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Conformal quantum mechanics of causal diamonds

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ABSTRACT: It is shown that a general radial conformal Killing vector in Minkowski space-time can be associated to a generator of time evolution in conformal quantum mechanics. Among these conformal Killing vectors there is a class which maps causal diamonds in Minkowski space-time into themselves. The flow of such Killing vectors describes worldlines of accelerated observers with a finite lifetime within a causal diamond. Time evolution of static diamond observers is equivalent to time evolution in conformal quantum mechanics governed by a hyperbolic Hamiltonian and covering only a segment of the time axis. This indicates that the Unruh temperature perceived by static diamond observers in the vacuum state of inertial observers in Minkowski space-time can be obtained from the behaviour of the two-point functions of conformal quantum mechanics. The results presented suggest a group theoretical description of the recently proposed *light-cone temperature* associated to null surfaces defined by light fronts in Minkowski space-time.

KEYWORDS: Space-Time Symmetries, Conformal and W Symmetry

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1 Introduction

Do static observers with a finite lifetime in Minkowski space-time perceive the vacuum for inertial observers as a thermal state? In this note we unveil a connection between the symmetries of *causal diamonds*, regions in Minkowski space-time defined by the intersection of past and future light cones of two events, and time evolution in conformal quantum mechanics, a $0 + 1$ -dimensional field theory with conformal invariance [1]. This connection provides group theoretical evidence for a *diamond temperature* associated to static observers within the diamond. Such temperature can be linked to the *light-cone temperature* associated to covariant regions of Minkowski space-time with null boundaries discussed in [2, 3], which appears to be directly connected with the Hawking temperature of black holes.

Our results are based on a simple observation: the most general radial conformal Killing vector in Minkowski space-time, in dimensions $d \geq 3$, is equivalent, at a group theoretic level, to the most general Hamiltonian governing time evolution in conformal quantum mechanics. A special class of *hyperbolic* radial conformal Killing vectors generates time evolution within the causal diamond corresponding to a choice of Hamiltonian in conformal quantum mechanics alternative to ordinary time translations. These translations cover only a finite region on the time domain of the theory whose size matches the size of the causal diamond in Minkowski space. The existence of different notions of time translations on both sides, leads to two-point functions which are thermal for observers whose time evolution is governed by a hyperbolic Hamiltonian.

Radial conformal symmetries of Minkowski space-time have been studied and classified in [4]. These conformal Killing vectors, according to the sign of their norm, divide Minkowski space-time into subregions with null boundaries. It has been recently appreciated [2] that the structure of these regions is analogous to the space-time associated to the inner and outer horizons of a Reissner-Nordstrom black hole. In fact one can associate to the null boundary of the causal complement of the diamond generated by the radial conformal Killing vectors thermodynamical relations analogous to the one satisfied by black

holes and thus a notion of entropy and temperature [2]. The conformal Killing vectors generating transformations which map diamonds into themselves are a special class of radial conformal Killing vectors given by a combination of the generators of time translations and special conformal transformations. The integral curves of these diamond Killing vectors are worldlines of observers moving with constant acceleration and indeed their worldlines can be obtained from a conformal mapping of the trajectories followed by accelerated observers in the Rindler wedge [5, 6]. As a consequence one naturally expects these diamond observers to experience phenomena like the Unruh effect whereby the vacuum state for inertial observers appears thermal to accelerated observers. Arguments in support of the existence of a diamond temperature associated to these observers were given for the first time in [5] based on a conjecture known as *thermal time hypothesis*. More recently, in [7], thermal effects associated to the restriction of the space-time domain to a causal diamond where studied introducing a set of Rindler-like diamond coordinates and analyzing the response function of a Unruh-DeWitt detector with Hamiltonian appropriately rescaled in order to be a constant of motion under the diamond time evolution. In this note we illustrate how the connection between diamond time evolution and hyperbolic time evolution in conformal quantum mechanics provides further evidence for the existence of a diamond temperature. Our analysis also shows that, within the framework of conformal quantum mechanics, thermal effects associated to a notion of time evolution which effectively restricts the time domain to a finite region can be derived starting from representations of the $SL(2, \mathbb{R})$ group, thus providing an example of quantum system in which a phenomenon in essence analogous to the Unruh effect can be rooted on group theoretic grounds.

