Gravitational couplings in $\mathcal{N} = 2$ string compactifications and Mathieu Moonshine

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ABSTRACT: We evaluate the low energy gravitational couplings, F_g in the heterotic $E_8 \times E_8$ string theory compactified on orbifolds of $K3 \times T^2$ by g' which acts as a $\mathbb{Z}_N$ automorphism on $K3$ together with a $1/N$ shift along T^2 . The orbifold g' corresponds to the conjugacy classes of the Mathieu group M_{24} . The holomorphic piece of F_g is given in terms of a polylogarithm with index $3 - 2g$ and predicts the Gopakumar-Vafa invariants in the corresponding dual type II Calabi-Yau compactifications. We show that low lying Gopakumar-Vafa invariants for each of these compactifications including the twisted sectors are integers. We observe that the conifold singularity for all these compactifications occurs only when states in the twisted sectors become massless and the strength of the singularity is determined by the genus zero Gopakumar-Vafa invariant at this point in the moduli space.

KEYWORDS: Superstrings and Heterotic Strings, Discrete Symmetries, Topological Strings

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1 Introduction

The study of the low energy effective action of string compactifications has proved to be important for various reasons. Higher derivative terms in the low energy effective action is relevant for evaluating sub-leading corrections to the Hawking-Bekenstein entropy of black holes in these theories. The one-loop low energy effective action encodes the quantum symmetries of the theory which are not visible from the classical effective action. Usually the one loop effective actions for string compactifications are automorphic functions in the moduli of the theory and therefore encode the underlying symmetries of the theory. It is often easier to evaluate one-loop effective actions in one duality frame where it is exact and it leads to non-trivial predictions in another frame. One simple example which has these features is the R^2 term in $\mathcal{N} = 4$ theories in $d = 4$ which can be evaluated by a one-loop

computation in the type II frame, but yields predictions for an infinite sum of space time instanton effects due to five branes in the heterotic frame [1, 2].

Let us focus on the gravitational coupling F_g of the low energy effective action of $\mathcal{N} = 2$ string theories in $d = 4$. These couplings appear as the following terms in the effective action

$$S = \int F_g(y, \bar{y}) F^{2g-2} R^2, \quad (1.1)$$

where F, R are the self dual part of the graviphoton and the Riemann curvature. The coupling F_g depends on the vector multiplets $y, \bar{y}$ of the theory. The canonical and well studied example of such a theory is the $E_8 \times E_8$ heterotic string theory compactified on $K3 \times T^2$ with the standard embedding of the spin connection in to a $SU(2)$ of one of the E_8 . Building on the earlier works [3–8], a detailed study of these couplings for this theory has been carried out by [9] for $g = 1$ and in [10] for $g > 1$ by explicitly evaluating the one loop threshold integrals. This compactification is also the standard and well studied example of $\mathcal{N} = 2$ string duality [11, 12]. The result from the heterotic side is one loop exact. Moduli dependence of the one loop threshold corrections in this model have also been studied earlier in [13–17]. On the type II A side, the theory is compactified on the Calabi-Yau manifold X with Euler number $\chi(X) = -480$. In fact the Euler number of the dual Calabi-Yau is predicted from F_g .

Though the full expression of F_g is intricate, the holomorphic¹ part of this coupling, which can be obtained by sending $y \rightarrow \infty$ is simple and predicts certain topological invariants of the dual Calabi-Yau three fold X . Lets call this holomorphic part $\bar{F}_g(y)$, the genus g topological amplitude. The dual Calabi-Yau is generally a $K3$ fibration over a base T^2 . The heterotic side corresponds to the semi-classical limit when the volume of the base is large. The topological amplitude $\bar{F}_g(y)$ predicts the number of genus g holomorphic curves in the $K3$ fibers of the dual Calabi-Yau X in the semi-classical limit. The most convenient way to extract this information is using the observation of Gopakumar and Vafa who showed that a generating function for $\bar{F}_g(y)$ for any Calabi-Yau can be written in terms of integer invariants which are called Gopakumar-Vafa invariants [18, 19]. Therefore the explicit calculation of $\bar{F}_g(y)$ from the heterotic side yields a prediction for the Gopakumar-Vafa invariants of the dual Calabi-Yau X in the semi-classical limit.

In this paper we generalize the evaluation of F_g to the heterotic $E_8 \times E_8$ string theory compactified on orbifolds of $K3 \times T^2$. The orbifold g' acts as a $\mathbb{Z}_N$ automorphism on $K3$ together with the $1/N$ shift on one of the circles of T^2 . We consider the standard embedding in which the $SU(2)$ spin connection of the $K3$ is embedded in one of the E_8 of the theory. The orbifold g' corresponds to the conjugacy classes of Mathieu group M_{24} listed in table 1 and they all preserve $\mathcal{N} = 2$ supersymmetry. These compactifications were introduced in [20, 21] with the motivation of exploring the role of M_{24} in string compactifications which was first studied in the original $K3 \times T^2$ model by [22]. We would like to emphasise that these compactifications are distinct from that studied by [23]. In this work the $K3$ was realized as a $\mathbb{Z}_N$ orbifold of T^4 , with $N = 2, 3, 4, 5$ and the $1/N$ shifts

¹It is actually the anti-holomorphic part of F_g but we will take the complex conjugate and refer to it as the holomorphic part.

where restricted to lie along the internal $E_8 \times E_8$ lattice and not on the external T^2 . Note that the results of our paper does not hinge on the realization of $K3$ as an orbifold of T^4 and the shifts we consider are along the external T^2 . One of our goals in evaluating the F_g for the orbifolds under consideration in this paper is to determine the properties of the dual Calabi-Yau geometry on the type II A side and to initiate a study of these geometries and understand if at all M_{24} plays a role in these geometries. Recently the role of sporadic symmetry groups in the elliptic genera of Calabi-Yau 5-folds has been investigated [24].

We now summarize our main result for the holomorphic gravitational coupling F_g for the orbifolds studied in this paper. Consider the twisted elliptic genus of $K3$ by an automorphism g' of order N , which is defined as

$$F^{(r,s)}(\tau, z) = \frac{1}{N} \text{Tr}_{RRg'^r} [(-1)^{F_{K3} + \tilde{F}_{K3}} g'^s e^{2\pi i z F_{K3}} q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}]. \quad (1.2)$$

Here $F_{K3}, \tilde{F}_{K3}$ is left and the right moving the fermion number. The trace is taken over the Ramond-Ramond sector twisted by g'^r . The twisted elliptic genus can be written in the form

$$\begin{aligned} F^{(0,0)}(\tau, z) &= \alpha_{g'}^{(0,0)} A(\tau, z), \\ F^{(r,s)}(\tau, z) &= \alpha_{g'}^{(r,s)} A(\tau, z) + \beta_{g'}^{(r,s)}(\tau) B(\tau, z), \\ r, s &\in \{0, 1, \dots, N-1\} \text{ with } (r, s) \neq (0, 0) \end{aligned} \quad (1.3)$$

where

$$A(\tau, z) = \frac{\theta_2^2(\tau, z)}{\theta_2^2(\tau, 0)} + \frac{\theta_3^2(\tau, z)}{\theta_3^2(\tau, 0)} + \frac{\theta_4^2(\tau, z)}{\theta_4^2(\tau, 0)}, \quad B(\tau, z) = \frac{\theta_1^2(\tau, z)}{\eta^6(\tau)} \quad (1.4)$$

are Jacobi forms which transform under $\text{SL}(2, \mathbb{Z})$ with index 1 and weight 0 and -2 respectively. $\alpha_{g'}^{(r,s)}$ are numerical constants and $\beta_{g'}^{(r,s)}$ is a weight 2 modular form under $\Gamma_0(N)$. After the discovery of the Mathieu moonshine symmetry in the elliptic genus of $K3$ [25], the twining character $F^{(0,1)}$ for all the M_{24} conjugacy classes was first found in [26–28]. For g' given in table 1, the corresponding to the full twisted elliptic genus M_{24} have been explicitly determined in [29, 30]. We list them in appendix E for completeness. Now given the twisted elliptic genus, consider the following weight $2g$ quasi-modular form under $\Gamma_0(N)$

$$f^{(r,s)}(\tau) \mathcal{P}_{2g}(G_2, G_4, G_6, \dots, G_{2g}) = \sum_{l \in \frac{\mathbb{Z}}{N}} c_{g-1}^{(r,s)}(l, 0) q^l, \quad (1.5)$$

$$\begin{aligned} \text{where} \quad f^{(r,s)}(\tau) &= \frac{1}{2\eta^{24}(\tau)} E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right], \\ G_{2k} &= 2\zeta(2k) E_{2k}. \end{aligned}$$

E_{2k} are Eisenstein series of weight $2k$ and ζ refers to the zeta function. $\mathcal{P}_{2g}$ is related to the Schur polynomial $\mathcal{S}$ of order g by

$$\mathcal{P}_{2g}(x_1, x_2, \dots, x_g) = -\mathcal{S} \left(x_1, \frac{1}{2}x_2, \dots, \frac{1}{g}x_g \right). \quad (1.6)$$

Then the topological amplitude $\bar{F}_g$ is given by

$$\bar{F}_g^{\text{hol}}(y) = \frac{(-1)^{g-1}}{\pi^2} \sum_{s=0}^{N-1} \times \left(\sum_{m>0} e^{-2\pi i n_2 s/N} c_{g-1}^{(r,s)}(m^2/2, 0) \text{Li}_{3-2g}(e^{2\pi i m \cdot y}) + \frac{1}{2} c_{g-1}^{(0,s)}(0, 0) \zeta(3-2g) \right). \quad (1.7)$$

The sum over lattice points $m > 0$ refers to the following lattice points (n_1, n_2) , $n_1 \in \frac{\mathbb{Z}}{N}$, $n_2 \in \mathbb{Z}$ with the restrictions

$$\begin{aligned} n_1, n_2 &\geq 0, & \text{but } (n_1, n_2) &\neq (0, 0), \\ (r/N, -n_2), & & \text{with } n_2 > 0 \text{ and } rn_2 &\leq N. \end{aligned} \quad (1.8)$$

$y = (T, U)$ is the Kähler and complex structure of the torus T^2 , $m^2 = 2n_1n_2$ and $m \cdot y = n_1T + n_2U$. The functions Li_{3-2g} are polylogarithm functions of order $3-2g$.

From comparing (1.5) and (1.7) we see that indeed it is the coefficients of the twisted elliptic genus of $K3$ which forms the basic input data for the topological amplitude $\bar{F}_g(y)$. We wish to emphasise that this observation is the key result of this paper. This is a generalization of the observation by [22] in which the elliptic genus of $K3$ which determines the factor E_4E_6/η^{24} , is the crucial input data for the topological amplitude for the unorbifolded model. Our result is also a generalisation of the work of [10] which evaluated the F_g for the un-orbifolded heterotic compactification on $K3 \times T^2$.

Now comparing the instanton contributions in $\bar{F}_g(y)$ with the form of the topological amplitude written in terms of the Gopakumar-Vafa invariants we can extract out these invariants. It is apriori not clear that the invariants will be integers since the coefficients $c_{g-1}^{(r,s)}$ in (1.5), themselves are not integers. However to the level we have verified the Gopakumar-Vafa invariants are all integral. This forms a simple consistency check of our result. In fact we will see that once the genus zero Gopakumar-Vafa invariants are integers, the higher genus invariants are assured to be integers. This is shown for genus $g = 1, 2, 3$. The constant term in (1.7) contains the information of the Euler character of the dual Calabi-Yau X . The Euler character of the Calabi-Yau dual to all the orbifolds considered in this paper is listed in table 4.

We then study the conifold singularities of the $\bar{F}_g^{\text{hol}}$ which correspond to points in moduli space of enhanced gauge symmetry. We observe that there are no conifold singularities from the untwisted sector and all conifold singularities are due to twisted sector states becoming massless. The strength of this singularity is proportional to the genus zero Gopakumar-Vafa invariant corresponding to this state. The list of low lying genus zero Gopakumar-Vafa invariants for g' which we refer to as CHL orbifolds² are provided in appendix D. For the rest of the g' in table 1. we provide an ancillary Mathematica code from which the genus zero Gopakumar-Vafa invariants can be evaluated and seen to be integral.

We briefly mention the method we adopt to evaluate F_g . In the heterotic frame it is given in terms of a one loop integral over the fundamental domain. We basically follow the

²These are geometric actions, which keep the Hodge number of $K3$; $h_{1,1} \geq 1$.

method of orbits adopted in many works in this subject starting from [31]. See [32] for a recent discussion of these methods. Therefore the methods used in this paper have been developed and established by several previous works. However we do not directly apply the lattice reduction theorem of Borchers [33] as done by earlier works [10, 34, 35] for evaluating F_g , $g > 1$. This is because the application of the reduction theorem when the integrand has modular forms of $\Gamma_0(N)$ is rather intricate and we find it is easier to proceed directly and carry out each of the steps involved in the integrations. We would like to emphasise that using the direct method we pursue, there are several manipulations which are straightforward, but nevertheless important to arrive the standard form of F_g .³ For this reason we find it instructive to provide the details of the way we implement the method of orbits in the appendices B, C. Since this area of research has a rich history with several contributions, it is important to mention that similar integrals involving modular forms of $\Gamma_0(N)$ have occurred earlier for the specific case of $N = 4$ in [36] and for $N = 2$ in [37] and [34]. Modular forms transforming under $\Gamma_0(N)$ for arbitrary N occurring in the theta lift of the twisted elliptic genus of $K3$ were done in [38] and [39]. As far as we are aware the present work is the first evaluation of gravitational couplings F_g for the orbifold compactifications introduced in [20] which generalizes the result of [10] to integrands involving $\Gamma_0(N)$ forms.

The organization of the paper is as follows. In section 2, we review aspects of the $\mathbb{Z}_N$ orbifold compactifications of the $E_8 \times E_8$ heterotic theory $K3 \times T^2$ we study. In particular we will present the new supersymmetric index which forms the basic ingredient for the gravitational threshold integral. Next in section 3 we write down the expressions for the coupling F_g for compactifications on orbifolds of $K3 \times T^2$. In section 4, as a warm up we first evaluate the gravitational coupling F_1 . We do this because the integrand as well as performing the integral is a straight forward generalization of the one done for the $K3 \times T^2$ theory by [9] and provides checks for our calculation. Next we present the result for the gravitational coupling $F_g(y, \bar{y})$ for $g > 1$ and extract the holomorphic coupling $\bar{F}_g^{\text{hol}}(y)$. In section 5 we study the properties of the genus g topological amplitude and observe that the Gopakumar-Vafa invariants of the dual Calabi-Yau predicted by our calculation. We then study the conifold singularities of $\bar{F}_g^{\text{hol}}(y)$. In section 6 we present our conclusions. The appendix A contains the modular properties of the integrand involved in evaluating $F_g(y, \bar{y})$. Appendices B, C contain the details of the integrations to obtain the gravitational thresholds. Appendix D lists the Gopakumar-Vafa invariants of for all g' corresponding to CHL orbifolds. Finally appendix E lists the elliptic genera of all the orbifolds considered in the paper for completeness.

2 Heterotic string on orbifolds of $K3 \times T^2$

In this section we briefly describe the general class of $\mathcal{N} = 2$ compactifications we will study. Consider the $E_8 \times E_8$ heterotic string theory compactified on $K3 \times T^2$ in which the $SU(2)$ spin connection of $K3$ is embedded in one of the E_8 's which is called the standard embedding. We then orbifold by a freely acting $\mathbb{Z}_N$ which acts as a g' automorphism on

³For example we obtain the form given in (C.7), which requires the next steps to cast in the standard form (C.11).

Conjugacy Class	Order
1A	1
2A	2
3A	3
5A	5
7A	7
11A	11
23A/B	23
4B	4
6A	6
8A	8
14A/B	14
15A/B	15
2B	2
3B	3

Table 1. Conjugacy classes of M_{24} studied in the paper. The twisted genera of classes 2B and 3B are such that the twining genera matches with the $F^{(0,1)}$ listed in [25]. These orbifolds are of order 4 and 9 respectively in our analysis. We refer to the classes 2A, 3A, 5A, 7A, 4B, 6A, 8A as the CHL orbifolds.

$K3$ together with a $1/N$ shift along one of the circles of $K3$. The g' action corresponds to any of the 26 conjugacy classes of Mathieu group listed in the table. In the standard embedding one of the E_8 lattice breaks to a $D6 \otimes D2$ lattice. The $SU(2)$ spin connection of $K3$ couples to the fermions of $D2$ lattice. These left moving fermions with the left moving bosons of the $K3$ combines with the right moving supersymmetric $K3$ CFT to form a $(6, 6)$ conformal field theory. The internal conformal field theory is given by

$$\mathcal{H}^{\text{internal}} = \mathcal{H}_{D2K3}^{(6,6)} \otimes \mathcal{H}_{D6}^{(6,0)} \otimes \mathcal{H}_{E_8}^{(8,0)} \otimes \mathcal{H}_{T^2}^{(2,3)} \quad (2.1)$$

The g' orbifold acts as a $\mathbb{Z}_N$ automorphism on the $(6, 6)$ CFT which preserves its $SU(2)$ R-symmetry together with a $1/N$ shift on one of the circles of the T^2 CFT. Thus these compactifications preserves $\mathcal{N} = 2$ supersymmetry in 4 dimensions. We refer to these theories as compactifications of the $E_8 \times E_8$ theory on the orbifold $K3 \times T^2/\mathbb{Z}_N$.

The starting point in evaluating both gauge and gravitational threshold corrections is to obtain the new supersymmetric index of the internal CFT. This is defined by

$$\mathcal{Z}_{\text{new}} = \frac{1}{\eta^2(\tau)} \text{Tr}_R [(-1)^F F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}]. \quad (2.2)$$

Here F refers to the right moving fermions number which is given by sum of the right moving fermion number of the T^2 CFT together with the right moving fermion number of the $K3$ CFT

$$F = F^{T^2} + F^{K3}. \quad (2.3)$$

The trace in (2.2) is taken over the Ramond sector of the right moving supersymmetric internal conformal field theory. Note that $(c, \bar{c}) = (22, 9)$.

In [21], the new supersymmetric index for compactifications on the orbifolds $K3 \times T^2/\mathbb{Z}_N$ was evaluated and it was shown that it can be written in terms of the twisted elliptic genus of $K3$. Let us briefly review the result. First we define the twisted elliptic genus of $K3$ which is given by

$$F^{(r,s)}(\tau, z) = \frac{1}{N} \text{Tr}_{RRg'^r} [(-1)^{F_{K3} + \tilde{F}_{K3}} g'^s e^{2\pi i z F_{K3}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}]. \quad (2.4)$$

Here g' is the $\mathbb{Z}_N$ automorphism on $K3$ corresponding to the conjugacy classes listed in table 1. However the construction goes through for any consistent orbifold on $K3$. The trace is taken in the Ramond-Ramond sector which is twisted by g'^r . The twisted elliptic genus can be written in the general form

$$\begin{aligned} F^{(0,0)}(\tau, z) &= \alpha_{g'}^{(0,0)} A(\tau, z), \\ F^{(r,s)}(\tau, z) &= \alpha_{g'}^{(r,s)} A(\tau, z) + \beta_{g'}^{(r,s)}(\tau) B(\tau, z), \\ r, s &\in \{0, 1, \dots, N-1\} \text{ with } (r, s) \neq (0, 0) \end{aligned} \quad (2.5)$$

where

$$A(\tau, z) = \frac{\theta_2^2(\tau, z)}{\theta_2^2(\tau, 0)} + \frac{\theta_3^2(\tau, z)}{\theta_3^2(\tau, 0)} + \frac{\theta_4^2(\tau, z)}{\theta_4^2(\tau, 0)}, \quad B(\tau, z) = \frac{\theta_1^2(\tau, z)}{\eta^6(\tau)} \quad (2.6)$$

are Jacobi forms which transform under $\text{SL}(2, \mathbb{Z})$ with index 1 and weight 0 and -2 respectively. $\alpha_{g'}^{(r,s)}$ are numerical constants and $\beta_{g'}^{(r,s)}$ is a weight 2 modular form under $\Gamma_0(N)$. For example for $g' \in pA$ with $p = 2, 3, 5, 7$ we have

$$\begin{aligned} \alpha_{pA}^{(0,0)} &= \frac{8}{p}, \quad \alpha_{pA}^{(r,s)} = \frac{8}{p(p+1)}, \quad (r, s) \neq (0, 0) \\ \beta^{(0,s)}(\tau) &= -\frac{2}{p+1} \mathcal{E}_p(\tau), \quad \text{for } 1 \leq s \leq p-1 \\ \beta^{(r,rk)}(\tau) &= \frac{2}{p(p+1)} \mathcal{E}_p\left(\frac{\tau+k}{p}\right), \\ &\text{for } 1 \leq r \leq p-1, \quad 1 \leq k \leq p-1 \end{aligned} \quad (2.7)$$

Here

$$\mathcal{E}_N = \frac{12i}{\pi(p-1)} \partial_\tau [\ln \eta(\tau) - \ln \eta(N\tau)] \quad (2.8)$$

is a modular form which transforms with weight 2 under $\Gamma_0(N)$. The twisted elliptic genus transforms under $\text{SL}(2, \mathbb{Z})$ as

$$F^{(r,s)}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \exp\left(2\pi i \frac{cz^2}{c\tau+d}\right) F^{(cs+ar, ds+br)}(\tau, z) \quad (2.9)$$

for

$$a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \quad (2.10)$$

The indices in (2.9) $cs + ar$ and $ds + br$ are taken to be mod N .

