RECEIVED: March 26, 2019

REVISED: April 21, 2019

ACCEPTED: May 3, 2019

PUBLISHED: May 14, 2019

Planar master integrals for four-loop form factors

Andreas von Manteuffel and Robert M. Schabinger

*Department of Physics and Astronomy, Michigan State University,
East Lansing, Michigan 48824, U.S.A.*

E-mail: manteuffel@pa.msu.edu, schabing@pa.msu.edu

ABSTRACT: We present the complete set of planar master integrals relevant to the calculation of three-point functions in four-loop massless Quantum Chromodynamics. Employing direct parametric integrations for a basis of finite integrals, we give analytic results for the Laurent expansion of conventional integrals in the parameter of dimensional regularization through to terms of weight eight.

KEYWORDS: Perturbative QCD, 1/N Expansion

ARXIV EPRINT: [1903.06171](https://arxiv.org/abs/1903.06171)

Contents

1	Introduction	1
2	Preliminaries	2
3	Computational details	3
4	Results	4
5	Outlook	11

1 Introduction

Due to their relevance to Drell-Yan lepton pair production [1] and Higgs boson production via gluon fusion [2–6], the basic quark and gluon form factors of massless Quantum Chromodynamics (QCD) have played a very important role in the development of the subject. For example, it has long been understood that the massless quark and gluon form factors provide a clean theoretical laboratory for the study of the dimensionally-regulated infrared singularities of perturbative scattering amplitudes in non-Abelian gauge theories [7]. In particular, the cusp and collinear anomalous dimensions can be conveniently extracted from the ϵ^{-2} and ϵ^{-1} poles of the bare form factors [8].

The four-loop cusp anomalous dimensions are especially relevant to cutting-edge analyses of Drell-Yan lepton production and gluon-fusion Higgs boson production because they are the last remaining ingredients required for a resummation of the next-to-next-to-next-to-leading Sudakov logarithms which are known only approximately [9, 10]. It is therefore unsurprising that the four-loop form factors and cusp anomalous dimensions of QCD have received significant recent attention [11–22]. Similar studies in $\mathcal{N} = 4$ super Yang-Mills theory [23–26] should ultimately provide a very useful cross-check on the QCD results via the principle of maximal transcendentality [27]. As a central building block for the complete calculation of the four-loop form factors of massless QCD, we extend previous results and present complete analytic expressions for all planar four-loop master integrals in this paper for the first time. The covering topologies are shown in figure 1.

This article is organized as follows. In section 2, we discuss our conventions, notation, and setup. We summarize our computational method in section 3 and our results for the most interesting integrals are provided in section 4. Finally, we give an outlook in section 5. We assemble the results for all 99 planar master integrals in the supplementary data `ff4l-ints-pl.m` attached to this paper.

Figure 1. A covering set of top-level integral topologies which, via edge contraction, generate the topologies of the ninety-nine irreducible planar four-loop form factor integrals.

2 Preliminaries

In this section, we establish some notation and describe our enumeration of the planar four-loop form factor master integrals. We use Minkowskian propagators and choose an absolute normalization of

$$\frac{\Gamma^4(d/2 - 1)}{\pi^{2d}} \quad (2.1)$$

for our four-loop Feynman integrals in d spacetime dimensions. Our integrals depend on the virtuality, $q^2 = (p_1 + p_2)^2$, in a trivial manner dictated by dimensional analysis. Therefore, without loss of generality, we set $q^2 = -1$ throughout this work. To understand our conventions, it suffices to study the four-loop generalized sunrise integral:

$$\begin{aligned} \text{Sunrise} &= \frac{\Gamma^4(1 - \epsilon)}{\pi^{8-4\epsilon}} \left(\prod_{i=1}^4 \int d^{4-2\epsilon} k_i \right) \frac{1}{k_2^2 k_3^2 k_4^2 (p_1 - k_1 + k_2 - k_3 + k_4)^2 (p_2 + k_1)^2} \Big|_{q^2=-1} \\ &= -\frac{\Gamma(-3 + 4\epsilon) \Gamma^9(1 - \epsilon)}{\Gamma(5 - 5\epsilon)} \\ &= \frac{1}{\epsilon} \left(\frac{1}{576} \right) + \frac{71}{2304} + \epsilon \left(\frac{26815}{82944} \right) + \epsilon^2 \left(\frac{872675}{331776} - \frac{5}{48} \zeta_3 \right) + \mathcal{O}(\epsilon^3). \end{aligned} \quad (2.2)$$

As a first non-trivial step, we construct a single **Reduze 2** [28–30] integral family (see table 1) which covers all planar sectors (or topologies). To achieve this, we make highly-symmetric choices for the auxiliary propagators of the four-loop planar ladder form factor integral topology. At this stage, **Reduze 2** allows for the construction of a compact sector selection encoding the minimal number of sectors for which integration by parts reductions are required. After carrying out integral reductions for all Feynman integrals with our in-house reduction code, we find just ninety-nine master integrals in ninety-seven sectors. In other words, only two of our master integral topologies are of the multi-component type. For these topologies, we prefer to work with squared propagators, marked with dotted edges at the level of graphs (see figure 2). Finally, we remark that we use the physicists’ convention for $\zeta_{5,3}$,

$$\zeta_{5,3} = \sum_{m=1}^{\infty} \frac{1}{m^5} \sum_{n=1}^{m-1} \frac{1}{n^3} \approx 0.0377076729848 \dots \quad (2.3)$$

Family A	
D_1	$(k_1)^2$
D_2	$(k_2)^2$
D_3	$(k_3)^2$
D_4	$(k_4)^2$
D_5	$(p_1 - k_1)^2$
D_6	$(p_1 - k_1 + k_2)^2$
D_7	$(p_1 - k_1 + k_2 - k_3)^2$
D_8	$(p_1 - k_1 + k_2 - k_3 + k_4)^2$
D_9	$(p_2 + k_1)^2$
D_{10}	$(p_2 + k_1 - k_2)^2$
D_{11}	$(p_2 + k_1 - k_2 + k_3)^2$
D_{12}	$(p_2 + k_1 - k_2 + k_3 - k_4)^2$
D_{13}	$(k_1 - k_2)^2$
D_{14}	$(k_2 - k_3)^2$
D_{15}	$(k_3 - k_4)^2$
D_{16}	$(k_1 - k_2 + k_3)^2$
D_{17}	$(k_2 - k_3 + k_4)^2$
D_{18}	$(k_1 - k_2 + k_3 - k_4)^2$

Table 1. A single **Reduze 2** integral family covers all planar four-loop form factor topologies. The permutation symmetry group of **A** above has order forty-eight. Using this family, integrals over $(\Pi_{i=1}^4 d^d k_i)(\Pi_{j=1}^{18} D_j^{-\nu_j})$ are indexed by $(\nu_1, \dots, \nu_{18})$.

Figure 2. The planar masters from irreducible multi-component topologies.

3 Computational details

In this section, we describe our computation of the master integrals. For all master integrals which diverge in four dimensions, we first derive dimensional recurrence relations [31, 32] and map to alternative finite integrals along the lines discussed in [11, 33]. The **Reduze 2** job `find_finite_integrals` efficiently generates a large number of finite integral candidates, which then facilitate the construction of the alternative finite basis. The change of basis to our alternative finite master integrals requires non-trivial integration by parts reductions for which we use finite field sampling and rational reconstruction [34, 35], see also [36]. A key advantage of **Finred**, the private reduction program developed by one of us which realizes these ideas, is that it may be run in a highly-distributed manner on a computer cluster.

