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Littlest mu-tau seesaw

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ABSTRACT: We propose a $\mu - \tau$ reflection symmetric Littlest Seesaw ($\mu\tau$ -LSS) model. In this model the two mass parameters of the LSS model are fixed to be in a special ratio by symmetry, so that the resulting neutrino mass matrix in the flavour basis (after the seesaw mechanism has been applied) satisfies $\mu - \tau$ reflection symmetry and has only one free adjustable parameter, namely an overall free mass scale. However the physical low energy predictions of the neutrino masses and lepton mixing angles and CP phases are subject to renormalisation group (RG) corrections, which introduces further parameters. Although the high energy model is rather complicated, involving $(S_4 \times U(1))^2$ and supersymmetry, with many flavons and driving fields, the low energy neutrino mass matrix has ultimate simplicity.

KEYWORDS: Discrete Symmetries, Neutrino Physics, Renormalization Group, Solar and Atmospheric Neutrinos

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1 Introduction

The nature of neutrino mass and lepton flavour mixing remains an intriguing puzzle [1–3], even as the parameters are being measured to increasing precision [4–6]. Indeed the latest neutrino data is consistent with the hypothesis of maximal atmospheric mixing and maximal CP violation in the lepton sector, corresponding to a $\mu - \tau$ reflection symmetry, namely $\nu_\mu \leftrightarrow \nu_\tau^*$, where the star indicates CP conjugation. For a review of $\mu\tau$ symmetry see e.g. [7] and references therein.

The smallness of neutrino mass could originate from the seesaw mechanism [8–12]. The minimal version, involving just two right-handed neutrinos (2RHN), was first proposed by one of us [13, 14]. In such a scheme the lightest neutrino is massless. Such a model with two texture zeros in the Dirac neutrino mass matrix [15] is consistent with cosmological leptogenesis [16–23], however it is incompatible with the normal hierarchy (NH) of neutrino masses, favoured by current data [22, 23]. On the other hand the 2RHN model with the more generic one texture zero, as originally proposed in [13, 14], is still compatible with data in the NH.

The Littlest Seesaw (LSS) model is an example of a 2RHN model with one texture zero, involving just two independent Yukawa couplings [24–30], leading to a highly predictive scheme characterised by near maximal atmospheric mixing and CP violation, as in $\mu - \tau$ reflection symmetry, but with additional predictions arising from tri-maximal nature of the first column of the PMNS matrix as well as a predicted reactor angle.

In a recent paper, the LSS model was shown to respect an approximate $\mu - \tau$ reflection symmetry, near the best fit region of parameter space, which was responsible for its approximate predictions of maximal atmospheric mixing and maximal CP violation in the lepton sector [31]. This was due to a ratio of input mass parameters of the LSS having a value close to that in which the model satisfied exact $\mu - \tau$ reflection symmetry, however no model which explained this apparent coincidence was proposed.

In the present paper we shall propose a version of the LSS model which satisfies exact $\mu - \tau$ reflection symmetry, which we refer to as the $\mu\tau$ -LSS model. We construct a supersymmetric model in the flavour symmetry $S_{4L} \times S_{4R} \times U(1) \times U(1)'$. We implement the idea of bi-multiplet to the non-Abelian discrete symmetries $S_{4L} \times S_{4R}$. Multiplets in S_{4L} are used to determine the direction in the flavour space, and multiplets in S_{4R} are crucial to fix the ratio of two right-handed neutrino masses and further fix the desired ratio of the two mass parameters of the LSS model. Two $U(1)$'s are imposed to give hierarchical and diagonal charged lepton masses and also to forbid unnecessary terms in the superpotential. The resulting neutrino mass matrix (after the seesaw mechanism has been applied) satisfies $\mu - \tau$ reflection symmetry and has only one free adjustable parameter, namely an overall free mass scale. However the physical low energy predictions of the neutrino masses and lepton mixing angles and CP phases are subject to renormalisation group (RG) corrections, which introduces further parameters. Although the high energy model is rather complicated, involving $(S_4 \times U(1))^2$ and supersymmetry (SUSY), with many flavons and driving fields, it leads to a neutrino mass matrix of ultimate simplicity and beauty capable of explaining all neutrino data in terms of one adjustable overall mass scale.

The layout of the remainder of the paper is as follows. In section 2, we briefly review the $\mu\tau$ -LSS mass texture and its prediction of oscillation parameters. In section 3, we consider corrections of these parameters by including radiative corrections for the first time. The concrete model is given in section 4, where all flavon vacuum alignments are realised explicitly. Section 5 is devoted to conclusions. In the appendices, we list the basis of S_4 used for model building and discuss the vacuum degeneracy of flavons.

2 The $\mu\tau$ -LSS mass matrix

There are two cases of the LSS neutrino mass matrix [29] (after the seesaw mechanism has been implemented) namely,

$$\begin{aligned} \text{Case I: } M_\nu &= \omega m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_s \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix}, \\ \text{Case II: } M_\nu &= \omega^2 m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_s \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & 9 \end{pmatrix}, \end{aligned} \quad (2.1)$$

where $\omega = e^{i2\pi/3}$. As observed in [31], if $m_{a,s}$ satisfy the special ratio $\frac{m_a}{m_s} = 11$ then this results in maximal atmospheric mixing and CP violation, as can be checked explicitly

using the analytic formulas in refs. [26, 29]. Inserting this ratio of masses, the neutrino mass matrix takes one of the two forms

$$\begin{aligned} \text{Case I: } M_\nu &= m_s \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 + 11\omega & 3 + 11\omega \\ 1 & 3 + 11\omega & 1 + 11\omega \end{pmatrix}, \\ \text{Case II: } M_\nu &= m_s \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 + 11\omega^2 & 3 + 11\omega^2 \\ 3 & 3 + 11\omega^2 & 9 + 11\omega^2 \end{pmatrix}. \end{aligned} \quad (2.2)$$

We refer to them as the $\mu\tau$ -LSS mass matrices. With the $\mu\tau$ conjugation [31],

$$\nu_e \rightarrow \nu_e^*, \quad \nu_\mu \rightarrow \nu_\tau^*, \quad \nu_\tau \rightarrow \nu_\mu^*, \quad (2.3)$$

one transforms the mass matrix from one case to the other. Both cases predict the same mixing angles $(\theta_{13}, \theta_{12}, \theta_{23})$, the same Dirac-type CP-violating phase (δ)

$$\begin{aligned} \theta_{13} &= \arcsin\left(\frac{c_-}{\sqrt{6}}\right) \approx 7.807^\circ, \\ \theta_{12} &= \arctan\left(\frac{c_+}{2}\right) \approx 34.50^\circ, \\ \theta_{23} &= 45^\circ, \\ \delta &= 270^\circ, \end{aligned} \quad (2.4)$$

where $c_- = \sqrt{1 - \frac{11}{3\sqrt{17}}}$ and $c_+ = \sqrt{1 + \frac{11}{3\sqrt{17}}}$, and the same neutrino masses

$$\begin{aligned} m_1 &= 0, \\ m_2 &= \sqrt{\Delta m_{21}^2} = \sqrt{\frac{33}{2} (13 - 3\sqrt{17})} m_s \approx 3.226 m_s, \\ m_3 &= \sqrt{\Delta m_{31}^2} = \sqrt{\frac{33}{2} (13 + 3\sqrt{17})} m_s \approx 20.46 m_s. \end{aligned} \quad (2.5)$$

Specifically the ratio

$$\frac{\Delta m_{21}^2}{\Delta m_{31}^2} = \frac{m_2^2}{m_3^2} = \frac{13 - 3\sqrt{17}}{13 + 3\sqrt{17}} \approx 0.0247 \quad (2.6)$$

is independent of the mass parameter m_s . The only difference is the Majorana phase. However, it is of little use because that phase cannot easily be accessed. Instead, we give the prediction of the effective neutrino mass parameter in neutrino-less double beta decay, which is the same in two cases, $m_{\beta\beta} = m_s$. The sum of neutrino masses is also the same, $m_1 + m_2 + m_3 = \sqrt{561} m_s$. The ratio of these two mass parameters is given by

$$\frac{m_{\beta\beta}}{m_1 + m_2 + m_3} \approx 0.0422. \quad (2.7)$$

Following [31], it is convenient to work with the Hermitian matrix $H_\nu = M_\nu^\dagger M_\nu$ instead of M_ν since H_ν preserves the $\mu - \tau$ reflection symmetry. H_ν is directly given by

$$\frac{H_\nu}{m_s^2} = 11 \begin{pmatrix} 1 & -1 \pm 2i\sqrt{3} & 1 \pm 2i\sqrt{3} \\ -1 \mp 2i\sqrt{3} & 19 & 17 + 4i\sqrt{3} \\ 1 \mp 2i\sqrt{3} & 17 - 4i\sqrt{3} & 19 \end{pmatrix} \quad (2.8)$$

for case I and case II, respectively. They satisfy the following structure

$$(H_\nu)_{12} = -(H_\nu)_{13}^*, \quad (H_\nu)_{22} = (H_\nu)_{33}, \quad (2.9)$$

from which one can directly prove $\theta_{23} = 45^\circ$ and $\delta = 270^\circ$. The difference of H_ν between two cases can be rotated away by redefinition of the unphysical phases in the charged lepton sector. Therefore, all oscillation parameters, including θ_{13} , θ_{12} , θ_{23} , δ , as well as mass parameters Δm_{21}^2 and Δm_{31}^2 , are predicted to be exactly the same, as have been obtained in eqs. (2.4) and (2.5). Without respecting the Majorana phase and unphysical phases, the PMNS matrix in both cases takes the same form as

$$U = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{c_+}{\sqrt{6}} & \frac{c_-}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} - \frac{c_+}{\sqrt{6}} - i\frac{c_-}{2} & -\frac{c_-}{\sqrt{6}} + i\frac{c_+}{2} \\ \frac{1}{\sqrt{6}} - \frac{c_+}{\sqrt{6}} + i\frac{c_-}{2} & -\frac{c_-}{\sqrt{6}} - i\frac{c_+}{2} \end{pmatrix}. \quad (2.10)$$

The mixing matrix respects $\mu - \tau$ reflection symmetry and is a special case of tri-maximal TM_1 mixing [32–38], with a fixed reactor angle and a fixed solar angle.

