A leading-order comparison between fluid-gravity and membrane-gravity dualities

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ABSTRACT: In this note, we have compared two different perturbation techniques that are used to generate dynamical black-brane solutions to Einstein's equations in the presence of negative cosmological constant. One is the 'derivative expansion', where the gravity solutions are in one-to-one correspondence with the solutions of relativistic Navier-Stokes equation. The second is the expansion in terms of inverse power of space-time dimensions and here the gravity solutions are dual to a co-dimension one dynamical membrane, embedded in AdS space and coupled to a velocity field. We have shown that in a large number of space-time dimensions, there exists an overlap regime between these two perturbation techniques and we matched the two gravity solutions along with their dual systems up to the first non-trivial order in the expansion parameter on both sides.

KEYWORDS: Black Holes, Classical Theories of Gravity

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1 Introduction

It is very hard to solve Einstein’s equations — the key equation governing the dynamics of space-time, even at the classical level. Only a few exact solutions are known, mostly being static or stationary. To handle non-trivial dynamics, we have to take recourse of perturbation.

Two such important perturbation schemes, which can handle dynamical fluctuations around static solutions even at non-linear level, are ‘derivative expansion’ [1–6] and expansion in inverse powers of dimension [7–16]. The first one generates ‘black-hole’ type solutions (i.e., space-time with singularity shielded behind the horizon) that are in one to one correspondence with the solutions of relativistic Navier Stokes equation¹ whereas the second one generates a similar ‘black hole type’ solutions, but dual to the dynamics of a codimension one membrane embedded in the asymptotic geometry.²

¹See [17] and references therein.

²See [18] and references therein.

It is natural to ask whether it is possible to apply both the perturbation techniques simultaneously in any regime(s) of the parameter space of the solutions, and if so, how the two solutions compare in those regimes. In this note, we would like to answer these two questions. In a nutshell, our final result is only what is expected.

- It is possible to apply both the perturbation techniques simultaneously. Further, in the regime where both D is large and derivatives are small in an appropriate sense, we could treat $(\frac{1}{D})$ and ∂_μ (with respect to some length scale) as two independent small parameters, with no constraint on their ratio.
- In other words, if the metric dual to hydrodynamics is further expanded in inverse powers of dimension, it matches with the metric dual to membrane-dynamics, again expanded in terms of derivatives.

However, this matching is not at all manifest. We could see it only after some appropriate gauge or coordinate transformation of one solution to the other. The whole subtlety of our computation lies in finding the appropriate coordinate transformation.

The ‘large- D ’ expansion technique, as described in [7], generates the dynamical black brane geometry in a ‘split form’ where the full metric could always be written as a sum of pure AdS metric and something else. In other words, the black-brane space-time, constructed through ‘large- D ’ approximation would always admit a very particular point-wise map to pure AdS geometry.

On the other hand, the space-time dual to fluid dynamics does not require any such map for its perturbative construction and apparently there is no guarantee that the particular map used in ‘large- D ’ technique, would also exist for the dynamical black-brane geometries, constructed in ‘derivative expansion’.

In this note, we have shown that the ‘hydrodynamic metric’³ indeed could be ‘split’ as required through an explicit computation up to first order in derivative expansion. This map could be constructed in any number of dimension and is independent of the ‘large- D ’ approximation. After determining this map, we have matched these two different gravity solutions up to the first subleading order on both sides. We believe it would get more non-trivial at next order but we leave that for the future.

One interesting outcome of this exercise is the matching of the dual theories of both sides. It essentially reduces to a rewriting of hydrodynamics in a large number of dimensions, in terms of the dynamics of the membrane. After implementing the correct gauge transformation, we finally get a field redefinition of the fluid variables (i.e., fluid velocity and the temperature) in terms of membrane velocity and its shape.⁴ We hope such a rewriting would lead to some new ways to view fluid and membrane dynamics and more ambitiously to a new duality between fluid and membrane dynamics in a large number of

³In this note, the black-brane solution dual to fluid dynamics would always be referred to as the ‘hydrodynamic metric’.

⁴Truly speaking, what we have actually worked with is the reverse of what we have stated here, i.e., we determined the membrane velocity and the shape in terms of fluid variables, up to corrections of order $\mathcal{O}(\frac{1}{D}, \partial^2)$. This is just for convenience. The relations we found are easily invertible within perturbation.

dimensions, where gravity has no role to play (see [16, 19] for a similar discussion on such field redefinition and rewriting of fluid equations though in [19] the authors have taken the large D limit in a little different way than ours).

The organization of this note is as follows.

In section 2 we first discussed the overlap regime of these two perturbation schemes. Next in the section 3 we discussed the map between the bulk of the ‘black-hole’ space-time and the pure AdS, mentioned above and described an algorithm to construct the map, whenever it exists. In section 4 we compared the two metrics and the two sets of dual equations (controlling the fluid-dynamics and the membrane dynamics respectively) within the overlap regime, up to the first subleading order on both sides. This section contains the main calculation of the paper. We worked out the map between these two sets of dual variables, leading to a map between large D relativistic hydrodynamics and the membrane dynamics. Finally, in the section 6 we concluded and discussed the future directions.

2 The overlap regime

In this section, we shall discuss whether we could apply both ‘derivative expansion’ and $(\frac{1}{D})$ expansion simultaneously. We shall first define the perturbation parameters for each of these two techniques in a precise way and also fix the range of their validity. We shall see that these two parameters are completely independent of each other and therefore their ratio could be tuned to any value, large or small.

Next, we shall compare the forms of the two metrics, determined using these two techniques, assuming the ratio (between the two perturbation parameters) to have any arbitrary value.

2.1 Perturbation parameter in ‘derivative expansion’

Here we shall very briefly describe the method of ‘derivative expansion’. See [17] for a more elaborate discussion.

The technique of ‘derivative expansion’ could be applied to construct a certain class of solutions to Einstein’s equations in the presence of negative cosmological constant in arbitrary dimension D .

$$\begin{aligned} &\textit{The key gravity equation:} \\ &\mathcal{E}_{AB} \equiv R_{AB} + (D-1)\lambda^2 g_{AB} = 0 \end{aligned} \tag{2.1}$$

λ is the inverse of AdS radius. From now on, we shall choose units such that λ is set to one.

These gravity solutions are of black hole’ type, meaning they would necessarily have a singularity shielded by some horizon [3]. They are in one-to-one correspondence with the solutions of relativistic Navier-Stokes equation in $(D-1)$ dimensional flat space-time (without any restriction on the value of D). In fact, we could use the hydrodynamic variables themselves to label the different gravity solutions, constructed using this technique of ‘derivative expansion’. The labeling hydrodynamic variables are

1. Unit normalized velocity: $u^\mu(x)$
2. Local temperature: $T(x) = (\frac{D-1}{4\pi}) r_H(x)$

At the moment r_H is just some arbitrary length scale, which would eventually be related to the horizon scale of the dual black-brane metric.

$\{x^\mu\}$, $\mu = \{0, 1, \dots, D-2\}$ are the coordinates on the flat space-time whose metric is simply given by the Minkowski metric, $\eta_{\mu\nu} = \text{Diag}\{-1, 1, 1, 1 \dots\}$.

‘Derivative expansion’ enters right into the definition of the hydrodynamic limit. The velocity and the temperature of fluid are functions of space-time but the functional dependence must be slow with respect to the length scale $r_H(x)$. For a generic fluid flow at a generic point, it implies the following.

Choose an arbitrary point x_0^μ ; scale the coordinates (or set the units) such that in the transformed coordinate $r_H(x_0) = 1$. Now the technique of derivative expansion would be applicable provided in this scaled coordinate system

$$\begin{aligned} |\bar{\partial}_{\alpha_1} \bar{\partial}_{\alpha_2} \dots \bar{\partial}_{\alpha_n} r_H|_{x_0} &\ll |\bar{\partial}_{\alpha_1} \bar{\partial}_{\alpha_2} \dots \bar{\partial}_{\alpha_{n-1}} r_H|_{x_0} \ll \dots \ll |\bar{\partial}_{\alpha_1} r_H|_{x_0} \ll 1 \quad \forall n, \alpha_i, x_0 \\ |\bar{\partial}_{\alpha_1} \bar{\partial}_{\alpha_2} \dots \bar{\partial}_{\alpha_n} u^\mu|_{x_0} &\ll |\bar{\partial}_{\alpha_1} \bar{\partial}_{\alpha_2} \dots \bar{\partial}_{\alpha_{n-1}} u^\mu|_{x_0} \ll \dots \ll |\bar{\partial}_{\alpha_1} u^\mu|_{x_0} \ll |u^\mu| \quad \forall n, \alpha_i, x_0 \end{aligned} \quad (2.2)$$

In other words, the number of ∂_α derivatives in a given term determines how suppressed the term is.⁵ In terms of original x^μ coordinates, each derivative ∂_μ corresponds to $r_H \bar{\partial}_\mu$. Therefore if we work in x^μ (which, unlike $\bar{x}^\mu$, are not defined around any given point) coordinates, the parameter that controls the perturbation is schematically $\sim r_H^{-1} \partial_\mu$.⁶

The starting point of this perturbation is a boosted black-brane in asymptotically AdS space. The metric has the following form (in coordinates denoted as $\{r, x^\mu\}$, $\mu = \{0, 1, \dots, D-2\}$). Units are chosen so that dimensionful constant, λ , appearing in equation (2.1) is set to one).⁷

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2 f(r/r_H) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\ \text{where } f(z) &= 1 - \frac{1}{z^{D-1}}, \quad P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu \end{aligned} \quad (2.3)$$

Equation (2.3) is an exact solution to equation (2.1) provided u_μ and r_H are constants.

Now the algorithm for ‘derivative expansion’ runs as follows. Suppose, u^μ and r_H are not constants but are functions of $\{x^\mu\}$. Equation (2.3) will no longer be a solution. If we evaluate the gravity equation $\mathcal{E}_{AB}$ on (2.3), the r.h.s. will certainly be proportional to

⁵The conditions as described in (2.2) are for a generic situation. For a particular fluid profile, it could happen that at a given point in space-time some n th order term is comparable to or even smaller than some $(n+1)$ th order term. One might have to rearrange the fluid expansion around such anomalous points if they exist, but they do not imply a ‘breakdown’ of hydrodynamic approximation. As long as all derivatives in appropriate dimensionless coordinates are suppressed compared to one, ‘derivative expansion’ could be applied.

⁶For a conformal fluid in finite dimension, there is only one length scale, set by the local temperature which also sets the scale of derivative expansion. But if we take $D \rightarrow \infty$, $T(x)$ and $r_H \sim \frac{T(x)}{D}$ are two parametrically separated scales and it becomes important to know which one among these two scales controls the derivative expansion. In the condition (2.2) we have chosen r_H to be the relevant scale and set it to order $\mathcal{O}(1)$. Indeed the results in [5] seem to indicate that terms of different derivative orders in hydrodynamic stress tensor, dual to gravity are weighted by factors of $r_H \sim \frac{T(x)}{D}$, and not T alone.

Note that here the temperature of the fluid would scale as D , which is different from the D scaling of the temperature, imposed in [19].

⁷Note that the scaling of λ with D is up to us. At finite D it is of no relevance, but it matters while taking the large D limit. Here λ would be fixed to one as we would take D to ∞ .

the derivatives of u_μ and r_H . But u_μ and r_H being the hydrodynamic variables, their derivatives are ‘small’ at every point in the sense described in (2.2). Therefore a ‘small’ correction in the leading ansatz could solve the equation.

The r dependence of these ‘small corrections’ could be determined exactly while the $\{x^\mu\}$ dependence would be treated in perturbation in terms of the labeling data $u^\mu(x)$ and $r_H(x)$ and their derivatives. $u^\mu(x)$ and $r_H(x)$ themselves would be constrained to satisfy the hydrodynamic equation, order by order in derivative expansion. While dealing with the full set of gravity equations (2.1), these equations on the hydrodynamic variables or the labeling data would emerge as the ‘constraint equations’ of the theory of classical gravity.