With these auxiliary integral reductions in hand, the problem reduces to one of finite Feynman integral evaluation. In all cases, the integrals are linearly reducible and accessible to **HyperInt** [37], a program for Feynman parametric integration, out of the box. The **HyperInt** program is capable of detecting and integrating out massless one-loop bubbles, a feature which pays off tremendously in a large number of cases. In fact, we find only twenty-eight integrals free of massless one-loop bubble insertions. Of these, five are two-point functions which were calculated already some time ago [38, 39]. This leaves us with just twenty-three non-trivial master integrals, for which we give explicit expressions through to weight eight in the next section.

Before proceeding, let us first stress one subtle point. It is essential for our workflow to avoid evaluating complicated finite four-loop Feynman integrals to excessively high orders in ϵ . In order to achieve this, it is necessary to select a basis of finite integrals for which complete weight eight information at the level of the finite basis integrals implies complete weight eight information at the level of the corresponding conventional basis integrals [22, 40]. What is perhaps surprising is how restrictive this requirement turns out to be in some cases. For example, the Feynman integral

$$\begin{aligned}
 (6-2\epsilon) \quad &= \frac{35}{2}\zeta_7 - \frac{15}{4}\zeta_5\zeta_2 + \frac{18}{5}\zeta_3\zeta_2^2 - \frac{17}{4}\zeta_3^2 - \frac{131}{140}\zeta_2^3 - \frac{75}{4}\zeta_5 + \frac{1}{2}\zeta_3\zeta_2 + \frac{1}{10}\zeta_2^2 \\
 &+ 3\zeta_3 + \epsilon \left(-\frac{383}{20}\zeta_{5,3} - \frac{29}{2}\zeta_5\zeta_3 - \frac{9}{2}\zeta_3^2\zeta_2 + \frac{83581}{5250}\zeta_2^4 + \frac{2677}{16}\zeta_7 + 43\zeta_5\zeta_2 + \frac{261}{20}\zeta_3\zeta_2^2 - 35\zeta_3^2 \right. \\
 &\left. - \frac{1943}{105}\zeta_2^3 - 322\zeta_5 - 6\zeta_3\zeta_2 + \frac{18}{5}\zeta_2^2 + 60\zeta_3 \right) + \mathcal{O}(\epsilon^2)
 \end{aligned} \tag{3.1}$$

appears to be the *only* one in this sector which is both finite in $d = 6 - 2\epsilon$ and allows for a faithful mapping of the weights in the sense just described.¹ In other words, if one would insist upon some other choice of finite integral in $6 - 2\epsilon$ dimensions not related to eq. (3.1) by symmetry, it would actually be necessary to Taylor expand a finite twelve-line integral through to $\mathcal{O}(\epsilon^2)$ in order to reproduce the weight eight information in eq. (4.1). From a computational point of view, it is usually desirable to avoid the evaluation of such higher-order expansion coefficients.

4 Results

In this section, we present explicit expressions through to weight eight for the subset of planar four-loop three-point master integrals which do not have any massless one-loop bubble insertions. Results for the first two twelve-line topologies of figure 1 were given, respectively, in references [15] and [12] and, very recently, results were given in reference [19] for the integrals of eq. (4.6), eq. (4.7), eqs. (4.12)–(4.16), and eqs. (4.19)–(4.23) below. To the best of our knowledge, the rest of the results which we present in this section are

¹For this sector, the **Reduze 2** job `find.finite.integrals` finds nine finite integral candidates in $6 - 2\epsilon$ dimensions which are not related by permutation symmetries.

new. Curiously, it turns out that some of the eleven-line master integrals are every bit as challenging to calculate as the twelve-line master integrals in our approach. Eq. (4.5), for example, is actually more convenient to derive indirectly by evaluating a reducible twelve-line finite integral. In the following, we define our conventional master integrals in $d = 4 - 2\epsilon$ dimensions as described in section 2 above and add a label to identify the topology in the conventions of **Reduze 2**.

A.12.121295

$$\begin{aligned}
 &= \frac{1}{\epsilon^7} \left(\frac{1}{576} \right) + \frac{1}{\epsilon^6} \left(-\frac{29}{1152} \right) + \frac{1}{\epsilon^5} \left(\frac{1}{32} \zeta_2 + \frac{25}{128} \right) + \frac{1}{\epsilon^4} \left(\frac{5}{32} \zeta_3 - \frac{95}{192} \zeta_2 \right. \\
 &\quad \left. - \frac{1297}{1152} \right) + \frac{1}{\epsilon^3} \left(\frac{4}{5} \zeta_2^2 - \frac{143}{192} \zeta_3 + \frac{183}{64} \zeta_2 + \frac{12913}{2304} \right) + \frac{1}{\epsilon^2} \left(\frac{87}{32} \zeta_5 + \frac{43}{16} \zeta_3 \zeta_2 - \frac{7481}{1440} \zeta_2^2 \right. \\
 &\quad \left. + \frac{1393}{192} \zeta_3 - \frac{6371}{576} \zeta_2 - \frac{121441}{4608} \right) + \frac{1}{\epsilon} \left(-\frac{275}{96} \zeta_3^2 + \frac{99551}{10080} \zeta_2^3 - \frac{8549}{192} \zeta_5 - \frac{7127}{288} \zeta_3 \zeta_2 + \frac{1607}{96} \zeta_2^2 \right. \\
 &\quad \left. - \frac{23291}{576} \zeta_3 + \frac{34891}{1152} \zeta_2 + \frac{381047}{3072} \right) - \frac{20803}{192} \zeta_7 + \frac{5935}{48} \zeta_5 \zeta_2 + \frac{1073}{40} \zeta_3 \zeta_2^2 - \frac{83135}{576} \zeta_3^2 \\
 &\quad - \frac{447551}{6720} \zeta_2^3 + \frac{38467}{192} \zeta_5 - \frac{1531}{96} \zeta_3 \zeta_2 - \frac{7639}{144} \zeta_2^2 + \frac{190099}{1152} \zeta_3 - \frac{46103}{2304} \zeta_2 - \frac{3710131}{6144} \\
 &\quad + \epsilon \left(\frac{6769}{60} \zeta_{5,3} - \frac{2377}{48} \zeta_5 \zeta_3 + \frac{4529}{16} \zeta_3^2 \zeta_2 + \frac{40670533}{1008000} \zeta_2^4 - \frac{160371}{128} \zeta_7 - \frac{10835}{96} \zeta_5 \zeta_2 \right. \\
 &\quad \left. - \frac{46127}{360} \zeta_3 \zeta_2^2 + \frac{17259}{64} \zeta_3^2 + \frac{2768177}{20160} \zeta_2^3 - \frac{150607}{192} \zeta_5 + \frac{14729}{32} \zeta_3 \zeta_2 + \frac{520043}{2880} \zeta_2^2 - \frac{1100159}{2304} \zeta_3 \right. \\
 &\quad \left. - \frac{2499101}{4608} \zeta_2 + \frac{12593941}{4096} \right) + \mathcal{O}(\epsilon^2)
 \end{aligned} \tag{4.1}$$

