An uplifting discussion of T-duality

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ABSTRACT: It is well known that string theory has a T-duality symmetry relating circle compactifications of large and small radius. This symmetry plays a foundational role in string theory. We note here that while T-duality is order two acting on the moduli space of compactifications, it is order four in its action on the conformal field theory state space. More generally, involutions in the Weyl group $W(G)$ which act at points of enhanced G symmetry have canonical lifts to order four elements of G , a phenomenon first investigated by J. Tits in the mathematical literature on Lie groups and generalized here to conformal field theory. This simple fact has a number of interesting consequences. One consequence is a reevaluation of a mod two condition appearing in asymmetric orbifold constructions. We also briefly discuss the implications for the idea that T-duality and its generalizations should be thought of as discrete gauge symmetries in spacetime.

KEYWORDS: Conformal Field Models in String Theory, Discrete Symmetries, Global Symmetries, String Duality

ARXIV EPRINT: [1707.08888](https://arxiv.org/abs/1707.08888)

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1 Introduction

This paper discusses the structure of, and consistency conditions for, group actions on two-dimensional conformal field theories (CFTs) defined by sigma models with toroidal target spaces. Such models are important building blocks in string theory. For simplicity we will discuss the bosonic string, but our considerations should have generalizations to heterotic and type II superstrings. We will see that some standard results in the literature have minor inaccuracies and we will indicate how these can be corrected. Some of the implications for general statements about string theory are also discussed. The central point of this paper is easily stated: some toroidal compactifications have symmetries of the lattice of momenta and winding beyond the trivial symmetry of reflection in the origin. Sometimes, these lattice symmetries act projectively on the CFT state space. When discussing symmetries of CFTs or constructing orbifolds this subtlety can be of some importance. This should come as no surprise: it is extremely common for symmetries of physical systems to be realized projectively on quantum Hilbert spaces. The surprise, perhaps, is the extent to which this elementary point has been overlooked in the literature.

There are two ways to detect the need for a projective action. One is based on modular covariance and is explained in section 2 below. The second is based on non-abelian symmetry and is more easily explained. The moduli space of toroidal compactifications is well known to have points of enhanced symmetry. For example there are points where the moduli space has an action of the Weyl group $W(G)$ where G is a Lie group whose Lie algebra is simply laced, that is of type A_n , D_n , E_6 , E_7 or E_8 (ADE). It is often assumed that the group that acts on the moduli space at points of enhanced symmetry is also the group that acts on the CFT space. However this is often not the case. As we will explain later, for some G the Weyl group $W(G)$ does not lift to a subgroup of G whose action on T by conjugation is isomorphic to that of $W(G)$, and which is isomorphic to $W(G)$. Rather one must choose a lift to a group $\widetilde{W}(G)$ where the order 2 elements that generate $W(G)$ lift to elements of order 4 in order to produce the desired action by conjugation. This subtlety is relevant for the theory of orbifolds since in an orbifold one gauges a subgroup of the group of automorphisms of the CFT, not a group of automorphisms of Narain moduli space. The orbifold construction plays an important role in string theory and Conformal Field Theory (CFT) [1–3] and we will see that this subtlety has implications for string-theoretic model building. For example we will show that a larger class of asymmetric orbifold constructions is allowed than is sometimes thought to be the case. The reason for this is that in the past some models have been discarded because they do not satisfy a certain mod-two consistency condition stated in the second part of equation (2.6) of [4]. What is less well-known is that this mod two condition was retracted in [5], just above equation (3.3), where it was suggested that one should double the order of the the group element. That reinterpretation is closely related to the discussion we give in this paper.

The outline of this paper is as follows. In section 2 we review the construction of toroidal CFTs and summarize the main points of the paper in more technical language than used here. The rest of the paper is a more leisurely exposition of this technical summary. In particular, we begin the story in section 3 by considering the basic example of the $c = 1$

Gaussian model at the self-dual point with affine level one $SU(2)_L \times SU(2)_R$ symmetry. We explain that the order two T-duality transformation which fixes this point in the moduli space lifts to an order four element of $SU(2)$, both from the point of view of group theory and from the point of view of modular covariance. We then use this point of view to analyze the consistency of orbifolds of products of self-dual Gaussian models by this order four lift of T-duality. In section 4 we extend our analysis of lifting of Weyl group symmetries to points with enhanced ADE symmetry and provide some illustrative examples. In section 5 we return to the $SU(2)$ analysis and explain how to understand the order four action of T-duality by carefully evaluating how the symmetry acts on vertex operators. Section 6 is devoted to a general analysis of consistency conditions of asymmetric orbifolds by the lifts of involutions of the Narain lattice and in section 7 we make some comments about more general asymmetric orbifolds. In CFT and Vertex Operator Algebras one must deal with an abelian extension of the Narain lattice and here we are also interested in the associated extension relating the automorphism group of the lattice to the automorphism group of its extension. The first appendix discusses the required mathematics. The remaining three appendices contain material on the transformation of orbifold boundary conditions under modular transformations, our conventions for theta functions, and a brief summary of the mathematical structure of lifts of the Weyl groups of compact simple Lie groups.

Note added for v3. In the first two versions of this paper posted on the arXiv we claimed that it is strictly necessary to modify the standard $\mathbb{Z}_2$ -valued cocycles in vertex operator algebras to $\mathbb{Z}_4$ -valued cocycles in order to understand the nontrivial lifting of T-duality discussed throughout the paper. This claim is erroneous. It was pointed out to us by the referee that one can perfectly well use the standard cocycles with a suitable modification of the lifting function See equation (5.25) and note added below for further details. We thank the referee for insisting on this point.

2 Technical summary of results

2.1 Review of toroidal CFT and T-duality

In order to state our results with more precision we first recall the essential elements of toroidal conformal field theories. As is well-known, two-dimensional CFTs of free scalar fields with toroidal target space are not isolated. There are actually two constructions of these CFTs which we may call the *vertex operator algebra (VOA) construction* and the *sigma model construction*. Each construction has its advantages, and each construction leads to a parameter space of conformal field theories which requires taking a quotient to obtain the moduli space of toroidal conformal field theories.

In the vertex operator algebra construction we begin with an embedding of the unique even unimodular lattice II^{d_L, d_R} , of signature $(+^{d_L}, -^{d_R})$ into a fixed real quadratic space V equipped with projection operators to a positive definite space of dimension d_L and a negative definite space of dimension d_R . We can identify V with the standard space $\mathbb{R}^{d_L; d_R}$ with diagonal quadratic form and projections onto the first d_L and last d_R coordinates, respectively. (The semi-colon is meant to remind us that this space comes with

definite projection operators.) We assume that $d_L, d_R > 0$ and $d_L - d_R = 0 \bmod 8$. We denote the image of II^{d_L, d_R} by $\Gamma \subset \mathbb{R}^{d_L; d_R}$. The moduli space of such embeddings is the homogeneous space

$$\mathfrak{L} := \mathcal{T} \backslash O(V) \quad (2.1)$$

where $\mathcal{T} \cong \text{Aut}(II^{d_L, d_R})$ is the T-duality group, usually written as $O(d_L, d_R; \mathbb{Z})$. (The latter notation is less precise, as it presupposes an integral quadratic form, but it is standard, so we will use it. With the same understanding it is also common to write $O(V)$ as $O(d_L, d_R; \mathbb{R})$.) Now for each $\Gamma \in \mathfrak{L}$ we can construct a 2d CFT $\mathcal{C}_\Gamma$ as follows. The vector space of left-moving creation oscillators (for any fixed positive integer frequency) can be identified with $V_L \otimes \mathbb{C}$, where $V := \Gamma \otimes \mathbb{R} \cong \mathbb{R}^{d_L; d_R}$ and V_L is its left-moving projection. Similarly the vector space of the right-moving creation operators (for a fixed frequency) is $V_R \otimes \mathbb{C}$. In these terms the CFT state space can be written as:

$$\mathcal{H}_\Gamma = S^\bullet(\oplus_{n>0} q^n V_L \otimes \mathbb{C}) \otimes S^\bullet(\oplus_{n>0} \bar{q}^n V_R \otimes \mathbb{C}) \otimes \mathbb{C}[\Gamma]. \quad (2.2)$$

Here $S^\bullet(\oplus_{n>0} q^n V_L \otimes \mathbb{C})$ denotes the symmetric algebra of the left-moving creation oscillators with positive frequency. The factor q^n is meant to indicate the space with frequency n . Similarly, $S^\bullet(\oplus_{n>0} \bar{q}^n V_R \otimes \mathbb{C})$ is the symmetric algebra of the right-moving creation oscillators. $\mathbb{C}[\Gamma]$ is the group algebra of the Narain lattice. As a vector space it is a direct sum of lines $L_p \cong \mathbb{C}$, one line associated to each momentum vector $p \in \Gamma$. The space $\mathcal{H}_\Gamma$ can be given the structure of a (in general, nonholomorphic) vertex operator algebra, although the details require some care, as recalled in section 5.1. Note that we therefore have a bundle of CFTs over $\mathfrak{L}$, with fiber $\mathcal{H}_\Gamma$ above $\Gamma \in \mathfrak{L}$.

Different embeddings $\Gamma, \Gamma' \subset V$ can lead to isomorphic conformal field theories. This happens if they are related by the action of the subgroup $O(d_L) \times O(d_R)$ of $O(d_L, d_R; \mathbb{R})$. For example, the Hamiltonian $H = \frac{1}{2}p_L^2 + \frac{1}{2}p_R^2 + H^{\text{osc}}$ commutes with this group. Therefore the true moduli space of conformal field theories is the quotient, known as Narain moduli space, and can be identified with the double coset:¹

$$\mathcal{N} := \mathcal{T} \backslash O(d_L, d_R; \mathbb{R}) / O(d_L) \times O(d_R). \quad (2.3)$$

We now recall briefly the sigma model construction. Since we do not wish to enter into the subtleties of quantizing the self-dual field we will limit considerations to theories with $d = d_L = d_R$. In this case one may easily write an action for the sigma model using the data of a flat metric, G , and B -field on the torus. Thus, the moduli space of sigma model data is

$$\mathfrak{B} := \{E = G + B | G = G^{tr} > 0 \quad \& \quad B = -B^{tr}\} \subset \text{Mat}_{d \times d}(\mathbb{R}). \quad (2.4)$$

¹We are actually being somewhat sloppy here from a mathematical viewpoint. (Most physicists will want to skip this footnote.) The “Narain moduli space” is an orbifold, and is more properly regarded as a global stack where the automorphism group of objects is always a finite group. However, it is not really the moduli stack of toroidal conformal field theories. In the latter stack, the automorphism group of an object will include continuous groups at, for example, the points of enhanced A-D-E symmetry, while in the Narain moduli stack the automorphism group of the A-D-E points is a finite group $F(\Gamma(\mathfrak{g}))$ discussed at length below. The moduli stack of conformal field theories maps to the Narain moduli space.

This space is isomorphic to $O(d, d; \mathbb{R})/O(d) \times O(d)$ as a smooth manifold. To illustrate we use a construction going back to [6, 7] (but here slightly modified from the original). Choose two invertible $d \times d$ matrices e_1, e_2 so that $e_1 e_1^{tr} = e_2 e_2^{tr} = G^{-1}$. Note that e_1 and e_2 are defined up to right action by an $O(d)$ matrix. Now define the $2d \times 2d$ matrix:

$$\mathcal{E} := \begin{pmatrix} \frac{1}{2}e_1 & \frac{1}{2}e_2 \\ E^{tr}e_1 & -Ee_2 \end{pmatrix} \quad (2.5)$$

The reader can readily check that this solves

$$\mathcal{E} Q_0 \mathcal{E}^{tr} = Q \quad (2.6)$$

where

$$Q_0 = \begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \quad (2.7)$$

Since Q and Q_0 are similar the space of matrices solving (2.6) is smoothly isomorphic to $O(V)$. Modding out by the right action on $\mathcal{E}$ of $O(d_L) \times O(d_R)$ produces, on the one hand, the space $\mathfrak{B}$ and on the other hand, the coset $O(d_L, d_R; \mathbb{R})/(O(d_L) \times O(d_R))$.

Now, quite similarly to the case of the bundle of CFT state spaces over $\mathfrak{L}$ we can likewise produce a bundle of state spaces $\mathcal{H}$ over $\mathfrak{B}$.² We denote the fiber over E by $\mathcal{H}_E$. It is produced by canonical quantization, and in the process of quantization one finds — after fixing the gauge for $O(d_L) \times O(d_R)$ — that the lattice of zero frequency momentum and winding modes is the embedded lattice in $\mathbb{R}^{d;d}$ generated by integer combinations of the rows of $\mathcal{E}$. By equation (2.6) this is an even unimodular lattice and hence defines an element of $\mathfrak{L}$. Tracing back the change of basis to an action of $\mathcal{T}$ on $\mathfrak{B}$ produces the familiar left action of $\mathcal{T}$ on $\mathfrak{B}$ via fractional linear transformations of E . The whole construction can be summarized in the diagram:

$$\begin{array}{ccc} & O(V) & \\ \swarrow & & \searrow \\ \mathfrak{L} & & \mathfrak{B} \\ \searrow & & \swarrow \\ & \mathcal{N} & \end{array} \quad (2.8)$$

where the left-hand path is the vertex operator algebra construction and the right-hand path is the sigma-model construction.

²Once again we are being somewhat sloppy from a strictly mathematical point of view. Canonical quantization only provides a projective Hilbert space because it is based on a choice of vacuum *line*, rather than a choice of vacuum *state*. Thus, what is canonically defined is a bundle of projective Hilbert spaces. One might dismiss this subtlety because $\mathfrak{B}$ is a contractible space. However, the space is not equivariantly contractible, so this issue will, doubtless, be of some importance in sorting out the issues of T-duality as a gauge symmetry mentioned below.

2.2 The enhanced symmetry locus

The space $\mathcal{N}$ has an important subspace $\mathcal{N}^{\text{ESP}}$ of points with *enhanced symmetry* that will be important in this paper. To define the enhanced symmetry locus first note that every embedded lattice $II^{d_L, d_R} \hookrightarrow \Gamma \subset \mathbb{R}^{d_L, d_R}$ has an automorphism corresponding to $p \rightarrow -p$. We will call this the *trivial involution*. Note that this is always in $O(d_L) \times O(d_R)$ for any embedding. However, on a positive codimension subvariety $\mathfrak{L}^{\text{ESP}} \subset \mathfrak{L}$ there will be nontrivial automorphisms of the CFT. To be precise, define the group:

$$F(\Gamma) := \text{Aut}(\Gamma) \cap (O(d_L) \times O(d_R)) . \quad (2.9)$$

Note that this group is both discrete and compact and hence is a finite group. The locus $\mathfrak{L}^{\text{ESP}} \subset \mathfrak{L}$ is defined to be the set of embeddings such that $F(\Gamma)$ is strictly larger than the central $\mathbb{Z}_2$ subgroup generated by the trivial involution. In a neighborhood of $\mathfrak{L}^{\text{ESP}}$ the action of $O(d_L) \times O(d_R)$ has fixed points, producing a complicated subvariety $\mathcal{N}^{\text{ESP}}$ of orbifold singularities where the orbifold group is, generically, $F(\Gamma)/\mathbb{Z}_2$. Thanks to equation (2.8) we know there is a corresponding locus $\mathfrak{B}^{\text{ESP}} \subset \mathfrak{B}$ where a finite subgroup of $\mathcal{T}$ acts with fixed points.³

The orbifold singularities at points $[\Gamma] \in \mathcal{N}$ signal the presence of nontrivial automorphisms of the conformal field theory $\mathcal{C}_\Gamma$ parametrized by Γ . In the string theory literature it is commonly assumed that $F(\Gamma)$ can be identified with a group of automorphisms of the CFT $\mathcal{C}_\Gamma$, but - *and this is the central point of this paper* - this is not always the case, and the distinction between $F(\Gamma)$ and $\text{Aut}(\mathcal{C}_\Gamma)$ can be important. How can this happen? To explain this point we note that the group $F(\Gamma)$ acts on the Narain lattice Γ and that action extends linearly to the vector space $V = \Gamma \otimes \mathbb{R}$ and commutes with the left-moving and right-moving projectors. Therefore it acts naturally on the left-moving and right-moving oscillators. However, we must *also* determine the action on $\mathbb{C}[\Gamma]$ and here is where a subtlety can arise. In physical terms, we choose a generating vector for each L_p (it is the ground state in the momentum sector p) and denote it by $|p\rangle$. In the literature one commonly finds the claim that we can choose a basis of momentum states $|p\rangle$ so that, for all $g \in F(\Gamma)$, there is an operator $U(g)$ on $\mathcal{C}_\Gamma$ such that⁴

$$U(g)|p\rangle = |g \cdot p\rangle . \quad (2.11)$$

While this is commonly assumed, it turns out that it is, in general, not consistent with the non-abelian global symmetry of special CFTs associated with special points in $\mathcal{N}$. It

³In the interest of technical accuracy we note that (2.9) for different Γ projecting to the same point in $\mathcal{N}$ will be conjugate groups. Similarly, we will often loosely speak of $F(\Gamma)$ when working with a point $E \in \mathfrak{B}$. What is meant here is that one fixes the $O(d_L) \times O(d_R)$ gauge by choosing inverse vielbeins e_1, e_2 as above and then constructs a particular Γ using the integer span of the rows of $\mathcal{E}$.

⁴In general it is also possible to include the action by *shift vectors*. Group elements are labeled by (g, s) where $s \in \Gamma \otimes \mathbb{Q}$ is known as a shift vector and we modify the action (2.11) by the (equally naive) action:

$$U(g, s)|p\rangle = e^{2\pi i p \cdot s} |g \cdot p\rangle . \quad (2.10)$$

We are not trying to be comprehensive and will, for the most part, ignore the inclusion of shift vectors in this paper. However, the incorporation of shift vectors will play a role in some examples below.

is also inconsistent with the same non-abelian global symmetry of the Operator Product Expansion (OPE), given the state-operator correspondence. More generally, at points where $F(\Gamma)$ is nontrivial it is, in general, inconsistent with *modular covariance*. (This term is explained below.)