This model is not fully consistent with the oscillation data since both the predicted θ_{13} and ratio of mass square differences $\Delta m_{21}^2/\Delta m_{31}^2$ are smaller than the current global data of neutrino oscillation in 3σ ranges. As a comparison, current data give $\theta_{13} \sim (8.09^\circ, 8.98^\circ)$ and $\Delta m_{21}^2/\Delta m_{31}^2 \sim (0.0262, 0.0334)$ in 3σ ranges [6]. The explicit flavour texture of the $\mu\tau$ -LSS model is corrected due to radiative corrections. We wonder if the $\mu\tau$ -LSS model can be compatible with current data after the RG running effect is included. Different from [31], where only case II is listed, here we write out both cases explicitly since radiative corrections have different contributions to μ and τ flavours.

3 Radiative corrections to the model

In this section, we are going to explore how the oscillation parameters are modified by including radiative corrections.

We assume the flavour structure of the $\mu\tau$ -LSS model is valid at a new scale $\Lambda_{\mu\tau}$. In order to gain a relatively large RG running effect, this scale should be sufficiently higher than the electroweak scale Λ_{EW} . $\Lambda_{\mu\tau}$ in principle could be different from the seesaw scale Λ_{ss} , but we assume they are close to each other, and thus running between $\Lambda_{\mu\tau}$ and Λ_{ss} is negligible. Once heavy degrees of freedom decouple from the theory below Λ_{ss} , the neutrino mass and flavour mixing is governed by the dimension-5 Weinberg operator

$$\mathcal{L} \supset \bar{\ell} \tilde{H} \kappa \ell^c \tilde{H} + \text{h.c.}, \quad (3.1)$$

where κ is a 3×3 coupling matrix and $\tilde{H} = i\sigma_2 H^*$. After the electroweak symmetry breaking, the Higgs gains the VEV $\langle H \rangle = v_H = 175 \text{ GeV}$, the neutrino mass is given by $M_\nu = \kappa v_H^2$. In our following discussion, we will always denote κv_H^2 by the effective mass matrix M_ν at any scale no matter lower or higher than the electroweak scale. For scale higher than the electroweak scale, M_ν should not be understood as neutrino masses, but just the coupling matrix with its unit normalised by v_H^2 .

RG running below Λ_{ss} do not need to include any heavy degrees of freedom in the RG running. The RG running of the coupling matrix κ was first discussed in [39, 40]. M_ν at two scales due to the radiative correction can be written as an integrated form as [19, 41–43]

$$M_\nu(\Lambda_{\text{EW}}) = I_\alpha \begin{pmatrix} I_e & 0 & 0 \\ 0 & I_\mu & 0 \\ 0 & 0 & I_\tau \end{pmatrix} M_\nu(\Lambda_{\mu\tau}) \begin{pmatrix} I_e & 0 & 0 \\ 0 & I_\mu & 0 \\ 0 & 0 & I_\tau \end{pmatrix}, \quad (3.2)$$

where

$$\begin{aligned} I_\alpha &= \exp \left[-\frac{1}{16\pi^2} \int_{\ln\Lambda_{\text{EW}}}^{\ln\Lambda_{\mu\tau}} \alpha(t) dt \right], \\ I_l &= \exp \left[-\frac{C}{16\pi^2} \int_{\ln\Lambda_{\text{EW}}}^{\ln\Lambda_{\mu\tau}} y_l^2(t) dt \right], \end{aligned} \quad (3.3)$$

for $l = e, \mu, \tau$. Here we have ignored the difference between M_ν at $\Lambda_{\mu\tau}$ and that just below Λ_{ss} . In the SM and the minimal supersymmetric model (MSSM), C and α are given by

$$\begin{aligned} C_{\text{SM}} &= -\frac{3}{2}, & \alpha_{\text{SM}} &\approx -3g_2^2 + \lambda + 6y_t^2, \\ C_{\text{MSSM}} &= 1, & \alpha_{\text{MSSM}} &\approx -\frac{6}{5}g_1^2 - 6g_2^2 + 6y_t^2, \end{aligned} \quad (3.4)$$

respectively, where $g_{1,2}$ denote the gauge couplings, λ denotes the quartic Higgs coupling in the SM, and y_t, y_l (for $l = e, \mu, \tau$) are Yukawa couplings of the top quark and charged leptons, respectively. In MSSM, the Higgs $\tilde{H}$ contributing to the Weinberg operator in eq. (3.1) should be replaced by H_u , and the VEV v_H contributing to the neutrino mass M_ν should be replaced by $v_{H_u} = v_H \sin \beta$.

We see that in eq. (3.2), I_α is an overall factor affecting the magnitudes of the absolute neutrino masses, and I_l are flavour-dependent corrections which may modify the mass structure and flavour mixing. Due to the different signs of C in SM and MSSM (cf. eq. (3.4)), the flavour-dependent corrections go to opposite directions in the SM and MSSM. We follow the approximation proposed in [44]: the Yukawa couplings y_e, y_μ are too small as compared with y_τ such that thus I_e and I_μ can be approximately set to be identities, and I_τ is re-parametrised as $1 + \epsilon$, where

$$\epsilon = I_\tau - 1 \approx -\frac{C}{16\pi^2} \int_{\ln\Lambda_{\text{EW}}}^{\ln\Lambda_{\mu\tau}} y_\tau^2(t) dt. \quad (3.5)$$

In the case of slowly varying Yukawa coupling, $y_\tau(t)$ can be replaced by $y_{\tau,\text{EW}} = m_\tau/v_H$ in SM (or $m_\tau/(v_H \sin \beta)$ in MSSM) and ϵ is approximated to $\epsilon \approx -\frac{C}{16\pi^2} y_{\tau,\text{EW}}^2 \ln \frac{\Lambda_{\mu\tau}}{\Lambda_{\text{EW}}}$ with $y_{\tau,\text{EW}}$ being the τ -lepton Yukawa coupling at the electroweak scale. Since C is negative in SM (positive in MSSM), the correction ϵ is positive in SM (negative in MSSM).

At the scale $\Lambda_{\mu\tau}$, $M_\nu(\Lambda_{\mu\tau})$ takes the exact form as in eq. (2.2). With the help of the above approximation, the Majorana mass matrix at the electroweak scale is represented by

$$\begin{aligned} \text{Case I: } M_\nu(\Lambda_{\text{EW}}) &= \tilde{m}_s \begin{pmatrix} 1 & 3 & 1+\epsilon \\ 3 & 9+11\omega & (3+11\omega)(1+\epsilon) \\ 1+\epsilon & (3+11\omega)(1+\epsilon) & (1+11\omega)(1+\epsilon)^2 \end{pmatrix}, \\ \text{Case II: } M_\nu(\Lambda_{\text{EW}}) &= \tilde{m}_s \begin{pmatrix} 1 & 1 & 3(1+\epsilon) \\ 1 & 1+11\omega^2 & (3+11\omega^2)(1+\epsilon) \\ 3(1+\epsilon) & (3+11\omega^2)(1+\epsilon) & (9+11\omega^2)(1+\epsilon)^2 \end{pmatrix}, \end{aligned} \quad (3.6)$$

where $\tilde{m}_s = I_\alpha m_s$. Only two real parameters are involved in the mass matrix M_ν at the electroweak scale. One of them, $\tilde{m}_s$ contributes only to the absolute values of neutrino masses. The exact value of m_s or I_α is not important at low energy theory. Only their combination $\tilde{m}_s$ is important. The other parameter ϵ , representing the RG running effect, is the only parameter contributing to flavour mixing and the ratio of mass square differences. It also violates the $\mu\tau$ conjugation relation between the two mass matrices.

The Hermitian matrix H_ν at the electroweak scale is directly obtained from eq. (3.6). In order to get the analytical approximate results of the oscillation parameters, we expand H_ν in order of ϵ as

$$\frac{H_\nu(\Lambda_{\text{EW}})}{\tilde{m}_s^2} = \frac{H_\nu(\Lambda_{\mu\tau})}{m_s^2} + \frac{\delta H_\nu}{\tilde{m}_s^2} \quad (3.7)$$

with

$$\begin{aligned} \frac{\delta H_\nu}{\tilde{m}_s^2} &= \begin{pmatrix} 2 & -5+11i\sqrt{3} & -20+33i\sqrt{3} \\ -5-11i\sqrt{3} & 194 & 391+66i\sqrt{3} \\ -20-33i\sqrt{3} & 391-66i\sqrt{3} & 640 \end{pmatrix} \epsilon + \mathcal{O}(\epsilon^2), \\ \frac{\delta H_\nu}{\tilde{m}_s^2} &= \begin{pmatrix} 18 & -15-33i\sqrt{3} & 32-55i\sqrt{3} \\ -15+33i\sqrt{3} & 194 & 351+110i\sqrt{3} \\ 32+55i\sqrt{3} & 351-110i\sqrt{3} & 624 \end{pmatrix} \epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \quad (3.8)$$

for case I and case II, respectively. On the right hand side of eq. (3.7), only one free parameter ϵ appears. The $\mu - \tau$ reflection symmetry is not preserved any more. The RG running effect specifies the τ sector, and thus two cases in eq. (2.2) gain totally different corrections.

