hat{A}^{(3)}_{U_3(L)} \end{bmatrix}. \tag{3.38}$$

Note that the position of each column is unimportant for computing the flavor factor (3.37),

$$\left[\hat{A}^{(i)}_{U_i(p)}\right] \;\simeq\; \left[\hat{A}^{(i)}_{U_i(\sigma(p))}\right], \qquad \forall\sigma\in S_L\,. \tag{3.39}$$

We fix the redundancy of V_0 in (3.34) as follows. Let us choose the position of the identity fields for each operator as

$$\begin{aligned} \Phi^{\hat{A}^{(1)}_p} &= \mathbf{1}_p\,, && (p=1,2,\dots,\overline{L}_1) \\ \Phi^{\hat{A}^{(2)}_p} &= \mathbf{1}_p\,, && (p=\overline{L}_1+1,\overline{L}_1+2,\dots,\overline{L}_1+\overline{L}_2) \\ \Phi^{\hat{A}^{(3)}_p} &= \mathbf{1}_p\,, && (p=\overline{L}_1+\overline{L}_2+1,\overline{L}_1+\overline{L}_2+2,\dots,L). \end{aligned} \tag{3.40}$$

[3]Each element of this matrix represents the flavor data. Note that this notation is slightly different from [18], where the Wick-contraction matrix is defined by the color data.

Here the subscript of $\mathbf{1}$ is a dummy index, which will disappear after the identification (3.39). The Wick-contraction matrix becomes

$$\left[\hat{A}^{(i)}_{U_i(p)}\right] = \begin{bmatrix} \mathbf{1}_1 & \dots & \mathbf{1}_{\overline{L}_1} & \hat{A}^{(1)}_{U_1(\overline{L}_1+1)} & \dots & \hat{A}^{(1)}_{U_1(L_3)} & \hat{A}^{(1)}_{U_1(L_3+1)} & \dots & \hat{A}^{(1)}_{U_1(L)} \\ \hat{A}^{(2)}_{U_2(1)} & \dots & \hat{A}^{(2)}_{U_2(\overline{L}_1)} & \mathbf{1}_{\overline{L}_1+1} & \dots & \mathbf{1}_{L_3} & \hat{A}^{(2)}_{U_2(L_3+1)} & \dots & \hat{A}^{(2)}_{U_2(L)} \\ \hat{A}^{(3)}_{U_3(1)} & \dots & \hat{A}^{(3)}_{U_3(\overline{L}_1)} & \hat{A}^{(3)}_{U_3(\overline{L}_1+1)} & \dots & \hat{A}^{(3)}_{U_3(L_3)} & \mathbf{1}_{L_3+1} & \dots & \mathbf{1}_L \end{bmatrix} . \tag{3.41}$$

The residual redundancy of V_0 is now $V_0' \in S_{\overline{L}_1} \otimes S_{\overline{L}_2} \otimes S_{\overline{L}_3}$.

After the partial gauge fixing (3.40), $\{U_i\}$ permute the non-identity fields only,

$$U_1 \;\in\; S_{L_1} \otimes \mathbf{1}_{\overline{L}_1} , \qquad U_2 \;\in\; S_{L_2} \otimes \mathbf{1}_{\overline{L}_2} , \qquad U_3 \;\in\; S_{L_3} \otimes \mathbf{1}_{\overline{L}_3} . \tag{3.42}$$

There is still residual redundancy generated by a combination of V_0' and V_i in (3.35),

$$\tilde{V} \;:\; \{U_i\} \;\mapsto\; \{U_i'\}, \qquad \hat{A}^{(i)}_{U_i'(p)} = \begin{cases} \mathbf{1}_p & \left(\text{if } \;\hat{A}^{(i)}_{U_i(p)} = \mathbf{1}_p\right) \\ \hat{A}^{(i)}_{\tilde{V}^{-1} U_i \tilde{V}(p)} & \left(\text{if } \;\hat{A}^{(i)}_{U_i(p)} \neq \mathbf{1}_p\right) \end{cases} \tag{3.43}$$

for any $\tilde{V} \in S_{\overline{L}_1} \otimes S_{\overline{L}_2} \otimes S_{\overline{L}_3}$. This map does not permute identity fields, but permutes the non-identity fields sitting in the same column.

3.3.3 Counting inequivalent Wick contractions

We pick up one set of partially gauge-fixed permutations $\{U_i^\bullet\}$ such that $\prod_{p=1}^L h^{\hat{A}^{(1)}_{U_1^\bullet(p)} \hat{A}^{(2)}_{U_2^\bullet(p)} \hat{A}^{(3)}_{U_3^\bullet(p)}} \neq 0$. We generate other $\{U_i\}$ by applying the flavor symmetry, $U_i^\bullet \to W_i U_i^\bullet$ in (3.36).

This procedure generates all non-vanishing Wick pairings. To show this, consider two sets of permutations $\{U_i^\bullet\}$ and $\{U_i^\circ\}$, both of which are subject to the partial gauge fixing (3.42) and giving the non-vanishing flavor factor (3.37). Define

$$U_i^\bullet \equiv W_i^{\bullet\circ} U_i^\circ , \qquad W_i^{\bullet\circ} \in S_{L_i} \otimes \mathbf{1}_{\overline{L}_i} . \tag{3.44}$$

Since any permutation consists of a product of transpositions, we may assume $(W_1^{\bullet\circ}, W_2^{\bullet\circ}, W_3^{\bullet\circ}) = ((ab), \mathbf{1}, \mathbf{1}) \in S_{L_1} \otimes S_{L_2} \otimes S_{L_3}$ without loss of generality. Let us represent the Wick contractions of $\{U_i^\bullet\}$ by

$$\begin{aligned} &\langle \operatorname{tr}(\Phi^{\hat{A}^{(1)}_a} \Phi^{\hat{A}^{(1)}_b} \dots) \operatorname{tr}(\Phi^{\hat{A}^{(2)}_c} \Phi^{\hat{A}^{(2)}_d} \dots) \operatorname{tr}(\Phi^{\hat{A}^{(3)}_e} \Phi^{\hat{A}^{(3)}_f} \dots) \rangle \\ &= \langle \Phi^{A^{(1)}_a} \Phi^{A^{(2)}_c} \Phi^{A^{(3)}_e} \rangle \langle \Phi^{A^{(1)}_b} \Phi^{A^{(2)}_d} \Phi^{A^{(3)}_f} \rangle \cdots \neq 0. \end{aligned} \tag{3.45}$$

Then, the Wick contractions of $\{U_i^\circ\}$ are written as

$$\begin{aligned} &\langle \operatorname{tr}(\Phi^{\hat{A}^{(1)}_a} \Phi^{\hat{A}^{(1)}_b} \dots) \operatorname{tr}(\Phi^{\hat{A}^{(2)}_c} \Phi^{\hat{A}^{(2)}_d} \dots) \operatorname{tr}(\Phi^{\hat{A}^{(3)}_e} \Phi^{\hat{A}^{(3)}_f} \dots) \rangle \\ &= \langle \Phi^{A^{(1)}_b} \Phi^{A^{(2)}_c} \Phi^{A^{(3)}_e} \rangle \langle \Phi^{A^{(1)}_a} \Phi^{A^{(2)}_d} \Phi^{A^{(3)}_f} \rangle \cdots \neq 0. \end{aligned} \tag{3.46}$$

JHEP05(2020)118

Since both (3.45) and (3.46) are non-zero, and since $\Phi = (X, Y, Z)$ have orthogonal inner products, we should have $\Phi^{A_a^{(1)}} = \Phi^{A_b^{(1)}}$. This implies that $W_i^{\bullet\circ} \in S_{l_i} \otimes S_{m_i} \otimes S_{n_i} \otimes \mathbf{1}_{\overline{L}_i}$, which is part of the flavor symmetry (3.36).

The range of $\{U_i\}$ in (3.42) now becomes

$$\begin{aligned} U_1 &\in S_{l_1} \otimes S_{m_1} \otimes S_{n_1} \otimes \mathbf{1}_{\overline{L}_1} \equiv \mathcal{S}_1 \\ U_2 &\in S_{l_2} \otimes S_{m_2} \otimes S_{n_2} \otimes \mathbf{1}_{\overline{L}_2} \equiv \mathcal{S}_2 \\ U_3 &\in S_{l_3} \otimes S_{m_3} \otimes S_{n_3} \otimes \mathbf{1}_{\overline{L}_3} \equiv \mathcal{S}_3 \end{aligned} \tag{3.47}$$

The sum over $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ counts each inequivalent Wick pairing more than once. The multiplicity comes from the residual redundancy (3.43),

$$\left| S_{\overline{L}_1} \otimes S_{\overline{L}_2} \otimes S_{\overline{L}_3} \right| = \overline{L}_1! \, \overline{L}_2! \, \overline{L}_3! \, . \tag{3.48}$$

The number of inequivalent Wick contractions is given by

$$|\text{Wick}| \equiv \left| \frac{\mathcal{S}_1 \otimes \mathcal{S}_2 \otimes \mathcal{S}_3}{S_{\overline{L}_1} \otimes S_{\overline{L}_2} \otimes S_{\overline{L}_3}} \right| = \prod_{i=1}^{3} \frac{l_i! \, m_i! \, n_i!}{\overline{L}_i} \tag{3.49}$$

3.3.4 The OPE coefficients simplified

We collected all non-vanishing Wick contractions by restricting the sum $\{U_i\}$ over the ranges (3.47). The OPE coefficient (3.32) becomes

$$\begin{aligned} \tilde{C}_{123} = &\left(\prod_{i=1}^{3} \frac{L_i!}{l_i! \, m_i! \, n_i! \, \overline{L}_i!} \right) \sum_{\hat{R} \vdash L} \frac{\text{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \times \\ &\sum_{\{T_i, \mu_i\}} \left(\prod_{i=1}^{3} \mathscr{P}^{\hat{R} \to \text{sub}}_{\hat{I}_i \hat{J}_i} \right) \sum_{U_1 \in \mathcal{S}_1} \sum_{U_2 \in \mathcal{S}_2} \sum_{U_3 \in \mathcal{S}_3} D^{\hat{R}}_{\hat{J}_1 \hat{I}_2}(U_1 U_2^{-1}) \, D^{\hat{R}}_{\hat{J}_2 \hat{I}_3}(U_2 U_3^{-1}) \, D^{\hat{R}}_{\hat{J}_3 \hat{I}_1}(U_3 U_1^{-1}). \end{aligned} \tag{3.50}$$

Recall that the projector is equal to the product of branching coefficients, $\mathscr{P} = \mathcal{B} \mathcal{B}^T$ as in (3.31). We can simplify the second line by using the identity of branching coefficients (A.21)

$$\sum_{\hat{J}} D^{\hat{R}}_{\hat{I} \hat{J}}(u \circ v \circ w) \, B^{\hat{R} \to (q,r,s)\nu}_{\hat{J} \to (j,k,l)} = \sum_{a,b,c} D^q_{aj}(u) \, D^r_{bk}(v) \, D^s_{cl}(w) \, B^{\hat{R} \to (q,r,s)\nu}_{\hat{I} \to (a,b,c)} \, . \tag{3.51}$$

If we bring $U_k = u_k \otimes v_k \otimes w_k$ and $U_k^{-1} = u_k^{-1} \otimes v_k^{-1} \otimes w_k^{-1}$ across the double branching coefficients $\mathcal{B}$ or $\mathcal{B}^T$, they annihilate each other; see (3.54).

Let us define a triple-projector product

$$\mathcal{I}^{\hat{R} \to \text{sub}}_{123} \equiv \mathscr{P}^{\hat{R} \to \text{sub}}_{\hat{I}_1 \hat{I}_2} \, \tilde{\mathscr{P}}^{\hat{R} \to \text{sub}}_{\hat{I}_2 \hat{I}_3} \, \tilde{\tilde{\mathscr{P}}}^{\hat{R} \to \text{sub}}_{\hat{I}_3 \hat{I}_1} \tag{3.52}$$

where we used the symbols $\tilde{\mathscr{P}}$ and $\tilde{\tilde{\mathscr{P}}}$ to keep in mind that the branching coefficients come from different restrictions of S_L. Then

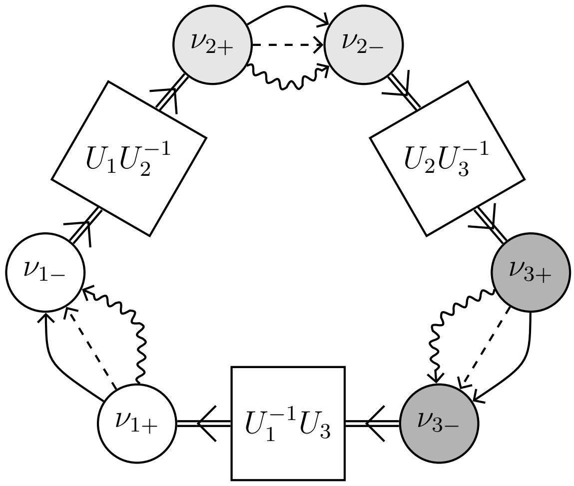
$$\begin{aligned} \tilde{C}_{123} &= \left(\prod_{i=1}^{3} \frac{L_i!}{l_i! \, m_i! \, n_i!} \right) |\text{Wick}| \sum_{\hat{R} \vdash L} \frac{\text{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \sum_{\{T_i, \mu_i\}} \mathcal{I}^{\hat{R} \to \text{sub}}_{123} \\ &= \left(\prod_{i=1}^{3} \frac{L_i!}{\overline{L}_i!} \right) \sum_{\hat{R} \vdash L} \frac{\text{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \sum_{\{T_i, \mu_i\}} \mathcal{I}^{\hat{R} \to \text{sub}}_{123} \end{aligned} \tag{3.53}$$

where we used (3.49).

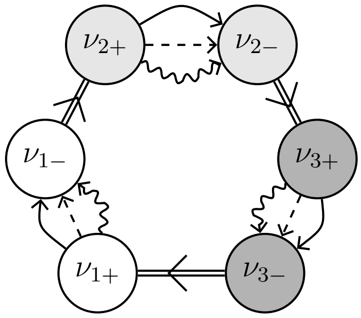
JHEP05(2020)118

In the notation of the quiver calculus in appendix B, we can express the above calculation as

$$\tilde{C}_{123} \sim \sum_{\hat{R}\vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \sum_{\{U_i \in \mathcal{S}_i\}}$$



$$\sim \sum_{\hat{R}\vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \, |\mathrm{Wick}| \tag{3.54}$$



From this diagram, we see that $\mathcal{I}_{123}^{\hat{R}\to\,\mathrm{sub}}$ in (3.52) is also a triple product of the transformation matrices (A.16).

3.4 Sum over the triple-projector products

We compute the OPE coefficients by evaluating a sum over the triple-projector products,

$$\sum_{\{T_i,\mu_i\}} \mathcal{I}_{123}^{\hat{R}\to\,\mathrm{sub}} = \sum_{T_1\vdash \overline{L}_1} \sum_{T_2\vdash \overline{L}_2} \sum_{T_3\vdash \overline{L}_3} \sum_{\mu_1,\mu_2,\mu_3} \mathscr{P}_{\hat{I}_1\hat{I}_2}^{\hat{R}\to\,\mathrm{sub}} \, \tilde{\mathscr{P}}_{\hat{I}_2\hat{I}_3}^{\hat{R}\to\,\mathrm{sub}} \, \tilde{\tilde{\mathscr{P}}}_{\hat{I}_3\hat{I}_1}^{\hat{R}\to\,\mathrm{sub}} \tag{3.55}$$

where the projector is given by (3.31). The main idea is to decompose each projector further into a sum of sub-projectors, so that we can make use of the orthogonality of the sub-projectors on the fully-split space, V_{FS}.

Below we discuss the two cases $\tilde{C}_{\circ\circ\circ}$ in (3.18) and $\tilde{C}_{\vec{h}}^{XYZ}$ in (3.20) separately.

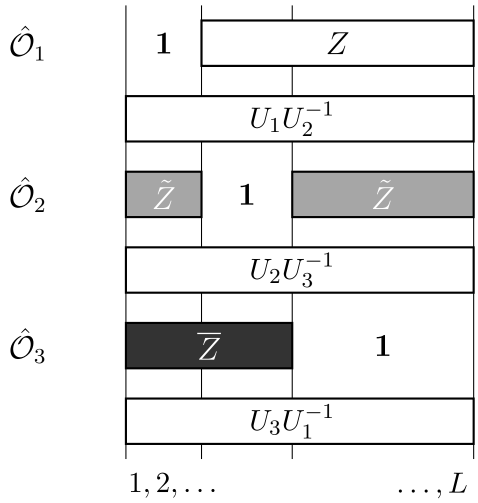
3.4.1 Case of $\tilde{C}_{\circ\circ\circ}$

Recall that $\tilde{C}_{\circ\circ\circ}$ is a linear combination of $C_{\circ\circ\circ}$ given in (3.7). The Wick-contraction matrix of $C_{\circ\circ\circ}$ after a partial gauge-fixing (3.41) is given by

$$\left[\hat{A}_{U_i(p)}^{(i)}\right] = \begin{bmatrix} \mathbf{1}_1 & \dots & \mathbf{1}_{\overline{L}_1} & Z_{U_1(\overline{L}_1+1)} & \dots & Z_{U_1(L_3)} & Z_{U_1(L_3+1)} & \dots & Z_{U_1(L)} \\ \tilde{Z}_{U_2(1)} & \dots & \tilde{Z}_{U_2(\overline{L}_1)} & \mathbf{1}_{\overline{L}_1+1} & \dots & \mathbf{1}_{L_3} & \tilde{Z}_{U_2(L_3+1)} & \dots & \tilde{Z}_{U_2(L)} \\ \overline{Z}_{U_3(1)} & \dots & \overline{Z}_{U_3(\overline{L}_1)} & \overline{Z}_{U_3(\overline{L}_1+1)} & \dots & \overline{Z}_{U_3(L_3)} & \mathbf{1}_{L_3+1} & \dots & \mathbf{1}_L \end{bmatrix} \tag{3.56}$$

JHEP05(2020)118

which shows that $\mathcal{S}_i = S_{L_i} \otimes S_{\overline{L}_i}$ in place of (3.47). We represent (3.56) as in the following figure,



(3.57)

Let us choose the fully-split space as

$$V_{FS} = V_{\overline{L}_1} \otimes V_{\overline{L}_2} \otimes V_{\overline{L}_3} \tag{3.58}$$

which induces the restriction $S_L \downarrow S_{FS}$, where

$$S_{FS} = S_{\overline{L}_1} \otimes S_{\overline{L}_2} \otimes S_{\overline{L}_3}\,. \tag{3.59}$$

On the space V_{FS}, the states decompose as

$$\left| \begin{matrix} \hat{R} \\ \hat{I} \end{matrix} \right\rangle = \left| \begin{matrix} R_i \; T_i \\ I_i \;\; c_i \end{matrix} \; \mu_i \right\rangle (B^T)^{\hat{R}\to(R_i,T_i),\mu_i}_{\hat{I}\to(I_i,c_i)} = \left| \begin{matrix} Q_i \; Q_i' \; T_i \\ b_i \;\; b_i' \;\; c_i \end{matrix} \; \mu_i \, \rho_i \right\rangle (B^T)^{\hat{R}\to(R_i,T_i),\mu_i}_{\hat{I}\to(I_i,c_i)} \, (B^T)^{R_i\to(Q_i,Q_i'),\rho_i}_{I_i\to(b_i,b_i')} \tag{3.60}$$

where we used (A.13). We introduce the fully-split branching coefficients by

$$\mathfrak{B}^{\hat{R}\to(R_i,T_i),\mu_i\to(Q_i,Q_i',T_i),(\mu_i,\rho_i)}_{\hat{I}\longrightarrow(b_i,b_i',c_i)} = \sum_{I_i=1}^{d_{R_i}} B^{\hat{R}\to(R_i,T_i),\mu_i}_{\hat{I}\to(I_i,c_i)} \, B^{R_i\to(Q_i,Q_i'),\rho_i}_{I_i\to(b_i,b_i')} \tag{3.61}$$

and the corresponding sub-projector by

$$\begin{aligned} &\mathfrak{P}^{\hat{R}\to(R_i,T_i),\mu_i\to(Q_i,Q_i',T_i),(\mu_i,\rho_i)}_{\hat{I}\hat{J}} \\ &= \sum_{b,b',c} \mathfrak{B}^{\hat{R}\to(R_i,T_i),\mu_i\to(Q_i,Q_i',T_i),(\mu_i,\rho_i)}_{\hat{I}\longrightarrow(b,b',c)} (\mathfrak{B}^T)^{\hat{R}\to(R_i,T_i),\mu_i\to(Q_i,Q_i',T_i),(\mu_i,\rho_i)}_{\hat{J}\longrightarrow(b,b',c)} \end{aligned} \tag{3.62}$$

We rewrite the original projectors in (3.31) as a sum over sub-projectors on V_{FS} as

$$\begin{aligned} \mathscr{P}^{\hat{R}\to(R_1,T_1),\mu_1,\rho_1}_{\hat{I}\hat{J}} &= \sum_{Q_1,Q_1',\rho_1} \mathfrak{P}^{\hat{R}\to(R_1,T_1),\mu_1\to(Q_1,Q_1',T_1),(\mu_1,\rho_1)}_{\hat{I}\hat{J}} \\ \tilde{\mathscr{P}}^{\hat{R}\to(R_2,T_2),\mu_2,\rho_2}_{\hat{I}\hat{J}} &= \sum_{Q_2,Q_2',\rho_2} \tilde{\mathfrak{P}}^{\hat{R}\to(R_2,T_2),\mu_2\to(Q_2,Q_2',T_2),(\mu_2,\rho_2)}_{\hat{I}\hat{J}} \\ \tilde{\tilde{\mathscr{P}}}^{\hat{R}\to(R_3,T_3),\mu_3,\rho_3}_{\hat{I}\hat{J}} &= \sum_{Q_3,Q_3',\rho_3} \tilde{\tilde{\mathfrak{P}}}^{\hat{R}\to(R_3,T_3),\mu_3\to(Q_3,Q_3',T_3),(\mu_3,\rho_3)}_{\hat{I}\hat{J}}\,. \end{aligned} \tag{3.63}$$