2.3 Non-Abelian symmetry

The simplest example of a conflict between equation (2.11) and non-abelian global symmetry is the Gaussian model at the self-dual radius. This CFT is, famously, equivalent to the level 1 $SU(2)$ WZW model [8–10].⁵

In order to avoid confusion it is important to specify precisely what the symmetries are of the Gaussian model and why we focus on a particular element that we call T-duality. At the self-dual point the Gaussian model has $su(2)_L \times su(2)_R$ affine symmetry. Focus for the moment on the $su(2)_R$ symmetry. There are two order 2 automorphisms of the $su(2)$ Lie algebra which are commonly used in the string theory literature. The first, a $\mathbb{Z}_2$ twist, acts on the currents as $\tilde{J}^3 \rightarrow -\tilde{J}_3$, $\tilde{J}^\pm \rightarrow \tilde{J}^\mp$. In the Frenkel-Kac-Segal construction of affine $su(2)_R$ this is implemented by the transformation $X_R \rightarrow -X_R$. Denote this transformation by σ_R . Clearly we can do the same thing but on holomorphic (left-moving) degrees of freedom. Denote this transformation by σ_L . In addition to these “twist” transformations we can consider “shifts.” An order 2 shift on the anti-holomorphic degrees of freedom at the self-dual radius acts on the bosonic coordinate as $\tilde{X}_R \rightarrow \tilde{X}_R + \pi/\sqrt{2}$ and takes $\tilde{J}_3 \rightarrow \tilde{J}_3$, $\tilde{J}^\pm \rightarrow -\tilde{J}^\pm$. There is an analogous order two symmetry acting on holomorphic degrees of freedom. Let us denote these by S_R, S_L . In the notation of the previous paragraph we take T-duality to be σ_R . This is a symmetry which exists only at the self-dual radius. The usual symmetric $\mathbb{Z}_2$ action used to construct the $\mathbb{Z}_2$ symmetric orbifold of the Gaussian model is $\sigma_L \sigma_R$ and exists at any radius. An alternative version of T-duality at the self-dual radius proposed in [14] is $S_L \sigma_R$. However it is easy to check that $S_L = g^{-1} \sigma_L g$ for $g \in SU(2)_L$. Now $SU(2)_L$ is a global symmetry of the CFT at the self-dual radius. Hence any two operators which are conjugate in $SU(2)_L$ will lead to identical physical predictions. In particular, any computation involving the $\mathbb{Z}_2$ operator $S_L \sigma_R$ will give physically identical results to a computation using the $SU(2)_L$ conjugate operator $\sigma_L \sigma_R$ which is the symmetric $\mathbb{Z}_2$ operator. Since this alternate “T-duality” is just the symmetric $\mathbb{Z}_2$ symmetry in disguise it is not surprising that the $\mathbb{Z}_2$ orbifold by it is consistent and that transformations acting on the CFT state space are order 2. However while the orbifold by this symmetry is consistent, it simply reproduces the usual symmetric orbifold of the Gaussian model. From now on we use T-duality to refer to the left-right asymmetric symmetry σ_R and its generalizations.

When acting on the currents of the model, T-duality acts trivially on (say) the left-moving currents but acts as a 180 degree rotation on the right-moving currents. On the other hand there are states in the model that transform as the tensor product of left- and right-moving spinor representations of $SU(2)_L \times SU(2)_R$. Therefore, in order to define an action on the Hilbert space we must lift the 180-degree rotation in the right-moving

⁵The following transparent example arose in discussions with N. Seiberg and has been quite important to our thinking. After submitting v1 of this paper it was pointed out to us that this particular example has been previously discussed by Aoki, D’Hoker, and Phong [11]. Related works include [12, 13].

$\mathrm{SO}(3)$ to an element in the right-moving $\mathrm{SU}(2)$. This lift to $\mathrm{SU}(2)$ is clearly of order four. This phenomena generalizes to the standard enhanced symmetry loci in $\mathcal{N}$ associated with semi-simple simply-laced Lie algebras. If $\mathfrak{g}$ is of full rank (and $d_L = d_R$) these are isolated points defined by

$$\Gamma(\mathfrak{g}) := \{(p_L; p_R) \in \Lambda_{wt}(\mathfrak{g}) \times \Lambda_{wt}(\mathfrak{g}) | p_L - p_R \in \Lambda_{rt}(\mathfrak{g})\}. \quad (2.12)$$

The corresponding CFT, $\mathcal{C}(\mathfrak{g}) := \mathcal{C}_{\Gamma(\mathfrak{g})}$ has $\widetilde{LG}_L^{(1)} \times \widetilde{LG}_R^{(1)}$ (dynamical) symmetry, where G is the compact simply connected Lie group with Lie algebra $\mathfrak{g}$ and $\widetilde{LG}^{(1)}$ is the level one $\mathrm{U}(1)$ central extension of the loop group LG . In particular it has an action of $G_L \times G_R$ corresponding to the constant loops. On the other hand, the crystallographic group $F(\Gamma(\mathfrak{g}))$ certainly contains

$$W(\mathfrak{g})_L \times W(\mathfrak{g})_R \quad (2.13)$$

as a subgroup, where $W(\mathfrak{g})$ is the Weyl group of $\mathfrak{g}$.

We must stress that $W(\mathfrak{g})$ is *not* a subgroup of G . This seemingly fastidious point will actually turn out to be important. This point has been noted before in the physics literature, see [15] where some of the material below is also discussed. Quite generally, the Weyl group $W(\mathfrak{g})$ is defined as follows. Choose a maximal torus $T \subset G$ and define the normalizer group $N(T) := \{g \in G | gTg^{-1} = T\}$. Of course $T \subset N(T)$, and in fact the conjugation action by T fixes every element pointwise, since T is abelian. However, the definition of $N(T)$ only requires conjugation to fix T setwise, and there are other elements of G which conjugate the maximal torus to itself but do not fix every element of T . In fact T is a normal subgroup of $N(T)$ and the Weyl group is *defined* as the quotient⁶

$$W(\mathfrak{g}) := N(T)/T. \quad (2.14)$$

Thus $W(\mathfrak{g})$ is not a subgroup of G but rather, it is a quotient of a subgroup of G - that is, it is a subquotient of G . It follows from (2.14) that $W(\mathfrak{g})$ fits in an exact sequence

$$1 \rightarrow T \rightarrow N(T) \rightarrow W(\mathfrak{g}) \rightarrow 1. \quad (2.15)$$

One can show that there are (many) discrete subgroups $\widetilde{W} \subset N(T) \subset G$ together with a homomorphism $\pi : \widetilde{W} \rightarrow W$ such that the conjugation action of $g \in \widetilde{W}$ on the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is identical to the Weyl group action of $\pi(g)$. Such a subgroup $\widetilde{W} \subset G$ is called a *lift of W* . In some cases ($G = \mathrm{SU}(2)$ is a case in point) there is no lift isomorphic to W . Thus, at enhanced symmetry points of the form $\mathcal{C}(\mathfrak{g})$, many discrete automorphism groups $\widetilde{W}_L \times \widetilde{W}_R$ of $\mathcal{C}(\mathfrak{g})$ induce the action of $F(\Gamma(\mathfrak{g}))$ on oscillators and momenta. Moreover, in some cases, no such lifting group is isomorphic to $W_L \times W_R$.

Since states are in one-one correspondence with operators in a CFT we expect that there will be an analogous story for the automorphisms of the vertex operator algebra,

⁶In fact, $W(\mathfrak{g})$ is intrinsically associated to the Lie algebra $\mathfrak{g}$ and does not depend on which Lie group G with Lie algebra $\mathfrak{g}$ we choose. Indeed, there are other, equivalent, definitions of the Weyl group which only make direct use of the root system of $\mathfrak{g}$, rendering this property obvious. We have chosen to emphasize the relation to the Lie group since it fits best with the main point of the paper.

and indeed this is the case. It is well-known that the naive expression $V^{\text{naive}}(p) =: e^{ip \cdot X}$: for the vertex operators associated to momentum vectors must be modified by “cocycle factors.” In section 5 we explain how this works for $\mathfrak{g} = \mathfrak{su}(2)$.

2.4 Modular covariance

Now let us turn to conflicts between (2.11) and modular covariance. We first explain the term “modular covariance.” Quite generally, if J is a global symmetry of a CFT $\mathcal{C}$ then we can “couple $\mathcal{C}$ to external J gauge fields.” What this means is that, if the worldsheet is Σ then we consider a principal J -bundle over Σ endowed with connection and couple the connection to the global symmetry currents of $\mathcal{C}$.⁷ If J is a discrete group then there is a unique connection on the principal bundle and coupling to the currents means imposing twisted boundary conditions by elements $g_a \in J$ around a set of generating cycles of $\pi_1(\Sigma, *)$. The diffeomorphism group acts on this picture relating different twisted partition functions. In the case of a torus we choose two commuting elements g_s, g_t for twisting around a choice of A and B cycles and form the partition function $Z(g_t, g_s; \tau)$. The “modular covariance” constraint is the statement that (see appendix B):⁸

$$Z\left(g_t, g_s; \frac{a\tau + b}{c\tau + d}\right) = e^{i\phi(\gamma)} Z(g_s^{-b} g_t^d, g_s^a g_t^{-c}; \tau) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \quad (2.16)$$

where $e^{i\phi(\gamma)}$ is some $\text{U}(1)$ -valued function of γ , reflecting the possibility of a modular anomaly. We will show that modular covariance is in conflict with the hypothesis that at enhanced symmetry points the group $F(\Gamma)$ is an automorphism group of the CFT $\mathcal{C}_\Gamma$.

The simplest example of a conflict between (2.11) and modular covariance appears, once again, in the Gaussian model at the self-dual radius. Recall that a single compact boson has a Lagrangian specified by the radius r of the target space circle and the T-duality group acts on the space of sigma models $O(1, 1; \mathbb{R})/O(1) \times O(1) \cong \mathbb{R}_+$ as $r \rightarrow \ell_s^2/r$ where ℓ_s is the string length. The T-duality group is isomorphic to $\mathbb{Z}_2$ and the self-dual radius is an orbifold point of order 2 in $\mathcal{N}$. (In this paper we will henceforth take $\ell_s = 1$ so the self-dual radius is $r = 1$.) This does *not* imply that there is an action of the T-duality group on the CFT space associated to the self-dual radius. In fact, only a two-fold covering group acts on the CFT space and the only action of T-duality on this state space consistent with modular invariance is order four. We will explain these statements in detail in section 3.

2.5 Doomed to fail

The phenomenon we have just described at the points $[\Gamma(\mathfrak{g})] \in \mathcal{N}$ arises more generally at the loci where $F(\Gamma)$ is larger than $\mathbb{Z}_2$. It is therefore useful to find a criterion for when (2.11) must be modified, or to put it colloquially, when implementing (2.11) in a naive way is “doomed to fail.” That is, we would like to know when this naive action of $F(\Gamma)$ on the

⁷This must be distinguished from gauging the J symmetry (as one does to form an orbifold). In that case one sums over isomorphism classes of J -bundles with connection.

⁸The term “modular covariance” was used in a slightly different way in [11] where the term is used for the same identity but with the phase $e^{i\phi(\gamma)}$ put to one.

state space is inconsistent and we must choose a nontrivial lift $\widetilde{F(\Gamma)}$ to act on the state space (or change the action (2.11)). Moreover, we could ask whether there is a canonical lift of $F(\Gamma)$ to $\text{Aut}(\mathcal{C}_\Gamma)$. In section 6.1 we will show that the $F(\Gamma)$ action defined by (2.11) is indeed inconsistent with modular covariance when there is a nontrivial involution,⁹ say g , such that there exists a vector $p \in \Gamma$ with $p \cdot g \cdot p$ an odd integer. More generally, as shown in section 6.4, there is an inconsistency with modular covariance when there are elements $g \in F(\Gamma)$ of even order ℓ such that:

$$\exists p \in \Gamma \quad s.t. \quad p \cdot g^{\ell/2} \cdot p = 1 \bmod 2 \quad (2.17)$$

(Of course, $g^{\ell/2}$ is an involution in $F(\Gamma)$, so the problem can always be traced to involutions.) The criterion (2.17) implies that the subgroup $\langle g \rangle \subset F(\Gamma)$ cannot be lifted to an isomorphic subgroup of $\text{Aut}(\mathcal{C}_\Gamma)$ inducing the action of $\langle g \rangle$ on Γ and satisfying (2.11). We hasten to add that the condition (2.17) does not rule out the existence of some lift $\widetilde{F(\Gamma)} \subset \text{Aut}(\mathcal{C}_\Gamma)$ isomorphic to $F(\Gamma)$. As we will show in section 4.1 in the explicit example of the $\text{SU}(3)$ level one WZW model, it is possible to modify the generators of the Tits lift by shift vectors so that there is a lift of $W(\mathfrak{g})_L \times W(\mathfrak{g})_R$ isomorphic to $W(\mathfrak{g})_L \times W(\mathfrak{g})_R$. What gives is that it is no longer true that $\hat{g}|p\rangle = |p\rangle$ where p is in the invariant lattice and $\hat{g} \in \text{Aut}(\mathcal{C}_\Gamma)$ is a lift of g . To summarize: the meaning of the criterion (2.17) is that either:

1. Equation (2.11) does not hold for some $p \in \Gamma^g$, or
2. Equation (2.11) does hold, but $\langle g \rangle \subset F(\Gamma)$ is lifted to an extension in $\text{Aut}(\mathcal{C}_\Gamma)$.

We further conjecture that there is in fact a canonical lift of $F(\Gamma)$ to

$$\widetilde{F(\Gamma)}^{\text{can}} \subset \text{Aut}(\mathcal{C}_\Gamma), \quad (2.18)$$

given by (6.35), (6.36). It satisfies the properties that there is a lifting $\hat{g} \in \widetilde{F(\Gamma)}^{\text{can}}$ of $g \in F(\Gamma)$ such that

$$\hat{g}|p\rangle = |p\rangle \quad \forall p \in \Gamma^g \quad (2.19)$$

where $\Gamma^g := \{p \in \Gamma | g \cdot p = p\}$ is the invariant sublattice of Γ and moreover

$$\hat{g}^\ell |p\rangle = e^{i\pi p \cdot g^{\ell/2} \cdot p} |p\rangle \quad \forall p \in \Gamma. \quad (2.20)$$

As already mentioned, in the case of CFTs based on $\Gamma(\mathfrak{g})$ with non-abelian symmetry there is a canonical lift based on the Tits lift described in appendix D. At the end of section 6 we provide some evidence that the canonical lift defined by (6.35), (6.36) is indeed a generalization of the Tits lift. If the conjecture made in section 6 is correct then lifting to $\widetilde{F(\Gamma)}^{\text{can}}$ at most doubles the order of any element $g \in F(\Gamma)$.

⁹We will refer to involutions in $F(\Gamma)$ that are not of the form $p \rightarrow -p$ as *nontrivial involutions*.

2.6 On T-duality as a target space gauge symmetry

The considerations of this paper have some interesting implications for the relation of the T-duality group to the gauge symmetries of string theory. Put briefly, it is believed that the symmetry groups $F(\Gamma(\mathfrak{g})) \subset O(d, d; \mathbb{Z})$ generate all of $O(d, d; \mathbb{Z})$ except for a $\mathbb{Z}_2$ transformation that corresponds to world-sheet parity. It is standard string-theory lore [16, 17] that $F(\Gamma(\mathfrak{g}))$ is a subgroup of the target space $G_L \times G_R$ gauge symmetry of the target space theory, and therefore $O(d, d; \mathbb{Z})$ is a gauge symmetry of string theory. Unfortunately, this is based on the misconception that $W(\mathfrak{g})$ is canonically a subgroup of G . Rather, there are subgroups

$$\widetilde{F(\Gamma(\mathfrak{g}))} \subset \text{Aut}(\mathcal{C}(\mathfrak{g})) \quad (2.21)$$

lifting $W(\mathfrak{g})_L \times W(\mathfrak{g})_R$. These do not fit (in any way obvious to us) as subgroups of a single common group and hence it is not clear what, if anything, the different groups $\widetilde{F(\Gamma(\mathfrak{g}))}$ generate. The main, open, issue can be phrased as follows.

The subgroup of $\mathcal{T}$ fixing a point $E \in \mathfrak{B}$ that projects to $[\Gamma]$ is isomorphic to $F(\Gamma)$. As we have just discussed at length, sometimes the group $F(\Gamma)$ does not lift to act on the fiber $\mathcal{H}_E$ over E . Only a covering group $\widetilde{F(\Gamma)}$ lifts. Thus, the bundle of CFT state spaces $\pi : \mathcal{H} \rightarrow \mathfrak{B}$ defined above does *not* admit the structure of an $O(d, d; \mathbb{Z})$ -equivariant bundle. This leaves us with two logical possibilities:

1. There is a group $\widetilde{\mathcal{T}}$ acting on $\mathcal{H}$, covering the $\mathcal{T}$ action on $\mathfrak{B}$, and inducing $\widetilde{F(\Gamma)}$ on the enhanced symmetry locus. Following the logic of [16, 17] it would actually be the group $\widetilde{\mathcal{T}}$, rather than $\mathcal{T}$, which would be a gauge symmetry of string theory.
2. There is no such group $\widetilde{\mathcal{T}}$. This is a reasonable possibility. Similar phenomena are quite standard in the study of twisted equivariant K-theory. If this is the case, the idea that “T-duality is a gauge symmetry of string theory” is in fact quite mistaken.

Which of the two possibilities is in fact the case is a very interesting question we leave to the future. The proper resolution of this question will involve an investigation into the moduli *stack* of toroidal CFTs. Moreover, one must take into account the existence of $U(1)^d \times U(1)^d$ automorphisms of the fiber, i.e. the possibility of combining the transformation with separate left and right $U(1)^d$ automorphisms. These left- and right- $U(1)^d$ automorphisms are also often represented by asymmetric shift vectors. They act trivially on the base. We thank D. Freed, D. Freidan, A. Tripathy, and G. Segal for useful discussions about this question.

2.7 Consistency conditions for orbifolds

Finally, we note that the considerations of this paper are very relevant to orbifold constructions, namely the gauging of discrete subgroups of the automorphism group of a CFT. It is important to bear in mind that the orbifold group is a subgroup of $\text{Aut}(\mathcal{C}_\Gamma)$ and is not a subgroup of $F(\Gamma)$, although much of the literature refers to the orbifold group as a subgroup of $F(\Gamma)$. In particular, we note that the criterion (2.17) is closely related to the work

of Lepowsky [18, 19] as well as to the work of Narain, Sarmadi, and Vafa [4] (see their equation (2.6)). The work of Lepowsky addresses a slightly different problem from that addressed here in that it is concerned with strictly chiral twisted affine Lie algebras and their modules. Our interpretation of (2.17) differs from [4], where it is suggested that the condition is a consistency condition in a sense similar to the level-matching constraints. We suggest instead the the correct interpretation is as stated above (2.18) and that one should only attempt to construct an orbifold by a subgroup of the lift of $F(\Gamma)$. This is consistent with the remarks above equation (3.3) of [5].

The consistency conditions for constructing orbifolds have been discussed by a number of authors [1, 2, 4, 20, 21]. A good example is “level-matching.” This is an anomaly cancellation condition that is closely related to modular covariance [2, 20]. The basic point is that the twisted partition functions $Z(g_t, g_s; \tau)$ described near equation (2.16) have a Hamiltonian interpretation. Namely, there is a space of twisted states $\mathcal{H}_{g_s}$ (a module for a twisted vertex operator algebra) and, for g_t in the centralizer of g_s , an action of g_t on $\mathcal{H}_{g_s}$. Then

$$Z(g_t, g_s; \tau) = \text{Tr}_{\mathcal{H}_{g_s}} g_t q^H \bar{q}^{\tilde{H}} \quad (2.22)$$

where $H = L_0 - c/24$ and $\tilde{H} = \tilde{L}_0 - \tilde{c}/24$. The partition function in the sector of the orbifold theory twisted by g_s is then

$$\frac{1}{|Z(g_s)|} \sum_{g_t \in Z(g_s)} \text{Tr}_{\mathcal{H}_{g_s}} g_t q^H \bar{q}^{\tilde{H}}. \quad (2.23)$$

Of course, $\langle g_s \rangle \subset Z(g_s)$ so if $g_{t,i}$ is any set of coset representatives for this subgroup then (2.23) can be written as

$$\frac{1}{|Z(g_s)|\ell} \sum_{g_{t,i}} \sum_{k=1}^{\ell} \text{Tr}_{\mathcal{H}_{g_s}} g_{t,i} g_s^k q^H \bar{q}^{\tilde{H}} \quad (2.24)$$

where the sum on k runs from 1 to ℓ , the order of g_s . But now

$$\sum_{k=1}^{\ell} \text{Tr}_{\mathcal{H}_{g_s}} g_{t,i} g_s^k q^H \bar{q}^{\tilde{H}} = \sum_{k=1}^{\ell} Z(g_{t,i}, g_s; \tau - k) \quad (2.25)$$

These averages will all vanish iff there is a modular anomaly in the untwisted sector for some congruence subgroup of $\text{PSL}(2, \mathbb{Z})$. The only way to have one of the averages be nonzero is for the spectrum of $H - \tilde{H}$ in $\mathcal{H}_{g_s}$ to contain an infinite number of integers. This is the level matching condition.