2.2 Perturbation parameter in $(\frac{1}{D})$ expansion

This is a perturbation technique, which is applicable only in a large number of space-time dimension (denoted as D), as a series expansion in powers of $(\frac{1}{D})$. Clearly $(\frac{1}{D})$ is the perturbation parameter (a dimensionless number to begin with) here, which must satisfy

$$\left(\frac{1}{D}\right) \ll 1$$

Unlike the derivative expansion, the $(\frac{1}{D})$ expansion does not necessarily need the presence of cosmological constant, but we could also apply it if the cosmological constant is present provided we keep λ , the AdS radius (see equation (2.1) in subsection 2.1) fixed as we take D large. Note that the choice $\lambda = 1$, as we have done in previous subsection, is consistent with this ‘ D -scaling’.

The starting point here is the following metric.

$$dS^2 \equiv \mathcal{G}_{AB} dX^A dX^B = \bar{G}_{AB} dX^A dX^B + \psi^{-D} (O_A dX^A)^2 \quad (2.4)$$

where $\bar{G}_{AB}$, ψ and O_A are defined as follows.

1. $\bar{G}_{AB}$ is a smooth metric of pure AdS geometry which we shall refer to as ‘background’.
We could choose any coordinate as long as the metric is smooth and all components of the Riemann curvature tensors are of order $\mathcal{O}(1)$ or smaller in terms of large D -order counting.
2. (ψ^{-D}) is a harmonic function with respect to the metric $\bar{G}_{AB}$.
3. O_A is the one-form dual to the tangent vector to a null geodesic in the background satisfying $O_A n_B \bar{G}^{AB} = 1$. Where, n_A is the unit normal on the constant ψ hypersurfaces (viewed as hypersurfaces embedded in the background).

The metric (2.4) would solve the Einstein’s equations (2.1) at leading order (which turns out to be of order $\mathcal{O}(D^2)$) provided the divergence of the $\mathcal{O}(1)$ vector field, $U^A \equiv n^A - O^A$ with respect to the background metric is also of order $\mathcal{O}(1)$.

$$\nabla \cdot U \equiv \left(\nabla \cdot n - \nabla \cdot O \right)_{\psi=1} = \mathcal{O}(1) \quad (2.5)$$

where $\nabla \equiv$ covariant derivative w.r.t. $\bar{G}_{AB}$

Naively equation (2.5) does not seem to constrain the vector field U^A since each of its components along with their derivatives in every direction are of order $\mathcal{O}(1)$ (this is what we mean by an ‘order $\mathcal{O}(1)$ vector field’). However, it is indeed a constraint within the validity-regime of $(\frac{1}{D})$ expansion. We could apply large D techniques provided for a generic $\mathcal{O}(1)$ vector field $V^A \partial_A$, its divergence is of order $\mathcal{O}(D)$.⁸

One easy way to ensure such scaling would be to assume that the dynamics is confined within a finite number of dimensions and the rest of the geometry is protected by some large symmetry [7].

From now on, we shall assume such symmetry to be present in all the dynamics we discuss, including the dual hydrodynamics, labeling the different geometries constructed in ‘derivative expansion’. For example, we shall assume that the divergence of the fluid velocity u^μ , which we shall denote by $\Theta (\equiv \partial_\mu u^\mu)$, is always of order $\mathcal{O}(D)$, whereas the velocity vector itself is of order $\mathcal{O}(1)$.

Now we shall briefly describe some general features of this leading geometry in (2.9). See [7] for a detailed discussion.

Firstly note that with the above conditions, the hypersurface $\psi = 1$ becomes null and we could identify this surface with the event horizon of the full space-time.

Also, if one is finitely away from the $\psi = 1$ hypersurface, the factor ψ^{-D} vanishes for large D and the metric reduces to its asymptotic form $\bar{G}_{AB}$.

Next, consider the region of thickness of the order of $\mathcal{O}(\frac{1}{D})$ around $\psi = 1$ hypersurface. This is the region,⁹ where $(\frac{1}{D})$ expansion would lead to a nontrivial correction to the leading geometry. To see why, let us do the following coordinate transformation.

$$X^A = X_0^A + \frac{\tilde{x}^A}{D} \quad \partial_A = D \tilde{\partial}_A$$

where $\{X_0^A\}$ is an arbitrary point on the $\psi = 1$ hypersurface. In these new coordinates

$$dS^2 = D^2 G_{AB} d\tilde{x}^A d\tilde{x}^B, \quad \text{where} \quad G_{AB} = \mathcal{G}_{AB} \left(X_0 + \frac{\tilde{x}}{D} \right) \quad (2.7)$$

⁸This requirement certainly restricts the allowed dynamics that could be handled using this method. But it is not as restrictive as it might seem to begin with. To see it explicitly, let us choose a coordinate system $\{z, y^\mu\}$ for the background.

$$\begin{aligned} \bar{G}_{zz} &= \frac{1}{z^2}, \quad \bar{G}_{\mu\nu} = z^2 \eta_{\mu\nu} \quad \text{Det}[\bar{G}] = -z^{(D-2)} \\ \nabla \cdot V &= z^{-(D-2)} \partial_z \left[z^{(D-2)} V^z \right] + \partial_\mu V^\mu \\ &= \partial_z V^z + \partial_\mu V^\mu + (D-2) \left(\frac{V_z}{z} \right) \end{aligned} \quad (2.6)$$

Here clearly the first term is of order $\mathcal{O}(1)$. The second term could potentially be of order $\mathcal{O}(D)$ since a large number of indices are summed over. Still to precisely cancel against the last term, which certainly is of order $\mathcal{O}(D)$ as long as $(\frac{V_z}{z})$ is not very small, it requires some fine tuning. Equation (2.5) says that $U^A \partial_A$ is such a fine-tuned vector field.

⁹Following [7], we shall refer to this region as ‘membrane region’.

Now, if $\tilde{x}^A$ is not as large as D , it is possible to expand ψ^{-D} , O_A and $\bar{G}_{AB}$ around X_0^A .

$$\begin{aligned} \psi^{-D}(X^A) &= e^{-\tilde{x}^A N_A} + \mathcal{O}\left(\frac{1}{D}\right), \quad \text{where } N_A = [\partial_A \psi]_{X_0^A} \\ O^A(X) &= O^A|_{X_0^A} + \mathcal{O}\left(\frac{1}{D}\right), \quad G_{AB}(X) = G_{AB}|_{X_0^A} + \mathcal{O}\left(\frac{1}{D}\right) \end{aligned} \quad (2.8)$$

Note that from the second condition (see the discussion below equation (2.4)) it follows that

$$\text{Extrinsic curvature of } (\psi = 1) \text{ surface} = K|_{\psi=1} = D\sqrt{N_A N_B \bar{G}^{AB}} + \mathcal{O}(1)$$

Substituting equation (2.8) in equation (2.7) we find

$$\begin{aligned} G_{AB} &= O_A(X_0) n_B(X_0) + O_B(X_0) n_A(X_0) + P_{AB}(X_0) \\ &\quad - \left(1 - e^{-\tilde{x}^A N_A}\right) O_A(X_0) O_B(X_0) + \mathcal{O}\left(\frac{1}{D}\right) \\ \text{where } P_{AB}(X^0) &\equiv \text{projector perpendicular to } n_A(X_0) \text{ and } O_A(X_0) \\ n_A &= \frac{\partial_A \psi}{\sqrt{(\partial_A \psi)(\partial_B \psi) \bar{G}^{AB}}} \end{aligned} \quad (2.9)$$

Clearly, at the very leading order, the metric will have non-trivial variation only along the direction of N_A — the normal to the $\psi = 1$ hypersurface at point X_0^A . Variations along all other directions are suppressed by factors of $(\frac{1}{D})$. This is very similar to the metric in equation (2.3) where at leading order the non-trivial variation is only along a single direction — r . Therefore, within this ‘membrane region’, $(\frac{1}{D})$ expansion would *almost* reduce to derivative expansion along directions other than N_A provided the metric (2.9) solves equation (2.1) at very leading order. The conditions, listed below equation (2.4) along with equation (2.5) ensure that this is the case.

Once the leading solution is found, the same algorithm, described in the previous subsection, would work and we could find the subleading corrections handling the variations of N_A and O_A along the constant ψ hypersurface. All such variations would be suppressed as long as none of the components of N_A , O_A and their derivatives (in the unscaled X^A coordinates) are as large as D . In other words, we should be able choose a coordinate system, along the horizon (or the hypersurface $\psi = 1$) such that

$$[\bar{G}^{AB} (\partial_A \psi^{-D}) (\partial_B \psi^{-D})]^{-\frac{1}{2}} \partial_A |_{\text{horizon}} \ll 1 \quad (2.10)$$

It is enough to impose this inequality only on the $\psi = 1$ hypersurface; the conditions listed below equation (2.4) will ensure that they are true on all constant ψ surfaces.

These conditions also specify the defining data (analogue of fluid-velocity and temperature in case of ‘derivative expansion’) for the class of metrics, generated by $(\frac{1}{D})$ expansion. Here, the gravity solutions are expressed in terms of the auxiliary function ψ and the one-form $O_A dX^A$. These two auxiliary fields satisfy the second and the third conditions, listed below equation (2.4). However, the above-mentioned conditions, being differential equations, could not fix the fields completely unless some boundary conditions are specified along any fixed surface. The most natural choice for this hypersurface is the surface

given by $\psi = 1$, which, by construction, is the horizon of the full space-time geometry. Different metric solutions are classified by the shape of this surface and the components of O_A projected along the surface. Just as in ‘derivative expansion’, we could solve for the metric correction only if these defining data (the projected O_A field and the shape of the surface, encoded in its extrinsic curvature) satisfy the constraint equation, which we shall refer to as the ‘membrane equation’.

2.3 Comparison between two perturbation schemes

In subsection 2.2, we have seen that within the membrane region, $\mathcal{O}(\frac{1}{D})$ expansion is *almost* like ‘derivative expansion’ as described in subsection 2.1. Still, it is also clear that they are not quite the same. The leading ansatz itself looks quite different for the two schemes, and there is no question of overlap if these two techniques compute perturbations around two entirely different geometries. So, to find an ‘overlap regime’, the first step would be to see where in the parameter-space and in what sense, equation (2.3) and (2.7) describe the same leading geometry.

Note that though the leading geometries look different algebraically, they both have similar geometric properties — namely the existence of a curvature singularity. In metric (2.3) it is located at $r = 0$ and the metric (2.7) is singular at $\psi = 0$. Also, the singularity is shielded by some event-horizon.¹⁰

To see the similarities more explicitly, let us first choose a coordinate system $X^A \equiv \{\rho, X^\mu\}$, such that the background metric- $\bar{G}_{AB}$ in equation (2.8) takes the form

$$\bar{G}_{AB} dX^A dX^B = \frac{d\rho^2}{\rho^2} + \rho^2 \eta_{\mu\nu} dX^\mu dX^\nu, \quad (2.11)$$

In this coordinate system, the following metric is an exact solution of equation (2.1)

$$ds^2 = \frac{d\rho^2}{\rho^2} + \rho^2 \eta_{\mu\nu} dX^\mu dX^\nu + \left(\frac{\rho}{r_H}\right)^{-(D-1)} \left(\frac{d\rho}{\rho} - \rho dt\right)^2 \quad (2.12)$$

This is simply the Schwarzschild black-brane solution, written in Kerr-Schild form. Now let us note the following features of this metric [7].

- The function $\left(\frac{\rho}{r_H}\right)^{-(D-1)}$ is harmonic with respect to the background up to correction of order $\mathcal{O}(\frac{1}{D})^2$.

$$\nabla^2 \left(\frac{\rho}{r_H}\right)^{-(D-1)} = \mathcal{O}\left(\frac{1}{D}\right)^2$$

Hence the function $\left(\frac{\rho}{r_H}\right)^{-(D-1)}$ could be identified with ψ^{-D} appearing in the metric (2.4) up to corrections of order $\mathcal{O}(\frac{1}{D})^2$.

¹⁰So far, the way both the techniques of ‘large- D expansion’ and ‘derivative expansion’ are developed, the existence of a horizon is a must. It would be interesting to know whether we could depart from this condition and still apply either of these two techniques to construct ‘horizon free’ or non-singular smooth geometries.

- The one form $\left(\frac{d\rho}{\rho} - \rho dt\right)$ is null and satisfies the geodesic equation. Further, contraction of this one-form with the unit normal to constant ρ hypersurfaces is one.

