Precision matching of circular Wilson loops and strings in $AdS_5 \times S^5$

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ABSTRACT: Previous attempts to match the exact $\mathcal{N} = 4$ super Yang-Mills expression for the expectation value of the $\frac{1}{2}$ -BPS circular Wilson loop with the semiclassical $AdS_5 \times S^5$ string theory prediction were not successful at the first subleading order. There was a missing prefactor $\sim \lambda^{-3/4}$ which could be attributed to the unknown normalization of the string path integral measure. Here we resolve this problem by computing the ratio of the string partition functions corresponding to the circular Wilson loop and the special $\frac{1}{4}$ -supersymmetric latitude Wilson loop. The fact that the latter has a trivial expectation value in the gauge theory allows us to relate the prefactor to the contribution of the three zero modes of the “transverse” fluctuation operator in the 5-sphere directions.

KEYWORDS: AdS-CFT Correspondence, Wilson, ’t Hooft and Polyakov loops

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1 Introduction

One of the central elements of the gauge-string correspondence is that the expectation value of the Wilson loop operator should be given by the string path integral [1]. In the context of the maximally supersymmetric AdS/CFT duality this translates, in particular, into the relation between the large N expectation value of the locally-supersymmetric Wilson loop in $\mathcal{N} = 4$ SYM theory [2, 3] and the $AdS_5 \times S^5$ superstring path integral on a disc with appropriate boundary conditions on a contour at the boundary of AdS_5 and in S^5 . Checking this precisely is non-trivial as it requires a careful normalization of the string path integral (which is subtle even in flat space, cf. [4]).

The simplest well-defined example is the $\frac{1}{2}$ -BPS circular WL [5, 6] the expectation value of which can be found exactly on the gauge theory side due to underlying superconformal symmetry [7–9]. Considering the planar limit and expanding at strong 't Hooft coupling $\lambda = g_{YM}^2 N_c$, one finds

$$W_C = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{(\sqrt{\lambda})^{3/2}} e^{\sqrt{\lambda}} \left(1 - \frac{3}{8\sqrt{\lambda}} + \dots \right), \quad \lambda \gg 1. \quad (1.1)$$

The corresponding $AdS_5 \times S^5$ superstring partition function expanded near the AdS_2 minimal surface ending on a circle at the boundary of AdS_5 should have the following expansion

in inverse string tension $T = \frac{\sqrt{\lambda}}{2\pi}$ [10]¹

$$W_C = n_0 \exp \left[\sqrt{\lambda} + c_1 + \frac{c_2}{\sqrt{\lambda}} + \mathcal{O}(\lambda^{-1/2}) \right], \quad (1.2)$$

where $\sqrt{\lambda}$ term comes from the classical string action, c_1 is one-loop string sigma-model correction, etc. Here n_0 is a potential overall factor of normalization of the string path integral measure on a disc.

The computation of the superstring fluctuation determinants in [11, 12] and [13] gave the one-loop value $c_1 = -\frac{1}{2} \log(2\pi)$, reproducing the π -factor in (1.1), with the remaining $\lambda^{-3/4}$ and power of 2 factor to be attributed to the presence of the n_0 normalization factor. It was suggested in [8] that $\lambda^{-3/4}$ may be related to the subtraction of the Möbius symmetry volume (or the ghost determinant zero modes) on the disc (see also [14]).

Our aim here will be to reproduce the exact prefactor in (1.1) and thus establish finally the precise matching between the gauge theory and string theory predictions at the two leading orders at strong coupling. Instead of computing the normalization n_0 directly we will consider the ratio of the circular WL expectation value to the expectation value of $\frac{1}{4}$ -supersymmetric WL [15] generalizing circular loop to the case when the string surface ends also on a big circle of S^5 . The residual global supersymmetry (this loop preserves 2 out of 8 Q -supercharges) implies the trivial expectation value for this loop for all values of λ

$$W_L = 1, \quad (1.3)$$

as anticipated in [15]. This non-renormalization theorem can be rigorously proven using superspace arguments in the $\mathcal{N} = 4$ gauge theory [16, 17]. While checking (1.3) directly on the string side would again require the knowledge of the string measure factor n_0 , one may consider the ratio of the two expectation values W_C/W_L in which the nuisance factor n_0 cancels out. Assuming non-renormalization of W_L , the 1-loop string theory computation should reproduce the precise factor appearing in (1.1). The appearance of $\lambda^{-3/4}$ can then be attributed to the normalization of the three zero modes of the S^5 fluctuation operator that arise due to degeneracies of the minimal surface for W_L [15].²

We used the label L in (1.3) to indicate that this $\frac{1}{4}$ -supersymmetric WL is a special case ($\theta_0 = \frac{\pi}{2}$) of a more general $\frac{1}{4}$ -BPS “latitude” WL [18, 19] depending on the latitude angle parameter θ_0 :

$$W_L \equiv W_L \left(\lambda, \theta_0 = \frac{\pi}{2} \right). \quad (1.4)$$

The case of $\theta_0 = 0$ corresponds to the $\frac{1}{2}$ -BPS circular WL and in general the exact expression for the expectation value of this loop $W_L(\theta_0)$ is given by (1.1) with the replacement $\lambda \rightarrow \lambda \cos^2 \theta_0$, i.e. $W_L(\lambda, \theta_0) = W_C(\lambda \cos^2 \theta_0)$ [19, 20]. The computation of the ratio $W_L(\lambda, \theta_0)/W_C$ on the string side was suggested in [21] and developed in [22] and later led

¹To simplify the notation we will not use separate letters for the gauge theory and string theory results as they should be eventually shown to be equal.