While level-matching is very powerful one should bear in mind that there can be other consistency conditions. Indeed the full set of consistency conditions for orbifolds is actually not known.¹⁰ Clearly, one necessary condition is that the one-loop partition function of

¹⁰One could imagine that modular invariance at higher genus involves new requirements, and this might be the case for non-abelian orbifold groups. However, in the abelian case it was shown that no new consistency conditions arise from anomaly cancellation at higher genus [21].

the orbifold theory should have a “good q -expansion.” This means that Z has a convergent expansion of the form

$$Z = \sum_{\mu, \tilde{\mu}} D_{\mu, \tilde{\mu}} q^{\mu} \bar{q}^{\tilde{\mu}} \quad (2.26)$$

which is not only modular invariant but moreover all the expansion coefficients $D_{\mu, \tilde{\mu}}$ are nonnegative integers.¹¹ Moreover the vacuum has degeneracy one, i.e. the coefficient of $q^{-c/24} \bar{q}^{-\tilde{c}/24}$ must be exactly one. Of course, given a consistent VOA acting on a unitary module $Z = \text{Tr}_{\mathcal{H}} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24}$ will automatically have a good q -expansion, but in our constructions we often fall short of defining the full VOA action on the twisted sectors, so the condition of having a good one-loop q -expansion is a useful one.

As we have just mentioned, we believe that (2.17) should not be interpreted as saying that the CFT orbifold is inconsistent, but rather that there is a nontrivial lift of the subgroup of $F(\Gamma)$ acting on the Narain lattice to the group of automorphisms of the CFT $\mathcal{C}_{\Gamma}$. In order to support our thesis we demonstrate in section 3 that orbifolding by the $\mathbb{Z}_4$ group of diagonal T-duality acting on d copies of the Gaussian model at the self-dual radius satisfies all known consistency conditions, so long as $d = 0 \bmod 4$. Similar remarks apply to chiral Weyl reflection orbifolds of the level one $\text{SU}(3)$ WZW model. In fact, given an involution in $F(\Gamma)$ satisfying some conditions stated at the beginning of section 7 we show that one can use the method of modular orbits to construct a one-loop partition function with a good q -expansion for the orbifold by $\langle \hat{g} \rangle \cong \mathbb{Z}_4$ provided that the associated *twisted characteristic vector* satisfies

$$W_g^2 = 0 \bmod 4. \quad (2.27)$$

A twisted characteristic vector is a vector $W_g \in \Gamma^g$ such that

$$p \cdot g^{\ell/2} \cdot p = W_g \cdot p \bmod 2 \quad \forall p \in \Gamma, \quad (2.28)$$

where ℓ is the (even) order of g . The vector W_g is only defined modulo 2Γ and generalizes the notion of a characteristic vector of an odd lattice. For more details see section 6.¹²

2.8 Future directions

The above discussion begs the question: what are the consistency conditions for toroidal orbifolds? It is possible that the application of recent ideas relevant to the classification of symmetry-protected topological phases of matter can be usefully applied to this problem. We have had some initial discussions about this idea with D. Gaiotto and N. Seiberg and we hope to develop this approach further in the future. Moreover, one can interpret many aspects of our discussion in the language of defects [37] and it might be fruitful to use the

¹¹The $\mu, \tilde{\mu}$ are arbitrary real numbers in general. The branch of the logarithm is defined by $q^{\mu} := \exp[2\pi i \mu \tau]$. It is important to note that this is true in the bosonic string, which contains no fermions. In superstring theories this must be modified to account for minus signs due to the presence of spacetime fermion fields. Nevertheless, one can impose the condition of a good q -expansion in the NS sector.

¹²The vector $W_g \bmod 2\Gamma$ should have a topological interpretation in terms of the G -equivariant E^4 -cohomology of BT for a suitable torus T , where $G = \langle g \rangle$. This interpretation should play a role when interpreting our results in terms of three-dimensional Chern-Simons theory. We leave such considerations to the future.

language of defects to approach the more general question of consistency conditions for asymmetric orbifolds.

Finally we discuss some possible consequences of our results. As mentioned earlier, the reinterpretation of (2.17) presented here and in [5] allows for a more general class of asymmetric orbifold constructions. We were in fact led to the considerations of this paper precisely by the study of such constructions in the context of work on moonshine and string duality which will appear in [22]. We expect that there will be additional consequences for the study of moonshine. For example one might wonder if there are interesting consequences for the “symmetry surfing” proposal of [24–26]. We hope to explore some of these potential consequences in future work.

3 Products of self-dual Gaussian models

We now use the Gaussian model at the self-dual radius as a simple model to diagnose the structure of T-duality, the conditions following from modular covariance, and the construction of asymmetric orbifolds by T-duality.¹³ We first consider a single Gaussian model and then in order to construct consistent asymmetric orbifolds, d copies of the Gaussian model.

The $c = 1$ Gaussian model (see [27] for a review) is described by a single real bosonic field X with action

$$S = \frac{r^2}{4\pi\ell_s^2} \int d\tau \int_0^{2\pi} d\sigma [(\partial_\tau X)^2 - (\partial_\sigma X)^2] \quad (3.1)$$

with periodicity $X \sim X + 2\pi$. In the context of string theory it describes string propagation on a target space circle of radius r . The momentum and winding zero modes of the Gaussian field are defined by the general solution of the equation of motion:

$$X = x_0 + \frac{p_L}{\sqrt{2}}(\tau + \sigma) + \frac{p_R}{\sqrt{2}}(\tau - \sigma) + X^{\text{osc}} \quad (3.2)$$

where we have set $\ell_s = 1$ and X^{osc} is the sum of solutions with nonzero Fourier modes. The zero modes have the property that the vectors (p_L, p_R) are valued in an even unimodular lattice embedded in $\mathbb{R}^{1,1}$. The lattice of zero modes can be written as

$$\Gamma(r) := \{ne_r + wf_r | n, w \in \mathbb{Z}\} \subset \mathbb{R}^{1,1} \quad (3.3)$$

where

$$e_r = \frac{1}{\sqrt{2}}(1/r; 1/r), \quad f_r = \frac{1}{\sqrt{2}}(r; -r) \quad (3.4)$$

Note that $e_r^2 = f_r^2 = 0$, $e_r \cdot f_r = 1$ so that $\Gamma(r)$ is indeed an embedding of the even unimodular (a.k.a. self-dual Lorentzian) lattice $II^{1,1}$ of rank 2 and signature (1, 1). Note that the CFT is invariant under $O(1)_L \times O(1)_R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Choose generators of this automorphism group:

$$\begin{aligned} \sigma_L : (X_L, X_R) &\rightarrow (-X_L, X_R) \\ \sigma_R : (X_L, X_R) &\rightarrow (X_L, -X_R) \end{aligned} \quad (3.5)$$

¹³This section has considerable overlap with section four of [11].

Then

$$\sigma_L \cdot \Gamma(r) = \sigma_R \cdot \Gamma(r) = \Gamma(1/r) \quad (3.6)$$

This proves that the moduli space $\mathcal{N}$ of CFTs is related to the space of sigma models $O(1, 1; \mathbb{R})/O(1) \times O(1) \cong \mathbb{R}_+$, parametrized by r , by the quotient by $r \rightarrow 1/r$. Note that $F(\Gamma(r)) \cong \mathbb{Z}_2$ for $r \neq 1$ and $F(\Gamma(r)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ at the self-dual radius $r = 1$. This is the $\mathbb{Z}_2$ orbifold point of the Narain moduli space $\mathcal{N} \cong \mathbb{R}_+/\mathbb{Z}_2 \cong [1, \infty)$. In this case the enhanced symmetry locus is a single (orbifold) point. We can say that σ_L and σ_R are left- and right-moving T-duality symmetries. Note that, with our particular choice of basis for $\Gamma(r = 1)$ the automorphism σ_R is just

$$e \leftrightarrow f \quad (3.7)$$

and we will focus on this T-duality below. Here and henceforth we simply denote e_r, f_r at $r = 1$ by e, f . Of course $\sigma_L \sigma_R = -1$ is the trivial involution. Note that if we identify the positive root of $\mathfrak{su}(2)$ with $\sqrt{2} \in \mathbb{R}$ and the dominant fundamental weight with $1/\sqrt{2}$ then we can identify $\Gamma(r = 1)$ with $\Gamma(\mathfrak{su}(2))$ defined in equation (2.12) above.

The easiest way to see that there is an order four action lifting the T-duality action is to consider the $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ current algebra symmetry of the Gaussian model at the self-dual point. The left- and right-moving currents are

$$J^3(z) = \frac{1}{\sqrt{2}} \partial X_L(z) \quad J^\pm(z) =: e^{\pm i\sqrt{2}X_L(z)} : \hat{c} \quad (3.8)$$

$$\tilde{J}^3(\bar{z}) = \frac{1}{\sqrt{2}} \bar{\partial} X_R(\bar{z}) \quad \tilde{J}^\pm(\bar{z}) =: e^{\pm i\sqrt{2}X_R(\bar{z})} : \hat{c} \quad (3.9)$$

where the tilde indicates right-moving symmetry and $\hat{c}$ is a cocycle factor discussed below. The T-duality transformation leaves the left-moving currents unchanged but takes $\tilde{J}^3 \rightarrow -\tilde{J}^3$ and $\tilde{J}^\pm \rightarrow \tilde{J}^\mp$. It therefore acts as a 180-degree rotation on the Lie algebra $\mathfrak{su}(2)_R$. On the other hand, the states with $(n = 0, w = \pm 1)$ and $(w = 0, n = \pm 1)$ transform in the $2_L \otimes 2_R$ of the $SU(2)_L \times SU(2)_R$ global symmetry. Hence, to define the action on the Hilbert space we must lift T-duality to an order four action. Thus, despite appearances, *T-duality of the Gaussian model at the self-dual point is order four!*¹⁴ We will generalize this discussion in section 4 below.

As an aside we note that while the $SU(2)$ level one WZW model has $SU(2)_L \times SU(2)_R$, the diagonally embedded center generated by $(-1, -1)$ acts ineffectively so in fact the symmetry is $SU(2)_L \times SU(2)_R/\mathbb{Z}_2$. The lift of the full enhanced symmetry group $F(\Gamma(r = 1))$ is then $(\mathbb{Z}_4 \times \mathbb{Z}_4)/\mathbb{Z}_2$.

Now we turn to the modular covariance approach. The Hilbert space of states has sectors labelled by n, w and each sector consists of the usual Fock space of states formed by acting on the vacuum with creation operators $\alpha_{-n}, \tilde{\alpha}_{-n}$, $n \in \mathbb{Z}$. The modular invariant

¹⁴A similar surprise was noted by W. Nahm and K. Wendland concerning mirror symmetry of Kummer surfaces in [28]. While similar in spirit the two remarks are different. In the example of mirror symmetry, the observation is that the action on the sigma model moduli space is order four. Here, the action on the sigma model moduli space is order two, but its action on the CFT state space is order four.

partition function is given by

$$Z(\tau) = B_+ \overline{B_+} \Theta_\Gamma = \frac{\sum_{(p_L, p_R) \in \Gamma^{1,1}} q^{p_L^2/2} \bar{q}^{p_R^2/2}}{\eta \bar{\eta}} \quad (3.10)$$

where $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ with $q = e^{2\pi i \tau}$ is the Dedekind eta function. We have also introduced the notation

$$B_{\pm} := \frac{1}{q^{1/24} \prod_{n=1}^{\infty} (1 \mp q^n)} \quad (3.11)$$

so that $B_+(\tau) = 1/\eta(\tau)$ and $B_-(\tau)$ is the trace of -1 acting on the chiral oscillators:

$$B_- = \frac{\eta(\tau)}{\eta(2\tau)} = \frac{\vartheta_4(2\tau)}{\eta(\tau)}. \quad (3.12)$$

and will be useful presently. Finally, Θ_Γ is the Siegel-Narain theta function. Our conventions for theta functions are spelled out in appendix C. In terms of modular functions the untwisted torus partition function can be written as

$$Z(1, 1) = \frac{1}{\eta \bar{\eta}} \left(\vartheta_3(2\tau) \overline{\vartheta_3(2\tau)} + \vartheta_2(2\tau) \overline{\vartheta_2(2\tau)} \right) \quad (3.13)$$

and this turns out to be modular invariant.

Now let us assume (counter-factually, as we have just seen using $SU(2)$ invariance) that there is a lift of $\langle \sigma_R \rangle$ to a $\mathbb{Z}_2$ group of automorphisms acting on the CFT. Let $\hat{g}$ be the generator of this purported lift.¹⁵ To evaluate $Z(\hat{g}, 1)$ use the relation between the one-loop partition function and a trace on the Hilbert space. The naive action on the Hilbert space is defined by choosing a basis

$$|A, \tilde{A}; p\rangle \quad (3.14)$$

where $A, \tilde{A}$ is shorthand for oscillator states, so that

$$\hat{g}|A, \tilde{A}; p\rangle = (-1)^{\tilde{N}} |A, \tilde{A}; g \cdot p\rangle \quad (3.15)$$

where $p = ne + wf$ and $\tilde{N}$ is the number of right-moving oscillators in the state. Invariance of the momentum forces $p_R = 0$, or equivalently $n = w$ so the momentum is purely left-moving. The phase then simplifies to $(-1)^{\tilde{N}}$ coming from the straightforward action on the oscillators. The resulting partition function is

$$Z(\hat{g}, 1) = \frac{\vartheta_3(2\tau) \overline{\vartheta_4(2\tau)}}{\eta(\tau) \bar{\eta}(\tau)}. \quad (3.16)$$

Modular covariance now forces

$$Z(1, \hat{g}) = 2^{-1} \frac{1}{\eta(\tau) \bar{\eta}(\tau)} \vartheta_3(\tau/2) \overline{\vartheta_2(\tau/2)} \quad (3.17)$$

¹⁵We are deprecating the notation $U(g)$ used in section 2 in favor of $\hat{g}$ for simplicity.

up to a phase, and then again using modular covariance we must have

$$\begin{aligned}
Z(1, \hat{g}) &= 2^{-1} \frac{1}{\eta(\tau)\bar{\eta}(\tau)} \vartheta_3(\tau/2) \bar{\vartheta}_2(\tau/2) \\
Z(\hat{g}, \hat{g}) &= 2^{-1} \frac{1}{\eta(\tau)\bar{\eta}(\tau)} \vartheta_3((\tau-1)/2) \bar{\vartheta}_2((\tau-1)/2) \\
Z(\hat{g}^2, \hat{g}) &= e^{i\pi/4} 2^{-1} \frac{1}{\eta(\tau)\bar{\eta}(\tau)} \vartheta_4(\tau/2) \bar{\vartheta}_2(\tau/2) \\
Z(\hat{g}^3, \hat{g}) &= e^{i\pi/4} 2^{-1} \frac{1}{\eta(\tau)\bar{\eta}(\tau)} \vartheta_4((\tau-1)/2) \bar{\vartheta}_2((\tau-1)/2) \\
Z(\hat{g}^4, \hat{g}) &= e^{i\pi/2} 2^{-1} \frac{1}{\eta(\tau)\bar{\eta}(\tau)} \vartheta_3(\tau/2) \bar{\vartheta}_2(\tau/2)
\end{aligned} \tag{3.18}$$

Note, particularly, that $Z(\hat{g}^2, \hat{g})$ is not proportional to $Z(1, \hat{g})$. Thus, there cannot be an order two action on the CFT space. It is true, however, that $Z(\hat{g}^4, \hat{g})$ is proportional to $Z(1, \hat{g})$ suggesting that T-duality might lift to an order four action on the CFT space. This will prove to be correct.

Indeed we can define an action of an order four lift of σ_R as follows:

$$\hat{g}|A, \tilde{A}; p\rangle = (-1)^{\tilde{N}} e^{\frac{i\pi}{2}(n+w)^2} |A, \tilde{A}; g \cdot p\rangle. \tag{3.19}$$

This can be derived using the discussion of cocycles in section 5 below. In particular this is order four. Note that

$$\hat{g}^2|p\rangle = e^{i\pi p \cdot g \cdot p} |p\rangle \quad \forall p \in \Gamma(\mathfrak{su}(2)) \tag{3.20}$$

in accord with our conjecture (2.20).

Now let us turn to the orbifold by T-duality, or rather, by the group (isomorphic to $\mathbb{Z}_4$) generated by $\hat{g}$. It follows immediately from (3.18) that there is a modular anomaly in the $\hat{g}$ -twisted sector (or equivalently, an anomaly under ST^4S in the untwisted sector). Therefore there is no consistent T-duality orbifold of the Gaussian model, assertions to the contrary in the literature notwithstanding. Equivalently, level matching fails for this model.

We could, however, consider a product of d copies of the Gaussian model at the self-dual radius and consider the orbifold by the simultaneous $r \rightarrow 1/r$ duality on all the circles. That is, we are considering the point in the moduli space of d bosons on a torus with $\mathfrak{g} = \mathfrak{su}(2)^{\oplus d}$. Concretely

$$\Gamma(A_1^d) = \{(r + \lambda; r' + \lambda)\} \tag{3.21}$$

where r, r' are root vectors and

$$\lambda = \frac{1}{2} \sum_i \epsilon_i \alpha_i \quad \epsilon_i \in \{0, 1\} \tag{3.22}$$

where α_i is the simple root for the i^{th} summand. We let g be the lattice automorphism taking $(p_L; p_R) \rightarrow (p_L; -p_R)$. Once again the lifted action $\hat{g}$ on the CFT defined by the diagonal action of the lift of T-duality on a single Gaussian model will be order four. We can

try to orbifold by the $\mathbb{Z}_4$ group $\langle \hat{g} \rangle$. This is of some interest since the alleged inconsistency condition (2.17) is met for such models for all values of d . To see this note that it is met by choosing all but one of the ϵ_i to vanish.

Let us examine first the level matching condition. A single $\mathbb{Z}_2$ -twisted boson has a ground state energy $+\frac{1}{16}$. Therefore in the twisted sector a state has quantum numbers N_L , the left moving oscillator level (this is an integer) and N_R the right-moving oscillator number (this is in $\frac{1}{2}\mathbb{Z}$ since the right-moving oscillators are half-integer moded: $\tilde{\alpha}_{-r}$, $r \in \mathbb{Z} + \frac{1}{2}$). In addition there is a momentum in the dual of the invariant sublattice: $p \in (\Gamma^g)^\vee$. For very general reasons explained below $2p^2$ is an integer, so $\frac{1}{2}p^2 \in \frac{1}{4}\mathbb{Z}$. In our specific example

$$(\Gamma^g)^\vee = \left\{ \left(\sum_i n_i \frac{\alpha_i}{2}; 0 \right) \mid n_i \in \mathbb{Z} \right\}. \quad (3.23)$$

Now, because the twisted sector ground state of a single real $\mathbb{Z}_2$ -twisted boson has energy $1/16$, a state with quantum numbers (N_L, N_R, p) will have equal left and right scaling dimensions if:

$$\left(N_L + \frac{1}{2}p_L^2 - \frac{d}{16} - N_R - \frac{1}{2}p_R^2 \right) = 0 \quad (3.24)$$

Since N_L can be an arbitrary nonnegative integer and $N_R \in \frac{1}{2}\mathbb{Z}$ this becomes a condition on p and d . That condition is

$$p^2 - \frac{d}{8} = 0 \bmod 1 \quad (3.25)$$

Now, since $p^2 \in \frac{1}{2}\mathbb{Z}$ we see that this condition can only be satisfied for $d = 0 \bmod 4$.