In the next section we exhibit the connection between radial conformal Killing symmetries in Minkowski space-time and time evolution in conformal quantum mechanics. In particular we show that conformal Killing vectors which map a causal diamond into itself are related to the generator of *hyperbolic* time evolution in conformal quantum mechanics. In section 3 we notice that two-point functions of conformal quantum mechanics, seen as a CFT_1 , which express correlations of quantum states whose time evolution is governed by a hyperbolic Hamiltonian have a thermal character. The correspondence between this time evolution and diamond time is then used to connect such two-point functions, for a particular value of the conformal weight, with the one for a massless field along the diamond static trajectory and thus connecting the diamond temperature with the hyperbolic temperature in CFT_1 . The final section 4 is devoted to a brief summary and outlook.

2 Radial conformal symmetry as time evolution

Let us consider the line element of Minkowski space-time in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \quad (2.1)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. The most general *radial conformal* Killing vector [4], i.e. a radial vector field ξ such that the Lie derivative $\mathcal{L}_\xi$ of the Minkowski metric $\eta_{\mu\nu}$ is proportional through a constant factor to the metric itself $\mathcal{L}_\xi \eta_{\mu\nu} \propto \eta_{\mu\nu}$, is given by

$$\xi = (a(t^2 + r^2) + bt + c) \partial_t + r(2at + b) \partial_r, \quad (2.2)$$

where a , b , and c are arbitrary *real* constants. Such conformal Killing vector can be written as a linear combination of generators of time translations

$$P_0 = \partial_t, \quad (2.3)$$

dilations¹

$$D_0 = r \partial_r + t \partial_t, \quad (2.4)$$

and special conformal transformations along the time direction

$$K_0 = 2tr \partial_r + (t^2 + r^2) \partial_t = 2tD_0 + (r^2 - t^2)P_0, \quad (2.5)$$

as

$$\xi = aK_0 + bD_0 + cP_0. \quad (2.6)$$

Notice that these generators close the $\mathfrak{sl}(2, \mathbb{R})$ algebra

$$[P_0, D_0] = P_0, \quad [K_0, D_0] = -K_0, \quad [P_0, K_0] = 2D_0. \quad (2.7)$$

The starting observation of this note is that time evolution in conformal quantum mechanics [1] is determined by a Hamiltonian which in its most general form is given in the literature as

$$G = uH + vD + wK \quad (2.8)$$

where H , D and K again belong to the $\mathfrak{sl}(2, \mathbb{R})$ algebra

$$[H, D] = iH, \quad [K, D] = -iK, \quad [H, K] = 2iD. \quad (2.9)$$

More precisely, the generators H , D and K are the “quantum mechanical” counterparts of P_0 , D_0 and K_0 in the sense that $H = iP_0$, $K = iK_0$, $D = iD_0$, and thus $G = i\xi$ if we make the identification $u = c$, $v = b$ and $w = a$. Such generators are strictly related to the isometries of AdS_2 and indeed P_0 , D_0 and K_0 , as given in (2.3), (2.4) and (2.5), can be identified with the Killing vectors of AdS_2 in Poincaré coordinates [8, 9].

There are three types of Hamiltonians which can not be connected to another via the adjoint action of $SL(2, \mathbb{R})$ [1, 8, 10] and thus belong to different conjugacy classes. Such classification can be obtained by noting that the invariance of the $\mathfrak{sl}(2, \mathbb{R})$ Casimir

$$\mathcal{C} = \frac{1}{2} (KH + HK) - D^2 \quad (2.10)$$

is reflected in the invariance of the determinant of the matrix

$$\begin{pmatrix} b & 2c \\ 2a & b \end{pmatrix}, \quad (2.11)$$

and the various Hamiltonians in conformal quantum mechanics can be classified according to the sign of $\Delta = b^2 - 4ac$:

¹We denote the generator of dilations with D_0 simply for notational convenience, the subscript 0, unlike for P_0 and K_0 , does not refer in this case to a time-like direction.