Hence this one form could be identified with the null one form $O_A dX^A$

Hence it follows that the metric in (2.12), which is an exact solution of (2.1), could be cast in the form of our leading ansatz up to corrections subleading in $\left(\frac{1}{D}\right)$ expansion. We could also expand the metric in equation (2.12) around a given point on the horizon $\rho = r_H$, the same way we have done (see equation (2.9)) in the previous subsection with the following set of identifications.

$$\begin{aligned} N_A dX^A|_{\rho=1} &= \frac{d\rho}{r_H}, & O_A dX^A|_{\rho=1} &= \frac{d\rho}{r_H} - r_H dt \\ n_A dX^A &= \frac{N_A dX^A}{\sqrt{N_A N^A}} = \frac{d\rho}{r_H} \end{aligned} \quad (2.13)$$

The very leading term in this expansion, once written in terms of N_A and O_A would have exactly the same form as that of the metric in equation (2.7). The main difference between our leading ansatz, equation (2.4) and equation (2.12) is that in the later N_A and O_A satisfy equation (2.13) everywhere along the horizon, in the same $\{\rho, y^\mu\}$ coordinates. For our leading ansatz (2.4) also, it is true that we could always choose a local $\{\rho, t\}$ coordinates by reversing the equations in (2.13). But for a generic ψ and O_A , this could not be done globally and this is the reason why our leading ansatz is not an exact solution of (2.1). However, the deviation from the exact solution would clearly be proportional to the derivatives of N_A and O_A and therefore subleading. So finally we conclude that locally around a point on the horizon, the leading ansatz for $\left(\frac{1}{D}\right)$ expansion looks like a Schwarzschild black-brane written in a Kerr-Schild form with the local ρ and t coordinates, respectively oriented along the direction of the normal N_A and the direction O_A projected along the membrane $\psi = 1$.

Now let us come to the leading ansatz for the metric in derivative expansion. As it is explained in detail in [2], the leading ansatz in derivative expansion, equation (2.3), reduces to Schwarzschild black-brane in Eddington-Finkelstein coordinates if we choose $r_H = \text{constant}$ and $u^\mu = \{1, 0, 0, \dots\}$. Also locally at any point $\{x_0^\mu\}$, we could always choose a coordinate system such that $u^\mu(x_0) = \{1, 0, 0, \dots\}$, or in other words by appropriate choice of coordinates locally the metric described in (2.3) could always be made to look like a Schwarzschild black-brane, though in a different gauge than in equation (2.4). Clearly, the starting point of these different expansions are ‘locally’ same and it is possible to have an overlap regime.

But the difference lies in the concept of ‘locality’ and also in the space of defining data. In case of ‘large- D ’ expansion, the classifying data of the metric is specified on the horizon whereas for ‘derivative expansion’ it is defined on the boundary of AdS space.

The range of validity for ‘large- D ’ expansion is given in equation (2.10). If we replace $\partial_A \psi^{-D}|_{\text{horizon}}$ by $(-DN_A)$ the condition (2.10) reduces to the existence of coordinate system such that

$$\partial_A |_{\text{horizon}} \ll D \quad (2.14)$$

which looks very similar to the validity regime for ‘derivative expansion’, as already mentioned in subsection 2.1

$$r_H^{-1} \partial_\mu \ll 1 \quad (2.15)$$

If we could somehow map each point on the boundary to a point on the horizon (viewed as a hypersurface embedded in the background), the same $\{x^\mu\}$ coordinates could be used as coordinates along the horizon. In that case, whenever r_H is of order $\mathcal{O}(1)$ in terms of ‘large- D ’ order counting, the inequality (2.15) would imply equation (2.14). In other words, as $D \rightarrow \infty$, all solutions of ‘derivative expansion’ could be legitimately expanded further in $(\frac{1}{D})$, though the reverse may not be true.

Now we know that ∂_A and ∂_μ are simply related (without any extra factor of D) for the case of exact Schwarzschild black-brane solutions. This is just the well-known coordinate transformation one should use to go from Kerr-Schild to Eddington-Finkelstein form of the black-brane metric. This transformation also gives the required map from the horizon to boundary coordinates. Once perturbations are introduced on both sides, we expect the relation between these two sets of coordinate systems would get corrected, but in a controlled and perturbative manner, thus maintaining the above argument for the existence of overlap.

So in summary, there does exist a region of overlap between these two perturbative techniques. In this note, our goal is to match them in the regime of overlap. As it is clear from the above discussion, the key step involves determining the map between ∂_A and ∂_μ , which we are going to elaborate in the next section.

3 Transforming to ‘large- D ’ gauge

From the discussion of section 2 it follows that if the space-time dimension D is very large, we could always apply ‘ $(\frac{1}{D})$ expansion’ whenever ‘derivative expansion’ is applicable. Therefore a metric, corrected in derivative expansion in arbitrary dimension, when further expanded in $(\frac{1}{D})$, should reproduce the metric generated independently using the method of ‘ $(\frac{1}{D})$ expansion’. More precisely if we take the metric of equation (4.1) from [5] and expand it in $(\frac{1}{D})$, it should match with the metric given in equation (8.1) of [7] after appropriate transformation.

In this section, our goal is to understand what these ‘appropriate transformations’ are. Let us explain it in little more detail.

As we have mentioned before, both of these two perturbative techniques generate black brane geometries, in terms of a set of ‘dynamical data’, confined to a codimension one hypersurface. In the first case, it is the boundary of the Asymptotic AdS space and in the second case, it is the event horizon viewed as a hypersurface embedded in pure AdS. So both the techniques require a map from the full space-time geometry to a codimension one membrane.

The details of this map are quite clear for the case of ‘derivative expansion’.

The data-set that distinguishes between different dynamical geometries, here is the profile of a relativistic conformal fluid (its velocity and temperature). In other words,

given a unit normalized velocity field and temperature, defined on a $(D - 1)$ dimensional flat space-time and satisfying the relativistic Navier-Stokes equation, we should be able to uniquely construct a D dimensional space-time with a dynamical event horizon such that its metric is a solution to (2.1). The $(D - 1)$ dimensional space is identified with the conformal boundary of this D dimensional black-brane geometry, which we shall refer to as bulk. This construction [5] uses a very specific coordinate system, that encodes how a point in the bulk could be associated with a point in the boundary. In [20], the authors have also explained how to reverse the construction of [2, 5]. They have given an algorithm to read off the dual fluid variables starting from any black-brane geometry that admits derivative expansion but written in arbitrary coordinates. This explicitly proves the claim of one-to-one correspondence between the dynamical black-brane geometry, admitting derivative expansion and the fluid profile, satisfying relativistic Navier-Stokes equation. This algorithm has been heavily used to cast the rotating black-holes in the ‘hydrodynamic form’ [5].

Similarly, according to [7], there exists a one-to-one correspondence between dynamical black-brane geometries in $(\frac{1}{D})$ expansion and a codimension-one ‘membrane dynamics’ in pure AdS space, though [7] shows the correspondence in only one direction. It starts from valid membrane data and integrates it outward towards infinity to construct the corresponding black-brane geometry. But to explicitly show this correspondence, we also need to know the reverse. In other words, we should know how to associate a point on the membrane to a point on the bulk and how to read off the membrane data, starting from a dynamical black-brane geometry that admits an expansion in $(\frac{1}{D})$, but written in some arbitrary coordinates.

In the next subsection, we shall formulate an algorithm to determine this ‘membrane-bulk map’, analogous to the discussion of [20] in the context of transforming the rotating black holes to the hydrodynamic form.

3.1 Bulk-membrane map¹¹

The ‘large-D expansion’ technique, as developed in [7], would always generate the dynamical black-brane metric G_{AB} in a ‘split’ form. This ‘split’ is specified in terms of an auxiliary function ψ and an auxiliary vector field $O^A \partial_A$. In terms of equation,

$$G_{AB} = \bar{G}_{AB} + G_{AB}^{(\text{rest})} \quad (3.1)$$

where $\bar{G}_{AB}$ is the background and $G_{AB}^{(\text{rest})}$ is such that there exists a null geodesic vector field $O^A \partial_A$ in the background, satisfying

$$O^A G_{AB} = O^A \bar{G}_{AB} \Rightarrow O^A G_{AB}^{(\text{rest})} = 0 \quad (3.2)$$

The normalization of this null geodesic vector is determined in terms of the function ψ , defined as follows.

¹¹This subsection has been worked out by Shiraz Minwalla in a different context. We sincerely thank him for explaining it in detail to us. This ‘bulk-membrane’ map is the key concept needed for the required ‘matching’ of the two perturbative gravity solutions.

1. (ψ^{-D}) is a harmonic function with respect to the metric $\bar{G}_{AB}$.
2. $\psi = 1$ hypersurface, when viewed as an embedded surface in full space-time, becomes the dynamical event horizon. This is how the boundary condition on ψ is specified.

After fixing ψ , the normalization of O^A is fixed through the following condition.

$$O^A n_A = 1.$$

where n_A is the unit normal on the constant ψ hypersurfaces (viewed as hypersurfaces embedded in the background).

The equations (3.1) and (3.2) together specify a map between two entirely different geometries, with metric $\bar{G}_{AB}$ and G_{AB} respectively, both satisfying equation (2.1). So if we want to recast an arbitrary dynamical black-brane metric, which admits $(\frac{1}{D})$ expansion, in the form as described in (3.1), the first step would be to figure out this map or the ‘split’ of the space-time between ‘background’ and the ‘rest’, so that the equation (3.2) is obeyed.

Now from the discussion of the previous subsection, we see that this ‘map’ is crucially dependent on the vector field $O^A \partial_A$ and the function ψ . But both of them are defined using the ‘background’ geometry and we immediately face a problem, since given an arbitrary black-brane metric, it is the ‘background’ that we are after.

For example, given a black-brane metric, we could always determine the location of the event horizon, but we would never know its embedding in the background, unless we know the ‘split’ and therefore we would not be able to construct the ψ function, by exploiting the harmonicity condition on ψ^{-D} . If we do not know ψ we would not be able to orient or normalize O^A , as required.

So we must have some equivalent formulation of this ‘split’ just in terms of the full space-time metric. The following observation allows us to do it. We could show that if G_{AB} admits a split between $\bar{G}_{AB}$ and $G_{AB}^{(\text{rest})}$ satisfying $O^A G_{AB}^{(\text{rest})} = 0$, then the vector — $O^A \partial_A$, which is a null geodesic with respect to $\bar{G}_{AB}$, is also a null geodesic with respect to G_{AB} .

Proof. We know that

$$(O \cdot \nabla) O^A = \kappa O^A$$

where ∇ denotes the covariant derivative with respect to $\bar{G}_{AB}$ and κ is the proportionality factor. We would like to show that

$$(O \cdot \bar{\nabla}) O^A \propto O^A, \quad \text{where } \bar{\nabla} \text{ is covariant derivative w.r.t. } G_{AB}$$

Suppose $\bar{\Gamma}_{BC}^A$ denotes the Christoffel symbol corresponding to $\bar{\nabla}_A$ and Γ_{BC}^A denotes the Christoffel symbol corresponding to ∇_A . These two would be related as follows [7].

$$\bar{\Gamma}_{BC}^A = \Gamma_{BC}^A + \underbrace{\frac{1}{2} \left(\nabla_B \left[G^{(\text{rest})} \right]_C^A + \nabla_C \left[G^{(\text{rest})} \right]_B^A - \nabla^A \left[G^{(\text{rest})} \right]_{BC} \right)}_{\delta \Gamma_{BC}^A} \quad (3.3)$$

Here all raising and lowering of indices have been done using $\bar{G}_{AB}$. Note that

$$\begin{aligned} O^B O^C \delta\Gamma_{BC}^A &= O^B (O \cdot \nabla) \left[G^{(\text{rest})} \right]_B^A - \frac{1}{2} O^B O^C \nabla^A \left[G^{(\text{rest})} \right]_{BC} \\ &= - \left[G^{(\text{rest})} \right]_B^A [(O \cdot \nabla) O^B] + \frac{1}{2} (\nabla^A O^C) \left[G^{(\text{rest})} \right]_{BC} O^B \\ &= \kappa \left(O^C \left[G^{(\text{rest})} \right]_C^A \right) = 0 \end{aligned} \quad (3.4)$$

What we want to show simply follows from equation (3.4)

$$(O \cdot \bar{\nabla}) O^A = (O \cdot \nabla) O^A = \kappa O^A \quad (3.5)$$

So we could determine O^A by solving the null geodesic equation with respect to the full space-time metric G_{AB} . But to determine it fully, we also need to know κ , fixed by the normalization of O^A . As mentioned before, the normalization used previously in the application of ‘large- D ’ technique is not suitable for our purpose, since it requires the knowledge of the ‘background’ beforehand. But luckily the form of the ‘split’, which is defined by the condition $\left[O^A G_{AB}^{(\text{rest})} = 0 \right]$ is independent of the normalization of O^A .