²The absence of the $\lambda^{-3/4}$ factor in (1.3) may be then understood as a result of the cancellation between the normalizations of the 3 Möbius symmetry zero modes of the ghost operator in conformal gauge and the 3 zero modes of the S^5 fluctuation operator.

to precise checks of the correspondence with gauge theory result, first in small θ_0 expansion [23] and then for general $0 < \theta_0 < \frac{\pi}{2}$ [24], reproducing

$$\frac{W_L(\lambda, \theta_0)}{W_C(\lambda)} = \exp \left[-\sqrt{\lambda}(1 - \cos \theta_0) - \frac{3}{2} \ln \cos \theta_0 + \mathcal{O}(\lambda^{-1/2}) \right], \quad \theta_0 < \frac{\pi}{2}. \quad (1.5)$$

The aim of the present paper is to extend the discussion in [24] to the special case of $\theta_0 = \frac{\pi}{2}$ when the S^5 fluctuation operator develops 3 zero modes and to show that in this case W_C/W_L is also in full agreement with (1.1), including the enigmatic $\lambda^{-3/4}$ and $\sqrt{2}$ factors.

We shall start in section 2 with a brief review of the definitions of W_C and W_L in gauge theory and the corresponding minimal string surfaces in $AdS_5 \times S^5$. Semiclassical expansion near a minimal string surface will be discussed in section 3, paying special attention to the contribution of zero modes of the quadratic fluctuation operator present in the special $\theta_0 = \frac{\pi}{2}$ latitude case. The relevant determinant will be computed in section 4 demonstrating that the string theory prediction for the ratio W_L/W_C is in perfect agreement with the gauge-theory results (1.1), (1.3) expanded at large λ . Properties of the conformal anomaly in the presence of zero modes, used in the intermediate steps of the derivation, are reviewed in the appendix.

2 Setup

2.1 Wilson loops

The Wilson loop expectation value in $\mathcal{N} = 4$ super-Yang-Mills theory [2] is given by

$$W(C; \mathbf{n}) = \frac{1}{N} \left\langle \text{tr P exp} \left[i \int_C d\tau (\dot{x}^\mu A_\mu + i|\dot{x}|n^I \Phi_I) \right] \right\rangle. \quad (2.1)$$

The spacial contour will always be the unit circle: $x^\mu = (\cos \tau, \sin \tau, 0, 0)$, while for the scalar coupling we consider two different cases: the pure circular loop with constant coupling to scalars and the latitude. The coupling of the Wilson loop to scalars is parameterized by a unit 6-vector n^I on S^5 , best represented for our purposes as an $S^1 \times S^3$ fibration over an interval,

$$\mathbf{n} = (\cos \theta \mathbf{k}, \sin \theta \cos \varphi, \sin \theta \sin \varphi), \quad (2.2)$$

where $\mathbf{k}$ is a unit 4-vector. The circular loop and the special latitude correspond to

$$\begin{aligned} \text{C :} \quad & \theta = 0, \quad \mathbf{k} = (1, 0, 0, 0), \quad \varphi = \text{any}, \\ \text{L :} \quad & \theta = \frac{\pi}{2}, \quad \mathbf{k} = \text{any}, \quad \varphi = \tau. \end{aligned} \quad (2.3)$$

On the north pole of the 5-sphere S^1 shrinks to zero size and consequently φ can take any value, while on the equator S^3 shrinks to zero size and $\mathbf{k}$ can be arbitrary. The vector $\mathbf{k}$ is not an extra parameter of the contour but will reappear as such in the string-theory calculation where the string will move away from the locus where S^3 is shrunk to a point.

As discussed above, the expectation value of the $\frac{1}{4}$ -supersymmetric latitude is trivial, and its ratio to the circular loop can be calculated exactly (cf. (1.1), (1.3))

$$\frac{W_L}{W_C} = \frac{\sqrt{\lambda}}{2I_1(\sqrt{\lambda})} \stackrel{\lambda \rightarrow \infty}{\simeq} \sqrt{\frac{\pi}{2}} \lambda^{\frac{3}{4}} e^{-\sqrt{\lambda}}. \quad (2.4)$$

2.2 String solutions

In string theory, the Wilson loop is represented by a disc partition function which at strong coupling is saturated by the area law in $AdS_5 \times S^5$:

$$W(C; \mathbf{n}) \stackrel{\lambda \rightarrow \infty}{\simeq} n_0 \text{Sdet}^{-\frac{1}{2}} \mathbb{K} e^{-\frac{\sqrt{\lambda}}{2\pi} A_{\min}}, \quad (2.5)$$

where $A_{\min}$ is the (regularized) minimal area, that depends on the contour at hand, $\mathbb{K}$ is the quadratic form of string fluctuations around the minimal surface and n_0 is the measure factor discussed in the introduction.³

The metric of $AdS_5 \times S^5$ that corresponds to the Poincaré coordinates in AdS_5 and to the parameterization (2.2) of the sphere is

$$ds^2 = \frac{dX_\mu^2 + dZ^2}{Z^2} + d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\Omega_{S^3}^2. \quad (2.6)$$

The AdS part of the string solution is the same for the circle and latitude:

$$\text{C/L :} \quad X^\mu = \frac{(\cos \tau, \sin \tau, 0, 0)}{\cosh \sigma}, \quad Z = \tanh \sigma. \quad (2.7)$$

For the circle the string always stays at $\theta = 0$ and the dynamics on S^5 are trivial, while for the latitude the string has a non-trivial profile in S^5 [15]:

$$\text{L :} \quad \cos \theta = \tanh \sigma, \quad \varphi = \tau, \quad (2.8)$$

where $\sigma \in (0, \infty)$ and $\tau \in (0, 2\pi)$ are the world-sheet coordinates. The unit 4-vector $\mathbf{k}$ remains arbitrary, meaning that there is a whole three-parametric family of solutions. All these solutions are different, because $\theta < \pi/2$ in the interior of the worldsheet and consequently S^3 inflates to a finite size away from the boundary at $\sigma = 0$ ($Z = 0$).

