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Bootstrapping non-commutative gauge theories from L_∞ algebras

Ralph Blumenhagen,[a] **Ilka Brunner,**[b] **Vladislav Kupriyanov**[a,c,d] **and Dieter Lüst**[a,b]

[a] *Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), Föhringer Ring 6, 80805 München, Germany*

[b] *Arnold Sommerfeld Center for Theoretical Physics, LMU, Theresienstr. 37, 80333 München, Germany*

[c] *CMCC-Universidade Federal do ABC, Santo André, SP, Brazil*

[d] *Tomsk State University, Tomsk, Russia*

E-mail: blumenha@mpp.mpg.de, Ilka.Brunner@physik.uni-muenchen.de, vladislav.kupriyanov@gmail.com, dieter.luest@lmu.de

Abstract: Non-commutative gauge theories with a non-constant NC-parameter are investigated. As a novel approach, we propose that such theories should admit an underlying L_∞ algebra, that governs not only the action of the symmetries but also the dynamics of the theory. Our approach is well motivated from string theory. We recall that such field theories arise in the context of branes in WZW models and briefly comment on its appearance for integrable deformations of AdS_5 sigma models. For the SU(2) WZW model, we show that the earlier proposed matrix valued gauge theory on the fuzzy 2-sphere can be bootstrapped via an L_∞ algebra. We then apply this approach to the construction of non-commutative Chern-Simons and Yang-Mills theories on flat and curved backgrounds with non-constant NC-structure. More concretely, up to the second order, we demonstrate how derivative and curvature corrections to the equations of motion can be bootstrapped in an algebraic way from the L_∞ algebra. The appearance of a non-trivial A_∞ algebra is discussed, as well.

Keywords: Non-Commutative Geometry, D-branes, Gauge Symmetry

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1 Introduction

It is one of the rather appealing features of string theory that the effective theory on D-branes in a two-form background is given by a non-commutative (NC) Yang-Mills theory [1]. For constant two-form flux this result can be explicitly derived by quantizing the open string and computing conformal field theory correlation functions [2]. In this case, the non-commutative theory is governed by the associative Moyal-Weyl star-product.

From string theory it is known that there also exist consistent D-brane solutions of the string equations of motion that wrap curved submanifolds and carry a non-constant two-form flux, thus leading to a non-constant non-commutativity structure[1] Θ^{ij}. Examples are branes in WZW models [3] or holographic duals of integrable deformations of AdS_5 sigma models [4]. In the latter case, the holographic dual gauge theory still lives on flat space and only receives a deformation in the non-commutativity structure. Therefore, one expect that one can formulate a non-commutative gauge theory also for such more general cases.

[1]In this paper we use the term non-commutativity structure for a non-constant Θ^{ij}.

Using techniques from conformal field theory, for the SU(2) WZW model it was shown that this theory is a non-commutative matrix valued gauge theory on the fuzzy 2-sphere. This theory is still associative, but in principle also this could be broken. Throughout this paper we will be agnostic about this point and admit also non-associative star-products.

There have been some attempts to provide a description of such gauge theories using the general Kontsevich [5] star-product [6–9] (for a recent application see [10]) and invoking techniques from Hopf-algebras [11, 12]. For more information and literature on these attempts please consult the review [13]. However, these approaches were motivated rather mathematically while missing a clear physical guiding principle for their construction. It is the aim of this paper, to take such a physical principle from string theory and to analyze whether it works and gives reasonable results.

We note that more recently there have also been proposals for the appearance of non-commutative and non-associative structures in the closed string sector [14, 15], in particular when one has a non-geometric flux background. Let us emphasize that in this paper we restrict to the open string case with D-branes.

