On the elliptic Calabi-Yau fourfold with maximal $h^{1,1}$

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ABSTRACT: In this paper, we explicitly construct the smooth compact base threefold for the elliptic Calabi-Yau fourfold with the largest known $h^{1,1} = 303148$. It is generated by blowing up a smooth toric “seed” base threefold with (E_8, E_8, E_8) collisions. The 4d F-theory compactification model over it has the largest geometric gauge group, $E_8^{2561} \times F_4^{7576} \times G_2^{20168} \times \text{SU}(2)^{30200}$, and the largest number of axions, 181 820, in the known 4d $\mathcal{N} = 1$ supergravity landscape. We also prove that there are at least $1100^{15048} \approx 7.5 \times 10^{45766}$ different flip and flop phases of this base threefold. Moreover, we find that many other base threefolds with large $h^{1,1}$ in the 4d F-theory landscape can be constructed in a similar way as well.

KEYWORDS: Differential and Algebraic Geometry, F-Theory, Superstring Vacua

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1 Introduction

In recent years, there have been a lot of activities to determine the boundary of the string landscape, see [1] for a brief overview. In even space-time dimensions, F-theory compactification models [2–6] give rise to a large class of supersymmetric vacuum solutions. In fact, the F-theory geometric landscape has the largest known number of vacuum solutions, along with the largest gauge groups and the largest number of moduli fields on a particular geometry.

For 6d (1,0) supergravity from F-theory on an elliptic Calabi-Yau threefold X_3 , the set of base complex surfaces has been studied in [7–11]. Especially, the total number of 2d toric base surfaces is computed to be 61 539 [8], under the condition that the generic elliptic fibration over them does not have any non-flat fiber over toric points.¹ The different fibrations over each base have been explored in e.g. [12–19]. However, the total number of elliptic Calabi-Yau threefolds has not been estimated yet.

¹In the Weierstrass model $y^2 = x^3 + fx + g$, the condition is that (f, g) does not vanish to order (4, 6) or higher at toric points.

In particular, the elliptic Calabi-Yau threefold with $(h^{1,1}, h^{2,1}) = (491, 11)$ has the largest known number of Kähler moduli, along with the largest geometric gauge group:

$$G = E_8^{17} \times F_4^{16} \times G_2^{32} \times \mathrm{SU}(2)^{32} \quad (1.1)$$

in the known 6d (1,0) supergravity landscape. It also has the largest $h^{1,1}$ in the known set of compact Calabi-Yau threefolds [20].

For 4d $\mathcal{N} = 1$ supergravity from F-theory on an elliptic Calabi-Yau fourfold X_4 , even the set of toric base threefolds has not been fully classified. If we allow bases that only support non-flat fibration over complex curves, then the lower bound of such toric threefolds was proved to be $\frac{4}{3} \times 2.96 \times 10^{755}$ in [21], and further estimated to be $\sim 10^{3,000}$ in [22]. In [22], the notion “good base” was introduced to describe the subset of bases that support a flat fibration over complex curves. This fibration cannot have a complex three-dimensional fiber over a point either. In the language of Weierstrass polynomials, it is required that (f, g) do not vanish to order (4, 6) or higher on any complex curve, and (f, g) do not vanish to order (8, 12) or higher over any point. Such bases are convenient for the computation of $h^{1,1}(X_4)$, because the Tate-Shioda-Wazir formula

$$h^{1,1}(X_4) = h^{1,1}(B_3) + \mathrm{rank}(G) + 1 \quad (1.2)$$

can be applied. Physically, a good base can support a 4d supergravity description, without strongly coupled matter sectors localized on the non-minimal loci.² The lower bound of the total number of good bases is estimated to be $\sim 10^{48}$ for the bases with $h^{1,1} < 130$ [24] and $\sim 10^{250}$ for the bases with $h^{1,1} > 1000$ [22]. It was also found that the $h^{1,1}$ of such good bases are concentrated at certain discrete values. Hence it was postulated that non-trivial structures exist on this subset.

These large exponential numbers arise from the large number of flip and flop operations on the base threefold, which potentially leads to different matter curves and 4d low energy physics [24]. The number of possible flips and flops grows as $h^{1,1}$ of the base grows. Hence to get the largest number of bases, the natural point of interest is the base with the largest $h^{1,1}$, which supports X_4 with the largest $h^{1,1}$ as well. Among the known set of Calabi-Yau fourfolds [25–30], such an X_4 has Hodge numbers

$$(h^{1,1}, h^{2,1}, h^{3,1}) = (303\,148, 0, 252). \quad (1.3)$$

It was originally constructed as the Calabi-Yau hypersurface in the dual polytope of weighted projective space $\mathbb{P}^{1,1,84,516,1204,1816}$ [31], and the 4d F-theory gauge group on X_4 was read off by toric top methods:

$$G = E_8^{2\,561} \times F_4^{7\,576} \times G_2^{20\,168} \times \mathrm{SU}(2)^{30\,200}. \quad (1.4)$$

However, the smooth base threefold B_3 for X_4 with $h^{1,1}(B_3) = 181\,819$ has not been constructed yet. Physically, it is interesting because the number of axions in the F-theory

²In the definition of a good base, we allow points where (f, g) vanish to order (4, 6) or higher. These points can give rise to higher order Yukawa coupling terms [23].

model on X_4 is the largest in the known 4d $\mathcal{N} = 1$ string landscape [32]:

$$\begin{aligned} N(\text{axion}) &= h^{1,1}(B_3) + 1 \\ &= 181\,820. \end{aligned} \tag{1.5}$$

The construction of B_3 will be the main focus of this paper, which is discussed in section 2, with the following steps:

1. We start with a smooth non-compact toric threefold B_{E_8} with 2561 rays and 5016 3d cones. On each of these 2561 toric divisors, we tune a Kodaira type II^* singular fiber, which corresponds to E_8 gauge group in 4d F-theory. Then each of the 5016 3d cones give rise to an “ $E_8 - E_8 - E_8$ Yukawa point” [33]. There are also 7576 2d cones with 4d (E_8, E_8) conformal matter.
2. Then we add two more rays into B_{E_8} to make a smooth compact toric threefold B_{seed} , which is called the “seed” of B_3 .
3. We blow up each of the 5016 3d cones, such that the resulting base does not have any toric curves where the (f, g) vanish to order to $(4, 6)$ or higher, or any toric points where (f, g) vanish to order $(8, 12)$ or higher. We arrive at a toric base B_{toric} with $h^{1,1}(B_{\text{toric}}) = 181\,200$ after this step.
4. Finally, it can be checked that for a generic fibration over B_{toric} , there are 619 toric divisors with a non-Higgsable E_8 gauge group, such that (f, g) vanish to order $(4, 6)$ over a non-toric curve on each of these divisors. We then blow up these 619 non-toric curves and get the final B_3 with $h^{1,1}(B_3) = 181\,819$.

In section 3, we discuss some physical aspects of the 4d F-theory on X_4 , including an argument for the “saturation” of the number of each geometric gauge group on the base B_3 . We also estimated the number of self-dual G_4 flux choices: $10^{194\,000}$ on X_4 . This number is much smaller than the number $10^{224\,000}$ on the elliptic Calabi-Yau fourfold with largest $h^{3,1}$ [34].³

In section 4, we give a lower bound on the number of different smooth base threefolds that are related by a number of flips and flops. This bound is proven to be $1100^{15\,048} \approx 7.5 \times 10^{45\,766}$, which is much larger than any previous estimations in the literature. In particular, there exists a base that support a truly flat and smooth fibration X_4 , see the local structure in figure 7.

In section 5, we revisit the “end point” bases with $1000 < h^{1,1} < 13\,000$ studied in [22]. It is found that a number of these end point bases can be constructed from a seed base with E_8 gauge groups in a similar way. We also explain the approximate ratio between the number of each gauge group and $h^{1,1}(B_3)$, which is observed in [22], as well as an approximate formula

$$\frac{h^{1,1}(X_4)}{h^{1,1}(B_3)} \approx \frac{5}{3} \tag{1.6}$$

³Note that in [34], the number of $10^{272\,000}$ flux vacua is counted without applying the self-duality condition on G_4 . After the self-duality condition is imposed, the number of flux vacua is reduced to $10^{224\,000}$.

for the elliptic Calabi-Yau fourfolds with large $h^{1,1}$. The number of base flips and flops is lower bounded by the approximate formula

$$N(\text{flip}) \gtrsim 10^{0.253 \times h^{1,1}(B)}. \quad (1.7)$$

Finally, in section 6, we discuss the interpretation of large rank conformal matter coupled to gravity, as well as standard model building aspects.

2 Construction of the maximal base

In this section, we construct the non-toric base B_3 with $h^{1,1}(B_3) = 181\,819$ that supports the elliptic Calabi-Yau fourfold X_4 with

$$(h^{1,1}, h^{2,1}, h^{3,1})(X_4) = (303\,148, 0, 252). \quad (2.1)$$

For the discussions of toric threefold bases and 4d F-theory models over the base, we use the notations in the section 2 of [24].