A.12.31455

$$\begin{aligned}
 &= \frac{1}{\epsilon^7} \left(-\frac{1}{288} \right) + \frac{1}{\epsilon^6} \left(\frac{13}{576} \right) + \frac{1}{\epsilon^5} \left(-\frac{19}{144} \zeta_2 - \frac{101}{576} \right) + \frac{1}{\epsilon^4} \left(-\frac{23}{72} \zeta_3 \right. \\
 &\quad \left. + \frac{253}{288} \zeta_2 + \frac{145}{96} \right) + \frac{1}{\epsilon^3} \left(-\frac{297}{160} \zeta_2^2 + \frac{149}{36} \zeta_3 - \frac{1181}{288} \zeta_2 - \frac{1669}{144} \right) + \frac{1}{\epsilon^2} \left(-\frac{61}{6} \zeta_5 - \frac{139}{18} \zeta_3 \zeta_2 \right. \\
 &\quad \left. + \frac{36289}{2880} \zeta_2^2 - \frac{7367}{288} \zeta_3 + \frac{67}{4} \zeta_2 + \frac{11243}{144} \right) + \frac{1}{\epsilon} \left(-\frac{4531}{144} \zeta_3^2 - \frac{313021}{10080} \zeta_2^3 + \frac{2231}{16} \zeta_5 + \frac{3587}{144} \zeta_3 \zeta_2 \right. \\
 &\quad \left. - \frac{81481}{1440} \zeta_2^2 + \frac{6347}{48} \zeta_3 - \frac{4351}{72} \zeta_2 - \frac{22757}{48} \right) - \frac{44097}{64} \zeta_7 - \frac{577}{6} \zeta_5 \zeta_2 - \frac{1993}{16} \zeta_3 \zeta_2^2 + \frac{9101}{72} \zeta_3^2 \\
 &\quad + \frac{240637}{1344} \zeta_2^3 - \frac{24965}{32} \zeta_5 - \frac{61}{144} \zeta_3 \zeta_2 + \frac{99523}{480} \zeta_2^2 - \frac{88745}{144} \zeta_3 + \frac{13319}{72} \zeta_2 + \frac{382375}{144} \\
 &\quad + \epsilon \left(\frac{4931}{30} \zeta_{5,3} - \frac{60385}{24} \zeta_5 \zeta_3 + \frac{1321}{36} \zeta_3^2 \zeta_2 - \frac{702196781}{1008000} \zeta_2^4 + \frac{21687}{8} \zeta_7 + \frac{9989}{24} \zeta_5 \zeta_2 \right. \\
 &\quad \left. + \frac{339821}{1440} \zeta_3 \zeta_2^2 - \frac{40519}{288} \zeta_3^2 - \frac{3681977}{5040} \zeta_2^3 + \frac{155087}{48} \zeta_5 - \frac{1457}{3} \zeta_3 \zeta_2 - \frac{179053}{288} \zeta_2^2 + \frac{381811}{144} \zeta_3 \right. \\
 &\quad \left. - \frac{10777}{24} \zeta_2 - \frac{2005247}{144} \right) + \mathcal{O}(\epsilon^2)
 \end{aligned} \tag{4.2}$$

A.11.30943

$$\begin{aligned}
 &= \frac{1}{\epsilon^3} \left(\frac{5}{32} \zeta_3 \right) + \frac{1}{\epsilon^2} \left(-\frac{19}{192} \zeta_2^2 \right) + \frac{1}{\epsilon} \left(\frac{475}{96} \zeta_5 - \frac{25}{12} \zeta_3 \zeta_2 + \frac{37}{24} \zeta_2^2 - \frac{25}{16} \zeta_3 \right) \\
 &- \frac{895}{96} \zeta_3^2 + \frac{24323}{4032} \zeta_2^3 + \frac{115}{6} \zeta_5 - \frac{5}{4} \zeta_3 \zeta_2 - \frac{413}{96} \zeta_2^2 + \frac{85}{16} \zeta_3 + \epsilon \left(\frac{125}{48} \zeta_7 + \frac{3085}{48} \zeta_5 \zeta_2 - \frac{1235}{96} \zeta_3 \zeta_2^2 \right. \\
 &+ \frac{415}{24} \zeta_3^2 + \frac{4079}{252} \zeta_2^3 - \frac{2265}{16} \zeta_5 + \frac{175}{4} \zeta_3 \zeta_2 - \frac{1015}{96} \zeta_2^2 + \frac{5}{16} \zeta_3 \left. \right) + \epsilon^2 \left(-\frac{1773}{8} \zeta_{5,3} - \frac{75815}{48} \zeta_5 \zeta_3 \right. \\
 &+ \frac{4055}{16} \zeta_3^2 \zeta_2 + \frac{67210487}{403200} \zeta_2^4 + \frac{91045}{96} \zeta_7 - \frac{310}{3} \zeta_5 \zeta_2 - \frac{217}{2} \zeta_3 \zeta_2^2 + \frac{1465}{48} \zeta_3^2 - \frac{67765}{672} \zeta_2^3 \\
 &+ \frac{5465}{16} \zeta_5 - \frac{1005}{4} \zeta_3 \zeta_2 + \frac{16945}{96} \zeta_2^2 - \frac{1655}{16} \zeta_3 \left. \right) + \mathcal{O}(\epsilon^3)
 \end{aligned} \tag{4.3}$$

A.11.27359

$$\begin{aligned}
 &= \frac{1}{\epsilon^4} \left(\frac{1}{16} \zeta_3 \right) + \frac{1}{\epsilon^3} \left(-\frac{19}{96} \zeta_2^2 \right) + \frac{1}{\epsilon^2} \left(\frac{29}{16} \zeta_5 + \frac{2}{3} \zeta_3 \zeta_2 + \frac{23}{30} \zeta_2^2 + \frac{1}{4} \zeta_3 \right) \\
 &+ \frac{1}{\epsilon} \left(\frac{265}{48} \zeta_3^2 + \frac{1621}{3360} \zeta_2^3 - \frac{3}{4} \zeta_5 - \zeta_3 \zeta_2 + \frac{3}{40} \zeta_2^2 - \frac{23}{4} \zeta_3 \right) - \frac{10381}{64} \zeta_7 + \frac{253}{12} \zeta_5 \zeta_2 + \frac{2523}{80} \zeta_3 \zeta_2^2 \\
 &- \frac{209}{12} \zeta_3^2 + \frac{12599}{630} \zeta_2^3 + \frac{11}{3} \zeta_5 + \frac{5}{3} \zeta_3 \zeta_2 - \frac{2639}{120} \zeta_2^2 + \frac{213}{4} \zeta_3 + \epsilon \left(\frac{139}{20} \zeta_{5,3} - \frac{11875}{24} \zeta_5 \zeta_3 \right. \\
 &+ \frac{171}{4} \zeta_3^2 \zeta_2 + \frac{426457}{144000} \zeta_2^4 + \frac{5045}{6} \zeta_7 + \frac{1715}{12} \zeta_5 \zeta_2 - \frac{1783}{20} \zeta_3 \zeta_2^2 + \frac{19}{4} \zeta_3^2 - \frac{28617}{280} \zeta_2^3 - \frac{292}{3} \zeta_5 \\
 &- \frac{31}{3} \zeta_3 \zeta_2 + \frac{24253}{120} \zeta_2^2 - \frac{1471}{4} \zeta_3 \left. \right) + \mathcal{O}(\epsilon^2)
 \end{aligned} \tag{4.4}$$