By perturbatively diagonalising H_ν , we obtain corrections to both θ_{13} and the ratio of mass square differences $\Delta m_{21}^2/\Delta m_{31}^2$, which are determined by ϵ . Including the other parameters, the corrected oscillation parameters are approximatively given by

$$\begin{aligned} \theta_{13} &\approx 7.807^\circ - 8.000^\circ \epsilon, \\ \theta_{12} &\approx 34.50^\circ - 12.30^\circ \epsilon, \\ \theta_{23} &\approx 45.00^\circ - 31.64^\circ \epsilon, \\ \delta &\approx 270.00^\circ + 3.23^\circ \epsilon, \\ \frac{\Delta m_{21}^2}{\Delta m_{31}^2} &\approx 0.0247 - 0.0147 \epsilon \end{aligned} \quad (3.9)$$

in case I, and

$$\begin{aligned}
\theta_{13} &\approx 7.807^\circ + 0.345^\circ \epsilon, \\
\theta_{12} &\approx 34.50^\circ - 13.96^\circ \epsilon, \\
\theta_{23} &\approx 45.00^\circ - 30.50^\circ \epsilon, \\
\delta &\approx 270.00^\circ + 2.33^\circ \epsilon, \\
\frac{\Delta m_{21}^2}{\Delta m_{31}^2} &\approx 0.0247 - 0.0249 \epsilon
\end{aligned} \tag{3.10}$$

in case II. Here again, I_α gives only an overall enhancement or suppression to masses and thus does not contribute to the above formulas.

Let us first have a look at case II. This case is not compatible with data after the RG running is included. Reasons are given below. In eq. (3.10), we can see that θ_{13} gains a very small correction from ϵ . In order to enhance θ_{13} by 0.2° , ϵ should be positive and not smaller than 0.5, in spite of validity of perturbation calculation. In MSSM, ϵ is always a negative parameter and thus, does not satisfy the requirement. In the SM, ϵ is positive, but the induced correction is too small. Furthermore, θ_{13} and $\Delta m_{21}^2/\Delta m_{31}^2$ always gain corrections in opposite directions. If one parameter runs closer to the experimental allowed range, the other runs farther away. Therefore, eq. (3.10) is not consistent with current oscillation data. In this work, we have assumed $\Lambda_{\mu\tau}$ close to the seesaw scale Λ_{ss} . If such an assumption is given up, e.g., $\Lambda_{\mu\tau} \gg \Lambda_{ss}$, heavy neutrinos may contribute to the running effect from $\Lambda_{\mu\tau}$ to Λ_{ss} , the RG running behaviour could be modified, and case II may be still allowed by data. We will not consider this possibility in our paper.

Then, we turn back to case I. Oscillation parameters as functions of the RG running parameter ϵ are shown in figure 1. In this case, all parameters can be compatible with current oscillation data in 3σ ranges with a suitable value for the RG running parameter ϵ . Specifically, both θ_{13} and $\Delta m_{21}^2/\Delta m_{31}^2$ are corrected in the same direction. To increase their values, ϵ has to be negative with value $-\epsilon \sim \mathcal{O}(0.1)$. However, these two parameters cannot be compatible with each other in 1σ ranges. We have compared the linear approximation in eq. (3.9) with the full one-loop RG running code in MSSM [19, 44] and confirm that it is valid to a very high precision level. By setting Λ_{EW} and $\Lambda_{\mu\tau}$ around 10^2 and 10^{14} GeV, respectively, we vary $\tan \beta$ in $(0, 100)$ and obtain correlation between $\tan \beta$ and ϵ . The value of ϵ of order 0.1 refers to a large $\tan \beta$, e.g., $\epsilon = 0.05, 0.1$ corresponding to $\tan \beta \sim 40, 66$, respectively. To summarise, the $\mu\tau$ -LSS model with RG correction in MSSM with large $\tan \beta$ is compatible with current oscillation data.

4 A concrete $\mu\tau$ -LSS model in $S_4 \times S_4$

In this section, we present a concrete flavour model to realise the $\mu\tau$ -LSS flavour structure in case I. We assume the flavour symmetry to be $S_{4L} \times S_{4R}$ in the SUSY framework. How leptons gain flavoured masses based on specified flavon vacua will be discussed in section 4.1 and how flavons gain the specified VEVs will be given in section 4.2. In addition, we introduce two $U(1)$'s to achieve diagonal and hierarchical masses for charged leptons and forbid unnecessary superpotential terms. An example of charge assignment in $S_{4L} \times S_{4R} \times U(1) \times U(1)'$ symmetries is given in section 4.3.

Figure 1. Oscillation parameters modified by radiative corrections in case I of the $\mu\tau$ -LSS model. Best-fit values, 1σ and 3σ ranges of global fits data of neutrino oscillation experiments are shown as comparisons.

4.1 Fermion masses

Three left-handed $SU(2)_L$ doublets of leptons ℓ transform as a triplet in S_{4L} but a trivial singlet in S_{4R} . We introduce two right-handed neutrinos N_1 and N_2 , transforming as a doublet in S_{4R} and blind in S_{4L} . In order to realise the flavour structure, we introduce four flavon multiplets ϕ'_t , ϕ_N , ξ_{TS} and ξ'_t . The flavon ϕ_N talks to both left-handed and right-handed fermions, and thus transforms non-trivially as a triplet-doublet $(\mathbf{3}_L, \mathbf{2}_R)$ in $S_{4L} \times S_{4R}$, where the subscripts L and R specifying groups S_{4L} and S_{4R} , respectively. The other flavons, ϕ'_t interacts with the SM leptons, arranged as a triplet $\mathbf{3}'_L$ of S_{4L} and non-trivial singlet $\mathbf{1}'_R$ of S_{4R} . ξ_{TS} and ξ'_t only interact with right-handed neutrinos, arranged as triplets $\mathbf{3}_R$, $\mathbf{3}'_R$ of S_{4R} , respectively. These representation arrangements are simply summarised as

$$\begin{aligned}
 \ell &= \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} \sim (\mathbf{3}_L, \mathbf{1}_R), \quad \phi'_t = \begin{pmatrix} \phi'_{t,1} \\ \phi'_{t,2} \\ \phi'_{t,3} \end{pmatrix} \sim (\mathbf{3}'_L, \mathbf{1}'_R), \\
 e^c &\sim (\mathbf{1}'_L, \mathbf{1}'_R), \quad \mu^c \sim (\mathbf{1}_L, \mathbf{1}_R), \quad \tau^c \sim (\mathbf{1}'_L, \mathbf{1}'_R), \\
 N &= (N_1, N_2) \sim (\mathbf{1}_L, \mathbf{2}_R), \\
 \phi_N \equiv (\phi_{\text{atm}}, \phi_{\text{sol}}) &= \begin{pmatrix} \phi_{\text{atm},1} & \phi_{\text{sol},1} \\ \phi_{\text{atm},2} & \phi_{\text{sol},2} \\ \phi_{\text{atm},3} & \phi_{\text{sol},3} \end{pmatrix} \sim (\mathbf{3}_L, \mathbf{2}_R), \\
 \xi_{TS} = (\xi_{TS,1}, \xi_{TS,2}, \xi_{TS,3}) &\sim (\mathbf{1}_L, \mathbf{3}_R), \quad \xi'_t = (\xi'_{t,1}, \xi'_{t,2}, \xi'_{t,3}) \sim (\mathbf{1}_L, \mathbf{3}'_R). \quad (4.1)
 \end{aligned}$$

We make the convention that S_{4L} acts on multiplets vertically and S_{4R} acts horizontally. Vacuum alignments are assumed (and later justified) to be

$$\begin{aligned}
 \langle \phi'_t \rangle &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} v_{\phi'_t}, \\
 \langle \phi_N \rangle &= \begin{pmatrix} 0 & \frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{11}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{11}} \end{pmatrix} v_{\phi_N}, \\
 \langle \xi_{TS} \rangle &= \left(\frac{1}{3}, -\frac{2}{3}\omega, -\frac{2}{3}\omega^2 \right) v_{\xi_{TS}}, \\
 \langle \xi'_t \rangle &= (0, 1, 0) v_{\xi'_t}. \quad (4.2)
 \end{aligned}$$

Note that ϕ'_t , ϕ_N , ξ_{TS} , ξ'_t and ℓ , N^c are not the only particles introduced in the model. More flavons, as well as driving fields, have to be introduced to achieve the vacuum alignment self-consistently, and will be discussed in detail in the next subsection.