Appendix E lists the twisted elliptic genus $F^{(r,s)}(\tau, z)$ for corresponding to the conjugacy classes g' in table. Given the twisted elliptic genus, we can read out $\alpha_{g'}^{(r,s)}, \beta_{g'}^{(r,s)}$ using the general form given in (2.5). Then the new supersymmetric index for the standard embedding compactifications of the $E_8 \times E_8$ heterotic string on the orbifold $K3 \times T^2/\mathbb{Z}_N$ is given by

$$\mathcal{Z}_{\text{new}}(q, \bar{q}) = -2 \frac{1}{\eta^{24}(\tau)} \Gamma_{2,2}^{(r,s)} E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right]. \quad (2.11)$$

Here it is understood that the indices r, s are summed from 0 to $N - 1$. In (2.11), $\Gamma_{2,2}^{(r,s)}$ is the lattice sum on T^2 which is defined as

$$\begin{aligned} \Gamma_{2,2}^{(r,s)}(q, \bar{q}) &= \sum_{\substack{m_1, m_2, n_2 \in \mathbb{Z} \\ n_1 = \mathbb{Z} + \frac{r}{N}}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} e^{2\pi i m_1 s / N}, \\ \frac{1}{2} p_R^2 &= \frac{1}{2T_2 U_2} | -m_1 U + m_2 + n_1 T + n_2 T U |^2, \\ \frac{1}{2} p_L^2 &= \frac{1}{2} p_R^2 + m_1 n_1 + m_2 n_2 \end{aligned} \quad (2.12)$$

and T, U are the Kähler and complex structure of the torus T^2 . It is also understood in (2.11), that there is a sum over the lattice momenta m_1, m_2 as well as the winding numbers n_1, n_2 . Note that in (2.11) its only the lattice sum that depends on both q and $\bar{q}$, the Eisenstein series E_4, E_6 as well as the $\Gamma_0(N)$ weight 2 form $\beta_{g'}^{(r,s)}$ depends only on the holomorphic co-ordinate q . It is instructive to examine the situation in which there is no orbifold performed. Then $p = 1$ and $\alpha_{1A} = 8$ and there are no twisted sectors. Thus the new supersymmetric index reduces to

$$\mathcal{Z}_{\text{new}}(q, \bar{q}) = -4 \frac{E_4(\tau) E_6(\tau)}{\eta^{24}(\tau)} \Gamma_{2,2}(q, \bar{q}) \quad (2.13)$$

Here $\Gamma_{2,2}$ is the usual lattice sum without phases and twists. For later purpose we define the following $\Gamma_0(N)$ form, which occurs in the new supersymmetric index given in (2.11).

$$\begin{aligned} f^{(r,s)}(\tau) &= \frac{1}{2\eta^{24}(\tau)} E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right], \\ &= \sum_{l \in \frac{\mathbb{Z}}{N}} c^{(r,s)}(l) q^l. \end{aligned} \quad (2.14)$$

Using the fact that $\beta_{g'}^{(r,s)}(\tau)$ is a $\Gamma_0(N)$ form and (2.9) it is easy to see that the $f^{(r,s)}(\tau)$ has the following transformation property under the $\text{SL}(2, \mathbb{Z})$ generators

$$\begin{aligned} f^{(r,s)}(\tau + 1) &= f^{(r,s+r)}(\tau), \\ f^{(r,s)}\left(-\frac{1}{\tau}\right) &= (-i\tau)^{-2} f^{(N-r,s)}(\tau) = (-i\tau)^{-2} f^{(r,N-s)}(\tau). \end{aligned} \quad (2.15)$$

Given the new supersymmetric index we can read out the difference between the number of vectors and hypers in the spectrum [9]. This is given by

$$N_h - N_v = - \sum_{s=0}^{N-1} c^{(0,s)}(0). \quad (2.16)$$

Here, we have also included the four $\text{U}(1)$'s resulting from the metric and the Neveu-Schwarz B -field with one index along the T^2 in the counting of the vectors. This data also gives the Euler number of the corresponding type II dual Calabi-Yau compactification which is given by

$$\chi(X) = -2(N_h - N_v) = 2 \sum_{s=0}^{N-1} c^{(r,s)}(0). \quad (2.17)$$

For the compactification on $K3 \times T^2$ the value of $N_h - N_v = 240$ and therefore the corresponding Euler number of the dual $K3$ -fibered Calabi-Yau manifold is $\chi(X) = -480$. These data for all the orbifolds considered in this paper is provided in table 4.

In the subsequent sections we will use the new supersymmetric index in (2.11) as a starting point and evaluate the gravitational one-loop corrections for heterotic compactifications on orbifolds $K3 \times T^2/\mathbb{Z}_N$.

3 The integral for gravitational thresholds

In this section we obtain the integral for gravitational thresholds involving $\mathcal{N} = 2$ heterotic compactifications on orbifolds considered in this paper. Though these integrals were obtained for the unorbifolded situation earlier in [8, 9] we find it instructive to go through the analysis. This is because we wish to arrive at the key result that the integrals which evaluate the thresholds for the orbifolds involve the twisted elliptic genus of $K3$.

It is first easy to discuss the one loop integral which captures the gravitational correction with no graviphoton insertions. The result of this integral provides the moduli dependence of the following higher derivative term in the $\mathcal{N} = 2$ effective action for these compactifications

$$I_1(T, U) = R_+^2 F_1(T, U). \quad (3.1)$$

Here R_+ refers to the anti-self-dual Riemann tensor. From the earlier works of [3, 4], a nice compact expression for the F_1 has been obtained in [9]. This is given by⁴

$$F_1(T, U) = -\frac{1}{4} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left\{ \frac{1}{\eta^2(\tau)} \text{Tr}_R[(-1)^F F q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}] \left[E_2 - \frac{3}{\pi\tau_2} \right] - b_{\text{grav}} \right\}. \quad (3.2)$$

⁴Note that we have normalized F_1 so that the result for the integral agrees with [9] for the $K3 \times T^2$ compactification for $N = 1$.

The subtraction by b_{grav} is to ensure that the integral is well defined for $\tau_2 \rightarrow \infty$. Note that the trace over the internal conformal field theory is precisely what occurs in the new supersymmetric index given in (2.2). Therefore to obtain the moduli dependence of the gravitational correction for the orbifold compactifications in this paper, we substitute the result for the new supersymmetric index (2.11) in (3.2). This results in

$$F_1(T, U) = \frac{1}{2} \int \frac{d^2\tau}{\tau_2} \frac{1}{\eta^{24}(\tau)} \Gamma_{2,2}^{(r,s)} E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right] \left[E_2 - \frac{3}{\pi\tau_2} \right], \quad (3.3)$$

where we have ignored the constant necessary to regulate the integral. From the properties (A.6) and (2.15) we see that the integrand is invariant under $SL(2, \mathbb{Z})$ transformations. Note that the key input that goes into the integral are the coefficients of the twisted elliptic genus of $K3$ represented by the functions $\alpha_{g'}^{(r,s)}, \beta_{g'}^{(r,s)}$. The next step is to perform the integral over the fundamental domain which is done in section 4.1.

Let us now discuss the integral which results in the moduli dependence of the following term in the one loop effective action

$$I_g(T, U) = R_+^2 F_+^{2g-2} F_g(T, U), \quad (3.4)$$

where F_+ is the anti-self dual field strength of the graviphoton and $g > 1$. From the world sheet analysis of [8] we see that the one loop integral is given by

$$F_g = \frac{1}{2\pi^2(g!)^2} \int \frac{d^2\tau}{\tau_2} \left\{ \frac{1}{\tau_2^2 \eta^2(\tau)} \text{Tr}_R \left[(i\bar{\partial}X)^{(2g-2)} (-1)^F F q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \right] \right. \\ \left. \times \left\langle \prod_{i=1}^g \int d^2x_i Z^1 \partial Z^2(x_i) \prod_{j=1}^g \int d^2y_j \bar{Z}^2 \partial \bar{Z}^1(y_j) \right\rangle \right\}. \quad (3.5)$$

Here X is the complex co-ordinate on the T^2 and Z^1, Z^2 and the complex co-ordinates of the transverse non-compact space time bosons. The above result is obtained by carrying out the Wick contractions of the vertex operators for the graviton and the graviphoton insertions as mentioned in [8]. Note that the correlators of the space time bosons and the internal conformal field theory factorize. The $2g - 2$ insertions of $\bar{\partial}X$ arise from the $(2g - 3)$ graviphoton vertices in the (0) -ghost picture and one $\bar{\partial}X$ that appears in the picture changing operator. Since the correlator is evaluated on the torus, the insertions of $\bar{\partial}X$ can contribute only through the zero modes. Therefore this can be replaced as follows

$$(i\bar{\partial}X)^{(2g-2)} \rightarrow \left(\frac{(p_R^{(r,s)})}{\sqrt{2T_2 U_2}} \right)^{(2g-2)}. \quad (3.6)$$

The super script (r, s) for p_R indicates the sector over which the right moving momentum belongs to. When $r \neq 0$ the winding number n_1 on one of the circles of the T^2 is quantized in units of $\mathbb{Z} + \frac{r}{N}$ since the right moving momentum belongs to the r th twisted sector of the

1/ N shift action. Then trace over the internal conformal field theory can be written as⁵

$$\begin{aligned} \frac{1}{\eta^2(\tau)} \text{Tr}_R \left[(i\partial X)^{(2g-2)} (-1)^F F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right] = \\ = 2 \frac{1}{\eta^{24}(\tau)} \Gamma_{2,2}^{(r,s)} \left(\frac{(p_R^{(r,s)})}{\sqrt{2T_2 U_2}} \right)^{(2g-2)} E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right]. \end{aligned} \quad (3.7)$$

This trace is evaluated using the input in (3.6) and repeating the steps which led to the result in (2.11). The only difference is the insertions of the lattice momenta $p_R^{(r,s)}$. Note that the sectors (r, s) are summed over, and it is always understood that the lattice momenta and winding are summed.

Now we need to simplify the correlators over the free non-compact bosons. To this end we follow [8] and use the generating function for these correlators

$$G(\lambda, \tau, \bar{\tau}) = \sum_{g=1}^{\infty} \frac{1}{(g!)^2} \left(\frac{\lambda}{\tau_2} \right)^{2g} \left\langle \prod_{i=1}^g \int d^2 x_i Z^1 \partial Z^2(x_i) \prod_{j=1}^g \int d^2 y_j \bar{Z}^2 \partial \bar{Z}^1(y_j) \right\rangle. \quad (3.8)$$

The correlation functions are normalized free field correlators of the space-time bosons. This generating function is given by

$$G(\lambda, \tau, \bar{\tau}) = \left(\frac{2\pi i \lambda \eta^3}{\theta_1(\lambda, \tau)} \right)^2 e^{-\frac{\pi \lambda^2}{\tau_2}}. \quad (3.9)$$

To use this generating function for the correlators, we consider

$$F(\lambda, T, U) = \sum_{g=1}^{\infty} \lambda^{2g} F_g(T, U). \quad (3.10)$$

Then using the result (3.7) and (3.9) in (3.5), the one loop amplitude simplifies to

$$F(\lambda, T, U) = \frac{1}{\pi^2} \int \frac{d^2 \tau}{\tau_2} \frac{1}{\eta^{24}(\tau)} \Gamma_{2,2}^{(r,s)} E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right] \left[\left(\frac{2\pi i \lambda \eta^3}{\theta_1(\tilde{\lambda}, \tau)} \right)^2 e^{-\frac{\pi \tilde{\lambda}^2}{\tau_2}} \right]^{(r,s)}. \quad (3.11)$$

where

$$\tilde{\lambda} = \frac{p_R^{(r,s)} \lambda}{\sqrt{2T_2 U_2}}. \quad (3.12)$$

Here again note that the key input that goes into the integrand (3.11) for the gravitational couplings for the orbifold compactifications considered in this paper is the twisted elliptic genus of $K3$.

To perform this integral we follow the approach in [10]. The reciprocal of the theta function admits the expansion

$$\left(\frac{2\pi \tilde{\lambda} \eta^3}{\theta_1(\tilde{\lambda}, \tau)} \right)^2 e^{-\frac{\pi \tilde{\lambda}^2}{\tau_2}} = \sum_{k=0}^{\infty} \tilde{\lambda}^{2k} \mathcal{P}_{2k}(\hat{G}_2, \dots, G_{2k}), \quad (3.13)$$

⁵In the analysis of [8], the trace over the internal conformal field theory was referred to as $C_\epsilon(\bar{\tau})$. Furthermore, what we call as left movers is right movers in [8] and vice-versa. We follow the notations of [10].

where $\mathcal{P}_{2k}$ is a polynomial related to the Schur polynomial by

$$\mathcal{P}_{2k}(\hat{G}_2, \dots, G_{2k}) = -\mathcal{S}_k\left(\hat{G}_2, \frac{1}{2}G_4, \frac{1}{3}G_6, \dots, \frac{1}{k}G_{2k}\right), \quad (3.14)$$

where the Schur polynomials are in turn defined by the expansion

$$\exp\left[\sum_{k=1}^{\infty} x_k z^k\right] = \sum_{k=0}^{\infty} \mathcal{S}_k(x_1, \dots, x_k) z^k, \quad (3.15)$$

$$\mathcal{S}_1(x_1) = x_1, \quad \mathcal{S}_2(x_1, x_2) = \frac{x_1^2}{2} + x_2, \quad \mathcal{S}_3(x_1, x_2, x_3) = \frac{x_1^3}{6} + x_1 x_2 + x_3.$$

Using these definitions we have

$$\begin{aligned} \mathcal{P}_2(\hat{G}_2) &= -\hat{G}_2, & \mathcal{P}_4(\hat{G}_2, G_4) &= -\frac{1}{2}(\hat{G}_2^2 + G_4), \\ \mathcal{P}_3(\hat{G}_2, G_4, G_6) &= -\frac{1}{6}(\hat{G}_2^3 + \hat{G}_2 G_4) - \frac{1}{3}G_6. \end{aligned} \quad (3.16)$$

where the G 's are normalized Eisenstein series given by

$$G_{2k} = 2\zeta(2k)E_{2k}, \quad \hat{E}_2(\tau) = E_2 - \frac{3}{\pi\tau_2}. \quad (3.17)$$

Note that $\mathcal{P}$ is holomorphic, except for the occurrence of $\hat{G}_2$ and it transforms as a modular form of weight $2k$. Substituting the expansion (3.13) in (3.11), we obtain

$$\begin{aligned} F(\lambda, T, U) &= \sum_{k=0}^{\infty} \frac{\lambda^{2(k+1)}}{\pi^2 (2T_2 U_2)^k} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \tau_2^{2k} \frac{1}{\eta^{24}(\tau)} \Gamma_{2,2}^{(r,s)} \\ &\quad \times E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right] (p_R^{(r,s)})^{2k} \mathcal{P}_{2k+2}. \end{aligned} \quad (3.18)$$

Note that the integrand is an invariant under $\text{SL}(2, \mathbb{Z})$. This can be seen using the properties in (A.9) and (2.15)

4 Evaluating the integral for gravitational thresholds

4.1 $g = 1$

We will first perform the integral in (3.3). As it will be clear subsequently, it is simpler to treat the case of $g = 1$ separately from the integral for $g > 1$. We will follow the unfolding method developed in [9] for performing this integral. However we generalize the discussion to integrands which contain modular forms transforming under $\Gamma_0(N)$ rather than $\text{SL}(2, \mathbb{Z})$. Let us first define the Fourier expansions of the expressions in the integrand.

$$\begin{aligned} \frac{1}{2\eta^{24}} E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right] \left[E_2 - \frac{3}{\pi\tau_2} \right] &= \sum_{l \in \mathbb{Z}/N} \left(\tilde{c}^{(r,s)}(l) q^l - \frac{3}{\pi\tau_2} c^{(r,s)}(l) q^l \right), \\ &= \tilde{h}(r, s)(\tau) - \frac{3}{\pi\tau_2} f^{(r,s)}(\tau). \end{aligned} \quad (4.1)$$

Note that as expected, the twisted sectors admit fractional q expansions. Substituting this expansion and the values of p_L^2 and p_R^2 from (2.12) into (3.3) we obtain

$$F_1(T, U) = \sum_{r,s=0}^{N-1} \mathcal{I}_{r,s}, \quad (4.2)$$

$$\mathcal{I}_{r,s} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{m_1, m_2, n_2 \in \mathbb{Z}, n_1 \in \mathbb{Z} + \frac{r}{N}} \exp[2\pi i \tau (m_1 n_1 + m_2 n_2)] \times$$

$$\times \exp \left[-\frac{\pi \tau_2}{T_2 U_2} |n_2 T U + n_1 T - U m_1 + m_2|^2 \right] e^{\frac{2\pi i s m_1}{N}} \left(h^{(r,s)}(\tau) - \frac{3}{\pi \tau_2} f^{(r,s)}(\tau) \right).$$

The first step to do the integral is to perform the Poisson re-summation over the momenta m_1, m_2 using the formula

$$\sum_{m \in \mathbb{Z}} f(m) e^{2\pi i s m / N} = \sum_{k \in \mathbb{Z} + \frac{s}{N}} \int_{-\infty}^{\infty} du f(u) \exp(2\pi i k u). \quad (4.3)$$

Using this identity and performing the integral over the corresponding variables u_1, u_2 , we obtain

$$\mathcal{I}_{r,s} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} T_2 \sum_{n_2, k_2 \in \mathbb{Z}, n_1 \in \mathbb{Z} + \frac{r}{N}, k_1 \in \mathbb{Z} + \frac{s}{N}} \left(\tilde{h}(r, s)(\tau) - \frac{3}{\pi \tau_2} f^{(r,s)}(\tau) \right) \exp \left[\mathcal{G}(\vec{n}, \vec{k}) \right], \quad (4.4)$$

where

$$\mathcal{G}(\vec{n}, \vec{k}) = -\frac{\pi T_2}{U_2 \tau_2} |\mathcal{A}|^2 - 2\pi i T (\det \mathcal{A}), \quad (4.5)$$

$$A = \begin{pmatrix} n_1 & k_1 \\ n_2 & k_2 \end{pmatrix}, \quad \mathcal{A} = (1, U) A \begin{pmatrix} \tau \\ 1 \end{pmatrix}. \quad (4.6)$$

Therefore using (4.4) in (4.2) we can write the integral as

$$F_1(T, U) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{r,s=0}^{N-1} \sum_{n_2, k_2 \in \mathbb{Z}, n_1, k_1 \in \mathbb{Z} + \frac{r}{N}} \mathcal{J}(A, \tau), \quad (4.7)$$

where

$$\mathcal{J}(A, \tau) = T_2 \exp \left(-\frac{\pi T_2}{U_2} |\mathcal{A}|^2 - 2\pi i T \det A \right) f^{(r,s)}(\tau) \hat{E}_2(\tau), \quad (4.8)$$

$$r = N n_1 \bmod N, \quad s = N k_1 \bmod N.$$

Using (4.6), we can think of $\mathcal{J}$ as a function of the matrix A . Then the sum over r, s and $\vec{n}, \vec{k}$ in the right hand side of (4.7) can be thought of as the sum over matrices of the form (4.6) with $n_2, k_2 \in \mathbb{Z}$ and $n_1, k_1 \in \frac{\mathbb{Z}}{N}$. Thus (4.7) can be written as

$$I_1(T, U) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_A \mathcal{J}(A, \tau). \quad (4.9)$$

Now using the modular transformation given in (2.15) and the definition of $\mathcal{J}$ in (4.8), it can be seen that

$$\mathcal{J}\left(A, \frac{a\tau + b}{c\tau + d}\right) = \mathcal{J}\left(A \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right). \quad (4.10)$$

This symmetry allows us to extend the integration over the fundamental domain to its images under $\text{SL}(2, \mathbb{Z})$ together with the restriction of the summation over A to its inequivalent $\text{SL}(2, \mathbb{Z})$ orbits. Lets denote the sum over inequivalent $\text{SL}(2, \mathbb{Z})$ orbits as $\sum'_A$, then F_1 becomes

$$\mathcal{I} = \sum'_A \int_{\mathcal{F}_A} \frac{d^2\tau}{\tau_2^2} T_2 \exp\left(-\frac{\pi T_2}{U_2 \tau_2} |\mathcal{A}|^2 - 2\pi i T \det A\right) f^{(r,s)}(\tau) \hat{E}_2(\tau). \quad (4.11)$$

Now the label r, s is to interpreted as $Nn_1 \bmod N$ and $Nk_1 \bmod N$ respectively. The region of integration $\mathcal{F}_A$ depends on the orbit represented by A .