We claim that all known consistency conditions are satisfied for the $\mathbb{Z}_4$ orbifold for $d = 0 \bmod 4$. We have just checked level matching. In addition to level-matching, one should check that the partition function has a good q -expansion in the sense explained in section 2.

We can easily compute the partition function for the orbifold of A_1^d using modular covariance and the partition functions computed above for the A_1 theory since

$$Z_{A_1^d}(g, h) = (Z_{A_1}(g, h))^d. \quad (3.26)$$

The only tricky point is the $\hat{g}^2$ -sector. Using modular covariance one easily computes

$$\begin{aligned} Z(\mathcal{H}_{\hat{g}^2}) &= \frac{1}{4} \frac{1}{(\eta\bar{\eta})^{4k}} [(\vartheta_3(2\tau)\bar{\vartheta}_2(2\tau) + \vartheta_2(2\tau)\bar{\vartheta}_3(2\tau))^{4k} \\ &\quad + (\vartheta_3(2\tau)\bar{\vartheta}_2(2\tau) - \vartheta_2(2\tau)\bar{\vartheta}_3(2\tau))^{4k} + 2(-1)^k \vartheta_2(2\tau)^{4k} \bar{\vartheta}_4(2\tau)^{4k}] \end{aligned} \quad (3.27)$$

where $d = 4k$. The second line contains contributions with minus signs which are potentially problematic. However, the terms in square brackets can be written as:

$$2 \sum_{s=0}^{2k-1} \binom{4k}{2s} (\vartheta_3\bar{\vartheta}_2)^{4k-2s} (\vartheta_2\bar{\vartheta}_3)^{2s} + 2\vartheta_2^{4k} \left(\bar{\vartheta}_3^{4k} + (-1)^k \bar{\vartheta}_4^{4k} \right) \quad (3.28)$$

The first sum manifestly has a good q -expansion. The only possibly problematic part is the second term. For $k = 1$ we note that positivity of this term follows from Jacobi's abstruse

identity, $\vartheta_3^4 - \vartheta_4^4 = \vartheta_2^4$. For general k we write this term as

$$\left(\bar{\vartheta}_3^{4k} + (-1)^k \bar{\vartheta}_4^{4k}\right) = \sum_{n_1, \dots, n_{4k}} \bar{q}^{\frac{1}{2}(n_1^2 + \dots + n_{4k}^2)} (1 + (-1)^{k+n_1+\dots+n_{4k}}) \quad (3.29)$$

and note that the coefficients of $\bar{q}^{\ell/2}$ in this expression are either 0 or 2. The entire expression in square brackets is of the form 2^{1+4k} times a good q -expansion and hence $Z(\mathcal{H}_{\hat{g}^2})$ has a good q -expansion. One can similarly check that $Z(\mathcal{H}_{\hat{g}})$ and $Z(\mathcal{H}_{\hat{g}^3})$ have good q -expansions.

4 Models with non-Abelian symmetry

Let $\mathfrak{g}$ be a semi-simple (but not necessarily simple) and simply-laced Lie algebra of full rank. The points $\Gamma(\mathfrak{g})$ of the Narain lattice defined in (2.12) are very special. The CFT $\mathcal{C}(\mathfrak{g})$ corresponding to these points is isomorphic to the WZW model at level one for the simply connected covering group G . When $\mathfrak{g}$ is simple the CFT space of the WZW model is

$$\mathcal{H} = \oplus_{\theta \cdot \lambda \leq 1} V_\lambda \otimes \overline{V}_\lambda \quad (4.1)$$

and is a representation of $\widetilde{LG}_L \times \widetilde{LG}_R$ although the diagonally embedded center of G acts trivially. Here θ is the highest root and V_λ is the integrable lowest weight representation. In particular, the subgroup of constant loops $G_L \times G_R$ acts. On the other hand, in the equivalent formulation in terms of free bosons on a torus, the crystallographic symmetry group given by (2.13) acts canonically on the oscillators and momenta of the theory. Nevertheless, as we have repeatedly stressed, this group must *not* be confused with a group of automorphisms of the CFT $\mathcal{C}(\mathfrak{g}) := \mathcal{C}_{\Gamma(\mathfrak{g})}$. In particular, there is no natural action of it on the state space (4.1) compatible with the action on the oscillators and momenta. We now discuss this in a little more detail.

First we compute $F(\Gamma(\mathfrak{g}))$. As is well-known, the automorphism group $\text{Aut}(\Lambda_{wt}(\mathfrak{g}))$ is the semidirect product $W(\mathfrak{g}) \rtimes \mathcal{D}(\mathfrak{g})$ where $\mathcal{D}(\mathfrak{g})$ is the group of outer automorphisms of $\mathfrak{g}$ ([29], Proposition D.40), ([30], section 16.5). The group $F(\Gamma(\mathfrak{g}))$ is thus the semidirect product

$$F(\Gamma(\mathfrak{g})) = (W(\mathfrak{g})_L \times W(\mathfrak{g})_R) \rtimes \mathcal{D}(\mathfrak{g}) \quad (4.2)$$

where $\mathcal{D}(\mathfrak{g})$ acts diagonally on $W(\mathfrak{g})_L \times W(\mathfrak{g})_R$.¹⁶

Let us begin by considering the lift of the subgroup $W(\mathfrak{g})_L \times W(\mathfrak{g})_R$. Recall the discussion around (2.14) and (2.15) of section 2. In order to define an automorphism group of the CFT $\mathcal{C}(\mathfrak{g})$ inducing the action of $F(\Gamma(\mathfrak{g}))$ on the oscillators we must lift $W(\mathfrak{g})$ to a subgroup of $N(T) \subset G$ and use the action defined by $G \subset \widetilde{LG}$. That is, we must choose a finite subgroup $\widetilde{W}(\mathfrak{g}) \subset N(T)$ so that if $\pi : \widetilde{W}(\mathfrak{g}) \rightarrow W(\mathfrak{g})$ then for every $\tilde{g} \in \widetilde{W}(\mathfrak{g})$ we have $\tilde{g}t\tilde{g}^{-1} = \pi(\tilde{g}) \cdot t$.

¹⁶It is worth noting that the definition of $\Gamma(\mathfrak{g})$ can be generalized to an even unimodular lattice $\Gamma(\mathfrak{g}, \sigma)$ defined by any element $\sigma \in \text{Aut}(\mathfrak{g})$ by choosing pairs $(p_L; p_R) \in \Lambda_{wt}(\mathfrak{g}) \times \Lambda_{wt}(\mathfrak{g})$ such that $p_L - \sigma(p_R) \in \Lambda_{rt}(\mathfrak{g})$. These lattices project to the same point in $\mathcal{N}$.

We now explain in more detail how subgroups of $N(T)_L \times N(T)_R$ act on the CFT space. We can choose a basis of states for $\mathcal{H}$ of the following form. We begin with the representation

$$\oplus_{\theta \cdot \lambda \leq 1} R_\lambda \otimes \overline{R_\lambda} \quad (4.3)$$

of the finite-dimensional group $G_L \times G_R$. Here R_λ is the irreducible representation of G with dominant weight λ . Now choose a weight basis for (4.3) and denote it:

$$|\mu_L\rangle \otimes \overline{|\mu_R\rangle}. \quad (4.4)$$

Note that μ_L, μ_R are weights in the same irreducible representation R_λ and hence $\mu_L - \mu_R$ is in the root lattice. Next we act on this basis with arbitrary monomials of raising operators for both the left and right-moving current algebra symmetry. The raising operators are either of the form $\alpha_I \cdot H_{-n}$ where α_I are simple roots and $n > 0$ labels the Fourier modes of the current, or they are of the form $E_{-\alpha}^n$ where again $n > 0$ labels a Fourier mode and α is a root. The resulting set of states is an overcomplete set in general (because of null vectors) but it will suffice to specify the group action on this set.

An element $(\hat{g}_L, \hat{g}_R) \in N(T)_L \times N(T)_R \subset G_L \times G_R$ preserves the currents. For example:

$$\begin{aligned} \hat{g}_L E_{-n}^\alpha \hat{g}_L^{-1} &= E_{-n}^{\bar{g}_L \cdot \alpha} \\ \hat{g}_L \alpha \cdot H_{-n} \hat{g}_L^{-1} &= (\bar{g}_L \cdot \alpha) \cdot H_{-n} \end{aligned} \quad (4.5)$$

where $\bar{g}_L \cdot \alpha$ is the induced action of the projection of $\hat{g}_L$ in $N(T)/T := W$ on the root lattice, and similarly for $\hat{g}_R$. The action (4.5) will map null vectors to null vectors so to define the action on the states we need only define the action on the states (4.4) and this is:

$$(\hat{g}_L, \hat{g}_R) \cdot (|\mu_L\rangle \otimes \overline{|\mu_R\rangle}) := R_\lambda(g_L)|\mu_L\rangle \otimes \overline{R_\lambda(g_R)|\mu_R\rangle}. \quad (4.6)$$

Note that states of the form (4.4) correspond to states $|p\rangle$ in the vertex operator algebra construction with momentum

$$p = (\mu_L; \mu_R) \quad (4.7)$$

so together with (4.5) we see that $(\hat{g}_L, \hat{g}_R)$ acts on the Narain lattice through the projection to the Weyl group.

Lifting the Weyl group to a subgroup of $N(T)$ has been studied in the mathematical literature and we review some relevant results in appendix D below. The key points are that there is always a canonical lift $\widetilde{W}(\mathfrak{g})^T$ called the *Tits lift*, but $\widetilde{W}(\mathfrak{g})^T$ is never isomorphic to the Weyl group: the lift of reflections in simple roots are elements of order four in $N(T)$. For some groups there do exist lifts isomorphic to $W(\mathfrak{g})$ but for some groups no such lift exists. It is possible to be quite explicit about the various possibilities, see for example [31–33] and appendix D.1.

4.1 Example: products of SU(3) level one

A very useful example is the model $\mathcal{C}(\mathfrak{su}(3))$ with g a right-moving involution corresponding to reflection in a simple root. In this case one can modify the generators of the Tits

lift by shift vectors so that there is a lift of $F(\Gamma)$ isomorphic to $F(\Gamma)$, even though the condition (2.17) is satisfied.

As discussed in appendix D below, if we take T to be the subgroup of diagonal $SU(3)$ matrices then lifts of the Weyl reflections in α_1, α_2 must have the form

$$\hat{g}_1 = \begin{pmatrix} 0 & x_1 & 0 \\ y_1 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix} \quad (4.8)$$

$$\hat{g}_2 = \begin{pmatrix} z_2 & 0 & 0 \\ 0 & 0 & x_2 \\ 0 & y_2 & 0 \end{pmatrix} \quad (4.9)$$

where $x_i y_i z_i = -1$. Conjugation on T by these matrices will induce the action of the Weyl reflections in α_1, α_2 , where we choose the standard simple roots. If we choose

$$\hat{g}_1^W = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (4.10)$$

$$\hat{g}_2^W = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (4.11)$$

then $\hat{g}_1, \hat{g}_2 \in SU(3)$ generate a subgroup of $N(T)$ isomorphic to S_3 . On the other hand the Tits lift is

$$\hat{g}_1^T = \exp \left[\frac{\pi}{2} (e_1 - f_1) \right] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.12)$$

$$\hat{g}_2^T = \exp \left[\frac{\pi}{2} (e_2 - f_2) \right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad (4.13)$$

(where e_i, f_i are Serre generators). Note that $\hat{g}_i^T$ are both of order four, so they generate an extension of S_3 by $\mathbb{Z}_2 \times \mathbb{Z}_2$. For later use note that if we compare the “Weyl lift” (4.10) and (4.11) with the Tits lift then we have

$$\hat{g}_1^W = \hat{g}_1^T t_1 \quad (4.14)$$

with

$$t_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}. \quad (4.15)$$

Note that this acts on the weight basis as

$$t_1 |\mu\rangle = e^{\pi i \theta \cdot \mu} |\mu\rangle \quad (4.16)$$

where $\theta = \alpha_1 + \alpha_2$ is the highest root. Similarly, one may check that $\hat{g}_2^W = \hat{g}_2^T t_2$ where $t_2|\mu\rangle = e^{i\pi\alpha_1 \cdot \mu}|\mu\rangle$ in the three-dimensional defining representation.

Turning now to the CFT $\mathcal{C}(\mathfrak{su}(3))$ the vectors in the Narain lattice are of the form

$$(n_1\alpha_1 + n_2\alpha_2 + r\lambda^2; \tilde{n}_1\alpha_1 + \tilde{n}_2\alpha_2 + r\lambda^2) \quad (4.17)$$

where α_i are the simple roots and λ^i the dual fundamental weights and $n_i, \tilde{n}_i \in \mathbb{Z}$ and $r = 0, 1, 2$. We are going to consider a symmetry which acts on the Narain lattice as a right-moving reflection in the simple root α_1 :

$$g \cdot (p_L; p_R) := (p_L; \sigma_{\alpha_1}(p_R)). \quad (4.18)$$

The condition (2.17) is satisfied iff $\tilde{n}_2$ is odd because:

$$p \cdot gp = p \cdot p + (2\tilde{n}_1 - \tilde{n}_2)(\tilde{n}_1\alpha_1 + \tilde{n}_2\alpha_2 + r\lambda^2) \cdot \alpha_1 = \tilde{n}_2 \bmod 2 \quad (4.19)$$

We can choose a twisted characteristic vector $W_g \in \Gamma^g$ (see equation (2.28) and section 6.1) to be

$$W_g = (0; \alpha_1 + 2\alpha_2) \quad (4.20)$$

so that $p \cdot gp = p \cdot W_g \bmod 2$ for all vectors $p \in \Gamma$.

The action of $\hat{g}_1^T$ on $\mathcal{C}(\mathfrak{su}(3))$ satisfies (2.19) and (2.20) so the discussion of modular covariance with respect to twisting by this action is very similar to that for T-duality in the Gaussian model. We have

$$Z(\hat{g}, 1) = B_+^2 \overline{B_+ B_-} \Theta_{\Gamma^g}(\tau, 0, 0) \quad (4.21)$$

$$Z(\hat{g}^2, 1) = B_+^2 \overline{B_+^2} \Theta_{\Gamma} \left(\tau, -\frac{1}{2} W_g, 0 \right) \quad (4.22)$$

where we recall that $B_{\pm}$ were defined in (3.11) and Θ_{Γ^g} is the theta function of the invariant sublattice under the action of g .

Applying the S -transformation to (4.21) we get

$$Z(1, \hat{g})(\tau) = B_+^2 \overline{B_+ T_-} \frac{1}{2} \Theta_{(\Gamma^g)^\vee}(\tau, 0, 0) \quad (4.23)$$

where

$$T_-(\tau) := \frac{\vartheta_2(\tau/2)}{\eta(\tau)}. \quad (4.24)$$

Using $T_-(\tau + 2) = e^{2\pi i/24} T_-(\tau)$ it is easy to check that under $\tau \rightarrow \tau + 2$ this function is not covariant, but

$$Z(\hat{g}^{-4}, \hat{g})(\tau) = Z(1, \hat{g})(\tau + 4) = e^{-4\pi i/8} \frac{1}{2} B_+^2 \overline{B_+^2} \bar{\vartheta}_2(\tau/2) \Theta_{(\Gamma^g)^\vee}(\tau, 0, 0) \quad (4.25)$$

As in the case of the Gaussian model, we can consider the orbifold of the direct product $\mathcal{C}(\mathfrak{su}(3))^d$ by the $\mathbb{Z}_4$ group generated by the diagonal action of $\hat{g}_1^T$. Level matching is only satisfied for $d = 0 \bmod 4$ and with a little patience one can check that the partition function indeed has a good q -expansion.

It is interesting to compare the above discussion with the analogous one for the Weyl lift $\hat{g}_1^W$. This differs from the Tits lift by a shift vector $e^{2\pi i \hat{p} \cdot s}$ with $s = (0; \frac{1}{2}\theta)$ and now we can compute

$$\begin{aligned} (\hat{g}_1^T) e^{2\pi i \hat{p} \cdot s} (\hat{g}_1^T) e^{2\pi i \hat{p} \cdot s} &= (\hat{g}_1^T)^2 e^{\pi i \hat{p} \cdot (0; \alpha_2 + \theta)} \\ &= (\hat{g}_1^T)^2 e^{\pi i \hat{p} \cdot (0; 2\alpha_2 + \alpha_1)} \end{aligned} \quad (4.26)$$

On the other hand

$$(\hat{g}_1^T)^2 = e^{\pi i \hat{p} \cdot W_g} = e^{\pi i \hat{p} \cdot (0; \alpha_1)} \quad (4.27)$$

and hence $\hat{g}_1^W = \hat{g}_1^T e^{2\pi i \hat{p} \cdot s}$ has order two acting on $\mathcal{C}(\mathfrak{su}(3))$. One can confirm that the partition functions have the correct modular covariance:

$$Z(1, \hat{g}_1^W)(\tau + 2) = -e^{-2\pi i/8} Z(1, \hat{g}_1^W)(\tau). \quad (4.28)$$

In checking this one must bear in mind that if $p \in \Gamma^g$ is in the invariant lattice then

$$\hat{g}_1^W |p\rangle = e^{i\pi(\tilde{n}_1 + r)} |p\rangle \quad (4.29)$$

in the parametrization used in (4.17). (In this parametrization the invariant lattice is defined by the condition $\tilde{n}_2 = 2\tilde{n}_1$.) Note that (4.29) violates (2.11), even for $p \in \Gamma^g$.

If we now consider the orbifold of the direct product $\mathcal{C}(\mathfrak{su}(3))^d$ by the $\mathbb{Z}_2$ group generated by the diagonal action of $\hat{g}_1^W$ then level-matching - or, equivalently, the absence of modular anomalies requires $d = 0 \bmod 8$. Once again, one can check that the partition function of the orbifold theory has a good q -expansion. Thus, the asymmetric orbifold by the $\mathbb{Z}_2$ group generated by $\hat{g}_1^W$ satisfies all known consistency conditions.

4.2 A nontrivial lift of an outer automorphism of $\mathfrak{g}$

Thus far we have discussed involutions in the subgroup $W(\mathfrak{g})_L \times W(\mathfrak{g})_R \subset F(\Gamma(\mathfrak{g}))$. It is also interesting to ask about group elements projecting to nontrivial members of $\mathcal{D}(\mathfrak{g})$. This group is described in [29]. For $\mathfrak{a}_n$ the diagram automorphism just corresponds to complex conjugation on $\mathfrak{su}(n+1)$. It acts as -1 on the lattice $\Gamma(\mathfrak{g})$ and is thus a trivial involution and the lift is order two. Similarly for $\mathfrak{d}_n$ the group is $\mathbb{Z}_2$. It corresponds to a parity transformation exchanging the two spinors, or equivalently to conjugation by an element of $O(2n)$ with determinant minus one. Moreover, for $\mathfrak{e}_6$ one can choose a lift of the Diagram automorphism by exchanging the appropriate simple roots. One can check that the condition (2.17) is never satisfied.