- $\Delta < 0$: these are generators of *elliptic* transformations. The generator of rotations

$$R = \frac{1}{2} \left(\alpha H + \frac{K}{\alpha} \right), \quad (2.12)$$

falls into this class. Notice the introduction of the constant α with dimensions of length, needed to render R dimensionless which will be crucial for what follows. The spectrum of such Hamiltonian is discrete and a basis of eigenstates of R was introduced in [1] and used in [11, 12] to construct the two-point functions of the theory which we will use below. One of the main upshots of this letter will be to give a geometrical interpretation of the constant α .

- $\Delta = 0$: these operators generate *parabolic* transformations or, if we look at the correspondence between $SL(2, \mathbb{R})$ and the three-dimensional Lorentz group $SO(2, 1)$, *null rotations*. Representatives of this class are H and K .
- $\Delta > 0$: these are generators of *hyperbolic* transformation or boosts from the Lorentz group perspective. The generator of dilations D and the combination

$$S = \frac{1}{2} \left(\alpha H - \frac{K}{\alpha} \right), \quad (2.13)$$

which will be central for what follows, fall into this class.

The corresponding classification from the point of view of radial conformal Killing vectors in Minkowski space was given in [4]. Such classification deals with the sign of the Minkowski norm of the conformal Killing vector which turns out to be determined by Δ . Indeed it can be shown that [4] (see also [2, 3])

- For $\Delta < 0$ the radial conformal Killing vector is *everywhere timelike*.
- For $\Delta = 0$ the radial conformal Killing vector is also everywhere timelike except that for the light cone emanating from $t = -(b/2a), r = 0$.
- For $\Delta > 0$ it is null on the light cones at the points $t = t_{\pm}, r = 0$ where $t_{\pm} = \frac{-b \pm \sqrt{\Delta}}{2a}$, time-like inside either lightcone or outside *both* light cones and space-like everywhere else.

Notice that setting $b = 0$ just makes everything symmetric about the origin $t = 0, r = 0$.

Now let us consider the time evolution determined by the three representative Hamiltonians H , R and S . These cases were already considered in [1]. If we write

$$H = i\partial_t \quad (2.14)$$

one can define new time variables T and τ such that

$$R = i\partial_T, \quad S = i\partial_{\tau}. \quad (2.15)$$

These three variables are related in conformal quantum mechanics through [1]

$$t = \alpha \tan(T/2) = \alpha \tanh(\tau/2). \quad (2.16)$$

An analogous association can be made with coordinates in Minkowski space. To see this let us notice that the generator $S = i\partial_\tau$ of translations in the time variable τ in conformal quantum mechanics can be written as a differential operator in Minkowski space-time in spherical coordinates as

$$S = \frac{1}{2\alpha} (\alpha^2 H - K) = \frac{i}{2\alpha} ((\alpha^2 - t^2 - r^2)\partial_t - 2tr\partial_r) . \quad (2.17)$$

This generator corresponds to the special class of radial conformal Killing vectors which map a causal diamond, i.e. the region $|t| + |r| < \alpha$, into itself [2, 3, 13]

$$\zeta = \frac{1}{2\alpha} ((\alpha^2 - t^2 - r^2)\partial_t - 2tr\partial_r) . \quad (2.18)$$

Through this correspondence we see that the length scale α introduced in conformal quantum mechanics is the radius of the spatial sphere defining the diamond in Minkowski space. Notice how 2α , the length of the vertical axis of the causal diamond, is the *finite lifetime* of a static observer sitting at the origin of Minkowski space-time.