From the analysis of [9] we see that there are three inequivalent orbits. These are as follows: the zero orbit

$$A = 0, \quad (4.12)$$

the non-degenerate orbit

$$A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix}, \quad p \neq 0, p \in \mathbb{Z}, \quad k, j \in \frac{1}{N}\mathbb{Z}, k > j \quad (4.13)$$

and the degenerate orbit

$$A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}, \quad (j, p) \neq (0, 0), \quad j \in \frac{1}{N}\mathbb{Z}, p \in \mathbb{Z}. \quad (4.14)$$

The contribution from each of these orbits has been evaluated in the appendix B. Taking the sum of these contributions the result for F_1 is given by

$$F_1 = T_2 \frac{\pi}{6} E_2^2(q) f^{(0,0)}(q) \Big|_{q^0} \quad (4.15)$$

$$+ \sum_{s=0}^{N-1} \left[\tilde{c}^{(0,s)}(0) \left(-\log(T_2 U_2) + \frac{2\zeta(2, s/N)}{\pi} U_2 - \kappa \right) - c^{(0,s)}(0) \frac{6\zeta(4, s/N)}{\pi^3} \frac{U_2^2}{T_2} + \frac{6}{\pi^2} c^{(0,s)}(0) \frac{\zeta(3)}{T_2 U_2} \right] \quad (4.16)$$

$$+ 4\text{Re} \left\{ \sum_{\substack{k \geq 0, l \geq -1 \\ (k,l) \neq (0,0) \\ k \in \mathbb{Z} + r/N}} \left[\tilde{c}^{(r,s)}(kl) e^{2\pi i s l / N} \text{Li}_1(e^{2\pi i (kT + lU)}) - \frac{3}{\pi T_2 U_2} c^{(r,s)}(kl) e^{2\pi i s l / N} \tilde{\mathcal{P}}(kT + lU) \right] \right\},$$

where $\tilde{\mathcal{P}}$ is given by

$$\tilde{\mathcal{P}}(x) = \text{Im}(x) \text{Li}_2(e^{2\pi i x}) + \frac{1}{2\pi} \text{Li}_3(e^{2\pi i x}). \quad (4.17)$$

Orbifold	$c^{(0,0)}(0)$	$c^{(0,1)}(0)$
2A	−120	136
3A	−80	109
5A	−48	77
7A	−240/7	829/14
11A	−240/11	442/11
23A	−240/23	473/23

Table 2. $c^{(0,0)}(0)$, $c^{(0,1)}(0)$ are listed for prime N .

From the appendix it will be clear that the above result holds for $T_2 > U_2$ as in the analysis of [9]. Let us examine the result for $N = 1$, the unorbifolded $K3 \times T^2$ compactification. We obtain For unorbifolded $K3$ this answer reduces to

$$\begin{aligned}
 F_1 = & -48\pi T_2 - 264(-\log(T_2 U_2) - \kappa) - 88\pi U_2 \\
 & \times 16\pi \frac{U_2^2}{T_2} + \frac{6}{\pi^2} c(0) \frac{\zeta(3)}{T_2 U_2} + \\
 & + 4\text{Re} \sum_{\substack{k \geq 0, l \geq -1 \\ (k,l) \neq (0,0) \\ k \in \mathbb{Z}}} \tilde{c}(kl) \text{Li}_1(e^{2\pi i(kT + lU)}) - \frac{3}{\pi T_2 U_2} c(kl) \mathcal{P}(kT + lU).
 \end{aligned} \tag{4.18}$$

Let us recall that for each of the orbifolds we can read out the coefficients $\tilde{c}^{(r,s)}$ and $c^{(r,s)}$ using their definition given in (4.1) and the explicit expressions for the twisted elliptic genera given in the appendix E. Let us make a few observations from the result for F_1 . Note that the coefficients which determine the moduli dependence of F_1 in the first two lines of (4.15) depends on low lying topological data of the new supersymmetric index. This dependence does not involve exponentials in moduli T, U . The low lying coefficients of the new supersymmetric index for the various orbifold compactifications corresponding to the Mathieu moonshine conjugacy classes are listed in tables 2 and 3.

One also observes that for an order N orbifold $c^{(0,s)}(-1) = 1/N$ and $\tilde{c}^{(0,s)}(0)$ can be written as

$$\tilde{c}^{(0,s)}(0) = c^{(0,s)}(0) - \frac{24}{N}. \tag{4.19}$$

Using this topological data we can evaluate the linear and quadratic dependences of the moduli T and U which result from the contribution of the integral along the zero orbit and the degenerate orbit. These coefficients for the various orbifolds are listed in table 2 and 3. Note that the coefficient of $\frac{\zeta(3)}{T_2 U_2}$ is determined by the difference $N_h - N_v$ or the Euler character. An interesting observation is the occurrence of the low lying topological data weighted with the Hurwitz-zeta function $\zeta(2, s/N)$ as well as $\zeta(4, s/N)$ in the orbifolds. This can be seen only for the orbifold compactifications. In fact if one is just given the

Orbifold	$c^{(0,0)}(0)$	$c^{(0,1)}(0)$	s_1	s_2	$c^{(0,s_1)}(0)$	$c^{(0,s_2)}(0)$
4B	−60	96	2	—	68	—
6A	−40	443/6	2	3	109/2	136/3
8A	−30	111/2	2	4	48	34
14A	−120/7	943/28	2	7	829/28	136/7
15A	−16	472/15	3	5	77/3	109/5
2B	−60	124	2	—	196	—
3B	−80/3	166/3	3	—	407/6	—

Table 3. $c^{(0,0)}(0)$, $c^{(0,1)}(0)$, $c^{(0,s_1)}(0)$, $c^{(0,s_2)}(0)$ are listed where s_1 and s_2 are divisors of the composite order N with $s_1 < s_2 < N$.

three coefficients of U_2 , and U_2^2/T_2 and $1/T_2 U_2$ in the gravitational threshold F_1 and the properties of the low lying coefficients in, we can determine $c^{(0,s)}$ for all the orbifold compactifications corresponding to the conjugacy classes of Mathieu Moonshine considered here. A summary of these coefficients, including the Euler number for all the orbifolds considered in this paper is given in table 4.

The higher level coefficients of the new supersymmetric index control the exponentially suppressed terms in the last line of (4.15). Note that the exponential dependence of T moduli carries the information of the twisted sectors.

4.2 $g > 1$

The gravitational threshold integral for $g > 1$ is given by

$$F_g = \frac{1}{\pi^2 (2T_2 U_2)^k} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \tau_2^{2k} \frac{1}{\eta^{24}(\tau)} \Gamma_{2,2}^{(r,s)} E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right] (p_R^{(r,s)})^{2k} \mathcal{P}_{2k+2}. \quad (4.20)$$

where $k = g - 1 > 0$. Note that here the summation over (r, s) is implied. The steps to perform the integral are similar to the case when $g = 1$, however one needs to keep track of the extra insertions of the momentum $p_R^{(r,s)}$. First we perform the Poisson re-summation over the variables (m_1, m_2) to obtain

$$F_g(T, U) = \sum_{r,s=0}^{N-1} \sum_{n_2, k_2 \in \mathbb{Z}, n_1, k_1 \in \mathbb{Z} + \frac{r}{N}} \left(\frac{T_2}{2U_2} \right)^k \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} \tilde{\mathcal{J}}(A, \tau), \quad (4.21)$$

$$\tilde{\mathcal{J}}(A, \tau) = T_2 \mathcal{A}^{2k} \exp \left(-\frac{\pi T_2}{U_2} |\mathcal{A}|^2 - 2\pi i T \det A \right) f^{(r,s)}(\tau) \mathcal{P}_{2k+2}(\tau), \quad (4.22)$$

$$r = N n_1 \bmod N, \quad s = N k_1 \bmod N.$$

Orbifold	$N_h - N_v$	χ	$\sum_{s=0}^{N-1} c^{0,s}(0) \left(\frac{2\zeta(2,s/N)}{\pi} \right)$	$-\sum_{s=0}^{N-1} c^{0,s}(0) \frac{6\zeta(4,s/N)}{\pi^3}$	$\frac{\pi}{6} E_2^2(q) F(q) _{q^0}$
1A	-240	-480	-88π	16π	-48π
2A	16	32	80π	-128π	-24π
3A	138	276	240π	-576π	-16π
5A	260	520	560π	-3200π	$-48\pi/5$
7A	321	642	880π	-9472π	$-48\pi/7$
11A	380	760	1512π	-39216π	$-48\pi/11$
23A	442	884	3432π	-383664π	$-48\pi/23$
4B	200	400	400π	-1600π	-12π
6A	262	524	720π	-6240π	-8π
8A	322	644	1040π	-15008π	-6π
14A	382	764	1992π	-85584π	$-24\pi/7$
15A	382	764	2152π	-105904π	$-16\pi/5$
2B	384	768	640π	-2176π	-12π
3B	441	882	1428π	-24264π	$-16\pi/3$

Table 4. $N_h - N_v$, χ ⁶ and coefficients of U_2 , U_2^2/T_2 and T_2 , in the limit $T_2 > U_2$ are shown.

⁶The Calabi Yau manifolds with χ values $-480, 32, 276, 400, 520, 524, 644, 768$ are known to exist and they are listed in [40].

Now similar to the $g = 1$ case we can think of the sum over r, s and $\vec{n}, \vec{k}$ as sum over the matrices of the form in (4.6) with $n_2, k_2 \in \mathbb{Z}$ and $n_1, k_1 \in \frac{\mathbb{Z}}{N}$. Thus (4.21) can be written as

$$I_g(T, U) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_A \tilde{\mathcal{J}}(A, \tau). \quad (4.23)$$

Now using the modular transformation given in (2.15) and the definition of $\tilde{\mathcal{J}}$ in (4.22), it can be seen that

$$\tilde{\mathcal{J}}\left(A, \frac{a\tau + b}{c\tau + d}\right) = \tilde{\mathcal{J}}\left(A \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right). \quad (4.24)$$

This symmetry allows us to extend the integration over the fundamental domain to its images under $\text{SL}(2, \mathbb{Z})$ together with the restriction of the summation over A to its inequivalent $\text{SL}(2, \mathbb{Z})$ orbits. Let us again denote the sum over inequivalent $\text{SL}(2, \mathbb{Z})$ orbits as $\sum'_A$, then F_g becomes

$$F_g = \sum'_A \int_{\mathcal{F}_A} \left(\frac{T_2}{2U_2}\right)^k \frac{d^2\tau}{\tau_2^2} T_2 \mathcal{A}^{2k} \exp\left(-\frac{\pi T_2}{U_2 \tau_2^2} |\mathcal{A}|^2 - 2\pi i T \det A\right) f^{(r,s)}(\tau) \mathcal{P}_{2k+2}(\tau). \quad (4.25)$$

Now the label r, s is to interpreted as $Nn_1 \bmod N$ and $Nk_1 \bmod N$ respectively. The region of integration $\mathcal{F}_A$ depends on the orbit represented by A .

We can now look at the contribution of the three inequivalent orbits. First note that the contribution of the zero orbit vanishes due to the presence of $\mathcal{A}^{2k}$ in the integrand. Thus we are left with the non-degenerate orbit which is characterized by the set of matrices

$$A = \begin{pmatrix} n_1 & j \\ 0 & p \end{pmatrix}, \quad p \in \mathbb{Z}, \quad n_1, j \in \frac{1}{N}\mathbb{Z}, \quad n_1 > j \quad (4.26)$$

and the degenerate orbit

$$A = \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix}, \quad j \in \frac{1}{N}\mathbb{Z}, \quad j \neq 0. \quad (4.27)$$

Note that here we have included the $p = 0$ case also in the non-degenerate orbit for convenience. This is because the $p = 0$ can be treated uniformly together with the $p \neq 0$ situation in the non-degenerate orbit.

The detailed evaluation of the integral in the two orbits is carried out in the appendix C. This integral was done in [10] for the case of $K3 \times T^2$ compactification and in [34] for the FHSV compactification. The latter situation involves only modular forms under $\Gamma_0(2)$ in the integrand. Here we have generalized the evaluation of the integral for integrands containing modular forms transforming under $\Gamma_0(N)$. Further more as it will be clear in the appendix C, we do not directly apply the reduction theorem of Borchers [33] as done in the earlier works to perform the integral. The application of the reduction theorem is rather intricate and it easier to proceed straightforwardly from (4.25) and carry out the steps involved in the integration.

To write the result in a convenient form, let us define the following two dimensional vectors and the inner products.

$$\begin{aligned}
 m &= (n_1, n_2), \quad n_1 \in \mathbb{Z} + \frac{r}{N}, \quad n_2 \in \mathbb{Z}, \quad \frac{m^2}{2} = n_1 n_2, \\
 y &= (T, U), \\
 m \cdot y &= n_1 T_1 + n_2 U_2 + i(n_1 T_1 + n_2 T_2), \\
 m \hat{\cdot} y &= n_1 T_1 + n_2 U_2 - i(n_1 T_1 + n_2 T_2).
 \end{aligned} \tag{4.28}$$

We also define the Fourier coefficients involved in the expansion of the modular forms as

$$f^{(r,s)}(\tau) \mathcal{P}_{2k+2}(\tau) = \sum_{l \in \frac{\mathbb{Z}}{N}, t=0}^{t=g} c_{g-1}^{(r,s)}(l, t) \tau_2^{-t} q^l. \tag{4.29}$$

Then, from the evaluation in appendix C, the result for the contribution from the non-degenerate orbit is given by

$$\begin{aligned}
 F_{g>1}^{\text{nondeg}} &= \frac{(-1)^{g-1}}{2^{2(g-1)} \pi^2} \sum_{r,s=0}^{N-1} \sum_{m \neq 0} \sum_{t=0}^g \sum_{h=0}^{2g-2} \sum_{j=0}^{g-1-h/2} \sum_{a=0}^{\hat{s}} e^{-2\pi i s n_2 / N} c^{(r,s)}(m^2/2, t) \times \\
 &\times \frac{(2g-2)!}{j! h! (2g-h-2j-2)!} \frac{(-1)^{j+h}}{(4\pi)^{j+a}} \frac{(\hat{s}+a)!}{(\hat{s}-a)! a!} (\text{sgn}(\text{Im}(m \cdot y)))^h \times \\
 &\times \frac{(\text{Im}(m \hat{\cdot} y))^{t-j-a}}{(T_2 U_2)^t} \text{Li}_{3+a+j+t-2g}(e^{2\pi i m \hat{\cdot} y}) \\
 &+ \sum_{s=0}^{N-1} \frac{c^{(0,s)}(0, g-1)}{2^{4g-5} \pi^{g+1}} \frac{1}{(T_2 U_2)^{g-1}} \sum_{\tilde{s}=0}^{g-1} (-1)^{\tilde{s}} \frac{(2g-2)!}{\tilde{s}! (g-1-\tilde{s})!} \psi(1/2 + \tilde{s}) \\
 &+ \sum_{s=0}^{N-1} \sum_{t=0}^{g-2} \frac{c^{(0,s)}(0, t)}{\pi^{t+5/2}} \frac{\zeta(3+2(t-g))}{(T_2 U_2)^t} \times \\
 &\times \sum_{\tilde{s}=0}^{g-1} (-1)^{\tilde{s}} 2^{2(\tilde{s}-2g+2)} \frac{(2g-2)!}{(2\tilde{s})! (g-1-\tilde{s})!} \Gamma(3/2 + \tilde{s} + t - g).
 \end{aligned} \tag{4.30}$$

Here

$$\hat{s} + 1/2 = |\nu|, \quad |\nu| = 2k - h - j - t - 1/2, \quad k = g - 1 \tag{4.31}$$

and $m \neq 0$ refers to any of the following cases,

$$\begin{aligned}
 n_1 &> 0, n_2 > 0; \quad n_1 < 0, n_2 < 0; \\
 n_1 &= 0, n_2 > 0 \text{ or } n_2 < 0; \quad n_2 = 0, n_1 > 0 \text{ or } n_1 < 0; \\
 n_1 &= \frac{r}{N}, n_2 < 0 \text{ or } n_1 = -\frac{r}{N}, n_2 > 0 \text{ with } r|n_2| \leq N, r > 0.
 \end{aligned} \tag{4.32}$$

In (4.30), the index r in $c^{(r,s)}$ is related to n_1 by $r = N n_1 \pmod{N}$ and $n_2 \in \mathbb{Z}$. The result for the non-degenerate orbit is valid for $T_2, U_2 > 0$.⁷ Performing the integral for the

⁷Note that the result for the integral also contains the complex conjugate $\bar{F}_g$, which we have suppressed for simplicity.

degenerate orbit we obtain for $T_2 < U_2$

$$F_{g>1}^{\text{deg}} = \frac{1}{T_2^{2g-3}} \sum_{s=0}^{N-1} \sum_{t=0}^g c^{(0,s)}(0,t) \frac{t!}{2^{2g-2} \pi^{t+3}} \left(\frac{T_2}{U_2} \right)^t \zeta \left(2(2+t-g), \frac{s}{N} \right). \quad (4.33)$$

The result for the integral in the degenerate orbit for $T_2 > U_2$ is obtained by interchanging $T_2 \leftrightarrow U_2$ in (4.33).

4.3 Extracting the holomorphic F_g

We now follow [10] and extract the part of the one loop gravitational coupling given in (4.30) and (4.33) which does not mix the holomorphic and anti-holomorphic parts in the moduli T, U . This part of the gravitational coupling can be written in terms of Gopakumar-Vafa invariants and essentially counts holomorphic curves in the target space [18, 19]. Note the non-degenerate contribution given in (4.30) along includes its complex conjugate. It is clear that the degenerate orbit does indeed mixes the holomorphic and anti-holomorphic parts of the moduli, and therefore cannot be written as a purely holomorphic term. Examining the non-degenerate orbit in (4.30), we see that the holomorphic term can only arise from $j = t = a = 0$. Then only the function Li_{3-2g} contributes.

Now if $n_1 \geq 0, n_2 \geq 0$ then one obtains the sum

$$\sum_{h=0}^{2g-2} \frac{(2g-2)!}{h!(2g-2-h)!} (-1)^h (\text{sgn}(\text{Im}(m \cdot y)))^h = 0. \quad (4.34)$$

Thus for these lattice points there is no holomorphic contribution.⁸ There is a non-zero contribution from $n_1 < 0, n_2 < 0$ from the F_g as given in (4.30). One can then change the sum over n_1, n_2 by flipping their sign and the result is that one obtains a purely anti-holomorphic term.⁹ Thus the contribution from these lattice points can be written as a purely anti-holomorphic term. The lattice points $n_1 = 0, n_2 \leq 0$ as well as $n_1 < 0, n_2 = 0$ also contribute to terms that do not have any mixing.

The remaining contribution which does not have mixing arises from the lattice points

$$\begin{aligned} n_1 &= \frac{r}{N}, n_2 < 0, r > 0 & \text{if } \frac{r}{N} T_2 < n_2 U_2; \\ n_1 &= -\frac{r}{N}, n_2 > 0, r > 0 & \text{if } \frac{r}{N} T_2 > n_2 U_2; \end{aligned} \quad (4.35)$$

with $r|n_2| \leq N$.

Note that these lattice points also include $(-1, 1)$ and $(1, -1)$. The contribution from points $(n_1, n_2) = (r/N, n_2); n_2 < 0, r > 0$ is non-vanishing only for $r/NT_2 < n_2 U_2$ and result in $\text{Li}_{3-2g}(q)$ where $q = \exp[2\pi i(r/NT_1 - n_2 U_1) + 2\pi(r/NT_2 - n_2 U_2)]$. Similarly the contribution from the points $(n_1, n_2) = (r/N, n_2); n_2 > 0, r > 0$ is non-vanishing only for $r/NT_2 < n_2 U_2$ and results in $\text{Li}_{3-2g}(q^{-1})$. These two contributions can be combined using the relation

$$\text{Li}_a(z) = (-1)^{|a|+1} \text{Li}_a(z^{-1}). \quad (4.36)$$

⁸Similarly for $n_1 \leq 0, n_2 \leq 0$, then there is no anti-holomorphic contribution from the complex conjugate of F_g .