Both of these solutions are in the conformal gauge, i.e. the induced world-sheet metric is

$$ds_{\text{ind}}^2 = \Omega^2 (d\tau^2 + d\sigma^2), \quad (2.9)$$

with the conformal factors being

$$\Omega_C^2 = \frac{1}{\sinh^2 \sigma}, \quad \Omega_L^2 = \frac{1}{\sinh^2 \sigma} + \frac{1}{\cosh^2 \sigma}. \quad (2.10)$$

³This semiclassical formula gets modified if the minimal surface is degenerate and $\mathbb{K}$ develops zero modes. Modifications are non-trivial and discussed in detail later.

3 Semiclassical string quantization

The semiclassical formula (2.5) can be readily applied to the circle, but has to be modified for the latitude to take into account the three-parametric degeneracy in the minimal surface. We briefly review the general formalism of collective coordinates and then apply it to the case of the latitude Wilson loop.

Suppose that a path integral of some field theory is saturated by a classical field configuration Φ_{cl} that depends on n parameters φ_i . These parameters are traded for collective coordinates by the change of variables in the path integral. The general formula for the semiclassical partition function is

$$\int \mathcal{D}\Phi \, e^{-\frac{1}{\hbar} S[\Phi]} \stackrel{\hbar \rightarrow 0}{\simeq} \int \prod_{i=1}^n \frac{d\varphi_i}{\sqrt{2\pi\hbar}} \det_{ij}^{\frac{1}{2}} \left\langle \frac{\partial \Phi_{\text{cl}}}{\partial \varphi_i}, \frac{\partial \Phi_{\text{cl}}}{\partial \varphi_j} \right\rangle \text{Sdet}'^{-\frac{1}{2}} \mathbb{K} \, e^{-\frac{1}{\hbar} S[\Phi_{\text{cl}}]}, \quad (3.1)$$

where $\mathbb{K}$ is the quadratic form of the action expanded around the classical solution:

$$\mathbb{K} = \frac{\delta^2 S}{\delta \Phi \delta \Phi}. \quad (3.2)$$

The quadratic form has zero modes, because

$$\mathbb{K} \cdot \frac{\partial \Phi_{\text{cl}}}{\partial \varphi_i} = \frac{\partial}{\partial \varphi_i} \frac{\delta S}{\delta \Phi} = 0, \quad (3.3)$$

and those should be omitted (which is marked by prime on Sdet), as they are already accounted for by the collective coordinates. The first determinant factor in (3.1) is the Jacobian of this transformation. Finally, the factor of $\frac{1}{\sqrt{2\pi\hbar}}$ comes from the missing Gaussian integral over each of the n zero modes.⁴

We can now apply this general result to the latitude normalized by the expectation value of the circle. According to the AdS/CFT dictionary, the $\hbar$ of string theory is the inverse string tension

$$\hbar = \frac{2\pi}{\sqrt{\lambda}}. \quad (3.4)$$

From (2.2), (2.8) we find:

$$\frac{\partial \mathbf{n}_{\text{cl}}}{\partial \varphi_i} = \left(\tanh \sigma \frac{\partial \mathbf{k}}{\partial \varphi_i}, 0, 0 \right), \quad (3.5)$$

where φ_i are the three angles on the three-sphere. Since no other factor depends on $\mathbf{k}$, the integral over the collective coordinates just gives the volume of S^3 :

$$\int_{S^3} d^3\varphi \det_{ij}^{\frac{1}{2}} \frac{\partial \mathbf{k}}{\partial \varphi_i} \cdot \frac{\partial \mathbf{k}}{\partial \varphi_j} = \text{Vol}(S^3) = 2\pi^2. \quad (3.6)$$

⁴The standard normalization of the measure is such that a Gaussian integral over each quadratic fluctuation mode should not depend on $\hbar$.

Taking into account that the (regularized) minimal area $S_{\text{cl}} = -2\pi$ for the circle [5, 6] and 0 for the latitude [15],⁵ one finds:

$$\frac{W_L}{W_C} = \frac{2\pi^2}{(2\pi)^3} \lambda^{\frac{3}{4}} e^{-\sqrt{\lambda}} \langle \psi_0 | \Omega_L^2 | \psi_0 \rangle^{\frac{3}{2}} \frac{\text{Sdet}^{\frac{1}{2}} \mathbb{K}_C}{\text{Sdet}'^{\frac{1}{2}} \mathbb{K}_L}, \quad (3.7)$$

where

$$\psi_0 = \tanh \sigma \quad (3.8)$$

is the zero mode of the fluctuation operator on S^5 .

The norm of the zero mode is defined with respect to the induced metric on the string worldsheet and contains the scale factor Ω . We reserve the bracket notation for the conventional unit norm, thus

$$\langle \psi_1, \psi_2 \rangle = \int d\tau d\sigma \Omega^2 \psi_1^* \psi_2 = \langle \psi_1 | \Omega^2 | \psi_2 \rangle. \quad (3.9)$$

The explicit form of the one-loop contribution has been worked out by applying the general formalism of [10, 26] to the minimal surface of the circular loop [10, 11] and the latitude [21, 22]:

$$\text{Sdet } \mathbb{K} = \frac{\det^3 \mathcal{K}_1 \det^3 \mathcal{K}_2 \det \mathcal{K}_{3+} \det \mathcal{K}_{3-}}{\det^4 \mathcal{D}_+ \det^4 \mathcal{D}_-}, \quad (3.10)$$

where $\mathcal{K}_1$ corresponds to three fluctuation modes on AdS_5 , $\mathcal{K}_2$ describes three “transverse” modes on S^5 , while the remaining two S^5 modes mix and result in $\mathcal{K}_{3\pm}$. The Dirac operators $\mathcal{D}_{\pm}$ originate from expanding the Green-Schwarz action on $AdS_5 \times S^5$ to quadratic order in fermions in a particular kappa-symmetry gauge [10].