So we shall first determine another null geodesic field (let us denote it by $\bar{O}_A$ to remind ourselves of the difference in normalization) which is affinely parametrized and whose inner-product with the normal to event horizon (which, up to normalization, could again be determined without any knowledge of the ‘split’) is one.

Now we are at a stage to define the map between the ‘background’ and the full space-time geometry.

Suppose $\{Y^A\}$ denote the coordinates in the background geometry (in our case pure AdS, the metric is denoted by $\bar{G}_{AB}$) and $\{X^A\}$ are the coordinates of the full space-time (the dynamical black-brane, the metric is denoted by $\mathcal{G}_{AB}$). Let us denote the invertible functions that give a one to one correspondence between these two spaces as $\{f^A\}$.

$$Y^A = f^A(\{X\}) \quad (3.6)$$

The equations that will determine f^A s are the following

$$\bar{O}^A \mathcal{G}_{AB}|_{\{X\}} = \bar{O}^A \left(\frac{\partial f^C}{\partial X^A} \right) \left(\frac{\partial f^{C'}}{\partial X^B} \right) \bar{G}_{CC'}|_{\{X\}}.^{12} \quad (3.7)$$

Here $\bar{O}^A$ are affinely parametrized the null geodesics in the full space-time geometries i.e.,

$$\bar{O} \cdot \bar{\nabla} \bar{O}^A = 0 \quad (3.8)$$

Equation (3.8) would fix $\bar{O}_A$ completely once we specify the angles it would make with the tangents of the horizons, which is effectively a set of $(D - 1)$ numbers. Now what we are

¹²The subscript $\{X\}$ in equation (3.7) denotes that both l.h.s. and r.h.s. of equation (3.7) have to be evaluated in terms $\{X^A\}$ coordinates.

actually interested in is not $\bar{O}_A$ but O_A which is related to $\bar{O}_A$ with a normalization. Therefore we are free to choose the normalization of $\bar{O}_A$, since anyway, we have to re-normalize it again. This will fix one of the $(D - 1)$ initial conditions. Rest we shall keep arbitrary.

We shall assume

$$\begin{aligned}\bar{O}^A N_A|_{\text{horizon}} &= 1 \\ \bar{O}^A l_A^{(i)}|_{\text{horizon}} &= \text{some arbitrary functions of horizon coordinates}\end{aligned}\tag{3.9}$$

where N_A is the null normal to the event horizon (with some arbitrary normalization) and $l_{(i)}^A$ s are the unit normalized space-like tangent vectors to the horizon.

It turns out that the hydrodynamic metric could be split for a very specific choice of these spatial initial conditions and we shall fix them order by order in derivative expansion by matching the hydrodynamic and the ‘large- D ’ metric. Once $\bar{O}^A$ is fixed (in terms of these arbitrary angles), we could determine f^A s up to some integration constants by solving equation (3.7).

Equation (3.7) further says that if we apply the map (3.6) as a coordinate transformation on the ‘background’, then in the new $\{X^A\}$ coordinates the map would just be an ‘identity’ map and the full space-time metric $\mathcal{G}_{AB}$ would admit the split as given in equation (3.1) satisfying (3.2).¹³

Once we have figured out how to split the full space-time metric into ‘background’ and the ‘rest’, we know how to view the event horizon as a surface embedded in the ‘background’ and therefore the auxiliary function ψ (by solving the harmonicity of ψ^{-D} w.r.t. the background) everywhere. Now we can normalize $\bar{O}^A$ as it has been done in [7]. Using these ψ and O^A (appropriately normalized) one should be able to recast any arbitrary metric, that admits large- D expansion, exactly in the form of [7].

4 Bulk-membrane map in metric dual to hydrodynamics

In this subsection, we shall implement the above algorithm, described in the previous subsection, for the metric dual to hydrodynamics. For convenience, we are summarizing the steps again.

- Determine the equation for the event horizon.
- Determine the null normal to the horizon.
- Solve equation (3.8) to determine $\bar{O}^A$ everywhere. We need the normal, derived in the previous step, to impose the boundary condition.
- Choose any arbitrary coordinate system $\{Y^A\}$, where the ‘background’ has a smooth metric G_{AB} .
- Now solve the equation (3.7) to determine the mapping functions f^A ’s.

¹³We would also like to emphasize that what we are describing here is not just a gauge or coordinate transformation. The ‘split’ mentioned in equation (3.1) is a genuine point-wise map between two entirely different geometries. Once we have figured out the ‘map’, we are free to transform the coordinates further; both G_{AB} and $\tilde{G}_{AB}$ would change, but the ‘map’ will still be there.

For a generic dynamical metric, it is not easy to implement all these steps. But in this case what would help us is the ‘derivative expansion’ and the fact that f^A ’s are exactly known at zero derivative order; it is simply the coordinate transformation between Eddington-Finkelstein and Kerr-Schild form of a static black brane metric.

Though the zeroth order transformation is already known, as a ‘warm-up’ exercise we shall re-derive it using the above algorithm. The condition of ‘staticity’ and translational symmetry of the metric allow us to solve relevant equations exactly in this case.

4.1 Zeroth order in ‘derivative expansion’

At zeroth order in derivative expansion, the metric dual to hydrodynamics has the following form

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(r/r_H) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \quad (4.1)$$

where $P_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu$, $f(z) \equiv [1 - z^{-(D-1)}]$, $u_\mu u_\nu \eta^{\mu\nu} = -1$

We could read off the components of the metric and its inverse.

$$\begin{aligned} \mathcal{G}_{rr} &= 0, & \mathcal{G}_{\mu r} &= -u_\mu, & \mathcal{G}_{\mu\nu} &= -r^2 f(r/r_H) u_\mu u_\nu + r^2 P_{\mu\nu} \\ \mathcal{G}^{rr} &= r^2 f(r/r_H), & \mathcal{G}^{\mu r} &= u^\mu, & \mathcal{G}^{\mu\nu} &= \frac{1}{r^2} P^{\mu\nu} \end{aligned} \quad (4.2)$$

At zero derivative order, both r_H and u^μ could be treated as constants, The event horizon and the null normal to it are given by

$$\text{Event Horizon : } \mathcal{S} = r - r_H = 0, \quad N_A dX^A = dX^A \partial_A \mathcal{S} = dr \quad (4.3)$$

Now we shall figure out the ‘map’ that will lead to the desired ‘split’ between ‘background’ and ‘rest’.

We have already determined the event horizon. Next, we have to solve for $\bar{O}^A$, satisfying the conditions

$$\bar{O}^B \bar{\nabla}_B \bar{O}^A = 0, \quad \bar{O}^A \bar{O}^B \mathcal{G}_{AB} = 0, \quad \bar{O}^A N_A|_{r=r_H} = \bar{O}^r|_{r=r_H} = 1$$

At zero derivative order, $\mathcal{G}_{AB}$ has translational symmetry in all the x^μ . The conditions on $\bar{O}^A$ does not break this symmetry. Hence $\bar{O}^A$ must have the form

$$\bar{O}^A \partial_A = h_1(r) \partial_r + h_2(r) u^\mu \partial_\mu \quad (4.4)$$

Now we shall process the condition that O^A is a null vector field.

$$\begin{aligned} \bar{O}^A \bar{O}^B \mathcal{G}_{AB} &= 0 \\ \Rightarrow 2h_2(r)h_1(r)\mathcal{G}_{\mu r}u^\mu + h_2(r)^2 u^\mu u^\nu \mathcal{G}_{\mu\nu} &= 0 \\ \Rightarrow h_2(r) [2h_1(r) - r^2 f(r/r_H) h_2(r)] &= 0 \\ \Rightarrow h_2(r) &= 0 \end{aligned} \quad (4.5)$$

So finally $\bar{O}^A \partial_A = h_1(r) \partial_r$.¹⁴

¹⁴Actually, there is two solutions to (4.5). If we assume $h_2(r) \neq 0$ and finite everywhere, then

$$h_1(r) = \frac{r^2}{2} f(r/r_H) h_2(r)$$

This implies that $h_1(r)$ will vanish at the horizon $r = r_H$ (which is a zero of the function $f(r/r_H)$), contradicting the boundary condition on $\bar{O}^r$.

Substituting this form of $\bar{O}^A$ in the geodesic equation we could see that $h_1(r)$ has to be a constant and then boundary condition simply says that $h_1(r) = 1$

$$\bar{O}^A \partial_A = \bar{O}^r \partial_r = \partial_r \quad (4.6)$$

Now let us choose a coordinate system $Y^A = \{\rho, y^\mu\}$ for the ‘background’ where the metric takes the following form

$$ds_{\text{background}}^2 = \frac{d\rho^2}{\rho^2} + \rho^2 \eta_{\mu\nu} dy^\mu dy^\nu \quad (4.7)$$

Again the symmetries motivate us to take the following form for the mapping, which gives the one to one correspondence between the background coordinates $\{Y^A\} = \{\rho, y^\mu\}$ and black-brane coordinates $\{X^A\} = \{r, x^\mu\}$

$$y^\mu = x^\mu + g(r)u^\mu, \quad \rho = h(r) \quad (4.8)$$

Let us apply the map (4.8) as a coordinate transformation on the background. In the new coordinates (where the map is just an ‘identity’) the background metric takes the following form

$$\bar{\mathcal{G}}_{rr} = \left(\frac{h'}{h}\right)^2 - (g'h)^2, \quad \bar{\mathcal{G}}_{\mu r} = g'h^2 u_\mu, \quad \bar{\mathcal{G}}_{\mu\nu} = h^2 \eta_{\mu\nu} \quad (4.9)$$

Here we have suppressed the r dependence and derivative w.r.t. r is denoted by prime ($'$). In this coordinates equation (3.7) takes the form

$$\left(\frac{h'}{h}\right)^2 - (g'h)^2 = 0, \quad g'h^2 = -1 \quad (4.10)$$

These two equation could be solved very simply. The general solution

$$h(r) = \pm(r + c_1), \quad g(r) = \frac{1}{r + c_1} + c_2 \quad (4.11)$$

where c_1 and c_2 are two arbitrary constants.

We shall choose the plus sign in $h(r)$ to make sure that whenever r increases, ρ also increases.

Now we have to fix the integration constants. Note that once we know the map, we know the form of $\mathcal{G}_{AB}^{(\text{rest})}$, satisfying equation (3.2) by construction.

$$\begin{aligned} \mathcal{G}_{rr}^{(\text{rest})} &= \mathcal{G}_{r\mu}^{(\text{rest})} = 0 \\ \mathcal{G}_{\mu\nu}^{(\text{rest})} &= [(r + c_1)^2 - r^2 f(r/r_H)] u_\mu u_\nu + [r^2 - (r + c_1)^2] P_{\mu\nu} \end{aligned} \quad (4.12)$$

Now we further want that if $D \rightarrow \infty$, the metric should reduce to its asymptotic form at any finite distance from the event horizon or in other words, $\mathcal{G}_{\mu\nu}^{(\text{rest})}$ must vanish outside the ‘membrane region’ (a region with ‘thickness’ of the order of $\mathcal{O}(\frac{1}{D})$ around the ‘membrane’, see section 2.2). This condition will force us to set $c_1 = 0$. The other constant c_2 is not appearing in the final form of the metric at all, so this ambiguity will remain here at this order and it is simply a consequence of the translational symmetry in x^μ and y^μ directions. For simplicity, we shall also choose $c_2 = 0$. So the final form of the map at zeroth order

$$\rho = r, \quad y^\mu = x^\mu + \frac{u^\mu}{r} \quad (4.13)$$

4.2 First order in derivative expansion

In this subsection, we shall extend the computation of the previous subsection up to the first order in derivative expansion. Here u^μ and r_H depends on x^μ but any term that has more than one derivatives of u^μ and r_H has been neglected. All calculations presented in this subsection generically will have corrections at order $\mathcal{O}(\partial^2)$.