Finally we come to the special case of $\mathfrak{d}_4$. We view the root lattice as four-tuples of integers with the sum of coordinates an even integer. Then in addition to the parity involution $(x_1, x_2, x_3, x_4) \rightarrow (-x_1, x_2, x_3, x_4)$ there is a nontrivial involution known as the Hadamard involution

$$H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \quad (4.30)$$

There are root vectors such that $r \cdot H \cdot r$ is odd and hence the vector $p = (r; 0)$ will satisfy (2.17). Therefore, the orthogonal transformation $(p_L; p_R) \rightarrow (Hp_L; Hp_R)$, which is an involution of the Narain lattice will lift to an automorphism of the CFT which is either order four or violates (2.11).

5 Cocycles at ADE enhanced symmetry points

5.1 Review of cocycles

We first review the standard reason why cocycles are required in the construction of vertex operators for toroidal CFTs.¹⁷ One might naively expect that under the state-operator correspondence the states $|p\rangle$ defining a basis for $\mathbb{C}[\Gamma]$ in (2.2) correspond to the vertex operator:

$$V^{\text{naive}}(p, z, \bar{z}) := e^{ipX} := e^{i(p_L X_L + p_R X_R)} : \quad (5.1)$$

where $p = (p_L; p_R)$ is the decomposition of p into its left- and right-moving projections. However the usual OPE

$$V^{\text{naive}}(p_1, z_1, \bar{z}_1) V^{\text{naive}}(p_2, z_2, \bar{z}_2) = z_{12}^{p_L^1 \cdot p_L^2} \bar{z}_{12}^{p_R^1 \cdot p_R^2} : e^{i(p_1 X(z_1, \bar{z}_1) + p_2 X(z_2, \bar{z}_2))} : \quad (5.2)$$

shows that the operators $V^{\text{naive}}(p, z, \bar{z})$ are not quite the right operators to use in a consistent CFT because they are not mutually local. In radial quantization we have the braiding relation:

$$V^{\text{naive}}(p_1, z_1) V^{\text{naive}}(p_2, z_2) = e^{i\pi p_1 \cdot p_2} V^{\text{naive}}(p_2, z_2) V^{\text{naive}}(p_1, z_1). \quad (5.3)$$

The problem is with the factor $e^{ix_0 \cdot p}$ in the vertex operator. This is a shift operator on $\mathbb{C}[\Gamma]$ taking $L_{p'} \rightarrow L_{p+p'}$ and these operators generate the commutative group algebra $\mathbb{C}[\Gamma]$. In order to cancel the phase in (5.3) we introduce an extra operator $\hat{c}(p)$ on $\mathbb{C}[\Gamma]$ which is diagonal in the direct sum decomposition $\oplus_{p'} L_{p'}$ and acts as a multiplication by a phase $\varepsilon(p, p')$ on $L_{p'}$ where the phases are valued in some subgroup $A \subset U(1)$. Then, if we define

$$\hat{C}(p) := e^{ix_0 \cdot p} \hat{c}(p) \quad (5.4)$$

these operators generate a noncommutative algebra

$$\hat{C}(p_1) \hat{C}(p_2) = \varepsilon(p_1, p_2) \hat{C}(p_1 + p_2), \quad (5.5)$$

where we have used the cocycle identity for ε . The correct vertex operators:

$$V(p, z, \bar{z}) := V^{\text{naive}}(p, z, \bar{z}) \hat{c}(p) \quad (5.6)$$

will be mutually local if ε satisfies the condition:

$$s(p_1, p_2) := \frac{\varepsilon(p_1, p_2)}{\varepsilon(p_2, p_1)} = e^{i\pi p_1 \cdot p_2} \quad (5.7)$$

¹⁷See [34], section 6, for a particularly lucid account of the cocycles for chiral vertex operator algebras associated with lattices.

because

$$\hat{C}(p_1)\hat{C}(p_2) = s(p_1, p_2)\hat{C}(p_2)\hat{C}(p_1). \quad (5.8)$$

It is useful to interpret these formulae in terms of a central extension of the group Γ . Associativity of the operators $\hat{C}(p)$ implies that ε defines an A -valued group cocycle on Γ , and hence defines a central extension:

$$1 \rightarrow A \rightarrow \hat{\Gamma} \rightarrow \Gamma \rightarrow 1. \quad (5.9)$$

This central extension acts on $\mathbb{C}[\Gamma]$ with A acting as scalars. The central extension is characterized, up to isomorphism of central extensions, by the commutator function $s(p^1, p^2)$. Changing ε by a coboundary corresponds to a redefinition of the operators $\hat{C}(p)$ by a phase valued in A , and the commutator function is gauge-invariant. Note, however, that a choice of A is part of the definition of the central extension. Once A has been chosen, valid coboundaries must be A -valued. In much of the literature the group $A = \{\pm 1\}$ has been chosen, but we will find that it is often more appropriate to let A be the group of fourth roots of unity.

5.2 Detailed cocycles for the SU(2) point

We now demonstrate that the standard choice of cocycle is incompatible with SU(2) symmetry at the SU(2) enhanced symmetry point of a single Gaussian model. It is important to note that the inconsistency does not arise at the level of vertex operators for the currents generating the affine SU(2) algebra. It is well known that the standard cocycle gives the correct commutation relations [8, 9, 34]. Rather as we will see, the problem arises in the OPE of currents with states transforming in the fundamental representation of SU(2).

Our SU(2) conventions are that we use anti-Hermitian generators

$$T^a = -\frac{i}{2}\sigma^a \quad [T^a, T^b] = \epsilon^{abc}T^c \quad (5.10)$$

so that the current $\times$ current OPE should be

$$J^a(z_1)J^b(z_2) = \frac{-\frac{k}{2}\delta^{ab}}{z_{12}^2} + \frac{\epsilon^{abc}}{z_{12}}J^c(z_2) + \dots \quad (5.11)$$

where in general k is the level and in our case $k = 1$.

Now in the two-dimensional representation of $\mathfrak{su}(2)$ we have

$$T^+ := T^1 + iT^2 = -i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T^- := T^1 - iT^2 = -i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (5.12)$$

So

$$\begin{aligned} [T^3, T^+] &= -iT^+ \\ [T^3, T^-] &= +iT^- \\ [T^+, T^-] &= -2iT^3. \end{aligned} \quad (5.13)$$

Now $J^3 = \partial X_L / \sqrt{2}$ gives the OPE

$$J^3(z_1) e^{\pm i\sqrt{2}X_L}(z_2) = \frac{\mp i}{z_{12}} e^{\pm i\sqrt{2}X_L}(z_2) + \dots \quad (5.14)$$

so that up to normalization $J^\pm(z) =: e^{\pm i\sqrt{2}X_L} : \hat{c}(\pm(e+f))$ and $\tilde{J}^\pm(z) =: e^{\pm i\sqrt{2}X_R} : \hat{c}(\pm(e-f))$.

To determine the cocycle $\varepsilon(p_1, p_2)$ we first determine the usual constraint coming from the current-current OPEs

$$J^+(z_1)J^-(z_2) = \varepsilon(e+f, -e-f) \left(\frac{1}{z_{12}^2} + \frac{2i}{z_{12}} J^3(z_2) + \dots \right). \quad (5.15)$$

We thus require

$$\varepsilon(e+f, -e-f) = \varepsilon(-e-f, e+f) = \varepsilon(e-f, -e+f) = \varepsilon(-e+f, e-f) = -1. \quad (5.16)$$

However we should also demand that we get the matrix elements of $T^\pm$ when acting on the vertex operators with $n = \pm 1, w = 0$ and $n = 0, w = \pm 1$. These vertex operators create states in the $(2, 2)$ of $SU(2)_L \times SU(2)_R$. Thus we consider

$$V_{\epsilon_L, \epsilon_R} :=: e^{\frac{i}{\sqrt{2}}(\epsilon_L X_L + \epsilon_R X_R)} : \hat{c}_{\epsilon_L, \epsilon_R} \quad (5.17)$$

where different choices of signs ϵ_L, ϵ_R give the four distinct vectors $\pm e, \pm f$. We now compute the OPE

$$J^+(z_1)V_{-, \pm}(z_2) = -\frac{i}{z_{12}} V_{+, \pm}(z_2) + \dots \quad (5.18)$$

and so on.

Continuing in this way we find that¹⁸

$$\begin{aligned} \varepsilon(e+f, -f) &= -i & \varepsilon(e-f, f) &= i \\ \varepsilon(e+f, -e) &= -i & \varepsilon(e-f, -e) &= i \\ \varepsilon(-e-f, e) &= -i & \varepsilon(-e+f, e) &= i \\ \varepsilon(-e-f, f) &= -i & \varepsilon(-e+f, -f) &= i \end{aligned} \quad (5.19)$$

To solve (5.19) we consider the general class of cocycles

$$\varepsilon(n_1 e + w_1 f, n_2 e + w_2 f) = e^{i\pi(\alpha n_1 n_2 + \beta w_1 w_2 + \gamma n_1 w_2 + \delta w_1 n_2)} \quad (5.20)$$

with $\alpha, \beta, \gamma, \delta$ defined mod 2 and impose (5.19) to obtain two solutions:

$$\varepsilon_1(n_1 e + w_1 f, n_2 e + w_2 f) = e^{(i\pi/2)(-2w_1 w_2 - n_1 w_2 + w_1 n_2)}, \quad (5.21)$$

$$\varepsilon_2(n_1 e + w_1 f, n_2 e + w_2 f) = e^{(i\pi/2)(-2n_1 n_2 + n_1 w_2 - w_1 n_2)}. \quad (5.22)$$

¹⁸It is crucial in obtaining the signs below to recall that the spectrum of the WZW model consists of states with left-moving part in the representation $\mathcal{R}_\lambda$ corresponding to weights $\lambda - r$ with r in the root lattice and with right-moving part in the representation $\overline{\mathcal{R}_\lambda}$ with weights $-\lambda + r$. See equations (4.3) and (4.4).

These cocycles are in fact equivalent since they are related by a coboundary. Explicitly we have

$$\varepsilon_2(n_1e + w_1f, n_2e + w_2f) = \varepsilon_1(n_1e + w_1f, n_2e + w_2f) e^{-i\pi(n_1-w_1)(n_2+w_2)} \quad (5.23)$$

and the factor on the right above is equal to

$$e^{-i\pi(n_1+w_1)(n_2+w_2)} = e^{-2\pi i p_L^1 \cdot p_L^2} = \frac{b(p^1 + p^2)}{b(p_1)b(p_2)} \quad (5.24)$$

with $b(p) = \exp(-i\pi p_L^2)$. From now on we work with the cocycle ε_1 .

We now show that this choice of cocycle ensures that the lift of the Weyl group element is order four. In appendix A we discuss a general formalism for lifting automorphisms of abelian extensions of lattices. In the notation used there we have a lattice extension $\hat{\Gamma}$ defined by the cocycle ε_1 and can solve (A.5) by choosing

$$\xi_g(p) = \exp\left((i\pi/2)(n+w)^2\right). \quad (5.25)$$

We then check that

$$\xi_g(p)\xi_g(gp) = \exp(i\pi(n+w)) \quad (5.26)$$

which shows that the lift of the Weyl reflection is order four.

We close with two remarks. The first (pointed out to us by K. Wendland) is that our cocycles do not satisfy the condition $\varepsilon(-p, p) = 1$ enforced in [34]. That condition is based on the choice of gauge $\varepsilon(0, p) = \varepsilon(p, 0) = 1$ together with the condition $V(p)^\dagger = V(-p)$. In fact, one could change the cocycle by a $(\mathbb{Z}_4\text{-valued})$ coboundary to enforce $\varepsilon(-p, p) = 1$. Moreover, the Hermitian structure on the Hilbert space of states and the state-operator correspondence is consistent with the more general Hermiticity condition to $V(p)^\dagger = (\varepsilon(-p, p))^{-1}V(-p)$, when $\varepsilon(-p, p) \neq 1$. The second remark is that the generalization of the above discussion to all the points $\Gamma(\mathfrak{g})$ associated with simply laced Lie algebras is not entirely trivial, and we hope to return to this question on a future occasion.

Note added for v3. In the first two versions of this paper on the arXiv we claimed that it is strictly necessary to modify the standard $\mathbb{Z}_2$ -valued cocycles in vertex operator algebras to $\mathbb{Z}_4$ -valued cocycles in order to understand the nontrivial lifting of T -duality discussed throughout the paper. This claim is erroneous. While the above formulae are correct, so far as we know, one could perfectly well use the standard cocycle,

$$\varepsilon(n_1e + w_1f, n_2e + w_2f) = (-1)^{n_1w_2} \quad (5.27)$$

Indeed, this $\mathbb{Z}_2$ -valued cocycle can be obtained from ε_1 by a $\mathbb{Z}_4$ -valued coboundary $b(ne + wf) = e^{\frac{i\pi}{2}(w^2 - nw)}$. If we use the standard $\mathbb{Z}_2$ -valued cocycle then we must modify the lifting function (5.25) to

$$\xi_g(ne + wf) = (-1)^{n(w+1)} \quad (5.28)$$

The matrix elements of $J^\pm$ acting on the half-spin modules are accounted for by a simple rescaling of the vertex operators $V_{\epsilon_L, \epsilon_R}$ given above (as is indeed implied by using the coboundary). We thank the referee for pointing this out.

6 Criterion for nontrivial lifting

In this section we discuss the modular covariance approach to determining when non-trivial elements $g \in F(\Gamma)$ must lift to elements $\hat{g} \in \text{Aut}(\mathcal{C}_\Gamma)$ of twice the order of g , or violate (2.11). We are discussing points in $\mathcal{N}^{\text{ESP}}$ that typically do not have non-abelian symmetry so we cannot use the crutch of the level one WZW model for a non-abelian group. We will derive the criterion (2.17).

6.1 Inconsistency with modular covariance

We begin by supposing that $g \in F(\Gamma)$ is an involution. Suppose that there is a lift $\hat{g}$ so that $\langle \hat{g} \rangle \subset \text{Aut}(\mathcal{C}_\Gamma)$ is isomorphic to $\mathbb{Z}_2$ and has an action (2.11) for $p \in \Gamma^g$. We are going to show that if (2.17) is satisfied then there is an inconsistency with modular covariance.

Using the methods of section 7 below it is easy to see that $Z(1, \hat{g})$ has a q -expansion which has the form of a sum over $p \in (\Gamma^g)^\vee$ of

$$(q\bar{q})^{-d/24} \exp[2\pi i(n_- \tau - \tilde{n}_- \bar{\tau})/16] q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}, \quad (6.1)$$

where n_- is the number of twisted left-moving bosons and $\tilde{n}_-$ is the number of twisted right-moving bosons, times a power series in integral powers of $q^{1/2}, \bar{q}^{1/2}$ with nonnegative integral coefficients. Under $\tau \rightarrow \tau + 2$ this transforms to

$$(q\bar{q})^{-d/24} e^{i\pi(n_- - \tilde{n}_-)/4} \exp[2\pi i(n_- \tau - \tilde{n}_- \bar{\tau})/16] q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} e^{i\pi 2p^2} \quad (6.2)$$

Now the key point is that (as we will show presently) for every vector p in the dual of the invariant lattice $2p^2$ is an integer, but it can be even or odd. If there are vectors for which it is odd, the sum over p will produce a new function in the sense that:

$$Z(1, \hat{g})(\tau + 2) \neq e^{i\phi} Z(1, \hat{g})(\tau) \quad (6.3)$$

for any phase $e^{i\phi}$. Therefore, if there are vectors in $(\Gamma^g)^\vee$ with $2p^2$ an odd integer then in the $\hat{g}$ -twisted sector $\hat{g}$ cannot be order two. One can then check that modular covariance implies that $\hat{g}$ cannot be order two in the untwisted sector either.

Note that for all $p \in (\Gamma^g)^\vee$ it is true that $4p^2$ is even. It follows that under a transformation $\tau \rightarrow \tau + 4$ (6.1) transforms to

$$(q\bar{q})^{-d/24} e^{i\pi(n_- - \tilde{n}_-)/2} \exp[2\pi i(n_- \tau - \tilde{n}_- \bar{\tau})/16] q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \quad (6.4)$$

and therefore this analysis of modular covariance indicates that it is consistent to assume that there is a lift $\hat{g}$ of g that is order four.

Now we show that for $p \in (\Gamma^g)^\vee$, $2p^2$ is an integer, and in fact, the existence of vectors such that it is an odd integer is precisely equivalent to the condition (2.17). To prove this let $\mathfrak{J} = \Gamma^g$ be the sublattice of invariant vectors. Then, for every $v \in \mathfrak{J}^\vee$ we have $v^2 \in \frac{1}{2}\mathbb{Z}$. Indeed we have the usual decomposition of Γ using glue vectors for

$$(\Gamma^g) \oplus (\Gamma^g)^\perp := \mathfrak{J} \oplus \mathfrak{N} \quad (6.5)$$

and the discriminant groups of the invariant lattice $\mathfrak{I}$ and its orthogonal complement $\mathfrak{N}$ are isomorphic. So if $v \in \mathfrak{I}^\vee$ then there is a $u \in \mathfrak{N}^\vee$ with $w = v + u \in \Gamma$. Conversely, every $w \in \Gamma$ can be written in this form. Therefore, since $g \cdot u = -u$ for $u \in \mathfrak{N}^\vee$,

$$\begin{aligned} w(1+g)w &= (v+u)^2 + (v+u)(v-u) \\ &= (v^2 + u^2) + (v^2 - u^2) \\ &= 2v^2. \end{aligned} \tag{6.6}$$

Therefore, $2v^2 \in \mathbb{Z}$, and, moreover, there is a $w \in \Gamma$ so that $w \cdot g \cdot w$ is odd iff there is a vector $v \in \mathfrak{I}^\vee$ so that $2v^2$ is odd.

6.2 Level matching for asymmetric orbifolds by involutions

The discussion of the previous section is closely related to the level matching constraint in an asymmetric orbifold using a nontrivial involution of the Narain lattice. Thus, consider the asymmetric orbifold corresponding to the action $X \rightarrow gX + s$ where $g^2 = 1$ and for simplicity we assume $g \cdot s = s$ so $2s \in \Gamma^g$.

The level matching constraint is

$$2 \times \left(\frac{n_-}{16} - \frac{\tilde{n}_-}{16} + \frac{1}{2}(p+s)^2 \right) = 0 \bmod \mathbb{Z} \tag{6.7}$$

where our convention for Narain lattices is $p^2 = p_L^2 - p_R^2$. Here $p \in (\Gamma^g)^\vee$. Since $2s \in \Gamma^g$ this can be simplified to

$$\frac{n_-}{8} - \frac{\tilde{n}_-}{8} + p^2 + s^2 = 0 \bmod \mathbb{Z}. \tag{6.8}$$

When (6.8) is satisfied for *every* vector $p \in (\Gamma^g)^\vee$ it follows (by subtracting the equation with $p = 0$) that $p^2 = 0 \bmod \mathbb{Z}$ for every vector $p \in (\Gamma^g)^\vee$, hence $2p^2$ is always even and hence for every vector $P \in \Gamma$ we have $P \cdot g \cdot P = 0 \bmod 2$. This is the condition for the modular covariance of an order two lift $\hat{g}$ of g . On the other hand, suppose we just know that (6.8) is satisfied for *some* vector $p_0 \in (\Gamma^g)^\vee$. Then we can conclude, first of all that for *every* vector $p \in (\Gamma^g)^\vee$ the modular covariance condition for an order four lift $\hat{g}$ is satisfied:

$$\frac{n_-}{4} - \frac{\tilde{n}_-}{4} + 2p^2 + 2s^2 = 0 \bmod \mathbb{Z}. \tag{6.9}$$

The reason is that we need only check that

$$2p^2 - 2p_0^2 = 0 \bmod \mathbb{Z} \tag{6.10}$$

but we have seen that $2p^2 \in \mathbb{Z}$ for every $p \in (\Gamma^g)^\vee$.