Following [21, 22, 24] we get rid of the conformal factor in the induced metric (2.9) by a Weyl transformation. Because the conformal factors at hand (2.10) diverge at the symmetry point of the minimal surface, this transformation is actually singular and changes the topology of the worldsheet from the disk to a semi-infinite cylinder. Singularity in the Weyl transformation induces an IR anomaly which affects the one-loop corrections and has to be carefully taken into account [24].

Assuming the standard dependence of the fluctuation operators on the conformal factor,

$$\mathcal{K} = \frac{1}{\Omega^2} \tilde{\mathcal{K}}, \quad \mathcal{D} = \frac{1}{\Omega^{\frac{3}{2}}} \tilde{\mathcal{D}} \Omega^{\frac{1}{2}}, \quad (3.11)$$

the flat-metric expressions⁶ for the circle take the form:

$$\begin{aligned} \text{C :} \quad \tilde{\mathcal{K}}_1 &= -\partial_\tau^2 - \partial_\sigma^2 + \frac{2}{\sinh^2 \sigma}, \\ \tilde{\mathcal{K}}_2 &= \tilde{\mathcal{K}}_{3\pm} = -\partial_\tau^2 - \partial_\sigma^2, \\ \tilde{\mathcal{D}}_{\pm} &= -i\partial_\tau \tau_2 + i\partial_\sigma \tau_1 + \frac{1}{\sinh \sigma} \tau_3, \end{aligned} \quad (3.12)$$

⁵The supersymmetric latitude belongs to a more general class of BPS string solutions, owing their existence to a non-integrable almost complex structure on $AdS_5 \times S^5$. All of these surfaces can be shown to have zero regularized area [25].

⁶We adopt notations and conventions of [21].

where τ_i are the standard Pauli matrices. For the latitude we have:

$$\begin{aligned}
 \text{L:} \quad \tilde{\mathcal{K}}_1 &= -\partial_\tau^2 - \partial_\sigma^2 + \frac{2}{\sinh^2 \sigma}, \\
 \tilde{\mathcal{K}}_2 &= -\partial_\tau^2 - \partial_\sigma^2 - \frac{2}{\cosh^2 \sigma}, \\
 \tilde{\mathcal{K}}_{3\pm} &= -\partial_\tau^2 - \partial_\sigma^2 \pm 2i (\tanh 2\sigma - 1) \partial_\tau + (\tanh 2\sigma - 1) (1 + 3 \tanh 2\sigma), \\
 \tilde{\mathcal{D}}_\pm &= - \left[i\partial_\tau \mp \frac{1}{2} (1 - \tanh 2\sigma) \right] \tau_2 + i\partial_\sigma \tau_1 + \frac{1}{\Omega \sinh^2 \sigma} \tau_3 \mp \frac{1}{\Omega \cosh^2 \sigma}.
 \end{aligned} \tag{3.13}$$

The operator $\mathcal{K}_2$, which describes the three transverse fluctuations on the sphere, has a zero mode associated with the three-parametric degeneracy of the classical string solution.⁷ It is easy to see directly that ψ_0 defined in (3.8) satisfies

$$\mathcal{K}_2 \psi_0 = 0. \tag{3.14}$$

As was shown in [10] and explicitly checked in [24], the one-loop partition function for the circle, $\text{Sdet}^{-\frac{1}{2}} \mathbb{K}_C$, is Weyl-invariant. For the latitude the situation is more complicated because $\mathcal{K}_2$ has a zero mode. In appendix A we show that the ratio $\langle \psi_0 | \Omega^2 | \psi_0 \rangle / \det' \mathcal{K}_2$ shifts under Weyl rescalings by the conventional conformal anomaly, which is eventually canceled by the contributions of other string modes when they are combined together. From this we conclude that for the latitude $\langle \psi_0 | \Omega_L^2 | \psi_0 \rangle^3 / \text{Sdet}' \mathbb{K}_L$ is Weyl invariant.

Conformal anomaly cancellation allows us to drop the conformal factor in the ratio (3.7) and to deal with the operators (3.12), (3.13) defined on a flat half-cylinder:

$$\frac{W_L}{W_C} = \frac{1}{4\pi} \lambda^{\frac{3}{4}} e^{-\sqrt{\lambda}} \langle \psi_0 | \psi_0 \rangle^{\frac{3}{2}} \frac{\text{Sdet}^{\frac{1}{2}} \tilde{\mathbb{K}}_C}{\text{Sdet}'^{\frac{1}{2}} \tilde{\mathbb{K}}_L}. \tag{3.15}$$

This expression, taken at face value, is ill-defined due to IR divergences at large values of σ . A way to regularize these divergences is to artificially impose some boundary conditions (e.g. Dirichlet or Neumann) at a large $\sigma = R_{C,L}$ and send $R_{C,L} \rightarrow \infty$ at the end of the calculation. As shown in [24], such regularization does not spoil conformal anomaly cancellation. While the cutoff dependence eventually cancels in the ratio (3.15), the diffeomorphism invariance requires to choose different cutoffs for the circle and the latitude. The difference leaves a finite remnant in the ratio of partition functions [24].