At first order in derivative expansion the metric dual to hydrodynamics has the following form [5]

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(r/r_H) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu + r \left[- (u_\mu a_\nu + u_\nu a_\mu) + \frac{2\Theta}{D-2} u_\mu u_\nu + 2F(r/r_H) \sigma_{\mu\nu} \right] dx^\mu dx^\nu \quad (4.14)$$

Where,

$$F(r) = r \int_r^\infty dx \frac{x^{D-2} - 1}{x(x^{D-1} - 1)}$$

And¹⁵

$$a_\mu = (u \cdot \partial) u_\mu, \quad \Theta = \partial \cdot u, \quad \sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \left(\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} \right) - \left(\frac{\Theta}{D-2} \right) P^{\mu\nu} \quad (4.15)$$

We shall often refer to this metric, described in equation (4.14), as ‘hydrodynamic metric’. Here both r_H and u_μ are functions of x^μ ; but they are not completely arbitrary. the hydrodynamic metric will solve the Einstein’s equations (up to corrections of order $\mathcal{O}(\partial^2)$) provided the derivatives of r_H and u_μ satisfies the following equations.¹⁶

$$\frac{(u \cdot \partial) r_H}{r_H} + \frac{\Theta}{D-2} = 0, \quad P^{\mu\nu} \left(\frac{\partial_\mu r_H}{r_H} \right) + a^\nu = 0 \quad (4.16)$$

We read off the components of the metric and its inverse

$$\begin{aligned} \mathcal{G}_{\mu r} &= -u_\mu, & \mathcal{G}_{rr} &= 0 \\ \mathcal{G}_{\mu\nu} &= -r^2 f(r/r_H) u_\mu u_\nu + r^2 P_{\mu\nu} \\ &+ r \left[- (u_\mu a_\nu + u_\nu a_\mu) + \left(\frac{2\Theta}{D-2} \right) u_\mu u_\nu + 2F(r/r_H) \sigma_{\mu\nu} \right] \end{aligned} \quad (4.17)$$

$$\begin{aligned} \mathcal{G}^{rr} &= r^2 f(r/r_H) - r \left(\frac{2\Theta}{D-2} \right), & \mathcal{G}^{\mu r} &= u^\mu - \frac{a^\mu}{r} \\ \mathcal{G}^{\mu\nu} &= \frac{P^{\mu\nu}}{r^2} - \frac{2F(r/r_H)}{r^3} \sigma^{\mu\nu} \end{aligned} \quad (4.18)$$

The horizon is still given by the surface (no correction at first order in derivative, though the normal gets corrected since $\partial_\mu r_H$ is not negligible now.)

$$\text{Event Horizon : } \mathcal{S} = r - r_H = 0, \quad N_A dX^A = dX^A \partial_A \mathcal{S} = dr - dx^\mu \partial_\mu r_H \quad (4.19)$$

¹⁵Here ‘.’ denotes contraction with respect to $\eta_{\mu\nu}$.

¹⁶These two equations are just the stress tensor conservation equation for a $(D-1)$ dimensional ideal conformal fluid.

We need the Christoffel symbols to compute the geodesic equation.

$$\begin{aligned}
 \Gamma_{rr}^r &= 0, & \Gamma_{rr}^\mu &= 0 \\
 \Gamma_{\alpha r}^r &= \left[r f(r/r_H) + \frac{r^2}{2r_H} f'(r/r_H) - \frac{\Theta}{D-2} \right] u_\alpha \\
 \Gamma_{r\delta}^\mu &= \frac{1}{2r^2} \left[2r P_\delta^\mu - \partial_\delta u^\mu - u_\delta a^\mu + \partial^\mu u_\delta + u^\mu a_\delta - 2F(r/r_H) \sigma_\delta^\mu + 2(r/r_H) F'(r/r_H) \sigma_\delta^\mu \right]
 \end{aligned} \tag{4.20}$$

At first order in derivative expansion, the most general correction that could be added to $\bar{O}^A$, maintaining it as a null vector with respect to the first order corrected metric:

$$\bar{O}^A \partial_A = \partial_r + w_1(r) \Theta \partial_r + w_2(r) a^\mu \partial_\mu \tag{4.21}$$

We shall fix $w_1(r)$ and $w_2(r)$ using the geodesic equation.

The r component of the geodesic equation gives the following.

$$\begin{aligned}
 (\bar{O} \cdot \bar{\nabla}) \bar{O}^r &= 0 \\
 \Rightarrow \bar{O}^r \bar{\nabla}_r \bar{O}^r + \bar{O}^\mu \bar{\nabla}_\mu \bar{O}^r &= 0 \\
 \Rightarrow \bar{O}^r \partial_r \bar{O}^r + \Gamma_{rr}^r \bar{O}^r \bar{O}^r + 2\bar{O}^r \bar{O}^\alpha \Gamma_{\alpha r}^r &= 0 \\
 \Rightarrow (1 + w_1(r)\Theta) w_1'(r)\Theta + 2(1 + w_1(r)\Theta)(w_2(r)a^\alpha) \Gamma_{\alpha r}^r &= 0 \\
 \Rightarrow w_1'(r) &= 0 \\
 \Rightarrow w_1(r) &= A_1, \quad \text{where } A_1 \text{ is a constant}
 \end{aligned}$$

From the μ component of the geodesic equation we find

$$\begin{aligned}
 (\bar{O} \cdot \bar{\nabla}) \bar{O}^\mu &= 0 \\
 \Rightarrow \bar{O}^r \bar{\nabla}_r \bar{O}^\mu + \bar{O}^\lambda \bar{\nabla}_\lambda \bar{O}^\mu &= 0 \\
 \Rightarrow \bar{O}^r \partial_r \bar{O}^\mu + \bar{O}^r \bar{O}^\alpha \Gamma_{\alpha r}^\mu + 2\bar{O}^r \bar{O}^\delta \Gamma_{r\delta}^\mu &= 0 \\
 \Rightarrow \left[w_2'(r) + \frac{2w_2(r)}{r} \right] a^\mu &= 0 \\
 \Rightarrow w_2(r) &= \left(\frac{A_2}{r^2} \right), \quad \text{where } A_2 \text{ is another integration constant}
 \end{aligned}$$

At this stage

$$\bar{O}^A \partial_A = \partial_r + A_1 \Theta \partial_r + \left(\frac{A_2}{r^2} \right) a^\mu \partial_\mu \tag{4.22}$$

We could partially fix the integration constants using the boundary conditions.

At horizon

$$\begin{aligned}
 \bar{O}^A N_A|_{r=r_H} = 1 &\Rightarrow (1 + A_1 \Theta) = 1 \Rightarrow A_1 = 0 \\
 \bar{O}^\mu \partial_\mu r_H = \mathcal{O}(\partial^2) &\Rightarrow \text{No constraint on } A_2
 \end{aligned} \tag{4.23}$$

Hence it follows that

$$\begin{aligned}
 \bar{O}^A \partial_A &= \partial_r + \left(\frac{A_2}{r^2} \right) a^\mu \partial_\mu + \text{terms 2nd order in derivative expansion} \\
 \Rightarrow \bar{O}_A dX^A &= -u_\mu dx^\mu + A_2 a_\mu dx^\mu + \text{terms 2nd order in derivative expansion}
 \end{aligned} \tag{4.24}$$

Next, we have to solve for the ‘mapping functions’. Let us choose the same coordinates $\{Y^A\}$, as in the previous subsection so that the background takes the form of equation (4.7). We expect that the mapping functions (4.13) will get corrected by first order terms in derivative expansion.

$$y^\mu = x^\mu + \frac{u^\mu(x)}{r} + f_1(r)\Theta u^\mu(x) + f_2(r) a^\mu(x), \quad \rho = r + f_3(r) \Theta \quad (4.25)$$

As before, we shall apply the map (4.25) as a coordinate transformation on the background. In the new coordinates (where the map is just an ‘identity’) the background metric takes the following form

$$\begin{aligned} \bar{\mathcal{G}}_{rr} &= 2 \left(f_1'(r) + \frac{f_3'(r)}{r^2} - \frac{2f_3(r)}{r^3} \right) \Theta \\ \bar{\mathcal{G}}_{\mu r} &= - \left[1 - \left(r^2 f_1'(r) - \frac{2f_3(r)}{r} \right) \Theta \right] u_\mu + r^2 f_2'(r) a_\mu \\ \bar{\mathcal{G}}_{\mu\nu} &= r^2 \left(1 + \frac{2f_3(r)}{r} \Theta \right) \eta_{\mu\nu} + r (\partial_\nu u_\mu + \partial_\mu u_\nu) \end{aligned} \quad (4.26)$$

Substituting equation (4.26) in equation (3.7) we find

$$\begin{aligned} \bar{\mathcal{G}}_{\mu r} + \left(\frac{A_2}{r^2} \right) a^\nu \bar{\mathcal{G}}_{\nu\mu} &= -u_\mu + A_2 a_\mu + \mathcal{O}(\partial^2), \quad \bar{\mathcal{G}}_{rr} = 0 \\ \Rightarrow \quad r^2 f_1'(r) - \frac{2f_3(r)}{r} &= 0, \quad f_2'(r) = 0, \quad f_1'(r) + \frac{f_3'(r)}{r^2} - \frac{f_3(r)}{r^3} = 0 \end{aligned} \quad (4.27)$$

The general solution for equation (4.27):

$$\begin{aligned} f_3(r) &= C_3, \quad f_2(r) = C_2, \quad f_1(r) = C_1 - \frac{C_3}{r^2} \\ \text{where } C_1, C_2 \text{ and } C_3 &\text{ are arbitray constants} \end{aligned} \quad (4.28)$$

In the new $X^A = \{r, x^\mu\}$ coordinates the metric of the background takes the following form

$$\begin{aligned} ds_{\text{background}}^2 &= \bar{\mathcal{G}}_{AB} dX^A dX^B \\ &= -2u_\mu dx^\mu dr + r^2 \eta_{\mu\nu} dx^\mu dx^\nu \\ &\quad + r [2C_3 \Theta \eta_{\mu\nu} + (\partial_\mu u_\nu + \partial_\nu u_\mu)] dx^\mu dx^\nu \\ &= -2u_\mu dx^\mu dr + r^2 \eta_{\mu\nu} dx^\mu dx^\nu \\ &\quad + 2r \left[-C_3 \Theta u_\mu u_\nu + \left(C_3 + \frac{1}{D-2} \right) \Theta P_{\mu\nu} - \left(\frac{a_\mu u_\nu + a_\nu u_\mu}{2} \right) + \sigma_{\mu\nu} \right] dx^\mu dx^\nu \end{aligned} \quad (4.29)$$

In the last step we have rewritten $\mathcal{G}_{\mu\nu}$ using the following identity

$$\partial_\mu u_\nu + \partial_\nu u_\mu = 2\sigma_{\mu\nu} + \left(\frac{2\Theta}{D-2} \right) P_{\mu\nu} - (a_\mu u_\nu + a_\nu u_\mu) \quad (4.30)$$

Once we know the background, we could determine $\bar{\mathcal{G}}_{AB}^{\text{rest}}$.

$$\begin{aligned}\mathcal{G}_{rr}^{(\text{rest})} &= 0, \quad \mathcal{G}_{\mu r}^{(\text{rest})} = 0 \\ \mathcal{G}_{\mu\nu}^{(\text{rest})} &= r^2 \left(\frac{r_H}{r} \right)^{D-1} u_\mu u_\nu - 2r \tilde{C}_3 \Theta \eta_{\mu\nu} + 2r [F(r/r_H) - 1] \sigma_{\mu\nu} \\ \text{where } \tilde{C}_3 &\equiv C_3 + \frac{1}{D-2}\end{aligned}\tag{4.31}$$

5 Hydrodynamic metric in $(\frac{1}{D})$ expansion

In this section, we would like to expand the ‘hydrodynamic metric’ (already split into ‘background’ and ‘rest’ in the previous section) in an expansion in $(\frac{1}{D})$ and compare it against the metric described in [7].

This comparison involves two steps. The first one is, of course, an exact match of the two metrics up to the required order. The second step involves the mapping of the evolution of the data. Let us explain it in a little more detail.