In this paper we propose that the missing physically motivated guiding principle is the existence of an L_∞ (or A_∞) algebra. Before we investigate this idea in more detail, let us mention that these structures appeared for the first time in the context of string field theory [16]. Indeed, e.g. for bosonic closed string field theory, both the action of symmetries on the string field and their string equations of motion were governed by an L_∞ algebra. The latter can be considered as a generalization of a Lie algebra, where one allows field dependent gauge parameters. This weakens the closure constraint and motivates the introduction of in general infinitely many higher products satisfying generalized Jacobi identities. These are quadratic expressions involving for each n finitely many higher products. In particular, the usual Jacobi identity for the two-product (the commutator) can be violated by "derivative" terms, thus allowing a mild form of non-associativity. For this reason, in the mathematics literature such algebras have been called strong homotopy algebras [17].

In [18], the authors showed that L_∞ algebras do not only show up in string field theory, but also in much simpler field theories, like Chern-Simons (CS) and Yang-Mills (YM) theories. Here the structure is considerably truncated and only a finite number of higher products and relations were non-trivial. It is very tantalizing that again not only the action of the symmetry but also the dynamics of the whole gauge theory fit into such finite L_∞ algebras. The authors also proposed that every consistent gauge theory should be governed by such an underlying L_∞ algebra.

In [19, 20], motivated by the AdS_3-CFT_2 holographic duality, it was shown that $\mathcal{W}$-algebras, describing infinitely many global symmetries in two-dimensional conformal field theories, also feature an underlying, highly non-trivial L_∞ structure. Here, it was the non-linearity of the $\mathcal{W}$-algebra that induced higher products and relations. Turning the logic around, if they were not already known, $\mathcal{W}$-algebras could have been bootstrapped from the L_∞ algebra. Furthermore it was shown in [21] that the non-associative closed string R-flux algebra as well as the associated M-theory R-flux algebra of the seven octonions can be extended to a 2-term L_∞ algebra.

Thus, so far there exist a couple of physical examples that could be rewritten in terms of L_∞ algebras. The motivation for this work is to advance the symmetry concept of L_∞ algebras and actually exploit it to determine the structure of the above mentioned NC gauge theories with general NC-structure. For this purpose we will follow a bootstrap approach, where we take some initial lower order products, like one- and two-products, and bootstrap the remaining higher products by invoking the L_∞ relations. The initial data are essentially the first term in the gauge variation and in the equations of motion, i.e. the one resulting from a kinetic term in the action. All of the rest follows. For a general NC-structure, we will see that all the other higher products receive derivative ($\partial\Theta$)-corrections. In other words, imposing the guiding principle of an underlying L_∞ algebra, we can algebraically bootstrap the derivative corrections to the action of the NC gauge symmetry onto the gauge fields and their equations of motion.

In this paper, we explicitly show how this procedure can be carried out in detail up to second order in Θ. For this purpose, in section 2 we review some facts about NC gauge theories and recall the mathematical definitions of L_∞ and A_∞ algebras. As a first application of L_∞ algebras, we show that NC-CS and NC-YM theories on the Moyal-Weyl plane fit into this scheme.

In section 3 we remind the reader of concrete string theory settings where NC gauge theories with non-constant NC-structure appeared. These are branes in WZW models and holographic duals of integrable deformations of AdS_5 sigma models. For the SU(2) WZW model, we recall that an NC matrix valued gauge theory has been derived via CFT techniques [22]. As a compelling first result, we show that this unconventional NC gauge theory on the fuzzy 2-sphere can be bootstrapped by imposing the existence of an L_∞ algebra.

In section 4, we apply the same technique to the more general class of NC gauge theory on flat and curved space with non-constant NC-structure. This is done in a perturbative approach in Θ. First, we point out the essential problem arising for non-constant Θ and argue that it receives a natural solution in the context of L_∞ algebras. We bootstrap the derivative corrections to the action of the NC gauge symmetry onto the gauge fields. Then, we extend the L_∞ algebra to also include the equations of motion of a NC-CS and a NC-YM theory on flat and curved space.

We show that up to second order in Θ one can even find an A_∞ algebra. As expected, the graded symmetrization of the obtained structure results in the corresponding L_∞ algebra. Since this involves a lengthy and tedious computation, we have delegated this part to an appendix - not because it is less important but for not too much disturbing the main flow of the paper.