An invariant cutoff is the worldsheet area left out by regularization:

$$s = \int_{\sigma > R} d^2 \sigma \sqrt{h} = 2\pi \int_R^\infty d\sigma \Omega^2, \tag{3.16}$$

and this should be the same for the circle and the latitude. Using explicit form (2.10) of the scale factors for the respective minimal surfaces, we find that the coordinate cutoffs

⁷For the general case of the latitude WL with $\theta_0 \neq \frac{\pi}{2}$ (cf. (1.4)) the corresponding $\mathcal{K}_2$ operator has a zero mode corresponding to $\tanh(\sigma + \sigma_0)$ where σ_0 is defined by $\tanh \sigma_0 = \cos \theta_0$. This mode corresponding also to the $p \rightarrow 0$ limit of the continuum spectrum, is not, however, normalizable.

are related by a finite relative shift:

$$R_C = R_L - \frac{1}{2} \ln 2. \quad (3.17)$$

Later we will give another justification for this relationship.

The IR cutoff is necessary to define determinants on the half-cylinder. In addition, it regularizes the zero mode normalization (cf. (3.8), (3.9))

$$\langle \psi_0 | \psi_0 \rangle = 2\pi \int_0^R d\sigma \tanh^2 \sigma = 2\pi R + \mathcal{O}(1). \quad (3.18)$$

4 Determinants and phase-shifts

The determinants in (3.15) were calculated in [21, 22, 24] for a more general, non-supersymmetric (but superconformal) θ_0 -latitude minimal surface and the results relevant for our problem may be read off from there by taking the limit $\theta_0 = \frac{\pi}{2}$. More precisely, this works directly for all the operators except $\mathcal{K}_2$ which develops a zero mode in the supersymmetric limit. The determinant of $\tilde{\mathcal{K}}_2$, as a result, blows up in the limit and has to be reconsidered anew by carefully handling the zero mode.

4.1 Spectral problem for $\tilde{\mathcal{K}}_2$

The eigenvalues of $\tilde{\mathcal{K}}_2$ defined in (3.12), (3.13) are of the form $\omega^2 + p^2$ where ω is an integer “Matsubara” frequency (corresponding to the τ -circle) and p^2 is an eigenvalue of a one-dimensional Schrödinger operator defined on functions of σ . For the circle this operator is simply $-\partial_\sigma^2$, while for the latitude the Schrödinger equation is

$$\left(-\partial_\sigma^2 - \frac{2}{\cosh^2 \sigma} \right) \psi = p^2 \psi. \quad (4.1)$$

The solution satisfying the boundary condition $\psi(0) = 0$ is

$$\psi(\sigma) = p \sin p\sigma + \tanh \sigma \cos p\sigma. \quad (4.2)$$

Notice that for $p = 0$ we recover the zero mode eigenfunction ψ_0 in (3.8).

The operator (4.1) has a continuous spectrum in infinite volume. To define the determinant we need to regularize the problem which we do by imposing the Neumann boundary condition at $\sigma = R_L$:

$$\psi'(R_L) = 0. \quad (4.3)$$

The precise form of the boundary condition is not so important, but it has to be consistent with the existence of the zero mode. The Dirichlet condition $\psi(R_L) = 0$, for instance, eliminates the zero mode and, if imposed, spoils connection to the original spectral problem for the undeformed operator $\mathcal{K}_2$.

A more rigorous regularization prescription consists in gradually eliminating the scale factor by replacing Ω with Ω^α in (3.11), as done in the appendix A. Weyl invariance of the

string partition function guarantees that the answer does not depend on α , and one can eliminate the scale factor by taking $\alpha \rightarrow 0$. The scale factor for a very small but finite α is approximately equal to one on a wide interval $e^{-1/2\alpha} \ll \sigma \ll 1/\alpha$. The divergence of Ω^α at small sigma reinforces the boundary condition $\psi(0) = 0$, while exponential decay at $\sigma \sim 1/\alpha$ introduces an effective IR cutoff. Defining the cutoff scale through $\Omega^\alpha \simeq C$, we find that $R \simeq \ln C/\alpha + \nu/2$, where $\nu = 2 \ln 2$ for the circle and $\nu = 3 \ln 2$ for the latitude, giving an alternative explanation to the relationship (3.17) between the respective IR cutoffs. With some effort these arguments can be made more precise; here we will not go deeper into the technical details, which are relatively sophisticated, but at the end give the same result as the simple-minded Neumann boundary condition.

At large σ the wavefunction (4.2) assumes the asymptotic form, valid up to exponential corrections:

$$\psi(\sigma) \simeq \sqrt{p^2 + 1} \sin(p\sigma + \delta(p)), \quad (4.4)$$

with the phase-shift equal to

$$\delta(p) = \frac{\pi}{2} - \arctan p. \quad (4.5)$$

The Neumann boundary condition imposes momentum quantization

$$p_n R_L + \delta(p_n) = \pi \left(n + \frac{1}{2} \right). \quad (4.6)$$

In this language the zero mode ($p_0 = 0$) arises because $\delta(0) = \frac{\pi}{2}$.