In spherical coordinates the Hamiltonian $H = i\partial_t$ is simply the generator of translations in the inertial time coordinate t . A set of coordinates in Minkowski space “adapted” to the radial Killing vector $\zeta = -iS$, i.e. such that $i\zeta = S = i\partial_\tau$, is given by

$$t = \alpha \frac{\sinh \tau}{\cosh x + \cosh \tau} \quad (2.19)$$

$$r = \alpha \frac{\sinh x}{\cosh x + \cosh \tau} . \quad (2.20)$$

The Minkowski line element in this *diamond coordinates* reads [14]

$$ds^2 = \frac{\alpha^2}{(\cosh \tau + \cosh x)^2} (-d\tau^2 + dx^2 + \sinh^2 x d\Omega^2) . \quad (2.21)$$

Integral curves of a generic radial conformal Killing vector were obtained in [4]. In the special case of the conformal Killing vector (2.17) (i.e. when $a = -\frac{1}{2\alpha}$, $b = 0$ and $c = \frac{\alpha}{2}$ in (2.2)) they are given by the one-parameter family of hyperbolae in the $r - t$ plane

$$t^2 - (r - \alpha\omega)^2 = \alpha^2(1 - \omega^2) \quad (2.22)$$

parametrized by $\omega \neq 0$. Each one of these curves² can be seen as the worldline of an accelerated observer with finite lifetime confined to the causal diamond of size 2α . From the four-velocity of observers static with respect to diamond time, i.e. observers whose worldlines are given by the curves (2.22), one can derive [14] the acceleration

$$a(x) = \frac{\sinh x}{\alpha} = \frac{1}{\alpha \sqrt{\omega^2 - 1}} , \quad (2.23)$$

which is *constant* along each integral curve and is related to the parameter $\omega = \frac{1}{\tanh x}$. Thus a translation in the diamond variable x maps worldlines of observers with different

²The curves (2.22) correspond to the ones found in [5] by mapping the worldlines of Rindler observers, i.e. orbits of boosts, to the diamond via a special conformal transformation.

acceleration into another. Diamond observers passing through the origin $x = 0$ have vanishing acceleration and follow the same worldline of a static inertial observer sitting at the origin. From (2.19) we see that inertial time and diamond time at $x = 0$ are related by

$$t = \alpha \frac{\sinh \tau}{1 + \cosh \tau} = \alpha \tanh(\tau/2) \quad (2.24)$$

i.e. the same relationship between H -time and S -time variable in conformal quantum mechanics! It should be noted how this situation is reminiscent of the idea explored in [15] that de Sitter space can be holographically described by a *worldline quantum mechanics* at the origin.³

3 Diamond temperature on the real line

In Minkowski space-time the vacuum state associated to the free Hamiltonian of inertial observers is perceived as a thermal state populated of excitations associated to the “boost” Hamiltonian generating time evolution for uniformly accelerating observers. This is the celebrated Unruh effect [16]. In [17] it was shown how, also in the simplest case of a quantum system with affine symmetry living in *one dimension*, the freedom in the choice of translation generators leads to an analogous effect. In this case one considers particles as irreducible representations of the group of affine transformations of the real line, also known as the $ax + b$ group. This is comprised of translations and dilations. Particles can carry an “energy” associated to the generators of both transformations. It turns out that the vacuum state with no excitations with respect to the translation Hamiltonian contains a thermal distribution of excitations associated to the dilation Hamiltonian.

We ask here whether a similar effect can take place in a one dimensional quantum system with a larger group of symmetries. This is the case of conformal quantum mechanics [1]. Such model can be seen as a $0 + 1$ -dimensional field theory like the one considered in [17] in which, besides translations and dilations, one also considers special conformal transformations i.e. the group generated by the algebra (2.9). The standard construction [1] of this field theory starts from the introduction of a set of eigenstates $|n\rangle$ of the generator of rotations R , carrying an irreducible representation of $SL(2, \mathbb{R})$

$$R|n\rangle = r_n |n\rangle \quad (3.1a)$$

$$r_n = r_0 + n, \quad r_0 > 0, \quad n = 0, 1 \dots$$

$$L_{\pm}|n\rangle = \sqrt{r_n(r_n \pm 1) - r_0(r_0 - 1)} |n \pm 1\rangle, \quad (3.1b)$$