⁹A similar observation can be made for the complex conjugate of F_g .

which is valid for $a < 0$ into a single holomorphic function for the lattice points $(n_1, n_2) = (r/N, n_2)$; $n_2 > 0, r > 0$ and valid for any T_2, U_2 . Finally the last term in (4.30) contributes at $t = 0$ as a constant. We can perform the sum over $\tilde{s}$ using the identity

$$\sum_{\tilde{s}=0}^{g-1} \frac{(-1)^{\tilde{s}} (2g-2)! 2^{2(-2g+\tilde{s}+2)}}{(2s)!(g-\tilde{s}-1)!} \Gamma\left(-g+\tilde{s}+\frac{3}{2}\right) = \frac{1}{2} \sqrt{\pi} (-1)^{g-1}. \quad (4.37)$$

Using all these arguments we obtain the following expression for the purely holomorphic contribution to the gravitational couplings.

$$\begin{aligned} \bar{F}_g^{\text{hol}} &= \frac{(-1)^{g-1}}{\pi^2} \sum_{s=0}^{N-1} \\ &\times \left(\sum_{m>0} e^{-2\pi i n_2 s/N} c_{g-1}^{(r,s)}(m^2/2, 0) \text{Li}_{3-2g}(e^{2\pi i m \cdot y}) + \frac{1}{2} c_{g-1}^{(0,s)}(0, 0) \zeta(3-2g) \right). \end{aligned} \quad (4.38)$$

In the sum over lattice points, $m > 0$ refers to the following lattice points

$$\begin{aligned} n_1, n_2 &\geq 0, & \text{but } (n_1, n_2) &\neq (0, 0), \\ (r/N, -n_2), & & \text{with } n_2 > 0 \text{ and } r n_2 &\leq N. \end{aligned} \quad (4.39)$$

Though the holomorphic contribution was obtained by examining the situation for $g > 1$, the result can be extrapolated for $g = 1$ as well as $g = 0$. For $g = 1$ we have evaluated the gravitational coupling explicitly in (4.15). By examining this result we see that except for the constant, the holomorphic part in (4.38) agrees with (4.15) up to a normalization for $g = 1$. For this, note that from the definition of $\tilde{c}^{(r,s)}$ in (4.1) and $c_0^{(r,s)}(m^2/2, 0)$ in (4.29), they are related to each other by a normalization. The constant in (4.38) for $g = 1$ requires a regularization to agree with the constant in (4.15) proportional to $\kappa \sum_{s=0}^{N-1} \tilde{c}^{(0,s)}(0)$.

5 Gopakumar-Vafa invariants and conifold singularities

The $E_8 \times E_8$ heterotic string compactified on $K3 \times T^2$ is dual to type II A compactified on a known Calabi-Yau geometry [11] by $\mathcal{N} = 2$ string duality. The compactifications which are the focus of study in this paper are presumably dual to appropriate freely acting quotients of this space which needs to be discovered. However since we have evaluated the gravitational couplings $\bar{F}_g^{\text{hol}}$ from the heterotic side, we do have some topological information of the dual Calabi-Yau. To obtain this we cast the $\bar{F}_g^{\text{hol}}$ in to a form from which we can read out the Gopakumar-Vafa invariants associated with the Calabi-Yau. The generating functional for the F_g 's can be written as

$$\begin{aligned} F^{\text{GV}}(g_s, y) &= \sum_{g=0}^{\infty} F_g^{\text{GV}}(y) g_s^{2g-2}, \\ &= \sum_{g=0}^{\infty} \sum_{m>0} \sum_{d=1}^{\infty} n_m^g \frac{1}{d} \left(2 \sin \frac{d g_s}{2} \right)^{2g-2} e^{2\pi i d(m \cdot y)}. \end{aligned} \quad (5.1)$$

Here we have focused on only the $y = (T, U)$ moduli. In general they will include all the Kähler parameters of the dual Calabi-Yau. The quantities n_m^g are called Gopakumar-Vafa invariants and they are integer numbers which encodes the number of BPS states which arise in the compactification of the type IIA side of the theory. For a nice review see [41].¹⁰

It is not apriori obvious that the compactifications on orbifolds of $K3 \times T^2$ associated with moonshine symmetry we have considered here will admit integer Gopakumar-Vafa invariants. In this section we will observe that, this is indeed the case for low values of $m = (n_1, n_2)$ for genus 0, 1 and 2. We will then study the points in the moduli space at which the topological amplitude $\bar{F}_g^{\text{hol}}$ becomes singular. On the heterotic side, these are points at which the certain hyper multiplets become massless. While on the type II A side, these points are associated with the conifold singularity. We will determine the points in moduli space at which these singularities occur. We observe that the states which become massless at these points in moduli space occur only in the twisted sector for these compactifications and the strength is determined by the genus zero Gopakumar-Vafa invariant for these sectors.

5.1 Low lying coefficients of $\bar{F}_g^{\text{hol}}$

Before we proceed, it is useful to determine the properties of certain low lying coefficients $c_{g-1}^{(r,s)}$ that occur in $\bar{F}_g^{\text{hol}}$. For this we obtain equations relating the constant in (4.38) $\sum_{s=0}^{N-1} c_{g-1}^{(0,s)}(0,0)$ and the Fourier coefficients $\sum_{s=0}^{N-1} c^{(0,s)}(0)$ in the expansion (2.14). Note that as we are in the untwisted sector the Fourier expansion is in integer powers of q . To extract this coefficient we need to examine the terms independent of τ_2 in the expansion (4.29). Therefore we can ignore the τ_2 dependence in the generating function (3.13). The generating function reduces as

$$\begin{aligned} \sum_{k=0}^{\infty} \tilde{\lambda}^{2k} \mathcal{P}_{2k}(G_2, \dots, G_{2k}) &= - \left(\frac{2\pi\eta^3 \tilde{\lambda}}{\theta_1(\tilde{\lambda}, \tau)} \right)^2, \\ &= - \left(\frac{\pi \tilde{\lambda}}{\sin \pi \tilde{\lambda}} \right)^2 + 8\pi^2 \tilde{\lambda}^2 q + \dots, \\ &= \sum_{g=0}^{\infty} \frac{(2g-1)}{(2g)!} (-1)^g (2\pi \tilde{\lambda})^{2g} B_{2g} + 8\pi^2 \tilde{\lambda}^2 q + \dots, \end{aligned} \quad (5.2)$$

where B_{2g} are Bernoulli numbers. These can be related to the ζ function using

$$B_{2g} = (-1)^{g+1} \zeta(2g) \frac{2(2g)!}{(2\pi)^{2g}}. \quad (5.3)$$

Using (5.2) we see that

$$\mathcal{P}_0 = -1, \quad \mathcal{P}_2(G_2)|_q = 8\pi^2, \quad \mathcal{P}_{2g}(G_2, \dots, G_{2g})|_q = 0, \text{ for } g > 1. \quad (5.4)$$

Now in the untwisted sector, we have the expansion

$$f^{(0,s)}(q) = c^{(0,s)}(-1)q^{-1} + c^{(0,s)}(0) + c^{(0,s)}(1)q + \dots \quad (5.5)$$

¹⁰We thank Rajesh Gopakumar for bringing this review to our attention.

Now using the definition (4.29) and the equations (5.2), (5.4) and (5.5) we obtain

$$\begin{aligned} \sum_{s=0}^{N-1} c_{g-1}^{(0,s)}(0,0) &= \sum_{s=0}^{N-1} c^{(0,s)}(0) \frac{(2g-1)}{(2g)!} (-1)^g (2\pi)^{2g} B_{2g}, \\ &= \frac{\chi(X)}{2} \frac{(2g-1)}{(2g)!} (-1)^g (2\pi)^{2g} B_{2g}, \quad g > 1. \end{aligned} \quad (5.6)$$

In the second line we have used the definition of the Euler character. Now for $g = 1$ we obtain

$$\begin{aligned} \sum_{s=0}^{N-1} c_0^{(0,s)}(0,0) &= -2\pi^2 B_2 \sum_{s=0}^{N-1} c^{(0,s)}(0) + 8\pi^2 \sum_{s=0}^{N-1} c^{(0,s)}(-1), \\ &= -\frac{\pi^2}{12} \chi + 8\pi^2, \end{aligned} \quad (5.7)$$

where we have used $c^{(0,s)}(-1) = \frac{1}{N}$, which can be shown to be true for all the orbifolds. And finally for $g = 0$ we obtain

$$\sum_{s=0}^{N-1} c_{-1}^{(0,s)}(0,0) = - \sum_{s=0}^{N-1} c^{(0,s)}(0) = -\frac{\chi}{2}. \quad (5.8)$$

Still being within the untwisted sector and using the same methods we obtain the relation

$$\begin{aligned} \sum_{s=0}^{N-1} e^{\frac{2\pi i s}{N}} c_{g-1}^{(0,s)}(-1,0) &= \sum_{s=0}^{N-1} e^{\frac{2\pi i s}{N}} c^{(0,s)}(-1) \frac{(2g-1)}{(2g)!} (-1)^g (2\pi)^{2g} B_{2g}, \\ &= 0, \quad \text{for all } g. \end{aligned} \quad (5.9)$$

To obtain the last line we have used the fact that $c^{(0,s)}(-1) = \frac{1}{N}$ for all the orbifolds.

Let us now proceed similarly and relate the coefficients of the low lying twisted sector. From examining the expansions of $f^{(r,s)}$ defined in (2.14), we see from the explicit evaluation of the q -expansions that the non-zero leading terms are given by

$$\sum_{s=0}^{N-1} e^{2\pi i \frac{n_2 s}{N}} f^{(r,s)}(q) = \sum_{s=0}^{N-1} e^{2\pi i \frac{n_2 s}{N}} c^{(r,s)} \left(-\frac{rn_2}{N} \right) q^{-\frac{rn_2}{N}} + O(q^{1-\frac{rn_2}{N}}). \quad (5.10)$$

Here $r, n_2 > 0$ and $rn_2 \leq N$. Therefore using the same methods we obtain the relation

$$\sum_{s=0}^{N-1} e^{2\pi i \frac{n_2 s}{N}} c_{g-1}^{(r,s)} \left(-\frac{rn_2}{N}, 0 \right) = \sum_{s=0}^{N-1} e^{2\pi i \frac{n_2 s}{N}} c^{(r,s)} \left(-\frac{rn_2}{N} \right) \frac{(2g-1)}{(2g)!} (-1)^g (2\pi)^{2g} B_{2g}. \quad (5.11)$$

Let us define the following quantity

$$n_{\left(\frac{r}{N}, -n_2\right)}^0 \equiv -2 \sum_{s=0}^{N-1} e^{2\pi i \frac{n_2 s}{N}} c^{(r,s)} \left(-\frac{rn_2}{N} \right). \quad (5.12)$$

Evaluating this for all the orbifold compactification of the heterotic string considered in this paper and in each of the twisted sectors, we see that $n_{\left(\frac{r}{N}, -n_2\right)}^0$ is an integer. We will see that this quantity will turn out to be the genus zero Gopakumar-Vafa invariant at the point $m = \left(\frac{r}{N}, -n_2\right)$. We list them in appendix D.

5.2 Integrality of the Gopakumar-Vafa invariants

To begin we first write down the Gopakumar-Vafa form of the genus g topological amplitude from (5.1). For $g > 1$, this is given by

$$F_g^{\text{GV}} = \frac{(-1)^g |B_{2g} B_{2g-2}| \chi(X)}{4g(2g-2)(2g-2)!} + \sum_{\beta} \left[\frac{|B_{2g}| n_m^0}{2g(2g-2)!} + \frac{2(-1)^g n_m^2}{(2g-2)!} \pm \dots - \frac{g-2}{12} n_m^{g-1} + n_m^g \right] \text{Li}_{3-2g}(e^{2\pi i m \cdot y}). \quad (5.13)$$

Here we have included the constant term which is the contribution to the topological amplitude due to holomorphic maps from genus g surface to a single point. For $g = 0$ we have we obtain

$$F_0^{\text{GV}} = \zeta(3) \frac{\chi(X)}{2} + \sum_{m>0} n_m^0 \text{Li}_3(e^{2\pi i m \cdot y}), \quad (5.14)$$

where we can included the contribution due to the Euler characteristic of the Calabi-Yau target. Finally for $g = 1$ we have

$$F_1^{\text{GV}} = \sum_{m>0} \left(\frac{1}{12} n_m^0 + n_m^1 \right) \text{Li}_1(e^{2\pi i m \cdot y}). \quad (5.15)$$

Here we have kept only the instanton contribution to the topological amplitude. The reason we have not kept the constant is that we need to regulate the constant term in (4.38) to extract this contribution for $g = 1$ to compare with the expected results from the type II Calabi-Yau side. Note that on examining the constant term in the calculation done to obtain F_1 in (4.15) the regulator we used essentially involves subtracting the divergence in $\zeta(1)$ to obtain the Euler-Mascheroni constant.

We can now compare the constant term in (4.38) and (5.13) to fix the overall normalization relating $\bar{F}_g^{\text{hol}}$ and F_g^{GV} for $g > 1$. For this we need to use the relation (5.6) and identities relating $\zeta(3-2g)$ to Bernoulli numbers. Requiring the constant term to agree results in the following relation between the topological amplitudes

$$F_g^{\text{GV}} = \frac{(-1)^{g+1}}{2(2\pi)^{2g-2}} \bar{F}_g^{\text{hol}}. \quad (5.16)$$

Though this relation was obtained for $g > 1$ we will also extrapolate it for $g = 1, 0$. Using the relation in (5.16) we can relate the Gopakumar-Vafa invariants n_m^g to the coefficients $c_{g-1}^{(r,s)}(m^2/2, 0)$. We will now perform this explicitly for $g = 0, 1, 2, 3$ and will observe that the resulting Gopakumar-Vafa invariants are integral.

$g = 0$. For $g = 0$ we have the relation

$$F_0^{\text{GV}} = -2\pi^2 \bar{F}_0^{\text{hol}}. \quad (5.17)$$

We see that using (5.8), the constant term in (5.14) and (4.38) agrees. Furthermore we can identify

$$\begin{aligned} n_{(n_1, n_2)}^0 &= 2 \sum_{s=0}^{N-1} e^{-\frac{2\pi i n_2 s}{N}} c_{-1}^{(r, s)}(m^2/2, 0), \quad r = n_2 N \bmod N \\ &= -2 \sum_{s=0}^{N-1} e^{-\frac{2\pi i n_2 s}{N}} c^{(r, s)}(n_1 n_2). \end{aligned} \quad (5.18)$$

To obtain the last line we have used (4.29) and the fact that $\mathcal{P}_0 = -1$. Note that a special case of this identification is that given in (5.12). We explicitly evaluate the genus zero low lying Gopakumar-Vafa invariants for all the $\mathbb{Z}_N$ orbifolds of $K3 \times T^2$ considered in this paper and observe that these are integers. A listing of these are provided in tables.

Now we will demonstrate that if the n_m^0 invariants are integers, then it is guaranteed that Gopakumar-Vafa invariants for $g = 1, 2, 3$ are integers. This analysis most likely extends to all genera.

$g = 1$. For $g = 1$ the relation between the Gopakumar-Vafa form of the topological amplitude and that obtain from the threshold integral is given by

$$F_1^{\text{GV}} = \frac{1}{2} \bar{F}_1^{\text{hol}}. \quad (5.19)$$

Comparing the instanton contributions of (4.38) and (5.15), we see that we can identify

$$n_{(n_1, n_2)}^1 = \frac{1}{2\pi^2} \sum_{s=0}^N e^{-\frac{2\pi i n_2 s}{N}} c_0^{(r, s)}(m^2/2, 0) - \frac{1}{12} n_{(n_1, n_2)}^0. \quad (5.20)$$

Now using (5.18) and the definition (4.29) for $g = 0, 1$ together with the relation

$$\mathcal{P}_2(G_2) = -2 \frac{\pi^2}{6} E_2, \quad (5.21)$$

we find that the genus one Gopakumar-Vafa invariants are guaranteed to be integral if the following are satisfied.

1. The Fourier coefficients of the following quasi-modular form are integral.

$$-\frac{1}{6}(E_2(\tau) - 1) = 4 \sum_{l=1}^{\infty} \sigma_1(l) q^l. \quad (5.22)$$

Here $\sigma_1(l)$ is the sum of divisors of l .

2. The genus zero Gopakumar-Vafa invariants n_m^0 are integers.

The condition (1) is assured by the definition of $E_2(\tau)$. We observe that the condition (2) holds true for all the orbifolds we consider. Therefore the genus one Gopakumar-Vafa invariants are assured to be integers.

$g = 2$. For this case the relation between the amplitudes becomes

$$F_2^{\text{GV}} = -\frac{1}{8\pi^2} \bar{F}_2^{\text{hol}}. \quad (5.23)$$

Comparing the instanton contributions it is easy to see that

$$n_{(n_1, n_2)}^2 = \frac{1}{8\pi^4} \sum_{s=0}^{N-1} e^{-\frac{2\pi i n_2}{N}} c_1^{(r,s)}(m^2/2, 0) - \frac{|B_4|}{8} n_{(n_1, n_2)}^0. \quad (5.24)$$

Using the definitions we get

$$\mathcal{P}_4 = -\pi^4 \left(\frac{1}{18} E_2^2 + \frac{1}{90} E_4 \right). \quad (5.25)$$

Then from (5.18) and the definition (4.29) for $g = 0, 2$ we can see that n_m^2 are integers provided the genus zero invariants n_m^0 are integral and the following quasi-modular form admits integral Fourier coefficients

$$\frac{1}{720} (6 - 5E_2^2 - E_4) = \sum_l n_l q^l. \quad (5.26)$$

For $n_l \in \mathbb{Z}$, we must have that the Fourier coefficients of $(6 - 5E_2^2 - E_4)$ be divisible by 720. This can be verified easily from the definition of the Eisenstein series

$$6 - 5E_2^2 - E_4 = 240 \sum_k (\sigma_1(k) - \sigma_3(k)) q^k + 2880 \sum_{k_1, k_2} \sigma_1(k_1) \sigma_1(k_2) q^{k_1 + k_2}. \quad (5.27)$$

The last term is obviously divisible by 720. The first term can be written as

$$240 \sum_k (\sigma_1(k) - \sigma_3(k)) = 240 \sum_{d|k} (d-1)(d)(d+1). \quad (5.28)$$

Now $(d-1)(d)(d+1)$ is always a multiple of 6 and therefore the first term is also divisible by 720. Thus the integrality of n_m^2 is assured by the integrality of n_m^0 .

$g = 3$. The relation between the topological amplitudes is given by

$$F_3^{\text{GV}} = \frac{1}{32\pi^4} \bar{F}_g^{\text{hol}}. \quad (5.29)$$

Going through a similar analysis for $g = 0$ it can be seen that the integrality of n_m^3 is ensured by the integrality of n_m^0 and the integrality of the Fourier coefficient of the following quasi-modular form

$$\begin{aligned} & \frac{1}{2} \left[\frac{1}{3024} - \frac{E_2 E_4}{270 \times 32} - \frac{E_6}{48 \times 945} - \frac{E_2^3}{6 \times 27 \times 32} + \frac{1}{12 \times 720} (6 - 5E_2^2 - E_4) \right] \\ & = 4q^3 + 30q^2 + 120q^5 + 350q^6 + 840q^7 + O(q^8). \end{aligned} \quad (5.30)$$

In the second line we have performed the q -expansion and retained a few terms to demonstrate that the Fourier coefficients are integers.

5.3 Conifold singularities

In the heterotic description, conifold singularities are points in moduli space where vectors becomes massless. This will be reflected by singularities in the gravitational threshold. Given the topological amplitude $\bar{F}_g^{\text{hol}}$, we can determine where the function becomes singular. Note that the poly-logarithm function $\text{Li}_a(z)$ is singular at $z = 1$. The leading pole is for $a < 0$

$$\text{Li}_a(z) = \left(z \frac{d}{dz} \right)^{|a|} \frac{1}{1-z} \Big|_{z \rightarrow 1} \sim a! \frac{1}{(1-z)^{|a|+1}}. \quad (5.31)$$

Therefore the leading divergence for $g > 1$ of $\bar{F}_g^{\text{hol}}$, is given by

$$\bar{F}_g^{\text{hol}}|_{m \cdot y \rightarrow 0} \sim \frac{(-1)^{g-1}}{\pi^2} \sum_{s=0}^{N-1} e^{-2\pi i n_2 s/N} c_{g-1}^{(r,s)}(m^2/2, 0) (2g-3)! \frac{1}{\{1 - e^{2\pi i m \cdot y}\}^{2g-2}}. \quad (5.32)$$

Note that we have removed the sum over the lattice point, since we are looking at lattice points where $m \cdot y = 0$. This requirement results in the equations

$$n_1 T_1 + n_2 U_1 = 0, \quad n_1 T_2 + n_2 U_2 = 0. \quad (5.33)$$

Since $T_2, U_2 > 0$ and $m = (n_1, n_2) > 0$ we must have $n_2 < 0$. Therefore in the untwisted sector a possible singularity lies at

$$m = (1, -1), \quad m^2/2 = -1 \quad (5.34)$$

and in the twisted sector they lie at

$$m = \left(\frac{r}{N}, -n_2 \right), \quad r, n_2 > 0, \quad \frac{m^2}{2} = -\frac{r n_2}{N}, \quad r n_2 \leq N. \quad (5.35)$$

However from (5.9) we see that the sum

$$\sum_{s=0}^{N-1} e^{2\pi i s/N} c_{g-1}^{(0,s)}(-1, 0) = 0. \quad (5.36)$$

Therefore there is no conifold singularity from the untwisted sector. In the twisted sector using (5.11) and (5.12) we see that the sum

$$\sum_{s=0}^{N-1} e^{2\pi i n_2 s/N} c_{g-1}^{(r,s)} \left(-\frac{r n_2}{N}, 0 \right) = n_{(\frac{r}{N}, n_2)}^0 \times \frac{2g-1}{2(2g)!} (-1)^g (2\pi)^g B_{2g}. \quad (5.37)$$

Now substituting this result into (5.32) we obtain

$$\begin{aligned} \bar{F}_g^{\text{hol}}|_{m \cdot y \rightarrow 0} &\sim -\frac{1}{2\pi^2} (2\pi)^g \frac{B_{2g}}{2g(2g-2)} \times n_{(\frac{r}{N}, -n_2)}^0, \\ &= \frac{1}{2\pi^2} (2\pi)^g \chi(\mathcal{M}_g) \times n_{(\frac{r}{N}, -n_2)}^0. \end{aligned} \quad (5.38)$$

In the second line we have used the definition of the Euler characteristic of the moduli space of genus g Riemann surfaces.