2 Preliminaries

For self-consistency, in this section we introduce some of the salient features of known NC gauge theories and the formal definitions of L_∞ and A_∞ algebras. In addition, we analyze the NC gauge theory based on the Moyal-Weyl star product with respect to an underlying L_∞ algebra.

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2.1 Non-commutative gauge theories

First, let us recall that the conformal field theory of an open string ending on a D-brane supporting a non-trivial gauge flux $\mathcal{F} = B + 2\pi F$ features a non-commutative geometry. In this paper we choose $\alpha' = 1$. Indeed, by computing the disc level scattering amplitude of N-tachyons, certain relative phases appear which for constant gauge flux can be described by the Moyal-Weyl star-product

$$(f \star g)(x) = \exp\left(\frac{i}{2}\,\Theta^{ij}\,\partial_i^{x_1}\,\partial_j^{x_2}\right) f(x_1)\,g(x_2)\Big|_x\,. \tag{2.1}$$

The open string quantities governing the theory on the D-brane are related to the initial closed string variables g and $\mathcal{F}$ via $G^{-1} + \Theta = (g + \mathcal{F})^{-1}$, where the anti-symmetric bi-vector Θ^{ij} is the one appearing in the star product (2.1). In the Seiberg-Witten limit the OPE exactly becomes the Moyal-Weyl star-product. This non-trivial product of functions leads to the non-commutative Moyal-Weyl plane with $[x^i, x^j]_\star = i\,\Theta^{ij}$. In [5] it has been shown that for every Poisson structure Θ^{ij} that by definition satisfies

$$\Pi^{ijk} := 3\,\Theta^{[im}\partial_m\Theta^{jk]} = 0 \tag{2.2}$$

one can define a corresponding associative star-product, which will also involve derivatives of the Poisson structure. The same product can also be considered for a quasi Poisson structure, but then leads to a non-associative star-product, which up to second order in Θ reads

$$\begin{aligned} f \bullet g = f \cdot g + \frac{i}{2}\Theta^{ij}\,\partial_i f\,\partial_j g - \frac{1}{8}\Theta^{ij}\Theta^{kl}\,\partial_i\partial_k f\,\partial_j\partial_l g \\ - \frac{1}{12}\big(\Theta^{im}\partial_m\Theta^{jk}\big)\big(\partial_i\partial_j f\,\partial_k g + \partial_i\partial_j g\,\partial_k f\big) + O(\Theta^3)\,. \end{aligned} \tag{2.3}$$

For the higher order expression see [23–26]. Often we will write this as

$$f \bullet g = f \star g - \frac{1}{12}\big(\Theta^{im}\partial_m\Theta^{jk}\big)\big(\partial_i\partial_j f\,\partial_k g + \partial_i\partial_j g\,\partial_k f\big) + O(\Theta^3) \tag{2.4}$$

that separates the derivative $\partial\Theta$-corrections from the standard Moyal-Weyl terms. The associator for this product becomes

$$(f \bullet g) \bullet h - f \bullet (g \bullet h) = \frac{1}{6}\,\Pi^{ijk}\,\partial_i f\,\partial_j g\,\partial_k h + O(\Theta^3)\,. \tag{2.5}$$

In [1], for the Moyal-Weyl case with constant open string metric and NC-parameter, it was shown that the effective theory on a stack of N branes is given by a non-commutative gauge theory with gauge group U(N). In the following we stick to the U(1) case. As in usual YM theory there is a gauge field $A_a(x)$ behaving under a gauge transformation as

$$\delta_f A_a = \partial_a f + i[f, A_a]_*\,. \tag{2.6}$$

Using the Leibniz rule for the star-bracket $[.,.]_\star$ and its associativity,[2] one can show that two gauge transformations close off-shell in the sense

$$[\delta_f, \delta_g]A_a = \delta_{-i[f,g]_\star} A_a\,. \tag{2.7}$$

[2]Note that both assumptions might not be satisfied for non-constant Θ.