In [31], the elliptic Calabi-Yau fourfold X_4 is constructed as the anticanonical hypersurface of the dual polytope of $\mathbb{P}^{1,1,84,516,1204,1806}$. Here we first perform an $\text{SL}(5, \mathbb{Z})$ rotation on $\mathbb{P}^{1,1,84,516,1204,1806}$, and get a polytope Δ_5 with the following vertices:

$$V(\Delta_5) = \{(0, 0, 0, -1, 1), (0, 0, 0, 2, -1), (1, 0, 0, -1, -1), (0, 1, 0, -1, -1), \\ (0, 0, 1, -1, -1), (-1, -84, -516, -1, -1)\}. \quad (2.2)$$

Its dual polytope Δ_5° has the following vertices:

$$V(\Delta_5^\circ) = \{(0, 0, 0, 1, 0), (0, 0, 0, 0, 1), (-6, -6, 1, -2, -3), (-6, 37, -6, -2, -3), \\ (-6, -6, -6, -2, -3), (3606, -6, -6, -2, -3)\}, \quad (2.3)$$

which has a structure of a $\mathbb{P}^{2,3,1}$ bundle fibered over a 3d polytope Δ_3° with vertices:

$$V(\Delta_3^\circ) = \{(-6, -6, 1), (-6, 37, -6), (-6, -6, -6), (3606, -6, -6)\}. \quad (2.4)$$

To construct a smooth toric base threefold, we only select the subset $S \subset \Delta_3^\circ$ of lattice points $v_i = (v_{i,x}, v_{i,y}, v_{i,z})$ with

$$\gcd(v_{i,x}, v_{i,y}, v_{i,z}) = 1, \quad (2.5)$$

since if there exists $p > 1$ with $p|v_{i,x}, v_{i,y}, v_{i,z}$, then any 3d cone containing v_i has volume greater than one, which breaks the smoothness condition. In total, there are $|S| = 181,203$ lattice points in this set, which correspond to the 1d rays of a compact toric threefold B_{toric} with

$$h^{1,1}(B_{\text{toric}}) = 181\,200. \quad (2.6)$$

In order to construct the list of 3d cones $\Sigma_3(B_{\text{toric}})$ of B_{toric} ,⁴ we first pick the subset $S_{E_8} \subset S$ that corresponds to the divisors supporting Kodaira type II^* singular fibers in

⁴Note that B_{toric} is not weak-Fano, and the list of 3d cones is not a triangulation of the full polytope Δ_3° .

X_4 (which carry E_8 geometric gauge groups in the 4d F-theory picture). To determine these rays, we consider the $\mathcal{F}$ and $\mathcal{G}$ polytope of B_{toric} , defined as:

$$\mathcal{F} = \{u \in \mathbb{Z}^3 | \forall v_i \in V(\Delta_3^\circ), \langle u, v_i \rangle \geq -4\}, \quad (2.7)$$

$$\mathcal{G} = \{u \in \mathbb{Z}^3 | \forall v_i \in V(\Delta_3^\circ), \langle u, v_i \rangle \geq -6\}. \quad (2.8)$$

Especially, the polytope $\mathcal{G}$ has vertices:

$$V(\mathcal{G}) = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -84, -516)\}. \quad (2.9)$$

The order of vanishing of the Weierstrass polynomials $f \in \mathcal{O}(-4K_{B_{\text{toric}}})$ and $g \in \mathcal{O}(-6K_{B_{\text{toric}}})$ on v_i are given by

$$\text{ord}_{v_i}(f) = \min_{u \in \mathcal{F}}(\langle u, v_i \rangle + 4), \quad (2.10)$$

$$\text{ord}_{v_i}(g) = \min_{u \in \mathcal{G}}(\langle u, v_i \rangle + 6). \quad (2.11)$$

There are in total 2561 $v_i \in S$ with $\text{ord}_{v_i}(f) = 4$, $\text{ord}_{v_i}(g) = 5$, which carries a type II^* singular fiber and non-Higgsable E_8 gauge group [7]. We define the set of these v_i to be S_{E_8} . More explicitly, this set includes the following lattice points:

$$\begin{aligned} S_{E_8} = \{ & (-1, 0, 0), (1, 0, 0), (m, -1, 0) \ (-1 \leq m \leq 85), \\ & (m, n, -1) \ (-1 \leq n \leq 6, -1 \leq m \leq 517 - 84n) \}. \end{aligned} \quad (2.12)$$

Note that there does not exist a point in S_{E_8} on the $z > 0$ half-plane, hence the rays of S_{E_8} only form a non-compact toric threefold B_{E_8} . Nonetheless, there exists a triangulation of the convex hull of S_{E_8} , which gives rise to the set $\Sigma_3(B_{E_8})$ of 5016 3d cones with unit volume (not including the $z = 0$ plane). The intersection of 3d cones in $\Sigma_3(B_{E_8})$ gives the set $\Sigma_2(B_{E_8})$ of 7576 2d cones. We present the detailed list of the lattice points and 3d cones in the supplementary material as a Mathematica file (it can also be found in ref. [35]).

As B_{E_8} is still non-compact, we add two more rays

$$v_{2562} = (-6, -6, 1), \ v_{2563} = (-6, 37, -6) \quad (2.13)$$

into the toric fan of B_{E_8} . We also add the following 3d cones

$$\begin{aligned} & \{(1, 0, 0), (-1, -1, 0), (-6, -6, 1)\}, \{(m, -1, 0), (m+1, -1, 0), (-6, -6, 1)\} \ (-1 \leq m \leq 84), \\ & \{(-1, 0, 0), (-6, -6, 1), (-6, 37, -6)\}, \{(-1, 0, 0), (-1, 6, -1), (-6, 37, -6)\}, \\ & \{(m, 6, -1), (m+1, 6, -1), (-6, 37, -6)\} \ (-1 \leq m \leq 12), \{(13, 6, -1), (1, 0, 0), (-6, 37, -6)\} \\ & \{(1, 0, 0), (-6, 37, -6), (-6, -6, 1)\}. \end{aligned} \quad (2.14)$$

With these 2563 rays and the 3d cones, we define a compact toric threefold denoted as the “seed” threefold B_{seed} . Note that B_{seed} does not correspond to a reflexive polytope, and it is not weak Fano. Hence it can have a much larger $h^{1,1}$ than the ones discussed in [21, 36]. From the 4d F-theory perspective, we tune 2561 type II^* singular fibers (E_8 gauge groups) on the rays in the set S_{E_8} . Then on the base B_{seed} , there are 5016 toric points where (f, g) vanishes to order equal or higher than $(8, 12)$ ($(8, 12)$ -points) and 7576 toric curves where

(f, g) vanishes to order equal or higher than $(4, 6)$ ($(4, 6)$ -curves). Thus the elliptic fibration is not flat, and we need to perform a sequence of base blow-ups. Note that the blow-up of $(4, 6)$ -curve or $(8, 12)$ -point does not change the number of complex structure moduli of the elliptic Calabi-Yau fourfold over it, because the set of Weierstrass monomials is unchanged after the blow up. In [33], this type of base structure was called an “ $E_8 - E_8 - E_8$ Yukawa point” along with 4d (E_8, E_8) conformal matter, and the base blow-up sequence was already constructed. We will use such a blow-up procedure in this section.

For each of the 5016 3d cones $\sigma_{i,3} = v_{i_1}v_{i_2}v_{i_3} \in \Sigma_3(B_{E_8})$, we denote the linear combination $av_{i_1} + bv_{i_2} + cv_{i_3}$ by abc . The blow-up of a toric point $v_{i_1}v_{i_2}v_{i_3}$ is then denoted by $(100, 010, 001; 111)$, and the blow-up of a toric curve $v_{i_1}v_{i_2}$ is denoted by $(100, 010; 110)$. We perform the following sequences of blow-ups:

$$\begin{aligned} Blp_1 = \{ & (100, 010, 001; 111), (100, 010, 111; 221), (100, 001, 111; 212), \\ & (010, 001, 111; 122), (100, 111; 211), (010, 111; 121), (001, 111; 112), \\ & (100, 211; 311), (010, 121, 131), (001, 112, 113) \\ & (100, 311; 411), (010, 131; 141), (001, 113; 114) \}, \end{aligned} \quad (2.15)$$

$$\begin{aligned} Blp_2 = \{ & (100, 010; 110), (100, 110; 210), (010, 110; 120), (110, 210; 320), \\ & (110, 120; 230), (100, 210; 310), (100, 310; 410), (100, 410; 510), \\ & (010, 120; 130), (010, 130; 140), (010, 140; 150), (100, 001; 101), \\ & (100, 101; 201), (001, 101; 102), (101, 201; 302), (101, 102; 203), \\ & (100, 201; 301), (100, 301; 401), (100, 401; 501), (001, 102; 103), \\ & (001, 103; 104), (001, 104; 105), (010, 001; 011), (010, 011; 021), \\ & (001, 011; 012), (011, 021; 032), (011, 012; 023), (010, 021; 031), \\ & (010, 031; 041), (010, 041; 051), (001, 012; 013), (001, 013; 014), \\ & (001, 014; 015) \}, \end{aligned} \quad (2.16)$$

$$\begin{aligned} Blp_3 = \{ & (100, 221; 321), (100, 212; 312), (010, 221; 231), (010; 122; 132), \\ & (001, 212; 213), (001, 122; 123) \}. \end{aligned} \quad (2.17)$$

The final cones from blowing up σ_3 is shown in figure 1, which is the dual graph of figure. 8 in [33]. There is no toric $(8, 12)$ point or toric $(4, 6)$ curve after the blow-up sequence.

The Kodaira singular fiber type and geometric non-Higgsable gauge groups on each toric divisor are given by (we also labelled the type II singular fiber, which does not have a non-Higgsable gauge group):

$$\begin{aligned} II^*, E_8 : & 100, 010, 001 \\ IV_{ns}^*, F_4 : & 110, 101, 011 \\ I_{0,ns}^*, G_2 : & 111, 210, 120, 201, 102, 021, 012 \\ IV_{ns}, SU(2) : & 211, 121, 112, 310, 130, 103, 301, 031, 013 \\ II : & 410, 320, 230, 140, 041, 032, 023, 014, 104, 203, 302, 401, 221, 311, 212, 113, 122, 131. \end{aligned} \quad (2.18)$$

Figure 1. The final 3d cones after blowing up the 3d cone $v_1v_2v_3$, where there are three E_8 geometric gauge groups on v_i . Each vertex abc denotes an 1d ray $av_1 + bv_2 + cv_3$. It can be checked that all the 3d cones have unit volume if the original cone $v_1v_2v_3$ has unit volume. The geometric non-Higgsable gauge groups are also labelled on each vertex. We also label the Kodaira type II singular fiber on the divisors.