Thus these models illustrate that the strength of the conifold singularity is also determined by the genus zero Gopakumar-Vafa invariant at that lattice point. For the unorbifolded $K3 \times T^2$ compactification studied in [10], this is not transparent. This is because, for this case the conifold singularity lies at the lattice point $m = (1, -1)$ and $n_{(1,-1)}^0 = -2$.

There is one more observation of interest. From the relation between the Gopakumar-Vafa form of the topological amplitude and that obtained from the threshold integral in (5.16) together with (5.37) We can show that at the lattice points corresponding to the conifold singularity given in (5.35) all the higher genus topological Gopakumar-Vafa invariants vanish, that is

$$n_{(r/N, -n_2)}^g = 0, \quad \text{for } r, n_2, g > 0 \text{ } rn_2 \leq N. \quad (5.39)$$

Let us end this section by a discussion on the role of M_{24} on organising Gopakumar-Vafa invariants. By the explicit evaluation of the gravitational threshold F_g for the orbifolds considered in this paper we have shown that the direct relation between the coefficients of the twisted elliptic genus and the Gopakumar-Vafa invariants. This is seen by comparing the equations (1.5) and (1.7). We see that it is indeed the coefficients of the twisted elliptic genus of $K3$ corresponding to the $\mathbb{Z}_N$ orbifold which are the input for the Gopakumar-Vafa invariants. As far as we are aware this direct relation of the twisted elliptic genus to the Gopakumar-Vafa invariants in these $\mathbb{Z}_N$ orbifold compactifications of the heterotic theory is new. This observation fits well other observations and proposals in the literature. For example the generating functions for refined Gopakumar-Vafa invariants of $K3$ is conjectured to be the generating function for the Hodge polynomials of the symmetric product of $K3$ [42]. In fact in [43] there is a generalization of this conjecture to when the $K3$ is orbifolded by the action of a discrete group g' just as considered in this paper. Our calculation of F_g for the orbifold compactifications and the observation that the Gopakumar-Vafa invariants are essentially determined by the twisted elliptic of $K3$ supports these conjectures. However we are not claiming that this observation implies that the M_{24} symmetry organises the BPS states in these compactifications. It has been noted recently that this link is difficult to establish [44]. One reason that we evaluated the Gopakumar-Vafa invariants explicitly for the orbifolds on the heterotic side is to extract the topological data which will help to identify the dual Calabi-Yau geometry corresponding to these orbifolds on the type II side.

6 Conclusions

The evaluation of F_g for the compactification of the $E_8 \times E_8$ heterotic string on $\mathbb{Z}_N$ orbifolds of $K3 \times T^2$ by g' generalizes the computation first done in [10] for the unorbifolded model. The results of this paper show that it is the twisted elliptic genus of $K3$ by g' which forms the crucial input which determines F_g in these models. It is in this sense M_{24} symmetry plays a role in these compactifications. Our analysis was restricted to the standard embedding. It will be interesting to generalize these results to nonstandard embeddings. Certain non-standard embeddings that were studied in [29] can form a starting point. Recently gauge threshold corrections and the F_1 gravitational threshold for $K3 \times T^2$ model with

non-zero 3-form flux were evaluated [45]. It will be also interesting to generalize this to the orbifolds considered here.

Recently there has been several studies of $\mathcal{N} = 4$ CHL models realized as orbifolds of type IIA compactified on $K3 \times T^2$ which is dual to orbifolds of heterotic on T^6 . These models have been shown to exhibit an enhanced Fricke symmetry that is beyond the expected S -duality symmetry $\Gamma_0(N)$ of these models [46, 47]. This symmetry has also been seen at the level of BPS saturated $\mathcal{N} = 4$ couplings in [48]. It is interesting to investigate if the $\mathcal{N} = 2$ compactifications considered in this paper also exhibit this symmetry.¹¹ Since we are examining BPS saturated thresholds in the heterotic frame, this symmetry should reflect as an enhanced T -duality corresponding to the torus T^2 for which the $\Gamma_0(N)$ symmetry is manifest. As a preliminary step we examined the difference in gauge thresholds evaluated in [20] for the $\mathbb{Z}_2$ orbifold and did indeed discover that $U \rightarrow -1/2U$ to be a symmetry.¹² This symmetry is the analog of the Fricke symmetry observed in the $\mathcal{N} = 4$ models. It is interesting to investigate this symmetry in detail for both the gravitational as well as the gauge thresholds for the $\mathcal{N} = 2$ models studied in this paper.

Another interesting detail worth studying is the property of the gravitational threshold F_g given as the sum of (4.30) and (4.33) across walls of marginal stability. In [10], it was shown that upto genus five, the corresponding gravitational threshold for the $K3 \times T^2$ model was a differentiable function across these walls. This involved details of the expansion coefficients $c_{g-1}(l, t)$. It will be interesting to repeat this analysis for the models considered here.

What is perhaps more interesting regarding the orbifolds of $K3 \times T^2$ inspired by M_{24} symmetry is the prediction of dual Calabi-Yau geometries on the type II A side. The analysis in this paper predicts the Euler numbers for each of these $K3$ fibered Calabi-Yau geometries. It also predicts the Gopakumar-Vafa invariants for the limit when the base T^2 of these geometries is large. It will be interesting to explicitly construct these Calabi-Yau. In this context it will be interesting to look at freely acting quotients of Calabi-Yau geometries such as that considered in [49] as well as Calabi-Yau obtained by non-geometric $K3$ fibrations in [50].

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¹¹We thank the referee for raising this point.

¹²This is essentially due to the fact that the U dependence in the threshold appears as the combination $\log(\eta(U)\eta(2U))$.

A Modular transformations

Modular properties result from the basic formula for Poisson resummation which is given by

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dy f(y) e^{2\pi i m y}. \quad (\text{A.1})$$

Using the above formulae on m_1, m_2 we arrive at

$$\sum_{m_1, m_2, n_1, n_2} p_R^{2k} q^{\frac{|p_L|^2}{2}} \bar{q}^{\frac{|p_R|^2}{2}} = \sum_{n_1, k_1, n_2, k_2} \left(\frac{-2\pi T_2}{\tau_2 \sqrt{2T_2 U_2}} \right)^{2k} (n_1 \tau + k_1 + U(n_2 \tau + k_2))^{2k} e^{\mathcal{G}}, \quad (\text{A.2})$$

where $\mathcal{G}$ is given by

$$\mathcal{G} = -\frac{\pi T_2}{U_2 \tau_2} |\mathcal{A}|^2 - 2\pi i T \det(A), \quad (\text{A.3})$$

where

$$A = \begin{pmatrix} n_1 & k_1 \\ n_2 & k_2 \end{pmatrix} \quad (\text{A.4})$$

$$\mathcal{A} = \begin{pmatrix} 1 & U \end{pmatrix} A \begin{pmatrix} \tau \\ 1 \end{pmatrix}. \quad (\text{A.5})$$

Using the Fourier transformation formulae one can now do the Poisson re-summation on all the four variables m_1, m_2, n_1, n_2 in each of the (r, s) sectors and arrive at the following results:

$$\begin{aligned} \Gamma_{2,2}^{(0,0)} \left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right) &= \tau \bar{\tau} \Gamma_{2,2}^{(0,0)}(\tau, \bar{\tau}), \\ \Gamma_{2,2}^{(0,s)} \left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right) &= \tau \bar{\tau} \Gamma_{2,2}^{(s,0)}(\tau, \bar{\tau}), \\ \Gamma_{2,2}^{(r,s)} \left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right) &= \tau \bar{\tau} \Gamma_{2,2}^{(s, N-r)}(\tau, \bar{\tau}). \end{aligned} \quad (\text{A.6})$$

Now changing the variable $\tau \rightarrow -\frac{1}{\tau}$ and $\bar{\tau} \rightarrow -\frac{1}{\bar{\tau}}$ in A.2 and performing a Poisson resummation where we take all the possible values of n_1, n_2, k_1, k_2 from $-\infty$ to $+\infty$ one can write the expression as

$$\begin{aligned} &\sum_{n_1, k_1, n_2, k_2 \in \mathbb{Z}} e^{\mathcal{G}(\frac{-1}{\tau}, \frac{-1}{\bar{\tau}})} \left(\frac{-2\pi T_2 \tau \bar{\tau}}{\tau_2 \sqrt{2T_2 U_2}} \right)^{2k} \left(\frac{-n_1}{\tau} + k_1 + U \left(-\frac{n_2}{\tau} + k_2 \right) \right), \\ &= \sum_{n_1, k_1, n_2, k_2 \in \mathbb{Z}} e^{\mathcal{G}(\tau, \bar{\tau})} \left(\frac{-2\pi T_2}{\tau_2 \sqrt{2T_2 U_2}} \right)^{2k} (-n_1 \tau + k_1 + U(-n_2 \tau + k_2)) \frac{(\tau \bar{\tau})^{2k+1}}{\tau^{2k}}, \\ &= \sum_{n_1, k_1, n_2, k_2 \in \mathbb{Z}} p_R^{2k} e^{\mathcal{G}} \frac{(\tau \bar{\tau})^{2k+1}}{\tau^{2k}} \end{aligned} \quad (\text{A.7})$$

Now assuming that n_1, k_1 are instead shifted from integers by r/N and s/N the roles of n_1, k_1 would be reversed in the re-summed result, thus the shifts would become s/N for n_1 and $(N-r)/N$ for k_1 as expected. Thus we have,

$$\begin{aligned} & \sum_{\substack{n_1 \in \mathbb{Z}+r/N, \\ k_1 \in \mathbb{Z}+s/N, \\ n_2, k_2 \in \mathbb{Z}}} e^{\mathcal{G}(\frac{-1}{\tau}, \frac{-1}{\bar{\tau}})} \left(\frac{-2\pi T_2 \tau \bar{\tau}}{\tau_2 \sqrt{2T_2 U_2}} \right)^{2k} \left(\frac{-n_1}{\tau} + k_1 + U \left(-\frac{n_2}{\tau} + k_2 \right) \right) \\ &= \sum_{\substack{n_1 \in \mathbb{Z}+s/N, \\ k_1 \in \mathbb{Z}+(N-r)/N, \\ n_2, k_2 \in \mathbb{Z}}} e^{\mathcal{G}(\tau, \bar{\tau})} \left(\frac{-2\pi T_2}{\tau_2 \sqrt{2T_2 U_2}} \right)^{2k} (-n_1 \tau + k_1 + U(-n_2 \tau + k_2)) \frac{(\tau \bar{\tau})^{2k+1}}{\tau^{2k}}. \end{aligned} \quad (\text{A.8})$$

The above equations lead to,

$$\sum_{m_1, m_2, n_1, n_2 \in \mathbb{Z}} (p_R^{(0,0)})^{2k} e^{-\frac{2\pi i}{\tau} \frac{|p_L|^2}{2}} e^{\frac{2\pi i}{\bar{\tau}} \frac{|p_R|^2}{2}} = \quad (\text{A.9})$$

$$\begin{aligned} &= \tau \bar{\tau}^{2k+1} \sum_{m_1, m_2, n_1, n_2 \in \mathbb{Z}} (p_R^{(0,0)})^{2k} q^{\frac{|p_L|^2}{2}} \bar{q}^{\frac{|p_R|^2}{2}}, \\ &\sum_{m_1, m_2, n_1, n_2 \in \mathbb{Z}} (p_R^{(0,s)})^{2k} e^{2\pi i m_1 s/N} e^{-\frac{2\pi i}{\tau} \frac{|p_L|^2}{2}} e^{\frac{2\pi i}{\bar{\tau}} \frac{|p_R|^2}{2}} = \quad (\text{A.10}) \\ &= \tau \bar{\tau}^{2k+1} \sum_{\substack{m_1, m_2, n_2 \in \mathbb{Z}, \\ n_1 \in \mathbb{Z}+r/N}} (p_R^{(s,0)})^{2k} q^{\frac{|p_L|^2}{2}} \bar{q}^{\frac{|p_R|^2}{2}}, \\ &\sum_{\substack{m_2, n_2 \in \mathbb{Z} \\ m_1 \in \mathbb{Z}+s/N \\ n_1 \in \mathbb{Z}+r/N}} (p_R^{(r,s)})^{2k} e^{2\pi i m_1 s/N} e^{-\frac{2\pi i}{\tau} \frac{|p_L|^2}{2}} e^{\frac{2\pi i}{\bar{\tau}} \frac{|p_R|^2}{2}} = \\ &= \tau \bar{\tau}^{2k+1} \sum_{\substack{m_1, m_2, n_2 \in \mathbb{Z} \\ n_1 \in \mathbb{Z}+s/N}} (p_R^{(s, N-r)})^{2k} e^{2\pi i m_1 (N-r)/N} q^{\frac{|p_L|^2}{2}} \bar{q}^{\frac{|p_R|^2}{2}}. \end{aligned}$$

Here $p_R^{(r,s)}$ refers to

$$p_R^{(r,s)} = \left(\frac{-2\pi T_2}{\tau_2 \sqrt{2T_2 U_2}} \right) (n_1 \tau + k_1 + U(n_2 \tau + k_2)), \quad (\text{A.11})$$

where $n_1 \in \mathbb{Z} + r/N$, $k_1 \in \mathbb{Z} + s/N$, $n_2, m_2 \in \mathbb{Z}$.

Note that these are the modular properties of lattice sectors that are required to ensure that the integrand involved in evaluating the gravitational threshold is an invariant function under $\text{SL}(2, \mathbb{Z})$ transformations.

B Details for $g = 1$ threshold integral

As discussed in section 4.1, the one loop integral can be reduced to the integral given in (4.11). This then receives contribution from three inequivalent orbits which are given by

1. The zero orbit, ie, $A = 0$,
2. The non-degenerate orbit, ie, $A = \begin{pmatrix} n_1 & j \\ 0 & p \end{pmatrix}$, $p \neq 0, \in \mathbb{Z}$, $n_1, j \in \frac{\mathbb{Z}}{N}$, $n_1 > j$.
3. The degenerate orbit, $A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}$, $j \in \frac{\mathbb{Z}}{N}$, $p \in \mathbb{Z}$.

We write the contributions as

$$\begin{aligned} \mathcal{I} &= \sum_A' \int_{\mathcal{F}_A} \frac{d^2\tau}{\tau_2^2} T_2 \exp\left(-\frac{\pi T_2}{U_2 \tau_2} |\mathcal{A}|^2 - 2\pi i T \det A\right) f^{(r,s)}(\tau) \hat{E}_2(\tau), \\ &= \mathcal{I}^{\text{zero}} + \mathcal{I}^{\text{nondegen}} + \mathcal{I}^{\text{degen}}. \end{aligned} \quad (\text{B.1})$$

We now discuss in detail the contribution from each of the orbits.

Zero orbit. For $A = 0$ we have $(r, s) = (0, 0)$. Now note that $f^{(0,0)} \times \hat{E}_2$ is an almost modular form of weight zero. The integral is over the fundamental domain. Then the only contribution to the integral comes from

infinity [51]. The result for integrals of this form is given by the formula

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(q) \hat{G}_2^k(q) = \frac{1}{\pi(k+1)} G_2^{k+1}(q) F(q) \Big|_{q^0}, \quad (\text{B.2})$$

where $\hat{G}_2(q) = 2\zeta(2)(E_2 - \frac{3}{\pi\tau_2})$. This gives the first term of 4.15 which is $\frac{\pi}{6} T_2 E_2^2 f^{(0,0)} \Big|_{q^0}$. Using the explicit expressions for $f^{(0,0)}$ for each of the orbifolds considered in this paper we obtain

$$\frac{\pi}{6} T_2 E_2^2 f^{(0,0)}(q) \Big|_{q^0} = -48\pi \frac{T_2}{N}. \quad (\text{B.3})$$

Note that the analysis is done based on $T_2 > U_2$ limit. Therefore we have

$$\mathcal{I}^{\text{zero}} = \frac{\pi}{6} T_2 E_2^2 f^{(0,0)}(q) \Big|_{q^0} = -48\pi \frac{T_2}{N}. \quad (\text{B.4})$$

Non-degenerate orbit. For the non-degenerate orbit we apply the unfolding technique, choose a representative of the matrix A and integrate over two copies of the upper half plane. In this case A becomes

$$A = \begin{pmatrix} n_1 & j \\ 0 & p \end{pmatrix}, \quad p > 0, \in \mathbb{Z}, \quad n_1, j \in \frac{\mathbb{Z}}{N}, \quad n_1 > j \geq 0. \quad (\text{B.5})$$

The restriction to positive values in A comes due to the fact that the negative values contribute equally and this is taken care of by the two copies of the upper half plane. We have $\det(A) = n_1 p$ and $\mathcal{A} = n_1 \tau + j + pU$, where p is non-zero. The integration domain $\mathcal{F}_A$ is given by

$$-\infty < \tau_1 < \infty, \quad 0 \leq \tau_2 < \infty. \quad (\text{B.6})$$

In the $(0,0)$ sector one has and $j < n_1$, $j = 0, 1, \dots, n_1 - 1$ and $p \in \mathbb{Z}$. So we can proceed for the τ_1 integral as done in [9] for the un-orbifolded $K3$. We write $\tau'_1 = j + pU_1 + n_1\tau_1$, then the relevant part for the τ_1 integration in (B) becomes

$$\hat{I}(\tau_1) = T_2 \int \frac{d^2\tau}{\tau_2^2} \exp \left\{ 2\pi i n \tau_1 - \frac{\pi T_2}{U_2 \tau_2} \tau_1'^2 \right\} \left(\tilde{c}^{(0,0)}(n) - \frac{3}{\pi \tau_2} c^{(0,0)}(n) \right). \quad (\text{B.7})$$

Here we have focused on the τ_1 dependent terms, and kept a generic term using the Fourier expansion of $f^{(0,0)}(q)\hat{E}_2(\tau)$.

Replacing τ_1 by τ'_1 we see that the only j dependence comes from

$$e^{2\pi i n \frac{1}{n_1}(\tau'_1 - j - pU_1)}. \quad (\text{B.8})$$

Now performing the sum over j forces $n_1 n_2 = n$ where n_2 is an integer. Rest of the τ_1 integration in the untwisted sector is Gaussian and the result is given by,

$$\begin{aligned} T_2 \int_{-\infty}^{\infty} \frac{d\tau'_1}{\tau_2^2} \exp \left\{ 2\pi i n_2 \tau'_1 - \frac{\pi T_2}{U_2 \tau_2} \tau_1'^2 \right\} & \left(\tilde{c}^{(0,0)}(n_1 n_2) - \frac{3}{\pi \tau_2} c^{(0,0)}(n_1 n_2) \right), \\ &= \frac{\sqrt{T_2 U_2 \tau_2}}{\tau_2^2} \left(\tilde{c}^{(0,0)}(n_1 n_2) - \frac{3}{\pi \tau_2} c^{(0,0)}(n_1 n_2) \right) e^{-\pi n_2^2 \tau_2 U_2 / T_2}, \\ &= \frac{\sqrt{T_2 U_2 \tau_2}}{\tau_2^2} \left(\tilde{c}^{(0,0)}(m^2/2) - \frac{3}{\pi \tau_2} c^{(0,0)}(m^2/2) \right) e^{-\pi n_2^2 \tau_2 U_2 / T_2}. \end{aligned} \quad (\text{B.9})$$

If we are in the untwisted sectors $(0, s)$, we have $j = j' + s/N$ with j' being an integer the above result holds but with an additional factor of $e^{-2\pi i n_2 s/N}$ and replacing $c^{(0,0)}(n_1 n_2)$ and $\tilde{c}^{(0,0)}(n_1 n_2)$ by $c^{(0,s)}(n_1 n_2)$ and $\tilde{c}^{(0,s)}(n_1 n_2)$.