By construction, all sub-projectors follow from the same restriction

$$S_L \downarrow S_{FS}, \qquad \hat{R} = \bigoplus_{Q,Q',T} \bigoplus_{\eta=1}^{g(Q,Q',T;\hat{R})} (Q \otimes Q' \otimes T)_\eta \tag{3.64}$$

and all sub-representations should be synchronized when evaluating $\mathcal{I}_{123}^{\hat{R}\to \text{sub}}$ in (3.55). The states can also be decomposed as

$$\left| \begin{matrix} \hat{R} \\ \hat{I} \end{matrix} \right\rangle = \left| \begin{matrix} Q\ Q'\ T \\ b\ \ b'\ \ c \end{matrix}\ \eta \right\rangle (B^T)^{\hat{R}\to(Q,Q',T),\eta}_{\hat{I}\to(b,b',c)} \tag{3.65}$$

in addition to (3.60). The consistency of the two decompositions suggests that the multiplicity labels can be rewritten as

$$\xi_i \equiv \{\mu_i\,, \rho_i\}, \qquad 1 \le \xi_i \le g(Q_i, Q_i'; R_i)\, g(R_i, T_i; \hat{R}). \tag{3.66}$$

In (3.63), the representations T_i come from the Fourier transform of identity fields $\mathbf{1}$, and Q_i, Q_i' come from the non-identity fields, $Z, \tilde{Z}, \overline{Z}$. Since the OPE coefficient $C_{\circ\circ\circ}$ has the Wick-contraction structure given in (3.57), we should identify the representations $\{Q_i\,, Q_i'\,, T_i\}$ with those acting on the constituent of V_{FS} as

$$\begin{aligned} T_1 = Q_2' = Q_3 \qquad &\in \mathrm{Hom}(V_{\overline{L}_1}) \\ Q_1 = T_2 = Q_3' \qquad &\in \mathrm{Hom}(V_{\overline{L}_2}) \\ Q_1' = Q_2 = T_3 \qquad &\in \mathrm{Hom}(V_{\overline{L}_3}). \end{aligned} \tag{3.67}$$

We can show (3.67) from another argument. The triple-projector product is equal to the product of generalized Racah-Wigner tensors in appendix C,

$$\mathrm{tr}_{\hat{R}} \left(\mathfrak{P}^{\hat{R}\to\cdots\to(Q_1,Q_1',T_1),\xi_1}_{\hat{I}\hat{J}}\ \tilde{\mathfrak{P}}^{\hat{R}\to\cdots\to(Q_2,Q_2',T_2),\xi_2}_{\hat{I}\hat{J}}\ \tilde{\tilde{\mathfrak{P}}}^{\hat{R}\to\cdots\to(Q_3,Q_3',T_3),\xi_3}_{\hat{I}\hat{J}} \right) = \mathrm{tr}\,(U_{\hat{R}} \tilde{U}_{\hat{R}} \tilde{\tilde{U}}_{\hat{R}}) \tag{3.68}$$

which we conjecture as (C.20),

$$\sum_{\xi_1,\xi_2,\xi_3} \mathrm{tr}\,(U_{\hat{R}} \tilde{U}_{\hat{R}} \tilde{\tilde{U}}_{\hat{R}}) = \delta^{T_1 Q_2'}\, \delta^{Q_2' Q_3}\, \delta^{Q_1 T_2}\, \delta^{T_2 Q_3'}\, \delta^{Q_1' Q_2}\, \delta^{Q_2 T_3} \left(\prod_{i=1}^{3} d_{Q_i} \right) \mathcal{G}_{123} \tag{3.69}$$

$$\mathcal{G}_{123} = \frac{g(Q_1, Q_2; R_1)\, g(R_1, Q_3; \hat{R})\, g(Q_2, Q_3; R_2)\, g(R_2, Q_1; \hat{R})\, g(Q_3, Q_1; R_3)\, g(R_3, Q_2; \hat{R})}{g(Q_1, Q_2, Q_3; \hat{R})^2}\,.$$

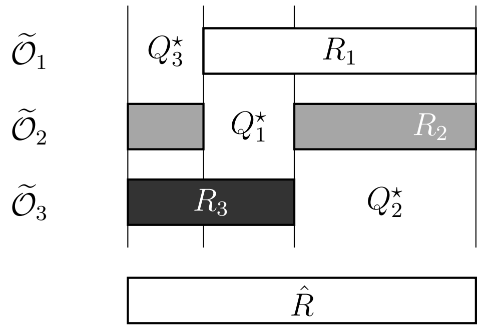
The three-point function (3.53) becomes

$$\tilde{C}_{\circ\circ\circ} = \left(\prod_{i=1}^{3} \frac{L_i!}{\overline{L}_i!} \right) \sum_{\hat{R}\vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \sum_{Q_1\vdash\overline{L}_2} \sum_{Q_2\vdash\overline{L}_3} \sum_{Q_3\vdash\overline{L}_1} \left(\prod_{i=1}^{3} d_{Q_i} \right) \mathcal{G}_{123}\,. \tag{3.70}$$

Here, the Littlewood-Richardson coefficients in $\mathcal{G}_{123}$ put constraints on the sum over $\{Q_i\}$. In other words, we should find all $\{Q_i\} = \{Q_i^\star\}$ such that

$$R_1 = Q_1^\star \otimes Q_2^\star\,, \quad R_2 = Q_2^\star \otimes Q_3^\star\,, \quad R_3 = Q_3^\star \otimes Q_1^\star\,, \quad \hat{R} = Q_1^\star \otimes Q_2^\star \otimes Q_3^\star \tag{3.71}$$

The conditions (3.71) can be summarized as



(3.72)

Extremal case. As a check, consider the situation $L_1 + L_2 = L_3 = L$. From (3.72), this corresponds to

$$Q_2 = \emptyset, \qquad R_1 = Q_1\,, \qquad R_2 = Q_3\,, \qquad \hat{R} = R_3\,. \tag{3.73}$$

We get

$$\mathcal{G}_{123} = \frac{g(R_1, Q_3; \hat{R})\; g(R_2, Q_1; \hat{R})\, g(Q_3, Q_1; R_3)}{g(Q_1, Q_3; \hat{R})^2} = g(R_1, R_2; R_3) \tag{3.74}$$

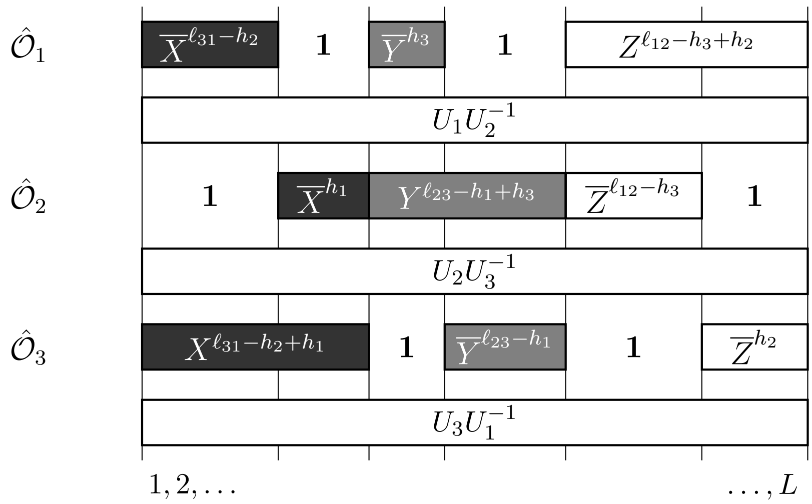
and therefore

$$\tilde{C}_{\circ\circ\circ} = L_3!\, \frac{\mathrm{Dim}_{N_c}(R_3)}{d_{R_3}}\, g(R_1, R_2; R_3). \tag{3.75}$$

This result agrees with the literature [19] including the normalization of the two-point function given in (2.13).

3.4.2 Case of $\tilde{C}^{XYZ}_{\vec{h}}$

Our discussion is quite parallel to section 3.4.1. Recall that $\tilde{C}^{XYZ}_{\vec{h}}$ is a linear combination of $C^{XYZ}_{\vec{h}}$ given in (3.9). We represent the Wick-contraction matrix by



(3.76)

where h_i are constrained by (3.11),

$$0 \le h_1 \le \ell_{23} = \overline{L}_1\,, \qquad 0 \le h_2 \le \ell_{31} = \overline{L}_2\,, \qquad 0 \le h_3 \le \ell_{12} = \overline{L}_3\,. \tag{3.77}$$

JHEP05(2020)118

We choose the fully-split space as

$$V_{FS} = V_{\ell_{31}-h_2} \otimes V_{h_1} \otimes V_{h_3} \otimes V_{\ell_{23}-h_1} \otimes V_{\ell_{12}-h_3} \otimes V_{h_2} \tag{3.78}$$

and decompose the original projectors (3.31). From (3.76), one finds that the new branch coefficients are needed for

$$\begin{array}{llll} S_{\ell_{12}-h_3+h_2} \downarrow (S_{\ell_{12}-h_3} \otimes S_{h_2}) & \text{and} & S_{\ell_{23}} \downarrow (S_{h_1} \otimes S_{\ell_{23}-h_1}) & \text{for } \mathcal{O}_1 \\ S_{\ell_{23}-h_1+h_3} \downarrow (S_{\ell_{23}-h_1} \otimes S_{h_3}) & \text{and} & S_{\ell_{31}} \downarrow (S_{\ell_{31}-h_2} \otimes S_{h_2}) & \text{for } \mathcal{O}_2 \\ S_{\ell_{31}-h_2+h_1} \downarrow (S_{\ell_{31}-h_2} \otimes S_{h_1}) & \text{and} & S_{\ell_{12}} \downarrow (S_{h_3} \otimes S_{\ell_{12}-h_3}) & \text{for } \mathcal{O}_3\,. \end{array} \tag{3.79}$$

For example, we rewrite the states for $\mathcal{O}_1$ on the space V_{FS} as

$$\begin{aligned} \left|\begin{matrix}\hat{R}\\ \hat{I}\end{matrix}\right\rangle &= \left|\begin{matrix}R_1\ T_1\\ I_1\ \ c_1\end{matrix}\ \mu_1\right\rangle (B^T)^{\hat{R}\to(R_1,T_1),\mu_1}_{\hat{I}\to(I_1,c_1)} \\ &= \left|\begin{matrix}q_1\ r_1\ s_1\ T_1\\ j_1\ k_1\ l_1\ c_1\end{matrix}\ \mu_1\,\nu_{1\mp}\right\rangle (B^T)^{\hat{R}\to(R_1,T_1),\mu_1}_{\hat{I}\to(I_1,c_1)} (B^T)^{R_1\to(q_1,r_1,s_1),\nu_{1\mp}}_{I_1\to(j_1,k_1,l_1)} \\ &= \left|\begin{matrix}q_1\ r_1\ s'_1\ s''_1\ t'_1\ t''_1\\ j_1\ k_1\ l'_1\ l''_1\ c'_1\ c''_1\end{matrix}\ \mu_1\,\nu_{1\mp}\,\rho_1\,\zeta_1\right\rangle \times \\ &\qquad (B^T)^{\hat{R}\to(R_1,T_1),\mu_1}_{\hat{I}\to(I_1,c_1)} (B^T)^{R_1\to(q_1,r_1,s_1),\nu_{1\mp}}_{I_1\to(j_1,k_1,l_1)} (B^T)^{s_1\to(s'_1,s''_1),\rho_1}_{I_1\to(l'_1,l''_1)} (B^T)^{T_1\to(t'_1,t''_1),\zeta_1}_{c_1\to(c'_1,c''_1)} \end{aligned} \tag{3.80}$$

and introduce the fully-split branching coefficients by

$$\begin{aligned} &\mathfrak{B}^{\hat{R}\to\cdots\to(q_1,r_1,s'_1,s''_1,t'_1,t''_1),\mu_1,\nu_{1\mp},\rho_1,\zeta_1}_{\hat{I}\longrightarrow(j_1,k_1,l'_1,l''_1,c'_1,c''_1)} \\ &= B^{\hat{R}\to(R_1,T_1),\mu_1}_{\hat{I}\to(I_1,c_1)} B^{R_1\to(q_1,r_1,s_1),\nu_{1\mp}}_{I_1\to(j_1,k_1,l_1)} B^{s_1\to(s'_1,s''_1),\rho_1}_{I_1\to(l'_1,l''_1)} B^{T_1\to(t'_1,t''_1),\zeta_1}_{c_1\to(c'_1,c''_1)}\,. \end{aligned} \tag{3.81}$$

The original projector (3.31) becomes a sum over the sub-projectors $\mathfrak{P} = \mathfrak{B}\,\mathfrak{B}^T$,

$$\mathscr{P}^{\hat{R}\to\boldsymbol{RT}_{1-,1+}}_{\hat{I}_1\hat{J}_1} = \sum_{s'_1,s''_1,t'_1,t''_1,\rho_1,\zeta_1} \mathfrak{P}^{\hat{R}\to\cdots\to(q_1,r_1,s'_1,s''_1,t'_1,t''_1),\mu_1,\nu_{1\mp},\rho_1,\zeta_1}_{\hat{I}\hat{J}} \tag{3.82}$$

and similarly

$$\begin{aligned} \tilde{\mathscr{P}}^{\hat{R}\to\boldsymbol{RT}_{2-,2+}}_{\hat{I}_2\hat{J}_2} &= \sum_{r'_2,r''_2,t'_2,t''_2,\rho_2,\zeta_2} \mathfrak{P}^{\hat{R}\to\cdots\to(q_2,r'_2,r''_2,s_2,t'_2,t''_2),\mu_2,\nu_{2\mp},\rho_2,\zeta_2}_{\hat{I}\hat{J}} \\ \tilde{\tilde{\mathscr{P}}}^{\hat{R}\to\boldsymbol{RT}_{3-,3+}}_{\hat{I}_3\hat{J}_3} &= \sum_{q'_3,q''_3,t'_3,t''_3,\rho_3,\zeta_3} \mathfrak{P}^{\hat{R}\to\cdots\to(q'_3,q''_3,r_3,s_3,t'_3,t''_3),\mu_3,\nu_{3\mp},\rho_3,\zeta_3}_{\hat{I}\hat{J}}\,. \end{aligned} \tag{3.83}$$

When summing over $\{t'_i, t''_i\}$ we can forget the constraint $t'_i \otimes t''_i \simeq T_i$, because the OPE coefficient (3.50) contains sums over $\{T_i\}$.

All sub-projectors come from the irreducible decompositions of $\hat{R}$ under the restriction $S_L \downarrow S_{FS}$,

$$\hat{R} = \bigoplus_{q',q'',r',r'',s',s''} \bigoplus_{\eta=1}^{g(q',q'',r',r'',s',s'';\hat{R})} (q' \otimes q'' \otimes r' \otimes r'' \otimes s' \otimes s'')_\eta \tag{3.84}$$

Since the OPE coefficient $C^{XYZ}_{\vec{h}}$ has the Wick contraction structure of (3.76), we should identify the representations as

$$\begin{aligned} q_1 &= t'_2 = q'_3 \ \in \mathrm{Hom}(V_{\ell_{31}-h_2}), \qquad & t'_1 &= q_2 = q''_3 \ \in \mathrm{Hom}(V_{h_1}) \\ r_1 &= r'_2 = t'_3 \ \in \mathrm{Hom}(V_{h_3}), & t''_1 &= r''_2 = r_3 \ \in \mathrm{Hom}(V_{\ell_{23}-h_1}) \\ s'_1 &= s_2 = t''_3 \ \in \mathrm{Hom}(V_{\ell_{12}-h_3}), & s''_1 &= t''_2 = s_3 \ \in \mathrm{Hom}(V_{h_2}) \end{aligned} \tag{3.85}$$

and replace the multiplicity labels by

$$\xi_{i\mp} = \{\mu_i, \nu_{i\mp}, \rho_i, \xi_i\} \,. \tag{3.86}$$

Again, the trace over the product of sub-projectors is given by the generalized Racah-Wigner tensors (C.28),

$$\begin{aligned} &\mathrm{tr}_{\hat{R}} \Big(\mathfrak{P}^{\hat{R} \to \cdots \to (q_1, r_1, s'_1, s''_1, t'_1, t''_1), \xi_{1-}, \xi_{1+}}_{\hat{I}_1 \hat{I}_2} \, \mathfrak{P}^{\hat{R} \to \cdots \to (q_2, r'_2, r''_2, s_2, t'_2, t''_2), \xi_{2-}, \xi_{2+}}_{\hat{I}_2 \hat{I}_3} \, \mathfrak{P}^{\hat{R} \to \cdots \to (q'_3, q''_3, r_3, s_3, t'_3, t''_3), \xi_{3-}, \xi_{3+}}_{\hat{I}_3 \hat{I}_1} \Big) \\ &= \mathrm{tr}\, (W_{\hat{R}} \tilde{W}_{\hat{R}} \tilde{\tilde{W}}_{\hat{R}}). \end{aligned} \tag{3.87}$$

From the identity of the projectors (A.46), this becomes

$$\begin{aligned} \mathrm{tr}\, (W_{\hat{R}} \, \tilde{W}_{\hat{R}} \, \tilde{\tilde{W}}_{\hat{R}}) &= \Big(\mathcal{D}_{123} \, d_{q_1} \, d_{q_2} \, d_{r_1} \, d_{r_3} \, d_{s_2} \, d_{s_3} \Big) \delta^{\xi_{1-} \, \xi_{2+}} \, \delta^{\xi_{2-} \, \xi_{3+}} \, \delta^{\xi_{3-} \, \xi_{1+}} \\ \mathcal{D}_{123} &= \delta^{q_1 t'_2} \, \delta^{q_1 q'_3} \, \delta^{t'_1 q_2} \, \delta^{q_2 q''_3} \, \delta^{r_1 r'_2} \, \delta^{r_1 t'_3} \, \delta^{t''_1 r_3} \, \delta^{{}''_2 r_3} \, \delta^{s'_1 s_2} \, \delta^{s_2 t''_3} \, \delta^{s''_1 s_3} \, \delta^{t''_2 s_3} \,. \end{aligned} \tag{3.88}$$

We need to sum over the representations and multiplicity labels. We conjecture that the result is given by (C.39),

$$\begin{aligned} \sum_{\xi_\mp, \xi'_\mp, \xi''_\mp} \mathrm{tr}\, (W_{\hat{R}} \, \tilde{W}_{\hat{R}} \, \tilde{\tilde{W}}_{\hat{R}}) &= \Big(\mathcal{D}_{123} \, d_{q_1} \, d_{q_2} \, d_{r_1} \, d_{r_3} \, d_{s_2} \, d_{s_3} \Big) \overline{\delta}^{\nu_{1-} \, \nu_{2+}} \, \overline{\delta}^{\nu_{2-} \, \nu_{3+}} \, \overline{\delta}^{\nu_{3-} \, \nu_{1+}} \, \mathcal{G}_{123} \\ \mathcal{G}'_{123} &= \frac{\left|\mathcal{M}_{R_1, s_1, \nu_{1-}}\right| \left|\mathcal{M}_{R_1, s_1, \nu_{1+}}\right| \left|\mathcal{M}_{R_2, r_2, \nu_{2-}}\right| \left|\mathcal{M}_{R_2, r_2, \nu_{2+}}\right| \left|\mathcal{M}_{R_3, q_3, \nu_{3-}}\right| \left|\mathcal{M}_{R_3, q_3, \nu_{3+}}\right|}{\left|\mathcal{M}_{\mathrm{tot}}\right|^3} \end{aligned} \tag{3.89}$$

where $\mathcal{M}_{R,r,\nu}$ is the slice of the total multiplicity space constrained by (R, r, ν).

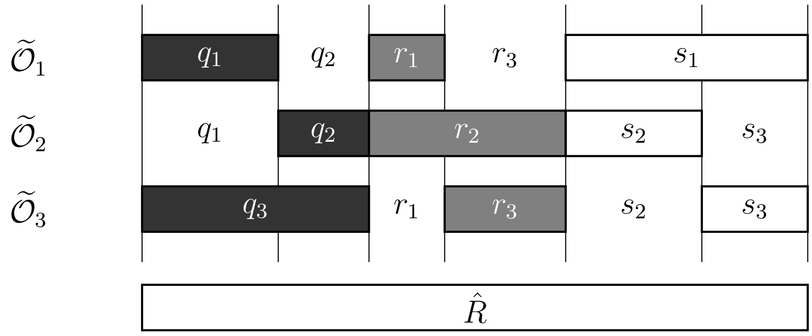
The three-point function (3.53) becomes

$$\tilde{C}^{XYZ}_{\vec{h}} = \left(\prod_{i=1}^{3} \frac{L_i!}{\overline{L}_i!} \right) \sum_{\hat{R} \vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \, (d_{q_1} \, d_{q_2} \, d_{r_1} \, d_{r_3} \, d_{s_2} \, d_{s_3}) \, \overline{\delta}^{\nu_{1-} \, \nu_{2+}} \, \overline{\delta}^{\nu_{2-} \, \nu_{3+}} \, \overline{\delta}^{\nu_{3-} \, \nu_{1+}} \, \mathcal{G}'_{123} \,. \tag{3.90}$$

Here $\{q_i, r_i, s_i\}$ must be consistent with $\boldsymbol{R}_i$ in (3.14). This condition is implicitly included in the definition of $\overline{\delta}$ in (C.37). In other words, the OPE coefficients are non-zero only if $(q_1, q_2, r_1, r_3, s_2, s_3)$ satisfy

$$\begin{aligned} &q_1 \otimes q_2 = q_3, \quad r_1 \otimes r_3 = r_2, \quad s_2 \otimes s_3 = s_1, \quad q_1 \otimes q_2 \otimes r_1 \otimes r_3 \otimes s_2 \otimes s_3 = \hat{R} \\ &(R_1)_{\nu_{1\mp}} = q_1 \otimes r_1 \otimes (s_2 \otimes s_3) \,, \ (R_2)_{\nu_{2\mp}} = q_2 \otimes (r_1 \otimes r_3) \otimes s_2, \ (R_3)_{\nu_{3\mp}} = (q_1 \otimes q_2) \otimes r_3 \otimes s_3 \end{aligned} \tag{3.91}$$

which can be represented by



(3.92)

We find some difference from the case of $\tilde{C}_{\circ\circ\circ}$ in (3.70). First, we do not have a sum over $(q_1^\star, q_2^\star, r_1^\star, r_3^\star, s_2^\star, s_3^\star)$. This is because $\tilde{C}_{\vec{h}}^{XYZ}$ has the same structure of the Wick contractions as the extremal correlators for each flavor X, Y, Z.[4] Thus, the first line of (3.91) is trivial. Second, there is no sum over $\{\nu_{i\mp}\}$, because $\{\nu_{i\mp}\}$ are part of the operator data $\boldsymbol{R}_i = \{R_i, (q_i, r_i, s_i), \nu_{i-}, \nu_{i+}\}$. We should pick up the right combination of multiplicities consistent with $\boldsymbol{R}_i$.