The order of vanishing of g over each divisor is given by the following table:

	$\emptyset$	II	$SU(2)$	G_2	F_4	E_8
$\text{ord}(g)$	0	1	2	3	4	5

(2.19)

Note that the sequence of non-Higgsable gauge groups on each edge v_1v_2 , v_1v_3 , v_2v_3 are exactly the same as the tensor branch of 6d minimal (E_8, E_8) conformal matter [37]:

$$E_8 - \emptyset - II - SU(2) - G_2 - II - F_4 - II - G_2 - SU(2) - II - \emptyset - E_8. \quad (2.20)$$

Finally, we get the smooth toric threefold B_{toric} after all the 5016 3d cones are blown up in this way. More precisely, if all the 5016 3d cones are blown up into figure 1, one

needs to first perform the blow-ups (2.15) for all of these 3d cones. Then one performs the blow-ups (2.16) for all the 7576 2d cones. Finally, one performs the blow-ups (2.17) for all the 5016 3d cones again. Note that the cones in (2.14) are also subdivided in the process. We provide the full list of 181 203 1d rays and 362 402 3d cones of B_{toric} in the supplementary material as a Mathematica file (it can also be found in ref. [35]).

The total numbers of non-Higgsable gauge groups are computed as follows. There is a single F_4 on each of the 7576 edges on B_{E_8} , so the total number of F_4 is 7,576. For G_2 , there are two of them on each of the 7576 edges, and one in each of the 5016 3d cones of B_{E_8} , hence its total number is 20 168. For $SU(2)$, there are two of them on each of the 7576 edges, and three of them in each of the 5016 3d cones of B_{E_8} , hence its total number is 30 200.

On B_{toric} , there are still a number of non-toric $(4,6)$ -curves located on E_8 divisors, which needs to be blown up. They are analogous to the $(4,6)$ -points on the $(-9)/(-10)/(-11)$ curves, in the elliptic Calabi-Yau threefold cases [7, 8]. Given a ray v_i with a non-Higgsable E_8 gauge group, we construct the set

$$\mathcal{G}_5(v_i) = \{u \in \mathbb{Z}^3 | \langle u, v_i \rangle + 6 = 5\}, \quad (2.21)$$

which corresponds to the monomials in the polynomial $g_5(v_i)$ of the expansion of the Weierstrass polynomial g :

$$g = g_5(v_i)z_i^5 + \mathcal{O}(z_i^6). \quad (2.22)$$

Here $z_i = 0$ is the local hypersurface equation of the divisor corresponding to v_i . If there is more than one monomial in $g_5(v_i)$, then the equation

$$g_5(v_i) = z_i = 0 \quad (2.23)$$

defines a $(4,6)$ -curve on the base B_{toric} , which needs to be blown up to get a fully flat elliptic fibration. On the other hand, if $g_5(v_i)$ only has a single monomial, then it has to be a constant complex number, as all the toric $(4,6)$ -curves are already blown up. In total, there are 619 different v_i s with $|\mathcal{G}_5(v_i)| > 1$, and we blow up the corresponding 619 $(4,6)$ -curves (which are all irreducible). Note that the locations of these non-toric curves depend on the coefficients of f and g , which correspond to the complex structure moduli of the elliptic Calabi-Yau fourfold.

After these non-toric blown ups, we get the non-toric base threefold B_3 with

$$\begin{aligned} h^{1,1}(B_3) &= h^{1,1}(B_{\text{toric}}) + 619 \\ &= 181\,819. \end{aligned} \quad (2.24)$$

The generic elliptic fibration over B_3 has no codimension-two non-flat fiber. However, there are still codimension-three non-flat fiber over the points where (f, g) vanish to order $(4,6)$ or higher. In figure 1, such points locate at the intersection of three divisors with $F_4(IV_{ns}^*) - II - II$ and $G_2(I_{0,ns}^*) - SU(2) - II$ singular fiber. Similar to the codimension-three non-flat fibers studied in the literature [23, 38–42], they will potentially lead to a tower of massless states and new Yukawa coupling terms. It is also notable that because of the $II - II$ collisions, the elliptic Calabi-Yau fourfold will have terminal singularities after the crepant resolution.

Nonetheless, in section 4, we will show that there exists another configuration of 3d cones, such that the aforementioned loci are all absent. Thus it is possible to construct a smooth threefold base that supports a flat fibration X_4 .

3 4d F-theory on X_4

3.1 X_4 and the physical fields

The 4d axions in the supergravity theories are given by the imaginary part of the Kähler moduli for the divisor classes on B_3 , along with the reduction of 10d axiodilaton. The total number of axion fields is given by [32]:

$$\begin{aligned} N(\text{axion}) &= h^{1,1}(B_3) + 1 \\ &= 181\,820. \end{aligned} \quad (3.1)$$

As X_4 is a generic elliptic fibration over B_3 , the geometric non-Higgsable gauge groups are already given in section 2:

$$G_{\text{nH}} = E_8^{2\,561} \times F_4^{7\,576} \times G_2^{20\,168} \times \text{SU}(2)^{30\,200}. \quad (3.2)$$

The Hodge number $h^{1,1}(X_4)$ can be verified as:

$$\begin{aligned} h^{1,1}(X_4) &= h^{1,1}(B_3) + \text{rank}(G_{\text{nH}}) + 1 \\ &= 303\,148. \end{aligned} \quad (3.3)$$

Finally, there are also a number of D3-branes in the 4d F-theory picture that can carry Abelian and non-Abelian gauge groups. The total number N_{D_3} is bounded by the tadpole cancellation equation in the M-theory dual picture:⁵

$$N_{D_3} + \frac{1}{2} \int_{X_4} G_4 \wedge G_4 = \frac{\chi(X_4)}{24}. \quad (3.4)$$

$\chi(X_4)$ is the Euler characteristic of X_4 :

$$\begin{aligned} \chi(X_4) &= 6(8 + h^{1,1}(X_4) + h^{3,1}(X_4) - h^{2,1}(X_4)) \\ &= 1\,820\,448. \end{aligned} \quad (3.5)$$

The self-dual G_4 flux satisfies

$$\int_{X_4} G_4 \wedge G_4 \geq 0, \quad (3.6)$$

hence we have

$$N_{D_3} \leq \frac{\chi(X_4)}{24} = 75\,852. \quad (3.7)$$

⁵We assume that the M-theory effective action can be approximated by the lowest order terms. It is possible that eight or higher derivative terms are significant, which breaks this assumption [43].

3.2 Saturation of gauge groups

On B_3 , one cannot tune any larger non-Abelian gauge group on any toric divisor, because of the “conformal matter” structure in figure 1. Namely, the line (100,010), (100,001) and (010,001) has a sequence of gauge groups (2.20) that is the same as the tensor branch of 6d minimal (E_8, E_8) conformal matter. The line (100,011), (010,101) and (001,110) also has the structure of 6d minimal (E_8, F_4) conformal matter. Then any further tuning of gauge groups on a toric divisor will lead to additional toric $(4, 6)$ -curves on the base, which need to be blown up. For example, if we tune any non-Abelian gauge group on the divisor labelled by 131, then the curve (010,141) would be a $(4, 6)$ -curve. For the points $v_{2562} = (-6, -6, 1)$ and $v_{2563} = (-6, 37, -6)$ that are not in B_{E_8} , they also intersect divisors D_i with E_8 . Hence the presence of non-Abelian gauge group on these divisors would lead to toric $(4, 6)$ -curves $D_i \cdot D_{2562}$ and $D_i \cdot D_{2563}$ as well.

However, one can check that the base B_3 (and B_{toric}) has an “end point” property (see [22]). Namely, any further toric blow-up leads to an invalid base with codimension-one locus with $\text{ord}(f, g) \geq (4, 6)$, for any fibration on the base. Hence we conclude that it is impossible to tune any larger non-Abelian gauge groups on the toric divisors.

For the non-toric divisors on B_3 , they either lie in the interior of the effective cone of B_3 or intersect a divisor with non-Higgsable E_8 gauge group. Thus it is expected that the tuning of any non-Abelian gauge group on these non-toric divisors would remove monomials in f and g and lead to a codimension-one $(4, 6)$ locus as well. Similarly, if one tunes Abelian gauge groups generated by a non-trivial rational sections, it is also necessary to remove a number of monomials in f and g [12, 44]. Hence we conjecture that any tuning of Abelian or non-Abelian gauge groups would lead to an invalid base with codimension-one $(4, 6)$ loci.

Moreover, if one conjectures that the $h^{1,1}$ of elliptic Calabi-Yau fourfold (or even Calabi-Yau fourfold) is bounded by 303 148, then any tuning of gauge groups on B_3 will exceed this bound. It is very interesting but difficult to prove this bound mathematically.

3.3 Number of flux choices

In [34], the total number of G_4 flux choices on the Calabi-Yau fourfold $\mathcal{M}_{\text{max}}$ with largest $h^{3,1}$ has been estimated following the logics of [45–48]. Namely, the different self-dual G_4 flux choices on a Calabi-Yau fourfold X_4 can be thought as lattice points in a sphere with radius $\sqrt{2Q}$ and dimension $b_4/2$, where

$$Q = \frac{\chi(X_4)}{24} \quad (3.8)$$

and

$$\begin{aligned} b_4 &= 2 + 2h^{3,1} + h^{2,2} \\ &= 4h^{1,1} - 2h^{2,1} + 6h^{3,1} + 46 \end{aligned} \quad (3.9)$$

is the fourth Betti number of X_4 .