Terms for generating charged lepton and neutrino masses are given by

$$\begin{aligned}
 w_\ell &= \frac{y_e}{\Lambda^3} \ell (\phi'_t)^3 e^c H_d + \frac{y_\mu}{\Lambda^2} \ell (\phi'_t)^2 \mu^c H_d + \frac{y_\tau}{\Lambda} \ell \phi'_t \tau^c H_d, \\
 w_\nu &= \frac{y}{\Lambda} \ell H_u \phi_N N^c + \frac{\lambda}{\Lambda} (N^c N^c)_{\mathbf{2}_R} (\xi_{TS} \xi'_t)_{\mathbf{2}_R}, \quad (4.3)
 \end{aligned}$$

where $(\phi'_t)^3 = a_1(\phi'_t\phi'_t)_{(\mathbf{1}_L, \mathbf{1}_R)}\phi'_t + a_2((\phi'_t\phi'_t)_{(\mathbf{2}_L, \mathbf{1}_R)}\phi'_t)_{(\mathbf{3}'_L, \mathbf{1}'_R)} + a_3((\phi'_t\phi'_t)_{(\mathbf{3}_L, \mathbf{1}_R)}\phi'_t)_{(\mathbf{3}'_L, \mathbf{1}'_R)}$ represents any $(\mathbf{3}'_L, \mathbf{1}_R)$ contractions of trilinear couplings of ϕ'_t with $a_{1,2,3}$ being dimensionless coefficients, $(\phi'_t)^2 = (\phi'_t\phi'_t)_{(\mathbf{3}_L, \mathbf{1}_R)}$ is a bilinear $(\mathbf{3}_L, \mathbf{1}_R)$ contractions of ϕ'_t , and H_u is a trivial singlet in both S_{4L} and S_{4R} . After the ϕ'_t gains the VEV, we arrive at

$$\langle(\phi'_t)^3\rangle = (a_2 - 2a_3)v_{\phi'_t}^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \langle(\phi'_t)^2\rangle = v_{\phi'_t}^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \langle\phi'_t\rangle = v_{\phi'_t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad (4.4)$$

Its VEV, as well as the Higgs VEV, results in diagonal charged lepton mass matrix with diagonal entries given by

$$m_e = y_e(a_2 - 2a_3)\frac{v_{\phi'_t}^3}{\Lambda^3}v_{H_d}, \quad m_\mu = y_\mu\frac{v_{\phi'_t}^2}{\Lambda^2}v_{H_d}, \quad m_\tau = y_\tau\frac{v_{\phi'_t}}{\Lambda}v_{H_d}. \quad (4.5)$$

Below, we will focus on mass matrices in the neutrino sector.

Based on the ϕ_N vacuum alignment, we obtain the Dirac mass matrix as

$$M_D = y \begin{pmatrix} \frac{1}{\sqrt{11}} & 0 \\ \frac{-1}{\sqrt{11}} & \frac{-1}{\sqrt{2}} \\ \frac{3}{\sqrt{11}} & \frac{1}{\sqrt{2}} \end{pmatrix} \frac{v_{\phi_N}}{\Lambda} v_{H_u}. \quad (4.6)$$

Here, the minus sign in the last row of M_D is unphysical, which can be absorbed by re-defining $\ell_\tau \rightarrow -\ell_\tau$. The doublet contraction of ξ_{TS} and ξ'_t gives rise to

$$\langle(\xi_{TS}\xi'_t)_{\mathbf{2}_R}\rangle = (2\omega, 1)\frac{-v_{\xi_{TS}}v_{\xi'_t}}{3}, \quad (4.7)$$

leading to the Majorana mass matrix

$$M_M = \lambda \begin{pmatrix} 2\omega & 0 \\ 0 & 1 \end{pmatrix} \frac{-v_{\xi_{TS}}v_{\xi'_t}}{3\Lambda}. \quad (4.8)$$

After right-handed neutrinos are integrated out, according to the seesaw mechanism, the active neutrino coupling matrix is given by

$$M_\nu = m_s \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 + 11\omega & 3 + 11\omega \\ 1 & 3 + 11\omega & 1 + 11\omega \end{pmatrix} \quad (4.9)$$

which reproduces Case I of eq. (2.2), with

$$m_s = \frac{3\omega^2 g^2 v_{\phi_N}^2 v_{H_u}^2}{22\lambda v_{\xi_T} v_{\xi'_t} \Lambda}. \quad (4.10)$$

4.2 Vacuum alignments

The vacuum alignment, in particular for ϕ_N in eq. (4.2), is not obvious. We set up this subsection for a detailed analysis of how these flavons gain the required VEVs. More flavons, which do not directly contribute to lepton masses, but influence on the other flavon VEVs, have to be introduced. First of all, we introduce three flavons $\phi_T \sim (\mathbf{3}_L, \mathbf{1}_R)$, $\phi_S \sim (\mathbf{3}_L, \mathbf{1}_R)$, $\phi_U \sim (\mathbf{3}'_L, \mathbf{1}_R)$ and require their VEVs invariant under the transformation of generators T, S, U of S_4 , respectively,

$$\langle \phi_T \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} v_{\phi_T}, \quad \langle \phi_S \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} v_{\phi_S}, \quad \langle \phi_U \rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} v_{\phi_U}. \quad (4.11)$$

The generators S, T and U are given in appendix A. These VEVs can be easily obtained and have been discussed in a lot of S_4 models (see e.g., [45–48]). Here, we give an example, with driving terms for these VEVs given by

$$w_d \supset \phi_T^d (\phi_T \phi_T)_{(\mathbf{2}_L, \mathbf{1}_R)} + \phi_S^d (\phi_S \phi_S)_{(\mathbf{3}_L, \mathbf{1}_R)} + \phi_U^d \left[A_U \phi_U + (\phi_T \phi_S)_{(\mathbf{3}'_L, \mathbf{1}_R)} \right]. \quad (4.12)$$

Here and in the following, we only consider renormalisable couplings. And any dimensionless coefficients which do not influence to our later discussion are ignored. A_U is a normalised parameter with a mass unit. The driving fields are arranged as suitable multiplets to keep each term satisfying the flavour symmetry. Minimisation of the superpotential respect to the driving fields ϕ_T^d , ϕ_S^d and ϕ_U^d gives the following equations

$$\begin{aligned} (\phi_T \phi_T)_{(\mathbf{2}_L, \mathbf{1}_R)} &= \begin{pmatrix} \phi_{T,2}^2 + 2\phi_{T,1}\phi_{T,3} \\ \phi_{T,3}^2 + 2\phi_{T,1}\phi_{T,2} \end{pmatrix} = 0, \\ (\phi_S \phi_S)_{(\mathbf{3}_L, \mathbf{1}_R)} &= 2 \begin{pmatrix} \phi_{S,1}^2 - \phi_{S,2}\phi_{S,3} \\ \phi_{S,3}^2 - \phi_{S,1}\phi_{S,2} \\ \phi_{S,2}^2 - \phi_{S,1}\phi_{S,3} \end{pmatrix} = 0, \\ A_U \phi_U + (\phi_T \phi_S)_{(\mathbf{3}'_L, \mathbf{1}_R)} &= A_U \begin{pmatrix} \phi_{U,1} \\ \phi_{U,2} \\ \phi_{U,3} \end{pmatrix} + \begin{pmatrix} \phi_{T,2}\phi_{S,3} - \phi_{T,3}\phi_{S,2} \\ \phi_{T,1}\phi_{S,2} - \phi_{T,2}\phi_{S,1} \\ \phi_{T,3}\phi_{S,1} - \phi_{T,1}\phi_{S,3} \end{pmatrix} = 0, \end{aligned} \quad (4.13)$$

respectively. The first two equations determine directions of $\langle \phi_T \rangle$ and $\langle \phi_S \rangle$ in eq. (4.11) with v_{ϕ_T} and v_{ϕ_S} undetermined. Taking $\langle \phi_T \rangle$ and $\langle \phi_S \rangle$ to the third equation $\langle \phi_T \rangle$ is determined with correlation $v_{\phi_U} = -v_{\phi_T}v_{\phi_S}/A_U$ satisfied.

It is worth noting that full solutions for $(\phi_T \phi_T)_{(\mathbf{2}_L, \mathbf{1}_R)} = 0$ are given by

$$(1, 0, 0)^T v_{\phi_T}, \quad \left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3} \right)^T v_{\phi_T}, \quad \left(\frac{1}{3}, -\frac{2}{3}\omega, -\frac{2}{3}\omega^2 \right)^T v_{\phi_T}, \quad \left(\frac{1}{3}, -\frac{2}{3}\omega^2, -\frac{2}{3}\omega \right)^T v_{\phi_T} \quad (4.14)$$

with v_{ϕ_T} undetermined. All these solution are related with each other by S_{4L} conjugacy transformation. By randomly choosing one of these solutions as the VEV, one can always

rotate it into the $(1, 0, 0)^T$ direction (see appendix B). Therefore, we fix the flavon VEV at the first solution without loss of generality.

The VEV of ϕ'_t can be obtained by evolving ϕ_T in its driving terms

$$w_d \supset \phi_t^d (\phi'_t \phi'_t)_{(\mathbf{1}_L, \mathbf{1}_R)} + \phi_t^{d'} (\phi_T \phi'_t)_{(\mathbf{1}'_L, \mathbf{1}_R)}. \quad (4.15)$$

Taking $\langle \phi_T \rangle$ into account, minimisation of these terms is explicitly written out as

$$\begin{aligned} (\phi'_t \phi'_t)_{(\mathbf{1}_L, \mathbf{1}_R)}|_{\langle \phi_T \rangle} &= \phi_{t,1}'^2 + 2\phi_{t,2}'\phi_{t,3}' = 0, \\ (\phi_T \phi'_t)_{(\mathbf{1}'_L, \mathbf{1}_R)}|_{\langle \phi_T \rangle} &= v_{\phi_T} \phi_{t,1}' = 0, \end{aligned} \quad (4.16)$$

which leads to $\phi_{t,1}' = 0$ and $\phi_{t,2}'\phi_{t,3}' = 0$. Without lose of generality, we choose $\phi_{t,3}' = 0$ and $\phi_{t,2}' = v_{\phi_t'}$ with $v_{\phi_t'}$ undetermined. We introduce another triplet flavon $\tilde{\phi}'_t$ for our later use. With similar constructions of the driving terms as in eq. (4.15),

$$w_d \supset \tilde{\phi}_t^d (\tilde{\phi}'_t \tilde{\phi}'_t)_{(\mathbf{1}_L, \mathbf{1}_R)} + \tilde{\phi}_t^{d'} (\phi_T \tilde{\phi}'_t)_{(\mathbf{1}'_L, \mathbf{1}_R)} \quad (4.17)$$

and adding one more term

$$w_d \supset \varphi_T^d \left[\mu_x^2 + (\phi'_t \tilde{\phi}'_t)_{(\mathbf{1}_L, \mathbf{1}_R)} \right], \quad (4.18)$$

$\langle \tilde{\phi}'_t \rangle = (0, 0, 1)^T v_{\tilde{\phi}'_t}$ can be determined and the correlation $\langle \mu_x^2 \rangle + v_{\phi_t'} v_{\tilde{\phi}'_t} = 0$ is obtained. Here, μ_x^2 is not a free parameter but a contraction of some other flavons. Its exact expression will be given later after the charge assignment is complete.