As we have mentioned before, both ‘hydrodynamic metric’ and ‘large- D ’ metric is determined in terms of data, defined on a codimension one hypersurfaces — in the first case it is the velocity and temperature of a relativistic fluid living on the boundary of asymptotic AdS and in the second case it is the horizon viewed as a membrane embedded in the background with fluctuating shape and velocity. However, we cannot choose the data arbitrarily. The hydrodynamic metric or the large D metric will solve Einstein’s equations only if the corresponding data satisfy certain evolution equation. For matching of these two metrics, the evolution of the data also should match. More precisely, we should be able to re-express the membrane velocity and shape in terms of fluid velocity and temperature and further, we have to show that once hydrodynamic equations are satisfied, the membrane equation is also true up to the required order.

Below we shall first compare the two metrics and in the next subsection, we shall prove the equivalence of the evolution of these two sets of defining data.

5.1 Comparison between the two metrics

If the hydrodynamic metric has to match with the final metric described in [7], the first requirement is that $\bar{\mathcal{G}}_{\mu\nu}^{\text{rest}}$ must vanish as one goes finitely away from the horizon. This is possible provided $\tilde{C}_3$ is zero and also the function $[F(r/r_H) - 1]$ has a certain type of fall-off behavior at large r . Now $\tilde{C}_3$ being an integration constant we could easily set it to zero. In appendix A we have analyzed the integral (4.2) and therefore the function $[F(r/r_H) - 1]$. It turns out that at large D this integral could be approximated as follows.

$$F(z) = F\left(1 + \frac{Z}{D}\right) = 1 - \left(\frac{1}{D}\right)^2 \sum_{m=1} \left(\frac{1+mZ}{m^2}\right) e^{-mZ} + \mathcal{O}\left(\frac{1}{D}\right)^3\tag{5.1}$$

Hence $[F(r/r_H) - 1]$ vanishes¹⁷ up to corrections of order $\mathcal{O}(\frac{1}{D})^2$.

¹⁷Also, note that the vanishing has appropriate fall-off behavior (exponential decay in the scaled Z variable) as required by large D corrections.

After substituting equation (5.1) and the value for the integration constant $\tilde{C}_3$, the black-brane metric dual to hydrodynamics takes the following form

$$dS^2 = dS_{\text{background}}^2 + r^2 \left(\frac{r_H}{r} \right)^{D-1} (u_\mu dx^\mu)^2 + \mathcal{O} \left(\frac{1}{D} \right)^2 \quad (5.2)$$

where $dS_{\text{background}}^2$ is given by equation (4.29)

As we have mentioned before, the metric in [7] is described in terms of one auxiliary function ψ and one auxiliary null one-form $O_A dX^A$. For convenience we are quoting the metric here again.

$$dS^2 = dS_{\text{background}}^2 + \psi^{-D} (O_A dX^A)^2 + \mathcal{O} \left(\frac{1}{D} \right)^2 \quad (5.3)$$

Here ψ^{-D} is harmonic with respect to the background with $\psi = 1$ being the event horizon of the full space-time and O_A is simply proportional to $\bar{O}_A$ determined in the previous subsection. The proportionality factor (let us denote it by the scalar function $\Phi(X)$) is fixed using the condition that the component of O_A along the unit normal of $\psi = \text{constant}$ hypersurfaces is one everywhere. In terms of equations, the above conditions could be expressed as

$$\bar{O}^A = \Phi(X) O^A, \quad \Phi(X) = \frac{\bar{O}^A \partial_A \psi}{\sqrt{(\partial_A \psi)(\partial^A \psi)}} \quad \text{where} \quad \partial^A \psi \equiv \bar{g}^{AB} \partial_B \psi \quad (5.4)$$

Rewriting (5.3) in terms of $\bar{O}_A$,

$$\begin{aligned} dS^2 &= dS_{\text{background}}^2 + \left(\frac{\psi^{-D}}{\Phi^2} \right) (\bar{O}_A dX^A)^2 + \mathcal{O} \left(\frac{1}{D} \right)^2 \\ &= dS_{\text{background}}^2 + \left(\frac{\psi^{-D}}{\Phi^2} \right) (u_\mu - A_2 a_\mu) (u_\nu - A_2 a_\nu) dx^\mu dx^\nu + \mathcal{O} \left(\frac{1}{D} \right)^2 \end{aligned} \quad (5.5)$$

The metric in (5.5) will match exactly with the metric in (5.2) provided we set A_2 to zero and identify $\left[\Phi^2 r^2 \left(\frac{r_H}{r} \right)^{D-1} \right]$ with the harmonic function ψ^{-D} up to corrections of order $\left(\frac{1}{D} \right)^2$. Hence in terms of equation, what we finally have to verify is the following

$$\psi^{-D} - \Phi^2 r^2 \left(\frac{r_H}{r} \right)^{D-1} = \mathcal{O} \left(\frac{1}{D} \right)^2 \quad (5.6)$$

where ψ satisfies

$$\nabla^2 \psi^{-D} = 0 \quad (5.7)$$

with the boundary condition that $\psi = 1$ should reduce to the horizon, i.e., the hypersurface given by $r = r_H$, in an expansion in $\left(\frac{1}{D} \right)$.

Now we shall first determine ψ and then Φ . Note that both ψ and the norm of $\partial_A \psi$ are scalar functions and it is much easier to compute them in a coordinate system where the background metric has a simple form. Therefore we shall solve the equation in the $\{\rho, y^\mu\}$ coordinate system and then transform the answer to the $\{r, x^\mu\}$ coordinates for final

matching. First, we need to know the position of the horizon in $\{Y^A\}$ coordinates since that will provide the required boundary condition for ψ . We know that in $\{X^A\} = \{r, x^\mu\}$ coordinates the horizon is at $r = r_H(x) + \mathcal{O}(\partial^2)$. Now $\{X^A\}$ and $\{Y^A\}$ coordinates are related as follows.

$$\begin{aligned}\rho &= r - \frac{\Theta(x)}{D-2} + \mathcal{O}(\partial^2), \\ y^\mu &= x^\mu + \frac{u^\mu(x)}{r} + \left(\frac{\Theta(x)}{D-2}\right) \left(\frac{u^\mu(x)}{r^2}\right) + C_1 \Theta(x) u^\mu(x) + C_2 a^\mu(x) + \mathcal{O}(\partial^2)\end{aligned}\quad (5.8)$$

The inverse transformation:

$$\begin{aligned}r &= \rho + \frac{\Theta(y)}{D-2} + \mathcal{O}(\partial^2) \\ x^\mu &= y^\mu - \frac{u^\mu(y)}{\rho} - C_1 \Theta(y) u^\mu(y) - C_2 a^\mu(y) + \mathcal{O}(\partial^2) \\ &= y^\mu - \frac{u^\mu(y)}{\rho} + \frac{a^\mu(y)}{\rho^2} - C_1 \Theta(y) u^\mu(y) - C_2 a^\mu(y) + \mathcal{O}(\partial^2)\end{aligned}\quad (5.9)$$

Therefore in terms of $\{Y^A\}$ coordinates the horizon is at

$$\begin{aligned}\rho &= r_H(x^\mu) - \left(\frac{\Theta}{D-2}\right) + \mathcal{O}(\partial^2) \\ &= r_H(y^\mu) - \frac{(u \cdot \partial) r_H}{r_H} - \left(\frac{\Theta}{D-2}\right) + \mathcal{O}(\partial^2) = r_H(y^\mu) + \mathcal{O}(\partial^2)\end{aligned}\quad (5.10)$$

Here, for any term that is of first order in derivative to begin with, this coordinate transformation will generate change of order $\mathcal{O}(\partial^2)$ and therefore negligible in our computation. In the last line, we have used equation (4.16).

Once we know the position of the horizon, we could solve for ψ . In $\{\rho, y^\mu\}$ coordinates the expressions for ψ and its norm are as follows (see appendix (B for derivation).

$$\begin{aligned}\psi(\rho, y^\mu) &= 1 + \left(1 - \frac{1}{D}\right) \left(\frac{\rho}{r_H(y)} - 1\right) + \mathcal{O}\left(\frac{1}{D}\right)^3 \\ \Rightarrow dY^A \partial_A \psi &= \left(1 - \frac{1}{D}\right) \left(\frac{d\rho}{r_H(y)}\right) - \rho \left(1 - \frac{1}{D}\right) \left(\frac{\partial_\mu r_H(y)}{r_H^2(y)}\right) dy^\mu \\ \Rightarrow \partial^A \psi \partial_A \psi &= \left(\frac{\rho}{r_H(y)}\right)^2 \left(1 - \frac{1}{D}\right)^2 + \mathcal{O}(\partial)^2\end{aligned}\quad (5.11)$$

Clearly this solution satisfies the boundary condition that $\psi = 1 \Rightarrow \rho = r_H(y) + \mathcal{O}(\partial^2)$.

Now we have to transform these quantities in $\{X^A\}$ coordinates. We shall first transform the quantity $\left[\frac{\rho}{r_H(y)}\right]$.

$$\begin{aligned}\frac{\rho}{r_H(y)} &= \frac{r - \frac{\Theta}{D-2}}{r_H(x) + \frac{(u \cdot \partial) r_H}{r}} + \mathcal{O}(\partial^2) \\ &= \left(\frac{1}{r_H(x)}\right) \left(r - \frac{\Theta}{D-2}\right) \left(1 - \frac{(u \cdot \partial) r_H}{r r_H}\right) + \mathcal{O}(\partial^2) \\ &= \left(\frac{1}{r_H(x)}\right) \left(r - \frac{\Theta}{D-2} - \frac{(u \cdot \partial) r_H}{r_H}\right) + \mathcal{O}(\partial^2) = \frac{r}{r_H(x)} + \mathcal{O}(\partial^2)\end{aligned}\quad (5.12)$$

From equation (5.12) it follows that

$$\begin{aligned}
\psi(r, x^\mu) &= 1 + \left(1 - \frac{1}{D}\right) \left(\frac{r}{r_H(x)} - 1\right) + \mathcal{O}\left(\frac{1}{D^3}, \partial^2\right) \\
\Rightarrow dX^A \partial_A \psi &= \left(1 - \frac{1}{D}\right) \left(\frac{dr}{r_H}\right) - r \left(1 - \frac{1}{D}\right) \left(\frac{\partial_\mu r_H}{r_H^2}\right) dx^\mu + \mathcal{O}\left(\frac{1}{D^2}, \partial^2\right) \\
\Rightarrow \partial^A \psi \partial_A \psi &= \left(\frac{r}{r_H}\right)^2 \left(1 - \frac{1}{D}\right)^2 + \mathcal{O}\left(\frac{1}{D^2}, \partial^2\right)
\end{aligned} \tag{5.13}$$

Substituting this solution in equation (5.4) we find $\Phi(X) = \frac{1}{r}$.

Now we have all the ingredients to verify equation (5.6). Let us introduce a new $\mathcal{O}(1)$ variable R such that

$$\frac{r}{r_H} = 1 + \frac{R}{D}$$

In terms of R we find

$$\begin{aligned}
\psi^{-D} - \Phi^2 r^2 \left(\frac{r_H}{r}\right)^{D-1} &= \psi^{-D} - \left(\frac{r}{r_H}\right)^{-(D-1)} \\
&= \left[1 + \left(1 - \frac{1}{D}\right) \left(\frac{R}{D}\right)\right]^{-D} - \left(1 + \frac{R}{D}\right)^{-(D-1)} \\
&= -\frac{1}{2} \left(\frac{R}{D}\right)^2 e^{-R} + \mathcal{O}\left(\frac{1}{D}\right)^3
\end{aligned} \tag{5.14}$$

This is exactly what is required to have a match between the ‘hydrodynamic metric’ and the ‘large- D ’ metric up to the expected order.