The determinant of $\tilde{\mathcal{K}}_2$ requires also a UV regularization at large momenta. We will use Pauli-Villars regularization, following [27], and define the regularized determinant by

$$\det_{\text{reg}} \tilde{\mathcal{K}}_2^L = \frac{\det' \tilde{\mathcal{K}}_2^L}{\det(\tilde{\mathcal{K}}_2^L + M^2)} = \frac{1}{M^2} \prod_{(\omega, n) \neq (0, 0)} \frac{\omega^2 + p_n^2}{\omega^2 + p_n^2 + M^2}, \quad (4.7)$$

which can also be written as

$$\det_{\text{reg}} \tilde{\mathcal{K}}_2^L = \frac{1}{M^2} \prod_{\omega \neq 0} \prod_{n=0}^{\infty} \frac{\omega^2 + p_n^2}{\omega^2 + p_n^2 + M^2} \prod_{n=0}^{\infty} \frac{p_{n+1}^2}{p_{n+1}^2 + M^2}. \quad (4.8)$$

Eventually we will divide this by the corresponding determinant for the circle, the wavefunction for which is $\psi(\sigma) = \sin p\sigma$ and the momentum quantization condition is just

$$\bar{p}_n R_C = \pi \left(n + \frac{1}{2} \right). \quad (4.9)$$

This has no zero mode, and the term with $(\omega, n) = (0, 0)$ need not be excluded from the product:

$$\det_{\text{reg}} \tilde{\mathcal{K}}_2^C = \prod_{\omega, n} \frac{\omega^2 + \bar{p}_n^2}{\omega^2 + \bar{p}_n^2 + M^2}. \quad (4.10)$$

The difference between R_L and R_C , given by (3.17), can be absorbed into the redefinition of the phase-shift:

$$\delta(p) \rightarrow \delta(p) + \frac{p}{2} \ln 2, \quad (4.11)$$

after which R_C can be used as an IR cutoff for both the circle and the latitude; we will denote it simply by R to simplify the notations. The momentum quantization condition (4.6) then reads:

$$p_n - \bar{p}_n = -\frac{1}{R} \left(\delta(p_n) + \frac{p_n}{2} \ln 2 \right). \quad (4.12)$$

The product in (4.8) receives contributions from the two type of modes, the regularized continuous spectrum with $n \sim R$ and the near-zero modes with $n \ll R$. To separate the two we introduce an intermediate scale $N \gg 1$ such that

$$\varepsilon = \frac{\pi N}{R} \ll 1, \quad (4.13)$$

and treat separately the modes with $n < N$ and $n \geq N$. The near-zero modes are only important at zero Matsubara frequency $\omega = 0$. Their contribution to the ratio of determinants is

$$\prod_{n=0}^{N-1} \frac{p_{n+1}^2}{\bar{p}_n^2} = \prod_{n=0}^N \frac{(n+1)^2}{(n+\frac{1}{2})^2} = \frac{\Gamma^2(\frac{1}{2}) \Gamma^2(N+1)}{\Gamma^2(N+\frac{1}{2})} \simeq \pi N. \quad (4.14)$$

The meaning of this formula is simple. For the circle, the wavefunction is $\sin p\sigma$, and the Neumann boundary condition picks up modes with $\bar{p}_n = \pi(n + \frac{1}{2})/R$, $n = 0, 1, 2, \dots$. The first term in the latitude wavefunction (4.2) is suppressed at low momenta, while the second term asymptotes to $\cos p\sigma$, which gives $p_n = \pi n/R$, $n = 0, 1, 2, \dots$. The difference in the quantization conditions can be attributed to the non-vanishing phase-shift at zero momentum, for the latitude: $\delta(0) = \frac{\pi}{2}$. The zero mode for the latitude should be dropped from the product which amounts to shifting the mode number: $n \rightarrow n + 1$. This shift affects only the zero Matsubara frequency and can be equivalently described as a phase-shift redefinition:

$$\delta(p) \rightarrow \delta(p) - \pi. \quad (4.15)$$

For the contribution of the continuous spectrum, the summation over n can be replaced by integration, and with the help of (4.12) we get:

$$\sum_{n=N}^{\infty} (f(p_n) - f(\bar{p}_n)) = - \int_{\varepsilon}^{\infty} \frac{dp}{\pi} f'(p) \left(\delta(p) + \frac{p}{2} \ln 2 \right), \quad (4.16)$$

up to corrections that vanish at $R \rightarrow \infty$. Applying this formula to (4.8), taking into account an extra shift (4.15) for the zero Matsubara frequency and adding the low-momentum contribution (4.14), we get for the ratio of determinants:

$$\begin{aligned} \frac{\det_{\text{reg}} \tilde{\mathcal{K}}_2^{\text{L}}}{\det_{\text{reg}} \tilde{\mathcal{K}}_2^{\text{C}}} &= \frac{\pi N}{M^2} \exp \left[-2 \int_{\varepsilon}^{\infty} \frac{dp}{\pi} \left(\delta(p) + \frac{p}{2} \ln 2 \right) \sum_{\omega \neq 0} \left(\frac{p}{\omega^2 + p^2} - \frac{p}{\omega^2 + p^2 + M^2} \right) \right. \\ &\quad \left. - 2 \int_{\varepsilon}^{\infty} \frac{dp}{\pi} \left(\delta(p) + \frac{p}{2} \ln 2 - \pi \right) \left(\frac{1}{p} - \frac{p}{p^2 + M^2} \right) \right]. \end{aligned} \quad (4.17)$$

As a next step, we perform summation over ω and explicitly integrate the π term in the last integral. The UV logarithm from this integral neatly cancels the UV divergence due to the zero mode ($1/M^2$ in the prefactor), which is not surprising as they both originate from

the “spectral flow”: omitted zero mode leaves an uncompensated mode in the regulator determinant and, at the same time, shifts up mode numbers by one, which is ultimately responsible for the extra UV log. The IR part of the log combines with the auxiliary scale N to form the physical IR cutoff R . Using the definition (4.13), we find:

$$\frac{\det_{\text{reg}} \tilde{\mathcal{K}}_2^{\text{L}}}{\det_{\text{reg}} \tilde{\mathcal{K}}_2^{\text{C}}} = R \exp \left[-2 \int_{\varepsilon}^{\infty} dp \left(\delta(p) + \frac{p}{2} \ln 2 \right) \right. \\ \left. \times \left(\coth \pi p - \frac{p}{\sqrt{p^2 + M^2}} \coth \pi \sqrt{p^2 + M^2} \right) - \ln \varepsilon \right]. \quad (4.18)$$

The dependence on ε in the exponent is fake, it actually cancels out because the momentum integral is log divergent on the lower limit.