Extremal case. Consider the situation where the operators consist of Z or $\overline{Z}$ only. This means

$$\begin{aligned} 0 = h_1 = \ell_{31} - h_2 = h_3\,, \qquad \ell_{23} = 0, \qquad V_{FS} = V_{\ell_{12}} \otimes V_{\ell_{31}} \\ q_i = r_i = \emptyset, \qquad R_i = s_i\,, \qquad \hat{R} = R_1\,. \end{aligned} \tag{3.93}$$

In particular, we do not need to specify $\nu_{i\mp}$.

The quantity $\mathcal{G}'_{123}$ becomes

$$\mathcal{G}'_{123} = \frac{|\mathcal{M}_{R_1}|^2 \, |\mathcal{M}_{R_2}|^2 \, |\mathcal{M}_{R_3}|^2}{|\mathcal{M}_{\rm tot}|^3} = g(R_2, R_3; R_1) \tag{3.94}$$

where we used

$$|\mathcal{M}_{R_1}| = 1, \qquad |\mathcal{M}_{R_2}| = |\mathcal{M}_{R_3}| = |\mathcal{M}_{\rm tot}| = g(R_2, R_3; R_1). \tag{3.95}$$

The three-point function (3.90) becomes

$$\tilde{C}_{\vec{h}}^{XYZ} = L_1! \, \frac{{\rm Dim}_{N_c}(R_1)}{d_{R_1}} \, g(R_2, R_3; R_1) \tag{3.96}$$

which agrees with (3.75) after relabeling.

In appendix C.3 we consider the restricted Littlewood-Richardson coefficients, which are related to the extremal three-point functions of different type.

4 Background independence at large N_c

We study the tree-level three-point functions in the representation basis, and check the background independence conjectured in [43]. Our proof is based on the conjectured relations for the generalized Racah-Wigner tensor in appendix C.

[4] Recall that $\langle \overline{Z} \overline{Z} \rangle = 0$ whereas any of $\langle Z, \tilde{Z} \rangle, \langle \tilde{Z} \overline{Z} \rangle, \langle \overline{Z} Z \rangle$ are non-zero.

4.1 The LLM operators

Let us review the argument on the large-N_c background independence [43]. They mapped the $\mathcal{N}=4$ SYM operators with the $\mathcal{O}(N_c^0)$ canonical dimensions to those with the $\mathcal{O}(N_c^2)$ canonical dimensions by attaching a large number of background boxes. We call the latter LLM operators, because they correspond to stringy excitations on the LLM geometry. Recall that the LLM geometries are the half-BPS solutions of IIB supergravity. This implies that the addition of $\mathcal{O}(N_c^2)$ boxes should consist of a single holomorphic scalar like $\sim Z^{N_c^2}$.

For simplicity, we consider the operator mixing in the $\mathfrak{su}(2)$ sector, at one-loop in λ at any N_c. We expand the dilatation eigenstates in terms of the restricted Schur basis as

$$\mathfrak{D}_1\, \mathcal{O}_\Delta = \Delta_1\, \mathcal{O}_\Delta\,, \qquad \mathcal{O}_\Delta = \sum_{R,r,s,\nu_\mp} c_{R,(r,s),\nu_-,\nu_+}\, \mathcal{O}^{R,(r,s),\nu_-,\nu_+}\,. \tag{4.1}$$

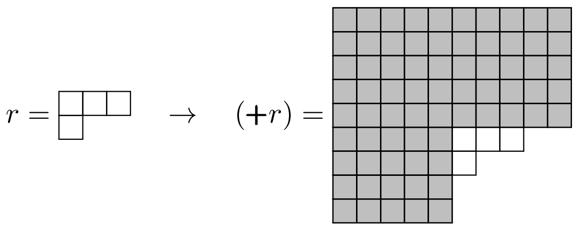
We denote the action of the one-loop dilatation on the restricted Schur basis by

$$\mathfrak{D}_1\, \mathcal{O}^{R,(r,s),\nu_-,\nu_+} = \sum_{T,t,u,\mu_-,\mu_+} N^{R,(r,s),\nu_-,\nu_+}_{T,(t,u),\mu_-,\mu_+}\, \mathcal{O}^{T,(t,u),\mu_-,\mu_+} \tag{4.2}$$

and define the LLM operator by

$$\mathcal{O}_\Delta \;\to\; \mathcal{O}_\Delta^{\rm LLM} = \sum_{R,(r,s),\nu_\mp} c_{R,(r,s),\nu_-,\nu_+}\, \mathcal{O}^{+R,(+r,s),\nu_-,\nu_+}\,. \tag{4.3}$$

The operation $r \to (+r)$ can be exemplified as



(4.4)

Here there are $\mathcal{O}(1)$ white boxes, and $\mathcal{O}(N_c^2)$ gray boxes in total. Each edge of the gray block has the length of $\mathcal{O}(N_c)$. The general form of the background Young diagram $\mathscr{B}$ is shown in figure 1.

We specify a corner of the background Young diagram $\mathscr{B}$, and consider a set of all Young diagrams attached to that corner. This set of states has many interesting properties. First, from the Littlewood-Richardson rule, we find

$$g(r,s;R) \simeq g(+r,s;+R), \qquad (N_c \gg 1). \tag{4.5}$$

This allows us to use the same multiplicity labels $\nu_\mp$ before and after the $+$ operation. Note that the tensor product $(+r)\otimes s$ contains representations in which boxes are attached to multiple corners of $\mathscr{B}$. However, the overlap between such states and $(+r)$ is suppressed by $1/N_c$. Second, the hook length of $(+r)$ factorizes as [43]

$$\frac{\text{hook}_{+r}}{\text{hook}_r\, \text{hook}_{\mathscr{B}}} \simeq (\eta_{\mathscr{B}})^{|r|} \qquad (N_c \gg 1) \tag{4.6}$$

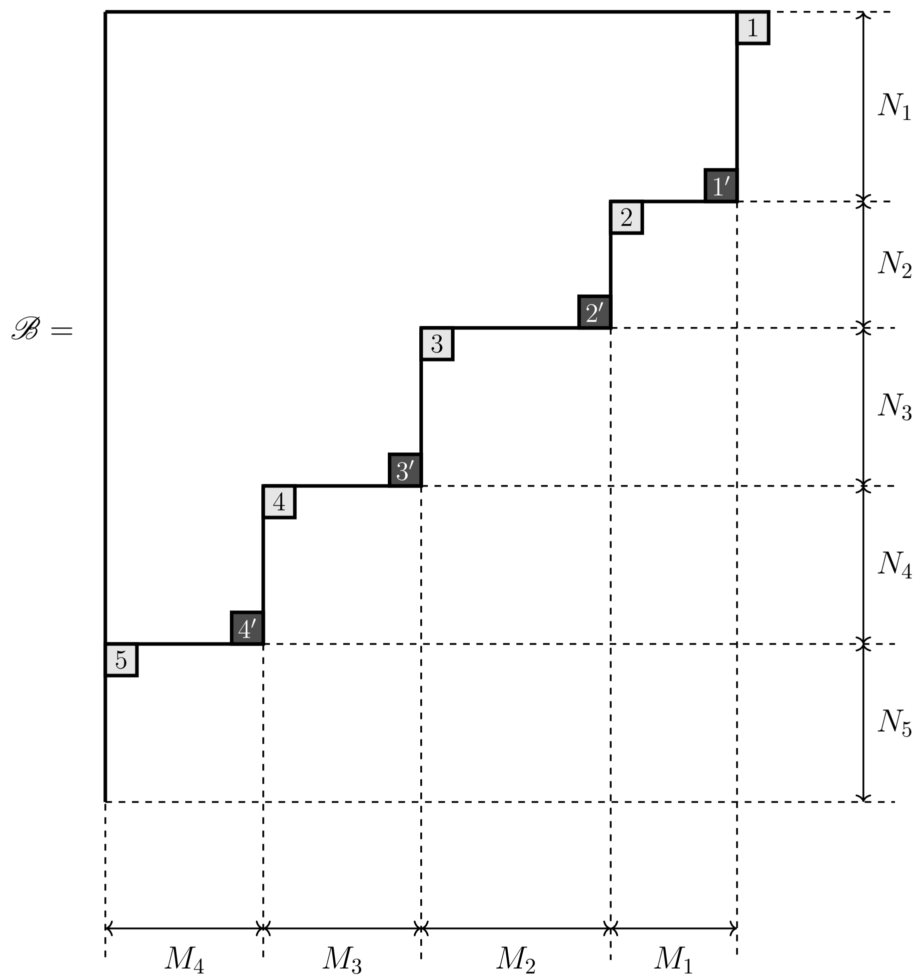


Figure 1. The general background Young diagram $\mathscr{B}$ having a staircase shape, which corresponds to the LLM geometry of concentric shapes by AdS/CFT. All M_i and N_i are $\mathcal{O}(N_c)$, and $\sum_i N_i = N_c$. The gray and black boxes represent localized string excitations. To define the operation $+$ we should choose one gray box.

where $\eta_{\mathscr{B}}$ is the factor which depends only on $\mathscr{B}$,

$$\eta_{\mathscr{B}} \equiv \prod_{k=1}^{C} \frac{L(k,C)}{L(k,C) - N_k} \prod_{l=C+1}^{D} \frac{L(C+1,l)}{L(C+1,l) - M_l}\,, \qquad L(a,b) = \sum_{k=a}^{b} (M_k + N_k) \tag{4.7}$$

assuming that the small diagram r is put at the C-th corner of $\mathscr{B}$ in figure 1. It follows that

$$\frac{(|\mathscr{B}| + |r|)!}{|\mathscr{B}|!} \simeq |\mathscr{B}|^{|r|}\,, \qquad \frac{d_{+r}}{d_r\, d_{\mathscr{B}}} \simeq \frac{1}{|r|!} \left(\frac{|\mathscr{B}|}{\eta_{\mathscr{B}}} \right)^{|r|} \qquad (N_c \gg 1). \tag{4.8}$$

Since position of the C-th corner is $(i,j) = (1 + \sum_{l=C+1}^{D} M_l, 1 + \sum_{k=1}^{C} N_k)$, from (A.5) we get

$$\frac{\mathrm{Dim}_{N_c}(+R)}{\mathrm{Dim}_{N_c}(\mathscr{B})} \simeq \mathrm{Dim}_{N_c'}(R), \qquad N_c' = N_c + \sum_{l=C+1}^{D} M_l - \sum_{k=1}^{C} N_k\,. \tag{4.9}$$

In [43] they found that the operator mixing coefficients satisfy the identity

$$N^{+R,(+r,s),\nu_-,\nu_+}_{+T,(+t,u),\mu_-,\mu_+} \simeq N^{R,(r,s),\nu_-,\nu_+}_{T,(t,u),\mu_-,\mu_+} \qquad (N_c \gg 1) \tag{4.10}$$

showing that

$$\mathfrak{D}_1\, \mathcal{O}^{\rm LLM}_{\Delta} \simeq \Delta_1\, \mathcal{O}^{\rm LLM}_{\Delta} \qquad (N_c \gg 1). \tag{4.11}$$

4.2 Tree-level OPE coefficients

We revisit two types of OPE coefficients in section 3. We will show that the OPE coefficients of non-extremal three-point functions in $\mathcal{N}=4$ SYM are essentially same as those of the LLM operators, after redefinition of N_c.

4.2.1 Adding a background tableau to $\tilde{C}_{\circ\circ\circ}$

Recall that $\tilde{C}_{\circ\circ\circ}$ is given by (3.70),

$$\begin{aligned}\tilde{C}_{\circ\circ\circ} &= \left\langle \hat{\mathcal{O}}_1^{R_1(\overline{L}_1)}[Z,\mathbf{1}]\, \hat{\mathcal{O}}_2^{R_2(\overline{L}_2)}[\tilde{Z},\mathbf{1}]\, \hat{\mathcal{O}}_3^{R_3(\overline{L}_3)}[\overline{Z},\mathbf{1}] \right\rangle \\ &= \left(\prod_{i=1}^{3} \frac{L_i!}{\overline{L}_i!}\right) \sum_{\hat{R}\vdash L} \frac{{\rm Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \sum_{Q_1\vdash\overline{L}_2} \sum_{Q_2\vdash\overline{L}_3} \sum_{Q_3\vdash\overline{L}_1} \left(\prod_{i=1}^{3} d_{Q_i}\right) \mathcal{G}_{123}\,.\end{aligned} \tag{4.12}$$

We obtain the OPE coefficients of the LLM operators by the substitution $Q_1 \to (+Q_1)$, while leaving $Q_2\,, Q_3$ as before. From (3.71) it follows that

$$\begin{gathered}(+R_1) = (+Q_1)\otimes Q_2\,, \qquad R_2 = Q_2 \otimes Q_3\,, \qquad (+R_3) = Q_3 \otimes (+Q_1) \\ (+\hat{R}) = (+Q_1)\otimes Q_2 \otimes Q_3\end{gathered} \tag{4.13}$$

and thus

$$\tilde{C}^{\rm LLM}_{\circ\circ\circ} \equiv \left\langle \hat{\mathcal{O}}_1^{+R_1(\overline{L}_1)}[Z,\mathbf{1}]\, \hat{\mathcal{O}}_2^{R_2(\overline{L}_2)}[\tilde{Z},\mathbf{1}]\, \hat{\mathcal{O}}_3^{+R_3(\overline{L}_3)}[\overline{Z},\mathbf{1}] \right\rangle \tag{4.14}$$

$$= \frac{(+L_1)!L_2!(+L_3)!}{\overline{L}_1!(+\overline{L}_2)!\overline{L}_3!} \sum_{(+\hat{R})\vdash(+L)} \frac{{\rm Dim}_{N_c}(+\hat{R})}{d_{+R_1} d_{R_2} d_{+R_3}} \sum_{(+Q_1)\vdash(+\overline{L}_2)} \sum_{Q_2\vdash\overline{L}_3} \sum_{Q_3\vdash\overline{L}_1} \left(d_{+Q_1} d_{Q_2} d_{Q_3}\right) \mathcal{G}^{\rm LLM}_{123}\,.$$

By using the identities in section 4.1, we find

$$\tilde{C}^{\rm LLM}_{\circ\circ\circ} \simeq (\eta_{\mathscr{B}})^L\, {\rm Wt}_{N_c}(\mathscr{B})\, \frac{L_1!L_2!L_3!}{\overline{L}_1!\overline{L}_2!\overline{L}_3!} \sum_{\hat{R}\vdash L} \frac{{\rm Dim}_{N_c'}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \sum_{Q_1\vdash\overline{L}_2} \sum_{Q_2\vdash\overline{L}_3} \sum_{Q_3\vdash\overline{L}_1} \left(d_{Q_1} d_{Q_2} d_{Q_3}\right) \mathcal{G}_{123}\,. \tag{4.15}$$

If we remove the $\mathscr{B}$-dependent prefactor $(\eta_{\mathscr{B}})^L\, {\rm Wt}_{N_c}(\mathscr{B})$, the OPE coefficient $\tilde{C}^{\rm LLM}_{\circ\circ\circ}$ agrees with $\tilde{C}_{\circ\circ\circ}$ up to the redefinition of $N_c \to N_c'$ in (4.9).

4.2.2 Adding a background tableau to $\tilde{C}^{XYZ}_{\vec{h}}$

Recall that $\tilde{C}^{XYZ}_{\vec{h}}$ is given by (3.90),

$$\begin{aligned}\tilde{C}^{XYZ}_{\vec{h}} &= \left\langle \hat{\mathcal{O}}_1^{\boldsymbol{R}_1(\overline{L}_1)}[\overline{X},\overline{Y},Z,\mathbf{1}]\, \hat{\mathcal{O}}_2^{\boldsymbol{R}_2(\overline{L}_2)}[\overline{X},Y,\overline{Z},\mathbf{1}]\, \hat{\mathcal{O}}_3^{\boldsymbol{R}_3(\overline{L}_3)}[X,Y,\overline{Z},\mathbf{1}] \right\rangle \\ &= \left(\prod_{i=1}^{3} \frac{L_i!}{\overline{L}_i!} \right) \sum_{\hat{R} \vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \left(d_{q_1}\, d_{q_2}\, d_{r_1}\, d_{r_3}\, d_{s_2}\, d_{s_3} \right) \overline{\delta}^{\nu_{1-}\,\nu_{2+}}\, \overline{\delta}^{\nu_{2-}\,\nu_{3+}}\, \overline{\delta}^{\nu_{3-}\,\nu_{1+}}\, \mathcal{G}'_{123}\end{aligned} \tag{4.16}$$

where $\boldsymbol{R}_i$ is defined in (3.14) as

$$\boldsymbol{R}_i = \{R_i, (q_i, r_i, s_i), \nu_{i-}, \nu_{i+}\}\,, \qquad (R_i \vdash L_i)\,. \tag{4.17}$$

We obtain the OPE coefficients in the LLM background by the substitution $(s_1\,, s_2\,, s_3) \to (+s_1\,, +s_2, s_3)\,,$ while q_i, r_i are the same as before. From (3.91) we find

$$\begin{gathered} q_1 \otimes q_2 = q_3, \quad r_1 \otimes r_3 = r_2, \quad (+s_2) \otimes s_3 = (+s_1), \quad q_1 \otimes q_2 \otimes r_1 \otimes r_3 \otimes (+s_2) \otimes s_3 = \hat{R} \\ (+R_1)_{\nu_{1\mp}} = q_1 \otimes r_1 \otimes \Big((+s_2) \otimes s_3 \Big) \\ (+R_2)_{\nu_{2\mp}} = q_2 \otimes \Big(r_1 \otimes r_3 \Big) \otimes (+s_2) \\ (R_3)_{\nu_{3\mp}} = \Big(q_1 \otimes q_2 \Big) \otimes r_3 \otimes s_3\,. \end{gathered} \tag{4.18}$$

It follows that

$$\begin{aligned}(\tilde{C}^{XYZ}_{\vec{h}})^{\mathrm{LLM}} = \left(\frac{(+L_1)!(+L_2)!L_3!}{\overline{L}_1!\overline{L}_2!(+\overline{L}_3)!} \right) \sum_{\hat{R} \vdash L} \frac{\mathrm{Dim}_{N_c}(+\hat{R})}{d_{+R_1} d_{+R_2} d_{R_3}} \left(d_{q_1}\, d_{q_2}\, d_{r_1}\, d_{r_3}\, d_{+s_2}\, d_{s_3} \right) \times \\ \overline{\delta}^{\nu_{1-}\,\nu_{2+}}\, \overline{\delta}^{\nu_{2-}\,\nu_{3+}}\, \overline{\delta}^{\nu_{3-}\,\nu_{1+}}\, \mathcal{G}'^{\,\mathrm{LLM}}_{123}\,.\end{aligned} \tag{4.19}$$

At large N_c, we can simplify this results following our discussion in section 4.1 as

$$\begin{aligned}(\tilde{C}^{XYZ}_{\vec{h}})^{\mathrm{LLM}} &= \frac{\overline{L}_3!}{(\overline{L}_3 - |r_1|)!} \left(\frac{\eta_{\mathscr{B}}}{|\mathscr{B}|} \right)^{|r_1|} \eta_{\mathscr{B}}^{L}\, \mathrm{Wt}_{N_c}(\mathscr{B}) \times \\ &\quad \frac{L_1!L_2!L_3!}{\overline{L}_1!\overline{L}_2!\overline{L}_3!} \sum_{\hat{R} \vdash L} \frac{\mathrm{Dim}_{N'_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \left(d_{q_1}\, d_{q_2}\, d_{r_1}\, d_{r_3}\, d_{s_2}\, d_{s_3} \right) \overline{\delta}^{\nu_{1-}\,\nu_{2+}} \overline{\delta}^{\nu_{2-}\,\nu_{3+}} \overline{\delta}^{\nu_{3-}\,\nu_{1+}} \mathcal{G}'_{123}\,.\end{aligned} \tag{4.20}$$

The first line is a numerical prefactor, and the second line agrees with $(\tilde{C}^{XYZ}_{\vec{h}})$ by the redefinition of $N_c \to N'_c$ in (4.9).

5 Conclusion and outlook

In this paper, we have studied general non-extremal three-point functions of scalar multi-trace operators at tree level valid for any values of N_c in gauge theory including $\mathcal{N} = 4$ SYM, by using the representation theory of symmetric groups.

We made full use of various new mathematical techniques. The quiver calculus of [29] gives a collection of diagrammatic method which simplifies various objects in the representation theory. The generalized Racah-Wigner tensor is introduced as an extension of the $6j$ symbols. We conjectured formulae about the invariant products of the generalized Racah-Wigner tensors, written in terms of the Littlewood-Richardson coefficients.

With these formulae, we provide strong evidence on the large N_c background independence, a correspondence between small $(\mathcal{O}(N_c^0))$ and huge $(\mathcal{O}(N_c^2))$ operators of $\mathcal{N}=4$ SYM. The background independence has been checked for two-point functions as well as extremal three-point functions. Our argument demonstrates that it extends to non-extremal three-point functions. These results will clarify the properties of stringy excitations on the LLM backgrounds, particularly how they differ from the usual strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

Let us comment on some important future directions.