A.11.14831

$$\begin{aligned}
 &= \frac{1}{\epsilon^7} \left(-\frac{1}{144} \right) + \frac{1}{\epsilon^6} \left(\frac{1}{96} \right) + \frac{1}{\epsilon^5} \left(-\frac{19}{144} \zeta_2 + \frac{1}{16} \right) + \frac{1}{\epsilon^4} \left(-\frac{25}{144} \zeta_3 + \frac{53}{144} \zeta_2 \right. \\
 &- \frac{37}{48} \left. \right) + \frac{1}{\epsilon^3} \left(-\frac{149}{360} \zeta_2^2 + \frac{125}{144} \zeta_3 - \frac{49}{72} \zeta_2 + \frac{263}{48} \right) + \frac{1}{\epsilon^2} \left(\frac{25}{48} \zeta_5 + \frac{37}{24} \zeta_3 \zeta_2 + \frac{185}{36} \zeta_2^2 + \frac{43}{36} \zeta_3 \right. \\
 &- \frac{65}{72} \zeta_2 - \frac{519}{16} \left. \right) + \frac{1}{\epsilon} \left(-\frac{71}{16} \zeta_3^2 + \frac{66499}{5040} \zeta_2^3 + \frac{791}{48} \zeta_5 + \frac{415}{72} \zeta_3 \zeta_2 - \frac{3539}{144} \zeta_2^2 - \frac{1423}{36} \zeta_3 + \frac{1403}{72} \zeta_2 \right. \\
 &+ \frac{8383}{48} \left. \right) + \frac{38779}{192} \zeta_7 + \frac{6359}{24} \zeta_5 \zeta_2 - \frac{24941}{360} \zeta_3 \zeta_2^2 - \frac{9035}{144} \zeta_3^2 + \frac{32783}{1008} \zeta_2^3 - \frac{863}{8} \zeta_5 - \frac{307}{18} \zeta_3 \zeta_2 \\
 &+ \frac{60949}{720} \zeta_2^2 + \frac{12433}{36} \zeta_3 - \frac{10481}{72} \zeta_2 - \frac{42557}{48} + \epsilon \left(-378 \zeta_{5,3} - \frac{7813}{8} \zeta_5 \zeta_3 + \frac{3089}{72} \zeta_3^2 \zeta_2 \right. \\
 &+ \frac{46907263}{100800} \zeta_2^4 - \frac{51555}{64} \zeta_7 + \frac{7987}{24} \zeta_5 \zeta_2 - \frac{67651}{360} \zeta_3 \zeta_2^2 + \frac{4322}{9} \zeta_3^2 - \frac{490657}{1680} \zeta_2^3 + \frac{12059}{24} \zeta_5 \\
 &- \frac{2105}{18} \zeta_3 \zeta_2 - \frac{140239}{720} \zeta_2^2 - \frac{81907}{36} \zeta_3 + \frac{62291}{72} \zeta_2 + \frac{69181}{16} \left. \right) + \mathcal{O}(\epsilon^2)
 \end{aligned} \tag{4.5}$$

A.10.31149

$$\begin{aligned}
 &= \frac{1}{\epsilon^6} \left(\frac{1}{96} \zeta_2 \right) + \frac{1}{\epsilon^5} \left(-\frac{5}{96} \zeta_3 \right) + \frac{1}{\epsilon^4} \left(\frac{19}{240} \zeta_2^2 \right) + \frac{1}{\epsilon^3} \left(\frac{233}{96} \zeta_5 - \frac{29}{24} \zeta_3 \zeta_2 \right) \\
 &+ \frac{1}{\epsilon^2} \left(\frac{25}{16} \zeta_3^2 - \frac{6269}{5040} \zeta_2^3 \right) + \frac{1}{\epsilon} \left(-\frac{3881}{192} \zeta_7 - \frac{107}{4} \zeta_5 \zeta_2 - \frac{169}{240} \zeta_3 \zeta_2^2 \right) + \frac{257}{40} \zeta_{5,3} - \frac{729}{8} \zeta_5 \zeta_3 \\
 &+ \frac{839}{16} \zeta_3^2 \zeta_2 - \frac{17086001}{252000} \zeta_2^4 + \mathcal{O}(\epsilon)
 \end{aligned} \tag{4.6}$$

A.10.30155

$$\begin{aligned}
 &= \frac{1}{\epsilon^5} \left(-\frac{1}{24} \zeta_3 \right) + \frac{1}{\epsilon^4} \left(\frac{19}{240} \zeta_2^2 \right) + \frac{1}{\epsilon^3} \left(-\frac{17}{24} \zeta_5 + \frac{11}{24} \zeta_2 \zeta_3 \right) + \frac{1}{\epsilon^2} \left(\frac{1}{3} \zeta_3^2 \right. \\
 &- \frac{311}{1260} \zeta_2^3 \left. \right) + \frac{1}{\epsilon} \left(-\frac{5353}{384} \zeta_7 - \frac{237}{16} \zeta_5 \zeta_2 + \frac{1069}{240} \zeta_3 \zeta_2^2 \right) + \frac{2551}{60} \zeta_{5,3} + \frac{1417}{12} \zeta_5 \zeta_3 - \frac{13}{24} \zeta_3^2 \zeta_2 \\
 &- \frac{3320159}{84000} \zeta_2^4 + \mathcal{O}(\epsilon)
 \end{aligned} \tag{4.7}$$

A.10.30123

$$\begin{aligned}
 &= \frac{1}{\epsilon^8} \left(\frac{1}{144} \right) + \frac{1}{\epsilon^6} \left(\frac{1}{8} \zeta_2 \right) + \frac{1}{\epsilon^5} \left(\frac{11}{24} \zeta_3 \right) + \frac{1}{\epsilon^4} \left(\frac{23}{16} \zeta_2^2 \right) + \frac{1}{\epsilon^3} \left(\frac{41}{4} \zeta_5 \right. \\
 &+ \frac{21}{8} \zeta_2 \zeta_3 \left. \right) + \frac{1}{\epsilon^2} \left(\frac{27}{16} \zeta_3^2 + \frac{13259}{840} \zeta_2^3 \right) + \frac{1}{\epsilon} \left(\frac{8071}{64} \zeta_7 + \frac{193}{2} \zeta_5 \zeta_2 - 16 \zeta_3 \zeta_2^2 \right) + \frac{327}{10} \zeta_{5,3} \\
 &+ \frac{1117}{8} \zeta_5 \zeta_3 - \frac{1577}{8} \zeta_3^2 \zeta_2 + \frac{5548241}{42000} \zeta_2^4 + \mathcal{O}(\epsilon)
 \end{aligned} \tag{4.8}$$

A.10.30095

$$\begin{aligned}
 &= \frac{1}{\epsilon^3} \left(-\frac{5}{16} \zeta_3 \right) + \frac{1}{\epsilon^2} \left(-\frac{7}{32} \zeta_2^2 + \frac{35}{16} \zeta_3 \right) + \frac{1}{\epsilon} \left(-\frac{125}{16} \zeta_5 + \frac{5}{6} \zeta_3 \zeta_2 + \frac{49}{32} \zeta_2^2 \right. \\
 &- \frac{185}{16} \zeta_3 \left. \right) - \frac{605}{48} \zeta_3^2 - \frac{7289}{672} \zeta_2^3 + \frac{875}{16} \zeta_5 - \frac{35}{6} \zeta_3 \zeta_2 - \frac{259}{32} \zeta_2^2 + \frac{875}{16} \zeta_3 + \epsilon \left(-\frac{13565}{32} \zeta_7 \right. \\
 &+ \frac{2165}{24} \zeta_5 \zeta_2 - \frac{1253}{48} \zeta_3 \zeta_2^2 + \frac{4235}{48} \zeta_3^2 + \frac{7289}{96} \zeta_2^3 - \frac{4625}{16} \zeta_5 + \frac{185}{6} \zeta_3 \zeta_2 + \frac{1225}{32} \zeta_2^2 - \frac{3905}{16} \zeta_3 \left. \right) \\
 &+ \epsilon^2 \left(\frac{1479}{4} \zeta_{5,3} + \frac{23795}{24} \zeta_5 \zeta_3 + \frac{4925}{24} \zeta_3^2 \zeta_2 - \frac{66262951}{201600} \zeta_2^4 + \frac{94955}{32} \zeta_7 - \frac{15155}{24} \zeta_5 \zeta_2 \right. \\
 &+ \frac{8771}{48} \zeta_3 \zeta_2^2 - \frac{22385}{48} \zeta_3^2 - \frac{269693}{672} \zeta_2^3 + \frac{21875}{16} \zeta_5 - \frac{875}{6} \zeta_3 \zeta_2 - \frac{5467}{32} \zeta_2^2 + \frac{16835}{16} \zeta_3 \left. \right) \\
 &+ \mathcal{O}(\epsilon^3)
 \end{aligned} \tag{4.9}$$