We then consider flavons which transform non-trivially in S_{4R} . We introduced additional ξ_T . This flavon, together with ξ_{TS} and ξ'_t , are arranged as $(\mathbf{1}_L, \mathbf{3}_R)$, $(\mathbf{1}_L, \mathbf{3}_R)$ and $(\mathbf{1}_L, \mathbf{3}'_R)$, respectively. The driving terms are given by

$$w_d \supset \xi_T^d (\xi_T \xi_T)_{(\mathbf{1}_L, \mathbf{2}_R)} + \xi_{TS}^d (\xi_{TS} \xi_{TS})_{(\mathbf{1}_L, \mathbf{2}_R)} + \xi_t^d (\xi'_t \xi'_t)_{(\mathbf{1}_L, \mathbf{1}_R)} + \xi_t^{d'} (\xi_T \xi'_t)_{(\mathbf{1}_L, \mathbf{1}'_R)}. \quad (4.19)$$

Minimisation of the first two terms lead to $(\xi_T \xi_T)_{(\mathbf{1}_L, \mathbf{2}_R)} = (\xi_{TS} \xi_{TS})_{(\mathbf{1}_L, \mathbf{2}_R)} = 0$. Full solutions for ξ_T are given by

$$(1, 0, 0)v_{\xi_T}, \quad \left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3} \right) v_{\xi_T}, \quad \left(\frac{1}{3}, -\frac{2}{3}\omega, -\frac{2}{3}\omega^2 \right) v_{\xi_T}, \quad \left(\frac{1}{3}, -\frac{2}{3}\omega^2, -\frac{2}{3}\omega \right) v_{\xi_T} \quad (4.20)$$

with v_{ξ_T} undetermined. Those for ξ_{TS} can be similarly written out. VEVs of ξ_T and ξ_{TS} could be any of them, respectively. In the case that both flavons preserve Z_3 symmetries, there is a larger probability that the direction $\langle \xi_{TS} \rangle$ is different from that of $\langle \xi_T \rangle$. And therefore, the S_4 transformation cannot rotate both directions to $(1, 0, 0)$. Instead, we can fix $\langle \xi_T \rangle$ at $(1, 0, 0)v_{\xi_T}$, and ξ_{TS} at $(\frac{1}{3}, -\frac{2}{3}\omega, -\frac{2}{3}\omega^2)v_{\xi_{TS}}$ as in eq. (4.2). The later is invariant under a different Z_3 symmetry generated by TS . For more detail of how to determine these VEVs, please see appendix B. Note that our model with current setup cannot fully determine the ξ_{TS} VEV, but leaves a large possibility for ξ_{TS} to take such a required VEV. In order to determine the ξ_{TS} VEV, another way could be to consider cross couplings between ξ_{TS} and ξ_T . With suitable small cross couplings between ξ_{TS} and ξ_T , the global

vacuum may be obtained when $\langle \xi_{TS} \rangle$ and $\langle \xi_T \rangle$ take different directions, and this vacuum degeneracy may be avoided. The last two driving terms determine the VEV $\langle \xi'_t \rangle$. Once $\langle \xi_T \rangle$ is fixed to be $\propto (1, 0, 0)$, we derive $\langle \xi'_t \rangle = (0, 1, 0)v_{\xi'_t}$, following similar discussion as that for ϕ'_t .

To achieve the VEV for ϕ_N is a non-trivial task. Let us first denote ϕ_N by $\phi_N = (\phi_{\text{atm}}, \phi_{\text{sol}})$. Both ϕ_{atm} and ϕ_{sol} are triplets $\mathbf{3}_L$ of S_{4L} , while ϕ_{atm} and ϕ_{sol} form a doublet $\mathbf{2}_R$ of S_{4R} . For convenience, we denote VEVs of ϕ_{atm} and ϕ_{sol} respectively as

$$\langle \phi_{\text{atm}} \rangle = \begin{pmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} v_{\phi_{\text{atm}}}, \quad \langle \phi_{\text{sol}} \rangle = \begin{pmatrix} \frac{1}{\sqrt{11}} \\ \frac{-1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{pmatrix} v_{\phi_{\text{sol}}}. \quad (4.21)$$

With this notation, we now address question of how to obtain the required ϕ_N VEV into three steps:

- I. To construct superpotential terms to separate ϕ_{atm} and ϕ_{sol} from the same doublet of S_{4R} .
- II. To drive ϕ_{atm} and ϕ_{sol} separately to gain VEVs with different directions, i.e., one in $(0, -1, 1)^T$ and the other in $(1, -1, 3)$.
- III. To force $v_{\phi_{\text{atm}}}$ identical to $v_{\phi_{\text{sol}}}$, i.e., $v_{\phi_{\text{atm}}} = v_{\phi_{\text{sol}}} \equiv v_{\phi_N}$.

How to achieve each step is given as following.

For the first step, as ϕ_{atm} and ϕ_{sol} form a $\mathbf{2}_R$ of S_{4R} , we need to take care of the correlation between directions of the ϕ_{atm} VEV and ϕ_{sol} VEV. One way to separate them is introducing two flavons ρ and $\tilde{\rho}$, which are doublets of S_{4R} and gain VEVs $\propto (1, 0)$ and $(0, 1)$, respectively. The singlet contraction in S_{4R} leaves ϕ_{sol} and ϕ_{atm} separately. For the convenience of step II, we arrange these flavons also as doublets of S_{4L} , i.e., $\rho \sim \tilde{\rho} \sim (\mathbf{2}_L, \mathbf{2}_R)$ and their VEVs in the following form

$$\langle \rho \rangle = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} v_{\rho}, \quad \langle \tilde{\rho} \rangle = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} v_{\tilde{\rho}}. \quad (4.22)$$

In S_{4L} , these VEVs take the direction $(1, 1)^T$, invariant under the generator U . This is prepared for our later use in step II. In order to realise these VEVs, we construct the driving terms as

$$w_d \supset \rho^d (\rho \rho)_{(\mathbf{2}_L, \mathbf{1}_R)} + \tilde{\rho}^d (\tilde{\rho} \tilde{\rho})_{(\mathbf{2}_L, \mathbf{1}_R)} + \rho^{d'} [(\rho \tilde{\rho})_{(\mathbf{2}_L, \mathbf{1}_R)} + g_U (\phi_U \phi_U)_{(\mathbf{2}_L, \mathbf{1}_R)}] + \rho^{d''} (\rho \tilde{\rho})_{(\mathbf{1}'_L, \mathbf{1}_R)}, \quad (4.23)$$

where g_U corresponds the ratio of coefficients between $(\rho\tilde{\rho})_{(2_L,1_R)}$ and $(\phi_U\phi_U)_{(2_L,1_R)}$ terms. Minimisation of these terms gives rise to

$$\begin{aligned}
 (\rho\rho)_{(2_L,1_R)} &= \begin{pmatrix} 2\rho_{21}\rho_{22} \\ 2\rho_{11}\rho_{12} \end{pmatrix} = 0, \\
 (\tilde{\rho}\tilde{\rho})_{(2_L,1_R)} &= \begin{pmatrix} 2\tilde{\rho}_{21}\tilde{\rho}_{22} \\ 2\tilde{\rho}_{11}\tilde{\rho}_{12} \end{pmatrix} = 0, \\
 (\rho\tilde{\rho})_{(2_L,1_R)} + (\phi_U\phi_U)_{(2_L,1_R)}|_{\langle\phi_U\rangle} &= \begin{pmatrix} \rho_{21}\tilde{\rho}_{22} + \rho_{22}\tilde{\rho}_{21} \\ \rho_{11}\tilde{\rho}_{12} + \rho_{12}\tilde{\rho}_{11} \end{pmatrix} + g_U \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_{\phi_U}^2 = 0, \\
 (\rho\tilde{\rho})_{(1'_L,1_R)} &= \rho_{11}\tilde{\rho}_{22} - \rho_{21}\tilde{\rho}_{12} + \rho_{12}\tilde{\rho}_{21} - \rho_{22}\tilde{\rho}_{11} = 0, \quad (4.24)
 \end{aligned}$$

where the VEV of ϕ_U , $\langle\phi_U\rangle = (0, 1, -1)^T v_{\phi_U}$, has been used. These equations determine eq. (4.22) (or in turn) with

$$v_\rho v_{\tilde{\rho}} = -g_U v_{\phi_U}^2 \quad (4.25)$$

satisfied. Note that v_ρ and $v_{\tilde{\rho}}$ cannot be determined by the above equation. They will be determined later once other minimisation conditions are satisfied.