Moreover, the field-strength

$$F_{ab} = \partial_a A_b - \partial_b A_a - i[A_a, A_b]_\star \tag{2.8}$$

transforms covariantly, i.e.

$$\delta_f F_{ab} = i[f, F_{ab}]_\star \,. \tag{2.9}$$

Then, the vacuum equation of motion for the non-commutative U(1) Yang-Mills theory reads

$$\begin{aligned} 0 = \mathcal{F}_a &= \partial^b F_{ba} - i[A^b, F_{ba}]_\star \\ &= \Box A_a - \partial_a(\partial \cdot A) - i\,\partial^b[A_b, A_a]_\star - i[A^b, \partial_b A_a - \partial_a A_b]_\star \\ &\quad - [A^b, [A_b, A_a]_\star]_\star \,. \end{aligned} \tag{2.10}$$

In section 2.3 we will come back to these relations and study their implementation into an L_∞ algebra.

Similarly one can also define a non-commutative Chern-Simons theory in three-dimensions, whose equation of motion is

$$0 = \mathcal{F}_c = \epsilon_c{}^{ab}\left(\partial_a A_b - \frac{i}{2}[A_a, A_b]_\star\right). \tag{2.11}$$

For usual CS and YM-theory, it was explicitly shown in [18] that both their symmetries and their dynamics are governed in an algebraic way by the objects and relations of an L_∞ algebra. Before reviewing this, in the next section we give a brief introduction into the general notion of L_∞ and also of A_∞ algebras.

2.2 L_∞ and A_∞ algebras and gauge symmetries

Following [18], let us review the basis notion of L_∞ and A_∞ algebras and the generic relation of the first to the description of gauge symmetries and their dynamics.

Definition of L_∞ algebra. L_∞ algebras are generalized Lie algebras where one has not only a two-product, the commutator, but more general multilinear n-products with n inputs

$$\begin{aligned} \ell_n : \qquad X^{\otimes n} &\to X \\ x_1, \ldots, x_n &\mapsto \ell_n(x_1, \ldots, x_n)\,, \end{aligned} \tag{2.12}$$

defined on a graded vector space $X = \bigoplus_n X_n$, where n denotes the grading. These products are graded anti-symmetric

$$\ell_n(\ldots, x_1, x_2, \ldots) = (-1)^{1+\deg(x_1)\deg(x_2)}\,\ell_n(\ldots, x_2, x_1, \ldots)\,, \tag{2.13}$$

with

$$\deg\big(\,\ell_n(x_1, \ldots, x_n)\big) = n - 2 + \sum_{i=1}^{n} \deg(x_i)\,. \tag{2.14}$$

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The set of higher products ℓ_n define an L_∞ algebra, if they satisfy the infinitely many relations

$$\begin{aligned}\mathcal{J}_n(x_1,\ldots,x_n) := \sum_{i+j=n+1} (-1)^{i(j-1)} \sum_\sigma (-1)^\sigma \chi(\sigma;x) \\ \ell_j\big(\,\ell_i(x_{\sigma(1)}\,,\ldots,x_{\sigma(i)})\,,x_{\sigma(i+1)},\ldots,x_{\sigma(n)}\,\big) = 0\,.\end{aligned} \tag{2.15}$$

The permutations are restricted to the ones with

$$\sigma(1) < \cdots < \sigma(i), \qquad \sigma(i+1) < \cdots < \sigma(n)\,, \tag{2.16}$$

and the sign $\chi(\sigma;x) = \pm 1$ can be read off from (2.13). The first relations $\mathcal{J}_n$ with $n = 1,2,3,\ldots$ can be schematically written as

$$\begin{aligned}&\mathcal{J}_1 = \ell_1\ell_1\,, \qquad \mathcal{J}_2 = \ell_1\ell_2 - \ell_2\ell_1\,, \qquad \mathcal{J}_3 = \ell_1\ell_3 + \ell_2\ell_2 + \ell_3\ell_1\,,\\ &\mathcal{J}_4 = \ell_1\ell_4 - \ell_2\ell_3 + \ell_3\ell_2 - \ell_4\ell_1\,,\end{aligned} \tag{2.17}$$