For $\mathcal{M}_{\text{max}}$ with

$$(h^{1,1}, h^{2,1}, h^{3,1}) = (252, 0, 303\,148), \quad (3.10)$$

the Euler characteristic is $\chi = 1\,820\,448$ and $b_4 = 1\,819\,942$. While for the X_4 with largest $h^{1,1}$, with

$$(h^{1,1}, h^{2,1}, h^{3,1}) = (303\,148, 0, 252), \quad (3.11)$$

it has the same Euler characteristic as $\mathcal{M}_{\max}$, but a smaller $b_4 = 1\,214\,150$. Thus we expect that the total number of G_4 flux choices on X_4 is much smaller than that on $\mathcal{M}_{\max}$.

More explicitly, the counting of lattice points can be computed by [49]:

$$N(b_4/2, Q) = \frac{1}{2\pi i} \int \frac{dt}{t} e^{-Qt} \vartheta_3(0, e^{t/2})^{b_4/2}, \quad (3.12)$$

where the integration goes from $i\infty$ to $-i\infty$, and ϑ_3 is the Jacobi Theta function. We use a saddle point approximation $N(b_4/2, Q) \approx e^{S(t_*)}$, where t_* is the critical point of the function

$$S(t) = -\ln(-t) - Qt + \frac{b_4}{2} \ln \vartheta_3(0, e^{t/2}). \quad (3.13)$$

In our case, $b_4 \approx 16Q$, hence the critical point is

$$t_* \approx -3.63, \quad (3.14)$$

and we can estimate

$$N(b_4/2, Q) \approx 10^{2.56Q} \approx 10^{194\,000}. \quad (3.15)$$

This number is much smaller than the estimated self-dual flux choices on $\mathcal{M}_{\max}$, which is $10^{224\,000}$ [34].

Note that we have not taken into account the non-trivial metric on the space of self-dual G_4 flux,⁶ and we have not computed the number of flux vacua associated to each flux choice either. Again, if the higher derivative terms with G_4 in the M-theory effective action are included, the number is going to be corrected.

4 Flip and flop phases

In this section, we give a lower bound on the total number of topologically different smooth bases that are related to B_3 by a sequence of toric flips and flops. Such a local operation leads to a different set of 3d cones, while the set of 1d rays remains the same, see figure 2. In general, it is required that the four 1d rays involved satisfy $av_1 + bv_3 = cv_2 + dv_4$, ($a, b, c, d \in \mathbb{Z}$). In particular, this operation is a flop if and only if $a = b = c = d = 1$. In this section, we simply consider the flips and flops of the toric base B_{toric} , and then perform the 619 blow-ups along the non-toric curves.

First, note that in figure 1, the divisors on the three (E_8, E_8) edges are completely fixed. This means that we can consider the flips and flops inside such an (E_8, E_8, E_8) triangle, and any set of 3d cones will be compatible with the other adjacent (E_8, E_8, E_8) triangles (the philosophy is similar to the counting in [21]). Moreover, we subdivide the triangle $(100, 010, 001)$ into six subsets: the three triangles

⁶It was shown that in certain cases of CY3 and CY4, the volume of moduli space can be significantly smaller than the naive estimation, which leads to a smaller number of flux vacua [50].

Figure 2. The toric flip (flop) operation. For a flop, it is required that $v_1 + v_3 = v_2 + v_4$.

Figure 3. All the possible sets of 3d cones of the triangle $(010, 110, 221)$, which is a subset of the full triangle in figure 1.

$(100, 010, 221)$, $(010, 001, 122)$, $(100, 001, 212)$ and the three polygons $(100, 221, 010, 111)$, $(100, 111, 001, 212)$, $(010, 111, 001, 122)$. We can individually evaluate the number of different sets of 3d cones in each of these subsets, and multiply these numbers together to get a lower bound on the total number of different base configurations within a single (E_8, E_8, E_8) triangle.

For the smaller triangle $(100, 010, 221)$, it can be further subdivided into two identical triangles $(010, 110, 221)$ and $(110, 100, 221)$. We plot all the five possible configurations of 3d cones of the triangle $(010, 110, 221)$ in figure 3. Note that the map from the bottom left to the bottom right configuration is a combination of two toric flips, instead of a simple toric flop. One can check that the volume of each 3d cone always equal to one, thus the base threefold is always smooth. Since there are in total six identical triangles of this shape in the full triangle $(100, 010, 001)$, there are 5^6 different 3d cone configurations if the vertices 100, 010 and 001 are taken to be inequivalent.

For the polygon $(100, 221, 010, 111)$, we also plot 44 possible configurations of 3d cones in figure 6. One can again check that the volume of each 3d cone is always equals to one. As there are three identical polygons of this shape in the full triangle $(100, 010, 001)$, they contribute to 44^3 different 3d cone configurations. Finally, the total number of 3d cone

Figure 4. The 44 sets of 3d cones of the polygon $(100, 221, 010, 111)$, which is a subset of the full triangle in figure 1.

configurations within one of the (E_8, E_8, E_8) triangle is lower bounded by

$$\begin{aligned} N_{\text{flip, single}} &= 44^3 \times 5^6 \\ &= 1100^3. \end{aligned} \quad (4.1)$$

For the whole base B_{toric} with 5016 (E_8, E_8, E_8) triangles, every toric ray is considered as inequivalent since there is no reflexive toric automorphism. Thus the lower bound of the different flip and flop phases of B_{toric} is given by

$$\begin{aligned} N_{\text{flip}}(B_{\text{toric}}) &= N_{\text{flip, single}}^{5016} \\ &= 1100^{15\,048} \\ &\approx 7.5 \times 10^{45\,766}. \end{aligned} \quad (4.2)$$

Figure 5. The 44 sets of 3d cones of the polygon $(100, 221, 010, 111)$, which is a subset of the full triangle in figure 1. (*cont.*)

It is amusing that after we multiply this number by the estimation of self-dual flux choices in (3.15), we get a number $\approx 10^{240\,000}$. It is bigger than the estimated number of self-dual flux choices on $\mathcal{M}_{\text{max}}$, which is $10^{224\,000}$ [34]. We will briefly comment on the standard model building aspects of this model in section 6.2.

It is also notable that if one chooses the bottom right configuration of 3d cones in figure 3 and figure 6 for all these polygons, then the base supports a flat and smooth fibration X_4 . One can check that all the codimension-three (4,6) loci and $II-II$ collisions are absent. We plot the subdivision of the (E_8, E_8, E_8) triangle in figure 7. It is possible to directly generate this set of 3d cones from blowing up the triangle (100,010,001):

$$Blp_1 = \{(100, 010, 001; 111)\}, \quad (4.3)$$

$$Blp_2 = \{(100, 010; 110), (100, 001; 101), (010, 001; 011)\}, \quad (4.4)$$

$$Blp_3 = \{(010, 110, 111; 231), (100, 110, 111; 321), (100, 101, 111; 312), (001, 101, 111; 213),$$

Figure 6. The 44 sets of 3d cones of the polygon $(100, 221, 010, 111)$, which is a subset of the full triangle in figure 1. (*cont.*)

$$\begin{aligned}
 & (001, 011, 111; 123), (010, 011, 111; 132), (110, 111; 221), (101, 111; 212), \\
 & (011, 111; 122), (010, 111; 121), (010, 121; 131), (010, 131; 141), (100, 111; 211), \\
 & (100, 211; 311), (100, 311; 411), (001, 111; 112), (001, 112; 113), (001, 113; 114)\}, \quad (4.5)
 \end{aligned}$$

$$\begin{aligned}
 Blp_4 = \{ & (100, 110; 210), (110, 210; 320), (100, 210; 310), (100, 310; 410), (100, 410; 510), \\
 & (010, 110; 120), (110, 120; 230), (010, 120; 130), (010, 130; 140), (010, 140; 150), \\
 & (100, 101; 201), (101, 201; 302), (100, 201; 301), (100, 301; 401), (100, 401; 501), \\
 & (001, 101; 102), (101, 102; 203), (001, 102; 103), (001, 103; 104), (001, 104; 105), \\
 & (010, 011; 021), (011, 021; 032), (010, 021; 031), (010, 031; 041), (010, 041; 051), \\
 & (001, 011; 012), (011, 012; 023), (001, 012; 013), (001, 013; 014), (001, 014; 015)\}. \quad (4.6)
 \end{aligned}$$

Figure 7. The configuration of 3d cones in an (E_8, E_8, E_8) triangle, such that all the codimension-three non-minimal loci and $II-II$ collision are absent. Each vertex abc denotes an 1d ray $av_1 + bv_2 + cv_3$. It can be checked that all the 3d cones have unit volume if the original cone $v_1v_2v_3$ has unit volume. The geometric non-Higgsable gauge groups and the Kodaira type II singular fiber are labelled.

To construct the full B_{toric} from B_{seed} , one needs to perform (4.3) for all the 5016 3d cones. Then one perform (4.4) for all the 7576 2d cones. Consequently, one performs (4.5) for all the 5016 3d cones again. Finally, one performs (4.6) for all the 7576 2d cones again.