Let us now look at the τ_2 integration. The τ_2 integrand is given by,

$$\mathcal{I}_2 = \sqrt{T_2 U_2} e^{-2\pi i n_2 s/N} \int_0^{\infty} \frac{d\tau_2}{\tau_2^{3/2}} \left(\tilde{c}^{(0,s)}(m^2/2) - \frac{3}{\pi \tau_2} c^{(0,s)}(m^2/2) \right) \exp\{\mathcal{F}\}, \quad (\text{B.10})$$

where $\mathcal{F}$ is given by,

$$\mathcal{F} = -2\pi \tau_2 n_2 n_1 - \frac{\pi T_2}{U_2 \tau_2} (n_1 \tau_2 + pU_2)^2 - 2\pi i T n_1 p - 2\pi p n_2 U_1 - \frac{\pi n_2^2 U_2 \tau_2}{T_2}. \quad (\text{B.11})$$

The τ_2 integration is of the Bessel form and we use the following formulae to evaluate it,

$$\begin{aligned} \int \frac{dx}{x^{3/2}} e^{-ax+b/x} &= \sqrt{\pi/b} e^{-2\sqrt{ab}}, \\ \int \frac{dx}{x^{5/2}} e^{-ax+b/x} &= \sqrt{\pi} e^{-2\sqrt{ab}} \left(\frac{1 + 2\sqrt{ab}}{2b^{3/2}} \right). \end{aligned} \quad (\text{B.12})$$

The result of this exercise yields for the untwisted sectors

$$\begin{aligned} \mathcal{I}^{\text{nondegen}}|_{\text{untwisted}} &= 4\text{Re} \sum_{\substack{n_1 > 0, n_2 \geq -1 \\ n_1 n_2 \neq 0 \\ n_1 \in \mathbb{Z}}} \left[\tilde{c}^{(0,s)}(m^2/2) e^{-2\pi i s n_2/N} \text{Li}_1(e^{2\pi i (n_1 T + n_2 U)}) \right. \\ &\quad \left. - \frac{3}{\pi T_2 U_2} c^{(0,s)}(m^2/2) e^{-2\pi i s n_2/N} \tilde{\mathcal{P}}(n_1 T + n_2 U) \right], \end{aligned}$$

where $\tilde{\mathcal{P}}$ is given by

$$\tilde{\mathcal{P}}(x) = \text{Im}(x) \text{Li}_2(e^{2\pi i x}) + \frac{1}{2\pi} \text{Li}_3(e^{2\pi i x}). \quad (\text{B.13})$$

Let us go over to the twisted sectors. First consider an order N orbifold when N is prime. In the twisted sectors, the n in the Fourier coefficients $c^{(r,s)}(n)$ and $\tilde{c}^{(r,s)}(n)$ is such that $n \in \mathbb{Z}/N$. While $j = j' + s/N$, $j' \in \mathbb{Z}$ and $n_1 \in r/N + \mathbb{Z}$. The modular forms $f^{(r,s)}$ has the property that the transformations of $\tau \rightarrow \tau + 1$ can be used to relate it to $f^{(r,0)}$.

After doing this, one effectively works in the $(r,0)$ twisted sector but now j runs from 0 to $n_1 - 1/N$ in steps of $1/N$.¹³ Thus from (B.8) we see that the sum over all the values of j for a prime N would be Nn_1 iff $n/(n_1N) = n_2$ is an integer, else it is zero. Therefore $n = n_1n_2N$, only integral values of n are picked up. However one observes that the coefficients of q^n in $f^{(r,s)}$ are related to the coefficients of q^n in $f^{(r,0)}$ under transformation $\tau \rightarrow \tau + 1$. If n is fractional the relation is through a multiplicative phase. Using this property one can write the result in terms of the Fourier coefficients in the $f^{(r,s)}$ sector.

The final result for a prime N for the non-degenerate orbit can be given by

$$\begin{aligned} \mathcal{I}^{\text{nondegen}}|_{\text{twisted}} = 4\text{Re} \sum_{\substack{n_1=0, n_2 \geq -1 \\ n_1n_2 \neq 0 \\ n_1 \in \mathbb{Z} + r/N}} \left[\tilde{c}^{(r,s)}(m^2/2) e^{-2\pi i s n_2/N} \text{Li}_1(e^{2\pi i(n_1T + n_2U)}) \right. \\ \left. - \frac{3}{\pi T_2 U_2} c^{(r,s)}(m^2/2) e^{-2\pi i s n_2/N} \tilde{\mathcal{P}}(n_1T + n_2U) \right]. \end{aligned} \quad (\text{B.14})$$

The above argument also holds when N is composite and the twisted sector r is such that r, N are co-prime. However if r divides N , then there are various sub-orbits for the sectors (r,s) under transformations $\tau \rightarrow \tau + 1$. For example if $N = 4$, then the sectors $(2,0), (2,2)$ and the sectors $(2,1), (2,3)$ form distinct sub-sectors not related by transformations of the kind $\tau \rightarrow \tau + 1$. One then uses the argument developed for N prime and carries it out for each of the sub-orbits. The end result works out to be the same as that given in (B.14).

Degenerate orbit. Here the determinant of A is 0. So we choose the matrix A as

$$A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix},$$

where $j \in \mathbb{Z}/N, p \in \mathbb{Z}$ and $(j,p) \neq (0,0)$. Since $n_1 = 0$ twisted sectors don't contribute to the degenerate orbit. There are logarithmic divergences due to the constant term in the expansion of $f^{(0,s)} E_2(\tau)$ which needs to be renormalized. We have

$$\hat{\mathcal{I}} = \sum_{A', n, s} \int_{\mathcal{F}_A} \frac{d^2\tau}{\tau_2^2} T_2 \exp\left(-\frac{\pi T_2}{U_2 \tau_2} |\mathcal{A}|^2\right) \left(\tilde{c}^{(0,s)}(n) - \frac{3}{\pi \tau_2} c^{(0,s)}(n) \right) q^n, \quad (\text{B.15})$$

where $|\mathcal{A}|^2 = |j + pU|^2$, $p \in \mathbb{Z}$, $j \in \mathbb{Z} + s/N$ in $(0,s)$ sector. To regularize the result we multiply the integrand for the $\tilde{c}^{(0,s)}$ with a factor of $(1 - e^{-\Lambda/\tau_2})$.

¹³See [38] for a discussion of this step for integrands which involve a $\Gamma_0(2)$ from.

The integration domain $\mathcal{F}_A$ in this orbit is given by

$$-1/2 \leq \tau_1 < 1/2, \quad 0 \leq \tau_2 < \infty. \quad (\text{B.16})$$

The only dependence of τ_1 comes from the q^n and since it is integrated from $-1/2 \leq \tau_1 < 1/2$ it picks up only the q^0 term from $f^{(0,s)} \hat{E}_2$. Performing the τ_2 integration for the term containing the coefficient $\tilde{c}^{(0,s)}$ we obtain

$$\tilde{c}^{(0,s)}(0) \left[\frac{U_2}{\pi} \left(\sum_{j,p} \frac{1}{|j+pU|^2} - \frac{1}{|j+pU|^2 + \Lambda U_2/(\pi T_2)} \right) - \int_{\mathcal{F}_A} \frac{d\tau_2}{\tau_2} (1 - e^{-\Lambda/\tau_2}) \right]. \quad (\text{B.17})$$

The result of the integral from the second term is given by

$$- \tilde{c}^{(0,s)}(0) \left(\log(\Lambda) + \gamma_E + 1 + \log(2/3\sqrt{3}) \right). \quad (\text{B.18})$$

Now for coefficient of $c^{(0,s)}$ one gets after integrating τ_2 the result

$$- \sum_{j,p} \frac{6}{\pi^3} \frac{U_2^2}{T_2} c^{(0,s)} \frac{1}{|j+pU|^4}. \quad (\text{B.19})$$

Note that here $j = j' + s/N$ where $j' \in \mathbb{Z}$. This is the difference as compared to the calculations of [9]. To evaluate these sums on j, p we need to divide the sum in $j' \neq 0$, $p = 0$ and $p > 0$, $j' \in \mathbb{Z}$ and use the following results [9]

$$\sum_{j' \in \mathbb{Z}} \frac{1}{(j' + B)^2 + C^2} = \frac{\pi}{C} \left[1 + \frac{e^{2\pi i(B+iC)}}{1 - e^{2\pi i(B+iC)}} + \frac{e^{-2\pi i(B-iC)}}{1 - e^{-2\pi i(B-iC)}} \right] \quad (\text{B.20})$$

$$-\frac{1}{2C} \partial_C \sum_{j' \in \mathbb{Z}} \frac{1}{(j' + B)^2 + C^2} = \sum_{j' \in \mathbb{Z}} \frac{1}{((j' + B)^2 + C^2)^2}. \quad (\text{B.21})$$

Note that in applying these formula, the information of s is present in B as now $B = U_1 p + s/N$. When $p = 0$, the terms in the sum that contribute in the $\Lambda \rightarrow \infty$ limit from (B.17) and (B.19) result in

$$\sum_{s=0}^{N-1} \left(\tilde{c}^{(0,s)}(0) \frac{2\zeta(2, s/N)}{\pi} U_2 - c^{(0,s)}(0) \frac{6\zeta(4, s/N)}{\pi^3} \frac{U_2^2}{T_2} \right). \quad (\text{B.22})$$

Now we carry out the sum over j' for $p \neq 0$ in (B.19) using (B.21). When one does this, there is a term that results from the action of the derivative on the first term of (B.20), that is the term independent of B . This results in

$$\sum_{s=0}^{N-1} \frac{6}{\pi^2} c^{(0,s)}(0) \frac{\zeta(3)}{T_2 U_2}. \quad (\text{B.23})$$

Now, let's carry out the sum over j' for $p \neq 0$ in the first two terms of (B.17). Applying (B.20) from the B independent part we get

$$\sum_{p=1}^{\infty} \left(\frac{2}{p} - \frac{2}{\sqrt{p^2 + \Lambda/\pi T_2 U_2}} = -\ln \frac{\pi T_2 U_2}{\Lambda} - \ln 4 + 2\gamma_E \right). \quad (\text{B.24})$$

This again comes in all the sectors of $(0, s)$ and taking care of the moduli dependence and combining (B.18) we obtain

$$\sum_{s=1}^{N-1} \tilde{c}^{(0,s)}(-\log T_2 U_2 - \kappa), \quad (\text{B.25})$$

where $\kappa = 1 - \gamma_E + \log(\frac{8\pi}{3\sqrt{3}})$.

Finally keeping track of all the second and third terms in the r.h.s. of B.20 and its derivative B.21, that is the B dependent part both in the sums (B.17) and (B.19) we obtain

$$4\text{Re} \sum_{\substack{l \neq 0, l \geq -1 \\ l \in \mathbb{Z}}} \left[\tilde{c}^{(0,s)}(0) e^{2\pi i s l / N} \text{Li}_1(e^{2\pi i(lU)}) - \frac{3}{\pi T_2 U_2} c^{(0,s)}(0) e^{2\pi i s l / N} \tilde{\mathcal{P}}(lU) \right]. \quad (\text{B.26})$$

Combining all these results

$$\mathcal{I}^{\text{degen}} = (\text{B.22}) + (\text{B.23}) + (\text{B.25}) + (\text{B.26}). \quad (\text{B.27})$$

C Details for the $g > 1$, threshold integral

The integral we have to do is the one in (4.21). As discussed there is no zero orbit. We have to perform the integral over the non-degenerate orbit and the degenerate orbit.

Non-degenerate orbit. This orbit are characterized by matrices of the form For the non-degenerate orbit we apply the unfolding technique and choose a representative of the matrix A and integrate over two copies of the upper half plane. In this case A becomes

$$A = \begin{pmatrix} n_1 & j \\ 0 & p \end{pmatrix}, \quad p \in \mathbb{Z}, n_1, j \in \frac{\mathbb{Z}}{N}, n_1 \geq j \geq 0. \quad (\text{C.1})$$

This gives $\det A = n_1 p$ and $\mathcal{A} = n_1 \tau + j + pU$. The domain for the integration now is two copies of the upper half plane. In the $(0, s)$ sectors and (r, s) sectors one can verify that the limits on j, n_1 are as in the last section for $g = 1$. The j dependence is in the phase given in (B.8). Again after shifting τ_1 to $\tau'_1 = j + pU_1 + n_1 \tau_1$ which forces the n of the Fourier coefficient to be integer as before.

The relevant part for the τ_1 integration in (4.21) becomes

$$I(\tau_1) = T_2 \int \frac{d^2 \tau}{\tau_2^{2+t}} \left(\frac{T_2}{2U_2} \right)^k (\tau'_1 + i(pU_2 + n_1 \tau_2))^{2k} \times \exp \left\{ 2\pi i n \tau_1 - \frac{\pi T_2}{U_2 \tau_2} \tau_1'^2 \right\}. \quad (\text{C.2})$$

Note that the coefficients here would be written as $c^{(r,s)}(m^2/2, t)$ in different sectors and a factor of $e^{-2\pi i s n_2 / N}$ will come as an extra phase to the integral in the corresponding sectors along with $e^{-2\pi i T n_1 p}$. Also a factor of 2 would be present due to the two copies of the upper half plane. We shall not write these in the integral to avoid cumbersome

notation and will only reinstate them back in the final result. Let us denote the vector $\vec{m}$ with the components

$$\vec{m} = (n_1, n_2), \quad m^2/2 = n_1 n_2, \quad (\text{C.3})$$

where $n_2 \in \mathbb{Z}$ and $n_1 \in \mathbb{Z} + r/N$.

This integral in (C.2) can be done by using binomial expansion for $(\tau'_1 + i(pU_2 + n_1\tau_2))^{2k}$ and shifting the variable τ'_1 to $\tilde{\tau}_1 = \tau'_1 - in_2 U_2 \tau_2 / T_2$. Following through these replacements in C.2, we obtain

$$I(\tau_1) = T_2 \int \frac{d^2\tau}{\tau_2^{2+t}} \left(\frac{T_2}{2U_2} \right)^k (\tilde{\tau}_1 + in_2 U_2 \tau_2 / T_2 + i(pU_2 + n_1\tau_2))^{2k} \times \exp \left\{ -\frac{\pi T_2}{U_2 \tau_2} \tilde{\tau}_1^2 - \frac{\pi U_2 \tau_2 n_2^2}{T_2} \right\}. \quad (\text{C.4})$$

The $\tilde{\tau}_1$ can be integrated and we are left with

$$I(\tau_1) = T_2 \frac{1}{\tau_2^{2+t}} \left(\frac{T_2}{2U_2} \right)^k \sum_{h=0}^{2k} \sum_{j=0}^{[h/2]} \binom{h}{2j} \binom{2k}{h} (in_2 U_2 \tau_2 / T_2)^{h-2j} (ipU_2 + in_1 \tau_2)^{2k-h} \times \Gamma(j+1/2) \frac{1}{(\pi T_2 / U_2 \tau_2)^{j+1/2}} e^{-\pi U_2 \tau_2 n_2^2 / T_2}. \quad (\text{C.5})$$

Here we have written down only the terms that involve τ_1 dependence. Now putting back all the terms which have τ_2 dependence we expand the complete integral as a polynomial in τ_2 which gives

$$\tilde{I} = \sum_{h=0}^{2k} \sum_{j=0}^{[h/2]} \sum_{l=0}^{2k-h} \binom{h}{2j} \binom{2k}{h} \binom{2k-h}{l} T_2 \left(\frac{T_2}{2U_2} \right)^k e^{-2\pi i T_1 n_1 p} \Gamma(j+1/2) (-1)^{k-j} \times \int_0^\infty d\tau_2 (\tau_2)^{l+h-j-3/2-t} \exp[-\pi T_2 U_2 p^2 / \tau_2 - \pi \tau_2 / (T_2 U_2) (n_1 T_2 + n_2 U_2)^2] \times n_1^l (pU_2)^{2k-h-l} (n_2 U_2 / T_2)^{h-2j} (U_2 / (\pi T_2))^{j+1/2}. \quad (\text{C.6})$$

Integrating term by term and replacing h by $2k-h$, which still keeps the limits as 0 to $2k$, we obtain

$$\tilde{I} = \sum_{h=0}^{2k} \sum_{l=0}^{2k-h} \sum_{j=0}^{[k-h/2]} \binom{2k-h}{2j} \binom{2k}{h} \binom{h}{l} T_2 \left(\frac{T_2}{2U_2} \right)^k e^{-2\pi i T_1 n_1 p} \Gamma(j+1/2) (-1)^{k-j} \times n_1^l (pU_2)^{h-l} (n_2 U_2 / T_2)^{2k-h-2j} (U_2 / (\pi T_2))^{j+1/2} \times 2 \left(\frac{|p| T_2 U_2}{n_1 T_2 + n_2 U_2} \right)^{l+2k-h-j-t-1/2} 2c^{(r,s)}(r^2/2, t) e^{-2\pi i s n_2 / N} \times K_{l+2k-h-j-t-1/2}(2\pi i |p| (n_1 T_2 + n_2 U_2)). \quad (\text{C.7})$$

The result of this integral (C.7) seems manifestly different from that obtained in [10] at this step. To bring into in the form we perform changes in the order of summations and limits. We outline these now. First we change the order of summation on l and h which gives

$$\sum_{h=0}^{2k} \sum_{l=0}^{2k-h} \sum_{j=0}^{[k-h/2]} \rightarrow \sum_{l=0}^{2k} \sum_{h=0}^{2k-l} \sum_{j=0}^{[k-h/2]}. \quad (\text{C.8})$$

Now we call $h - l$ as h^+ and change the orders of the sums on h^+ and l .

$$\sum_{l=0}^{2k} \sum_{h=0}^{2k-l} \sum_{j=0}^{[k-h/2]} \rightarrow \sum_{h^+=0}^{2k} \sum_{l=0}^{2k-h^+} \sum_{j=0}^{[k-(h^++l)/2]} . \quad (\text{C.9})$$

Now one needs to change the order of the sums j and l which can be done as

$$\sum_{h^+=0}^{2k} \sum_{l=0}^{2k-h^+} \sum_{j=0}^{[k-(h^++l)/2]} \rightarrow \sum_{h^+=0}^{2k} \sum_{j=0}^{[k-h^+/2]} \sum_{l=0}^{2k-2j-h^+} . \quad (\text{C.10})$$

The final sum over l is doable and this gives us

$$\begin{aligned} \tilde{I} = & 2^{1-k} \sum_{t=0}^{k+1} \sum_{n_1 n_2 \neq 0, p \neq 0} e^{-2\pi i s n_2 / N} c^{(r,s)}(m^2/2, t) \sum_{h^+=0}^{2k} \sum_{j=0}^{[k-h^+/2]} \frac{(2k)!}{j! h^+! (2k - h^+ - 2j)! (4\pi)^j} \\ & \times p^{h^+} (T_2 U_2)^{j+h^++1/2-k} (n_2 U_2 + n_1 T_2)^{2k-h^+-2j} (-1)^{k-j} \left(\frac{|p| T_2 U_2}{n_1 T_2 + n_2 U_2} \right)^{2k-h^+-j-t-1/2} \\ & \times K_{2k-h^+-j-t-1/2}(2\pi|p|(n_1 T_2 + n_2 U_2)) e^{-2\pi i p(n_1 T_2 + n_2 U_2)} . \end{aligned} \quad (\text{C.11})$$

The result (C.11) agrees with equation (4.29) of [10] when $K^3 \times T^2$ is not orbifolded. In (C.11), the summation over the orbifold sectors is implied.