To achieve step II, the U -invariant direction of $\langle\rho\rangle$ and $\langle\tilde{\rho}\rangle$ in S_{4L} is important. We write out driving terms to determine directions of $\langle\phi_{\text{sol}}\rangle$ and $\langle\phi_{\text{atm}}\rangle$ with ρ and $\tilde{\rho}$ involved,

$$w_d \supset \phi_N^d \left[(\phi_N \rho)_{(3_L,1'_R)} + g_t (\tilde{\phi}'_t \phi_U)_{(3_L,1'_R)} \right] + \phi_N^d \left[(\phi_N \tilde{\rho})_{(3_L,1_R)} + g_\eta \eta \phi_U \right]. \quad (4.26)$$

Once ρ , $\tilde{\rho}$ and ϕ_U get the VEVs, the minimisation leads to

$$\begin{aligned}
 (\phi_N \rho)_{(3_L,1'_R)}|_{\langle\rho\rangle} + g_t (\tilde{\phi}'_t \phi_U)_{(3_L,1'_R)}|_{\langle\tilde{\phi}'_t\rangle, \langle\phi_U\rangle} &= \begin{pmatrix} \phi_{\text{sol},2} + \phi_{\text{sol},3} \\ \phi_{\text{sol},3} + \phi_{\text{sol},1} \\ \phi_{\text{sol},1} + \phi_{\text{sol},2} \end{pmatrix} v_\rho - g_t \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} v_{\tilde{\phi}'_t} v_{\phi_U} = 0, \\
 (\phi_N \tilde{\rho})_{(3_L,1_R)}|_{\langle\tilde{\rho}\rangle} + g_\eta \eta \phi_U|_{\langle\phi_U\rangle, \langle\eta\rangle} &= \begin{pmatrix} \phi_{\text{atm},2} + \phi_{\text{atm},3} \\ \phi_{\text{atm},3} + \phi_{\text{atm},1} \\ \phi_{\text{atm},1} + \phi_{\text{atm},2} \end{pmatrix} v_{\tilde{\rho}} + g_\eta \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} v_\eta v_{\phi_U} = 0, \quad (4.27)
 \end{aligned}$$

where $\phi_{\text{sol}} = (\phi_{\text{sol},1}, \phi_{\text{sol},2}, \phi_{\text{sol},3})^T$ and $\phi_{\text{atm}} = (\phi_{\text{atm},1}, \phi_{\text{atm},2}, \phi_{\text{atm},3})^T$. As shown above, the contractions $(\phi_N \rho)_{(3_L,1'_R)}$ and $(\phi_N \tilde{\rho})_{(3_L,1'_R)}$ select ϕ_{sol} and ϕ_{atm} , respectively. After ρ and $\tilde{\rho}$ gain their VEVs, ϕ_{sol} and ϕ_{atm} are separated and gain the required VEV directions in eq. (4.21) separately, with correlations

$$2 \frac{v_{\phi_{\text{sol}}}}{\sqrt{11}} v_\rho = g_t v_{\tilde{\phi}'_t} v_{\phi_U}, \quad \frac{v_{\phi_{\text{atm}}}}{\sqrt{2}} v_{\tilde{\rho}} = -g_\eta v_\eta v_{\phi_U} \quad (4.28)$$

satisfied. Here, we have applied the technique developed in ref. [29] to achieve the direction $(1, -1, 3)^T$.

Finally, we consider how to achieve $v_{\phi_{\text{sol}}} = v_{\phi_{\text{atm}}}$ in step III. We introduce another flavon $\tilde{\phi}_N \equiv (\tilde{\phi}_{\text{atm}}, \tilde{\phi}_{\text{sol}})$, which transforms as $(\mathbf{3}_L, \mathbf{2}_R)$, the same as ϕ_N . Given the following driving terms similar to those for $\tilde{\phi}_N$,

$$w_d \supset \tilde{\phi}_N^d \left[(\tilde{\phi}_N \tilde{\rho})_{(\mathbf{3}_L, \mathbf{1}'_R)} + g_t' (\phi_t' \phi_U)_{(\mathbf{3}_L, \mathbf{1}'_R)} \right] + \tilde{\phi}_N^d \left[(\tilde{\phi}_N \rho)_{(\mathbf{3}_L, \mathbf{1}'_R)} + g_{\tilde{\eta}} \tilde{\eta} \phi_U \right] \quad (4.29)$$

and following a similar analysis, we arrive at

$$\langle \tilde{\phi}_{\text{atm}} \rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} v_{\tilde{\phi}_{\text{atm}}}, \quad \langle \tilde{\phi}_{\text{sol}} \rangle = \begin{pmatrix} \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \\ -\frac{1}{\sqrt{11}} \end{pmatrix} v_{\tilde{\phi}_{\text{sol}}} \quad (4.30)$$

with $v_{\tilde{\phi}_{\text{sol}}}$ and $v_{\tilde{\phi}_{\text{atm}}}$ satisfying

$$2 \frac{v_{\tilde{\phi}_{\text{sol}}}}{\sqrt{11}} v_{\tilde{\rho}} = g_t' v_{\phi_t'} v_{\phi_U}, \quad \frac{v_{\tilde{\phi}_{\text{atm}}}}{\sqrt{2}} v_{\rho} = -g_{\tilde{\eta}} v_{\tilde{\eta}} v_{\phi_U}. \quad (4.31)$$

Then, we construct the driving terms

$$w_d \supset \varphi_N^d (\phi_N \tilde{\phi}_N)_{(\mathbf{1}_L, \mathbf{1}'_R)} + \varphi_N^d \left[(\phi_N \tilde{\phi}_N)_{(\mathbf{1}_L, \mathbf{2}_R)} + A_{\sigma} \sigma \right]. \quad (4.32)$$

These terms result in

$$\begin{aligned} (\phi_N \tilde{\phi}_N)_{(\mathbf{1}_L, \mathbf{1}'_R)} &= v_{\phi_{\text{atm}}} v_{\tilde{\phi}_{\text{atm}}} - v_{\phi_{\text{sol}}} v_{\tilde{\phi}_{\text{sol}}} = 0, \\ (\phi_N \tilde{\phi}_N)_{(\mathbf{1}_L, \mathbf{2}_R)} + A_{\sigma} \sigma|_{\langle \sigma \rangle} &= \frac{4}{\sqrt{22}} (v_{\phi_{\text{sol}}} v_{\tilde{\phi}_{\text{atm}}}, v_{\phi_{\text{atm}}} v_{\tilde{\phi}_{\text{sol}}}) = (1, 1) A_{\sigma} v_{\sigma}, \end{aligned} \quad (4.33)$$

where $\langle \sigma \rangle = (1, 1) v_{\sigma}$ has been used. Following a straightforward calculation, we obtain

$$v_{\phi_{\text{atm}}} = v_{\phi_{\text{sol}}}, \quad v_{\tilde{\phi}_{\text{atm}}} = v_{\tilde{\phi}_{\text{sol}}}, \quad v_{\phi_{\text{atm}}} v_{\tilde{\phi}_{\text{atm}}} = \frac{\sqrt{22}}{4} A_{\sigma} v_{\sigma}. \quad (4.34)$$

Combining the above equation with eqs. (4.25) and (4.31), we further determine v_{ρ} and $v_{\tilde{\rho}}$,

$$v_{\rho} = \sqrt{\frac{11 g_t g_U v_{\phi_t'}}{2 \sqrt{2} g_{\eta} v_{\eta}}} v_{\phi_U}, \quad v_{\tilde{\rho}} = \sqrt{\frac{11 g_t' g_U v_{\phi_t'}}{2 \sqrt{2} g_{\tilde{\eta}} v_{\tilde{\eta}}}} v_{\phi_U}. \quad (4.35)$$

4.3 Charge assignment of the model

Finally, we list our particle content in table 1. Representations of all fields in $S_{4L} \times S_{4R}$ are explicitly the same as introduced in the last subsection. In order to forbid unnecessary terms, e.g., $\phi_t^d \phi_U \phi_U$, which may violate the required directions but cannot be forbidden by two S_4 's, we introduce two U(1) symmetries. The unnecessary Goldstone bosons accompanying with U(1) breakings can be avoided by considering small U(1)-explicit-breaking terms, which will not be discussed here. Table 1 guarantees superpotential terms w_{ℓ} and w_{ν} in eq. (4.3) to generate lepton masses. No extra higher-dimensional operators up to $d = 6$ should be considered. These U(1) symmetries can in principle be replaced by several Z_n symmetries with a careful arrangement of all field charges. For the driving superpotential, we only consider renormalisable terms. The full driving superpotential w_d is the collection of eqs. (4.12), (4.15), (4.17), (4.18), (4.19), (4.23), (4.26), (4.29), and (4.32), but re-expressing μ_x^2 in eq. (4.18) as

$$\mu_x^2 = g_{t1} \eta \tilde{\eta} + g_{t2} (\phi_N \phi_N)_{(\mathbf{1}_L, \mathbf{1}_R)}, \quad (4.36)$$

where g_{t1} and g_{t2} are dimensionless coefficients.