Now suppose that (6.8) is satisfied for *some* vector $p_0 \in (\Gamma^g)^\vee$ but not for $p = 0$. Then since $p_0^2 \in \frac{1}{2}\mathbb{Z}$ it must be that $2p_0^2$ is odd and hence there is some vector w in Γ satisfying (2.17). As we have seen, this means there is no order two lift $\hat{g}$ consistent with modular covariance. Moreover, even if $\hat{g}$ has order four, level matching would be violated by *some* momentum sectors in the first twisted sector (of the equivariant theory). Nevertheless, the relevant criterion for level matching for an order four element is that infinitely many states satisfy (6.8) for *some* vector $p_0 \in (\Gamma^g)^\vee$.

We conclude that the condition (2.17) should not be interpreted as a consistency condition for an orbifold by a covering group of $\langle g \rangle$ based on considerations of level-matching in the first twisted sector. We remark that the argument here did not use any special properties of the formula for the right-moving ground state energy $\tilde{n}_-/16$ so exactly the same reasoning will apply to the heterotic string.

6.3 Twisted characteristic vectors

In preparation for section 7 we note that the phase $e^{i\pi p \cdot g \cdot p}$ can be written as a character on the Narain lattice. That is, there is a vector $W_g \in \Gamma$ so that

$$e^{i\pi p \cdot g \cdot p} = e^{i\pi p \cdot W_g}. \quad (6.11)$$

To prove this note that, using that g is an orthogonal involution:

$$e^{i\pi(p_1+p_2) \cdot g(p_1+p_2)} = e^{i\pi p_1 \cdot g p_1} e^{i\pi p_2 \cdot g p_2} \quad (6.12)$$

so the map $\Gamma \rightarrow \mathbb{Z}_2$ given by $p \mapsto e^{i\pi p \cdot g p}$ is a group homomorphism, and by Pontryagin duality¹⁹ there must be a vector $W_g \in \Gamma$ so that

$$p \cdot g p = p \cdot W_g \text{ mod } 2 \quad (6.13)$$

thus proving equation (6.11). The vector W_g is only defined up to addition by a vector in 2Γ . It is the analog of an integral lift of a Stiefel-Whitney class.²⁰

As an example, for the Gaussian model at the self-dual radius taking $g = \sigma_R$, which exchanges e and f we have

$$p \cdot g p = (ne + wf) \cdot (nf + we) = n^2 + w^2 = n \pm w \text{ mod } 2. \quad (6.14)$$

So this is indeed the same as the sign from before. We could take

$$W_g = \pm e \pm f \quad (6.15)$$

or any translate by an element of 2Γ . Note that we could choose a representative $W_g = e + f \in \Gamma^g$ which is orthogonal to $(\Gamma^g)^\perp$.

We can easily generalize this example by considering a product of Gaussian models, all at the self-dual radius and with no B -field. Then Γ is a direct sum of d copies of $\Gamma(r=1)$ with basis vectors $e_i, f_i, i = 1, \dots, d$. Then if $g : e_i \leftrightarrow f_i$ we have

$$\Gamma^g = \left\{ \sum_i n_i (e_i + f_i) \right\} \quad (6.16)$$

$$\Gamma^{g,\perp} = \left\{ \sum_i w_i (e_i - f_i) \right\} \quad (6.17)$$

¹⁹The Pontryagin dual is $\Gamma^{PD} = \text{Hom}(\Gamma, \text{U}(1))$ and here we are defining an order 2 element of Γ^{PD} . But for a locally compact abelian group $(G^{PD})^{PD} = G$ and $G \times G^{PD} \rightarrow \text{U}(1)$ is a perfect pairing. In particular every homomorphism in Γ^{PD} is of the form $\chi_{\bar{k}} = p \mapsto e^{2\pi i k \cdot p}$ where $\bar{k} \in (\mathbb{R} \otimes \Gamma^\vee)/\Gamma^\vee$ and k is any lift to $\mathbb{R} \otimes \Gamma^\vee$. Since our homomorphism is two-torsion and since Γ is self-dual there is a vector $W_g \in \Gamma$ so that $k = \frac{1}{2}W_g$.

²⁰A characteristic vector would satisfy $p^2 = p \cdot W \text{ mod } 2$. It should not be confused with W_g .

and we can choose the representative

$$W_g = \sum_i (e_i + f_i) \in \Gamma^g \quad (6.18)$$

In section 7 we will also need a similar twisted characteristic vector relevant to the orbifold theory by $\langle \hat{g} \rangle$. We claim that there is a vector $W_g^{tw} \in \Gamma^g$ so that

$$2p^2 = W_g^{tw} \cdot p \bmod 2 \quad \forall p \in (\Gamma^g)^\vee. \quad (6.19)$$

Moreover we claim there is a choice of $W_g^{tw} \in \Gamma^g$ (as usual, modulo $2\Gamma^g$). In fact, we claim that there are representatives of W_g so that we can take $W_g^{tw} = W_g$.

To prove these statements about W_g^{tw} we use the decomposition of vectors $w \in \Gamma$ as $w = p + p'$ with $p \in (\Gamma^g)^\vee$ and $p' \in ((\Gamma^g)^\perp)^\vee$ and the identity

$$w(1+g)w = 2p^2 \quad (6.20)$$

derived above. Now we note that

$$(w_1 + w_2)(1+g)(w_1 + w_2) = w_1(1+g)w_1 + w_2(1+g)w_2 + [2w_1(1+g)w_2] \quad (6.21)$$

where we used the fact that g is an involution. The term in square brackets is even so

$$(w_1 + w_2)(1+g)(w_1 + w_2) = w_1(1+g)w_1 + w_2(1+g)w_2 \bmod 2 \quad (6.22)$$

Then we see that $p \mapsto \exp[i\pi 2p^2]$ is a group homomorphism $(\Gamma^g)^\vee \rightarrow \text{U}(1)$ of order two so must be given by a homomorphism in the torus $(\Gamma^g \otimes \mathbb{R})/\Gamma^g$ of order two. In fact, we can do better: since $W_g^{tw} \in \Gamma^g$, if we write $w = p + p' \in (\Gamma^g)^\vee \oplus (\Gamma^{g,\perp})^\vee$ then (all equations taken modulo two):

$$w \cdot W_g^{tw} = p \cdot W_g^{tw} = 2p^2 = w(1+g)w = w \cdot W_g \bmod 2 \quad (6.23)$$

so we can take $W_g = W_g^{tw}$.

6.4 Generalization to elements of arbitrary even order

We can generalize the above discussion to elements $g \in F(\Gamma)$ of arbitrary order as follows. We investigate modular covariance under the $\mathbb{Z}_\ell$ subgroup generated by g . In order to do this we need the generalization of equation (6.1) above. The action of g on $V_L \otimes \mathbb{C}$ can be diagonalized so that it takes the form

$$g \sim +1^{n_+} \oplus -1^{n_-} \oplus_a (e^{2\pi i \theta_a} \oplus e^{-2\pi i \theta_a}) \quad (6.24)$$

where a labels the eigenvalues of g that are not ± 1 so we can take $0 < \theta_a < 1$. There is a similar diagonalization of the action of g on $V_R \otimes \mathbb{C}$ with $n_+ \rightarrow \tilde{n}_+$, etc. Then, assuming equation (2.11) one can compute that $Z(1, \hat{g})$ is a sum over terms $p \in (\Gamma^g)^\vee$:

$$(q\bar{q})^{-d/24} q^{E_0} \bar{q}^{\tilde{E}_0} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}\tilde{p}_R^2} S(q^{1/\ell}, \bar{q}^{1/\ell}) \quad (6.25)$$

where S is a series in nonnegative powers of $q^{1/\ell}$ and $\bar{q}^{1/\ell}$. The ground state energies are

$$E_0 = \frac{n_-}{16} + \sum_a \frac{1}{2} \theta_a (1 - \theta_a) \quad (6.26)$$

$$\tilde{E}_0 = \frac{\tilde{n}_-}{16} + \sum_{\tilde{a}} \frac{1}{2} \tilde{\theta}_{\tilde{a}} (1 - \tilde{\theta}_{\tilde{a}}) \quad (6.27)$$

We now ask if it is consistent to assume that $\hat{g}$ has order ℓ . Once again, the crucial point is that for $p \in (\Gamma^g)^\vee$ we have

$$\ell p^2 = P \cdot g^{\ell/2} \cdot P \bmod 2 \quad (6.28)$$

where P is a vector $P \in \Gamma$ constructed below. It follows that if g has even order ℓ and (2.17) is satisfied, then modular covariance of $Z(1, \hat{g})$ is violated for $\tau \rightarrow \tau + \ell$ if we apply (2.11). However, modular covariance is consistent with the existence of a lift $\hat{g}$ of order 2ℓ , provided the $2\ell(E_0 - \tilde{E}_0) = 0 \bmod 1$. On the other hand, if $P \cdot g^{\ell/2} \cdot P = 0 \bmod 2$ for all $P \in \Gamma$ then the existence of a lift $\hat{g}$ of g of order ℓ is consistent with modular covariance, provided the standard level-matching constraint $\ell(E_0 - \tilde{E}_0) = 0 \bmod 1$ is satisfied.

We now prove equation (6.28). We first note that for all $P \in \Gamma$, we have

$$P \cdot \left(1 + g + g^2 + \dots + g^{\ell-1}\right) P = \begin{cases} 0 \bmod 2 & \ell \text{ odd} \\ P g^{\ell/2} P \bmod 2 & \ell \text{ even} \end{cases} \quad (6.29)$$

To prove this note that we can group terms so that

$$\begin{aligned} P \cdot \left(1 + g + g^2 + \dots + g^{\ell-1}\right) P &= P^2 + P \cdot (g + g^{\ell-1})P + P \cdot (g^2 + g^{\ell-2})P + \dots \\ &+ \begin{cases} P \cdot (g^{(\ell-1)/2} + g^{(\ell+1)/2})P & \ell \text{ odd} \\ P g^{\ell/2} P & \ell \text{ even} \end{cases} \end{aligned} \quad (6.30)$$

Now P^2 is an even integer and

$$P g^k P + P g^{\ell-k} P = P g^k P + P g^{-k} P = P g^k P + (g^k P) \cdot P = 2 P g^k P \in 2\mathbb{Z}. \quad (6.31)$$

Therefore all the paired terms are even. The only thing left is the unpaired term when ℓ is even.

Now, when we tensor over the complex numbers to consider Γ embedded in the complex vector space $\Gamma \otimes \mathbb{C}$ we can apply projection operators onto sublattices transforming according to the irreducible characters of χ of $\mathbb{Z}_\ell$:

$$\otimes_{\chi \in \text{Irrep}(\mathbb{Z}_\ell)} \mathfrak{I}_\chi \quad (6.32)$$

where $\mathfrak{I}_\chi = P_\chi \Gamma$ and P_χ is a projection operator. Then every vector $P \in \Gamma$ has a decomposition

$$P = \sum_{\chi} p_\chi \quad (6.33)$$

with $p_\chi \in \mathfrak{I}_\chi$. Now note that

$$\left(1 + g + g^2 + \cdots + g^{\ell-1}\right) p_\chi = \begin{cases} \ell p_\chi & \chi = 1 \\ 0 & \chi \neq 1 \end{cases} \quad (6.34)$$

Taking an inner produce with P proves equation (6.28) with $p_{\chi=1} = p$. To complete the story we need to know that in fact every vector $p \in (\Gamma^g)^\vee$ has a completion (6.33) with $P \in \Gamma$. To prove this we simply apply Nikulin's theorem to the primitively embedded sublattice Γ^g .

These considerations suggest a natural conjecture for a canonical lift of g to $\hat{g}$ in the automorphism group of the CFT that acts as

$$\hat{g}|p\rangle = e^{i\pi\phi}|g \cdot p\rangle \quad (6.35)$$

where

$$\phi = \frac{1}{\ell} p \cdot \left(1 + g + g^2 + \cdots + g^{\ell-1}\right) p \quad (6.36)$$

We can check then that

$$\hat{g}^\ell |p\rangle = \begin{cases} |p\rangle & \ell \text{ odd} \\ e^{i\pi p g^{\ell/2} p} |p\rangle & \ell \text{ even} \end{cases} \quad (6.37)$$

As a check on this proposal consider the ADE point $\Gamma(\mathfrak{g})$ and let $g = (\sigma_\alpha, 1)$ be a left-moving reflection in a root. Acting on the states of the form (4.4) our conjecture becomes:

$$\hat{g}|(\mu_L; \mu_R)\rangle = e^{-\frac{i\pi}{2}(\alpha \cdot \mu_L)^2} |(\sigma_\alpha(\mu_L); \mu_R)\rangle \quad (6.38)$$

In particular, when $\sigma_\alpha(\mu_L) = \mu_L$ the eigenvalue is $+1$, exactly what we expect for the Tits lift. Moreover, one can check explicitly for reflections in simple roots acting on the fundamental representation of $SU(N)$ that there is a basis of weight vectors such that equation (6.38) holds. Thus, our conjectured canonical lift appears to be a generalization of the Tits lift for finite-dimensional groups acting on toroidal CFTs.

7 General discussion of partition functions

In this section we consider a point in Narain moduli space with a nontrivial involution in $F(\Gamma)$ which satisfies the condition (2.17). We assume that there is a lift of the involution $\hat{g}$ so that

$$\hat{g}|p\rangle = |p\rangle \quad \forall p \in \Gamma^g \quad (7.1)$$

$$\hat{g}^2|p\rangle = e^{i\pi p \cdot W_g} |p\rangle \quad \forall p \in \Gamma \quad (7.2)$$

where $p \cdot g \cdot p = p \cdot W_g \bmod 2$ for all $p \in \Gamma$ and we take $W_g \in \Gamma^g$ and not equivalent to zero. We are interested in whether the orbifold by the $\mathbb{Z}_4$ subgroup of $\text{Aut}(\mathcal{C}_\Gamma)$ generated by $\hat{g}$ is consistent. Just using the assumptions (7.1) and (7.2) and the method of modular orbits we will construct the partition function and in this section we will ask if the resulting partition function has a good q -expansion in the sense of section 2. Of course, if we had an action of

$\hat{g}$ on the full Hilbert space then it would follow trivially that we have a good q expansion, but we have not constructed a consistent vertex operator algebra action on the various twisted sectors (including the untwisted sector) and therefore it is useful to check whether the untwisted sector partition function is consistent with an operator interpretation, which necessarily implies there is a good q -expansion. In fact, we will find a new consistency condition, equation (7.22) below, just from this necessary condition.

To write the partition functions we will use the lattice theta functions defined in appendix C. From (7.1) we have:

$$Z(\hat{g}, 1) = \frac{1}{\eta^{n_+}} \left(\frac{\vartheta_4(2\tau)}{\eta} \right)^{n_-} \frac{1}{\bar{\eta}^{\tilde{n}_+}} \left(\frac{\bar{\vartheta}_4(2\tau)}{\bar{\eta}} \right)^{\tilde{n}_-} \Theta_{\Gamma^g}(\tau, 0, 0) \quad (7.3)$$

where Γ^g is the sublattice of vectors fixed by g , and $n_+ + n_- = d$. From (7.2) we get:

$$Z(\hat{g}^2, 1) = \frac{1}{\eta^d} \frac{1}{\bar{\eta}^d} \sum_{p \in \Gamma} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} e^{2\pi i(p \cdot \frac{1}{2}W_g)} \quad (7.4)$$

From (7.3) a modular transformation gives:

$$Z(1, \hat{g}) = 2^{-(n_- + \tilde{n}_-)/2} |\mathcal{D}|^{-1/2} \frac{1}{\eta^{n_+}} \left(\frac{\vartheta_2(\tau/2)}{\eta} \right)^{n_-} \frac{1}{\bar{\eta}^{\tilde{n}_+}} \left(\frac{\bar{\vartheta}_2(\tau/2)}{\bar{\eta}} \right)^{\tilde{n}_-} \Theta_{(\Gamma^g)^\vee}(\tau; 0, 0) \quad (7.5)$$

where $\mathcal{D}$ is the discriminant group of $(\Gamma^g)^\vee$. Now we want to average this over shifts of τ to construct the partition function in the first twisted sector.

When checking that we get good q -expansions it will be useful to define

$$\begin{aligned} \vartheta_2(\tau/2) &= q^{1/16} \sum_{n \in \mathbb{Z}} e^{i\pi \frac{\tau}{2}(n^2 + n)} = 2q^{1/16} \left(1 + \sum_{n=1}^{\infty} q^{\frac{n(n+1)}{4}} \right) \\ &:= 2q^{1/16} S(\tau) \end{aligned} \quad (7.6)$$

Note that S is a power series in positive powers of $q^{1/2}$ with positive integral coefficients. In these terms we can write:

$$Z(1, \hat{g}) = D(B_+ \bar{B}_+)^d q^{n_-/16} \bar{q}^{\tilde{n}_-/16} S^{n_-} \bar{S}^{\tilde{n}_-} \Theta_{(\Gamma^g)^\vee}(\tau; 0, 0) \quad (7.7)$$

where $B_+ = 1/\eta$ and

$$D := \sqrt{\frac{2^{(n_- + \tilde{n}_-)/2}}{|\mathcal{D}|}} \quad (7.8)$$

is an integer, according to [4, 18].

Next, $\tau \rightarrow \tau + 2$ gives the partition function:

$$Z(\hat{g}^2, \hat{g}) = e^{i\pi(n_- - \tilde{n}_-)/4} 2^{-(n_- + \tilde{n}_-)/2} |\mathcal{D}|^{-1/2} \frac{1}{\eta^{n_+}} \left(\frac{\vartheta_2(\tau/2)}{\eta} \right)^{n_-} \frac{1}{\bar{\eta}^{\tilde{n}_+}} \left(\frac{\bar{\vartheta}_2(\tau/2)}{\bar{\eta}} \right)^{\tilde{n}_-} \Theta_{(\Gamma^g)^\vee}(\tau; \alpha, 0) \quad (7.9)$$

where $\alpha = -\frac{1}{2}W_g$. Now we can again use a modular transform to get

$$Z(\hat{g}, \hat{g}^2) = e^{i\pi(n_- - \tilde{n}_-)/4} \frac{1}{\eta^{n_+}} \left(\frac{\vartheta_4(2\tau)}{\eta} \right)^{n_-} \frac{1}{\bar{\eta}^{\tilde{n}_+}} \left(\frac{\bar{\vartheta}_4(2\tau)}{\bar{\eta}} \right)^{\tilde{n}_-} \Theta_{\Gamma^g} \left(\tau; 0; \frac{1}{2}W_g \right) \quad (7.10)$$

Modular invariance (and level matching) requires $n_- - \tilde{n}_- = 0 \bmod 4$. Equation (7.10) shows that if $n_- - \tilde{n}_- = 4 \bmod 8$ then we get bad signs that can potentially spoil the operator interpretation. Level matching is not strong enough to guarantee a good q -expansion.