A.10.27343

$$\begin{aligned}
 &= \frac{1}{\epsilon^2} \left(\frac{5}{4} \zeta_5 \right) + \frac{1}{\epsilon} \left(\frac{7}{8} \zeta_3^2 + \frac{13}{21} \zeta_2^3 - 5 \zeta_5 \right) - \frac{3775}{64} \zeta_7 - \frac{155}{8} \zeta_5 \zeta_2 - \frac{7}{40} \zeta_3 \zeta_2^2 \\
 &- \frac{7}{2} \zeta_3^2 - \frac{52}{21} \zeta_2^3 + 20 \zeta_5 + \epsilon \left(-\frac{19}{2} \zeta_{5,3} - \frac{2373}{4} \zeta_5 \zeta_3 - \frac{183}{4} \zeta_3^2 \zeta_2 - \frac{33959}{600} \zeta_2^4 + \frac{3775}{16} \zeta_7 + \frac{155}{2} \zeta_5 \zeta_2 \right. \\
 &\left. + \frac{7}{10} \zeta_3 \zeta_2^2 + 14 \zeta_3^2 + \frac{208}{21} \zeta_2^3 - 80 \zeta_5 \right) + \mathcal{O}(\epsilon^2)
 \end{aligned} \tag{4.10}$$

A.10.14767

$$\begin{aligned}
 &= \frac{1}{\epsilon^7} \left(-\frac{5}{576} \right) + \frac{1}{\epsilon^6} \left(\frac{5}{144} \right) + \frac{1}{\epsilon^5} \left(-\frac{5}{96} \zeta_2 - \frac{5}{36} \right) + \frac{1}{\epsilon^4} \left(-\frac{67}{96} \zeta_3 + \frac{5}{24} \zeta_2 \right. \\
 &\left. + \frac{5}{9} \right) + \frac{1}{\epsilon^3} \left(-\frac{269}{480} \zeta_2^2 + \frac{67}{24} \zeta_3 - \frac{5}{6} \zeta_2 - \frac{20}{9} \right) + \frac{1}{\epsilon^2} \left(\frac{223}{96} \zeta_5 - \frac{313}{48} \zeta_3 \zeta_2 + \frac{269}{120} \zeta_2^2 - \frac{67}{6} \zeta_3 \right. \\
 &\left. + \frac{10}{3} \zeta_2 + \frac{80}{9} \right) + \frac{1}{\epsilon} \left(-\frac{535}{96} \zeta_3^2 - \frac{39181}{5040} \zeta_2^3 - \frac{223}{24} \zeta_5 + \frac{313}{12} \zeta_3 \zeta_2 - \frac{269}{30} \zeta_2^2 + \frac{134}{3} \zeta_3 - \frac{40}{3} \zeta_2 \right. \\
 &\left. - \frac{320}{9} \right) + 307 \zeta_7 - \frac{9863}{48} \zeta_5 \zeta_2 - \frac{193}{48} \zeta_3 \zeta_2^2 + \frac{535}{24} \zeta_3^2 + \frac{39181}{1260} \zeta_2^3 + \frac{223}{6} \zeta_5 - \frac{313}{3} \zeta_3 \zeta_2 + \frac{538}{15} \zeta_2^2 \\
 &- \frac{536}{3} \zeta_3 + \frac{160}{3} \zeta_2 + \frac{1280}{9} + \epsilon \left(\frac{9779}{60} \zeta_{5,3} + \frac{17521}{16} \zeta_5 \zeta_3 + \frac{5311}{48} \zeta_3^2 \zeta_2 + \frac{8076847}{252000} \zeta_2^4 - 1228 \zeta_7 \right. \\
 &\left. + \frac{9863}{12} \zeta_5 \zeta_2 + \frac{193}{12} \zeta_3 \zeta_2^2 - \frac{535}{6} \zeta_3^2 - \frac{39181}{315} \zeta_2^3 - \frac{446}{3} \zeta_5 + \frac{1252}{3} \zeta_3 \zeta_2 - \frac{2152}{15} \zeta_2^2 + \frac{2144}{3} \zeta_3 \right. \\
 &\left. - \frac{640}{3} \zeta_2 - \frac{5120}{9} \right) + \mathcal{O}(\epsilon^2)
 \end{aligned} \tag{4.11}$$

A.9.30877

$$\begin{aligned}
 &= \frac{1}{\epsilon^2} \left(-\frac{3}{2} \zeta_3^2 - \frac{433}{840} \zeta_2^3 \right) + \frac{1}{\epsilon} \left(-\frac{157}{4} \zeta_7 + \frac{51}{4} \zeta_5 \zeta_2 - \frac{157}{40} \zeta_3 \zeta_2^2 \right) + 46 \zeta_{5,3} \\
 &+ \frac{21}{4} \zeta_5 \zeta_3 + \frac{89}{8} \zeta_3^2 \zeta_2 - \frac{162131}{4200} \zeta_2^4 + \mathcal{O}(\epsilon)
 \end{aligned} \tag{4.12}$$

A.9.29899

$$\begin{aligned}
 &= \frac{1}{\epsilon} \left(\frac{441}{64} \zeta_7 + \frac{5}{2} \zeta_5 \zeta_2 - \frac{7}{40} \zeta_3 \zeta_2^2 \right) + \frac{13}{2} \zeta_{5,3} + 26 \zeta_5 \zeta_3 + \frac{39}{8} \zeta_3^2 \zeta_2 + \frac{21039}{5600} \zeta_2^4 \\
 &+ \mathcal{O}(\epsilon)
 \end{aligned} \tag{4.13}$$

A.9.27339

$$\begin{aligned}
 &= -\frac{441}{8} \zeta_7 - 15 \zeta_5 \zeta_2 - \frac{9}{5} \zeta_3 \zeta_2^2 + \epsilon \left(\frac{36}{5} \zeta_{5,3} + 51 \zeta_5 \zeta_3 - \frac{74954}{2625} \zeta_2^4 - 441 \zeta_7 \right. \\
 &\left. - 120 \zeta_5 \zeta_2 - \frac{72}{5} \zeta_3 \zeta_2^2 \right) + \mathcal{O}(\epsilon^2)
 \end{aligned} \tag{4.14}$$

A.9.14765

$$\begin{aligned}
 &= \frac{1}{\epsilon^2} \left(-20\zeta_5 \right) + \frac{1}{\epsilon} \left(-2\zeta_3^2 - \frac{272}{21}\zeta_2^3 - 160\zeta_5 \right) - \frac{1607}{4}\zeta_7 + 170\zeta_5\zeta_2 \\
 &- \frac{182}{5}\zeta_3\zeta_2^2 - 16\zeta_3^2 - \frac{2176}{21}\zeta_2^3 - 1280\zeta_5 + \epsilon \left(\frac{464}{5}\zeta_{5,3} + 2432\zeta_5\zeta_3 - 120\zeta_3^2\zeta_2 - \frac{1036}{125}\zeta_2^4 \right. \\
 &\left. - 3214\zeta_7 + 1360\zeta_5\zeta_2 - \frac{1456}{5}\zeta_3\zeta_2^2 - 128\zeta_3^2 - \frac{17408}{21}\zeta_2^3 - 10240\zeta_5 \right) + \mathcal{O}(\epsilon^2) \quad (4.15)
 \end{aligned}$$

A.9.10959

$$= \frac{72}{5}\zeta_{5,3} - 15\zeta_5\zeta_3 - 3\zeta_3^2\zeta_2 - \frac{49151}{5250}\zeta_2^4 + \mathcal{O}(\epsilon) \quad (4.16)$$