To rewrite the result in terms of poly-logarithms we use the relations

$$K_{s+1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{k=0}^s \frac{(s+k)!}{k!(s-k)!} \frac{1}{(2x)^k}, \quad (\text{C.12})$$

$$\text{Li}_m(x) = \sum_{l=1}^{\infty} \frac{x^l}{l^m}. \quad (\text{C.13})$$

This leads to

$$\begin{aligned} \tilde{I} = & 2^{1-k} \sum_{t=0}^{k+1} \sum_{n_1, n_2 \neq 0} \sum_{h^+=0}^{2k} \sum_{j=0}^{[k-h^+/2]} \sum_{a=0}^s \frac{(2k)!}{j! h^+! (2k - h^+ - 2j)!} \frac{(s+a)!}{a!(s-a)!} \frac{(-1)^{k-j+h^+}}{(4\pi)^{j+a}} \\ & \times (T_2 U_2)^{k-t} (\text{sgn Im}(m \cdot y))^{h^+} c^{(r,s)}(m^2/2, t) e^{-2\pi i s n_2 / N} \text{Im}(m \cdot y)^{t-j-a} \text{Li}_{1+a+j+t-2k}(e^{2\pi i m \cdot y}). \end{aligned} \quad (\text{C.14})$$

The case $m = 0$ needs to be treated separately. From (C.2) and (C.6) we see that we can perform the integral also for the limit $n_1 = n_2 = 0$. After the τ_1 integration, for the $m = 0$ limit we obtain

$$\begin{aligned} \tilde{I}|_{m=0} = & \sum_{j=0}^k \binom{2k}{2j} T_2 \left(\frac{T_2}{2U_2} \right)^k \Gamma(j+1/2) (-1)^{k-j} \\ & \times \int_0^\infty d\tau_2 (\tau_2)^{j-3/2-t} \exp[-\pi T_2 U_2 p^2 / \tau_2] (p U_2)^{2k-2j} (U_2 / (\pi T_2))^{j+1/2}. \end{aligned} \quad (\text{C.15})$$

One can shift to the variable $j' = k - j$ which also runs from 0 to k . We will re-label this variable j . The resultant integral is a gamma function and since there is a sum over p it

forms a zeta function. A factor of 2 arises on taking two copies of the upper half plane. However for values $t = k$ the zeta function blows up so one can use a regularization by adding $-\epsilon$ to the power of τ_2 . We have

$$\int e^{-\pi T_2 U_2 p^2 / \tau_2} \tau_2^{k-j-t+1/2-\epsilon} \frac{d\tau_2}{\tau_2^2} = \left(\frac{1}{\pi T_2 U_2 p^2} \right)^{j-k+t+1/2+\epsilon} \Gamma(j-k+t+1/2+\epsilon). \quad (\text{C.16})$$

So from C.15 the contribution from $m = 0$ to the non-degenerate part of the integral is given by,

$$\begin{aligned} \mathcal{I}^{\text{nondegen}}|_{m=0} &= 2 \sum_{j=0}^k \sum_{t=0}^{k+1} (-1)^j 2^{2j-4k+t} \frac{(2k)!}{(2j)!(k-j)!} e^{-2\pi i s n_2 / N} \left(\frac{c^{(0,s)}(0, t)}{\pi^{t+1/2}} \right) \\ &\quad \times (\pi T_2 U_2)^{-\epsilon} \Gamma(1/2 + j + t - k + \epsilon) \zeta(1 + 2\epsilon + 2t - 2k). \end{aligned} \quad (\text{C.17})$$

In the equation (C.17) we have used the fact

$$\Gamma(n + 1/2) = \frac{(2n)! \sqrt{\pi}}{n! 4^n}. \quad (\text{C.18})$$

We now use the equation (C.17) and consider $\epsilon \rightarrow 0$ limit to extract out the finite contribution as follows,

$$\begin{aligned} (\pi T_2 U_2)^{-\epsilon} &= 1 - \epsilon \log(\pi T_2 U_2), \\ \Gamma(1/2 + s + \epsilon) &= \Gamma(1/2 + s)(1 + \epsilon \psi(1/2 + s)), \\ \zeta(1 + 2\epsilon) &= \frac{1}{2\epsilon} + \gamma_E, \end{aligned} \quad (\text{C.19})$$

where $\psi(x)$ is the logarithmic derivative of the gamma function. The resulting sums in C.17 can be explicitly computed for some cases. We use the results $\Gamma(1/2+n) = \pi^{1/2} 2^{-n} (2n+1)!!$ and $\Gamma(1/2-n) = (-1)^n \pi^{1/2} 2^n / (2n-1)!!$ with $n \geq 0$ to arrive at

$$\sum_{j=0}^k (-1)^j 2^j \frac{(2k)!}{(2j)!(k-j)!} \Gamma(1/2 + j) = \pi^{1/2} \delta_{k,0}. \quad (\text{C.20})$$

Thus results for $k = 0$ that is genus one and $k \neq 0$ are different. For $k > 0$ we can write the full non-degenerate orbit result as,

$$\begin{aligned} \mathcal{I}_{k>0, m=0}^{\text{nondegen}} &= 2 \sum_{s=0}^{N-1} \sum_{t=0}^{k-1} \frac{c^{(0,s)}(m^2/2, t)}{\pi^{t+1/2}} \zeta(1 + 2(t-k)) (2T_2 U_2)^{(k-t)} \\ &\quad \times \sum_{\tilde{s}=0}^k (-1)^{\tilde{s}} 2^{2(\tilde{s}-2k)+t} \frac{(2k)!}{(2\tilde{s})!(k-\tilde{s})!} \Gamma(1/2 + \tilde{s} + t - k) \\ &\quad + \sum_{s=0}^{N-1} \frac{c^{(0,s)}}{2^{3k} \pi^k} \sum_{\tilde{s}=0}^k (-1)^s \frac{(2k)!}{(2\tilde{s})!(k-\tilde{s})!} \psi(1/2 + \tilde{s}). \end{aligned} \quad (\text{C.21})$$

Putting all the contributions together the contribution for $F_{g>1}^{\text{nondegen}}$ is given by

$$\begin{aligned}
 F_{g>1}^{\text{nondegen}} = & \frac{(-1)^{g-1}}{2^{2(g-1)}\pi^2} \sum_{r,s} \sum_{m \neq 0} \sum_{t=0}^g \sum_{h=0}^{2g-2} \sum_{j=0}^{g-1-h/2} \sum_{a=0}^{\hat{s}} e^{-2\pi i s n_2/N} c^{(r,s)}(m^2/2, t) \quad (\text{C.22}) \\
 & \times \frac{(2g-2)!}{j!h!(2g-h-2j-2)!} \frac{(-1)^{j+h}}{(4\pi)^{j+a}} \frac{(\hat{s}+a)!}{(\hat{s}-a)!a!} (\text{sgn}(\text{Im}(m \cdot y)))^h \\
 & \times \frac{(\text{Im}(m \cdot y))^{t-j-a}}{(T_2 U_2)^t} \text{Li}_{3+a+j+t-2g}(e^{2\pi i m \cdot y}) \\
 & + \sum_{s=0}^{N-1} \frac{c^{(0,s)}(0, g-1)}{2^{4g-5} \pi^{g+1}} \frac{1}{(T_2 U_2)^{g-1}} \sum_{\tilde{s}=0}^{g-1} (-1)^s \frac{(2g-2)!}{\tilde{s}!(g-1-\tilde{s})!} \psi(1/2 + \tilde{s}) \\
 & + \sum_{s=0}^{N-1} \sum_{t=0}^{g-2} \frac{c^{(0,s)}(0, t)}{\pi^{t+5/2}} \frac{\zeta(3+2(t-g))}{(T_2 U_2)^t} \\
 & \times \sum_{\tilde{s}=0}^{g-1} (-1)^{\tilde{s}} 2^{2(\tilde{s}-2g+2)} \frac{(2g-2)!}{(2\tilde{s})!(g-1-\tilde{s})!} \Gamma(3/2 + \tilde{s} + t - g).
 \end{aligned}$$

Here $m \neq 0$ is defined in (4.32) and $\hat{s}$ is defined in (4.31).

Degenerate orbit. The degenerate orbit is characterised by matrices of the form

$$A = \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix}, \quad j \in \frac{\mathbb{Z}}{N}, \quad j \neq 0. \quad (\text{C.23})$$

The evaluation of the integral in the degenerate orbit proceeds as follows. Let us first assume $T_2 > U_2$ for this part of the calculation we have

$$I^{\text{deg}} = \frac{T_2^{k+1}}{(2U_2)^k} \int j^{2k} \tau_2^{-t-2} d\tau_2 e^{-\pi j^2 T_2/U_2 \tau_2 - 2\pi n' \tau_2} \int_{-1/2}^{1/2} d\tau_1 e^{2\pi i n' \tau_1}. \quad (\text{C.24})$$

Note that since we are in the un-twisted sector for this orbit $n' \in \mathbb{Z}$. Furthermore to obtain a non-zero result due to the integration we require $n' = 0$. Then the result for I^{degen} is given by

$$I^{\text{deg}} = \sum_{s=0}^{N-1} \sum_t \pi^{-t-1} U_2 (T_2/U_2)^{k-t} c^{(0,s)}(0, t) t! \zeta(2(t+1-k), s/N). \quad (\text{C.25})$$

If $U_2 > T_2$, we need to in fact we need start by performing at T-duality and then we obtain [9, 33]¹⁴

$$I^{\text{degen}} = \sum_{s=0}^{N-1} \sum_t \pi^{-t-1} T_2 (U_2/T_2)^{k-t} c^{(0,s)}(0, t) t! \zeta(2(t+1-k), s/N). \quad (\text{C.26})$$

Now reinstating all factors we obtain

$$F_{g>1}^{\text{degen}} = \frac{1}{T_2^{2g-3}} \sum_{s=0}^{N-1} \sum_{t=0}^g c^{(0,s)}(0, t) \frac{t!}{2^{2g-2} \pi^{t+3}} \left(\frac{T_2}{U_2} \right)^t \zeta(2(2+t-g), s/N). \quad (\text{C.27})$$

¹⁴The reason this has to be done has to do with the convergence of the integral for Fourier coefficient when $n' = -1$.

(n_1, n_2)	$(1, -1)$	$(1, 0)$	$(1, 1)$	$(1, 2)$	$(1, 3)$
$n_{(n_1, n_2)}^0$	-2	480	282888	17058560	477516780

(n_1, n_2)	$(1, 4)$	$(1, 5)$	$(1, 6)$	$(1, 7)$	$(1, 8)$
$n_{(n_1, n_2)}^0$	8606976768	115311621680	1242058447872	11292809553810	89550084115200

Table 5. $N = 1$, $K3$ itself.

(n_1, n_2)	$(0, n_2)$	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$
$n_{(n_1, n_2)}^0$	-32	151552	8387328	47890048	4294949632

(n_1, n_2)	$(1/2, 0)$	$(1/2, 2)$	$(1/2, 4)$	$(1/2, 6)$	$(1/2, 8)$
$n_{(n_1, n_2)}^0$	512	151552	8671232	240009216	4312027136

(n_1, n_2)	$(1/2, -1)$	$(1/2, 1)$	$(3/2, 1)$	$(5/2, 1)$	$(7/2, 1)$
$n_{(n_1, n_2)}^0$	16	8128	1212576	47890048	1055720304

Table 6. $N = 2$, $2A$ orbifold.

D Genus zero GV invariants for CHL orbifolds

In this appendix we list the genus zero Gopakumar-Vafa invariants n_m^0 corresponding to the CHL orbifolds. We observe that they are integers. For the remaining orbifolds in table 1, we provide an ancillary Mathematica code from which these invariants can be evaluated and seen to be integers. One point worth mentioning, is that all the remaining orbifolds especially the case of $2B$ and $3B$ do not have any geometric interpretation as actions on the $K3$ surface. Therefore the fact that the Gopakumar-Vafa invariants turn out to be integers is interesting.

E List of twisted elliptic genera

In this appendix we provide the list of the twisted elliptic genera used to determine the $f^{(r,s)}$ defined in (2.14) in this paper. We call the classes $2A, 3A, 5A, 7A, 4B, 6A, 7A$, CHL

(n_1, n_2)	$(0, n_2)$	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$
$n_{(n_1, n_2)}^0$	-276	110646	6063822	149000544	2918573208

(n_1, n_2)	$(r/3, 0)$	$(1/3, 3)$	$(1/3, 6)$	$(1/3, 9)$	$(1/3, 12)$
$n_{(n_1, n_2)}^0$	378	110646	6063822	164258118	2918573208

(n_1, n_2)	$(1/3, -1)$	$(1/3, 1)$	$(4/3, 1)$	$(7/3, 1)$	$(10/3, 1)$
$n_{(n_1, n_2)}^0$	18	3066	-463800	19292628	445321248

(n_1, n_2)	$(2/3, -1)$	$(2/3, 1)$	$(5/3, 1)$	$(8/3, 1)$	$(11/3, 1)$
$n_{(n_1, n_2)}^0$	6	9108	-1345986	51211656	1089299772

Table 7. $N = 3$, 3A orbifold.

orbifolds. When g' corresponding to these classes act on $K3$, it can be seen that the values of the Hodge number $h_{1,1} \geq 1$. The twisted elliptic genus of classes $2B$, $3B$ is such that their twining character coincides with that constructed in [26–28]. But they do not correspond to M_{24} symmetry. In fact in M_{24} , the classes have order 2 and 3, however the orbifolds we construct have order 4 and 9. The $2B$ twisted elliptic genus has been constructed in [52] by considering $K3$ as 6, $SU(2)$ WZW models at level 1. While $3B$ twisted elliptic genus we give here was constructed in [29].

Conjugacy class pA , $p = \{1, 2, 3, 5, 7\}$.

$$\begin{aligned}
 F^{(0,0)} &= \frac{8}{p} A(\tau, z), \\
 F^{(0,s)} &= \frac{8}{p(p+1)} A(\tau, z) - B(\tau, z) \left(\frac{2}{p+1} \mathcal{E}_p(\tau) \right) \\
 F^{(r,rs)} &= \frac{8}{p(p+1)} A(\tau, z) + B(\tau, z) \left(\frac{2}{p(p+1)} \mathcal{E}_p \left(\frac{\tau+s}{p} \right) \right)
 \end{aligned} \tag{E.1}$$

(n_1, n_2)	$(0, n_2)$	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$
$n_{(n_1, n_2)}^0$	-520	72750	1490560	103674750	1820321750

(n_1, n_2)	$(r/5, 0)$	$(1/5, 5)$	$(2/5, 5)$	$(3/5, 5)$	$(4/5, 5)$
$n_{(n_1, n_2)}^0$	250	72750	3892000	103674750	1820321750

(n_1, n_2)	$(1/5, -1)$	$(1/5, 1)$	$(6/5, 1)$	$(11/5, 1)$	$(16/5, 1)$
$n_{(n_1, n_2)}^0$	14	1032	159648	7172574	175994068

(n_1, n_2)	$(2/5, -1)$	$(2/5, 1)$	$(7/5, 1)$	$(12/5, 1)$	$(17/5, 1)$
$n_{(n_1, n_2)}^0$	8	3040	450704	17378724	375804020

(n_1, n_2)	$(3/5, -1)$	$(3/5, 1)$	$(8/5, 1)$	$(13/5, 1)$	$(18/5, 1)$
$n_{(n_1, n_2)}^0$	6	4062	605260	11893715	260500896

(n_1, n_2)	$(4/5, -1)$	$(4/5, 1)$	$(9/5, 1)$	$(14/5, 1)$	$(19/5, 1)$
$n_{(n_1, n_2)}^0$	2	7082	1045848	39672656	840212212

Table 8. $N = 5$, 5A orbifold.

Conjugacy class 11A. Going through these steps we obtain the following formula for the twisted elliptic genus for 11A.

$$\begin{aligned}
 F^{(0,0)} &= \frac{8}{11} A(\tau, z), \\
 F^{(0,s)} &= \frac{2}{33} A(\tau, z) - B(\tau, z) \left(\frac{1}{6} \mathcal{E}_{11}(\tau) - \frac{2}{5} \eta^2(\tau) \eta^2(11\tau) \right), \\
 F^{(r,rs)} &= \frac{2}{33} A(\tau, z) + B(\tau, z) \left(\frac{1}{66} \mathcal{E}_{11} \left(\frac{\tau+s}{11} \right) - \frac{2}{55} \eta^2(\tau+s) \eta^2 \left(\frac{\tau+s}{11} \right) \right).
 \end{aligned} \tag{E.2}$$

(n_1, n_2)	$(0, n_2)$	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$
$n_{(n_1, n_2)}^0$	-642	54309	2881360	76296885	1333671600

(n_1, n_2)	$(r/7, 0)$	$(1/7, 7)$	$(2/7, 7)$	$(3/7, 7)$	$(4/7, 7)$
$n_{(n_1, n_2)}^0$	187	54309	-2881360	76296885	1333671600

(n_1, n_2)	$(1/7, -1)$	$(1/7, 1)$	$(8/7, 1)$	$(15/7, 1)$	$(22/7, 1)$
$n_{(n_1, n_2)}^0$	12	519	81348	3813632	96969840

(n_1, n_2)	$(2/7, -1)$	$(2/7, 1)$	$(9/7, 1)$	$(16/7, 1)$	$(23/7, 1)$
$n_{(n_1, n_2)}^0$	6	1525	227875	9059700	201601562

(n_1, n_2)	$(3/7, -1)$	$(3/7, 1)$	$(10/7, 1)$	$(17/7, 1)$	$(24/7)$
$n_{(n_1, n_2)}^0$	7	2034	304146	12117044	268891664

(n_1, n_2)	$(4/7, -1)$	$(4/7, 1)$	$(11/7, 1)$	$(18/7, 1)$	$(25/7, 1)$
$n_{(n_1, n_2)}^0$	4	3540	522435	19770135	418523073

(n_1, n_2)	$(5/7, -1)$	$(5/7, 1)$	$(12/7, 1)$	$(19/7, 1)$	$(26/7, 1)$
$n_{(n_1, n_2)}^0$	3	3052	456716	9122877	202745998

(n_1, n_2)	$(6/7, -1)$	$(6/7, 1)$	$(13/7, 1)$	$(20/7, 1)$	$(27/7, 1)$
$n_{(n_1, n_2)}^0$	1	6062	892266	33394384	697917622

Table 9. $N = 7$, 7A orbifold.

(n_1, n_2)	$(0, n_2)$	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$
$n_{(n_1, n_2)}^0$	−400	75776	5323136	120004608	1941741952

(n_1, n_2)	$(r/4, 0), r = 1, 3$	$(1/4, 4)$	$(1/4, 8)$	$(1/4, 12)$	$(1/4, 16)$
$n_{(n_1, n_2)}^0$	256	75776	4335616	120004608	2156013568

(n_1, n_2)	$(1/4, -1)$	$(1/4, -2)$	$(1/4, -3)$	$(1/4, 1)$	$(5/4, 1)$
$n_{(n_1, n_2)}^0$	16	8	4	2040	307204

(n_1, n_2)	$(1/2, -1)$	$(1/2, 0)$	$(1/2, 1)$	$(1/2, 2)$	$(3/2, 1)$
$n_{(n_1, n_2)}^0$	8	368	4064	45952	606288

(n_1, n_2)	$(3/4, -1)$	$(3/4, 1)$	$(7/4, 1)$	$(11/4, 1)$	$(15/4, 1)$
$n_{(n_1, n_2)}^0$	4	8096	1196400	45505376	965922096

Table 10. $N = 4$, 4B orbifold.

Conjugacy class 23A/B.

$$\begin{aligned}
 F^{(0,k)}(\tau, z) &= \frac{1}{23} \left(\frac{1}{3} A - B \left(\frac{23}{12} \mathcal{E}_{23}(\tau) - \frac{23}{22} f_{23,1}(\tau) - \frac{161}{22} \eta^2(\tau) \eta^2(23\tau) \right) \right), \\
 F^{(r,rk)}(\tau, z) &= \frac{1}{23} \left[\frac{1}{3} A + B \left(\frac{1}{12} \mathcal{E}_{23} \left(\frac{\tau+k}{23} \right) - \frac{1}{22} f_{23,1} \left(\frac{\tau+k}{23} \right) - \frac{7}{22} \eta^2(\tau+k) \eta^2 \left(\frac{\tau+k}{23} \right) \right) \right].
 \end{aligned} \tag{E.3}$$

where

$$f_{23,1}(\tau) = 2 \frac{\eta^3(\tau) \eta^3(23\tau)}{\eta(2\tau) \eta(46\tau)} + 8\eta(\tau) \eta(2\tau) \eta(23\tau) \eta(46\tau) + 8\eta^2(2\tau) \eta^2(46\tau) + 5\eta^2(\tau) \eta^2(23\tau) \tag{E.4}$$

(n_1, n_2)	$(0, n_2)$	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$	$(1, 5)$
$n_{(n_1, n_2)}^0$	-524	39912	3596922	97260000	1599680040	17990607384

(n_1, n_2)	$(r/6, 0), r = 1, 5$	$(1/6, 6)$	$(1/6, 12)$	$(1/6, 18)$	$(1/6, 24)$	$(1/6, 30)$
$n_{(n_1, n_2)}^0$	132	39912	2466900	71374608	1318893168	17990607384

(n_1, n_2)	$(r/3, 0), r = 1, 2$	$(1/2, 0)$	$(1/3, 3)$	$(1/2, 2)$	$(1/3, 6)$	$(1/2, 4)$
$n_{(n_1, n_2)}^0$	246	248	70734	71728	3596922	3737432

(n_1, n_2)	$(1/6, -1)$	$(5/6, -1)$	$(1/6, 1)$	$(5/6, 1)$	$(7/6, 1)$	$(11/6, 1)$
$n_{(n_1, n_2)}^0$	12	2	1024	6072	155620	897300

(n_1, n_2)	$(1/6, -2)$	$(1/6, -3)$	$(1/6, -4)$	$(1/2, -1)$	$(1/2, 1)$	$(3/2, 1)$
$n_{(n_1, n_2)}^0$	8	6	4	6	3054	457740

(n_1, n_2)	$(1/3, -1)$	$(2/3, -1)$	$(1/3, 1)$	$(2/3, 1)$	$(4/3, 1)$	$(5/3, 1)$
$n_{(n_1, n_2)}^0$	8	2032	303152	4	4056	602240

 Table 11. $N = 6$, 6A orbifold.