To compute the partition function in the $\hat{g}^2$ -twisted sector we begin with

$$Z(\hat{g}^2, 1) = \frac{1}{\eta^d} \frac{1}{\bar{\eta}^d} \sum_{p \in \Gamma} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} e^{2\pi i(p \cdot \frac{1}{2}W_g)} \quad (7.11)$$

and then

$$Z(1, \hat{g}^2) = \frac{1}{\eta^d} \frac{1}{\bar{\eta}^d} \sum_{p \in \Gamma} e^{i\pi\tau(p_L + \frac{1}{2}W_{g,L})^2} e^{-i\pi\bar{\tau}(p_R + \frac{1}{2}W_{g,R})^2} \quad (7.12)$$

Now taking $\tau \rightarrow \tau + 1$ we get:

$$Z(\hat{g}^2, \hat{g}^2) = e^{2\pi i \frac{W_g^2}{8}} \frac{1}{\eta^d} \frac{1}{\bar{\eta}^d} \sum_{p \in \Gamma} e^{i\pi\tau(p_L + \frac{1}{2}W_{g,L})^2} e^{-i\pi\bar{\tau}(p_R + \frac{1}{2}W_{g,R})^2} e^{i\pi p \cdot W_g} \quad (7.13)$$

We now have all the ingredients to write the full partition functions. We would like to check that all coefficients in the $q, \bar{q}$ -expansion in all four sectors are nonnegative integers.

We first consider the untwisted sector and this is just:

$$\begin{aligned} Z(\mathcal{H}_1) = \frac{1}{4} \frac{1}{\eta^d \bar{\eta}^d} & \left[\sum_{p \in \Gamma} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2} (1 + e^{i\pi p \cdot W_g}) \right. \\ & \left. + 2(\vartheta_4(2\tau))^{n_-} (\bar{\vartheta}_4(2\tau))^{\tilde{n}_-} \sum_{p \in \Gamma^g} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2} \right] \end{aligned} \quad (7.14)$$

The potential problem here are the minus signs from the factors ϑ_4 and $\bar{\vartheta}_4$. Also the coefficients are potentially half-integral. (The vacuum is easily seen to have degeneracy 1.)

We claim there is a good operator interpretation. To show this define $\Gamma_0 := \Gamma^g \oplus \Gamma^{g,\perp}$. Then we can write

$$\Gamma = \amalg_{i=0}^d (\Gamma_0 + \gamma_i) \quad (7.15)$$

where the glue vectors γ_i project to representatives of the discriminant group. Then we can write²¹

$$\begin{aligned} Z(\mathcal{H}_1) = \frac{1}{2} \frac{1}{\eta^d \bar{\eta}^d} & \left[\left(\sum_{p \in \Gamma^{g,\perp}} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2} + (\vartheta_4(2\tau))^{n_-} (\bar{\vartheta}_4(2\tau))^{\tilde{n}_-} \right) \sum_{p \in \Gamma^g} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2} \right. \\ & \left. + \sum_{\gamma_i \neq 0} \frac{1 + e^{i\pi\gamma_i \cdot W_g}}{2} \sum_{p \in \Gamma_0 + \gamma_i} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2} \right] \end{aligned} \quad (7.16)$$

Regarding the sum over glue vectors we note that if $\frac{1}{2}(1 + e^{i\pi\gamma_i \cdot W_g}) = +1$ then $\frac{1}{2}(1 + e^{-i\pi\gamma_i \cdot W_g}) = +1$ so we can pair the terms with p and $-p$ and that cancels the overall factor

²¹Note that this step uses the fact that if $p_1 \in \Gamma^g$ and $p_2 \in (\Gamma^g)^\vee$ then not only is $p_1 \cdot p_2 = 0$, but also $p_{1,L} \cdot p_{2,L} = 0$. This follows since $g(p_L; p_R) = (g_L p_L; g_R p_R)$ with g_L, g_R both involutions. We thank K. Wendland for a clarifying remark on this point.

of $1/2$ and yields a series with nonnegative integer coefficients. If $-\gamma_i = \gamma_i \bmod \Gamma_0$ then there is only one term in the sum over γ_i but then $\sum_{p \in \Gamma_0 + \gamma_i} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2}$ has degeneracies which are multiples of 2.

For the remaining terms it would suffice to prove that

$$\frac{1}{2} \frac{1}{\eta^{n_-} \bar{\eta}^{\tilde{n}_-}} \left(\sum_{p \in \Gamma^{g, \perp}} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2} + (\vartheta_4(2\tau))^{n_-} (\bar{\vartheta}_4(2\tau))^{\tilde{n}_-} \right) \quad (7.17)$$

is a positive $q, \bar{q}$ expansion with nonnegative integer coefficients. But note that the lattice $\Gamma^{g, \perp}$ is even and signature $(n_-; \tilde{n}_-)$. This expression is manifestly the untwisted sector partition function of a system of bosons on $\Gamma^{g, \perp}$ with the orbifold action $p \rightarrow -p$. It therefore has an operator interpretation.

The partition function in the $\hat{g}^2$ -twisted sector is

$$\begin{aligned} Z(\mathcal{H}_{\hat{g}^2}) = & \frac{1}{4} \frac{1}{\eta^d \bar{\eta}^{\tilde{d}}} \left[\sum_{p \in \Gamma} e^{i\pi\tau(p + \frac{1}{2}W_g)_L^2 - i\pi\bar{\tau}(p + \frac{1}{2}W_g)_R^2} \left(1 + e^{2\pi i \frac{W_g^2}{8}} e^{i\pi p \cdot W_g} \right) \right. \\ & \left. + e^{i\pi \frac{(n_- - \tilde{n}_-)}{4}} (\vartheta_4(2\tau))^{n_-} (\bar{\vartheta}_4(2\tau))^{\tilde{n}_-} \sum_{p \in \Gamma^g} e^{i\pi\tau(p + \frac{1}{2}W_g)_L^2 - i\pi\bar{\tau}(p + \frac{1}{2}W_g)_R^2} \left(1 + e^{2\pi i \frac{W_g^2}{8}} e^{i\pi p \cdot W_g} \right) \right] \end{aligned} \quad (7.18)$$

Now again we have to worry about potential signs and half-integers.

Now to make progress note that $e^{2\pi i \frac{W_g^2}{8}}$ is always a fourth root of unity since $W_g \in \Gamma^g$ is in an even lattice. We will now argue that $e^{2\pi i W_g^2/8}$ should be a sign. Let us define

$$\xi := e^{2\pi i W_g^2/8} \quad \xi' = e^{i\pi(n_- - \tilde{n}_-)/4} \quad (7.19)$$

We know that ξ' is ± 1 by the cancellation of modular anomalies.

In analogy to (7.16), we can write (7.18) as

$$\begin{aligned} Z(\mathcal{H}_{\hat{g}^2}) = & \frac{1}{2} \frac{1}{\eta^d \bar{\eta}^{\tilde{d}}} \left[\frac{1+\xi}{2} \left\{ \sum_{p \in \Gamma^{g, \perp}} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2} + \xi' (\vartheta_4(2\tau))^{n_-} (\bar{\vartheta}_4(2\tau))^{\tilde{n}_-} \right\} \left(\sum_{p \in \Gamma^g + \frac{1}{2}W_g} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2} \right) \right. \\ & \left. + \sum_{\gamma_i \neq 0} \frac{1 + \xi e^{i\pi\gamma_i \cdot W_g}}{2} \sum_{p \in \Gamma_0 + \gamma_i + \frac{1}{2}W_g} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2} \right] \end{aligned} \quad (7.20)$$

If ξ is $\pm i$ then it is clear that we will not get an integral expansion in (7.20). For example, we could choose W_g to be minimal length among its representatives and then the leading term in the q expansion will involve

$$\frac{1}{2} (1 + \xi)(1 + \xi') e^{i\pi\tau W_{g,L}^2/4} e^{-i\pi\bar{\tau} W_{g,R}^2/4} \quad (7.21)$$

If $\xi' = 1$ then it is clear that we cannot have $\xi = \pm i$. If $\xi' = -1$ we must look at the next-to-leading terms and again it is clear we cannot have $\xi = \pm i$. Therefore we must have $\xi^2 = 1$. Thus a consistency condition for asymmetric orbifolds is the requirement that

$$\xi^2 = e^{2\pi i W_g^2/4} = 1. \quad (7.22)$$

We believe this condition has not appeared in the literature before.

Given that $\xi^2 = 1$ the argument that $Z(\mathcal{H}_{\hat{g}^2})$ has a good q -expansion is very similar to that for the untwisted sector. In the sum over γ_i we pair up terms with γ_i and $-\gamma_i - W_g$ (and when these are the same in the discriminant group then the shifted theta function has even degeneracies). What we need to check is that

$$\frac{1}{2} \frac{1}{\eta^{n_-} \bar{\eta}^{\tilde{n}_-}} \left(\sum_{p \in \Gamma^{g, \perp}} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2} + \xi' (\vartheta_4(2\tau))^{n_-} (\bar{\vartheta}_4(2\tau))^{\tilde{n}_-} \right) \quad (7.23)$$

is a positive $q, \bar{q}$ expansion with nonnegative integer coefficients. Again as with (7.17) we interpret this in terms of a system of bosons on $\Gamma^{g, \perp}$ with the orbifold action $p \rightarrow -p$. It therefore has an operator interpretation. Depending on ξ' we might be projecting to the anti-invariant subspace, but it still has a good q -expansion.

Finally, we must check that the operator interpretation is sensible in the $\hat{g}$ -twisted sector $\mathcal{H}_{\hat{g}}$:

$$Z(\mathcal{H}_{\hat{g}}) = \frac{1}{4} \left(Z(1, \hat{g})(\tau) + Z(1, \hat{g})(\tau + 1) + Z(1, \hat{g})(\tau + 2) + Z(1, \hat{g})(\tau + 3) \right) \quad (7.24)$$

To do this we return to the equation:

$$Z(1, \hat{g}) = D(B_+ \bar{B}_+)^d q^{n_-/16} \bar{q}^{\tilde{n}_-/16} S^{n_-} \bar{S}^{\tilde{n}_-} \Theta_{(\Gamma^g)^\vee}(\tau; 0, 0). \quad (7.25)$$

Now write the terms in the theta function as a sum over

$$q^{\frac{1}{2}p^2} (q\bar{q})^{\frac{1}{2}p_R^2}. \quad (7.26)$$

But $q^{\frac{1}{2}p^2}$ is q^μ where $\mu \in \frac{1}{4}\mathbb{Z}$. Similarly we can write:

$$q^{n_-/16} \bar{q}^{\tilde{n}_-/16} = q^{(n_- - \tilde{n}_-)/16} (q\bar{q})^{\tilde{n}_-/16}. \quad (7.27)$$

Since $q\bar{q}$ is inert under $\tau \rightarrow \tau + 1$, when $n_- - \tilde{n}_- = 0 \bmod 4$ we can write the whole partition function in the form:

$$Z(1, \hat{g})(\tau) = D \sum_{\mu, \nu \in \frac{1}{4}\mathbb{Z}} \wp_{\mu, \nu}((q\bar{q})^{1/\ell}) q^\mu \bar{q}^\nu \quad (7.28)$$

where ℓ is some integer (for rational theories) and $\wp_{\mu, \nu}(x)$ is a power series in x with positive integer coefficients. Now the sum over shifts of τ just projects to the subset of terms with $\mu - \nu = 0 \bmod 1$. This concludes the proof. ♠

To conclude we remark that the consistency condition (7.22) is satisfied for (6.18) since $W_g^2/4 = (\tilde{n}_- - n_-)/4$ in this example.

Acknowledgments

We would like to thank T. Banks, D. Freed, D. Friedan, M. Gaberdiel, D. Gaiotto, P. Goddard, Y. Z. Huang, J. Lepowsky, K.S. Narain, I. Runkel, C. Schweigert, G. Segal, A. Taormina, W. Taylor, A. Tripathy, C. Vafa, and R. Volpato for helpful discussions

and correspondence. We are very grateful to J. Cushing and K. Wendland for providing detailed feedback that helped us correct minor errors and improve the presentation. We are especially grateful to N. Seiberg for some collaboration on these matters and for the essential remark that T duality of the Gaussian model is the lifting of a 180 degree rotation. This collaboration owes a great deal to the hospitality of the Aspen Center for Physics (under NSF Grant No. PHY-1066293). JH acknowledges support from the NSF²² under grant PHY 1520748 and from the Simons Foundation (#399639). GM thanks Institute for Advanced Study in Princeton for support from the IBM Einstein Fellowship of the Institute for Advanced Study. GM also acknowledges support by the DOE under grant DOE-SC0010008 to Rutgers University.

A Automorphism groups of extensions of lattices

As is well known, and as discussed in section 5, locality of the OPE for vertex operators requires that we consider a group extension of the momentum lattice Γ . This takes the form

$$1 \rightarrow A \rightarrow \hat{\Gamma} \rightarrow \Gamma \rightarrow 0. \quad (\text{A.1})$$

As discussed in the text, much of the literature takes A to be isomorphic to $\mathbb{Z}/2\mathbb{Z}$, but we have argued that group invariance at enhanced symmetry points requires $A = \mathbb{Z}/4\mathbb{Z}$. Here we just assume that A is a finite abelian group. We write the group law in Γ additively and the group law in A multiplicatively. We now discuss how to lift automorphisms of Γ to automorphisms of $\hat{\Gamma}$.

We begin by constructing a group $\widehat{\text{Aut}(\Gamma)}$ that is a subgroup of the group of automorphisms of $\hat{\Gamma}$ and covers the action of $\text{Aut}(\Gamma)$ on Γ . Our group will fit in an extension of the form

$$1 \rightarrow \text{Hom}(\Gamma, A) \rightarrow \widehat{\text{Aut}(\Gamma)} \rightarrow \text{Aut}(\Gamma) \rightarrow 1. \quad (\text{A.2})$$

We denote elements of Γ by p , elements of A by a and the action of $g \in \text{Aut}(\Gamma)$ on $p \in \Gamma$ by gp . Elements of $\hat{\Gamma}$ are pairs (a, p) with composition law

$$(a_1, p_1) \cdot (a_2, p_2) = (a_1 a_2 \varepsilon(p_1, p_2), p_1 + p_2) \quad (\text{A.3})$$

with ε a cocycle. For each $g \in \text{Aut}(\Gamma)$ we wish to define an element $T_g \in \widehat{\text{Aut}(\Gamma)}$ of the form

$$T_g(a, p) = (a \xi_g(p), gp), \quad (\text{A.4})$$

where ξ_g is a function from Γ to A . Demanding that $T_g \in \widehat{\text{Aut}(\Gamma)}$ gives a constraint on ξ_g :

$$\frac{\xi_g(p_1 + p_2)}{\xi_g(p_1) \xi_g(p_2)} = \frac{\varepsilon(gp_1, gp_2)}{\varepsilon(p_1, p_2)}. \quad (\text{A.5})$$

For each $g \in \text{Aut}(\Gamma)$ we choose a solution of (A.5) (we assume it exists). Note that given one solution we can multiply ξ_g by any element $\ell_g \in \text{Hom}(\Gamma, A)$ to produce another solution.

²²Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

Note that if we change ε by a coboundary b then ξ_g will be replaced by

$$\tilde{\xi}_g(p) = \xi_g(p) \frac{b(p)}{b(gp)} \quad (\text{A.6})$$

The set of operators T_g for $g \in \text{Aut}(\Gamma)$ generate a subgroup of $\text{Aut}(\hat{\Gamma})$ which is an extension of $\text{Aut}(\Gamma)$. Now a small computation shows that

$$T_{g_1 g_2}^{-1} \circ T_{g_1} \circ T_{g_2}(a, p) = (a \cdot (\xi_{g_1 g_2}(p))^{-1} \xi_{g_2}(p) \xi_{g_1}(g_2 p), p) . \quad (\text{A.7})$$

Now for each g_1, g_2 define a function $\ell_{g_1, g_2} : \Gamma \rightarrow A$ by

$$\ell_{g_1, g_2}(p) := (\xi_{g_1 g_2}(p))^{-1} \xi_{g_2}(p) \xi_{g_1}(g_2 p) . \quad (\text{A.8})$$

Another short computation using (A.5) shows that ℓ_{g_1, g_2} is a linear function:

$$\ell_{g_1, g_2}(p_1 + p_2) = \ell_{g_1, g_2}(p_1) \ell_{g_1, g_2}(p_2) \quad (\text{A.9})$$

and hence $\ell_{g_1, g_2} \in \text{Hom}(\Gamma, A)$. Now, for each $\ell \in \text{Hom}(\Gamma, A)$ define an automorphism $L_\ell \in \text{Aut}(\hat{\Gamma})$ by

$$L_\ell(a, p) = (a\ell(p), p) \quad (\text{A.10})$$

Applying (A.5) shows that L_ℓ is indeed an automorphism of $\hat{\Gamma}$. We have shown that

$$T_{g_1} \circ T_{g_2} = T_{g_1 g_2} \circ L_{\ell_{g_1, g_2}} \quad (\text{A.11})$$

But now note that $\text{Hom}(\Gamma, A)$ is itself a group under pointwise multiplication: $(\ell_1 \cdot \ell_2)(p) := \ell_1(p) \ell_2(p)$ where the r.h.s. is defined by multiplication in A and clearly

$$L_{\ell_1} \circ L_{\ell_2} = L_{\ell_1 \cdot \ell_2} \quad (\text{A.12})$$

Moreover, $\text{Aut}(\Gamma)$ acts on $\text{Hom}(\Gamma, A)$ via $g \cdot \ell(p) := \ell(g \cdot p)$ and one can check that

$$L_\ell \circ T_g = T_g \circ L_{g \cdot \ell} . \quad (\text{A.13})$$

The equations (A.11), (A.12), and (A.13) show that the set of automorphisms

$$\widehat{\text{Aut}(\Gamma)} := \{T_{g, \ell} := T_g L_\ell\} \quad (\text{A.14})$$

labeled by $(g, \ell) \in \text{Aut}(\Gamma) \times \text{Hom}(\Gamma, A)$ form a group with multiplication law:

$$T_{g_1, \ell_1} T_{g_2, \ell_2} = T_{g_1 g_2, \ell_{g_1, g_2} \cdot (g_2 \cdot \ell_1) \cdot \ell_2} \quad (\text{A.15})$$

The injection $\ell \mapsto L_\ell$ and projection $T_g L_\ell \mapsto g$ show that the group fits in the exact sequence (A.2).

Now we can restrict to the subgroup of $\widehat{\text{Aut}(\Gamma)}$ that projects to $F(\Gamma) = \text{Aut}(\Gamma) \cap (O(d)_L \times O(d)_R)$. Or we can even restrict to a subgroup of $F(\Gamma)$. The main example in

the text is the case where $g \in F(\Gamma)$ is a nontrivial involution that generates a $\mathbb{Z}_2$ -subgroup of $F(\Gamma)$. In this case the square of T_g is given by

$$T_g \cdot T_g(a, p) = (a\xi_g(p)\xi_g(gp), p) \quad (\text{A.16})$$

The element $\xi_g(p)\xi_g(gp)$ is invariant under a change of cocycle by a coboundary, as one easily checks using (A.6). In this sense it is gauge invariant. Thus, T_g squares to the identity only if

$$\xi_g(p)\xi_g(gp) = 1. \quad (\text{A.17})$$

In the examples of section 5 we find rather that $\xi_g(p)\xi_g(gp)$ is a $\mathbb{Z}_2$ -valued linear function that is, moreover, g -invariant, so in this case the restriction of the extension to $\langle g \rangle \subset F(\Gamma)$ is just an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$, consistent with the $\mathbb{Z}_4$ lift we found using $\text{SU}(2)$ invariance.