³In light of this observation and in view of the connection between the radial direction in AdS and the energy scale in CFT in the AdS/CFT correspondence one might wonder how the radial direction in the diamond is “holographically” encoded in conformal quantum mechanics. At first sight there appears to be no clear correspondence between radial translations in the diamond and some type of evolution in conformal quantum mechanics. However it should be pointed out that if one writes the general radial conformal Killing vector $\xi = (a(t^2 + r^2) + bt + c) \partial_t + r(2at + b) \partial_r$ in terms of light-cone coordinates $v = t + r$ and $u = t - r$, one gets $\xi = (au^2 + bu + c) \partial_u + (av^2 + bv + c) \partial_v$, and thus we see that on the light cones we have a formal correspondence between such Killing vector and the most general Hamiltonian of conformal quantum mechanics if light-cone coordinates are identified with the time variable in the conformal theory. Thus light-like directions, unlike the radial direction, seem to have a clear dual in conformal quantum mechanics.

with inner product $\langle n|n'\rangle = \delta_{nn'}$, where

$$L_{\pm} \equiv \frac{1}{2} \left(\frac{K}{\alpha} - \alpha H \right) \pm i D, \quad (3.2)$$

and $|n=0\rangle$ is the “R-vacuum”. The constant r_0 is related to the eigenvalue of the Casimir operator of $\mathfrak{sl}(2, \mathbb{R})$

$$\mathcal{C} = \frac{1}{2} (KH + HK) - D^2 = r_0 (r_0 - 1). \quad (3.3)$$

The use of the eigenstates $|n\rangle$ of the R operator was motivated in [1] by the fact that they form a discrete spectrum and thus from a quantum mechanical standpoint R would be preferable as the Hamiltonian of the system in contrast with H and its associated continuous spectrum and non-normalizable ground state.

We now look for states $|t\rangle$ labelled by a time variable t on which H acts as a generator of translations [1]

$$H |t\rangle = -i \frac{d}{dt} |t\rangle. \quad (3.4)$$

The action of the other generators on these states in terms of time derivatives will be given by

$$D |t\rangle = -i \left(t \frac{d}{dt} + r_0 \right) |t\rangle, \quad (3.5)$$

$$K |t\rangle = -i \left(t^2 \frac{d}{dt} + 2r_0 t \right) |t\rangle \quad (3.6)$$

so that they close the algebra (2.9) and are related through the Casimir (3.3). Notice how translations generated by H can cover the entire time line $\mathbb{R}$, in analogy with the free Hamiltonian of inertial observers who have access to the whole Minkowski space-time.

The $|t\rangle$ states can be characterized by looking at their overlap with $|n\rangle$ states. Using the fact that the latter are diagonal under the action of the generator R one can write the following differential equation

$$\langle t|R|n\rangle = r_n \langle t|n\rangle = \frac{i}{2} \left[\left(\alpha + \frac{t^2}{\alpha} \right) \frac{d}{dt} + 2r_0 \frac{t}{\alpha} \right] \langle t|n\rangle \quad (3.7)$$

which can be solved to give [1]

$$\langle t|n\rangle = (-1)^n \left[\frac{\Gamma(2r_0 + n)}{n!} \right]^{\frac{1}{2}} \left(\frac{\alpha - it}{\alpha + it} \right)^{r_n} \left(1 + \frac{t^2}{\alpha^2} \right)^{-r_0}. \quad (3.8)$$

Alternatively [12], the kets $|t\rangle$ can be obtained directly from the R -vacuum through a “complex” time translation

$$|t\rangle = e^{iHt} |t=0\rangle = \frac{\Gamma^{1/2}(2r_0)}{2^{2r_0}} e^{(\alpha+it)H} |n=0\rangle. \quad (3.9)$$