5 Other base threefolds

5.1 End point bases and mirror pairs

In [22], the set of toric base threefolds is probed by random blow-up sequences from a starting point base, such as $\mathbb{P}^3$. The random blow-up sequences terminate at an “end point” base, where any further toric blow-up would lead to an invalid base with codimension-one (4,6) loci in the generic fibration. It is found that the $h^{1,1}$ of end point bases are

concentrated at certain numbers. For example, about 10% of the random blow-up sequences from $\mathbb{P}^3$ end up with toric bases with $h^{1,1}(B_{\text{toric}}) = 1943$.

In this paper, we show that a number of end point bases in [22] can be constructed by blowing up compact toric “seed bases” B_{seed} with tuned E_8 on the toric divisors. For any 3d cones with E_8 on all the three 1d rays, the structures of rays and cones can be chosen as figure 1. Then one can perform flips and flops to get more topologically distinct bases, as in section 4.

For example, we can start with a weak Fano toric threefold B_{seed} with the following 29 rays:

$$\begin{aligned} \{v_i\} = \{ & (0, 0, 1), (0, 1, 0), (1, 0, 0), (-1, -1, -1), (1, 1, 1), (0, 1, 1), (-1, 0, 1), (2, 2, 1), \\ & (0, 0, -1), (1, 0, 1), (1, -1, 0), (0, 1, -1), (0, -2, -1), (-1, 0, -1), (1, 2, 1), \\ & (-2, -1, 0), (-3, -2, -1), (2, 0, 1), (-1, -1, 0), (0, -1, 0), (2, 1, 1), (1, 1, 0), \\ & (-2, -2, -1), (0, -1, -1), (1, 2, 0), (-1, -2, -1), (-1, 0, 0), (-2, -1, -1), (2, 3, 1) \}. \end{aligned} \quad (5.1)$$

We can choose the following set of 54 3d cones (where a number i denotes the ray v_i):

$$\begin{aligned} \{\sigma_3\} = \{ & (13, 4, 26), (26, 19, 13), (10, 5, 18), (22, 25, 12), (25, 2, 12), (8, 5, 15), \\ & (9, 3, 12), (12, 3, 22), (8, 25, 22), (20, 19, 1), (9, 4, 24), (27, 14, 2), \\ & (10, 1, 5), (10, 18, 11), (18, 3, 11), (8, 22, 21), (21, 5, 8), (25, 29, 15), \\ & (15, 29, 8), (8, 29, 25), (28, 17, 23), (28, 16, 17), (17, 16, 23), (28, 14, 16), \\ & (4, 14, 28), (13, 20, 11), (11, 24, 13), (16, 14, 27), (11, 1, 10), (5, 1, 6), \\ & (11, 3, 24), (24, 3, 9), (20, 1, 11), (9, 14, 4), (22, 3, 21), (18, 5, 21), \\ & (21, 3, 18), (2, 14, 12), (12, 14, 9), (15, 5, 6), (6, 2, 15), (15, 2, 25), \\ & (24, 4, 13), (13, 19, 20), (19, 16, 7), (7, 16, 27), (19, 7, 1), (1, 7, 6), \\ & (6, 7, 2), (2, 7, 27), (28, 23, 4), (4, 23, 26), (23, 16, 26), (26, 16, 19) \}. \end{aligned} \quad (5.2)$$

One can check that the number of 2d cones is 81. After each of the 54 3d cones are blown up (for example according to figure 1), there are 19 new rays in the interior of each of 54 3d cones. On each of the 81 2d cones, there are 11 new rays. In total, we count the number of 1d rays in the blown up base B_{toric} :

$$N_1(B_{\text{toric}}) = 29 + 54 \times 19 + 81 \times 11 = 1946. \quad (5.3)$$

Thus $h^{1,1}(B_{\text{toric}}) = 1943$ exactly. The non-Higgsable gauge group on B_{toric} can be counted as:

$$G_{\text{nH}} = E_8^{29} \times F_4^{81} \times G_2^{216} \times \text{SU}(2)^{324}. \quad (5.4)$$

After the toric blow-ups, there are still 21 E_8 divisors with non-toric $(4, 6)$ -curves. Again this can be checked from the fact that the $\mathcal{G}_5$ polytopes (2.21) associated to these E_8 divisors have more than one lattice point. In this case, we can also check that these $(4, 6)$ -curves are all irreducible. From B_{toric} to B_3 , one then needs to blow up these non-toric curves. Hence the total $h^{1,1}(X_4)$ is

$$\begin{aligned} h^{1,1}(X_4) &= h^{1,1}(B_{\text{toric}}) + \text{rank}(G_{\text{nH}}) + 21 + 1 \\ &= 3277. \end{aligned} \quad (5.5)$$

It was also observed in [22] that this X_4 has exactly the mirror Hodge number of the generic elliptic CY4 over $\mathbb{P}^1 \times \mathbb{P}^2$, which has $(h^{1,1}, h^{3,1}) = (3, 3277)$.⁷ Here we further observe that the convex hull of rays in (5.1) is exactly the dual polytope of $\mathbb{P}^1 \times \mathbb{P}^2$, after an $\text{SL}(3, \mathbb{Z})$ rotation. A similar phenomenon has been observed in the case of elliptic CY3 [52]. In fact, the base for the elliptic CY3 with $(h^{1,1}, h^{2,1}) = (272, 2)$ can be generated by blowing up a 2d seed base with the following nine rays:

$$\{v_i\} = \{(1, 0), (0, 1), (-1, 2), (-1, 1), (-1, 0), (-1, -1), (0, -1), (1, -1), (2, -1)\}. \quad (5.6)$$

Its dual polytope form the toric fan of a $\mathbb{P}^2$. After one tune nine E_8 gauge groups on these rays and blow up all the (E_8, E_8) point into the full tensor branch, one gets the base in figure 2 of [52]. Then one needs to blow up the three (-11) -curves corresponding to rays $(-1, 2)$, $(-1, -1)$ and $(2, -1)$ to get a non-toric base that supports a flat fibration.

Note that the seed base for $h^{1,1}(B_{\text{toric}}) = 1943$ is not unique. One can also start with a base with the same number of rays

$$\begin{aligned} \{v_i\} = \{ & (0, 0, 1), (0, 1, 0), (1, 0, 0), (-1, -1, -1), (0, 1, 1), (1, 1, 1), (2, 1, 1), \\ & (0, -1, -1), (-1, 0, 0), (-1, 1, 1), (2, 1, 2), (-1, -1, 0), (1, 1, 0), (-1, 0, -1), \\ & (-1, -2, -1), (0, 1, 2), (2, 1, 0), (-1, 1, 2), (1, 1, 2), (1, 0, 1), (0, 0, -1), (0, 1, -1), \\ & (-1, 0, 1), (-1, 1, -1), (1, 0, -1), (1, 1, -1), (-1, 1, 0), (2, 1, -1), (0, -1, 0) \} \end{aligned} \quad (5.7)$$

and 3d cones:

$$\begin{aligned} \{\sigma_3\} = \{ & (2, 10, 27), (10, 18, 23), (23, 18, 1), (6, 2, 13), (1, 19, 20), (13, 28, 17), \\ & (28, 3, 17), (13, 2, 26), (3, 29, 20), (20, 29, 1), (17, 6, 13), (1, 18, 16), \\ & (16, 18, 5), (19, 11, 20), (19, 6, 11), (11, 7, 20), (11, 6, 7), (7, 3, 20), \\ & (7, 6, 17), (17, 3, 7), (25, 28, 26), (25, 3, 28), (26, 28, 13), (9, 12, 4), \\ & (5, 18, 10), (19, 5, 6), (6, 5, 2), (29, 12, 1), (1, 12, 23), (23, 12, 9), \\ & (27, 10, 9), (14, 21, 22), (5, 10, 2), (27, 24, 2), (27, 9, 24), (9, 14, 24), \\ & (24, 22, 2), (24, 14, 22), (10, 23, 9), (29, 15, 12), (12, 15, 4), (4, 15, 8), \\ & (8, 15, 29), (1, 16, 19), (19, 16, 5), (25, 8, 3), (25, 21, 8), (26, 21, 25), \\ & (2, 22, 26), (22, 21, 26), (4, 14, 9), (4, 21, 14), (3, 8, 29), (8, 21, 4) \}. \end{aligned} \quad (5.8)$$

After we tune E_8 on each of the 29 rays and perform the toric blow-ups, the resulting base is also an end point base with $h^{1,1}(B_{\text{toric}}) = 1943$. Nonetheless, the convex hull of (5.7) is different from (5.1). In fact, the dual polytope of (5.7) has vertices

$$\{v_i^\circ\} = \{(1, 0, 0), (0, -1, 0), (-1, 1, 0), (0, 0, 1), (0, 1, -1)\}, \quad (5.9)$$

which is the toric rays of a twisted $\mathbb{P}^2$ fibered over $\mathbb{P}^1$.

For the end points with $h^{1,1}(B_{\text{toric}}) = 1727, 2015, 2303$ and 2591 , we have checked that they can be generated by a simple seed base as well. Note that for $h^{1,1}(B_{\text{toric}}) = 2591$, the seed base is exactly given by the maximal reflexive polytope in [21, 36]. The details of the seed bases will be presented in appendix A, and they form mirror pairs in a similar way.

⁷For the computations of Hodge numbers of elliptic CY4 over simple bases, also see [25, 51].

5.2 Seed bases with F_4

For other end point bases in [22], they may be generated in a similar way with different seed bases. In general, one can pick an end point base with large $h^{1,1}$, and select the 1d rays that carry non-Higgsable E_8 . Such rays will naturally form the rays of the toric seed base, and the list of 3d cones can be generated by the triangulation of the convex hull of these rays. However, one need to check if the computed $N_1(B_{\text{toric}})$ after the blow-ups matches the correct number of 1d ray. If they do not match, then the seed base needs to include more rays with other non-Higgsable gauge groups as well.