B Transformation of boundary conditions

Suppose our field on the torus has twisted boundary conditions

$$\begin{aligned} X(\sigma_1 + 1, \sigma_2) &= g_s \cdot X(\sigma_1, \sigma_2) \\ X(\sigma_1, \sigma_2 + 1) &= g_t \cdot X(\sigma_1, \sigma_2) \end{aligned} \quad (\text{B.1})$$

with modular parameter:

$$|dz|^2 = |d\sigma_1 + \tau d\sigma_2|^2 \quad (\text{B.2})$$

For an $\text{SL}(2, \mathbb{Z})$ transformation define

$$\begin{aligned} \sigma_1 &= d\sigma'_1 + b\sigma'_2 \\ \sigma_2 &= c\sigma'_1 + a\sigma'_2 \end{aligned} \quad (\text{B.3})$$

so that

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad (\text{B.4})$$

Now, under $(\Delta\sigma'_1 = 1, \Delta\sigma'_2 = 0)$ we have $(\Delta\sigma_1 = d, \Delta\sigma_2 = c)$ etc. So

$$X(\sigma'_1 + 1, \sigma'_2) = g_s^d g_t^c X(\sigma'_1, \sigma'_2) \quad (\text{B.5})$$

and so on. In this way we derive

$$Z(g_s^b g_t^a, g_s^d g_t^c; \tau') = Z(g_t, g_s; \tau) \quad (\text{B.6})$$

(This just says we should get the same answer working in σ' -variables.) Making a few trivial change of variables this means:

$$Z(g_t, g_s; \tau') = Z(g_s^{-b} g_t^d, g_s^a g_t^{-c}; \tau) \quad (\text{B.7})$$

Note that the action on functions of τ must descend to $\text{PSL}(2, \mathbb{Z})$, but the action on the boundary conditions:

$$(g_t, g_s) \rightarrow (g_s^{-b} g_t^d, g_s^a g_t^{-c}) \quad (\text{B.8})$$

does not descend. Therefore equation (B.7) only makes sense if:

$$Z(g_t^{-1}, g_s^{-1}; \tau) = Z(g_t, g_s; \tau) \quad (\text{B.9})$$

for all commuting pairs g_s, g_t .

C Theta functions

Suppose that $\mathbb{R}^{b_+, b_-}$ is Euclidean space with quadratic form $\eta_{AB} = (+1^{b_+}, -1^{b_-})$. We use indices $a, b, \dots = 1, \dots, b_+$ for the Euclidean coordinates on the positive definite space and $s, t, \dots = 1, \dots, b_-$ for Euclidean coordinates on the negative definite space, while $A, B, \dots$ run from 1 to $d := b_+ + b_-$.

Now suppose that $\Lambda \subset \mathbb{R}^{b_+, b_-}$ is an embedded lattice. It is the integral span of vectors e^A_i so we have vectors with coordinates x^A :

$$x^A = \sum_{i=1}^d n^i e^A_i \quad A = 1, \dots, d \quad (\text{C.1})$$

The Gram matrix is

$$G_{ij} = e^A_i \eta_{AB} e^B_j \quad (\text{C.2})$$

At this point we are not making any integrality assumptions about G_{ij} . It is just a non-degenerate symmetric real matrix. We consider the theta function:

$$\begin{aligned} \Theta_\Lambda(\tau, \alpha, \beta) &:= \sum_{\lambda \in \Lambda} e^{i\pi\tau(\lambda+\beta)_+^2 + i\pi\bar{\tau}(\lambda+\beta)_-^2 - 2\pi i(\lambda + \frac{1}{2}\beta, \alpha)} \\ &= \sum_{n^i \in \mathbb{Z}} e^{(n^i + \beta^i)(n^j + \beta^j)Q_{ij}(\tau) - 2\pi i(n^i + \frac{1}{2}\beta^i)\alpha^j G_{ij}} \end{aligned} \quad (\text{C.3})$$

with

$$Q_{ij}(\tau) = \sum_{a=1}^{b_+} i\pi\tau e^a_i e^a_j - \sum_{s=1}^{b_-} i\pi\bar{\tau} e^s_i e^s_j \quad (\text{C.4})$$

The Poisson summation formula gives:

$$\Theta_\Lambda(-1/\tau, \alpha, \beta) = (-i\tau)^{b_+/2} (i\bar{\tau})^{b_-/2} |\det e^i_A| \Theta_{\Lambda^\vee}(\tau, \beta, -\alpha) \quad (\text{C.5})$$

where $\Lambda^\vee$ is the lattice spanned by the vectors with coordinates

$$x_A = \sum_{i=1}^d m_i e^i_A \quad (\text{C.6})$$

with $m_i \in \mathbb{Z}$ and e^i_A is the inverse matrix of e^A_i . Note that consistency with making two S transformations requires

$$\Theta_\Lambda(\tau, \alpha, \beta) = \Theta_\Lambda(\tau, -\alpha, -\beta) \quad (\text{C.7})$$

which is indeed the case.

Up to this point we have *not* assumed G_{ij} is an integral matrix. In particular $\Theta_\Lambda(\tau, \alpha, \beta)$ does not have any special properties under $\tau \rightarrow \tau + 1$. Now assume that G_{ij} is an integral matrix. Then

$$|\det e^i_A| = \sqrt{|\det G^{ij}|} = \frac{1}{\sqrt{|\det G_{ij}|}} = \frac{1}{\sqrt{|\mathcal{D}|}} \quad (\text{C.8})$$

and $|\mathcal{D}|$ is the order of the discriminant group. So for integral lattices we have the S-transformation

$$\Theta_{\Lambda}(-1/\tau, \alpha, \beta) := (-i\tau)^{b_+/2} (i\bar{\tau})^{b_-/2} |\mathcal{D}|^{-1/2} \Theta_{\Lambda^\vee}(\tau, \beta, -\alpha) \quad (\text{C.9})$$

In the text we sometimes use the standard theta functions:

$$\begin{aligned} \vartheta_2 &= \sum_{n \in \mathbb{Z}} e^{i\pi\tau(n+\frac{1}{2})^2} \\ \vartheta_3 &= \sum_{n \in \mathbb{Z}} e^{i\pi\tau n^2} \\ \vartheta_4 &= \sum_{n \in \mathbb{Z}} e^{i\pi\tau n^2} (-1)^n \end{aligned} \quad (\text{C.10})$$

D Lifting Weyl groups of compact simple Lie groups

The lifting of Weyl groups to subgroups of the normalizer $N(T)$ or a maximal torus is well studied in the mathematical literature and goes back to work of Tits [35]. To state the general problem more formally, let G be a compact Lie group of rank r and choose a maximal torus T in G . Let $N(T)$ be the normalizer of T in G . As explained in section 1 the Weyl group is defined as $N(T)/T$ and hence fits in a short exact sequence

$$1 \rightarrow T \rightarrow N(T) \xrightarrow{\pi} W \rightarrow 1 \quad (\text{D.1})$$

We say this short exact sequence of groups splits if there is a group homomorphism $W \rightarrow N(T)$ such that $W \rightarrow N(T) \rightarrow W$ is the identity map on W . When the sequence splits we can use this homomorphism to define a subgroup of $N(T)$ isomorphic to W such that the conjugation action of this subgroup on T induces the Weyl group action on T . In general, the sequence (D.1) does not split, although there are examples of groups for which it does. In general, we say that a subgroup $\widetilde{W} \subset N(T)$ is a *lifting* of W if there is a surjective homomorphism $\pi : \widetilde{W} \rightarrow W$ such that for all $\tilde{g} \in \widetilde{W}$ and all $t \in T$, $\tilde{g}t\tilde{g}^{-1} = \pi(\tilde{g}) \cdot t$. There are infinitely many liftings of W , but there is a canonical lifting, known as the Tits lift. If G is the compact simply connected group with Lie algebra $\mathfrak{g}$ then for the Tits lift the Weyl reflections of simple roots lift to order 4 elements of G . In particular, the Tits lift is never isomorphic to $W(\mathfrak{g})$.

It is worth explaining the situation with respect to $SU(2)$ in more detail since much of it carries over to more general G . In $SU(2)$ we can choose the maximal torus $T \simeq S^1$ to consist of the diagonal matrices

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \quad \alpha \in \mathbb{R} \quad (\text{D.2})$$

The normalizer of T then has two connected components. The first component contains the identity and consists of T itself. The second component consists of the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot T = \begin{pmatrix} 0 & -e^{-i\alpha} \\ e^{i\alpha} & 0 \end{pmatrix} \quad (\text{D.3})$$

Note that, for all α these elements square to -1 and are hence of order four. Thus this makes it clear that there are two elements in $N(T)/T$ and that $N(T)/T$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. It is also clear that there is no homomorphism from the Weyl group $\mathbb{Z}/2\mathbb{Z}$ to $N(T)$ because the first component of $N(T)$ has no elements that act as a Weyl reflection on T and the second component of $N(T)$ has such elements and all such elements have order four.

In order to discuss the general case of a simple Lie algebra $\mathfrak{g}$ with simply connected Lie group G and maximal torus T we introduce a set of Chevalley-Serre generators: e_i, f_i, h_i , $i = 1, \dots, r$ satisfying:

$$\begin{aligned}
 [h_i, h_j] &= 0 \\
 [e_i, f_j] &= \delta_{ij} h_i \\
 [h_i, e_j] &= C_{ji} e_j \\
 [h_i, f_j] &= -C_{ji} f_j \\
 \text{ad}(e_i)^{1-C_{ji}}(e_j) &= 0 \quad i \neq j \\
 \text{ad}(f_i)^{1-C_{ji}}(f_j) &= 0 \quad i \neq j \\
 C_{ij} &:= \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \alpha_i(h_j)
 \end{aligned} \tag{D.4}$$

where C_{ij} is the Cartan matrix of $\mathfrak{g}$ and the simple coroots h_i define a basis of the Cartan subalgebra $\mathfrak{t}$. For each $i = 1, \dots, r$ there is an embedding of $\mathfrak{sl}(2) \rightarrow \mathfrak{g}$ defined by e_i, f_i, h_i , $e \rightarrow e_i$ etc. where

$$[e, f] = h \quad [h, e] = 2e \quad [h, f] = -2f \tag{D.5}$$

For each simple root α_i we have an order 2 element of T given by

$$m_i = \exp(i\pi h_i) \tag{D.6}$$

Tits showed that there is a canonical abelian extension $\hat{W}$ of W by an abelian group $\mathbb{Z}_2^r$ which does embed in G [35]. His work has been extended in a number of directions. Our description below is based on [31–33, 35, 36]. Recall that the action of a Weyl reflection in a root α on an element $h \in \mathfrak{t}$ of the Cartan subalgebra is

$$\sigma_\alpha(h) = h - \langle \alpha, h \rangle h_\alpha \tag{D.7}$$

where h_α is the coroot canonically assigned to α . Denoting reflections in the simple roots, σ_{α_i} , by s_i we have

$$s_i(h_j) = h_j - C_{ji} h_i \tag{D.8}$$

where C_{ji} is the Cartan matrix of G .

The Weyl group W is generated by the reflections s_i , $i = 1, \dots, r$. These obey the relations

$$\begin{aligned}
 s_i^2 &= 1 \\
 (s_i s_j)^{m_{i,j}} &= 1, \quad i \neq j
 \end{aligned} \tag{D.9}$$

where $m_{i,j}$ is the i, j element of the Coxeter matrix. Note that for simple laced G which is our main case of interest $m_{i,j} = 2$ if $i \neq j$ and the roots α_i, α_j are orthogonal and $m_{i,j} = 3$ if $i \neq j$ and the roots α_i, α_j make an angle of $2\pi/3$.

Following [35] the latter relation can be replaced by

$$s_i s_j s_i s_j \cdots s_i s_j = s_j s_i s_j s_i \cdots s_j s_i \quad (\text{D.10})$$

where on the l.h.s. there are $m_{i,j}$ terms $s_i s_j$ and on the r.h.s. there are $m_{i,j}$ terms $s_j s_i$. This relation follows from the second relation in (D.9) by successively multiplying the l.h.s. by $s_i^{-1}, s_j^{-1} s_i^{-1}$ and so on and using $s_i^{-1} = s_i$.

Tits shows that the extension $\hat{W}$ has generators a_i , one for each simple reflection which act on the h_i as

$$a_i h_j a_i^{-1} = \sigma_{\alpha_i}(h_j) \quad (\text{D.11})$$

and obey the relations

$$a_i^2 = m_i \quad (\text{D.12})$$

$$a_i a_j a_i a_j \cdots a_i a_j = a_j a_i a_j a_i \cdots a_j a_i \quad (\text{D.13})$$

where on the l.h.s. there are $m_{i,j}$ terms $a_i a_j$ and on the r.h.s. there are $m_{i,j}$ terms $a_j a_i$.

The m_i generate an abelian 2-group T_2 which is a subgroup of T and the map from a_i to m_i induces an exact sequence

$$1 \rightarrow T_2 \rightarrow \hat{W} \rightarrow W \rightarrow 1 \quad (\text{D.14})$$

When G is the simple and simply connected Lie group associated with $\mathfrak{g}$ we can identify $T_2 \cong \mathbb{Z}^r$, where r is the rank of G with the subgroup of T of points of order two.

In general it appears to be a complicated problem to figure out which conjugacy classes of Weyl group elements have orders which double when lifted to $\hat{W}$, but several examples which are relevant to Narian compactifications are discussed in [33]. We will content ourselves here with a general discussion for $\text{SU}(N)$.

D.1 Example: $G = \text{SU}(N)$

We consider $\text{SU}(N)$ matrices acting on the defining N -dimensional representation. We choose the standard system of simple roots and denote the highest weight of the fundamental representation by λ^1 . Then, up to the action of a diagonal matrix, a weight basis with weights

$$\lambda^1, \lambda^1 - \alpha_1, \lambda^1 - \alpha_1 - \alpha_2, \dots, \lambda^1 - (\alpha_1 + \dots + \alpha_{N-1}) \quad (\text{D.15})$$

corresponds to the standard Euclidean basis $e_1, \dots, e_N$ of $\mathbb{C}^N$. Labeling the weight vectors by $1, 2, \dots, N$ the Weyl reflection $\bar{g}_i = \sigma_{\alpha_i}$ acts on these weights as the permutation $(i, i+1)$. Therefore, any lift to $\text{SU}(N)$ must have the form:

$$\hat{g}_i = \sum_{k \neq i, i+1} z_k^{(i)} e_{k,k} + (x_i e_{i,i+1} + y_i e_{i+1,i}) \quad i = 1, \dots, N-1 \quad (\text{D.16})$$

where $x_i, y_i, z_k^{(i)}$ are phases and the $SU(N)$ condition implies

$$z_i x_i y_i = -1 \quad z_i := \prod_{k \neq i, i+1} z_k^{(i)}. \quad (\text{D.17})$$

We claim that *any* choice of $x_i, y_i, z_k^{(i)}$ has the correct conjugation properties to project to an element of the Weyl group:

$$\hat{g}_i h_j \hat{g}_i^{-1} = \begin{cases} h_j & i \neq j, j \pm 1 \\ -h_i & i = j \\ h_j + h_{j+1} & j = i - 1 \\ h_{j-1} + h_j & j = i + 1 \end{cases} \quad (\text{D.18})$$

as one easily checks by direct computation with (D.16) and $h_i = -\frac{i}{2}(e_{i,i} - e_{i+1,i+1})$.

Note that

$$\hat{g}_i^2 = \sum_{k \neq i, i+1} (z_k^{(i)})^2 e_{k,k} + (x_i y_i)(e_{i,i} + e_{i+1,i+1}) \quad (\text{D.19})$$

$$(\hat{g}_i \hat{g}_{i+1})^3 = \sum_{k \neq i, i+1, i+2} (z_k^{(i)} z_k^{(i+1)})^3 e_{k,k} + (x_i y_i z_{i+2}^{(i)})(x_{i+1} y_{i+1} z_i^{(i+1)})(e_{i,i} + e_{i+1,i+1} + e_{i+2,i+2}) \quad (\text{D.20})$$

Definition. Let $\widetilde{W}(x, y, z) \subset N(T)$ be the subgroup of $N(T)$ generated by the elements $\hat{g}_i$, where the $x_i, y_i, z_k^{(i)}$ are arbitrary phases subject only to the constraints (D.17).

Remarks:

1. The subgroups $\widetilde{W}(x, y, z)$ map surjectively to the Weyl group under the conjugation action.
2. They are finite subgroups iff $z_k^{(i)}$ and $x_i y_i$ are all roots of unity.
3. All such subgroups are related by right-multiplication of the generators by suitable elements of T . That is, for any two such groups determined by (x, y, z) and (x', y', z') there are elements $t_i \in T$ with $\hat{g}'_i = \hat{g}_i t_i$.
4. The Tits lift is

$$g_i = \exp \left[\frac{\pi}{2} (e_i - f_i) \right] \quad (\text{D.21})$$

where e_i, f_i are Serre generators and is given by taking $z_k^{(i)} = 1$, $x_i = 1$, and $y_i = -1$. According to [38] the expression (D.21) is true in much greater generality than discussed here.

Now let us ask if we can have a subgroup $W(x, y, z)$ isomorphic to $W(\mathfrak{su}(N))$. Since we want $\hat{g}_i^2 = 1$ we must choose $z_k^{(i)} \in \{\pm 1\}$ as well as $x_i y_i = 1$. Then the constraints (D.17) show that $z_i = -1$. Therefore we cannot take all $z_k^{(i)} = 1$.

Next we need to check the braid relations:

$$\hat{g}_i \hat{g}_{i+1} \hat{g}_i = \hat{g}_{i+1} \hat{g}_i \hat{g}_{i+1} \quad (\text{D.22})$$

For order two elements $(\hat{g}_i \hat{g}_{i+1})^3$ simplifies to

$$(\hat{g}_i \hat{g}_{i+1})^3 = \sum_{k \neq i, i+1, i+2} (z_k^{(i)} z_k^{(i+1)}) e_{k,k} + (z_{i+2}^{(i)} z_i^{(i+1)}) (e_{i,i} + e_{i+1,i+1} + e_{i+2,i+2}) \quad (\text{D.23})$$

so for a group isomorphic to the Weyl group we need this to be = 1, putting some further constraint on the $z_k^{(i)}$.

Finally, for $N > 3$ we also must check the relations

$$\hat{g}_i \hat{g}_j = \hat{g}_j \hat{g}_i \quad |i - j| > 1 \quad (\text{D.24})$$

This is very constraining and shows that $z_i^{(j)} = z_{i+1}^{(j)}$ for $|i - j| > 1$. Therefore

$$z_1^{(j)} = \dots = z_{j-1}^{(j)} = z_-^{(j)} \quad z_{j+2}^{(j)} = \dots = z_N^{(j)} = z_+^{(j)} \quad (\text{D.25})$$

Now combining these constraints with the constraints (D.23) from the braid relations shows that in fact all

$$z_k^{(i)} = z \quad (\text{D.26})$$

must have a common value. Since the $z_i = -1$ this common value must be $z = -1$. But this is only compatible with the second equation in (D.17) when N is odd.

We conclude that for N odd we can take all $z_k^{(j)} = -1$ for $k \neq j, j+1$ and $x_j = y_j = 1$. This gives an explicit subgroup $W(x, y, z)$ satisfying all the relations. For N even there is no subgroup of $N(T)$ isomorphic to the Weyl group and the sequence does not split.

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