The first direction is to find a connection with the integrability results of the planar $\mathcal{N}=4$ SYM. Clearly, the operators in the representation basis are not the eigenstates of the dilatation operator of $\mathcal{N}=4$ SYM. One should think of the representation basis as a tool for the finite N_c computation. The two-point functions of single-trace operators in the $\mathfrak{su}(2)$ sector have been computed in this way [27, 46], generalizing the old results of the complex matrix model [47, 48]. A particularly interesting question is to determine the so-called octagon frame, namely the tree-level part of the "simplest" four-point functions of $\mathcal{N}=4$ SYM in the large charge limit [11]. The finite group methods developed in this paper can be used for the exact finite-N_c computation, because it is a generalization of the character expansion methods familiar in the matrix models [49–51].

The second direction is to refine our computation. The conjectured formula for the invariant products of generalized Racah-Wigner tensor should be proven. The computation of the n-point functions in the representation basis is also important. It is interesting to ask whether one can bootstrap four-point functions out of two- and three-point data.

The third direction is to investigate a possible relation between quiver calculus and knot theory. The $6j$ symbol of the unitary group has been extensively studied in the context of knot theory and integrable systems [52]. Since the $6j$ symbols of symmetrical groups are related to those of unitary groups, the quiver calculus could give a new insight into the study of knot polynomials. For example, some non-trivial conjectures about the $6j$ symbols have been made [53–55], though most of them discuss the multiplicity-free cases only. Since the new invariants $\mathcal{G}_{123}$ and $\mathcal{G}'_{123}$ discussed in this paper are closely related to the multiplicity structure, studying similar quantity in the case of unitary groups is a fascinating problem.

Finally, we hope to find a clear understanding of the AdS/CFT correspondence of the operators with huge anomalous dimensions, including giant gravitons [56, 57] and the fluctuation in the LLM geometry [43, 58, 59]. Some correlation functions have been studied such as three giants [60–62], two giants and one single-trace [63–70].

JHEP05(2020)118

Acknowledgments

RS thanks Robert de Mello Koch and Sanjaye Ramgoolam for their comments on the manuscript, and is obliged to Korea Institute for Advanced Study where this research has been initiated.

A Survey of finite-group representation theory

We explain our notation and formulae used in the main text, while providing a brief survey of the representation theory of finite groups. Our notation is similar to the one used in [22]. For more details on finite groups, see textbooks like [71, 72].

A.1 Basic notation

The symmetric group permuting L elements is denoted by S_L. We denote the conjugacy class of S_L by

$$\mathtt{C}_\alpha = \frac{1}{|S_L|} \sum_{\gamma \in S_L} \gamma \alpha \gamma^{-1}, \tag{A.1}$$

The δ-function over S_L (or $\mathbb{C}[S_L]$) is defined by

$$\delta(\beta) = \begin{cases} 1 & (\beta = \mathbf{1} \in S_L) \\ 0 & (\text{otherwise}). \end{cases} \tag{A.2}$$

A permutation cycle is denoted by $(12 \dots L) \in \mathbb{Z}_L$. Any element of S_L consists of permutation cycles. The number of length-k cycles in $\sigma \in S_L$ is denoted by $\mathrm{Cyc}_k(\sigma)$. The number of cycles in σ is

$$C(\sigma) = \sum_k \mathrm{Cyc}_k(\sigma) \tag{A.3}$$

so that $C(\mathbf{id}) = C((1)(2) \dots (L)) = L$.

A partition of L, or equivalently a Young diagram with L boxes, is denoted by $R \vdash L$. Define

$$d_R = \frac{L!}{\mathrm{hook}_R}, \qquad \mathrm{hook}_R = \prod_{(i,j) \in R} \Big(\text{hook length at } (i,j)\Big) \tag{A.4}$$

$$\mathrm{Dim}_N(R) = \frac{d_R}{L!} \mathrm{Wt}_N(R), \qquad \mathrm{Wt}_N(R) = \prod_{(i,j) \in R} (N + i - j) \tag{A.5}$$

where d_R is the dimension of R as the representation of S_L, and $\mathrm{Dim}_N(R)$ is the dimension of R as the representation of $\mathrm{U}(N)$.[5] For example, hook_R and $\mathrm{Wt}_N(R)$ of the Young tableau

[5] $\mathrm{Wt}_N(R)$ is also denoted by f_R in the literature, e.g. [23].

$R = \yng(4,2)$ are given by

$$\begin{array}{|c|c|c|c|}\hline 5 & 4 & 2 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array} \quad \Rightarrow \quad \text{hook}_{\yng(4,2)} = 5 \times 4 \times 2 \times 2 \times 1 \times 1$$

$$\begin{array}{|c|c|c|c|}\hline N & N+1 & N+2 & N+3 \\ \hline N-1 & N \\ \cline{1-2}\end{array} \quad \Rightarrow \quad \mathrm{Wt}_N \left(\yng(4,2)\right) = (N-1)\, N^2\, (N+1)\, (N+2)\, (N+3)\,. \tag{A.6}$$

We assume that all representations are real and orthogonal.[6] Denote the I-th component of the irreducible representation R of S_L by $\left|\begin{smallmatrix} R \\ I \end{smallmatrix}\right\rangle$, with $I = 1, 2, \ldots, d_R$. Introduce the dual basis by

$$\left\langle \begin{matrix} R \\ I \end{matrix} \middle| \begin{matrix} S \\ J \end{matrix} \right\rangle = \delta^{RS}\, \delta_{IJ}\,. \tag{A.7}$$

Let $D^R_{IJ}(\sigma)$ be the representation matrix of $\sigma \in S_{m+n}$ of the representation $R \vdash L$,

$$D^R_{IJ}(\sigma) = \left\langle \begin{matrix} R \\ I \end{matrix} \middle| \sigma \middle| \begin{matrix} R \\ J \end{matrix} \right\rangle = D^R_{JI}(\sigma^{-1}). \tag{A.8}$$

The character of the representation R for the group element σ is denoted by[7]

$$\chi^R(\sigma) = \sum_{I=1}^{d_R} D^R_{II}(\sigma). \tag{A.9}$$

By restricting $\sigma \in S_L = S_{m+n}$ to $S_m \otimes S_n$, we obtain the irreducible decomposition[8]

$$R = \bigoplus_{\substack{r \vdash m \\ s \vdash n}} g(r, s; R)\, (r \otimes s) = \bigoplus_{\substack{r \vdash m \\ s \vdash n}} \bigoplus_{\nu=1}^{g(r,s;R)} (r \otimes s)_\nu \tag{A.10}$$

where $g(r, s; R)$ is the Littlewood-Richardson coefficient. It counts the number of $r \otimes s$ appearing in the irreducible decomposition of R. The subscript ν is called the multiplicity label. With an appropriate change of basis,[9] we can transform the representation matrix into a block-diagonal form,

$$D^R_{IJ}(\sigma) = B \begin{pmatrix} D^{r^{(1)} \otimes s^{(1)}}_{i_1 j_1}(\sigma) & & & \\ & D^{r^{(2)} \otimes s^{(2)}}_{i_2 j_2}(\sigma) & & \\ & & D^{r^{(3)} \otimes s^{(3)}}_{i_3 j_3}(\sigma) & \\ & & & \ddots \end{pmatrix} B^T \qquad (\sigma \in S_m \otimes S_n) \tag{A.11}$$

[6]The orthogonal form of the Young-Yamanouchi basis satisfies these conditions.
[7]Often we sum over the repeated indices of matrices. The symbol $\sum$ is written explicitly in appendix A.
[8]The restriction to a subgroup is also called subduction in the literature.
[9]This appropriate basis is called the split basis.

such that it matches (A.10). By definition of the irreducible decomposition, there are no off-block-diagonal elements including the multiplicity labels. For general $\sigma \in S_{m+n}$, the matrix (A.11) has off-block-diagonal elements.[10]

Let $\left| \begin{smallmatrix} r,s \\ i,j \end{smallmatrix} \nu \right\rangle$ be an orthonormal basis of $r \otimes s$ at the ν-th multiplicity, satisfying

$$\left\langle \begin{matrix} r_1 \; s_1 \\ i_1 \; j_1 \end{matrix} \; \nu_1 \middle| \begin{matrix} r_2 \; s_2 \\ i_2 \; j_2 \end{matrix} \; \nu_2 \right\rangle = \delta^{r_1 r_2} \, \delta^{s_1 s_2} \, \delta^{\nu_1 \nu_2} \, \delta_{i_1 i_2} \, \delta_{j_1 j_2} \tag{A.12}$$

for $\nu_k = 1, 2, \ldots, g(r_k, s_k; R)$. The rotation matrix is called the branching coefficients, defined by

$$B^{R \to (r,s),\nu}_{I \to (i,j)} = \left\langle \begin{matrix} R \\ I \end{matrix} \middle| \begin{matrix} r \; s \\ i \; j \end{matrix} \; \nu \right\rangle, \qquad (B^T)^{R \to (r,s),\nu}_{I \to (i,j)} = \left\langle \begin{matrix} r \; s \\ i \; j \end{matrix} \; \nu \middle| \begin{matrix} R \\ I \end{matrix} \right\rangle . \tag{A.13}$$

A.2 Branching coefficients

We find from (A.11) that the branching coefficients satisfy the completeness relations

$$\sum_{r,s,\nu} \sum_{i,j} B^{R \to (r,s),\nu}_{I \to (i,j)} \, (B^T)^{R \to (r,s),\nu}_{J \to (i,j)} = \delta_{I,J} \tag{A.14}$$

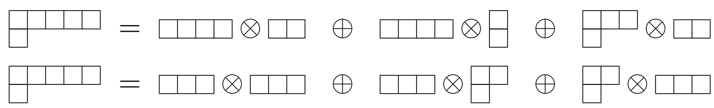
$$\sum_I (B^T)^{R \to (r_1,r_2),\nu}_{I \to (i_1,i_2)} \, B^{R \to (s_1,s_2),\mu}_{I \to (j_1,j_2)} = \delta^{r_1,s_1} \, \delta^{r_2,s_2} \, \delta^{\nu\mu} \, \delta_{i_1,j_1} \, \delta_{i_2,j_2} \, . \tag{A.15}$$

In (A.15), we assume that two product representations $r_1 \otimes r_2$ and $s_1 \otimes s_2$ descend from the same restriction $S_{m+n} \downarrow (S_m \otimes S_n)$. If they descend from different restrictions, then the two branching coefficients B and $\tilde{B}$ are unrelated, and we obtain another orthogonal matrix

$$\sum_I (B^T)^{R \to (r_1,r_2),\nu}_{I \to (i_1,i_2)} \, \tilde{B}^{R \to (s_1,s_2),\mu}_{I \to (j_1,j_2)} = \left\langle \begin{matrix} r_1 \; r_2 \\ i_1 \; i_2 \end{matrix} \; \nu \middle| \begin{matrix} s_1 \; s_2 \\ j_1 \; j_2 \end{matrix} \; \mu \right\rangle . \tag{A.16}$$

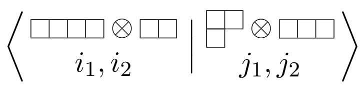
For example, given two irreducible decompositions

$S_6 \downarrow (S_4 \otimes S_2)$,
$S_6 \downarrow (S_3 \otimes S_3)$,



(A.17)

any pairs $r_1 \otimes r_2$ and $s_1 \otimes s_2$ from different restrictions can have non-vanishing overlap, e.g.



(A.18)

Sometimes we take the coordinates explicitly in order to distinguish $S_{m+n} \downarrow (S_m \otimes S_n)$ and $S_{m+n} \downarrow (S_n \otimes S_m)$. For example, the following two restrictions

$$\begin{aligned} S_{m+n} \downarrow (S_m \otimes S_n) \;&\sim\; \text{Permute}\,(\{1, 2, \ldots, m\}) \times \text{Permute}\,(\{m+1, \ldots m+n\}) \\ S_{m+n} \downarrow (S_n \otimes S_m) \;&\sim\; \text{Permute}\,(\{1, 2, \ldots, n\}) \times \text{Permute}\,(\{n+1, \ldots n+m\}) \end{aligned} \tag{A.19}$$

define different branching coefficients, $B^{R \to (r_1,r_2),\nu}_{I \to (i_1,i_2)}$ and $\tilde{B}^{R \to (s_1,s_2),\mu}_{I \to (j_1,j_2)}$.

[10] The restricted Schur basis should have off-block-diagonal elements with respect to the multiplicity labels, which can be checked by counting the dimensions [46].

JHEP05(2020)118

From (A.11), we obtain the following identities for the matrix elements of $\gamma = \gamma_1 \circ \gamma_2 \in S_m \otimes S_n$

$$D^R_{IJ}(\gamma_1 \circ \gamma_2) = \sum_{r_1,r_2,\nu} \sum_{i,j,k,l} D^{r_1}_{ik}(\gamma_1)\, D^{r_2}_{jl}(\gamma_2)\, B^{R\to(r_1,r_2)\nu}_{I\to(i,j)}\, (B^T)^{R\to(r_1,r_2)\nu}_{J\to(k,l)} \tag{A.20}$$

By multiplying $B^{R\to(r_1,r_2)\nu}_{J\to(k',l')}$ to (A.20) and summing over J, we find

$$\sum_J D^R_{IJ}(\gamma_1 \circ \gamma_2)\, B^{R\to(r_1,r_2)\nu}_{J\to(k,l)} = \sum_{i,j} D^{r_1}_{ik}(\gamma_1)\, D^{r_2}_{jl}(\gamma_2)\, B^{R\to(r_1,r_2)\nu}_{I\to(i,j)}\,. \tag{A.21}$$

Again, by multiplying $(B^T)^{R\to(r_1,r_2)\mu}_{I\to(i',j')}$ to (A.21) and summing over J, we find

$$\sum_{I,J} D^R_{IJ}(\gamma_1 \circ \gamma_2)\, (B^T)^{R\to(r_1,r_2)\mu}_{I\to(i,j)}\, B^{R\to(r_1,r_2)\nu}_{J\to(k,l)} = \delta^{\mu\nu}\, D^{r_1}_{ik}(\gamma_1)\, D^{r_2}_{jl}(\gamma_2). \tag{A.22}$$

In the r.h.s., the matrix elements of $\gamma_1 \circ \gamma_2$ in the split basis are independent of the multiplicity labels μ, ν. This can be understood also from the construction of the Young-Yamanouchi basis.

The branching coefficients (A.13) for general restriction $S_L \downarrow (S_{m_1} \otimes S_{m_2} \otimes \cdots \otimes S_{m_\ell})$ are given by

$$B^{R\to(r_1,r_2,\ldots,r_\ell),\nu}_{I\to(i_1,i_2,\ldots,i_\ell)} = \left\langle \begin{matrix} R \\ I \end{matrix} \middle| \begin{matrix} r_1\, r_2 \ldots r_\ell \\ i_1\, i_2 \ldots i_\ell \end{matrix}\; \nu \right\rangle, \quad (B^T)^{R\to(r_1,r_2,\ldots,r_\ell),\nu}_{I\to(i_1,i_2,\ldots,i_\ell)} = \left\langle \begin{matrix} r_1\, r_2 \ldots r_\ell \\ i_1\, i_2 \ldots i_\ell \end{matrix}\; \nu \middle| \begin{matrix} R \\ I \end{matrix} \right\rangle \tag{A.23}$$

for $\nu = 1, 2, \ldots, g(r_1, r_2, \ldots, r_\ell; R)$. The generalized split basis can be defined by the branching coefficients as in (A.11). The formula (A.20) is generalized as

$$\begin{aligned} &D^R_{IJ}(\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_\ell) \\ &= \sum_{r_1,r_2,\nu} \sum_{i,j,k,l} D^{r_1}_{i_1k_1}(\gamma_1)\, D^{r_2}_{i_2k_2}(\gamma_2) \,\ldots\, D^{r_\ell}_{i_\ell k_\ell}(\gamma_\ell)\, B^{R\to(r_1,r_2,\ldots,r_\ell),\nu}_{I\to(i_1,i_2,\ldots,i_\ell)}\, (B^T)^{R\to(r_1,r_2,\ldots,r_\ell),\nu}_{J\to(k_1,k_2,\ldots,k_\ell)}\,. \end{aligned} \tag{A.24}$$

for $\gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_\ell \in (S_{m_1} \otimes S_{m_2} \otimes \cdots \otimes S_{m_\ell})$.

A.3 Restricted Schur basis

Consider the restriction $S_M \downarrow (S_{m_1} \otimes S_{m_2} \otimes S_{m_3})$ with $M = m_1 + m_2 + m_3$, which corresponds to the multi-trace operators with three complex scalars in (2.2).

Define the restricted Schur characters by using the branching coefficients [29],

$$\chi^{R,(r_1,r_2,r_3),\nu_+,\nu_-}(\sigma) \equiv \sum_{I,J} \sum_{i,j,k} B^{R\to(r_1,r_2,r_3)\nu_+}_{I\to(i,j,k)}\, (B^T)^{R\to(r_1,r_2,r_3),\nu_-}_{J\to(i,j,k)}\, D^R_{IJ}(\sigma), \quad (\sigma \in S_M). \tag{A.25}$$

Define the operator in the restricted Schur basis by

$$\mathcal{O}^{R,(r_1,r_2,r_3),\nu_+,\nu_-}[X,Y,Z] = \frac{1}{m_1!\, m_2!\, m_3!} \sum_{\alpha \in S_M} \chi^{R,(r_1,r_2,r_3),\nu_+,\nu_-}(\alpha)\, \mathrm{tr}_M\!\left(\alpha X^{\otimes m_1} Y^{\otimes m_2} Z^{\otimes m_3}\right). \tag{A.26}$$

The inverse transformation from the restricted Schur basis to the permutation basis is

$$\begin{aligned}&\mathrm{tr}_M\left(\alpha\, X^{\otimes m_1}\, Y^{\otimes m_2}\, Z^{\otimes m_3}\right)\\&=\frac{m_1!\, m_2!\, m_3!}{M!}\sum_{R,r_1,r_2,r_3,\mu_+,\mu_-}\frac{d_R}{d_{r_1}d_{r_2}d_{r_3}}\,\chi^{R,(r_1,r_2,r_3),\mu_+,\mu_-}(\alpha)\,\mathcal{O}^{R,(r_1,r_2,r_3),\mu_+,\mu_-}\end{aligned} \tag{A.27}$$

which can be checked by the row orthogonality of the restricted characters (A.52),

$$\frac{1}{M!}\sum_{\sigma\in S_M}\chi^{R,(r_1,r_2,r_3),\nu_+,\nu_-}(\sigma)\chi^{S,(s_1,s_2,s_3),\mu_+,\mu_-}(\sigma)=\frac{d_{r_1}d_{r_2}d_{r_3}}{d_R}\,\delta^{RS}\delta^{r_1s_1}\delta^{r_2s_2}\delta^{r_3s_3}\delta^{\nu_+\mu_+}\delta^{\nu_-\mu_-}\,. \tag{A.28}$$

As discussed in section 2.2, the tree-level two-point function is

$$\begin{aligned}&\left\langle\mathcal{O}^{R,(r_1,r_2,r_3)(\nu_+,\nu_-)}[X,Y,Z](x)\,\mathcal{O}^{S,(s_1,s_2,s_3)(\mu_+,\mu_-)}[\overline{X},\overline{Y},\overline{Z}](0)\right\rangle\\&=\frac{\mathrm{Wt}_N(R)}{|x|^{2M}}\,\frac{\mathrm{hook}_R}{\mathrm{hook}_{r_1}\mathrm{hook}_{r_2}\mathrm{hook}_{r_3}}\,\delta^{RS}\,\delta^{r_1s_1}\,\delta^{r_2s_2}\,\delta^{r_3s_3}\,\delta^{\nu_+\mu_+}\,\delta^{\nu_-\mu_-}\,.\end{aligned} \tag{A.29}$$

A.4 Formulae

The formulae for the irreducible characters and the restricted characters will be summarized below. For simplicity, we mostly consider the restriction $S_{m+n}\downarrow(S_m\otimes S_n)$. Generalization to $S_M\downarrow(\otimes_k S_{m_k})$ is straightforward.

Character orthogonality. Let R,S be the irreducible representations of S_L. The representation matrices satisfy the grand orthogonality relation

$$\sum_{\sigma\in S_L}D^R_{ij}(\sigma)D^S_{kl}(\sigma^{-1})=\frac{L!}{d_R}\,\delta_{il}\delta_{jk}\,. \tag{A.30}$$

By taking the trace, we obtain the row (or first) orthogonality relation of irreducible characters,

$$\sum_{\sigma\in S_L}\chi^R(\sigma)\chi^S(\sigma^{-1})=L!\,\delta^{RS}\,. \tag{A.31}$$

The irreducible characters also satisfy the column (or second) orthogonality relation,

$$\sum_{R\vdash L}\chi^R(\sigma)\chi^R(\tau)=\sum_{\gamma\in S_L}\delta(\sigma\gamma\tau\gamma^{-1})=\begin{cases}|\mathtt{C}_\sigma| & (\mathtt{C}_\sigma=\mathtt{C}_\tau)\\ 0 & (\text{otherwise})\end{cases} \tag{A.32}$$

where $|\mathtt{C}_\sigma|$ is the number of elements in a given conjugacy class (A.1). This relation follows from the fact that any class function can be expanded by irreducible characters

$$f(\sigma)=f(\gamma\sigma\gamma^{-1}),\quad(\forall\gamma\in S_L)\qquad\Leftrightarrow\qquad f(\sigma)=\sum_{R\vdash L}\tilde{f}_R\,\chi^R(\sigma). \tag{A.33}$$

As a corollary, the δ-function can be written as

$$\delta(\beta)=\frac{1}{L!}\sum_{R\vdash L}d_R\,\chi^R(\beta). \tag{A.34}$$

Multiplicity label. There are several ways to understand Littlewood-Richardson coefficients.