Conjugacy class 4B. The twisted elliptic genus for this class is given by

$$\begin{aligned}
 F^{(0,0)}(\tau, z) &= 2A(\tau, z), \\
 F^{(0,1)}(\tau, z) &= F^{(0,3)}(\tau, z) = \frac{1}{4} \left[\frac{4A}{3} - B \left(-\frac{1}{3} \mathcal{E}_2(\tau) + 2\mathcal{E}_4(\tau) \right) \right], \\
 F^{(1,s)}(\tau, z) &= F^{(3,3s)} = \frac{1}{4} \left[\frac{4A}{3} + B \left(-\frac{1}{6} \mathcal{E}_2 \left(\frac{\tau+s}{2} \right) + \frac{1}{2} \mathcal{E}_4 \left(\frac{\tau+s}{4} \right) \right) \right], \\
 F^{(2,1)}(\tau, z) &= F^{(2,3)} = \frac{1}{4} \left(\frac{4A}{3} - \frac{B}{3} (5\mathcal{E}_2(\tau) - 6\mathcal{E}_4(\tau)) \right), \\
 F^{(0,2)}(\tau, z) &= \frac{1}{4} \left(\frac{8A}{3} - \frac{4B}{3} \mathcal{E}_2(\tau) \right), \\
 F^{(2,2s)}(\tau, z) &= \frac{1}{4} \left(\frac{8A}{3} + \frac{2B}{3} \mathcal{E}_2 \left(\frac{\tau+s}{2} \right) \right).
 \end{aligned} \tag{E.5}$$

(n_1, n_2)	$(0, n_2)$	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$	$(1, 5)$
$n_{(n_1, n_2)}^0$	-644	37888	2661568	60002304	1452283424	14426028032

(n_1, n_2)	$(r/8, 0), r = \text{odd}$	$(1/8, 8)$	$(1/8, 16)$	$(r/4, 0), r = \text{odd}$	$(1/4, 4)$	$(1/4, 8)$
$n_{(n_1, n_2)}^0$	128	37888	2167808	184	22976	2661568

(n_1, n_2)	$(1/4, -2)$	$(1/4, -3)$	$(1/2, -1)$	$(1/2, 0)$	$(1/2, 2)$	$(1/2, 4)$
$n_{(n_1, n_2)}^0$	6	2	4	244	22976	1880800

(n_1, n_2)	$(1/8, -1)$	$(1/8, 1)$	$(3/8, 1)$	$(5/8, 1)$	$(7/8, 1)$	$(9/8, 1)$
$n_{(n_1, n_2)}^0$	8	517	2028	3038	4056	80334

(n_1, n_2)	$(1/8, -2)$	$(1/8, -3)$	$(1/8, -4)$	$(1/8, -5)$	$(1/8, -6)$	$(1/8, -7)$
$n_{(n_1, n_2)}^0$	8	6	4	4	2	1

(n_1, n_2)	$(1/4, -1)$	$(3/4, -1)$	$(1/4, 1)$	$(3/4, 1)$	$(5/4, 1)$	$(7/4, 1)$
$n_{(n_1, n_2)}^0$	16	2	1020	8096	153602	1196400

Table 12. $N = 8$, 8A orbifold.

Conjugacy class 6A. The twisted elliptic genus for 6A are given by

$$F^{(0,0)} = \frac{4}{3}A; \quad F^{(0,1)} = F^{(0,5)}; \quad F^{(0,2)} = F^{(0,4)}; \quad (\text{E.6})$$

$$F^{(0,1)} = \frac{1}{6} \left[\frac{2A}{3} - B \left(-\frac{1}{6}\mathcal{E}_2(\tau) - \frac{1}{2}\mathcal{E}_3(\tau) + \frac{5}{2}\mathcal{E}_6(\tau) \right) \right],$$

$$F^{(0,2)} = \frac{1}{6} \left[2A - \frac{3}{2}B\mathcal{E}_3(\tau) \right],$$

$$F^{(0,3)} = \frac{1}{6} \left[\frac{8A}{3} - \frac{4}{3}B\mathcal{E}_2(\tau) \right].$$

$$F^{(1,k)} = F^{(5,5k)} = \frac{1}{6} \left[\frac{2A}{3} + B \left(-\frac{1}{12}\mathcal{E}_2\left(\frac{\tau+k}{2}\right) - \frac{1}{6}\mathcal{E}_3\left(\frac{\tau+k}{3}\right) + \frac{5}{12}\mathcal{E}_6\left(\frac{\tau+k}{6}\right) \right) \right], \quad (\text{E.7})$$

$$F^{(2,2k+1)} = \frac{A}{9} + \frac{B}{36} \left[\mathcal{E}_3 \left(\frac{\tau+2+k}{3} \right) + \mathcal{E}_2(\tau) - \mathcal{E}_2 \left(\frac{\tau+k+2}{3} \right) \right], \quad (\text{E.8})$$

$$\begin{aligned} F^{(4,4k+1)} &= \frac{A}{9} + \frac{B}{36} \left[\mathcal{E}_3 \left(\frac{\tau+1+k}{3} \right) + \mathcal{E}_2(\tau) - \mathcal{E}_2 \left(\frac{\tau+k+1}{3} \right) \right], \\ F^{(3,1)} = F^{(3,5)} &= \frac{A}{9} - \frac{B}{12} \mathcal{E}_3(\tau) - \frac{B}{72} \mathcal{E}_2 \left(\frac{\tau+1}{2} \right) + \frac{B}{8} \mathcal{E}_2 \left(\frac{3\tau+1}{2} \right), \\ F^{(3,2)} = F^{(3,4)} &= \frac{A}{9} - \frac{B}{12} \mathcal{E}_3(\tau) - \frac{B}{72} \mathcal{E}_2 \left(\frac{\tau}{2} \right) + \frac{B}{8} \mathcal{E}_2 \left(\frac{3\tau}{2} \right), \\ F^{(2r,2rk)} &= \frac{1}{6} \left[2A + \frac{1}{2} B \mathcal{E}_3 \left(\frac{\tau+k}{3} \right) \right], \\ F^{(3,3k)} &= \frac{1}{6} \left[\frac{8A}{3} + \frac{2}{3} B \mathcal{E}_2 \left(\frac{\tau+k}{2} \right) \right]. \end{aligned} \quad (\text{E.9})$$

Conjugacy class 8A.

$$\begin{aligned} F^{(0,0)}(\tau, z) &= A(\tau, z), \\ F^{(0,1)} &= F^{(0,3)} = F^{(0,5)} = F^{(0,7)}, \end{aligned} \quad (\text{E.10})$$

$$\begin{aligned} &= \frac{1}{8} \left[\frac{2A}{3} - B \left(-\frac{1}{2} \mathcal{E}_4(\tau) + \frac{7}{3} \mathcal{E}_8(\tau) \right) \right], \\ F^{(r,rk)}(\tau, z) &= \frac{1}{8} \left[\frac{2A}{3} + \frac{B}{8} \left(-\mathcal{E}_4 \left(\frac{\tau+k}{4} \right) + \frac{7}{3} \mathcal{E}_8 \left(\frac{\tau+k}{8} \right) \right) \right]. \end{aligned} \quad (\text{E.11})$$

where $r = 1, 3, 5, 7$.

$$\begin{aligned} F^{(2,1)} &= F^{(6,3)} = F^{(2,5)} = F^{(6,7)}, \\ &= \frac{1}{8} \left[\frac{2A}{3} + \frac{B}{3} \left(-\mathcal{E}_2(2\tau) + \frac{3}{2} \mathcal{E}_4 \left(\frac{2\tau+1}{4} \right) \right) \right]; \end{aligned} \quad (\text{E.12})$$

$$\begin{aligned} F^{(2,3)} &= F^{(6,5)} = F^{(2,7)} = F^{(6,1)}, \\ &= \frac{1}{8} \left[\frac{2A}{3} + \frac{B}{3} \left(-\mathcal{E}_2(2\tau) + \frac{3}{2} \mathcal{E}_4 \left(\frac{2\tau+3}{4} \right) \right) \right], \\ F^{(0,2)} = F^{(0,6)} &= \frac{1}{8} \left(\frac{4A}{3} - B \left(-\frac{1}{3} \mathcal{E}_2(\tau) + 2\mathcal{E}_4(\tau) \right) \right), \end{aligned} \quad (\text{E.13})$$

$$\begin{aligned} F^{(0,4)} &= \frac{1}{8} \left(\frac{8A}{3} - \frac{4B}{3} \mathcal{E}_2(\tau) \right), \\ F^{(2,2s)} = F^{(6,6s)} &= \frac{1}{8} \left(\frac{4A}{3} + B \left(-\frac{1}{6} \mathcal{E}_2 \left(\frac{\tau+s}{2} \right) + \frac{1}{2} \mathcal{E}_4 \left(\frac{\tau+s}{4} \right) \right) \right), \\ F^{(4,4s)} &= \frac{1}{8} \left(\frac{8A}{3} + \frac{2B}{3} \mathcal{E}_2 \left(\frac{\tau+s}{2} \right) \right), \\ F^{(4,2)} = F^{(4,6)} &= \frac{1}{8} \left(\frac{4A}{3} - \frac{B}{3} (3\mathcal{E}_2(\tau) - 4\mathcal{E}_2(2\tau)) \right), \\ F^{(4,2k+1)} &= \frac{1}{8} \left(\frac{2A}{3} + B \left(\frac{4}{3} \mathcal{E}_2(4\tau) - \frac{2}{3} \mathcal{E}_2(2\tau) - \frac{1}{2} \mathcal{E}_4(\tau) \right) \right). \end{aligned}$$

Conjugacy class 14A.

$$F^{(0,1)}(\tau, z) = F^{(0,3)} = F^{(0,5)} = F^{(0,9)} = F^{(0,11)} = F^{(0,13)}; \quad (\text{E.14})$$

$$= \frac{1}{14} \left[\frac{A}{3} - B \left(-\frac{1}{36} \mathcal{E}_2(\tau) - \frac{7}{12} \mathcal{E}_7(\tau) + \frac{91}{36} \mathcal{E}_{14}(\tau) - \frac{14}{3} \eta(\tau) \eta(2\tau) \eta(7\tau) \eta(14\tau) \right) \right];$$

$$F^{(r,rk)} = \frac{1}{14} \left[\frac{A}{3} + B \left(-\frac{1}{72} \mathcal{E}_2 \left(\frac{\tau+k}{2} \right) - \frac{1}{12} \mathcal{E}_7 \left(\frac{\tau+k}{7} \right) + \frac{13}{72} \mathcal{E}_{14} \left(\frac{\tau+k}{14} \right) - \frac{1}{3} \eta(\tau+k) \eta \left(\frac{\tau+k}{2} \right) \eta \left(\frac{\tau+k}{7} \right) \eta \left(\frac{\tau+k}{14} \right) \right) \right]; \quad (\text{E.15})$$

where $r=1,3,5,9,11,13$ and rk is Mod 14.

$$F^{(2r,2rk+7)} = \frac{1}{14} \left[\frac{A}{3} + B \left(-\frac{1}{6} \mathcal{E}_2(\tau) - \frac{1}{12} \mathcal{E}_7 \left(\frac{\tau+k}{7} \right) + \frac{1}{3} \mathcal{E}_7 \left(\frac{2\tau+2k}{7} \right) - \frac{2}{3} \eta(\tau+k) \eta(2\tau+2k) \eta \left(\frac{\tau+k}{7} \right) \eta \left(\frac{2\tau+2k}{7} \right) \right) \right]; \quad (\text{E.16})$$

where k runs from 0 to 6 and except 3 and r from 1 to 6.

$$F^{(7,2k+1)} = \frac{1}{14} \left[\frac{A}{3} + B \left(-\frac{7}{12} \mathcal{E}_7(\tau) + \frac{49}{72} \mathcal{E}_2 \left(\frac{7\tau+1}{2} \right) - \frac{1}{72} \mathcal{E}_2 \left(\frac{\tau+1}{2} \right) + \frac{7}{3} e^{i\pi 11/12} \eta(\tau) \eta(7\tau) \eta \left(\frac{\tau+1}{2} \right) \eta \left(\frac{7\tau+1}{2} \right) \right) \right]; \quad (\text{E.17})$$

$$F^{(7,2k)} = \frac{1}{14} \left[\frac{A}{3} + B \left(-\frac{7}{12} \mathcal{E}_7(\tau) + \frac{49}{72} \mathcal{E}_2 \left(\frac{7\tau}{2} \right) - \frac{1}{72} \mathcal{E}_2 \left(\frac{\tau}{2} \right) + \frac{7}{3} \eta(\tau) \eta(7\tau) \eta \left(\frac{\tau}{2} \right) \eta \left(\frac{7\tau}{2} \right) \right) \right].$$

$$F^{(0,0)} = \frac{4}{7} A. \quad (\text{E.18})$$

$$F^{(0,2k)} = \frac{1}{14} \left[A - \frac{7}{4} B \mathcal{E}_7(\tau) \right] \quad k \text{ runs from 1 to 6}, \quad (\text{E.19})$$

$$F^{(2r,2rk)} = \frac{1}{14} \left[A + \frac{1}{4} B \mathcal{E}_7 \left(\frac{\tau+k}{7} \right) \right]; \quad k \text{ runs from 0 to 6.}$$

$$F^{(0,7)} = \frac{1}{14} \left[\frac{8}{3} A - \frac{4}{3} B \mathcal{E}_2(\tau) \right], \quad (\text{E.20})$$

$$F^{(7,7k)} = \frac{1}{14} \left[\frac{8}{3} A + \frac{2}{3} B \mathcal{E}_2 \left(\frac{\tau+k}{2} \right) \right] \quad k \text{ runs from 0 to 1.}$$

Conjugacy class 15A.

$$F^{(0,1)}(\tau, z) = F^{(0,2)} = F^{(0,4)} = F^{(0,7)} = F^{(0,8)} = F^{(0,11)} = F^{(0,13)} = F^{(0,14)}; \quad (\text{E.21})$$

$$= \frac{1}{15} \left[\frac{A}{3} - B \left(-\frac{1}{16} \mathcal{E}_3(\tau) - \frac{5}{24} \mathcal{E}_5(\tau) + \frac{35}{16} \mathcal{E}_{15}(\tau) - \frac{15}{4} \eta(\tau) \eta(3\tau) \eta(5\tau) \eta(15\tau) \right) \right].$$

$$F^{(r,rk)} = \frac{1}{15} \left[\frac{A}{3} + B \left(-\frac{1}{48} \mathcal{E}_3 \left(\frac{\tau+k}{3} \right) - \frac{1}{24} \mathcal{E}_5 \left(\frac{\tau+k}{5} \right) + \frac{7}{48} \mathcal{E}_{15} \left(\frac{\tau+k}{15} \right) - \frac{1}{4} \eta(\tau+k) \eta \left(\frac{\tau+k}{3} \right) \eta \left(\frac{\tau+k}{5} \right) \eta \left(\frac{\tau+k}{15} \right) \right) \right]; \quad (\text{E.22})$$

where $r=1,2,4,7,8,11,13,14$ and rk is mod 15. The sectors belonging to the 5A and 3A sub-orbits are given by

$$F^{(0,0)} = \frac{8}{15} A. \quad (\text{E.23})$$

$$F^{(0,3k)} = \frac{1}{15} \left(\frac{4}{3} A - \frac{5}{3} B \mathcal{E}_5(\tau) \right) \quad k \text{ runs from 1 to 4}; \quad (\text{E.24})$$

$$F^{(3r,3rk)} = \frac{1}{15} \left(\frac{4}{3} A + \frac{1}{3} B \mathcal{E}_5 \left(\frac{\tau+k}{5} \right) \right); \quad k \text{ runs from 0 to 4}.$$

$$F^{(0,5k)} = \frac{1}{15} \left(2A - \frac{3}{2} B \mathcal{E}_3(\tau) \right); \quad (\text{E.25})$$

$$F^{(5r,5rk)} = \frac{1}{15} \left(2A + \frac{1}{2} B \mathcal{E}_3 \left(\frac{\tau+k}{3} \right) \right) \quad k \text{ runs from 0 to 2}.$$

Finally the remaining sectors are given by

$$F^{(3r,5+3rk)} = \frac{1}{15} \left(\frac{A}{3} + B \left(-\frac{1}{4} \mathcal{E}_3(\tau) - \frac{1}{24} \mathcal{E}_5 \left(\frac{\tau+k}{5} \right) + \frac{3}{8} \mathcal{E}_5 \left(\frac{3\tau+3k}{5} \right) - \frac{3}{4} \eta(\tau+k) \eta(3\tau+3k) \eta \left(\frac{\tau+k}{5} \right) \eta \left(\frac{3\tau+3k}{5} \right) \right) \right), \quad (\text{E.26})$$

$$F^{(3r,10+3rk)} = \frac{1}{15} \left(\frac{A}{3} + B \left(-\frac{1}{4} \mathcal{E}_3(\tau) - \frac{1}{24} \mathcal{E}_5 \left(\frac{\tau+k}{5} \right) + \frac{3}{8} \mathcal{E}_5 \left(\frac{3\tau+3k}{5} \right) - \frac{3}{4} e^{-\frac{2\pi i}{5}} \eta(\tau+k) \eta(3\tau+3k) \eta \left(\frac{\tau+k}{5} \right) \eta \left(\frac{3\tau+3k}{5} \right) \right) \right);$$

where k runs from 0 to 4 and $s=1$ to 4.

$$F^{(5r,3s+5rk)} = \frac{1}{15} \left(\frac{A}{3} + B \left(\frac{5}{24} \mathcal{E}_5(\tau) + \frac{1}{12} \mathcal{E}_3 \left(\frac{\tau+k}{3} \right) - \frac{5}{24} \mathcal{E}_5 \left(\frac{\tau+k}{3} \right) + \frac{5}{4} \eta(\tau+k) \eta(5\tau+5k) \eta \left(\frac{\tau+k}{3} \right) \eta \left(\frac{5\tau+5k}{3} \right) \right) \right); \quad (\text{E.27})$$

where k runs from 0 to 2 and $s=1$ to 2.

Conjugacy class 2B.

$$\begin{aligned}
F^{(0,0)}(\tau, z) &= 2A; & F^{(0,1)}(\tau, z) &= F^{(0,3)}(\tau, z), \\
F^{(0,1)}(\tau, z) &= \frac{B(\tau, z)}{2}(\mathcal{E}_2(\tau) - \mathcal{E}_4(\tau)), \\
F^{(0,2)}(\tau, z) &= -\frac{2A(\tau, z)}{3} - \frac{2B(\tau, z)}{3}\mathcal{E}_2(\tau), \\
F^{(1,s)}(\tau, z) &= F^{(3,3s)} = -\frac{B(\tau, z)}{4}\left(\mathcal{E}_2\left(\frac{\tau+s}{2}\right) - \mathcal{E}_4\left(\frac{\tau+s}{4}\right)\right), \\
F^{(2,1)}(\tau, z) &= F^{(2,3)} = \frac{B(\tau, z)}{2}\left(-\frac{1}{6}\mathcal{E}_2(\tau) + \frac{2}{3}\mathcal{E}_2(2\tau)\right), \\
F^{(2,2s)}(\tau, z) &= -\frac{2A(\tau, z)}{3} + \frac{B(\tau, z)}{3}\mathcal{E}_2\left(\frac{\tau+s}{2}\right).
\end{aligned} \tag{E.28}$$

Conjugacy class 3B.

$$\begin{aligned}
F^{(0,0)}(\tau, z) &= \frac{8A(\tau, z)}{9}, \\
F^{(0,1)}(\tau, z) &= F^{(0,2)} = F^{(0,4)} = F^{(0,5)} = F^{(0,7)} = F^{(0,8)}; \\
F^{(0,1)}(\tau, z) &= -\frac{2B(\tau, z)}{9}\frac{\eta^6(\tau)}{\eta^2(3\tau)}, \\
F^{(0,3)}(\tau, z) &= -\frac{A(\tau, z)}{9} - \frac{B(\tau, z)}{4}\mathcal{E}_3(\tau), \\
F^{(r,rs)}(\tau, z) &= \frac{2B(\tau, z)}{3}\frac{\eta^6(\tau+s)}{\eta^2(\frac{\tau+s}{3})}, \quad r = 1, 2, 4, 5, 7, 8 \\
F^{(3,1)}(\tau, z) &= -\frac{2B(\tau, z)}{9}e^{2\pi i/3}\frac{\eta^6(\tau)}{\eta^2(3\tau)}, \\
&= F^{(3,4)} = F^{(3,7)} = F^{(6,2)} = F^{(6,8)} = F^{(6,5)}; \\
F^{(3,2)}(\tau, z) &= -\frac{2B(\tau, z)}{9}e^{4\pi i/3}\frac{\eta^6(\tau)}{\eta^2(3\tau)}, \\
&= F^{(3,5)} = F^{(3,8)} = F^{(6,1)} = F^{(6,7)} = F^{(6,4)}; \\
F^{(3r,3rk)}(\tau, z) &= -\frac{A(\tau, z)}{9} + \frac{B(\tau, z)}{12}\mathcal{E}_3\left(\frac{\tau+k}{3}\right).
\end{aligned} \tag{E.29}$$

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