For example, the class of 3d bases with $h^{1,1}(B_{\text{toric}}) = 2249$ cannot be generated from a toric seed base with only E_8 non-Higgsable gauge groups. Nonetheless, we can start with a seed base B_{seed} with the following 34 rays:

$$\begin{aligned} \{v_i\} = \{ & (0, 0, 1), (0, 1, 0), (1, 0, 0), (-1, -1, -1), (0, 1, 1), (0, 0, -1), (1, 1, 1), \\ & (-1, 0, -2), (-2, -1, -3), (1, 2, 3), (2, 4, 5), (0, -1, -1), (-2, -1, -4), (1, 2, 2), \\ & (1, -1, -1), (1, 1, 2), (0, -1, 0), (2, 1, 2), (-1, -1, -2), (2, 3, 4), (0, -1, -2), \\ & (1, 3, 3), (-1, -1, -3), (1, 0, 1), (0, 2, 1), (-3, -1, -5), (-2, 0, -3), (2, 0, 1), \\ & (1, -1, 0), (2, -1, 0), (-1, 0, -1), (2, 2, 3), (-1, 1, -1), (-1, -2, 0)\} \end{aligned} \quad (5.10)$$

and the 64 3d cones

$$\begin{aligned} \{\sigma_3\} = \{ & (14, 7, 18), (23, 13, 9), (27, 13, 23), (11, 20, 10), (11, 22, 20), (10, 22, 11), \\ & (3, 30, 28), (28, 30, 24), (16, 28, 24), (15, 21, 12), (15, 6, 21), (10, 32, 18), \\ & (18, 32, 14), (10, 5, 25), (12, 29, 15), (12, 4, 17), (10, 18, 16), (25, 5, 33), \\ & (33, 2, 25), (15, 3, 6), (14, 32, 22), (24, 30, 29), (23, 8, 27), (10, 25, 22), \\ & (22, 25, 14), (4, 9, 27), (1, 4, 31), (23, 9, 19), (3, 28, 7), (29, 30, 15), \\ & (15, 30, 3), (10, 1, 5), (16, 1, 10), (22, 32, 20), (20, 32, 10), (27, 9, 26), \\ & (9, 13, 26), (26, 13, 27), (17, 29, 12), (25, 2, 7), (31, 4, 27), (14, 25, 7), \\ & (7, 28, 18), (18, 28, 16), (33, 8, 6), (5, 1, 33), (1, 31, 33), (6, 2, 33), \\ & (24, 1, 16), (3, 2, 6), (7, 2, 3), (17, 1, 29), (29, 1, 24), (21, 4, 12), \\ & (31, 27, 33), (27, 8, 33), (19, 9, 4), (23, 19, 21), (21, 8, 23), (6, 8, 21), \\ & (21, 19, 4), (1, 4, 34), (1, 17, 34), (4, 17, 34)\}. \end{aligned} \quad (5.11)$$

Now we tune 33 E_8 gauge groups on the rays $v_1, \dots, v_{33}$ and an F_4 gauge group on v_{34} . Then the first 61 3d cones of (5.11) can be blown up according to figure 1. But the last three of the 3d cones correspond to an (E_8, E_8, F_4) collision, which can be blown up to figure 8. We denote the two rays with E_8 by v_1, v_2 and the ray with F_4 by v_3 . We use

Figure 8. The final 3d cones after blowing up the 3d cone $v_1v_2v_3$, where there are two E_8 geometric gauge groups on v_1, v_2 and an F_4 on v_3 . Each vertex abc denotes an 1d ray $av_1 + bv_2 + cv_3$. The geometric non-Higgsable gauge groups are also labelled on each vertex. We also label the Kodaira type II singular fiber on the divisors.

abc to denote a ray $av_1 + bv_2 + cv_3$. Then the blow-up sequence is:

$$\begin{aligned}
 Blp = \{ & (100, 010, 001; 111), (100, 010, 111; 221), (111, 001; 112), (100, 010; 110), \\
 & (100, 110; 210), (010, 110; 120), (110, 210; 320), (110, 120; 230), \\
 & (100, 210; 310), (100, 310; 410), (100, 410; 510), (010, 120; 130), \\
 & (010, 130; 140), (010, 140; 150), (100, 111; 211), (100, 211; 311), \\
 & (010, 111; 121), (010, 121; 131), (100, 001; 101), (100, 101; 201), \\
 & (100, 201; 301), (100, 301; 401), (101, 001; 102), (010, 001; 011), \\
 & (010, 011; 021), (010, 021; 031), (010, 031; 041), (001, 011; 012) \}.
 \end{aligned} \tag{5.12}$$

Now we can count the total number of B_{toric} after the blow-ups: there are 34 rays in (5.10), 19 rays in the interior of each of the 61 (E_8, E_8, E_8) 3d cones, 7 rays in the interior of each of the 3 (E_8, E_8, F_4) 3d cones, 11 rays in the interior of each of the 93 (E_8, E_8) 2d cones and 5 rays in the interior of each of the 3 (E_8, F_4) 2d cones. In total, there are exactly 2252 rays in B_{toric} , which gives rise to the correct $h^{1,1}(B_{\text{toric}}) = 2249!$

The non-Higgsable gauge groups on B_{toric} can be counted:

$$G_{\text{nH}} = E_8^{33} \times F_4^{94} \times G_2^{250} \times \text{SU}(2)^{375}. \quad (5.13)$$

After the toric blow-ups, there are still 21 E_8 divisors with non-toric $(4, 6)$ -curve. From B_{toric} to B_3 , one needs to blow up these non-toric curves. Hence the total $h^{1,1}(X_4)$ is

$$\begin{aligned} h^{1,1}(X_4) &= h^{1,1}(B_{\text{toric}}) + \text{rank}(G_{\text{nH}}) + 21 + 1 \\ &= 3786. \end{aligned} \quad (5.14)$$

5.3 Estimated number of gauge factors

Here we provide an explanation of the gauge group counting formula (4.1) in [22]. In a seed base with a large number N of 1d rays, the number of 2d and 3d cones generally scale as:

$$\begin{aligned} N_2(B_{\text{seed}}) &\approx 3N \\ N_3(B_{\text{seed}}) &\approx 2N. \end{aligned} \quad (5.15)$$

Assuming that each 1d ray has a geometric E_8 gauge group, then after the seed base is fully blown up to the base B_{toric} with no toric codimension-two $(4, 6)$ locus, the total number of 1d rays is given by

$$\begin{aligned} N_1(B_{\text{toric}}) &\approx N + 19 \times 2N + 11 \times 3N \\ &\approx 72N. \end{aligned} \quad (5.16)$$

The number $N_1(B_{\text{toric}}) \approx h^{1,1}(B_{\text{toric}})$ if the number is large. The total numbers of geometric non-Higgsable gauge groups of each type are given by:

$$\begin{aligned} N(E_8) &= N \\ N(F_4) &= N_2(B_{\text{seed}}) \approx 3N \\ N(G_2) &= 2N_2(B_{\text{seed}}) + N_3(B_{\text{seed}}) \approx 8N \\ N(\text{SU}(2)) &= 2N_2(B_{\text{seed}}) + 3N_3(B_{\text{seed}}) \approx 12N. \end{aligned} \quad (5.17)$$

Thus we have the approximate formula for the number of each type of gauge groups

$$\begin{aligned} N(E_8) &\approx \frac{h^{1,1}(B_{\text{toric}})}{72} \\ N(F_4) &\approx \frac{h^{1,1}(B_{\text{toric}})}{24} \\ N(G_2) &\approx \frac{h^{1,1}(B_{\text{toric}})}{9} \\ N(\text{SU}(2)) &\approx \frac{h^{1,1}(B_{\text{toric}})}{6}. \end{aligned} \quad (5.18)$$

We can also estimate the ratio between $h^{1,1}(B_3)$ and $h^{1,1}(X_4)$ based on (5.18) and

$$h^{1,1}(X_4) = h^{1,1}(B_3) + \text{rank}(G_{\text{nH}}) + 1. \quad (5.19)$$

We use an approximation $h^{1,1}(B_3) \approx h^{1,1}(B_{\text{toric}})$, although in principle the B_3 base is generated by blowing up B_{toric} along non-toric curves. Then we arrive at the following approximate formula for the end point bases with large $h^{1,1}(B_3)$:

$$\frac{h^{1,1}(X_4)}{h^{1,1}(B_3)} \approx \frac{5}{3}. \quad (5.20)$$

For the number of flip and flop phases on B_{toric} , the lower bound can also be estimated with the methods in section 4:

$$\begin{aligned} N(\text{fp}) &\gtrsim (1100)^{3N_3} \\ &\approx (1100)^{h^{1,1}(B_{\text{toric}})/12} \\ &\approx 10^{0.253 \times h^{1,1}(B_{\text{toric}})}. \end{aligned} \quad (5.21)$$

Note that if there exists toric automorphism on the seed base B_{seed} , then this number will be reduced by $\mathcal{O}(10^1)$, which has no significant change on the exponential.

For example, for the number of B_{toric} with $h^{1,1}(B_{\text{toric}}) = 1943$, it is estimated to be

$$N(\text{fp}) \gtrsim 10^{493}. \quad (5.22)$$

This number is much bigger than the estimated number of “good bases” (the bases without codimension-two (4,6) locus) with the same $h^{1,1}$ in [22], which is around 10^{200} . This suggests that the statistical methods in [22] lead to a systematic underestimation.