The first way is by restriction $S_{m+n} \downarrow (S_m \otimes S_n)$ as in (A.10)

$$R = \bigoplus_{\substack{r \vdash m \\ s \vdash n}} g(r, s; R)\,(r \otimes s)\,. \tag{A.35}$$

The second way is by induction,

$$r \otimes s = \bigoplus_{R} g(r, s; R)\, R \tag{A.36}$$

Frobenius reciprocity guarantees the consistency between (A.36) and (A.35). Finally, the Littlewood-Richardson coefficient can be computed by

$$g(r, s; R) = \frac{1}{|S_m \otimes S_n|} \sum_{\alpha \in S_m} \sum_{\beta \in S_n} \chi^r(\alpha)\, \chi^s(\beta)\, \chi^R(\alpha \circ \beta) \tag{A.37}$$

where $\alpha \circ \beta \in S_m \otimes S_n \subset S_{m+n}$.

The generalized Littlewood-Richardson coefficient for $\otimes_{k=1}^{l} S_{m_k}$ is given by

$$g(r_1, r_2, \ldots, r_l; R) = \frac{1}{|\otimes_{k=1}^{l} S_{m_k}|} \sum_{\{\sigma_k \in S_{m_k}\}} \left(\prod_{k=1}^{l} \chi^{r_k}(\sigma_k) \right) \chi^R(\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_l). \tag{A.38}$$

They satisfy a recursion relation

$$\sum_{R \vdash M} g(r_1, r_2, \ldots, r_l; R)\, g(R, r_{l+1}; S) = g(r_1, r_2, \ldots, r_{l+1}; S), \qquad \left(M = \sum_{k=1}^{l} m_k \right) \tag{A.39}$$

which can be shown from (A.32). The equation (A.39) implies an important identity for multiple branching coefficients

$$B^{S \to (r_1, r_2, \ldots, r_{l+1}), \eta}_{I \to (a_1, a_2, \ldots, a_{l+1})} = \sum_{R} \sum_{A=1}^{d_R} B^{S \to (R, r_{l+1}), \mu}_{I \to (A, a_{l+1})} B^{R \to (r_1, r_2, \ldots, r_l), \rho}_{A \to (a_1, a_2, \ldots, a_l)} \tag{A.40}$$

$\eta = 1, 2, \ldots, g(r_1, r_2, \ldots, r_{l+1}; S)$, $\mu = 1, 2, \ldots, g(R, r_{l+1}; S)$, $\rho = 1, 2, \ldots, g(r_1, r_2, \ldots, r_l; R)$.

Schur-Weyl duality. The quantity $N^{C(\sigma)}$ is a class function. We obtain its irreducible decomposition (A.33) by using the Schur-Weyl duality [19] as

$$N^{C(\sigma)} = \sum_{R \vdash L} \mathrm{Dim}_N(R)\, \chi^R(\sigma). \tag{A.41}$$

Note that $\mathrm{Dim}_N(R) = 0$ if the height of the Young diagram R is larger than N, as can be seen from (A.5). By applying the grand orthogonality relation (A.30), we find

$$\sum_{\sigma \in S_L} D^S_{IJ}(\sigma) N^{C(\sigma)} = \delta_{IJ}\, \mathrm{Dim}_N(S)\, \mathrm{hooks}_S = \delta_{IJ}\, \mathrm{Wt}_N(S). \tag{A.42}$$

By multiplying the branching coefficients as in (A.44), we obtain another formula [23]

$$\sum_{\sigma \in S_{m+n}} \chi^{R,(r,s),\nu_+,\nu_-}(\sigma)\, N^{C(\sigma)} = \delta^{\nu_+ \nu_-}\, d_r d_s \mathrm{Wt}_N(R). \tag{A.43}$$

Restricted projector. We define the restricted projector

$$\mathscr{P}^{R,(r_1,r_2),\nu_+,\nu_-} = \frac{d_R}{(m+n)!} \sum_{\sigma \in S_{m+n}} \chi^{R,(r_1,r_2),\nu_+,\nu_-}(\sigma)\,\sigma \;\in\; \mathbb{C}[S_{m+n}] \tag{A.44}$$

so that [46]

$$\chi^{R,(r_1,r_2),\nu_+,\nu_-}(\sigma) = \chi^R\left(\mathscr{P}^{R,(r_1,r_2),\nu_+,\nu_-}\sigma\right) \tag{A.45}$$

$$\mathscr{P}^{R,(r_1,r_2),\nu_+,\nu_-}\;\mathscr{P}^{S,(s_1,s_2),\mu_+,\mu_-} = \delta^{RS}\,\delta^{r_1 s_1}\,\delta^{r_2 s_2}\,\delta^{\nu_-\mu_+}\,\mathscr{P}^{R,(r_1,r_2),\nu_+,\mu_-}\,. \tag{A.46}$$

By comparing (A.45) and (A.25), one finds

$$\mathscr{P}_{IJ}^{R,(r_1,r_2),\nu_+,\nu_-} \equiv D_{IJ}^R\left(\mathscr{P}^{R,(r_1,r_2),\nu_+,\nu_-}\right) = \sum_{i,j} B_{I\to(i,j)}^{R\to(r_1,r_2)\nu_+}\,(B^T)_{J\to(i,j)}^{R\to(r_1,r_2),\nu_-}\,. \tag{A.47}$$

It follows that

$$\chi^R\left(\mathscr{P}^{R,(r_1,r_2),\nu_+,\nu_-}\right) = \sum_I \sum_{i,j} B_{I\to(i,j)}^{R\to(r_1,r_2)\nu_+}\,(B^T)_{I\to(i,j)}^{R\to(r_1,r_2),\nu_-} = \delta^{\nu_+\nu_-}\,d_{r_1}\,d_{r_2}\,. \tag{A.48}$$

The restricted projector is useful for fixing the normalization. These formulae as well as the following identities can be proven by using the quiver calculus in appendix B.

Restricted character orthogonality. The restricted characters (A.25) satisfy the identities

$$\chi^{R,(r,s),\nu_+,\nu_-}(\sigma) = \chi^{R,(r,s),\nu_-,\nu_+}(\sigma^{-1}) \tag{A.49}$$

$$\chi^{R,(r,s),\nu_+,\nu_-}(\gamma\sigma\gamma^{-1}) = \chi^{R,(r,s),\nu_+,\nu_-}(\sigma) \qquad (\forall\gamma \in S_m \otimes S_n) \tag{A.50}$$

$$\chi^{R,(r,s),\nu_+,\nu_-}(\sigma_1\circ\sigma_2) = \delta^{\nu_+\nu_-}\,\chi^r(\sigma_1)\,\chi^s(\sigma_2) \qquad (\forall\sigma_1\circ\sigma_2 \in S_m \otimes S_n) \tag{A.51}$$

where the last relation is consistent with (A.22). The row and column orthogonality relations (A.32) are generalized as

$$\frac{1}{(m+n)!}\sum_{\sigma\in S_{m+n}} \chi^{R,(r_1,r_2),\nu_+,\nu_-}(\sigma)\chi^{S,(s_1,s_2),\mu_+,\mu_-}(\sigma) = \frac{d_{r_1}d_{r_2}}{d_R}\,\delta^{RS}\delta^{r_1s_1}\delta^{r_2s_2}\delta^{\nu_+\mu_+}\delta^{\nu_-\mu_-} \tag{A.52}$$

$$\sum_{R,r_1,r_2,\nu_+,\nu_-} \frac{d_R}{d_{r_1}d_{r_2}}\,\chi^{R,(r_1,r_2),\nu_+,\nu_-}(\sigma)\chi^{R,(r_1,r_2),\nu_+,\nu_-}(\tau) = \frac{(m+n)!}{m!n!}\sum_{\gamma\in S_m\otimes S_n} \delta(\gamma\sigma\gamma^{-1}\tau^{-1}). \tag{A.53}$$

One can generalize the grand orthogonality relation (A.30) with the branching coefficients in two ways. First, let R and S be the irreducible representations of S_{m+n}. A sum over S_{m+n} gives

$$\begin{aligned}&\frac{1}{(m+n)!}\sum_{\sigma\in S_{m+n}} D_{IJ}^R(\sigma)\,B^{\dagger\,R\to(r_1,r_2)\nu_+}_{I\to(i,j)}\,B_{J\to(k,l)}^{R\to(r_1,r_2)\nu_-}\,D_{MN}^S(\sigma)\,B^{\dagger\,S\to(s_1,s_2)\mu_+}_{M\to(m,n)}\,B_{N\to(p,q)}^{S\to(s_1,s_2)\mu_-}\\ &= \frac{\delta^{RS}}{d_R}\,\delta^{\nu_+\mu_+}\,\delta^{\nu_-\mu_-}\,\delta^{r_1,s_1}\,\delta^{r_2,s_2}\,\delta_{i,m}\,\delta_{j,n}\,\delta_{k,p}\,\delta_{l,q}\end{aligned} \tag{A.54}$$

which reduces to (A.52) by taking the trace over $r_1 \otimes r_2 = s_1 \otimes s_2$. Second, let (r_1, r_2) and (s_1, s_2) be the irreducible representations of $S_m \otimes S_n$. A sum over $S_m \otimes S_n$ gives

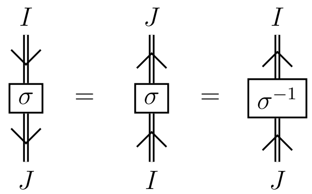
$$\frac{1}{m!\,n!} \sum_{\sigma \in S_m \otimes S_n} D^R_{IJ}(\sigma)\, B^{\dagger R \to (r_1,r_2)\nu_+}_{I \to (i,j)}\, B^{R \to (r_1,r_2)\nu_-}_{J \to (k,l)}\, D^S_{MN}(\sigma)\, B^{\dagger S \to (s_1,s_2)\mu_+}_{M \to (m,n)}\, B^{S \to (s_1,s_2)\mu_-}_{N \to (p,q)}$$
$$= \frac{\delta^{r_1 s_1} \delta^{r_2 s_2}}{d_{r_1} d_{r_2}}\, \delta^{\nu_+ \nu_-}\, \delta^{\mu_+ \mu_-}\, \delta_{i,m}\, \delta_{j,n}\, \delta_{k,p}\, \delta_{l,q} \tag{A.55}$$

where we used (A.22)

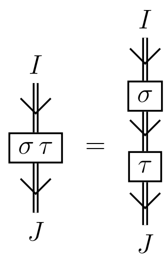
B Quiver calculus

Let us introduce a graphical notation of various representation-theoretical objects following [29]. We denote the indices of $R \vdash L = (m+n)$ by a double line, and those of $r_1 \vdash m$ or $r_2 \vdash n$ by a single line. We use different lines to distinguish two set of representations $\{R, (r_1, r_2)\}$ and $\{S, (s_1, s_2)\}$.

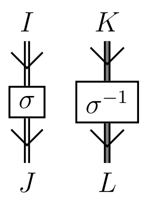
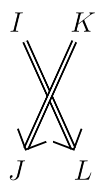
The matrix representation of a permutation group element is represented by

$D^R_{IJ}(\sigma) =$  (B.1)

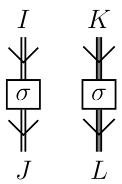
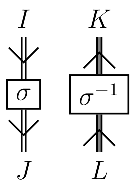
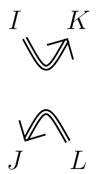
by using (A.8). Note that the matrix transposition is represented as flipping all the arrow directions. The composition of permutations is

$D^R_{IJ}(\sigma\tau) = \sum_{K=1}^{d_R} D^R_{IK}(\sigma) D^R_{KJ}(\tau) =$  (B.2)

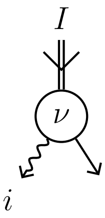
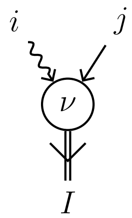
The grand orthogonality relation (A.30) is

$\frac{1}{L!} \sum_{\sigma \in S_L}$  $= \frac{\delta^{RS}}{d_R}$  $= \frac{\delta^{RS}}{d_R}\, \delta_{IL}\, \delta_{JK}$ (B.3)

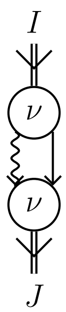
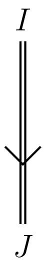
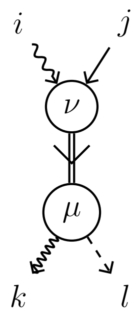
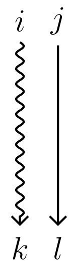
or equivalently

$\frac{1}{L!} \sum_{\sigma \in S_L}$  $= \frac{1}{L!} \sum_{\sigma \in S_L}$  $= \frac{\delta^{RS}}{d_R}$  $= \frac{\delta^{RS}}{d_R}\, \delta_{IK}\, \delta_{JL}\,.$ (B.4)

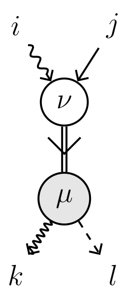
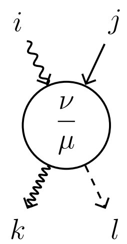
The branching coefficients (A.13) are represented as

$$B^{R\to(r_1,r_2)\nu}_{I\to(i,j)} =$$  $$(B^T)^{R\to(r_1,r_2)\nu}_{I\to(i,j)} =$$  (B.5)

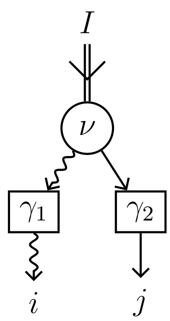
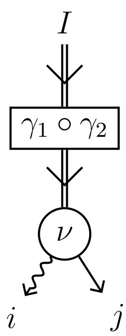
We use double lines for the indices of S_{m+n}, wavy lines for S_m and straight lines for S_n. The completeness relations of the branching coefficients (A.14), (A.15) are

$$\sum_{r_1,r_2,\nu}$$  $=$   $= \delta^{\nu\mu}\,\delta^{r_1 s_1}\,\delta^{r_2 s_2}$  (B.6)

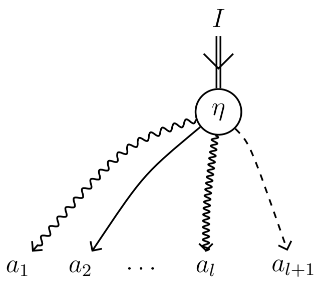
where we assumed that $r_1 \otimes r_2$ and $s_1 \otimes s_2$ follow from the same restriction of R. If the two product representations descend from different restrictions, we get the orthogonal matrix (A.16)

 $=$  (B.7)

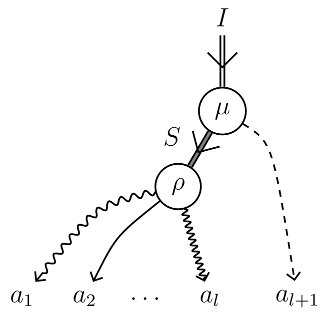
The relation (A.21) is expressed as

 $=$  (B.8)

The identity for multiple branching coefficients (A.40) is



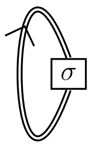
$$= \sum_S$$



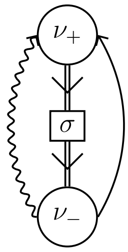
(B.9)

The character and the restricted characters are

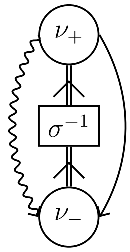
$$\chi^R(\sigma) = \chi^R(\sigma^{-1}) =$$



$$\chi^{R(r_1,r_2)(\nu_+,\nu_-)}(\sigma) =$$



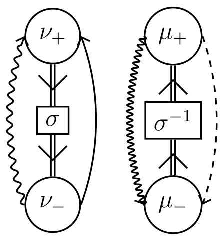
$$=$$



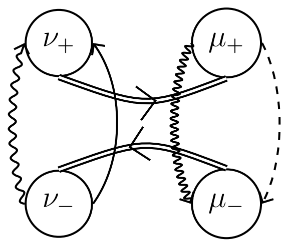
(B.10)

We can show the row orthogonality of the restricted character as

$$\frac{1}{L!}\sum_{\sigma\in S_L}$$



$$= \frac{\delta^{RS}}{d_R}$$



(B.11)

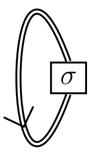
$$= \frac{d_{r_1} d_{r_2}}{d_R}\, \delta^{RS}\, \delta^{\nu_+\mu_+}\, \delta^{\nu_-\mu_-}\, \delta^{r_1 s_1}\, \delta^{r_2 s_2}.$$

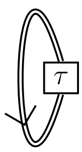
To show the column orthogonality, we insert the resolution of identity on the irreducible representation R by (A.30),

$$\delta_{il}\,\delta_{jk} = \frac{d_R}{L!}\sum_{\gamma\in S_L} D^R_{ij}(\gamma) D^R_{kl}(\gamma^{-1}), \qquad (i,j,k,l = 1,2,\ldots,d_R). \tag{B.12}$$

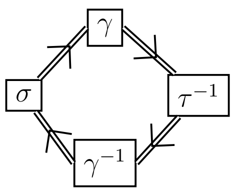
We obtain

$$\sum_{R\vdash L}$$





$$= \sum_{R\vdash L}\frac{d_R}{L!}\sum_{\gamma\in S_L}$$



$$= \sum_{\gamma\in S_L}\delta(\sigma\gamma\tau^{-1}\gamma^{-1}) \tag{B.13}$$

where we used (A.34). Note that

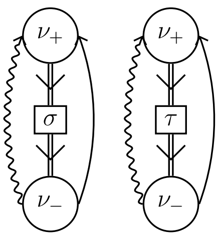
$$\sum_{\gamma \in S_L} \delta(\sigma\gamma\tau^{-1}\gamma^{-1}) = \sum_{\omega \in S_L} \delta(\sigma\omega\tau\omega^{-1}), \qquad (\omega\tau = \gamma \in S_L). \tag{B.14}$$

Similarly, we can derive the column orthogonality for the restricted characters (A.53). By using

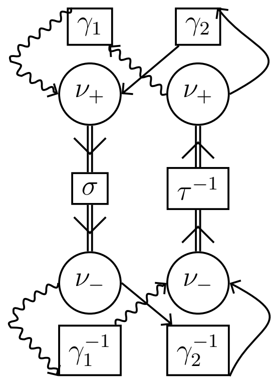
$$\begin{aligned}
\delta_{il}\,\delta_{jk} &= \frac{d_{r_1}}{m!} \sum_{\gamma \in S_m} D^{r_1}_{ij}(\gamma_1) D^{r_1}_{kl}(\gamma_1^{-1}), \qquad (i,j,k,l = 1,2,\ldots,d_{r_1}) \\
\delta_{mq}\,\delta_{np} &= \frac{d_{r_2}}{n!} \sum_{\gamma \in S_n} D^{r_2}_{mn}(\gamma_2) D^{r_2}_{pq}(\gamma_2^{-1}), \qquad (i,j,k,l = 1,2,\ldots,d_{r_2})
\end{aligned} \tag{B.15}$$

we find

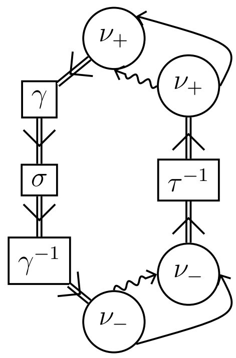
$$\sum_{R,r_1,r_2,\nu_+,\nu_-} \frac{d_R}{d_{r_1} d_{r_2}}$$



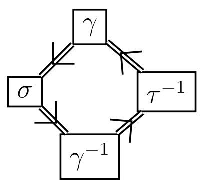
$$= \sum_{R,r_1,r_2,\nu_+,\nu_-} \frac{d_R}{m!n!} \sum_{\substack{\gamma_1 \in S_m \\ \gamma_2 \in S_n}}$$



$$= \sum_{R,r_1,r_2,\nu_+,\nu_-} \frac{d_R}{m!n!} \sum_{\gamma \in S_m \otimes S_n}$$



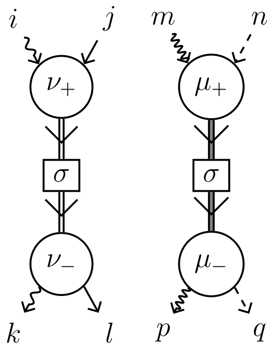
$$= \sum_{R \vdash L} \frac{d_R}{m!n!}$$

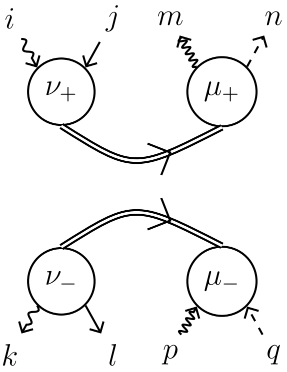


$$= \frac{(m+n)!}{m!n!} \sum_{\gamma \in S_m \otimes S_n} \delta(\sigma\gamma^{-1}\tau^{-1}\gamma). \tag{B.16}$$

In the last line, we cannot use (B.14), because $\gamma \in S_m \otimes S_n \subsetneq S_{m+n}$.