4.2 Phase-shifts and other operators

The determinants of the other operators in (3.13) are substantially simpler, because they have no zero modes, and can be expressed through the phase-shifts for the respective 1d operators. The explicit expressions for the phase-shifts (taken from [24]) are:

$$\begin{aligned} \text{C :} \quad & \delta_1(p) = -\arctan p, \\ & \delta_2(p) = 0, \\ & \delta_3(p) = 0, \\ & \delta_F(p) = -\arctan 2p, \end{aligned} \quad (4.19)$$

for the circle and

$$\begin{aligned} \text{L :} \quad & \delta_1(p) = -\arctan p, \\ & \delta_2(p) = \frac{\pi}{2} - \arctan p, \\ & \delta_3(p) = \frac{1}{2} \arctan p - \frac{1}{2} \arctan \frac{p}{2}, \\ & \delta_F(p) = -\frac{1}{2} \arctan 2p - \frac{1}{2} \arctan \frac{2p}{3}, \end{aligned} \quad (4.20)$$

for the latitude. The only mode with $\delta(0) \neq 0$ is the one corresponding to $\tilde{\mathcal{K}}_2$, all others have $\delta(0) = 0$ and do not produce an IR contribution similar to (4.14).

Combining the contributions of all the modes together we find:

$$\frac{\text{Sdet}' \mathbb{K}_{\text{L}}}{\text{Sdet} \mathbb{K}_{\text{C}}} = R^3 \exp \left[-2 \int_{\varepsilon}^{\infty} dp \sum_a (-1)^{F_a} \left(\delta_a^{\text{L}}(p) - \delta_a^{\text{C}}(p) + \frac{p}{2} \ln 2 \right) \right. \\ \left. \times \coth^{(-1)^{F_a}} \pi p - 3 \ln \varepsilon \right], \quad (4.21)$$

where the sum is over all $8_{\text{B}} + 8_{\text{F}}$ string modes counted with multiplicities. This formula takes into account anti-periodic boundary conditions for fermions in τ , which entails half-integer Matsubara frequencies and replacement of $\coth$ by $\tanh$ in the momentum integral. The UV divergences in the momentum integral happen to cancel, allowing us to drop the

regulator term. The IR scale R cancels in (3.15) the divergent norm of the zero-mode wavefunction (3.18).⁸

The final result for the semiclassical string theory expression for the ratio of the two Wilson loops is thus

$$\frac{W_L}{W_C} = \sqrt{\frac{\pi}{2}} \lambda^{\frac{3}{4}} e^{-\sqrt{\lambda}} Z_{1\text{-loop}} , \quad (4.22)$$

where $Z_{1\text{-loop}}$ given by

$$\ln Z_{1\text{-loop}} = \int_{\varepsilon}^{\infty} dp \sum_a (-1)^{F_a} \left(\delta_a^L(p) - \delta_a^C(p) + \frac{p}{2} \ln 2 \right) \coth^{(-1)^{F_a}} \pi p + \frac{3}{2} \ln \varepsilon \quad (4.23)$$

is the genuine contribution of the string fluctuations. It is interesting to note that the zero modes produce the factor $\sqrt{\frac{\pi}{2}} \lambda^{\frac{3}{4}}$ in (4.22), which itself happens to agree already with the prefactor in the gauge-theory prediction in (2.4).

4.3 String fluctuations

To compute the momentum integral in (4.23) we first see that

$$\begin{aligned} & \sum_a (-1)^{F_a} \left(\delta_a^L(p) - \delta_a^C(p) + \frac{p}{2} \ln 2 \right) \\ &= \left(\frac{3\pi}{2} - \arctan \frac{p}{2} - 2 \arctan p \right)_B + 4 \left(\arctan \frac{2p}{3} - \arctan 2p \right)_F . \end{aligned} \quad (4.24)$$

The combined expression decreases very fast at infinity, as $1/p^3$, and can be integrated just by itself:

$$J_I = \int_0^{\infty} dp (-1)^{F_a} \left(\delta_a^L(p) - \delta_a^C(p) + \frac{p}{2} \ln 2 \right) = 6 \ln \frac{3}{2} . \quad (4.25)$$

In the full integral (4.23), the bosonic/fermionic part of (4.24) is multiplied by $\coth \pi p / \tanh \pi p$. After subtracting 1 from the hyperbolic functions, each term in the sum converges individually and all the terms can be integrated one by one. The basic integrals are

$$\begin{aligned} J_{II} &= \int_0^{\infty} dp (\coth \pi p - 1) \arctan \frac{p}{z} = \ln \Gamma(z) - \left(z - \frac{1}{2} \right) \ln z + z - \frac{1}{2} \ln 2\pi , \\ J_{III} &= \int_0^{\infty} dp (\tanh \pi p - 1) \arctan \frac{p}{z} = \ln \Gamma\left(z + \frac{1}{2}\right) - z \ln z + z - \frac{1}{2} \ln 2\pi . \end{aligned} \quad (4.26)$$

The constant term produces an IR divergent integral:

$$J_{IV} = \pi \int_{\varepsilon}^{\infty} dp (\coth \pi p - 1) = -\ln(2\pi\varepsilon) . \quad (4.27)$$

⁸The IR cutoff in (3.18) is R_L while the prefactor in the ratio of determinants is R_C . Their difference, of order one, was important to keep in the exponent, but in the pre-factor this difference is immaterial as long as $R \rightarrow \infty$.