A.9.10671

$$\begin{aligned}
 &= \frac{1}{\epsilon} \left(\frac{441}{16}\zeta_7 + 10\zeta_5\zeta_2 - \frac{7}{10}\zeta_3\zeta_2^2 \right) + \frac{587}{10}\zeta_{5,3} + \frac{373}{2}\zeta_5\zeta_3 + 38\zeta_3^2\zeta_2 \\
 &+ \frac{214183}{21000}\zeta_2^4 + \mathcal{O}(\epsilon) \quad (4.17)
 \end{aligned}$$

A.9.2783

$$\begin{aligned}
 &= \frac{1}{\epsilon} \left(\frac{441}{16}\zeta_7 + \frac{5}{2}\zeta_5\zeta_2 - \frac{3}{5}\zeta_3\zeta_2^2 \right) - \frac{587}{10}\zeta_{5,3} - 58\zeta_3\zeta_5 - 16\zeta_3^2\zeta_2 + \frac{866647\zeta_2^4}{21000} \\
 &+ \mathcal{O}(\epsilon) \quad (4.18)
 \end{aligned}$$

A.8.29835

$$\begin{aligned}
 &= \frac{1}{\epsilon} \left(5\zeta_5 \right) + 3\zeta_3^2 + 4\zeta_2^3 + 55\zeta_5 + \epsilon \left(\frac{419}{4}\zeta_7 + 40\zeta_5\zeta_2 - \frac{92}{5}\zeta_3\zeta_2^2 + 33\zeta_3^2 \right. \\
 &+ 44\zeta_2^3 + 455\zeta_5 \left. \right) + \epsilon^2 \left(-\frac{14}{5}\zeta_{5,3} - 488\zeta_5\zeta_3 + 54\zeta_3^2\zeta_2 + \frac{6486}{125}\zeta_2^4 + \frac{4609}{4}\zeta_7 + 440\zeta_5\zeta_2 \right. \\
 &\left. - \frac{1012}{5}\zeta_3\zeta_2^2 + 273\zeta_3^2 + 364\zeta_2^3 + 3355\zeta_5 \right) + \mathcal{O}(\epsilon^3) \quad (4.19)
 \end{aligned}$$

A.8.25291

$$\begin{aligned}
 &= \frac{1}{\epsilon} \left(5\zeta_5 \right) + \frac{8\zeta_2^3}{3} - \frac{9\zeta_3^2}{2} + 55\zeta_5 + \epsilon \left(\frac{1039}{16}\zeta_7 - \frac{95}{2}\zeta_5\zeta_2 + \frac{77}{10}\zeta_3\zeta_2^2 - \frac{99}{2}\zeta_3^2 \right. \\
 &+ \frac{88}{3}\zeta_2^3 + 455\zeta_5 \left. \right) + \epsilon^2 \left(\frac{682}{5}\zeta_{5,3} - 175\zeta_5\zeta_3 + 33\zeta_3^2\zeta_2 - \frac{208239}{3500}\zeta_2^4 + \frac{11429}{16}\zeta_7 - \frac{1045}{2}\zeta_5\zeta_2 \right. \\
 &\left. + \frac{847}{10}\zeta_3\zeta_2^2 - \frac{819}{2}\zeta_3^2 + \frac{728}{3}\zeta_2^3 + 3355\zeta_5 \right) + \mathcal{O}(\epsilon^3) \quad (4.20)
 \end{aligned}$$

A.8.25291

$$\begin{aligned}
 &= \frac{1}{\epsilon^5} \left(\frac{1}{48} \zeta_2 \right) + \frac{1}{\epsilon^3} \left(\frac{109}{480} \zeta_2^2 \right) + \frac{1}{\epsilon^2} \left(\frac{9}{16} \zeta_5 - \frac{3}{4} \zeta_3 \zeta_2 \right) + \frac{1}{\epsilon} \left(-\frac{3}{2} \zeta_3^2 \right. \\
 &+ \left. \frac{2921}{3360} \zeta_2^3 \right) - \frac{487}{128} \zeta_7 - 22 \zeta_5 \zeta_2 - \frac{349}{40} \zeta_3 \zeta_2^2 + \epsilon \left(\frac{76}{5} \zeta_{5,3} - \frac{235}{4} \zeta_5 \zeta_3 + \frac{51}{4} \zeta_3^2 \zeta_2 - \frac{7374677}{336000} \zeta_2^4 \right) \\
 &+ \mathcal{O}(\epsilon^2)
 \end{aligned} \tag{4.21}$$

A.8.17567

$$\begin{aligned}
 &= \frac{1}{\epsilon^2} \left(-\frac{3}{2} \zeta_3 \right) + \frac{1}{\epsilon} \left(10 \zeta_5 + \zeta_3 \zeta_2 - \frac{11}{10} \zeta_2^2 - \frac{45}{2} \zeta_3 \right) + 20 \zeta_3^2 + \frac{447}{35} \zeta_2^3 \\
 &+ \frac{147}{2} \zeta_5 + 9 \zeta_3 \zeta_2 - \frac{33}{2} \zeta_2^2 - \frac{453}{2} \zeta_3 + \epsilon \left(\frac{3521}{8} \zeta_7 + 60 \zeta_5 \zeta_2 + \frac{32}{5} \zeta_3 \zeta_2^2 + \frac{561}{2} \zeta_3^2 + \frac{11929}{105} \zeta_2^3 \right. \\
 &+ \left. \frac{725}{2} \zeta_5 + 61 \zeta_3 \zeta_2 - \frac{1661}{10} \zeta_2^2 - \frac{3825}{2} \zeta_3 \right) + \epsilon^2 \left(-\frac{257}{5} \zeta_{5,3} + 132 \zeta_5 \zeta_3 - 70 \zeta_3^2 \zeta_2 + \frac{1633561}{5250} \zeta_2^4 \right. \\
 &+ \left. \frac{33133}{8} \zeta_7 + 500 \zeta_5 \zeta_2 + 225 \zeta_3 \zeta_2^2 + \frac{5455}{2} \zeta_3^2 + \frac{26567}{35} \zeta_2^3 + \frac{2397}{2} \zeta_5 + 369 \zeta_3 \zeta_2 - \frac{2805}{2} \zeta_2^2 \right. \\
 &- \left. \frac{29253}{2} \zeta_3 \right) + \mathcal{O}(\epsilon^3)
 \end{aligned} \tag{4.22}$$

A.8.10415

$$\begin{aligned}
 &= \frac{1}{\epsilon^2} \left(\frac{1}{2} \zeta_3 \right) + \frac{1}{\epsilon} \left(\frac{3}{10} \zeta_2^2 + \frac{11}{2} \zeta_3 \right) + 6 \zeta_5 + 5 \zeta_3 \zeta_2 + \frac{31}{10} \zeta_2^2 + \frac{71}{2} \zeta_3 \\
 &+ \epsilon \left(-8 \zeta_3^2 + \frac{379}{70} \zeta_2^3 + \frac{145}{2} \zeta_5 + 76 \zeta_3 \zeta_2 + \frac{173}{10} \zeta_2^2 + \frac{271}{2} \zeta_3 \right) + \epsilon^2 \left(\frac{3225}{16} \zeta_7 - 39 \zeta_5 \zeta_2 \right. \\
 &+ \left. \frac{244}{5} \zeta_3 \zeta_2^2 + \frac{7}{2} \zeta_3^2 + \frac{1243}{15} \zeta_2^3 + 556 \zeta_5 + 775 \zeta_3 \zeta_2 + \frac{311}{10} \zeta_2^2 - \frac{369}{2} \zeta_3 \right) + \epsilon^3 \left(-\frac{513}{5} \zeta_{5,3} \right. \\
 &- 189 \zeta_5 \zeta_3 - 344 \zeta_3^2 \zeta_2 + \frac{937697 \zeta_2^4}{7000} + \frac{24899}{8} \zeta_7 - 520 \zeta_5 \zeta_2 + 859 \zeta_3 \zeta_2^2 + 1262 \zeta_3^2 + \frac{178627}{210} \zeta_2^3 \\
 &+ \left. \frac{6515}{2} \zeta_5 + 6626 \zeta_3 \zeta_2 - \frac{6167}{10} \zeta_2^2 - \frac{19569}{2} \zeta_3 \right) + \mathcal{O}(\epsilon^4)
 \end{aligned} \tag{4.23}$$