We can show the restricted grand orthogonality (A.54) by

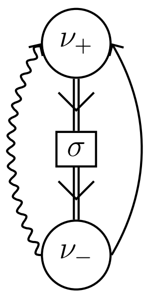
$$\frac{1}{L!}\sum_{\sigma\in S_L}$$ 

$$= \frac{\delta^{RS}}{d_R}$$ 

(B.17)

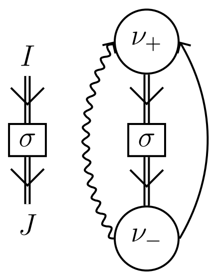
$$= \frac{\delta^{RS}}{d_R}\,\delta^{\nu_+\mu_+}\,\delta^{\nu_-\mu_-}\,\delta^{r_1,s_1}\,\delta^{r_2,s_2}\,\delta^{i,m}\,\delta^{j,n}\,\delta^{k,p}\,\delta^{l,q}\,.$$

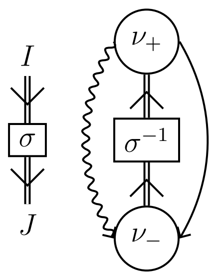
Restricted projector. The restricted projector (A.44) can be represented as

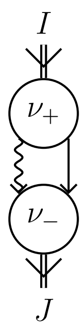
$$\mathscr{P}^{R,(r_1,r_2),\nu_+,\nu_-} = \frac{d_R}{(m+n)!}\sum_{\sigma\in S_{m+n}}\sigma\,\cdot$$ 

(B.18)

which is an element of $\mathbb{C}[S_{m+n}]$ and not a number. Its matrix elements are given by the branching coefficients (A.47), which can be shown by

$$\mathscr{P}^{R,(r_1,r_2),\nu_+,\nu_-}_{IJ} = \frac{d_R}{(m+n)!}\sum_{\sigma\in S_{m+n}}$$ 

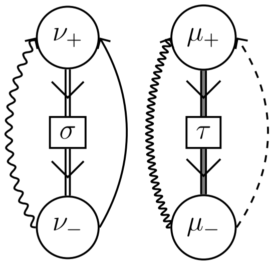
$$= \frac{d_R}{(m+n)!}\sum_{\sigma\in S_{m+n}}$$ 

$=$ 

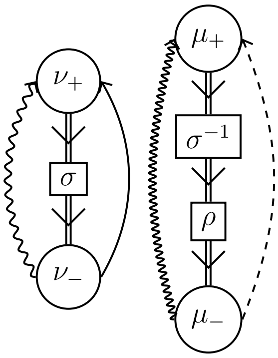
(B.19)

The identity (A.46) follows from the calculation

$$\frac{d_R d_S}{(m+n)!^2} \sum_{\sigma,\tau \in S_{m+n}} \sigma\, \tau \cdot$$

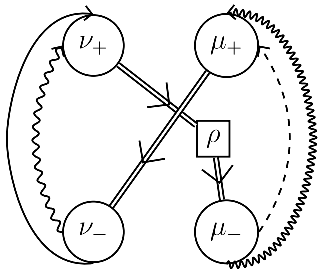


$$= \frac{d_R d_S}{(m+n)!^2} \sum_{\sigma,\rho \in S_{m+n}} \rho \cdot$$

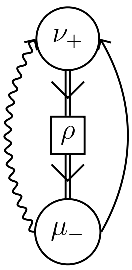


(B.20)

$$= \frac{\delta^{RS}\, d_R}{(m+n)!} \sum_{\rho \in S_{m+n}} \rho \cdot$$



$$= \delta^{RS}\, \delta^{r_1 s_1}\, \delta^{r_2 s_2}\, \delta^{\nu_- \mu_+} \frac{d_R}{(m+n)!} \sum_{\sigma \in S_{m+n}} \rho \cdot$$



C Generalized Racah-Wigner tensor

The associativity of triple tensor-product representations gives rise to the $6j$ symbols, which is also called Wigner's $6j$ invariants [73], Racah W-coefficients [74] or recoupling coefficients [75],

$$\begin{Bmatrix} j_1 & j_2 & j_{1+2} \\ j_3 & J & j_{2+3} \end{Bmatrix} : \, \mathrm{Hom}\Big((j_1 \otimes j_2) \otimes j_3, J\Big) \;\to\; \mathrm{Hom}\Big(j_1 \otimes (j_2 \otimes j_3), J\Big). \tag{C.1}$$

The problem of computing $6j$ symbol is called the Racah-Wigner calculus.

We construct a slightly general object from the branching coefficients. The generalized $6j$ symbol is covariant under the action of symmetric groups, and contains four multiplicity labels.

JHEP05(2020)118

C.1 Case of $\tilde{C}_{\circ\circ\circ}$

Consider two ways of the double restriction

$$S_L \downarrow (S_{L_1+L_2} \otimes S_{L_3}) \downarrow (S_{L_1} \otimes S_{L_2} \otimes S_{L_3}) \,, \quad S_L \downarrow (S_{L_1} \otimes S_{L_2+L_3}) \downarrow (S_{L_1} \otimes S_{L_2} \otimes S_{L_3}) \tag{C.2}$$

with $L = L_1 + L_2 + L_3$, which corresponds to the calculation of $\tilde{C}_{\circ\circ\circ}$ in section 3.4.1. They induce the irreducible decompositions

$$\begin{aligned} \hat{R} &= \bigoplus_{R_{12}, q_3} g(R_{12}, q_3; \hat{R})\, R_{12} \otimes q_3 = \bigoplus_{q_1,q_2,q_3} g(q_1, q_2; R_{12}) g(R_{12}, q_3; \hat{R})\, q_1 \otimes q_2 \otimes q_3 \\ \hat{R} &= \bigoplus_{R_{23}, q_1'} g(R_{23}, q_1'; \hat{R})\, q_1' \otimes R_{23} = \bigoplus_{q_1',q_2',q_3'} g(q_2', q_3'; R_{23}) g(R_{23}, q_1'; \hat{R})\, q_1' \otimes q_2' \otimes q_3' \,. \end{aligned} \tag{C.3}$$

The corresponding branching coefficients are

$$\begin{aligned} \left| \begin{matrix} \hat{R} \\ \hat{I} \end{matrix} \right\rangle &= \left| \begin{matrix} R_{12} & q_3 \\ I & c \end{matrix} \; \mu \right\rangle (B^T)^{\hat{R} \to (R_{12}, q_3), \mu}_{\hat{I} \to (I, c)} = \left| \begin{matrix} q_1 & q_2 & q_3 \\ a & b & c \end{matrix} \; \mu\, \rho \right\rangle (B^T)^{\hat{R} \to (R_{12}, q_3), \mu}_{\hat{I} \to (I, c)} (B^T)^{R_{12} \to (q_1, q_2), \rho}_{I \to (a, b)} \\ &= \left| \begin{matrix} q_1' & R_{23} \\ a' & I' \end{matrix} \; \mu' \right\rangle (\tilde{B}^T)^{\hat{R} \to (q_1', R_{23}), \mu'}_{\hat{I} \to (a', I')} = \left| \begin{matrix} q_1' & q_2' & q_3' \\ a' & b' & c' \end{matrix} \; \mu'\, \rho' \right\rangle (\tilde{B}^T)^{\hat{R} \to (q_1', R_{23}), \mu'}_{\hat{I} \to (a', I')} (\tilde{B}^T)^{R_{23} \to (q_2', q_3'), \rho'}_{I' \to (b', c')} \,. \end{aligned} \tag{C.4}$$

The multiplicity labels (μ, ρ) and (μ', ρ') run over the spaces

$$\begin{aligned} \xi &\equiv (\mu, \rho) \in \mathcal{M}_{12} \,, & |\mathcal{M}_{12}| &= g(q_1, q_2; R_{12})\, g(R_{12}, q_3; \hat{R}) \\ \xi' &\equiv (\mu', \rho') \in \mathcal{M}_{23} \,, & |\mathcal{M}_{23}| &= g(q_2, q_3; R_{23})\, g(R_{23}, q_1; \hat{R}) \end{aligned} \tag{C.5}$$

which are subsets of the total multiplicity space induced by the irreducible decomposition

$$\begin{gathered} \hat{R} = \bigoplus_{q_1,q_2,q_3} \bigoplus_{\eta \in \mathcal{M}_{1,2,3}} (q_1 \otimes q_2 \otimes q_3)_\eta \,, \qquad \left| \begin{matrix} \hat{R} \\ \hat{I} \end{matrix} \right\rangle = \sum_{q_1,q_2,q_3,\eta} \left| \begin{matrix} q_1 & q_2 & q_3 \\ a & b & c \end{matrix} \; \eta \right\rangle (B^T)^{\hat{R} \to (q_1,q_2,q_3), \eta}_{\hat{I} \to (a,b,c)} \\ \eta \in \mathcal{M}_{\rm tot} \,, \quad |\mathcal{M}_{\rm tot}| = g(q_1, q_2, q_3; \hat{R}). \end{gathered} \tag{C.6}$$

From the identity (A.40), we obtain the following relation between the branching coefficients in (C.4) and (C.6),

$$\begin{aligned} \left\langle \begin{matrix} \tilde{q}_1 & \tilde{q}_2 & \tilde{q}_3 \\ \tilde{a} & \tilde{b} & \tilde{c} \end{matrix} \; \tilde{\eta} \,\middle|\, \begin{matrix} q_1 & q_2 & q_3 \\ a & b & c \end{matrix} \; \mu\, \rho \right\rangle &= \sum_{\tilde{R}_{12}} \left\langle \begin{matrix} \tilde{q}_1 & \tilde{q}_2 & \tilde{q}_3 \\ \tilde{a} & \tilde{b} & \tilde{c} \end{matrix} \; \tilde{\mu}\, \tilde{\rho} \,\middle|\, \begin{matrix} q_1 & q_2 & q_3 \\ a & b & c \end{matrix} \; \mu\, \rho \right\rangle \\ &= \delta^{\tilde{q}_1 q_1}\, \delta^{\tilde{q}_2 q_2}\, \delta^{\tilde{q}_3 q_3}\, \delta^{\tilde{\mu}\mu}\, \delta^{\tilde{\rho}\rho}\, \delta_{\tilde{a}a}\, \delta_{\tilde{b}b}\, \delta_{\tilde{c}c} \end{aligned} \tag{C.7}$$

where the r.h.s. depends on R_{12} through the multiplicity space of (μ, ρ) in (C.5).

We define the orthogonal matrix (A.16) between the two states by

$$U_{\hat{R}} \begin{pmatrix} q_1 & q_2 & q_3 & R_{12} & \mu & \rho \\ q_1' & q_2' & q_3' & R_{23} & \mu' & \rho' \end{pmatrix}_{abc, a'b'c'} \equiv \left\langle \begin{matrix} q_1 & q_2 & q_3 \\ a & b & c \end{matrix} \; \mu\, \rho \,\middle|\, \begin{matrix} q_1' & q_2' & q_3' \\ a' & b' & c' \end{matrix} \; \mu'\, \rho' \right\rangle \tag{C.8}$$

$$= \sum_{\hat{I}=1}^{d_{\hat{R}}} \sum_{I=1}^{d_{R_{12}}} \sum_{I'=1}^{d_{R_{23}}} (B^T)^{\hat{R} \to (R_{12}, q_3), \mu}_{\hat{I} \to (I, c)} (B^T)^{R_{12} \to (q_1, q_2), \rho}_{I \to (a, b)} \tilde{B}^{\hat{R} \to (q_1', R_{23}), \mu'}_{\hat{I} \to (a', I')} \tilde{B}^{R_{23} \to (q_2', q_3'), \rho'}_{I' \to (b', c')} \tag{C.9}$$

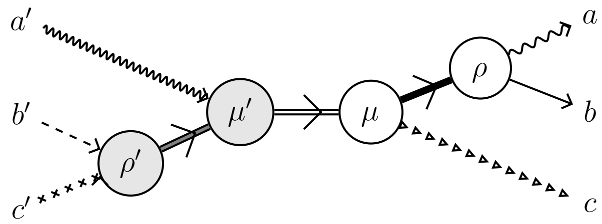
JHEP05(2020)118

and call it the *generalized Racah-Wigner tensor*. Our notation is slightly redundant because the generalized Racah-Wigner tensor is proportional to $\prod_{i=1}^{3}\delta^{q_i q_i'}$, which follows from (C.8). The usual $6j$ symbol for a symmetric group is given by

$$\mathrm{tr}\,(U_{\hat{R}}) \equiv \sum_{a,b,c} U_{\hat{R}} \left(\begin{array}{cccc|cc} q_1 & q_2 & q_3 & R_{12} & \mu & \rho \\ q_1 & q_2 & q_3 & R_{23} & \mu' & \rho' \end{array}\right)_{abc,abc} . \tag{C.10}$$

The generalized Racah-Wigner tensor can be depicted as

$$U_{\hat{R}} \left(\begin{array}{cccc|cc} q_1 & q_2 & q_3 & R_{12} & \mu & \rho \\ q_1' & q_2' & q_3' & R_{23} & \mu' & \rho' \end{array}\right)_{abc,a'b'c'} =$$



(C.11)

We want to compute the products of generalized Racah-Wigner tensors

$$\mathrm{tr}\,(U_{\hat{R}}\tilde{U}_{\hat{R}}) \equiv \sum_{\mu,\rho,\mu',\rho'} U_{\hat{R}} \left(\begin{array}{cccc|cc} q_1 & q_2 & q_3 & R_{12} & \mu & \rho \\ q_1' & q_2' & q_3' & R_{23} & \mu' & \rho' \end{array}\right)_{abc,a'b'c'} U_{\hat{R}} \left(\begin{array}{cccc|cc} q_1' & q_2' & q_3' & R_{23} & \mu' & \rho' \\ q_1 & q_2 & q_3 & R_{12} & \mu & \rho \end{array}\right)_{a'b'c',abc}$$

$$\mathrm{tr}\,(U_{\hat{R}}\tilde{U}_{\hat{R}}\tilde{\tilde{U}}_{\hat{R}}) \equiv \sum_{\mu,\rho,\mu',\rho',\mu'',\rho''} U_{\hat{R}} \left(\begin{array}{cccc|cc} q_1 & q_2 & q_3 & R_{12} & \mu & \rho \\ q_1' & q_2' & q_3' & R_{23} & \mu' & \rho' \end{array}\right)_{abc,a'b'c'} \times \tag{C.12}$$

$$U_{\hat{R}} \left(\begin{array}{cccc|cc} q_1' & q_2' & q_3' & R_{23} & \mu' & \rho' \\ q_1'' & q_2'' & q_3'' & R_{23} & \mu'' & \rho'' \end{array}\right)_{a'b'c',a''b''c''} U_{\hat{R}} \left(\begin{array}{cccc|cc} q_1'' & q_2'' & q_3'' & R_{23} & \mu'' & \rho'' \\ q_1 & q_2 & q_3 & R_{12} & \mu & \rho \end{array}\right)_{a''b''c'',abc}$$

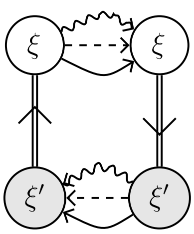
which are rewriting of the product of projectors (3.55),

$$\mathrm{tr}\,(U_{\hat{R}}\tilde{U}_{\hat{R}}) = \mathrm{tr}_{\hat{R}}\left(\mathfrak{P}^{\hat{R}\to\cdots\to(q_1,q_2,q_3),\mu\rho,\mu\rho}\,\tilde{\mathfrak{P}}^{\hat{R}\to\cdots\to(q_1',q_2',q_3'),\mu'\rho',\mu'\rho'}\right)$$

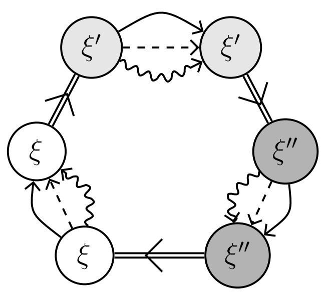
$$\mathrm{tr}\,(U_{\hat{R}}\tilde{U}_{\hat{R}}\tilde{\tilde{U}}_{\hat{R}}) = \mathrm{tr}_{\hat{R}}\left(\mathfrak{P}^{\hat{R}\to\cdots\to(q_1,q_2,q_3),\mu\rho,\mu\rho}\,\tilde{\mathfrak{P}}^{\hat{R}\to\cdots\to(q_1',q_2',q_3'),\mu'\rho',\mu'\rho'}\,\tilde{\tilde{\mathfrak{P}}}^{\hat{R}\to\cdots\to(q_1'',q_2'',q_3''),\mu''\rho'',\mu''\rho''}\right). \tag{C.13}$$

By using ξ, ξ', ξ'' in (C.5), we depict these products as

$\mathrm{tr}\,(U_{\hat{R}}\tilde{U}_{\hat{R}}) =$



$\mathrm{tr}\,(U_{\hat{R}}\tilde{U}_{\hat{R}}\tilde{\tilde{U}}_{\hat{R}}) =$



(C.14)

By grouping pairs of nodes with the same color, we obtain the projector representation (C.13). From the identity of the projectors (A.46), we get

$$\begin{aligned}\operatorname{tr}(U_{\hat{R}}\,\tilde{U}_{\hat{R}}) &= \left(\prod_{i=1}^{3}\delta^{q_i q_i'}\,d_{q_i}\right)\delta^{\xi_1\,\xi_2}\,\delta^{\xi_2\,\xi_1}\\ \operatorname{tr}(U_{\hat{R}}\,\tilde{U}_{\hat{R}}\,\tilde{\tilde{U}}_{\hat{R}}) &= \left(\prod_{i=1}^{3}\delta^{q_i q_i'}\,\delta^{q_i q_i''}\,d_{q_i}\right)\delta^{\xi_1\,\xi_2}\,\delta^{\xi_2\,\xi_3}\,\delta^{\xi_3\,\xi_1}\end{aligned} \tag{C.15}$$

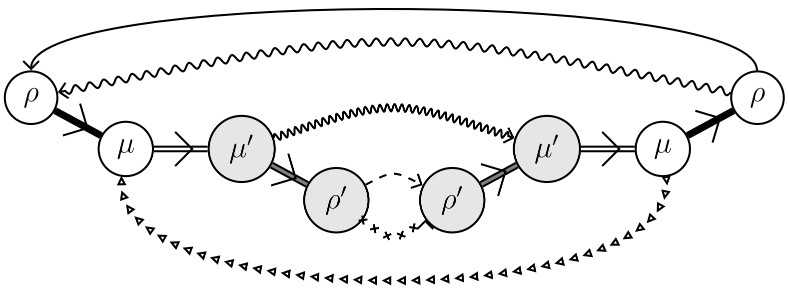
where we do not sum over the repeated indices (ξ_i's).

The product $\operatorname{tr}(U_{\hat{R}}\,\tilde{U}_{\hat{R}})$ satisfies the following sum rules,

$$\begin{aligned}\sum_{R_{23}}\sum_{\xi_1,\xi_2}\operatorname{tr}(U_{\hat{R}}\,\tilde{U}_{\hat{R}}) &= \left(\prod_{i=1}^{3}\delta^{q_i q_i'}\,d_{q_i}\right)g(q_1,q_2;R_{12})\,g(R_{12},q_3;\hat{R})\\ \sum_{R_{12}}\sum_{\xi_1,\xi_2}\operatorname{tr}(U_{\hat{R}}\,\tilde{U}_{\hat{R}}) &= \left(\prod_{i=1}^{3}\delta^{q_i q_i'}\,d_{q_i}\right)g(q_2,q_3;R_{23})\,g(R_{23},q_1;\hat{R}).\end{aligned} \tag{C.16}$$

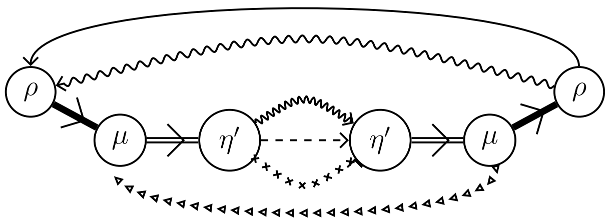
We can derive these sum rules by using the identities (A.40), (A.15) and (C.7), as

$$\sum_{R_{23}}\ \sum_{\mu,\rho,\mu',\rho'}$$



(C.17)

$$=\sum_{\mu,\rho,\eta'}$$



$$= \delta^{q_1' q_1}\,\delta^{q_2' q_2}\,\delta^{q_3' q_3}\,d_{q_1}d_{q_2}d_{q_3}\,g(R_{12},q_3;\hat{R})\,g(q_1,q_2;R_{12})\,.$$

A solution to the equations (C.16) is

$$\sum_{\xi_1,\xi_2}\operatorname{tr}(U_{\hat{R}}\tilde{U}_{\hat{R}}) \overset{?}{=} \left(\prod_{i=1}^{3}\delta^{q_i q_i'}\,d_{q_i}\right)\frac{g(q_1,q_2;R_{12})\,g(R_{12},q_3;\hat{R})\,g(q_2,q_3;R_{23})\,g(R_{23},q_1;\hat{R})}{g(q_1,q_2,q_3;\hat{R})}\,. \tag{C.18}$$

JHEP05(2020)118

We conjecture that both sides are equal, and continue the discussion below. Similarly, we find

$$\sum_{R_{31}}\sum_{\xi_1,\xi_2,\xi_3}\mathrm{tr}\,(U_{\hat{R}}\tilde{U}_{\hat{R}}\tilde{\tilde{U}}_{\hat{R}}) = \left(\prod_{i=1}^{3}\delta^{q''_i q_i}\,\delta^{q''_i q'_i}\right)\sum_{\mu,\rho,\mu',\rho'} U_{\hat{R}}\begin{pmatrix} q_1 & q_2 & q_3 & R_{12} \middle| \mu & \rho \\ q'_1 & q'_2 & q'_3 & R_{23} \middle| \mu' & \rho' \end{pmatrix}_{abc,a'b'c'} \times$$
$$U_{\hat{R}}\begin{pmatrix} q'_1 & q'_2 & q'_3 & R_{23} \middle| \mu' & \rho' \\ q_1 & q_2 & q_3 & R_{12} \middle| \mu & \rho \end{pmatrix}_{a'b'c',abc}$$

$$\sum_{R_{23}}\sum_{\xi_1,\xi_2,\xi_3}\mathrm{tr}\,(U_{\hat{R}}\tilde{U}_{\hat{R}}\tilde{\tilde{U}}_{\hat{R}}) = \left(\prod_{i=1}^{3}\delta^{q'_i q_i}\,\delta^{q'_i q''_i}\right)\sum_{\mu,\rho,\mu'',\rho''} U_{\hat{R}}\begin{pmatrix} q_1 & q_2 & q_3 & R_{12} \middle| \mu & \rho \\ q''_1 & q''_2 & q''_3 & R_{31} \middle| \mu'' & \rho'' \end{pmatrix}_{abc,a''b''c''} \times$$
$$U_{\hat{R}}\begin{pmatrix} q''_1 & q''_2 & q''_3 & R_{31} \middle| \mu'' & \rho'' \\ q_1 & q_2 & q_3 & R_{12} \middle| \mu & \rho \end{pmatrix}_{a''b''c'',abc}$$

$$\sum_{R_{12}}\sum_{\xi_1,\xi_2,\xi_3}\mathrm{tr}\,(U_{\hat{R}}\tilde{U}_{\hat{R}}\tilde{\tilde{U}}_{\hat{R}}) = \left(\prod_{i=1}^{3}\delta^{q''_i q_i}\,\delta^{q''_i q'_i}\right)\sum_{\mu',\rho',\mu'',\rho''} U_{\hat{R}}\begin{pmatrix} q''_1 & q''_2 & q''_3 & R_{31} \middle| \mu'' & \rho'' \\ q'_1 & q'_2 & q'_3 & R_{23} \middle| \mu' & \rho' \end{pmatrix}_{a''b''c'',a'b'c'} \times$$
$$U_{\hat{R}}\begin{pmatrix} q'_1 & q'_2 & q'_3 & R_{23} \middle| \mu' & \rho' \\ q''_1 & q''_2 & q''_3 & R_{31} \middle| \mu'' & \rho'' \end{pmatrix}_{a'b'c',a''b''c''}. \tag{C.19}$$