6 Discussions

6.1 Supergravity coupled to conformal matter

From the construction of base threefolds with large $h^{1,1}$, we see that the (E_8, E_8, E_8) collision is prevalent. In fact, the Calabi-Yau with largest known $h^{1,1}$ provides the known example of the highest rank conformal matter coupled to supergravity. For 6d (1,0) theories, the elliptic Calabi-Yau threefold X_3 with the largest $h^{1,1}(X_3) = 491$ has the following toric base geometry [9]:

$$(-12// -11//(-12//)^{13}, -11// -12, 0). \quad (6.1)$$

Each number denotes the self-intersection number of each $\mathbb{P}^1$ curve on the base, which intersects each other in a cyclic way. The symbol “//” denotes the following chain of curves in the tensor branch of minimal (E_8, E_8) conformal matter:

$$// \equiv -1, -2, -2, -3, -1, -5, -1, -3, -2, -2, -1. \quad (6.2)$$

To get a base without (4,6)-points, the two (-11) -curves need to be blown up at a non-toric point as well. One can see that the non-minimal (E_8, E_8) conformal matter [37, 53] with order $N = 16$ can be embedded into this base. Such a 6d (1,0) SCFT has the following tensor branch in the standard notation:

$$[E_8] - 1 - 2 - \overset{\text{su}(2)}{2} - \overset{\text{g}_2}{3} - 1 - \overset{\text{f}_4}{5} - 1 - \overset{\text{g}_2}{3} - \overset{\text{su}(2)}{2} - 2 - 1 - \overset{\text{e}_8}{12} - 1 - \cdots - 1 - [E_8], \quad (6.3)$$

where there are 15 (-12) -curves with e_8 gauge group in the middle.

For 5d $\mathcal{N} = 1$ theories, similarly one can consider M-theory on the resolved compact Calabi-Yau threefold X_3 . Then we can couple 5d supergravity with the KK reduction of the non-minimal (E_8, E_8) conformal matter with $N = 16$, which has the following 5d IR quiver gauge theory description with rank $r = 471$ [53]:

$$\begin{array}{c} \text{SU}(48) \\ | \\ \text{SU}(16) - \text{SU}(32) - \text{SU}(48) - \text{SU}(64) - \text{SU}(80) - \text{SU}(96) - \text{SU}(64) - \text{SU}(32). \end{array} \quad (6.4)$$

However, to get a genuine 5d SCFT fixed point, one needs to decouple an $\text{SU}(16)$ vector multiplet from the theory, which geometrically corresponds to decompactifying 15 divisors in X_3 [54, 55]. After the decompactification, the gravity sector will be decoupled again. Nonetheless, the 5d quiver (6.4) is still the quiver gauge theory with the known largest rank that can be coupled to 5d $\mathcal{N} = 1$ supergravity.

Finally, for the 4d $\mathcal{N} = 1$ theories, it is unclear whether the $E_8 - E_8 - E_8$ Yukawa point actually corresponds to an SCFT fixed point or not [33]. Nonetheless, the X_4 with largest $h^{1,1}(X_4) = 303\,148$ in this paper provides the example of 4d supergravity coupled to an E_8 quiver network with the largest known number (2561) of E_8 gauge groups.

It would be fascinating to have a swampland bound argument along the philosophy of [56–59], for the various cases discussed here: 6d (1,0), 5d $\mathcal{N} = 1$ and 4d $\mathcal{N} = 1$. Alternatively, one can also attempt to challenge these bounds in other parts of the string landscape with a supergravity sector.

6.2 Standard model building

Another interesting question is whether the 4d F-theory model on X_4 with the largest $h^{1,1}(X_4)$ has any model building implications. Since this model has the largest number of axions

$$N(\text{axions}) = 181\,820 \quad (6.5)$$

in the known 4d superstring landscape, it is potentially useful for the inflation models with a large number of axions, e.g. [60–63].

Nonetheless, it is hard to realize the standard model gauge group on X_4 geometrically, because of a similar problem to the one discussed in [34, 64]. If we embed $G_{\text{sm}} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ into a single E_8 , then it is implausible to get chiral families [65]. On the other hand, if we embed $G_{\text{sm}} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ into a single F_4 , the branching rule does not give the correct standard model hypercharges. Moreover, it is impossible to tune any larger gauge groups on the base, if we accept that $h^{1,1}(X_4)$ is already maximal (also see the discussions in section 3.2).

Thus it is more plausible to realize a part or all of the G_{sm} as the gauge bosons from D3 branes. Then the geometric gauge groups will be treated as dark matter sectors, if they do not intersect these D3 branes.

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A Other toric seed threefolds

In section 5, we have presented some toric seed threefolds for the end point bases in [22] with $h^{1,1}(B_{\text{toric}}) \sim \mathcal{O}(10^3)$. In this section, we will present four more toric seed threefolds with E_8 on each of the toric rays.

1. Toric seed threefold B_{seed} for $h^{1,1}(B_{\text{toric}}) = 1727$

There are 26 1d rays in B_{seed} :

$$\begin{aligned} \{v_i\} = \{ & (0,0,1), (1,0,0), (0,0,-1), (1,1,0), (1,1,2), (-1,0,1), (2,1,-1), \\ & (-1,-1,0), (-1,0,0), (-1,0,-2), (1,1,-1), (-2,-1,-1), (2,1,0), (3,1,0), \\ & (-3,-1,-1), (-1,0,2), (1,1,1), (-3,-1,-2), (-3,-1,0), (1,1,-2), (-1,0,-1), \\ & (-2,-1,0), (2,1,1), (-3,-1,1), (-2,-1,1), (-3,-1,2) \} \end{aligned} \quad (\text{A.1})$$

and 48 3d cones:

$$\begin{aligned} \{\sigma_3\} = \{ & (13,4,7), (13,7,14), (7,2,14), (4,23,17), (22,19,25), (13,23,4), \\ & (19,9,6), (6,9,4), (12,15,19), (19,15,9), (14,23,13), (14,2,23), \\ & (5,16,6), (1,16,5), (25,8,22), (9,15,21), (17,23,5), (23,1,5), \\ & (5,6,17), (4,21,11), (18,12,10), (18,15,12), (10,15,18), (2,1,23), \\ & (21,15,10), (7,4,11), (11,20,7), (20,2,7), (3,2,20), (17,6,4), \\ & (24,19,6), (25,19,24), (25,24,26), (24,6,26), (25,16,1), (1,8,25), \\ & (6,16,26), (26,16,25), (2,8,1), (3,8,2), (4,9,21), (8,12,22), \\ & (20,10,3), (11,10,20), (21,10,11), (3,12,8), (10,12,3), (22,12,19) \}. \end{aligned} \quad (\text{A.2})$$

After the full blow-up to $h^{1,1}(B_{\text{toric}})$, the non-Higgsable gauge group is

$$G_{\text{nH}} = E_8^{26} \times F_4^{72} \times G_2^{192} \times \text{SU}(2)^{288}. \quad (\text{A.3})$$

There are 19 E_8 divisors with a non-toric $(4,6)$ -curve. Among them, the non-toric $(4,6)$ -curve on the ray $(1,0,0)$ has two irreducible components. Hence from B_{toric} to B_3 , one needs to blow up these 20 non-toric curves, and the total $h^{1,1}(X_4)$ is

$$\begin{aligned} h^{1,1}(X_4) &= h^{1,1}(B_{\text{toric}}) + \text{rank}(G_{\text{nH}}) + 20 + 1 \\ &= 2916. \end{aligned} \quad (\text{A.4})$$

The dual polytope of (A.1) form a toric fan with rays $\{(0,-1,0), (0,1,0), (1,-2,0), (-1,2,-1), (-1,2,1), (-1,2,0)\}$. The generic elliptic CY4 on this toric threefold has $(h^{1,1}, h^{3,1}) = (4, 2916)$, which should be the mirror of (A.4).

2. Toric seed threefold B_{seed} for $h^{1,1}(B_{\text{toric}}) = 2015$

There are 30 1d rays in B_{seed} :

$$\begin{aligned} \{v_i\} = \{ & (0,0,1), (0,1,0), (1,0,0), (-1,-1,-1), (0,1,1), (0,-1,-1), (0,1,2), \\ & (0,-1,0), (0,0,-1), (-1,0,-1), (-1,1,-1), (-1,-1,0), (-1,-1,1), (-1,0,0), \\ & (-1,0,1), (-1,1,2), (-1,-1,-2), (-1,-2,-2), (1,1,1), (-1,1,0), (-1,1,3), \\ & (-1,0,-2), (-1,-2,-3), (-1,-2,-1), (-1,0,2), (-1,-2,-4), (-1,-2,0), \\ & (-1,1,1), (-1,-1,-3), (0,-1,-2) \} \end{aligned} \quad (\text{A.5})$$

and 56 3d cones:

$$\begin{aligned} \{\sigma_3\} = \{ & (29,17,23), (2,20,11), (11,20,10), (30,22,29), (22,17,29), (29,26,30), \\ & (29,23,26), (26,23,30), (9,30,3), (3,30,6), (8,13,1), (12,13,27), \\ & (27,13,8), (7,5,19), (19,1,7), (5,2,19), (19,2,3), (3,1,19), \\ & (8,3,6), (27,24,4), (6,24,27), (2,11,9), (5,28,2), (7,16,5), \\ & (17,18,23), (23,18,30), (8,1,3), (3,2,9), (4,12,27), (1,25,7), \\ & (14,15,12), (28,16,15), (9,22,30), (30,18,6), (22,11,10), (9,11,22), \\ & (16,21,25), (25,21,7), (7,21,16), (6,27,8), (22,10,4), (4,18,17), \\ & (12,15,13), (1,13,25), (13,15,25), (5,16,28), (28,15,14), (4,17,22), \\ & (15,16,25), (10,20,14), (4,14,12), (10,14,4), (6,18,24), (24,18,4), \\ & (28,20,2), (14,20,28) \}. \end{aligned} \quad (\text{A.6})$$

After the full blow-up to $h^{1,1}(B_{\text{toric}})$, the non-Higgsable gauge group is

$$G_{\text{nH}} = E_8^{30} \times F_4^{84} \times G_2^{224} \times \text{SU}(2)^{336}. \quad (\text{A.7})$$

There are 20 E_8 divisors with non-toric $(4,6)$ -curves. For the single divisor corresponding to the ray $(-1,-2,0)$, the $(4,6)$ -curve has two irreducible components. Hence there are in total 21 non-toric curves to be blown up in order to get B_3 . Hence the total $h^{1,1}(X_4)$ is

$$\begin{aligned} h^{1,1}(X_4) &= h^{1,1}(B_{\text{toric}}) + \text{rank}(G_{\text{nH}}) + 21 + 1 \\ &= 3397. \end{aligned} \quad (\text{A.8})$$

The dual polytope of (A.5) form the toric fan of generalized Hirzebruch surface $\tilde{\mathbb{F}}_1$. The generic elliptic fibration over $\tilde{\mathbb{F}}_1$ exactly has Hodge numbers $(h^{1,1}, h^{3,1}) = (3, 3397)$.