The new results given above and in the supplementary data `ff4l-ints-pl.m` attached to this paper were cross-checked either by computing them twice with separate sets of finite integrals or by evaluating them in closed form to all orders in ϵ [41]. After an integration by parts reduction, we see that the results of [12] and [15] agree completely with our eqs. (4.1) and (4.2). We also find complete agreement with the subset of planar master integrals made public recently by the authors of [19]. Finally, it is worth mentioning that many factorizable master integrals occur. These may be evaluated immediately by taking products of the well-known lower-loop results [11, 41–48]. For the convenience of the reader, all ninety-nine master integrals are given through to weight eight in `ff4l-ints-pl.m`, including expressions in our normalization for the previously-published, planar four-loop two-point master integrals [38, 39, 49].

5 Outlook

In this paper, we evaluated the previously-unpublished planar massless four-loop form factor master integrals in dimensional regularization using the finite integral method [11, 33]. In particular, we provided explicit expressions in section 4 for all planar three-point master integrals without one-loop massless bubble insertions, up to and including terms of weight eight. The computational strategy employed in this paper may also be fruitfully applied to treat Feynman integrals in non-trivial non-planar topologies (see reference [22]). However, it is usually far more challenging to evaluate non-planar finite Feynman integrals through to weight eight. Therefore, it remains unclear whether one should expect the finite integral method to be competitive in all cases with other approaches to single-scale Feynman integrals, such as the method of Henn, Smirnov, and Smirnov [48].

Acknowledgments

We give our heartfelt thanks to Ruth Britto and David O'Regan for providing, respectively, software and hardware which played an essential role in our analytic Feynman integral evaluations. The authors also acknowledge the DJEI/DES/SFI/HEA Irish Centre for High-End Computing (ICHEC) for the provision of computational facilities and support and acknowledge Trinity Centre for High Performance Computing and Science Foundation Ireland, for the maintenance and funding, respectively, of the Boyle cluster on which calculations were performed. We are especially grateful to Paddy Doyle and Sean McGrath for their technical support. This work was also supported by TCHPC (Research IT, Trinity College Dublin). We are indebted to Hubert Spiesberger for essential help, to the PRISMA excellence cluster for generous financial support, and to the Mogon team for technical support with our usage of the supercomputer Mogon at Johannes Gutenberg University Mainz for this work. We gratefully acknowledge Dalibor Djukanovic for generously providing additional computing resources at the Helmholtz Institute at Johannes Gutenberg University Mainz. This work employed computing resources provided by the High Performance Computing Center at Michigan State University, and we gratefully acknowledge the HPCC team for their help and support. This work was supported in part by the National Science Foundation under Grant No. 1719863. RMS was supported in part by the European Research Council through grant 647356 (CutLoops). Our figures were generated using Jaxodraw [50], based on AxoDraw [51].

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] S.D. Drell and T.-M. Yan, *Massive Lepton Pair Production in Hadron-Hadron Collisions at High-Energies*, *Phys. Rev. Lett.* **25** (1970) 316 [Erratum *ibid.* **25** (1970) 902] [[INSPIRE](#)].
- [2] H.M. Georgi, S.L. Glashow, M.E. Machacek and D.V. Nanopoulos, *Higgs Bosons from Two Gluon Annihilation in Proton Proton Collisions*, *Phys. Rev. Lett.* **40** (1978) 692 [[INSPIRE](#)].
- [3] F. Wilczek, *Decays of Heavy Vector Mesons Into Higgs Particles*, *Phys. Rev. Lett.* **39** (1977) 1304 [[INSPIRE](#)].
- [4] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Remarks on Higgs Boson Interactions with Nucleons*, *Phys. Lett.* **78B** (1978) 443 [[INSPIRE](#)].
- [5] J.R. Ellis, M.K. Gaillard, D.V. Nanopoulos and C.T. Sachrajda, *Is the Mass of the Higgs Boson About 10-GeV?*, *Phys. Lett.* **83B** (1979) 339 [[INSPIRE](#)].
- [6] T. Inami, T. Kubota and Y. Okada, *Effective Gauge Theory and the Effect of Heavy Quarks in Higgs Boson Decays*, *Z. Phys. C* **18** (1983) 69 [[INSPIRE](#)].
- [7] L. Magnea and G.F. Sterman, *Analytic continuation of the Sudakov form-factor in QCD*, *Phys. Rev. D* **42** (1990) 4222 [[INSPIRE](#)].
- [8] S. Moch, J.A.M. Vermaseren and A. Vogt, *Three-loop results for quark and gluon form-factors*, *Phys. Lett. B* **625** (2005) 245 [[hep-ph/0508055](#)] [[INSPIRE](#)].
- [9] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*, *JHEP* **10** (2017) 041 [[arXiv:1707.08315](#)] [[INSPIRE](#)].
- [10] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*, *Phys. Lett. B* **782** (2018) 627 [[arXiv:1805.09638](#)] [[INSPIRE](#)].
- [11] A. von Manteuffel, E. Panzer and R.M. Schabinger, *On the Computation of Form Factors in Massless QCD with Finite Master Integrals*, *Phys. Rev. D* **93** (2016) 125014 [[arXiv:1510.06758](#)] [[INSPIRE](#)].
- [12] J.M. Henn, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *A planar four-loop form factor and cusp anomalous dimension in QCD*, *JHEP* **05** (2016) 066 [[arXiv:1604.03126](#)] [[INSPIRE](#)].
- [13] B. Ruijl, T. Ueda, J.A.M. Vermaseren, J. Davies and A. Vogt, *First Forcer results on deep-inelastic scattering and related quantities*, *PoS(LL2016)071* (2016) [[arXiv:1605.08408](#)] [[INSPIRE](#)].
- [14] A. von Manteuffel and R.M. Schabinger, *Quark and gluon form factors to four-loop order in QCD: the N_f^3 contributions*, *Phys. Rev. D* **95** (2017) 034030 [[arXiv:1611.00795](#)] [[INSPIRE](#)].
- [15] J. Henn, A.V. Smirnov, V.A. Smirnov, M. Steinhauser and R.N. Lee, *Four-loop photon quark form factor and cusp anomalous dimension in the large- N_c limit of QCD*, *JHEP* **03** (2017) 139 [[arXiv:1612.04389](#)] [[INSPIRE](#)].