From (3.8), using the completeness of the states $|n\rangle$, one can derive the inner product of $|t\rangle$ states

$$\langle t_1|t_2\rangle = \frac{\Gamma(2r_0) \alpha^{2r_0}}{[2i(t_1 - t_2)]^{2r_0}}, \quad (3.10)$$

which are thus *not* orthonormal. In [11] the authors propose to identify the inner product (3.10) with the two-point functions of a CFT_1

$$G_2(t_1, t_2) \equiv \langle t_1 | t_2 \rangle = \frac{\Gamma(2r_0) \alpha^{2r_0}}{[2i(t_1 - t_2)]^{2r_0}}, \quad (3.11)$$

with conformal weight r_0 . For $r_0 = 1$ such two-point function is identical, modulo a constant factor, to the two-point function of a free massless scalar field in Minkowski space-time in the inertial vacuum, evaluated along the trajectory of a static inertial observer sitting at the origin. In other words two-point correlators for a massless field along the worldline of static observers in Minkowski space-time are in correspondence with two-point functions of conformal quantum mechanics for the states $|t\rangle$.

There is a subtlety concerning the states $|t\rangle$, discussed in detail in [11, 12]. As mentioned above, such states can be obtained by acting on the R -vacuum with the operator

$$\mathcal{O}(t) = \frac{\Gamma^{1/2}(2r_0)}{2^{2r_0}} e^{(\alpha+it)H}. \quad (3.12)$$

As it was already noticed in [1], the R -vacuum is not invariant under the action of the full conformal group $SL(2, \mathbb{R})$. At the same time the operator $\mathcal{O}(t)$ is not a conformal primary, i.e. it does not transform as $\mathcal{O}(t') = \left(\frac{dt}{dt'}\right)^{r_0} \mathcal{O}(t)$ under a conformal transformation. However, as discussed in [11], these two “flaws” compensate each other when $\mathcal{O}(t)$ acts on the R vacuum and creates states $|t\rangle$ which are conformally covariant. From a state-operator-correspondence point of view the operator $\mathcal{O}(t)$ when acting on the R -vacuum behaves as a conformal primary of weight r_0 . This observation has a crucial upshot for our discussion since it tells us that under a change of time variable from t to a new variable t' the quantum states transform as

$$|t'\rangle = \left(\frac{dt}{dt'}\right)^{r_0} |t = t(t')\rangle. \quad (3.13)$$

Let us now look at the action of the generator S on the kets $|t\rangle$

$$S|t\rangle = \frac{1}{2} \left(\alpha H - \frac{K}{\alpha} \right) |t\rangle = -i \left(\frac{1}{2} [t^2/\alpha - \alpha] \frac{d}{dt} + \frac{r_0 t}{\alpha} \right) |t\rangle. \quad (3.14)$$

As seen in section 2, S generates translations in the time variable τ related to t through (2.16)

$$t = \alpha \tanh(\tau/2). \quad (3.15)$$

Accordingly we will have new states $|\tau\rangle$ associated to this time variable

$$|\tau\rangle = \left(\frac{\alpha}{2}\right)^{r_0} (\cosh \tau/2)^{-2r_0} |t = \alpha \tanh(\tau/2)\rangle, \quad (3.16)$$

such that [12]

$$S|\tau\rangle = -i \frac{d}{d\tau} |\tau\rangle. \quad (3.17)$$

The two-point function associated to the new kets is then given by

$$G_2(\tau_1, \tau_2) = \langle \tau_1 | \tau_2 \rangle = \frac{\Gamma(2r_0) \alpha^{2r_0}}{[4i \sinh(\frac{\tau_1 - \tau_2}{2})]^{2r_0}}. \quad (3.18)$$

This two-point function is periodic in imaginary time and it is associated to a thermal state at temperature $T = \frac{1}{2\pi}$. Such thermal character has been linked in the literature to the temperature perceived by a static patch observer in de Sitter space [15, 18].