Fields	S_{4L}	S_{4R}	U(1)	U(1)'	Fields	S_{4L}	S_{4R}	U(1)	U(1)'	
Higgs & leptons	ℓ	$\mathbf{3}_L$	$\mathbf{1}_R$	1	6	ϕ_T^d	$\mathbf{2}_L$	$\mathbf{1}_R$	-4	-4
	e^c	$\mathbf{1}'_L$	$\mathbf{1}'_R$	-4	3	ϕ_S^d	$\mathbf{3}_L$	$\mathbf{1}_R$	4	-2
	μ^c	$\mathbf{1}_L$	$\mathbf{1}_R$	-3	0	ϕ_U^d	$\mathbf{3}'_L$	$\mathbf{1}_R$	0	-3
	τ^c	$\mathbf{1}'_L$	$\mathbf{1}'_R$	-2	-3	ϕ_t^d	$\mathbf{1}_L$	$\mathbf{1}_R$	-2	6
	N^c	$\mathbf{1}_L$	$\mathbf{2}_R$	-4	-3	$\phi_t^{d'}$	$\mathbf{1}'_L$	$\mathbf{1}_R$	-3	1
	H_u	$\mathbf{1}_L$	$\mathbf{1}_R$	0	0	$\tilde{\phi}_t^d$	$\mathbf{1}_L$	$\mathbf{1}_R$	-3	1
	H_d	$\mathbf{1}_L$	$\mathbf{1}_R$	0	0	$\tilde{\phi}_t^{d'}$	$\mathbf{1}'_L$	$\mathbf{1}_R$	-9	3
Flavons	ϕ_T	$\mathbf{3}_L$	$\mathbf{1}_R$	2	2	φ_T^d	$\mathbf{1}_L$	$\mathbf{1}_R$	-8	8
	ϕ_S	$\mathbf{3}_L$	$\mathbf{1}_R$	-2	1	ξ_T^d	$\mathbf{1}_L$	$\mathbf{2}_R$	12	-8
	ϕ_U	$\mathbf{3}'_L$	$\mathbf{1}_R$	0	3	ξ_{TS}^d	$\mathbf{1}_L$	$\mathbf{2}_R$	-10	-6
	ϕ'_t	$\mathbf{3}'_L$	$\mathbf{1}'_R$	1	-3	ξ_t^d	$\mathbf{1}_L$	$\mathbf{1}_R$	-6	-6
	$\tilde{\phi}'_t$	$\mathbf{3}'_L$	$\mathbf{1}'_R$	7	-5	$\xi_t^{d'}$	$\mathbf{1}_L$	$\mathbf{1}'_R$	3	-7
	ξ_T	$\mathbf{1}_L$	$\mathbf{3}_R$	-6	4	ρ^d	$\mathbf{2}_L$	$\mathbf{1}_R$	-8	-2
	ξ_{TS}	$\mathbf{1}_L$	$\mathbf{3}_R$	5	3	$\tilde{\rho}^d$	$\mathbf{2}_L$	$\mathbf{1}_R$	8	-10
	ξ'_t	$\mathbf{1}_L$	$\mathbf{3}'_R$	3	3	$\rho^{d'}$	$\mathbf{2}_L$	$\mathbf{1}_R$	0	-6
	ρ	$\mathbf{2}_L$	$\mathbf{2}_R$	4	1	$\rho^{d''}$	$\mathbf{1}'_L$	$\mathbf{1}_R$	0	-6
	$\tilde{\rho}$	$\mathbf{2}_L$	$\mathbf{2}_R$	-4	5	ϕ_N^d	$\mathbf{3}_L$	$\mathbf{1}'_R$	-7	2
	ϕ_N	$\mathbf{3}_L$	$\mathbf{2}_R$	3	-3	$\phi_N^{d'}$	$\mathbf{3}_L$	$\mathbf{1}'_R$	1	-2
	$\tilde{\phi}_N$	$\mathbf{3}_L$	$\mathbf{2}_R$	5	-5	$\tilde{\phi}_N^d$	$\mathbf{3}_L$	$\mathbf{1}'_R$	-1	0
	η	$\mathbf{1}'_L$	$\mathbf{1}_R$	-1	-1	$\tilde{\phi}_N^{d'}$	$\mathbf{3}_L$	$\mathbf{1}'_R$	-9	4
	$\tilde{\eta}$	$\mathbf{1}'_L$	$\mathbf{1}_R$	9	-7	φ_N^d	$\mathbf{1}_L$	$\mathbf{1}'_R$	-8	8
	σ	$\mathbf{1}_L$	$\mathbf{2}_R$	8	-8	$\varphi_N^{d'}$	$\mathbf{1}_L$	$\mathbf{2}_R$	-8	8

Table 1. Field arrangements of the $\mu\tau$ -LSS model in $S_{4L} \times S_{4R} \times \text{U}(1) \times \text{U}(1)'$. In addition, we assume a standard $\text{U}(1)_R$ symmetry with the charge assignments: +1 for lepton, 0 for Higgs and flavon fields, and +2 for driving fields.

5 Conclusion

In this paper we have proposed a $\mu - \tau$ reflection symmetric Littlest Seesaw ($\mu\tau$ -LSS) model. In this model the two mass parameters of the LSS model are fixed to be in a special ratio by symmetry, so that the resulting neutrino mass matrix in the flavour basis (after the seesaw mechanism has been applied) satisfies $\mu - \tau$ reflection symmetry and has only one free adjustable parameter, namely an overall free mass scale.

The resulting $\mu\tau$ -LSS model predicts $\theta_{23} = 45^\circ$, $\delta = -90^\circ$ and $\theta_{12} \approx 34.5^\circ$, which are compatible with data. The predicted θ_{13} and the ratio of mass square differences $\Delta m_{21}^2 / \Delta m_{31}^2$ are out of the 3σ ranges of the current global oscillation data. However, with radiative corrections included, assuming SUSY, all mixing parameters and the ra-

tio $\Delta m_{21}^2/\Delta m_{31}^2$ depend on one single free parameter, namely ϵ , which can bring all the observables within their 3σ ranges.

We have constructed a concrete lepton flavour model in $S_{4L} \times S_{4R} \times U(1) \times U(1)'$ to realise littlest mu-tau seesaw model, S_{4L} for left-handed fermions and S_{4R} for right-handed fermions. The two right-handed neutrinos are arranged as singlets in S_{4L} , in usual constrained sequential dominance. However they are arranged as a doublet S_{4R} , which is necessary to achieve the desired ratio of effective mass parameters $\frac{m_a}{m_s} = 11$ as required for $\mu - \tau$ reflection symmetry.

The desired ratio of effective mass parameters $\frac{m_a}{m_s} = 11$ also relies on special vacuum alignments which have been carefully realised with the help of SUSY driving fields. Specifically, the flavon $\phi_N = (\phi_{\text{atm}}, \phi_{\text{sol}})$, which contributing to the neutrino Dirac mass matrix, achieves VEV $\langle \phi_{\text{sol}} \rangle = (1, -1, 3)^T v_{\phi_{\text{sol}}}/\sqrt{11}$ and $\langle \phi_{\text{atm}} \rangle = (0, -1, 1)^T v_{\phi_{\text{atm}}}/\sqrt{2}$ separately in S_{4L} and $v_{\phi_{\text{sol}}} = v_{\phi_{\text{atm}}}$ due to the constraint of S_{4R} .

Although the high energy model is rather complicated, involving $(S_4 \times U(1))^2$, with many flavons and driving fields, the low energy neutrino mass matrix has ultimate simplicity, with a built-in $\mu - \tau$ reflection symmetry and tri-maximal mixing. Since the neutrino mass matrix only depends on one overall mass scale, the low energy observables are completely specified in terms of one radiative correction parameter, leading to testable predictions for all lepton mixing angles and CP phases, as well as neutrino mass ratios.

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A Group theory of S_4

S_4 is the permutation group of 4 objects. Its three generators S , T and U satisfying the equalities $T^3 = S^2 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = 1$, from which $(STU)^4 = 1$ is automatically obtained. The minimal number of generators of S_4 is actually two [49, 50]. However, we follow the presentation in [51] to emphasise the Z_2 residual symmetries generated by S and U , respectively. S_4 contains 5 irreducible representations (irreps), $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{3}'$. Throughout this paper, we work in the basis where the generator T diagonal. Generators of S_4 in different irreps are listed in table 2.

The Kronecker products between different irreps can be easily obtained:

$$\begin{aligned} \mathbf{1} \otimes \mathbf{r} &= \mathbf{r}, \quad \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}, \quad \mathbf{1}' \otimes \mathbf{2} = \mathbf{2}, \quad \mathbf{1}' \otimes \mathbf{3} = \mathbf{3}', \quad \mathbf{1}' \otimes \mathbf{3}' = \mathbf{3}, \quad \mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}, \\ \mathbf{2} \otimes \mathbf{3} &= \mathbf{2} \otimes \mathbf{3}' = \mathbf{3} \oplus \mathbf{3}', \quad \mathbf{3} \otimes \mathbf{3} = \mathbf{3}' \otimes \mathbf{3}' = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}', \quad \mathbf{3} \otimes \mathbf{3}' = \mathbf{1}' \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}', \end{aligned} \quad (\text{A.1})$$

where $\mathbf{r}$ represents any irrep of S_4 . The Clebsch-Gordan coefficients for products of any two irreps a and b are listed in table 3.

	T	S	U
1	1	1	1
1'	1	1	-1
2	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
3'	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Table 2. The representation matrices for the S_4 generators T , S and U , where $\omega = e^{2\pi i/3}$.