5.2 Comparison between the evolution of two sets of data

As mentioned before, the ‘hydrodynamic metric’ is defined in terms of the velocity and the temperature¹⁸ of the relativistic conformal fluid moving in a flat Minkowski space-time of dimension $(D - 1)$. In case of large- D expansion, the metric is given in terms of a $(D - 1)$ dimensional time-like fluctuating membrane embedded in pure AdS space-time with a dynamical velocity field on it. Both of these two sets of data are controlled by separate equations. For ‘derivative expansion’, the governing equation of data is given in (4.16). In ‘large- D ’ technique, the relevant equation is the following [7]

$$\hat{\nabla} \cdot U = 0, \quad \left[\frac{\hat{\nabla}^2 U_\alpha}{\mathcal{K}} - \frac{\hat{\nabla}_\alpha \mathcal{K}}{\mathcal{K}} + U^\beta \mathcal{K}_{\beta\alpha} - U \cdot \hat{\nabla} U_\alpha \right] \mathcal{P}_\gamma^\alpha = 0 \tag{5.15}$$

Here the equation is written as an intrinsic equation on the membrane world-volume. All raising, lowering and contraction of the indices are done with respect to the induced metric

¹⁸The temperature and the horizon radius are related by the following relation

$$r_H = \frac{4\pi T}{(D-1)}$$

In our choice of units

$$r_H \sim \mathcal{O}(1) \Rightarrow T \sim \mathcal{O}(D).$$

on the dynamical membrane. U_α is the velocity of the membrane, expressed in terms of its intrinsic coordinates. $\mathcal{K}_{\beta\alpha}$ is the extrinsic curvature of the membrane, expressed as a symmetric tensor on the membrane world-volume. $\mathcal{K}$ denotes its trace. $\mathcal{P}_\gamma^\alpha$ is the projector perpendicular to U^α .

In this subsection, our goal is to show that equation (4.16) implies equation (5.15) up to corrections of order $\mathcal{O}\left(\frac{1}{D}\right)^2$.

Our first job would be to express the U^α and $\mathcal{K}_{\alpha\beta}$ in terms of velocity u^μ and temperature (or r_H) of the relativistic fluid. Remember that though both u^μ and U^α are unit normalized velocity vector, they are defined on completely different spaces, one being a flat Minkowski metric and the other is the curved (both intrinsic and extrinsic curvature, being nonzero) membrane world volume.

For convenience, we shall work in $\{Y^A\} = \{\rho, y^\mu\}$ coordinates where the background metric is simple. We shall first compute the unit normal to the membrane and different components of its extrinsic curvature, to begin with in terms of background coordinates and then we shall re-express it as an intrinsic symmetric tensor on the membrane.

The unit normal to the membrane is given by

$$\begin{aligned} n_A dY^A|_{\text{membrane}} &\equiv dY^A \left[\frac{\partial_A \psi}{\sqrt{\partial^A \psi \partial_A \psi}} \right]_{\text{membrane}} \\ &= \frac{d\rho - dy^\mu \partial_\mu r_H(y)}{r_H(y)} \end{aligned} \quad (5.16)$$

The extrinsic curvature is defined as follows.

$$\begin{aligned} K_{AB} &= \Pi_A^C \nabla_C n_B = \Pi_A^C (\partial_C n_B - \Gamma_{CB}^D n_D) \\ \text{where } \Pi_A^B &= \delta_A^B - n_A n^B \text{ and } \nabla \text{ is the covariant derivative w.r.t. background} \end{aligned} \quad (5.17)$$

Now let us choose $\{y^\mu\}$ as the intrinsic coordinate on the membrane world volume. In this choice of coordinates, the extrinsic curvature $\mathcal{K}_{\alpha\beta}$ will have the following structure.

$$\mathcal{K}_{\alpha\beta} = K_{\rho\rho} (\partial_\alpha r_H) (\partial_\beta r_H) + [K_{\rho\alpha} (\partial_\beta r_H) + K_{\rho\beta} (\partial_\alpha r_H)] + K_{\alpha\beta} \quad (5.18)$$

Note that the first term in the r.h.s. of equation (5.18) does not contribute at first order derivative expansion.

After using equation (5.17) and (5.18), at this order the final expression for $\mathcal{K}_{\mu\nu}$ turns out to be very simple (see appendix C for the details of the computation).

$$\mathcal{K}_{\alpha\beta} = r_H^2 \eta_{\alpha\beta} + \mathcal{O}(\partial^2), \quad \mathcal{K} = (D-1) \quad (5.19)$$

The induced metric on the membrane is given by

$$g_{\alpha\beta} = r_H^2 \eta_{\alpha\beta} + \mathcal{O}(\partial^2) \quad (5.20)$$

Now we shall determine the velocity U^α . The velocity is defined as the projection of O^A on the membrane which, by construction, would be unit normalized with respect to the

induced metric of the membrane. In $\{Y^A\}$ coordinates, $O_A dY^A$ takes the following form

$$\begin{aligned}
 O_A dX^A|_{\text{membrane}} &= -[r u_\mu(x) dx^\mu]_{\text{membrane}} \\
 &= -\left(r_H(y) + \frac{\Theta}{D-2}\right) \left[u_\mu(y) - \frac{a_\mu(y)}{r_H}\right] \left[\left(\frac{\partial x^\mu}{\partial \rho}\right) d\rho + \left(\frac{\partial x^\mu}{\partial y^\nu}\right) dy^\nu\right]_{\rho=r_H(y)} \\
 &= -\left(r_H(y) + \frac{\Theta}{D-2}\right) \left[u_\mu(y) - \frac{a_\mu(y)}{r_H}\right] \left[\left(\frac{u^\mu(y)}{r_H^2(y)} - \frac{2a^\mu(y)}{r_H^3(y)}\right) d\rho + \left(\delta_\nu^\mu - \frac{\partial_\nu u^\mu}{r_H}\right) dy^\nu\right] \\
 &= \left(\frac{1}{r_H(y)} + \frac{\Theta}{(D-2)r_H^2}\right) d\rho + \left[-r_H(y) u_\mu(y) - \left(\frac{\Theta}{D-2}\right) u_\mu + a_\mu(y)\right] dy^\mu \\
 &= \left(\frac{1}{r_H(y)} + \frac{\Theta}{(D-2)r_H^2}\right) d\rho + \left[-r_H(y) u_\mu(y) - \left(\frac{\partial_\mu r_H}{r_H}\right)\right] dy^\mu
 \end{aligned} \tag{5.21}$$

In the last line, we have used equation (4.16), which is the governing equation for the data in the hydrodynamic side of the duality.

From equations (5.21) and (5.16) it follows that

$$U_A dY^A \equiv -dY^A [O_A - n_A]_{\text{membrane}} = -\left(\frac{1}{r_H^2}\right) \left(\frac{\Theta}{D-2}\right) d\rho + r_H u_\mu dy^\mu \tag{5.22}$$

Now U_α is just rewriting of U_A in terms of the intrinsic coordinates of the membrane. Following the same method as in equation (5.18) we find

$$U_\alpha dy^\alpha \equiv [r_H u_\alpha + \mathcal{O}(\partial^2)] dy^\alpha \tag{5.23}$$

Once we know $\mathcal{K}_{\alpha\beta}$, U^α and the induced metric on the membrane, we could compute each term in the equation (5.15).

$$\begin{aligned}
 \hat{\nabla} \cdot U &= \left(\frac{D-2}{r_H}\right) \left[\frac{\Theta}{D-2} + \frac{(u \cdot \partial) r_H}{r_H}\right] + \mathcal{O}(\partial^2) = \mathcal{O}(\partial^2) \\
 \hat{\nabla}^2 U_\alpha &= \mathcal{O}(\partial^2) \\
 (U \cdot \hat{\nabla}) U_\beta &= a_\beta + \frac{P_\beta^\alpha \partial_\alpha r_H}{r_H} + \mathcal{O}(\partial^2) = \mathcal{O}(\partial^2) \\
 U^\alpha \mathcal{K}_{\alpha\beta} \mathcal{P}_\gamma^\beta &= \mathcal{O}(\partial^2) \\
 \hat{\nabla}_\alpha \mathcal{K} &= \mathcal{O}(\partial^2)
 \end{aligned} \tag{5.24}$$

As it is clear from the notation, in the l.h.s. of each equation the relevant metric is the induced metric on the membrane whereas in r.h.s. it is the flat Minkowski metric $\eta_{\alpha\beta}$.

Substituting equations (5.24) in equation (5.15) we could easily show that membrane equation follows as a consequence of fluid equation.

In this context let us mention the work in [16]. Here the authors have computed the boundary stress tensor dual to a slowly varying membrane embedded in AdS. They have found the dual fluid velocity in terms of the membrane velocity. It could be easily checked that equation (5.23) is indeed the inverse of what they have found up to correction of order $\mathcal{O}(\partial^2)$.

6 Conclusion

In this note, we have compared dynamical black-brane solutions to Einstein's equations (in presence of negative cosmological constant) generated by two different perturbative schemes, namely 'derivative expansion' and Large-dimension expansion. In both the cases, the space-time necessarily has an event horizon. We have shown that in a large number of dimensions whenever 'derivative expansion' is applicable, we can expand the metric further in $(\frac{1}{D})$, (though the reverse may not be true always). We have found a perfect match in this overlap regime of these two perturbative techniques up to first subleading order on both sides.

One immediate interesting project would be to extend this calculation to the next order on both sides since we already know both the 'hydrodynamic metric' and the 'large D metric' up to the second subleading order [2, 18]. It would also be interesting to generalize this calculation to Einstein-Maxwell system in presence of negative cosmological constant, where also we know the metric on both sides up to the first subleading order [21–24].

In some sense, our analysis serves as a consistency test for these two methods. But this comparison could teach us something more. This is about the dual systems of these two gravity solutions.

The dynamical black-brane metric generated by 'derivative expansion' in D dimension is dual to the relativistic conformal hydrodynamics living in $(D - 1)$ dimensional flat space-time. The variables of hydrodynamics are fluid velocity and temperature, which are the data that label different black-brane solutions in derivative expansion.

On the other hand, the metric generated in 'large D expansion' is dual to a co-dimension one dynamical membrane embedded in pure AdS and coupled with a velocity field. Here also the labeling data of the metric live on a $(D - 1)$ dimensional hypersurface and they consist of a scalar function — the shape of the membrane and a unit normalized velocity field. This is very similar to hydrodynamics in terms of counting, though the governing equations and the physical significance of the variables are entirely different.

However, we have already seen that these two systems of equations are approximately equivalent after an appropriate field redefinition. In this note, we have verified it at the very leading order and we expect that the project of comparing the two metrics up to second subleading order would extend this equivalence to the next order on both sides.

In fact, it is expected that this equivalence is valid to all orders [16]. In other words, in the overlap regime, these two equations must be exactly equivalent to each other if we consider all orders on both sides [16], though to see this equivalence we need to re-express the variables of one side in terms of the other [16, 19, 25].

This equivalence actually involves some interesting resummation of one series into the other. Even the leading term in derivative expansion can encode many terms of $(\frac{1}{D})$ expansion and on the other hand, the leading membrane equation might have information about many higher order transport coefficients. At linearized level, this has been nicely captured in the analysis in [26]. The frequencies of Quasi-normal modes do exhibit such resummation. In [16], the authors have proposed a resummed stress tensor that could exactly reproduce the fluid stress tensor exactly up to the first order in derivative expansion.

It would be very interesting to understand this structure in full detail, at a non-linear level. This might lead to a fluid-membrane duality in large number of dimensions where gravity does not have any role to play.

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A Analysis of $F(r/r_H)$

In this section, we shall evaluate the integral (4.2) in large D limit. For convenience we are quoting the equation here.

$$F(y) = y \int_y^\infty dx \frac{x^{D-2} - 1}{x(x^{D-1} - 1)} \quad (\text{A.1})$$

We would like to evaluate this integral systematically for large D . Let us first evaluate the integral for $y \geq 2$. In this case, since D is very large, $x^D \gg 1$ throughout the range of integration. So we shall expand the integrand in the following way.

$$\begin{aligned} \frac{x^{D-2} - 1}{x(x^{D-1} - 1)} &= \left(\frac{1}{x^2}\right) \left(1 - x^{-(D-2)}\right) \left(1 - x^{-(D-1)}\right)^{-1} \\ &= \left(\frac{1}{x^2}\right) \left(1 - x^{-(D-2)}\right) \left(1 + \sum_{m=1}^\infty x^{-m(D-1)}\right) \\ &= \left(\frac{1}{x^2}\right) \left(1 + \sum_{m=1}^\infty \left[x^{-m(D-1)} - x^{-m(D-1)+1}\right]\right) \end{aligned} \quad (\text{A.2})$$

Integrating (A.2) we find

$$y \int_{y \geq 2}^\infty dx \frac{x^{D-2} - 1}{x(x^{D-1} - 1)} = 1 + \sum_{m=1}^\infty \left[\left(\frac{1}{(D-1)m+1}\right) y^{-(D-1)m} - \left(\frac{1}{(D-1)m}\right) y^{-(D-1)m+1} \right] \quad (\text{A.3})$$

Clearly, the sums in the r.h.s. of (A.3) are convergent for $y \geq 2$. Let us denote the r.h.s. as $k(y)$.