Here we show that there is a direct connection between the thermal two-point function of conformal quantum mechanics (3.18) and the one associated to a static diamond observer. Let us first note that the time variable τ associated to the S generator, as customarily defined in conformal quantum mechanics, i.e. as in equation (2.16), is *dimensionless*. The right dimensions can be recovered by defining a new time variable multiplied by a factor containing the length scale α . We can determine such factor by requiring the generator of diamond time translations to *match the generator of inertial time translations* H when the size of the diamond goes to infinity, i.e. defining a new generator S'

$$S' = \frac{2}{\alpha} S = H - \frac{K}{\alpha^2}. \quad (3.19)$$

This choice can be realized by introducing new diamond coordinates $\tau' = \frac{\alpha}{2} \tau$, $x' = \frac{\alpha}{2} x$ so that $S' = i\partial_{\tau'}$ and with inertial time related to the new diamond time for worldlines passing through the origin via (see (2.24))

$$t = \alpha \tanh(\tau'/\alpha). \quad (3.20)$$

The choice of Hamiltonian (3.19) is justified since S' generates time translations for the static diamond observer and when the observer's lifetime goes to infinity, $\alpha \rightarrow \infty$, her notion of time coincides with that of a static inertial observer $t = \tau'$ (as it is evident from (3.20)). Similarly for what happens with translations associated to the dilation generator in the case of the affine group [17], here the time translations associated to the generators S and S' only cover a portion $t \in (-\alpha, \alpha)$ of the time line when τ and τ' range from $-\infty$ to ∞ . In analogy with the change of variable from t to τ we can now define new kets $|\tau'\rangle$ whose two-point function for $r_0 = 1$ is given by

$$G_2(\tau'_1, \tau'_2) = -\frac{\alpha^2}{16 \sinh^2\left(\frac{\tau'_1 - \tau'_2}{\alpha}\right)}. \quad (3.21)$$

This is a thermal two-point function at temperature

$$T_{S'} = \frac{1}{\pi\alpha}. \quad (3.22)$$

In fact, the two-point function (3.21) coincides (modulo a $(2\alpha^2/\pi)^2$ factor) with the two-point function of a massless scalar field evaluated along the trajectory of a Rindler observer moving with constant acceleration $a = \frac{2}{\alpha}$ whose proper time is τ' [7, 19, 21]. The coincidence of the two-point function of a massless scalar field written in Rindler-like diamond

coordinates with such two point function was exploited in [7] in order to derive the temperature (3.22) using a Unruh-DeWitt detector with a rescaled Hamiltonian. The effect of such rescaling is to multiply the two-point function in the Minkowski vacuum used to calculate the detector's response function [7] by a non-trivial factor. This factor, from the point of view of the corresponding conformal quantum mechanics, results from the non-trivial transformation properties of the kets (3.16) when writing the two-point function in terms of the hyperbolic time variable τ . Compared to the time dependent diamond temperature derived in [5] via the *thermal time hypothesis*, the temperature (3.22) is that perceived by a static observer in the middle of her lifetime i.e. at $\tau' = 0$.

A natural question that one could ask at this point is whether the temperature (3.22) could be obtained by tracing over the degrees of freedom outside the region $t \in (-\alpha, \alpha)$ so that the density matrix associated to the vacuum for the H generator becomes a thermal state in terms of excitations associated to the S Hamiltonian. An answer to this question is not immediate as it might appear at first sight. Indeed, unlike the case of ordinary field theory in Minkowski space-time where the inertial vacuum is Poincaré invariant, the H -vacuum in conformal quantum mechanics is not invariant under the full $SL(2, \mathbb{R})$ group [20] and thus the vacuum density matrix would have the unpleasant feature of being not conformally invariant. This is analogous to what happens when considering the R -vacuum, as discussed in detail above, and it is likely that a proper understanding of the temperature (3.22) from a density matrix point of view should resort to arguments based on the “state-operator correspondence” discussed in [11, 12].