• $a \sim \mathbf{1}, b \sim \mathbf{r}$	$(ab)_{\mathbf{r}} = a(b_1, b_2, \dots)$
• $a \sim b \sim \mathbf{1}'$	$(ab)_{\mathbf{1}} = ab$
• $a \sim \mathbf{1}', b \sim \mathbf{2}$	$(ab)_{\mathbf{2}} = a(b_1, -b_2)$
• $a \sim \mathbf{1}', b \sim \mathbf{3}(\mathbf{3}')$	$(ab)_{\mathbf{3}'(\mathbf{3})} = a(b_1, b_2, b_3)$
• $a \sim b \sim \mathbf{2}$	$(ab)_{\mathbf{1}} = a_1b_2 + a_2b_1$ $(ab)_{\mathbf{1}'} = a_1b_2 - a_2b_1$ $(ab)_{\mathbf{2}} = (a_2b_2, a_1b_1)$
• $a \sim \mathbf{2}, b \sim \mathbf{3}(\mathbf{3}')$	$(ab)_{\mathbf{3}(\mathbf{3}')} = (a_1b_2 + a_2b_3, a_1b_3 + a_2b_1, a_1b_1 + a_2b_2)$ $(ab)_{\mathbf{3}'(\mathbf{3})} = (a_1b_2 - a_2b_3, a_1b_3 - a_2b_1, a_1b_1 - a_2b_2)$
• $a \sim b \sim \mathbf{3}$ or $\mathbf{3}'$	$(ab)_{\mathbf{1}} = a_1b_1 + a_2b_3 + a_3b_2$ $(ab)_{\mathbf{2}} = (a_2b_2 + a_1b_3 + a_3b_1, a_3b_3 + a_1b_2 + a_2b_1)$ $(ab)_{\mathbf{3}} = (2a_1b_1 - a_2b_3 - a_3b_2, 2a_3b_3 - a_1b_2 - a_2b_1, 2a_2b_2 - a_3b_1 - a_1b_3)$ $(ab)_{\mathbf{3}'} = (a_2b_3 - a_3b_2, a_1b_2 - a_2b_1, a_3b_1 - a_1b_3)$
• $a \sim \mathbf{3}, b \sim \mathbf{3}'$	$(ab)_{\mathbf{1}'} = a_1b_1 + a_2b_3 + a_3b_2$ $(ab)_{\mathbf{2}} = (a_2b_2 + a_1b_3 + a_3b_1, -a_3b_3 - a_1b_2 - a_2b_1)$ $(ab)_{\mathbf{3}} = (a_2b_3 - a_3b_2, a_1b_2 - a_2b_1, a_3b_1 - a_1b_3)$ $(ab)_{\mathbf{3}'} = (2a_1b_1 - a_2b_3 - a_3b_2, 2a_3b_3 - a_1b_2 - a_2b_1, 2a_2b_2 - a_3b_1 - a_1b_3)$

Table 3. Kronecker products and Clebsch-Gordan (CG) coefficients of S_4 . $\mathbf{r}$ represents any irrep of S_4 .

B Vacuum degeneracy

Degenerate vacua exist in theories of discrete symmetries. Discussion on structures and physical equivalence of degenerate vacua has been given in e.g., [52] based on the A_4 symmetry. This appendix is devoted to the discussion of vacuum degeneracy in S_4 . In section 4.2, we select the VEV for ϕ_T along $(1, 0, 0)^T$ direction in S_{4L} , and those for ξ_T and ξ_{TS} along the $(1, 0, 0)$ and $(\frac{1}{3}, -\frac{2}{3}\omega, -\frac{2}{3}\omega^2)$ directions in S_{4R} , respectively. Their validity is explained in the following. Given any S_4 triplet flavon $\Phi = (\Phi_1, \Phi_2, \Phi_3)^T$ and the superpotential $w = \Phi^d (\Phi\Phi)_2$ with Φ^d being an S_4 doublet driving field. The vacuum is solved via $\partial w / \partial \Phi^d = (\Phi\Phi)_2 = 0$, i.e.,

$$\begin{aligned}\frac{\partial w}{\partial \Phi_1^d} &= \Phi_2^2 + 2\Phi_1\Phi_3 = 0 \\ \frac{\partial w}{\partial \Phi_2^d} &= \Phi_3^2 + 2\Phi_1\Phi_2 = 0\end{aligned}\tag{B.1}$$

Straightforward calculation shows the full solution is given by

$$\langle \Phi \rangle^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} v_\Phi, \langle \Phi \rangle^{STS} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} v_\Phi, \langle \Phi \rangle^{TS} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3}\omega \\ -\frac{2}{3}\omega^2 \end{pmatrix} v_\Phi, \langle \Phi \rangle^{ST} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3}\omega^2 \\ -\frac{2}{3}\omega \end{pmatrix} v_\Phi\tag{B.2}$$

with v_Φ undetermined. These VEVs are invariant under the transformation of T , STS , TS and ST , respectively,

$$T \langle \Phi \rangle^T = \langle \Phi \rangle^T, STS \langle \Phi \rangle^{STS} = \langle \Phi \rangle^{STS}, TS \langle \Phi \rangle^{TS} = \langle \Phi \rangle^{TS}, ST \langle \Phi \rangle^{ST} = \langle \Phi \rangle^{ST},\tag{B.3}$$

and therefore, preserve residual symmetries generated by these elements,

$$Z_3^T = \{1, T, T^2\}, Z_3^{STS} = \{1, STS, ST^2S\}, Z_3^{TS} = \{1, TS, (TS)^2\}, Z_3^{ST} = \{1, ST, (ST)^2\},\tag{B.4}$$

respectively. Note that all these Z_3 symmetries are conjugate with each other. Their elements satisfy the following conjugacy transformations

$$STS = S T S^{-1}, TS = (ST)^{-1} T (ST), ST = (TS) T (TS)^{-1}.\tag{B.5}$$

Starting from one VEV, e.g., $\langle \Phi \rangle^T$, the rest degenerate VEVs are obtained via

$$T \langle \Phi \rangle^T = \langle \Phi \rangle^T \Rightarrow \begin{cases} STS S \langle \Phi \rangle^T = S \langle \Phi \rangle^T & \Rightarrow \langle \Phi \rangle^{STS} = S \langle \Phi \rangle^T \\ (ST)^{-1} T (ST) (ST)^{-1} \langle \Phi \rangle^T = (ST)^{-1} \langle \Phi \rangle^T & \Rightarrow \langle \Phi \rangle^{TS} = (ST)^{-1} \langle \Phi \rangle^T \\ (TS) T (TS)^{-1} (ST)^{-1} \langle \Phi \rangle^T = (TS) \langle \Phi \rangle^T & \Rightarrow \langle \Phi \rangle^{ST} = (TS) \langle \Phi \rangle^T \end{cases}.\tag{B.6}$$

Therefore, one vacuum maps to another under the transformation of some group element.

In flavour model building, if there is only one flavon whose VEV preserves a Z_3 , selecting one or another VEV, e.g., $\langle\Phi\rangle^T$ or $\langle\Phi\rangle^{STS}$, respectively, does not make a difference in the physical point of view. All VEVs connect with each other via the conjugacy transformation as discussed above. In other words, starting from $\langle\Phi\rangle^{STS}$, one can rotate it to $\langle\Phi\rangle^T$ via the reverse transformation of eq. (B.6). Therefore, it is always safe to use $\langle\Phi\rangle^T$ as the VEV for model building. This is what we have done for the VEV of ϕ_T , which is the only Z_3 -invariant VEV in S_{4L} .

However, If there are two flavons Φ and Φ' taking Z_3 -invariant VEVs at the same time, we must be careful about the VEV selection. Whether these VEVs preserve the same Z_3 or different Z_3 's are physically different.

The number of degenerate VEVs is four, as shown in (B.2). From a naive estimation, we know that there is a probability of $1/4$ that both $\langle\Phi\rangle$ and $\langle\Phi'\rangle$ preserve the same Z_3 . In this case, one can always rotate them to the Z_3^T one following the above discussion.

For the other case, with a probability of $3/4$, two flavon VEVs preserve different Z_3 symmetries. We argue that the discussion in the case that only one flavon VEV preserves a Z_3 does not hold. Instead, one can always rotate the Φ VEV to preserve Z_3^T and the Φ' VEV to preserve Z_3^{TS} . We explain this with the help of the following example. Without loss of generality, let us assume the VEV of Φ preserves Z_3^{STS} and that of Φ' preserves another one, Z_3^{TS} . Following the reverse transformation of eq. (B.6), the first flavon VEV $\langle\Phi\rangle^{STS}$ can always be rotated to $\langle\Phi\rangle^T$,

$$S^{-1}\langle\Phi\rangle^{STS} = \langle\Phi\rangle^T \quad (\text{B.7})$$

The same transformation acting on $\langle\Phi'\rangle^{TS}$

$$S^{-1}\langle\Phi'\rangle^{TS} = S^{-1}(ST)^{-1}\langle\Phi'\rangle^T = (TS)T\langle\Phi'\rangle^T = (TS)\langle\Phi'\rangle^T = \langle\Phi'\rangle^{ST}. \quad (\text{B.8})$$

Here, for the first and the fourth identities, we have applied eq. (B.6), and for the third identity, we applied eq. (B.3). Once $\langle\Phi\rangle^T$ has been fixed, one can perform rotations by acting T and T^2 which do not change the Z_3^T -invariant VEV $\langle\Phi\rangle^T$, but connect the Z_3^{ST} -invariant $\langle\Phi'\rangle^{ST}$ with Z_3^{TS} - and Z_3^{STS} -invariant VEVs, respectively,

$$\begin{aligned} T\langle\Phi'\rangle^{ST} &= T(TS)\langle\Phi'\rangle^T = (ST)^{-1}\langle\Phi'\rangle^T = \langle\Phi'\rangle^{TS}, \\ T^2\langle\Phi'\rangle^{ST} &= T^2(TS)\langle\Phi'\rangle^T = S\langle\Phi'\rangle^T = \langle\Phi'\rangle^{STS}. \end{aligned} \quad (\text{B.9})$$

Therefore, under the transformation of T and T^2 , the rest three VEVs are physically equivalent. One can always select the Z_3^{TS} -invariant VEV for Φ' . Applying this conclusion to our VEV alignment for ξ_T and ξ_{TS} , we have both VEVs of ξ_T and ξ_{TS} preserving Z_3 symmetries of S_{4R} , a smaller chance that both of them select the $(1, 0, 0)$ direction and a larger chance that $\langle\xi_T\rangle$ selects $(1, 0, 0)$ direction and $\langle\xi_{TS}\rangle$ selects $(\frac{1}{3}, -\frac{2}{3}\omega, -\frac{2}{3}\omega^2)$ direction. Involving more flavons may complicate the vacuum degeneracy problem and decrease the chance to achieve the required VEV, which will not be expanded here.

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