from which one can deduce the scheme for the higher $\mathcal{J}_n$. More concretely, the first L_∞ relations read

$$\begin{aligned}\ell_1\big(\,\ell_1(x)\,\big) &= 0\\ \ell_1\big(\,\ell_2(x_1,x_2)\,\big) &= \ell_2\big(\,\ell_1(x_1),x_2\,\big) + (-1)^{x_1}\ell_2\big(\,x_1,\ell_1(x_2)\,\big)\,,\end{aligned} \tag{2.18}$$

revealing that ℓ_1 must be a nilpotent derivation with respect to ℓ_2, i.e. that in particular the Leibniz rule is satisfied. Denoting $(-1)^{x_i} = (-1)^{deg(x_i)}$ the full relation $\mathcal{J}_3$ reads

$$\begin{aligned}0 = {}& \ell_1\big(\ell_3(x_1,x_2,x_3)\,\big)\\ &+\ell_2\big(\ell_2(x_1,x_2),x_3\,\big) + (-1)^{(x_2+x_3)x_1}\ell_2\big(\ell_2(x_2,x_3),x_1\,\big)\\ &+(-1)^{(x_1+x_2)x_3}\ell_2\big(\ell_2(x_3,x_1),x_2\,\big)\\ &+\ell_3\big(\ell_1(x_1),x_2,x_3\,\big) + (-1)^{x_1}\ell_3\big(x_1,\ell_1(x_2),x_3\,\big) + (-1)^{x_1+x_2}\ell_3\big(x_1,x_2,\ell_1(x_3)\,\big)\end{aligned} \tag{2.19}$$

and means that the Jacobi identity for the ℓ_2 product is mildly violated by ℓ_1 exact expressions.

Definition of $\mathbf{A}_\infty$ algebras. The definition of an A_∞ algebras is very similar to the definition of an L_∞ algebra. While L_∞ algebras are generalized differential graded Lie algebras with a mild violation of the Jacobi identity, A_∞ algebras generalize algebras with a mild violation of associativity. One has higher products $m_n(x_1,\ldots,x_n)$ of degree $n-2$, where x_i are again elements of a graded vector space. The quadratic relations for the higher products are

$$\mathcal{A}_{n-1}(x_1,\ldots,x_{n-1}) = \sum_{l=1}^{n-1}(-1)^{n(l+1)}\,m_l \circ m_{n-l} = 0 \tag{2.20}$$

with the second product defined as

$$m_p = \sum_{r=0}^{n-1-p} (-1)^{r(p+1)}\,1^r \otimes m_p \otimes 1^{n-1-p-r}\,. \tag{2.21}$$

The first three relations read

$$\begin{aligned}\mathcal{A}_1 &= m_1 \circ m_1\\ \mathcal{A}_2 &= m_1 \circ m_2 - m_2 \circ (m_1 \otimes 1 + 1 \otimes m_1)\\ \mathcal{A}_3 &= m_1 \circ m_3 + m_2 \circ (m_2 \otimes 1 - 1 \otimes m_2)\\ &\quad + m_3 \circ (m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)\end{aligned} \tag{2.22}$$

where whenever an odd degree m_n is exchanged with an odd degree x_m one gets an extra minus sign. This will become clearer in appendix B where we will consider explicit examples. There we also need the next relation

$$\begin{aligned}\mathcal{A}_4 = &\, m_1 \circ m_4 - m_2 \circ (m_3 \otimes 1 + 1 \otimes m_3)\\ &+ m_3 \circ (m_2 \otimes 1 \otimes 1 - 1 \otimes m_2 \otimes 1 + 1 \otimes 1 \otimes m_2)\\ &- m_4 \circ (m_1 \otimes 1^3 + 1 \otimes m_1 \otimes 1^2 + 1^2 \otimes m_1 \otimes 1 + 1^3 \otimes m_1)\,.\end{aligned} \tag{2.23}$$

Even though gauge theories arise for the open string and string field theory suggest that they are related by an A_∞ structure, [18] proposed that they also fit nicely into the structure of L_∞ algebras.