Let us now see how the temperature $T_{S'}$ derived above can be linked to the notion of light-cone temperature introduced in [2, 3]. We start from observing that the most general radial conformal Killing vector considered in [2] is given by

$$\xi = \frac{1}{r_0^2 - r_H^2} ((-r_H^2 + t^2 + r^2)\partial_t + 2tr\partial_r) . \quad (3.23)$$

where $r_H = \alpha$ is the radius of the sphere determining the causal diamond and r_0 is the radius of the “observers sphere” i.e. the sphere where $\xi = \partial_t$ on the $t = 0$ surface. Comparing this expression to (2.18) we see that the two Killing vectors when $r_0 = 0$ are related by

$$\xi = \frac{2}{r_H} \zeta . \quad (3.24)$$

We can thus generalize the time evolution defined by S' to the case $r_0 \neq 0$ using the Killing vector

$$\xi = -\frac{2r_H}{r_0^2 - r_H^2} \zeta . \quad (3.25)$$

Now, let us look at the change to new diamond coordinates $\tau' = \frac{\alpha}{2} \tau = \frac{r_H}{2} \tau$, $x' = \frac{\alpha}{2} x = \frac{r_H}{2} x$ we made above. A comparison with the diamond coordinates given in [2] shows that the generalization of the diamond coordinates (2.19) to the case $r_0 \neq 0$ is given by

$$t = r_H \frac{\sinh(-\sqrt{\Delta} \tau)}{\cosh(-\sqrt{\Delta} x) + \cosh(-\sqrt{\Delta} \tau)} \quad (3.26)$$

$$r = r_H \frac{\sinh(-\sqrt{\Delta} x)}{\cosh(-\sqrt{\Delta} x) + \cosh(-\sqrt{\Delta} \tau)} , \quad (3.27)$$

where $\Delta = \frac{4r_H^2}{(r_0^2 - r_H^2)^2}$. These coordinates differ from the ones given in [2] by the minus sign in front of $\sqrt{\Delta}$ in the argument of the hyperbolic functions. Without this sign the inertial and diamond time coordinates would have had opposite orientation at $x = 0$. One can now repeat all the steps of the derivation of $T_{S'}$ above to find that for the case $r_0 \neq 0$ it generalizes to

$$T_{S'} = -\frac{\sqrt{\Delta}}{2\pi}, \quad (3.28)$$

i.e., up to a crucial minus sign which ensures its positivity, the same *light cone temperature* found in [2] considering the causal complement of the diamond. According to the results of [3] such temperature can be interpreted as the Hawking temperature in a target black hole spacetime obtained via a Weyl map of Minkowski space-time. The temperature found in [2], with opposite sign, corresponds to the temperature associated to the mapping of the causal complement of the diamond to the “right” asymptotic region of the target black hole space-time while the temperature (3.28) to the Hawking temperature associated to the image of the diamond i.e. the “left” asymptotic region of the same black hole geometry. Thus our analysis offers a suggestive group theoretical interpretation of the conformal temperature associated to null surfaces in Minkowski space-time.

4 Outlook

Causal diamonds play a fundamental role in the understanding of thermodynamic properties of space-time and gravity [14, 22, 23]. According to the covariant entropy bound [24] they also have holographic properties: one can associate a Bekenstein-Hawking entropy to the null boundary of the causal diamond. In this note we exhibited a deep connection between the radial symmetries of causal diamonds and time evolution in conformal quantum mechanics, a one-dimensional quantum field theory which appears to play an important role in a variety of contexts from cosmology [25] to black hole thermodynamics [26], from holography [9, 11] to the understanding of quantum anomalies [27]. In particular we showed that the thermal behaviour of one-dimensional conformal two-point functions is directly connected with the two-point function along the worldline of a static diamond observer experiencing a temperature inversely proportional to her lifetime. We postpone to later studies an analysis of the significance of the effect discussed in this note for the disparate variety of contexts in which conformal quantum mechanics appears as an effective theory.

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