However, the expansion in (A.2) is not valid inside the ‘membrane region’, i.e., when $y - 1 \sim \mathcal{O}\left(\frac{1}{D}\right)$ and naively $k(y)$ is not the answer for the integral.

But consider the function $\tilde{k}(y) = F(y) - k(y)$. This function vanishes for all $y \geq 2$ and also by construction, it is a smooth function at $y = 2$ (none of the derivatives diverge).

Hence $\tilde{k}(y)$ must vanish for every y . So we conclude, for every allowed y (i.e., $y \geq 1$)

$$F(y) = 1 + \sum_{m=1} \left[\left(\frac{1}{(D-1)m+1} \right) y^{-(D-1)m} - \left(\frac{1}{(D-1)m} \right) y^{-(D-1)m+1} \right] \quad (\text{A.4})$$

Note that $F(y)$ reduces to 1 as $y \rightarrow \infty$ as required in section 4.2.

Now we would like to expand $F(y)$ in a series in $(\frac{1}{D})$, where y is in the membrane regime.

$$y = 1 + \frac{Y}{D}, \quad Y \sim \mathcal{O}(1)$$

In this regime $F(y)$ takes the following form

$$F(y) = F\left(1 + \frac{Y}{D}\right) = 1 - \left(\frac{1}{D}\right)^2 \sum_{m=1} \left(\frac{1+mY}{m^2} \right) e^{-mY} + \mathcal{O}\left(\frac{1}{D^3}\right) \quad (\text{A.5})$$

In this note, we consider only the first subleading correction in $(\frac{1}{D})$ expansion. Therefore $F(y)$ could be set to 1 for our purpose.

B Derivation of ψ in $\{Y^A\} = \{\rho, y^\mu\}$ coordinates

In this section, we shall give the derivation of ψ as mentioned in eq. (5.7). We want to solve ψ such that $\nabla^2 \psi^{-D} = 0$. Where ∇ is the covariant derivative with respect to the background metric

$$ds_{\text{background}}^2 = \frac{d\rho^2}{\rho^2} + \rho^2 \eta_{\mu\nu} dy^\mu dy^\nu \quad (\text{B.1})$$

we can expand ψ as follows

$$\psi = 1 + \left(A_{10} + \epsilon B_{10} + \frac{A_{11} + \epsilon B_{11}}{D} \right) (\rho - r_H) + (A_{20} + \epsilon B_{20}) (\rho - r_H)^2 + \mathcal{O}\left(\frac{1}{D^3}\right) \quad (\text{B.2})$$

Here ϵ denotes that B_{ij} 's are $\mathcal{O}(\partial)$ terms.

$$\begin{aligned} \nabla^2 (\psi^{-D}) &= 0 \\ \Rightarrow \psi (\nabla^2 \psi) - (D+1)(\nabla^A \psi)(\nabla_A \psi) &= 0 \\ \Rightarrow \psi \rho^2 \left[\partial_\rho \partial_\rho \psi - \Gamma_{\rho\rho}^\rho (\partial_\rho \psi) - \Gamma_{\rho\rho}^\mu (\partial_\mu \psi) \right] + \frac{\psi}{\rho^2} \eta^{\mu\nu} \left[-\Gamma_{\mu\nu}^\rho (\partial_\rho \psi) - \Gamma_{\mu\nu}^\alpha \partial_\alpha \psi \right] \\ &\quad - (D+1) \rho^2 (\partial_\rho \psi)^2 + \mathcal{O}(\partial)^2 = 0 \end{aligned} \quad (\text{B.3})$$

The required Christoffel symbols are

$$\Gamma_{\rho\rho}^\rho = -\frac{1}{\rho}; \quad \Gamma_{\rho\rho}^\mu = 0; \quad \Gamma_{\mu\nu}^\rho = -\rho^3 \eta_{\mu\nu}; \quad \Gamma_{\mu\nu}^\alpha = 0; \quad (\text{B.4})$$

Using the above Christoffel symbol we get

$$\psi \left[\rho^2 \partial_\rho^2 \psi + D \rho \partial_\rho \psi \right] - (D+1) \rho^2 (\partial_\rho \psi)^2 = 0 \quad (\text{B.5})$$

Now,

$$\begin{aligned}\partial_\rho\psi &= \left(A_{10} + \epsilon B_{10} + \frac{A_{11} + \epsilon B_{11}}{D}\right) + 2(A_{20} + \epsilon B_{20})(\rho - r_H) \\ \partial_\rho^2\psi &= 2(A_{20} + \epsilon B_{20})\end{aligned}\tag{B.6}$$

Solving, (B.5) order by order in derivative expansion we get the following solution

$$\psi(\rho, y^\mu) = 1 + \left(1 - \frac{1}{D}\right) \left(\frac{\rho}{r_H(y^\mu)} - 1\right) + \mathcal{O}\left(\frac{1}{D}\right)^3\tag{B.7}$$

C Computing different terms in membrane equation

In this section we shall give the details of calculations of different terms that appear in the membrane equation. The different components of the projector defined in (5.17) are given by

$$\Pi_\rho^\rho = 0; \quad \Pi_\mu^\rho = \partial_\mu r_H; \quad \Pi_\rho^\mu = \frac{1}{r_H^4}(\partial^\mu r_H); \quad \Pi_\nu^\mu = \delta_\nu^\mu\tag{C.1}$$

The different components of the Christoffel symbol of the background metric in $Y^A = \{\rho, y^\mu\}$ co-ordinates are given by

$$\Gamma_{\rho\rho}^\rho = -\frac{1}{\rho}; \quad \Gamma_{\mu\rho}^\rho = 0; \quad \Gamma_{\mu\nu}^\rho = -\rho^3\eta_{\mu\nu}; \quad \Gamma_{\mu\rho}^\nu = \frac{1}{\rho}\delta_\mu^\nu; \quad \Gamma_{\mu\nu}^\alpha = 0; \quad \Gamma_{\rho\rho}^\mu = 0;\tag{C.2}$$

From (5.18) it is clear that we need only $K_{\rho\alpha}$ and $K_{\alpha\beta}$ component of extrinsic curvature

$$\begin{aligned}K_{\rho\mu} &= \Pi_\rho^C \left(\partial_C n_\mu - \Gamma_{C\mu}^D n_D \right) \\ &= \Pi_\rho^\nu \left(\partial_\nu n_\mu - \Gamma_{\nu\mu}^\rho n_\rho \right) \\ &= \frac{\partial_\mu r_H}{r_H^2} \\ K_{\mu\nu} &= \Pi_\mu^C \left(\partial_C n_\nu - \Gamma_{C\nu}^D n_D \right) \\ &= \Pi_\mu^\rho \left(\partial_\rho n_\nu - \Gamma_{\rho\nu}^\rho n_\rho \right) + \Pi_\mu^\alpha \left(\partial_\alpha n_\nu - \Gamma_{\alpha\nu}^\rho n_\rho \right) \\ &= -\delta_\mu^\alpha \Gamma_{\alpha\nu}^\rho n_\rho \\ &= \rho^2 \eta_{\mu\nu}\end{aligned}\tag{C.3}$$

Now, as mentioned in (5.18) in terms of the intrinsic coordinates on the membrane the extrinsic curvature will have the structure

$$\begin{aligned}\mathcal{K}_{\alpha\beta} &= K_{\rho\rho}(\partial_\alpha r_H)(\partial_\beta r_H) + [K_{\rho\alpha}(\partial_\beta r_H) + K_{\rho\beta}(\partial_\alpha r_H)] + K_{\alpha\beta} \\ &= r_H^2 \eta_{\alpha\beta} + \mathcal{O}(\partial)^2\end{aligned}\tag{C.4}$$

The trace of the extrinsic curvature

$$\mathcal{K} = (D - 1) + \mathcal{O}(\partial^2) \quad (\text{C.5})$$

For the calculation of the extrinsic curvature we need background metric, where for the rest of the calculation we require induced metric on the horizon. The induced metric on the horizon is given by

$$g_{\alpha\beta} = r_H^2 \eta_{\alpha\beta} + \mathcal{O}(\partial^2) \quad (\text{C.6})$$

The Christoffel symbol of the induced metric

$$\Gamma_{\beta\alpha}^\delta = \left(\delta_\beta^\delta \frac{\partial_\alpha r_H}{r_H} + \delta_\alpha^\delta \frac{\partial_\beta r_H}{r_H} - \eta_{\alpha\beta} \frac{\partial^\delta r_H}{r_H} \right) \quad (\text{C.7})$$

Now we shall calculate all the terms mentioned in (5.24). First, we shall calculate

$$\begin{aligned} \hat{\nabla} \cdot U &= g^{\alpha\beta} \hat{\nabla}_\alpha U_\beta \\ &= \frac{\eta^{\alpha\beta}}{r_H^2} \left[\partial_\alpha U_\beta - \Gamma_{\alpha\beta}^\delta U_\delta \right] + \mathcal{O}(\partial)^2 \\ &= \frac{\eta^{\alpha\beta}}{r_H^2} \left[\partial_\alpha (r_H u_\beta) - (r_H u_\delta) \left(\delta_\beta^\delta \frac{\partial_\alpha r_H}{r_H} + \delta_\alpha^\delta \frac{\partial_\beta r_H}{r_H} - \eta_{\alpha\beta} \frac{\partial^\delta r_H}{r_H} \right) \right] + \mathcal{O}(\partial)^2 \\ &= (D - 2) \left(\frac{(u \cdot \partial) r_H}{r_H^2} \right) + \frac{\partial \cdot u}{r_H} + \mathcal{O}(\partial)^2 \end{aligned} \quad (\text{C.8})$$

Now we shall calculate $\hat{\nabla}^2 U_\mu$ and $(U \cdot \hat{\nabla}) U_\alpha$

$$\begin{aligned} \hat{\nabla}^2 U_\mu &= g^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta U_\mu \\ &= g^{\alpha\beta} \left[\partial_\alpha (\hat{\nabla}_\beta U_\mu) - \Gamma_{\alpha\beta}^\delta (\hat{\nabla}_\delta U_\mu) - \Gamma_{\alpha\mu}^\delta (\hat{\nabla}_\beta U_\delta) \right] \\ &= \mathcal{O}(\partial)^2 \end{aligned} \quad (\text{C.9})$$

$$\begin{aligned} (U \cdot \hat{\nabla}) U_\alpha &= U^\beta (\partial_\beta U_\alpha) - U^\beta \Gamma_{\beta\alpha}^\delta U_\delta \\ &= \frac{u^\beta}{r_H} \left(r_H (\partial_\beta u_\alpha) + u_\alpha (\partial_\beta r_H) \right) - \frac{u^\beta}{r_H} (r_H u_\delta) \left(\delta_\beta^\delta \frac{\partial_\alpha r_H}{r_H} + \delta_\alpha^\delta \frac{\partial_\beta r_H}{r_H} - \eta_{\alpha\beta} \frac{\partial^\delta r_H}{r_H} \right) + \mathcal{O}(\partial^2) \\ &= (u \cdot \partial) u_\alpha + u_\alpha \left(\frac{(u \cdot \partial) r_H}{r_H} \right) + \frac{\partial_\alpha r_H}{r_H} + \mathcal{O}(\partial^2) \end{aligned} \quad (\text{C.10})$$

Now,

$$\begin{aligned} U^\alpha \mathcal{K}_{\alpha\beta} \mathcal{P}_\gamma^\beta &= (\delta_\gamma^\beta + U^\beta U_\gamma) (U^\alpha r_H^2 \eta_{\alpha\beta}) + \mathcal{O}(\partial^2) \\ &= (\delta_\gamma^\beta + U^\beta U_\gamma) U_\beta + \mathcal{O}(\partial^2) \\ &= \mathcal{O}(\partial^2) \end{aligned} \quad (\text{C.11})$$

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