A solution to these equations is

$$\sum_{\xi_1,\xi_2,\xi_3}\mathrm{tr}\,(U_{\hat{R}}\tilde{U}_{\hat{R}}\tilde{\tilde{U}}_{\hat{R}}) = \left(\prod_{i=1}^{3}\delta^{q_i q'_i}\,\delta^{q'_i q''_i}\,d_{q_i}\right)\times$$
$$\frac{g(q_1,q_2;R_{12})\,g(R_{12},q_3;\hat{R})\,g(q_2,q_3;R_{23})\,g(R_{23},q_1;\hat{R})\,g(q_3,q_1;R_{31})\,g(R_{31},q_2;\hat{R})}{g(q_1,q_2,q_3;\hat{R})^2}. \tag{C.20}$$

In view of (C.15), our conjecture is summarized as

$$\sum_{\xi_1\in\mathcal{M}_{12}}\sum_{\xi_2\in\mathcal{M}_{23}}\delta^{\xi_1\,\xi_2}\,\delta^{\xi_2\,\xi_1} = \frac{|\mathcal{M}_{12}|\,|\mathcal{M}_{23}|}{|\mathcal{M}_{\mathrm{tot}}|}$$
$$\sum_{\xi_1\in\mathcal{M}_{12}}\sum_{\xi_2\in\mathcal{M}_{23}}\sum_{\xi_3\in\mathcal{M}_{31}}\delta^{\xi_1\,\xi_2}\,\delta^{\xi_2\,\xi_3}\,\delta^{\xi_3\,\xi_1} = \frac{|\mathcal{M}_{12}|\,|\mathcal{M}_{23}|\,|\mathcal{M}_{31}|}{|\mathcal{M}_{\mathrm{tot}}|^2}. \tag{C.21}$$

C.2 Case of $\tilde{C}_{\vec{h}}^{XYZ}$

Consider another set of restrictions

$$S_L \downarrow \Big(\big((S_{L_5}\otimes S_{L_6})\otimes S_{L_1}\otimes S_{L_3}\big)\otimes(S_{L_2}\otimes S_{L_4})\Big)$$
$$S_L \downarrow \Big(\big((S_{L_3}\otimes S_{L_4})\otimes S_{L_2}\otimes S_{L_5}\big)\otimes(S_{L_1}\otimes S_{L_6})\Big) \tag{C.22}$$
$$S_L \downarrow \Big(\big((S_{L_1}\otimes S_{L_2})\otimes S_{L_4}\otimes S_{L_6}\big)\otimes(S_{L_3}\otimes S_{L_5})\Big)$$

with $L = \sum_{i=1}^{6} L_i$, which correspond to the case of $\tilde{C}_{\vec{h}}^{XYZ}$ in section 3.4.2. They induce the irreducible decomposition

$$\hat{R} = \bigoplus_{Q,R,T} \bigoplus_{\{q_i\}} \Big\{ g(q_5, q_6; Q) g(Q, q_1, q_3; R) g(q_2, q_4; T) g(R, T; \hat{R}) \bigotimes_{i=1}^{6} q_i \Big\}$$
$$\hat{R} = \bigoplus_{Q',R',T'} \bigoplus_{\{q'_i\}} \Big\{ g(q'_3, q'_4; Q') g(Q', q'_2, q'_5; R') g(q'_1, q'_6; T') g(R', T'; \hat{R}) \bigotimes_{i=1}^{6} q'_i \Big\} \tag{C.23}$$
$$\hat{R} = \bigoplus_{Q'',R'',T''} \bigoplus_{\{q''_i\}} \Big\{ g(q''_1, q''_2; Q'') g(Q'', q''_4, q''_6; R'') g(q''_3, q''_5; T'') g(R'', T''; \hat{R}) \bigotimes_{i=1}^{6} q''_i \Big\}.$$

We fix the representations $(R, Q), (R', Q'), (R'', Q'')$ and the multiplicity labels ν, ν', ν'' according to the external operators. The space of multiplicities run over the spaces

$$\xi \in \mathcal{M}_{R,Q,\nu}\,, \qquad \xi' \in \mathcal{M}_{R',Q',\nu'}\,, \qquad \xi'' \in \mathcal{M}_{R'',Q'',\nu''} \tag{C.24}$$

where

$$\begin{aligned} |\mathcal{M}_{R,Q,\nu}| &= g(q_5, q_6; Q) g(q_2, q_4; T) g(R, T; \hat{R}) \\ \left|\mathcal{M}_{R',Q',\nu'}\right| &= g(q'_3, q'_4; Q') g(q'_1, q'_6; T') g(R', T'; \hat{R}) \\ \left|\mathcal{M}_{R'',Q'',\nu''}\right| &= g(q''_1, q''_2; Q'') g(q''_3, q''_5; T'') g(R'', T''; \hat{R}) \end{aligned} \tag{C.25}$$

They are subsets of the total multiplicity space

$$|\mathcal{M}_{\text{tot}}| \equiv g(q_1, q_2, q_3, q_4, q_5, q_6; \hat{R}), \tag{C.26}$$
$$\begin{aligned} |\mathcal{M}_{\text{tot}}| &= \sum_{R,Q} \sum_{\nu=1}^{g(Q,q_1,q_3;R)} |\mathcal{M}_{R,Q,\nu}| = \sum_{R',Q'} \sum_{\nu'=1}^{g(Q',q'_2,q'_5;R')} \left|\mathcal{M}_{R',Q',\nu'}\right| \\ &= \sum_{R'',Q''} \sum_{\nu'=1}^{g(Q'',q''_4,q''_6;R'')} \left|\mathcal{M}_{R'',Q'',\nu''}\right|. \end{aligned}$$

Since the restricted Schur characters have two multiplicity labels (A.25), we introduce

$$\xi_\pm \in \mathcal{M}_{R_\pm,Q_\pm,\nu_\pm}\,, \qquad \xi'_\pm \in \mathcal{M}_{R'_\pm,Q'_\pm,\nu'_\pm}\,, \qquad \xi''_\pm \in \mathcal{M}_{R''_\pm,Q''_\pm,\nu''_\pm} \tag{C.27}$$

where the $\pm$ signs are correlated.[11]

Let us define the generalized Racah-Wigner tensor by

$$W_{\hat{R}} \begin{pmatrix} q_1 \; q_2 \; \dots \; q_6 \, \Big| \, R_- \; \xi_- \\ q'_1 \; q'_2 \; \dots \; q'_6 \, \Big| \, R'_+ \; \xi'_+ \end{pmatrix}_{ab\dots f, a'b'\dots f'} \equiv \left\langle \begin{matrix} q_1 \; q_2 \dots \; q_6 \\ a \;\; b \; \dots \; f \end{matrix} \; \xi_- \, \Big| \, \begin{matrix} q'_1 \; q'_2 \dots \; q'_6 \\ a' \;\; b' \; \dots \; f' \end{matrix} \; \xi'_+ \right\rangle \tag{C.28}$$

which is again proportional to $\prod_{i=1}^{6} \delta^{q_i q'_i}$. The r.h.s. depends in R_-, R'_+ through the multiplicity space $\xi \in \mathcal{M}_{R_-,Q_-,\nu_-}, \xi'_+ \in \mathcal{M}_{R'_+,Q'_+,\nu'_+}$, as we discussed in (C.7). We want to

[11] Note that $(R_-, R'_-, R''_-) = (R_+, R'_+, R''_+)$ in the main text. We removed these constraints for convenience.

compute their products

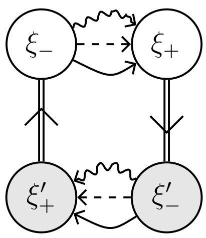
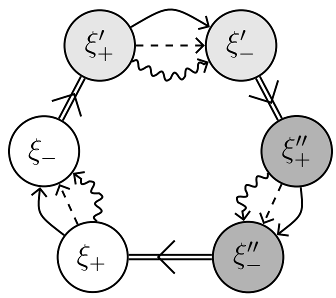
$$\operatorname{tr}(W_{\hat R}\tilde W_{\hat R}) \equiv \sum_{\xi_\mp,\xi'_\mp} W_{\hat R}\begin{pmatrix} q_1 & q_2 & \dots & q_6 \\ q'_1 & q'_2 & \dots & q'_6 \end{pmatrix}\left|\begin{matrix} R_- & \xi_- \\ R'_+ & \xi'_+ \end{matrix}\right)_{ab\dots f,a'b'\dots f'} \times$$
$$W_{\hat R}\begin{pmatrix} q'_1 & q'_2 & \dots & q'_6 \\ q_1 & q_2 & \dots & q_6 \end{pmatrix}\left|\begin{matrix} R'_- & \xi'_- \\ R_+ & \xi_+ \end{matrix}\right)_{a'b'\dots f',ab\dots f} \tag{C.29}$$

$$\operatorname{tr}(W_{\hat R}\tilde W_{\hat R}\tilde{\tilde W}_{\hat R}) \equiv \sum_{\xi_\mp,\xi'_\mp,\xi''_\mp} W_{\hat R}\begin{pmatrix} q_1 & q_2 & \dots & q_6 \\ q'_1 & q'_2 & \dots & q'_6 \end{pmatrix}\left|\begin{matrix} R_- & \xi_- \\ R'_+ & \xi'_+ \end{matrix}\right)_{ab\dots f,a'b'\dots f'} \times \tag{C.30}$$
$$W_{\hat R}\begin{pmatrix} q'_1 & q'_2 & \dots & q'_6 \\ q''_1 & q''_2 & \dots & q''_6 \end{pmatrix}\left|\begin{matrix} R'_- & \xi'_- \\ R''_+ & \xi''_+ \end{matrix}\right)_{a'b'\dots f',a''b''\dots f''} W_{\hat R}\begin{pmatrix} q''_1 & q''_2 & \dots & q''_6 \\ q_1 & q_2 & \dots & q_6 \end{pmatrix}\left|\begin{matrix} R''_- & \xi''_- \\ R_+ & \xi_+ \end{matrix}\right)_{a''b''\dots f'',ab\dots f}.$$

They are identical to the product of projectors (3.55),

$$\operatorname{tr}(W_{\hat R}\tilde W_{\hat R}) = \operatorname{tr}_{\hat R}\left(\mathfrak{P}^{\hat R\to\dots\to(q_1,q_2,\dots,q_6),\xi_-,\xi_+}_{\hat I_1\hat I_2}\,\mathfrak{P}^{\hat R\to\dots\to(q'_1,q'_2,\dots,q'_6),\xi'_-,\xi'_+}_{\hat I_2\hat I_1}\right) \tag{C.31}$$
$$\operatorname{tr}(W_{\hat R}\tilde W_{\hat R}\tilde{\tilde W}_{\hat R}) = \operatorname{tr}_{\hat R}\left(\mathfrak{P}^{\hat R\to\dots\to(q_1,q_2,\dots,q_6),\xi_-,\xi_+}_{\hat I_1\hat I_2}\,\mathfrak{P}^{\hat R\to\dots\to(q'_1,q'_2,\dots,q'_6),\xi'_-,\xi'_+}_{\hat I_2\hat I_3}\,\mathfrak{P}^{\hat R\to\dots\to(q''_1,q''_2,\dots,q''_6),\xi''_-,\xi''_+}_{\hat I_3\hat I_1}\right).$$

These products are depicted as

$\operatorname{tr}(W_{\hat R}\tilde W_{\hat R}) =$  $\operatorname{tr}(W_{\hat R}\tilde W_{\hat R}\tilde{\tilde W}_{\hat R}) =$  (C.32)

As a corollary of the identity of the projectors (A.46), we find that

$$\operatorname{tr}(W_{\hat R}\tilde W_{\hat R}) = \left(\prod_{i=1}^{6}\delta^{q_i q'_i}\, d_{q_i}\right)\delta^{\xi_-\,\xi'_+}\delta^{\xi'_-\,\xi_+}$$
$$\operatorname{tr}(W_{\hat R}\tilde W_{\hat R}\tilde{\tilde W}_{\hat R}) = \left(\prod_{i=1}^{6}\delta^{q_i q'_i}\,\delta^{q_i q''_i}\, d_{q_i}\right)\delta^{\xi_-\,\xi'_+}\,\delta^{\xi'_-\,\xi''_+}\,\delta^{\xi''_-\,\xi_+}\,. \tag{C.33}$$

By summing $\{\xi_\mp,\xi'_\mp,\xi''_\mp\}$ over the ranges $\{\mathcal{M}_{R_\mp,Q_\mp,\nu_\mp},\mathcal{M}_{R'_\mp,Q'_\mp,\nu'_\mp},\mathcal{M}_{R''_\mp,Q''_\mp,\nu''_\mp}\}$, we discover the overlap

$$\sum_{\xi_-\in\mathcal{M}_{R_-,Q_-,\nu_-}}\ \sum_{\xi'_+\in\mathcal{M}_{R'_+,Q'_+,\nu'_+}} \delta^{\xi_-\,\xi'_+} = \left|\mathcal{M}_{R_-,Q_-,\nu_-}\cap\mathcal{M}_{R'_+,Q'_+,\nu'_+}\right|\,. \tag{C.34}$$

The overlap satisfies the sum rules

$$\begin{aligned}\sum_{R_-,Q_-,\nu_-}\sum_{R'_+,Q'_+,\nu'_+}\left|\mathcal{M}_{R_-,Q_-,\nu_-}\cap\mathcal{M}_{R'_+,Q'_+,\nu'_+}\right| &= |\mathcal{M}_{\rm tot}| \\ \sum_{R_-,Q_-,\nu_-}\left|\mathcal{M}_{R_-,Q_-,\nu_-}\cap\mathcal{M}_{R'_+,Q'_+,\nu'_+}\right| &= \left|\mathcal{M}_{R'_+,Q'_+,\nu'_+}\right| \\ \sum_{R'_+,Q'_+,\nu'_+}\left|\mathcal{M}_{R_-,Q_-,\nu_-}\cap\mathcal{M}_{R'_+,Q'_+,\nu'_+}\right| &= \left|\mathcal{M}_{R_-,Q_-,\nu_-}\right| \ .\end{aligned} \tag{C.35}$$

As a solution to the sum rules, we conjecture that

$$\left|\mathcal{M}_{R_-,Q_-,\nu_-}\cap\mathcal{M}_{R'_+,Q'_+,\nu'_+}\right| = \overline{\delta}^{\nu_-\,\nu'_+}\,\frac{\left|\mathcal{M}_{R_-,Q_-,\nu_-}\right|\left|\mathcal{M}_{R'_+,Q'_+,\nu'_+}\right|}{|\mathcal{M}_{\rm tot}|} \tag{C.36}$$

where $\overline{\delta}^{\nu\nu'}$ should be understood as the intersection inside $\mathcal{M}_{\rm tot}$

$$\overline{\delta}^{\nu_+\,\nu'_-} = \begin{cases} 1 & \left(\mathcal{M}_{R_-,Q_-,\nu_-}\cap\mathcal{M}_{R'_+,Q'_+,\nu'_+}\neq\emptyset\right) \\ 0 & ({\rm otherwise})\,. \end{cases} \tag{C.37}$$

It follows that

$$\sum_{\xi_\mp,\xi'_\mp}{\rm tr}\,(W_{\hat R}\,\tilde W_{\hat R}) = \left(\prod_{i=1}^6\delta^{q_iq'_i}\,d_{q_i}\right)\overline{\delta}^{\nu_-\,\nu'_+}\,\overline{\delta}^{\nu'_-\,\nu_+}\ \times \\ \frac{\left|\mathcal{M}_{R_-,Q-,\nu_-}\right|\left|\mathcal{M}_{R_+,Q+,\nu_+}\right|\left|\mathcal{M}_{R'_-,Q'_-,\nu'_-}\right|\left|\mathcal{M}_{R'_+,Q'_+,\nu'_+}\right|}{|\mathcal{M}_{\rm tot}|^2} \tag{C.38}$$

$$\sum_{\xi_\mp,\xi'_\mp,\xi''_\mp}{\rm tr}\,(W_{\hat R}\,\tilde W_{\hat R}\,\tilde{\tilde W}_{\hat R}) = \left(\prod_{i=1}^6\delta^{q_iq'_i}\,\delta^{q_iq''_i}\,d_{q_i}\right)\overline{\delta}^{\nu_-\,\nu'_+}\,\overline{\delta}^{\nu'_-\,\nu''_+}\,\overline{\delta}^{\nu''_-\,\nu_+}\ \times \\ \frac{\left|\mathcal{M}_{R_-,Q-,\nu_-}\right|\left|\mathcal{M}_{R_+,Q+,\nu_+}\right|\left|\mathcal{M}_{R'_-,Q'_-,\nu'_-}\right|\left|\mathcal{M}_{R'_+,Q'_+,\nu'_+}\right|\left|\mathcal{M}_{R''_-,Q''_-,\nu''_-}\right|\left|\mathcal{M}_{R''_+,Q''_+,\nu''_+}\right|}{|\mathcal{M}_{\rm tot}|^3}. \tag{C.39}$$

C.3 Restricted Littlewood-Richardson coefficients

Let us compute the restricted Littlewood-Richardson coefficients in [27] in our method. We will find the perfect agreement. However, they considered multiplicity-free cases only. Thus, this agreement does not provide non-trivial checks of our conjectured formula.

We define the restricted Littlewood-Richardson coefficients by

$$F^{\{3\}}_{\{1\}\{2\}} = \frac{1}{L_1!\,L_2!}\sum_{\sigma_1\in S_{L_1}}\sum_{\sigma_2\in S_{L_2}}\chi^{\boldsymbol{R}_1}(\sigma_1)\,\chi^{\boldsymbol{R}_2}(\sigma_2)\,\chi^{\boldsymbol{R}_3}(\sigma_1\circ\sigma_2) \\ L_i = m_i+n_i\,,\quad \boldsymbol{R}_i = \{R_i,(r_i,s_i),(\nu_{i-}\,,\nu_{i+})\}\,. \tag{C.40}$$

The definition used in [27] is

$$f^{\{3\}}_{\{1\}\{2\}} = \frac{1}{m_1!n_1!m_2!n_2!}\,\frac{m_3!n_3!}{L_3!}\,\frac{d_{R_3}}{d_{r_3}\,d_{s_3}}\sum_{\sigma_1\in S_{L_1}}\sum_{\sigma_2\in S_{L_2}}\chi^{\boldsymbol{R}_1}(\sigma_1)\,\chi^{\boldsymbol{R}_2}(\sigma_2)\,\chi^{\boldsymbol{R}_3}(\sigma_1\circ\sigma_2). \tag{C.41}$$

The two definitions are related by

$$F^{\{3\}}_{\{1\}\{2\}} = \frac{m_1!n_1!m_2!n_2!}{m_3!n_3!}\,\frac{L_3!}{L_1!L_2!}\,\frac{d_{r_3}\,d_{s_3}}{d_{R_3}}\,f^{\{3\}}_{\{1\}\{2\}}\,. \tag{C.42}$$

The restricted Littlewood-Richardson coefficients $F^{\{3\}}_{\{1\}\{2\}}$ can be computed as follows. First, consider the restriction $S_{L_3} \downarrow (S_{L_1} \otimes S_{L_2})$, which gives

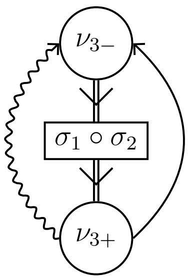
$$R_3 = \bigoplus_{T_1,T_2} g(T_1,T_2;R_3)\,(T_1 \otimes T_2)\,. \tag{C.43}$$

The restricted character in (C.40) becomes

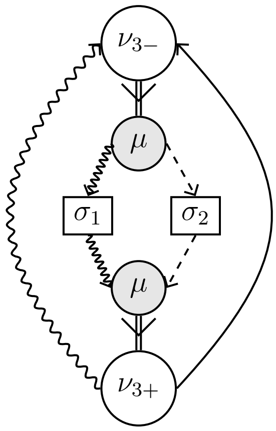
$$\chi^{\boldsymbol{R}_3}(\sigma_1 \circ \sigma_2) = \sum_{T_1,T_2}\sum_{\mu=1}^{g(T_1,T_2;R_3)} D^{T_1}_{h_1h_1'}(\sigma_1)\,D^{T_2}_{h_2h_2'}(\sigma_2)\,\tilde{B}^{R_3\to(T_1,T_2)\mu}_{I\to(h_1h_2)}\,(\tilde{B}^T)^{R_3\to(T_1,T_2)\mu}_{I'\to(h_1'h_2')}\,\times$$
$$B^{R_3\to(r_3,s_3),\nu_{3-}}_{I\to(i,j)}\,(B^T)^{R_3\to(r_3,s_3),\nu_{3+}}_{I'\to(i,j)}\,. \tag{C.44}$$

In the quiver notation, we can depict this equation as

$\chi^{R_3(r_3,s_3),(\nu_{3-},\nu_{3+})}(\sigma_1\circ\sigma_2) =$

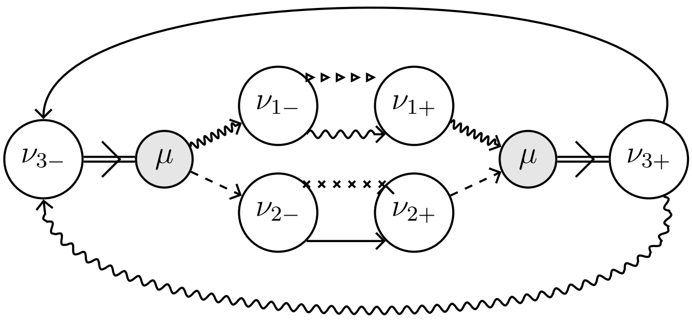


$= \sum_{T_1,T_2,\mu}$



(C.45)