Finally, there is also a contribution from the common auxiliary phase-shift originating from the difference (3.17) between R_L and R_C :

$$J_V = 8 \int_0^\infty dp p (\coth \pi p - \tanh \pi p) = 1. \quad (4.28)$$

Collecting all the pieces together we get:

$$\begin{aligned} \ln Z_{1\text{-loop}} = & \left(6 \ln \frac{3}{2} \right)_{(I)} + \left(\frac{3}{2} \ln 2\pi - \frac{1}{2} \ln 2 - 6 \ln \frac{3}{2} \right)_{(II)+(III)} \\ & + \left(-\frac{3}{2} \ln(2\pi\varepsilon) \right)_{(IV)} + \left(\frac{1}{2} \ln 2 \right)_{(V)} + \frac{3}{2} \ln \varepsilon, \end{aligned} \quad (4.29)$$

where subscripts indicate individual contributions of the integrals (4.25)–(4.28). When the dust settles, we find

$$Z_{1\text{-loop}} = 1, \quad (4.30)$$

so that (4.22) becomes

$$\frac{W_L}{W_C} = \sqrt{\frac{\pi}{2}} \lambda^{\frac{3}{4}} e^{-\sqrt{\lambda}}, \quad (4.31)$$

in agreement with the field-theory prediction (2.4).

5 Conclusions

We have shown how string theory in $AdS_5 \times S^5$ reproduces the strong-coupling asymptotics of the exact expectation value of the circular Wilson loop, which has been known for a long time [7–9], finally nailing down the exact prefactor which arises due to quantum fluctuations of the string. The derivation is not very simple and rests on an additional assumption that the supersymmetric latitude has trivial expectation value. The latter is a theorem in field theory [16, 17], but an intrinsic string-theory derivation of this statement is lacking. Perhaps it can be proven by extending the classical argument of [25], that applies to general BPS surfaces, to quantum theory.

It is interesting that the whole contribution to the prefactor comes from the zero modes, while the non-zero modes cancel, as they do for the straight line related to the circle by an (anomalous) conformal transformation. The nature of this cancellation may shed light on a non-perturbative disc amplitude, which is supposed to reproduce the whole Bessel function expression (1.1), valid at any coupling in planar super-Yang-Mills theory.

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A Conformal anomaly and zero modes

Consider a family of 2d scalar operators (ϕ and E are some given functions)

$$K(\alpha) = e^{2\alpha\phi} (-\partial^2 + E), \quad (\text{A.1})$$

each having some number of zero modes:

$$K(\alpha) |n\rangle = 0. \quad (\text{A.2})$$

The operator $K(\alpha)$ is Hermitian with respect to the measure

$$\langle \psi_1, \psi_2 \rangle_\alpha = \int d^2\sigma e^{-2\alpha\phi} \psi_1^* \psi_2. \quad (\text{A.3})$$

As in the main text, we reserve the bra-ket notations for the conventional ($\alpha = 0$) scalar product. The projector onto non-zero modes of $K(\alpha)$ then takes the form

$$P = \mathbb{1} - \sum_n \frac{|n\rangle \langle n| e^{-2\alpha\phi}}{\langle n| e^{-2\alpha\phi} |n\rangle}. \quad (\text{A.4})$$

The dependence of the determinant of K on α is governed by the conformal anomaly. Here we give a brief derivation, emphasizing the contribution of the zero modes, following [28].

The regularized determinant of K (with zero modes omitted) can be defined via its zeta-function:

$$\ln \det' K = - \lim_{s \rightarrow 0} \frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr} P e^{-tK}. \quad (\text{A.5})$$

Differentiating in α , and taking into account that

$$\frac{\partial}{\partial \alpha} \text{Tr} P e^{-tK} = 2t \frac{\partial}{\partial t} \text{Tr} P \phi e^{-tK}, \quad (\text{A.6})$$

we get

$$\begin{aligned} \frac{d}{d\alpha} \ln \det' K &= 2 \lim_{s \rightarrow 0} \frac{d}{ds} \frac{s}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr} P \phi e^{-tK} = 2 \lim_{t \rightarrow 0} \text{Tr} P \phi e^{-tK} \\ &= 2 \lim_{t \rightarrow 0} \text{Tr} (\phi e^{-tK}) - 2 \sum_n \frac{\langle n| \phi e^{-2\alpha\phi} |n\rangle}{\langle n| e^{-2\alpha\phi} |n\rangle}. \end{aligned} \quad (\text{A.7})$$

The first term can be expressed in terms of the second DeWitt-Seeley coefficient of K (see [28] for more details), giving

$$\frac{d}{d\alpha} \ln \det' K = -\frac{1}{2\pi} \int d^2\sigma \left(\frac{\alpha}{3} \partial_\mu \phi \partial^\mu \phi + \phi E \pm \frac{1}{2} \partial_\mu \partial^\mu \phi \right) - 2 \sum_n \frac{\langle n| \phi e^{-2\alpha\phi} |n\rangle}{\langle n| e^{-2\alpha\phi} |n\rangle}, \quad (\text{A.8})$$

where the $\pm$ sign reflects the difference between the Neumann/Dirichlet boundary conditions at the endpoints of the string. Integration over α gives:

$$\ln \frac{\det' K(1)}{\det' K(0)} = -\frac{1}{2\pi} \int d^2\sigma \left(\frac{1}{6} \partial_\mu \phi \partial^\mu \phi + \phi E \pm \frac{1}{2} \partial_\mu \partial^\mu \phi \right) + \sum_n \ln \frac{\langle n | e^{-2\phi} | n \rangle}{\langle n | n \rangle}. \quad (\text{A.9})$$

This result shows that the ratio

$$\frac{\det' K(1)}{\prod_n \langle n | e^{-2\phi} | n \rangle} \quad (\text{A.10})$$

transforms under Weyl rescalings the same way a determinant without zero modes would transform.

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