3. Toric seed threefold B_{seed} for $h^{1,1}(B_{\text{toric}}) = 2303$

There are 34 1d rays in B_{seed} :

$$\begin{aligned} \{v_i\} = \{ & (0, 0, 1), (0, 1, 0), (1, 0, 0), (-1, -1, -1), (1, 1, 1), (-1, 0, -1), (0, 1, 1), \\ & (0, 1, -1), (0, -1, 0), (1, 2, 2), (0, -1, -1), (1, -1, 1), (1, -1, 0), (1, 2, 3), \\ & (0, 0, -1), (-1, -1, -2), (1, 2, 1), (1, 1, 0), (1, 1, 2), (-1, -2, -2), (1, 2, 0), \\ & (-2, -1, -3), (1, -2, 0), (0, -2, -1), (1, 0, 1), (1, 0, 2), (0, 1, 2), (-2, -1, -2), \\ & (-1, 0, -2), (-3, -2, -4), (-1, 0, 0), (1, 1, 3), (1, 2, 4), (-2, -2, -3) \} \end{aligned} \quad (\text{A.9})$$

and 64 3d cones:

$$\begin{aligned} \{\sigma_3\} = \{ & (12, 26, 25), (12, 1, 26), (13, 23, 12), (12, 23, 9), (11, 3, 15), (8, 3, 18), \\ & (19, 32, 10), (7, 31, 2), (2, 31, 6), (11, 29, 16), (11, 23, 13), (8, 29, 15), \\ & (6, 29, 8), (4, 31, 9), (4, 24, 20), (20, 24, 11), (18, 21, 8), (21, 2, 8), \\ & (5, 26, 19), (11, 24, 23), (23, 24, 9), (9, 24, 4), (34, 20, 16), (34, 28, 20), \\ & (4, 28, 31), (31, 28, 6), (7, 17, 10), (10, 17, 5), (25, 26, 5), (15, 29, 11), \\ & (14, 27, 10), (10, 32, 14), (28, 22, 29), (29, 22, 16), (26, 1, 27), (30, 28, 34), \\ & (30, 22, 28), (34, 22, 30), (27, 32, 26), (26, 32, 19), (16, 20, 11), (15, 3, 8), \\ & (8, 2, 6), (13, 3, 11), (16, 22, 34), (27, 33, 32), (32, 33, 14), (14, 33, 27), \\ & (27, 31, 7), (27, 1, 31), (10, 27, 7), (5, 19, 10), (5, 21, 18), (20, 28, 4), \\ & (6, 28, 29), (7, 2, 17), (17, 21, 5), (17, 2, 21), (31, 1, 9), (25, 3, 12), \\ & (9, 1, 12), (12, 3, 13), (5, 3, 25), (18, 3, 5) \} . \end{aligned} \quad (\text{A.10})$$

After the full blow-up to $h^{1,1}(B_{\text{toric}})$, the non-Higgsable gauge group is

$$G_{\text{nH}} = E_8^{34} \times F_4^{96} \times G_2^{256} \times \text{SU}(2)^{384} . \quad (\text{A.11})$$

There are 22 E_8 divisors with a non-toric $(4, 6)$ -curve, which are all irreducible. From B_{toric} to B_3 , one needs to blow up these non-toric curves. Hence the total $h^{1,1}(X_4)$ is

$$\begin{aligned} h^{1,1}(X_4) &= h^{1,1}(B_{\text{toric}}) + \text{rank}(G_{\text{nH}}) + 22 + 1 \\ &= 3878 . \end{aligned} \quad (\text{A.12})$$

In fact, the dual polytope of (A.9) is exactly the convex hull of rays on $\mathbb{P}^3$. The Hodge number (A.12) exactly equals to the $h^{3,1}$ of the generic elliptic CY4 over $\mathbb{P}^3$.

4. Toric seed threefold B_{seed} for $h^{1,1}(B_{\text{toric}}) = 2591$

There are 38 1d rays in B_{seed} :

$$\begin{aligned} \{v_i\} = \{ & (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1), (1, 1, 2), (1, 2, 2), (1, 3, 2), \\ & (1, 2, 1), (-1, 0, -1), (0, 0, -1), (1, 1, 0), (0, -1, 0), (1, 0, 1), (0, -1, 1), \\ & (1, -1, 0), (1, -1, 2), (1, -2, 0), (0, -1, -2), (0, 1, 1), (1, 1, 3), (1, -1, -2), \\ & (1, -2, 3), (1, -2, 2), (1, 3, 3), (1, 0, -1), (1, 0, 2), (1, -2, -1), (1, -1, 1), \\ & (1, -2, 1), (1, 2, 3), (1, -2, -3), (1, 4, 3), (0, -1, -1), (1, -1, -1), (1, 0, 3), \\ & (1, -1, 3), (1, -2, -2), (0, 2, 1) \} \end{aligned} \quad (\text{A.13})$$

and 72 3d cones:

$$\begin{aligned} \{\sigma_3\} = \{ & (15, 29, 28), (3, 34, 15), (2, 11, 8), (8, 11, 4), (9, 10, 2), (16, 36, 35), \\ & (35, 36, 1), (4, 11, 3), (26, 35, 5), (25, 34, 3), (19, 20, 1), (18, 33, 27), \\ & (9, 18, 10), (9, 33, 18), (24, 32, 7), (24, 38, 32), (32, 38, 7), (3, 13, 4), \\ & (1, 14, 9), (9, 14, 12), (9, 19, 1), (28, 29, 16), (14, 29, 12), (21, 25, 10), \\ & (21, 34, 25), (16, 29, 23), (23, 29, 14), (16, 35, 26), (12, 33, 9), (2, 25, 11), \\ & (11, 25, 3), (12, 29, 17), (17, 29, 15), (1, 36, 14), (16, 23, 22), (22, 23, 14), \\ & (10, 25, 2), (14, 36, 22), (22, 36, 16), (16, 26, 13), (19, 30, 20), (20, 30, 5), \\ & (13, 28, 16), (9, 38, 19), (2, 38, 9), (27, 34, 21), (4, 26, 5), (13, 26, 4), \\ & (15, 27, 17), (17, 27, 12), (27, 33, 12), (15, 34, 27), (3, 28, 13), (15, 28, 3), \\ & (5, 35, 20), (20, 35, 1), (7, 8, 6), (7, 38, 8), (21, 37, 27), (27, 37, 18), \\ & (6, 24, 7), (8, 38, 2), (18, 31, 21), (18, 37, 31), (31, 37, 21), (18, 21, 10), \\ & (6, 8, 4), (6, 30, 24), (30, 38, 24), (5, 6, 4), (19, 38, 30), (5, 30, 6) \}. \end{aligned} \quad (\text{A.14})$$

After the full blow-up to $h^{1,1}(B_{\text{toric}})$, the non-Higgsable gauge group is

$$G_{\text{nH}} = E_8^{38} \times F_4^{108} \times G_2^{288} \times \text{SU}(2)^{432}. \quad (\text{A.15})$$

There are 22 E_8 divisors with a non-toric $(4, 6)$ -curve, which are all irreducible. From B_{toric} to B_3 , one needs to blow up these non-toric curves. Hence the total $h^{1,1}(X_4)$ is

$$\begin{aligned} h^{1,1}(X_4) &= h^{1,1}(B_{\text{toric}}) + \text{rank}(G_{\text{nH}}) + 22 + 1 \\ &= 4358. \end{aligned} \quad (\text{A.16})$$

The low bound of the number of flips is given by (5.21):

$$N(\text{flp}) \gtrsim 10^{681}. \quad (\text{A.17})$$

Note that the polytope (A.13) has vertices

$$\{V\} = \{(1, 4, 3), (1, -2, 3), (1, -2, -3), (-1, 0, -1)\}, \quad (\text{A.18})$$

which is exactly isomorphic to the maximal reflexive polytope S_1 in [21, 36].

The dual polytope of (A.13) has vertices

$$\{V^\circ\} = \{(-1, 0, 0), (2, 0, -1), (0, -1, 1), (1, 1, 0)\}. \quad (\text{A.19})$$

Along with a ray $(1, 0, 0)$, they form the rays of the generalized Hirzebruch threefold $\tilde{\mathbb{F}}_3$. The generic elliptic fibration over $\tilde{\mathbb{F}}_3$ exactly has the Hodge numbers $(h^{1,1}, h^{3,1}) = (3, 4358)$, which is the mirror of (A.16).

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