By summing over σ_1 and σ_2 in (C.40), we get $\delta^{T_1,R_1}\,\delta^{T_2,R_2}$ and another sets of branching coefficients in place of $\sigma_1\,,\sigma_2$ in (C.45), giving us



$= \mathrm{tr}\,(\mathscr{P}\,\tilde{\mathscr{P}})$. (C.46)

The restricted Littlewood-Richardson coefficient (C.40) becomes

$$F^{\{3\}}_{\{1\}\{2\}} = \frac{1}{d_{R_1}d_{R_2}}\sum_{\mu}\mathrm{tr}\Big(\mathscr{P}^{R_3\to(r_3,s_3),(\nu_{3-}\nu_{3+})}\,\tilde{\mathscr{P}}^{R_3\to(R_1,R_2),\mu\to(r_1,s_1,r_2,s_2),(\mu,(\nu_{1+},\nu_{2+}),(\nu_{1-},\nu_{2-}))}\Big). \tag{C.47}$$

To evaluate the projectors, we introduce the permutations on the fully-split space

$$S_{\rm FS} = S_{m_1} \otimes S_{m_2} \otimes S_{n_1} \otimes S_{n_2} \tag{C.48}$$

and consider sub-projectors. The total multiplicity space for the restriction $S_{L_3} \downarrow S_{\rm FS}$ is

$$|\mathcal{M}_{\rm tot}| = g(r_1, r_2, s_1, s_2; R_3). \tag{C.49}$$

The multiplicity space for the first projector $\mathscr{P}^{R_3 \to (r_3,s_3),(\nu_{3-}\nu_{3+})}$ is

$$\begin{gathered} \left|\mathcal{M}_{r_3,s_3,\nu_{3\mp}}\right| = g(r_1, r_2; r_3) g(s_1, s_2; s_3), \\ \sum_{r_3,s_3} \sum_{\nu_{3-}=1}^{g(r_3,s_3;R_3)} \left|\mathcal{M}_{r_3,s_3,\nu_{3-}}\right| = \sum_{r_3,s_3} \sum_{\nu_{3+}=1}^{g(r_3,s_3;R_3)} \left|\mathcal{M}_{r_3,s_3,\nu_{3+}}\right| = |\mathcal{M}_{\rm tot}|\,. \end{gathered} \tag{C.50}$$

The multiplicity space for the second projector $\tilde{\mathscr{P}}^{R_3 \to \cdots \to (r_1,s_1,r_2,s_2),(\mu,\nu_{1\mp},\nu_{2\mp})}$ is

$$\begin{aligned} \left|\mathcal{M}_{R_1,R_2,\nu_{1\mp},\nu_{2\mp}}\right| &= g(R_1, R_2; R_3) \\ \sum_{R_1,R_2} \sum_{\nu_{1-}=1}^{g(r_1,s_1;R_1)} \sum_{\nu_{2-}=1}^{g(r_2,s_2;R_2)} \left|\mathcal{M}_{R_1,R_2,\nu_{1-},\nu_{2-}}\right| &= \sum_{R_1,R_2} \sum_{\nu_{1+}=1}^{g(r_1,s_1;R_1)} \sum_{\nu_{2+}=1}^{g(r_2,s_2;R_2)} \left|\mathcal{M}_{R_1,R_2,\nu_{1+},\nu_{2+}}\right| \\ &= |\mathcal{M}_{\rm tot}|\,. \end{aligned} \tag{C.51}$$

From the identity of the projector (A.46), we obtain

$$\operatorname{tr}(\mathscr{P}\,\tilde{\mathscr{P}}) = \overline{\delta}^{\,\nu_{3+}\,(\nu_{1+},\nu_{2+})}\,\overline{\delta}^{\,(\nu_{1-},\nu_{2-})\,\nu_{3-}}\, d_{r_1} d_{r_2} d_{s_1} d_{s_2}\, \mathcal{G}_{\rm LR} \tag{C.52}$$

where we grouped $(\nu_{1\mp}, \nu_{2\mp})$ so that they can be compared with $\nu_{3\mp}$. Just like before, we conjecture that

$$\begin{aligned} \mathcal{G}_{\rm LR} &= \frac{\left|\mathcal{M}_{r_3,s_3,\nu_{3-}}\right| \left|\mathcal{M}_{r_3,s_3,\nu_{3+}}\right| \left|\mathcal{M}_{R_1,R_2,\nu_{1-},\nu_{2-}}\right| \left|\mathcal{M}_{R_1,R_2,\nu_{1+},\nu_{2+}}\right|}{|\mathcal{M}_{\rm tot}|^2} \\ &= \left(\frac{g(R_1, R_2; R_3) g(r_1, r_2; r_3) g(s_1, s_2; s_3)}{g(r_1, r_2, s_1, s_2; R_3)}\right)^2. \end{aligned} \tag{C.53}$$

In summary, we get

$$F^{\{3\}}_{\{1\}\{2\}} = \overline{\delta}^{\,\nu_{3+}\,(\nu_{1+},\nu_{2+})}\,\overline{\delta}^{\,(\nu_{1-},\nu_{2-})\,\nu_{3-}}\, \frac{d_{r_1} d_{r_2} d_{s_1} d_{s_2}}{d_{R_1} d_{R_2}} \left(\frac{g(R_1, R_2; R_3)\, g(r_1, r_2; r_3)\, g(s_1, s_2; s_3)}{g(r_1, r_2, s_1, s_2; R_3)}\right)^2. \tag{C.54}$$

Three cases have been considered in [27]. The first case is the antisymmetric representations,

$$(R_i, r_i, s_i) = \left([1^{m_i+n_i}], [1^{m_i}], [1^{n_i}]\right) \tag{C.55}$$

and the second case is the symmetric representations,

$$(R_i, r_i, s_i) = ([m_i + n_i], [m_i], [n_i])\,. \tag{C.56}$$

In both cases, all representations are one-dimensional and multiplicity-free. Therefore $F_{\{1\}\{2\}}^{\{3\}} = 1$, which means

$$f_{\{1\}\{2\}}^{\{3\}} = \frac{m_3!\, n_3!\, L_1!\, L_2!}{m_1!\, n_1!\, m_2!\, n_2!\, L_3!}\,. \tag{C.57}$$

The last case is $r_2 = s_1 = \emptyset$, implying that

$$R_1 = r_1 = r_3\,, \quad R_2 = s_2 = s_3\,, \quad F_{\{1\}\{2\}}^{\{3\}} = 1 \tag{C.58}$$

and hence

$$f_{\{1\}\{2\}}^{\{3\}} = \delta^{R_1,r_3}\,\delta^{R_2,s_3}\,\frac{L_1!L_2!}{L_3!}\,\frac{d_{R_3}}{d_{r_3}\,d_{s_3}}\,. \tag{C.59}$$

All the results agree with [27].

References

[1] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [hep-th/9711200] [INSPIRE].

[2] B. Basso, S. Komatsu and P. Vieira, *Structure Constants and Integrable Bootstrap in Planar $N = 4$ SYM Theory*, arXiv:1505.06745 [INSPIRE].

[3] T. Fleury and S. Komatsu, *Hexagonalization of Correlation Functions*, *JHEP* **01** (2017) 130 [arXiv:1611.05577] [INSPIRE].

[4] B. Eden and A. Sfondrini, *Tessellating cushions: four-point functions in $\mathcal{N} = 4$ SYM*, *JHEP* **10** (2017) 098 [arXiv:1611.05436] [INSPIRE].

[5] T. Fleury and S. Komatsu, *Hexagonalization of Correlation Functions II: Two-Particle Contributions*, *JHEP* **02** (2018) 177 [arXiv:1711.05327] [INSPIRE].

[6] T. Bargheer, J. Caetano, T. Fleury, S. Komatsu and P. Vieira, *Handling Handles: Nonplanar Integrability in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **121** (2018) 231602 [arXiv:1711.05326] [INSPIRE].

[7] B. Eden, Y. Jiang, D. le Plat and A. Sfondrini, *Colour-dressed hexagon tessellations for correlation functions and non-planar corrections*, *JHEP* **02** (2018) 170 [arXiv:1710.10212] [INSPIRE].

[8] T. Bargheer, J. Caetano, T. Fleury, S. Komatsu and P. Vieira, *Handling handles. Part II. Stratification and data analysis*, *JHEP* **11** (2018) 095 [arXiv:1809.09145] [INSPIRE].

[9] F. Coronado, *Perturbative four-point functions in planar $\mathcal{N} = 4$ SYM from hexagonalization*, *JHEP* **01** (2019) 056 [arXiv:1811.00467] [INSPIRE].

[10] F. Coronado, *Bootstrapping the simplest correlator in planar $\mathcal{N} = 4$ SYM at all loops*, *Phys. Rev. Lett.* **124** (2020) 171601 [arXiv:1811.03282] [INSPIRE].

[11] T. Bargheer, F. Coronado and P. Vieira, *Octagons I: Combinatorics and Non-Planar Resummations*, *JHEP* **08** (2019) 162 [arXiv:1904.00965] [INSPIRE].

JHEP05(2020)118

[12] T. Bargheer, F. Coronado and P. Vieira, *Octagons II: Strong Coupling*, arXiv:1909.04077 [INSPIRE].

[13] B. Basso, V. Goncalves, S. Komatsu and P. Vieira, *Gluing Hexagons at Three Loops*, Nucl. Phys. **B 907** (2016) 695 [arXiv:1510.01683] [INSPIRE].

[14] B. Eden and A. Sfondrini, *Three-point functions in $\mathcal{N} = 4$ SYM: the hexagon proposal at three loops*, JHEP **02** (2016) 165 [arXiv:1510.01242] [INSPIRE].

[15] B. Basso, V. Goncalves and S. Komatsu, *Structure constants at wrapping order*, JHEP **05** (2017) 124 [arXiv:1702.02154] [INSPIRE].

[16] B. Eden, Y. Jiang, M. de Leeuw, T. Meier, D. le Plat and A. Sfondrini, *Positivity of hexagon perturbation theory*, JHEP **11** (2018) 097 [arXiv:1806.06051] [INSPIRE].

[17] M. De Leeuw, B. Eden, D. Le Plat, T. Meier and A. Sfondrini, *Multi-particle finite-volume effects for hexagon tessellations*, arXiv:1912.12231 [INSPIRE].

[18] R. Suzuki, *Multi-trace Correlators from Permutations as Moduli Space*, JHEP **05** (2019) 168 [arXiv:1810.09478] [INSPIRE].

[19] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, Adv. Theor. Math. Phys. **5** (2002) 809 [hep-th/0111222] [INSPIRE].

[20] T.W. Brown, P.J. Heslop and S. Ramgoolam, *Diagonal multi-matrix correlators and BPS operators in $N = 4$ SYM*, JHEP **02** (2008) 030 [arXiv:0711.0176] [INSPIRE].

[21] T.W. Brown, P.J. Heslop and S. Ramgoolam, *Diagonal free field matrix correlators, global symmetries and giant gravitons*, JHEP **04** (2009) 089 [arXiv:0806.1911] [INSPIRE].

[22] Y. Kimura, S. Ramgoolam and R. Suzuki, *Flavour singlets in gauge theory as Permutations*, JHEP **12** (2016) 142 [arXiv:1608.03188] [INSPIRE].

[23] R. de Mello Koch, J. Smolic and M. Smolic, *Giant Gravitons — with Strings Attached (I)*, JHEP **06** (2007) 074 [hep-th/0701066] [INSPIRE].

[24] R. de Mello Koch, J. Smolic and M. Smolic, *Giant Gravitons — with Strings Attached (II)*, JHEP **09** (2007) 049 [hep-th/0701067] [INSPIRE].

[25] D. Bekker, R. de Mello Koch and M. Stephanou, *Giant Gravitons — with Strings Attached. III.*, JHEP **02** (2008) 029 [arXiv:0710.5372] [INSPIRE].

[26] R. Bhattacharyya, S. Collins and R. de Mello Koch, *Exact Multi-Matrix Correlators*, JHEP **03** (2008) 044 [arXiv:0801.2061] [INSPIRE].

[27] R. Bhattacharyya, R. de Mello Koch and M. Stephanou, *Exact Multi-Restricted Schur Polynomial Correlators*, JHEP **06** (2008) 101 [arXiv:0805.3025] [INSPIRE].

[28] S. Collins, *Restricted Schur Polynomials and Finite N Counting*, Phys. Rev. **D 79** (2009) 026002 [arXiv:0810.4217] [INSPIRE].

[29] J. Pasukonis and S. Ramgoolam, *Quivers as Calculators: Counting, Correlators and Riemann Surfaces*, JHEP **04** (2013) 094 [arXiv:1301.1980] [INSPIRE].

[30] P. Kramer, *Orbital Fractional Parentage Coefficients for the Harmonic Oscillator Shell Model*, Z. Phys. **205** (1967) 181.

[31] R.W. Hasse and P.H. Butler, *Symmetric and unitary group representations: I. Duality theory*, J. Phys. **A 17** (1984) 61.

[32] N. Drukker and J. Plefka, *Superprotected n-point correlation functions of local operators in $N = 4$ super Yang-Mills*, *JHEP* **04** (2009) 052 [arXiv:0901.3653] [INSPIRE].

[33] L.F. McAven, P.H. Butler and A.M. Hamel, *Split bases and multiplicity separations in symmetric group transformation coefficients*, *J. Phys.* **A 31** (1998) 8363.

[34] J.P. Elliott, J. Hope and H.A. Jahn, *Theoretical Studies in Nuclear Structure IV. Wave Functions for the Nuclear p-shell Part B. Fractional Parentage Coefficients*, *Phil. Trans. Roy. Soc.* **A 246** (1953) 241.

[35] J.-Q. Chen, D.F. Collinson and M.-J. Gao, *Transformation coefficients of permutation groups*, *J. Math. Phys.* **24** (198) 2695.

[36] F. Pan and J.-Q. Chen, *Irreducible representations of Hecke algebras in the non-standard basis and subduction coefficients*, *J. Phys.* **A 26** (1993) 4299.

[37] L.F. McAven and P.H. Butler, *Split-standard transformation coefficients: the block-selective conjecture*, *J. Phys.* **A 32** (1999) 7509.

[38] L.F. McAven and A.M. Hamel, *Calculating symmetric group split-standard transformation coefficients using the block selective method: a proof*, *J. Phys.* **A 35** (2002) 1719.

[39] V. Chilla, *On the linear equation method for the subduction problem in symmetric groups*, *J. Phys.* **A 39** (2006) 7657 [math-ph/0512011] [INSPIRE].

[40] V. Chilla, *Reduced subduction graph and higher multiplicity in S_n transformation coefficients*, *J. Phys.* **A 39** (2006) 12395 [math-ph/0606037] [INSPIRE].

[41] R. de Mello Koch, N. Ives and M. Stephanou, *On subgroup adapted bases for representations of the symmetric group*, *J. Phys.* **A 45** (2012) 135204 [arXiv:1112.4316] [INSPIRE].

[42] R. de Mello Koch, M. Dessein, D. Giataganas and C. Mathwin, *Giant Graviton Oscillators*, *JHEP* **10** (2011) 009 [arXiv:1108.2761] [INSPIRE].

[43] R. de Mello Koch, J.-H. Huang and L. Tribelhorn, *Exciting LLM Geometries*, *JHEP* **07** (2018) 146 [arXiv:1806.06586] [INSPIRE].

[44] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025 [hep-th/0409174] [INSPIRE].

[45] M. Kim and H.J.R. van Zyl, *Semiclassical* SL(2) *strings on LLM backgrounds*, *Phys. Lett.* **B 784** (2018) 62 [arXiv:1805.12460] [INSPIRE].

[46] P. Mattioli and S. Ramgoolam, *Permutation Centralizer Algebras and Multi-Matrix Invariants*, *Phys. Rev.* **D 93** (2016) 065040 [arXiv:1601.06086] [INSPIRE].

[47] J. Ginibre, *Statistical Ensembles of Complex, Quaternion and Real Matrices*, *J. Math. Phys.* **6** (1965) 440 [INSPIRE].

[48] M.L. Mehta, *Random Matrices*, third edition, Elsevier, Amsterdam The Netherlands (2004).

[49] I.K. Kostov and M. Staudacher, *Two-dimensional chiral matrix models and string theories*, *Phys. Lett.* **B 394** (1997) 75 [hep-th/9611011] [INSPIRE].

[50] I.K. Kostov, M. Staudacher and T. Wynter, *Complex matrix models and statistics of branched coverings of $2-D$ surfaces*, *Commun. Math. Phys.* **191** (1998) 283 [hep-th/9703189] [INSPIRE].

[51] C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *A New double scaling limit of $N = 4$ superYang-Mills theory and PP wave strings*, Nucl. Phys. **B 643** (2002) 3 [hep-th/0205033] [INSPIRE].

[52] A.N. Kirillov and N.Y. Reshetikhin, *Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links*, in *New Developments in the Theory of Knots*, World Scientific, Singapore (1990), pg. 202.

[53] H. Itoyama, A. Mironov, A. Morozov and A. Morozov, *Eigenvalue hypothesis for Racah matrices and HOMFLY polynomials for 3-strand knots in any symmetric and antisymmetric representations*, Int. J. Mod. Phys. **A 28** (2013) 1340009 [arXiv:1209.6304] [INSPIRE].

[54] S. Nawata, P. Ramadevi and Zodinmawia, *Multiplicity-free quantum 6-j symbols for $U_q(sl_N)$*, Lett. Math. Phys. **103** (2013) 1389 [arXiv:1302.5143] [INSPIRE].

[55] A. Morozov and A. Sleptsov, *New symmetries for the $U_q(sl_N)$ 6-j symbols from the Eigenvalue conjecture*, JETP Lett. **108** (2018) 697 [arXiv:1905.01876] [INSPIRE].

[56] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from Anti-de Sitter space*, JHEP **06** (2000) 008 [hep-th/0003075] [INSPIRE].

[57] V. Balasubramanian, M. Berkooz, A. Naqvi and M.J. Strassler, *Giant gravitons in conformal field theory*, JHEP **04** (2002) 034 [hep-th/0107119] [INSPIRE].

[58] R. de Mello Koch, C. Mathwin and H.J.R. van Zyl, *LLM Magnons*, JHEP **03** (2016) 110 [arXiv:1601.06914] [INSPIRE].

[59] R. de Mello Koch, M. Kim and H.J.R. Zyl, *Integrable Subsectors from Holography*, JHEP **05** (2018) 198 [arXiv:1802.01367] [INSPIRE].

[60] H. Takayanagi and T. Takayanagi, *Notes on giant gravitons on PP waves*, JHEP **12** (2002) 018 [hep-th/0209160] [INSPIRE].

[61] S. Hirano and Y. Sato, *Giant graviton interactions and M2-branes ending on multiple M5-branes*, JHEP **05** (2018) 065 [arXiv:1803.04172] [INSPIRE].

[62] R. de Mello Koch, E. Gandote and J.-H. Huang, *Non-Perturbative String Theory from AdS/CFT*, JHEP **02** (2019) 169 [arXiv:1901.02591] [INSPIRE].

[63] A. Bissi, C. Kristjansen, D. Young and K. Zoubos, *Holographic three-point functions of giant gravitons*, JHEP **06** (2011) 085 [arXiv:1103.4079] [INSPIRE].

[64] P. Caputa, R. de Mello Koch and K. Zoubos, *Extremal versus Non-Extremal Correlators with Giant Gravitons*, JHEP **08** (2012) 143 [arXiv:1204.4172] [INSPIRE].

[65] H. Lin, *Giant gravitons and correlators*, JHEP **12** (2012) 011 [arXiv:1209.6624] [INSPIRE].

[66] C. Kristjansen, S. Mori and D. Young, *On the Regularization of Extremal Three-point Functions Involving Giant Gravitons*, Phys. Lett. **B 750** (2015) 379 [arXiv:1507.03965] [INSPIRE].

[67] Y. Jiang, S. Komatsu and E. Vescovi, *Exact Three-Point Functions of Determinant Operators in Planar $N = 4$ Supersymmetric Yang-Mills Theory*, Phys. Rev. Lett. **123** (2019) 191601 [arXiv:1907.11242] [INSPIRE].

[68] Y. Jiang, S. Komatsu and E. Vescovi, *Structure Constants in $\mathcal{N} = 4$ SYM at Finite Coupling as Worldsheet g-Function*, arXiv:1906.07733 [INSPIRE].

[69] G. Chen, R. de Mello Koch, M. Kim and H.J.R. Van Zyl, *Absorption of closed strings by giant gravitons*, JHEP **10** (2019) 133 [arXiv:1908.03553] [INSPIRE].

JHEP05(2020)118

[70] K.-Y. Kim, M. Kim and K. Lee, *Structure Constants of a Single Trace Operator and Determinant Operators from Hexagon*, arXiv:1906.11515 [INSPIRE].

[71] D.M. Goldschmidt, *University Lecture Series. Vol. 4: Group Characters, Symmetric Functions and the Hecke Algebras*, AMS Press, Providence U.S.A. (1993).

[72] R.P. Stanley, *Cambridge Studies in Advanced Mathematics. Book 62: Enumerative Combinatorics: Volume 2*, Cambridge University Press, Cambridge U.K. (1999).

[73] E.P. Wigner, *Group theory and its application to the quantum mechanics of atomic spectra*, Academic Press, New York U.S.A. (1959).

[74] G. Racah, *Theory of Complex Spectra. II*, *Phys. Rev.* **62** (1942) 438 [INSPIRE].

[75] P. Kramer, *Recoupling Coefficients of the Symmetric Group for Shell and Cluster Model Configurations*, *Z. Phys.* **216** (1968) 68.