- [16] R.N. Lee, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *The n_f^2 contributions to fermionic four-loop form factors*, *Phys. Rev. D* **96** (2017) 014008 [[arXiv:1705.06862](#)] [[INSPIRE](#)].
- [17] A. Grozin, J. Henn and M. Stahlhofen, *On the Casimir scaling violation in the cusp anomalous dimension at small angle*, *JHEP* **10** (2017) 052 [[arXiv:1708.01221](#)] [[INSPIRE](#)].
- [18] A. Grozin, *Four-loop cusp anomalous dimension in QED*, *JHEP* **06** (2018) 073 [Addendum *ibid.* **01** (2019) 134] [[arXiv:1805.05050](#)] [[INSPIRE](#)].
- [19] R.N. Lee, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *Four-loop quark form factor with quartic fundamental colour factor*, *JHEP* **02** (2019) 172 [[arXiv:1901.02898](#)] [[INSPIRE](#)].
- [20] J.M. Henn, T. Peraro, M. Stahlhofen and P. Wasser, *Matter dependence of the four-loop cusp anomalous dimension*, [arXiv:1901.03693](#) [[INSPIRE](#)].
- [21] R. Brüser, A. Grozin, J.M. Henn and M. Stahlhofen, *Matter dependence of the four-loop QCD cusp anomalous dimension: from small angles to all angles*, [arXiv:1902.05076](#) [[INSPIRE](#)].
- [22] A. von Manteuffel and R.M. Schabinger, *Quark and gluon form factors in four loop QCD: the N_f^2 and $N_{q\gamma}N_f$ contributions*, [arXiv:1902.08208](#) [[INSPIRE](#)].
- [23] R.H. Boels, B.A. Kniehl, O.V. Tarasov and G. Yang, *Color-kinematic Duality for Form Factors*, *JHEP* **02** (2013) 063 [[arXiv:1211.7028](#)] [[INSPIRE](#)].
- [24] R. Boels, B.A. Kniehl and G. Yang, *Master integrals for the four-loop Sudakov form factor*, *Nucl. Phys. B* **902** (2016) 387 [[arXiv:1508.03717](#)] [[INSPIRE](#)].
- [25] R.H. Boels, T. Huber and G. Yang, *Four-Loop Nonplanar Cusp Anomalous Dimension in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **119** (2017) 201601 [[arXiv:1705.03444](#)] [[INSPIRE](#)].
- [26] R.H. Boels, T. Huber and G. Yang, *The Sudakov form factor at four loops in maximal super Yang-Mills theory*, *JHEP* **01** (2018) 153 [[arXiv:1711.08449](#)] [[INSPIRE](#)].
- [27] A.V. Kotikov and L.N. Lipatov, *DGLAP and BFKL equations in the $\mathcal{N} = 4$ supersymmetric gauge theory*, *Nucl. Phys. B* **661** (2003) 19 [Erratum *ibid.* **B 685** (2004) 405] [[hep-ph/0208220](#)] [[INSPIRE](#)].
- [28] A. von Manteuffel and C. Studerus, *Reduze 2 — Distributed Feynman Integral Reduction*, [arXiv:1201.4330](#) [[INSPIRE](#)].
- [29] C. Studerus, *Reduze-Feynman Integral Reduction in C++*, *Comput. Phys. Commun.* **181** (2010) 1293 [[arXiv:0912.2546](#)] [[INSPIRE](#)].
- [30] C.W. Bauer, A. Frink and R. Kreckel, *Introduction to the GiNaC framework for symbolic computation within the C++ programming language*, *J. Symb. Comput.* **33** (2000) 1 [[cs/0004015](#)] [[INSPIRE](#)].
- [31] O.V. Tarasov, *Connection between Feynman integrals having different values of the space-time dimension*, *Phys. Rev. D* **54** (1996) 6479 [[hep-th/9606018](#)] [[INSPIRE](#)].
- [32] R.N. Lee, *Presenting LiteRed: a tool for the Loop InTEgrals REDuction*, [arXiv:1212.2685](#) [[INSPIRE](#)].
- [33] A. von Manteuffel, E. Panzer and R.M. Schabinger, *A quasi-finite basis for multi-loop Feynman integrals*, *JHEP* **02** (2015) 120 [[arXiv:1411.7392](#)] [[INSPIRE](#)].
- [34] A. von Manteuffel and R.M. Schabinger, *A novel approach to integration by parts reduction*, *Phys. Lett. B* **744** (2015) 101 [[arXiv:1406.4513](#)] [[INSPIRE](#)].

- [35] W.B. Hart, *Fast Library for Number Theory: An Introduction*, Springer (2010) [<http://flintlib.org>].
- [36] T. Peraro, *Scattering amplitudes over finite fields and multivariate functional reconstruction*, *JHEP* **12** (2016) 030 [[arXiv:1608.01902](#)] [[INSPIRE](#)].
- [37] E. Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals*, *Comput. Phys. Commun.* **188** (2015) 148 [[arXiv:1403.3385](#)] [[INSPIRE](#)].
- [38] P.A. Baikov and K.G. Chetyrkin, *Four Loop Massless Propagators: An Algebraic Evaluation of All Master Integrals*, *Nucl. Phys. B* **837** (2010) 186 [[arXiv:1004.1153](#)] [[INSPIRE](#)].
- [39] R.N. Lee, A.V. Smirnov and V.A. Smirnov, *Master Integrals for Four-Loop Massless Propagators up to Transcendentality Weight Twelve*, *Nucl. Phys. B* **856** (2012) 95 [[arXiv:1108.0732](#)] [[INSPIRE](#)].
- [40] R.M. Schabinger, *Constructing multi-loop scattering amplitudes with manifest singularity structure*, [arXiv:1806.05682](#) [[INSPIRE](#)].
- [41] R.J. Gonsalves, *Dimensionally regularized two-loop on-shell quark form factor*, *Phys. Rev. D* **28** (1983) 1542 [[INSPIRE](#)].
- [42] T. Gehrmann, T. Huber and D. Maître, *Two-loop quark and gluon form-factors in dimensional regularisation*, *Phys. Lett. B* **622** (2005) 295 [[hep-ph/0507061](#)] [[INSPIRE](#)].
- [43] T. Gehrmann, G. Heinrich, T. Huber and C. Studerus, *Master integrals for massless three-loop form-factors: One-loop and two-loop insertions*, *Phys. Lett. B* **640** (2006) 252 [[hep-ph/0607185](#)] [[INSPIRE](#)].
- [44] G. Heinrich, T. Huber and D. Maître, *Master integrals for fermionic contributions to massless three-loop form-factors*, *Phys. Lett. B* **662** (2008) 344 [[arXiv:0711.3590](#)] [[INSPIRE](#)].
- [45] G. Heinrich, T. Huber, D.A. Kosower and V.A. Smirnov, *Nine-Propagator Master Integrals for Massless Three-Loop Form Factors*, *Phys. Lett. B* **678** (2009) 359 [[arXiv:0902.3512](#)] [[INSPIRE](#)].
- [46] R.N. Lee, A.V. Smirnov and V.A. Smirnov, *Dimensional recurrence relations: an easy way to evaluate higher orders of expansion in ϵ* , *Nucl. Phys. Proc. Suppl.* **205-206** (2010) 308 [[arXiv:1005.0362](#)] [[INSPIRE](#)].
- [47] R.N. Lee and V.A. Smirnov, *Analytic ϵ -expansions of Master Integrals Corresponding to Massless Three-Loop Form Factors and Three-Loop $g - 2$ up to Four-Loop Transcendentality Weight*, *JHEP* **02** (2011) 102 [[arXiv:1010.1334](#)] [[INSPIRE](#)].
- [48] J.M. Henn, A.V. Smirnov and V.A. Smirnov, *Evaluating single-scale and/or non-planar diagrams by differential equations*, *JHEP* **03** (2014) 088 [[arXiv:1312.2588](#)] [[INSPIRE](#)].
- [49] R.N. Lee and K.T. Mingulov, *Introducing SummerTime: a package for high-precision computation of sums appearing in DRA method*, *Comput. Phys. Commun.* **203** (2016) 255 [[arXiv:1507.04256](#)] [[INSPIRE](#)].
- [50] D. Binosi and L. Theussl, *JaxoDraw: A Graphical user interface for drawing Feynman diagrams*, *Comput. Phys. Commun.* **161** (2004) 76 [[hep-ph/0309015](#)] [[INSPIRE](#)].
- [51] J.A.M. Vermaseren, *Axodraw*, *Comput. Phys. Commun.* **83** (1994) 45 [[INSPIRE](#)].