Gauge theories and L_∞ algebras. The framework of L_∞ algebras is quite flexible and it has been suggested that every classical perturbative gauge theory (derived from string theory), including its dynamics, is organized by an underlying L_∞ structure [18]. For sure, the pure gauge algebra, called $\mathrm{L}_\infty^{\rm gauge}$, of such theories satisfies the L_∞ identities. To see this, let us assume that the field theory has a standard type gauge structure, meaning that the variations of the fields can be organized unambiguously into a sum of terms each of a definite power in the fields. Then we choose only two non-trivial vector spaces as

$$\begin{array}{cc} X_0 & X_{-1} \\ f & A_a \end{array}\,. \tag{2.24}$$

In this case, the only allowed non-trivial higher product are the ones with one and two gauge parameters $\ell_{n+1}(f, A^n) \in X_{-1}$ and $\ell_{n+2}(f, g, A^n) \in X_0$ and the only non-trivial relations are $\mathcal{J}_{n+2}(f, g, A^n) \in X_{-1}$ and $\mathcal{J}_{n+3}(f, g, h, A^n) \in X_0$. Then, the gauge variations are expanded as

$$\delta_f A = \sum_{n \geq 0} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}}\, \ell_{n+1}(f, \underbrace{A, \ldots, A}_{n \text{ times}})\,. \tag{2.25}$$

This allows to read off the higher products $\ell_{n+1}(f, A^n) \in X_{-1}$. It was shown in [18, 27, 28], that the off-shell closure of the symmetry variations

$$[\delta_f, \delta_g] A = \delta_{-C(f,g,A)} A\,, \tag{2.26}$$

and the Jacobi identity

$$\sum_{\rm cycl} \big[\delta_f, [\delta_g, \delta_h]\big] = 0 \tag{2.27}$$

are equivalent to the L_∞ relations with two and three gauge parameters. Here the closure relation allows for a field dependent gauge parameter which can be written in terms of L_∞ products as

$$C(f,g,A) = \sum_{n\geq 0} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \, \ell_{n+2}(f,g,\underbrace{A,\dots,A}_{n \text{ times}}) \,. \tag{2.28}$$

Thus, the action of gauge symmetries on the fundamental fields is governed by an L_∞^{gauge} algebra. However, this is not the end of the story, as string field theory suggests that also the dynamics of the theory, i.e. the equations of motion, are expected to fit into an extended L_∞^{full} algebra.

For this purpose one extends the vector space to $X_0 \oplus X_{-1} \oplus X_{-2}$

$$\begin{array}{ccc} X_0 & X_{-1} & X_{-2} \\ f & A_a & E_a \end{array} \tag{2.29}$$

where X_{-2} also contains the equations of motion, i.e. $\mathcal{F} \in X_{-2}$. Now many more higher products can be non-trivial and one has to check in a case by case study whether indeed the L_∞^{full} algebra closes. The higher products $\ell_n(A^n) \in X_{-2}$ are special as they give the equation of motion that is expanded as

$$\mathcal{F} = \sum_{n\geq 1} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \, \ell_n(A^n) = \ell_1(A) - \frac{1}{2}\ell_2(A^2) - \frac{1}{3!}\ell_3(A^3) + \dots \,. \tag{2.30}$$

Moreover, the structure admits that the closure condition (2.26) is only satisfied on-shell, i.e. there can be terms $\ell_{n+3}(f,g,\mathcal{F},A^n) \in X_{-1}$ on the right hand side. In case one has off-shell closure (like for the CS and YM theories considered in this paper) all these higher product are vanishing. Moreover, the gauge variation of $\mathcal{F}$ is given by

$$\delta_f \mathcal{F} = \ell_2(f,\mathcal{F}) + \ell_3(f,\mathcal{F},A) - \frac{1}{2}\ell_4(f,\mathcal{F},A^2) + \dots \tag{2.31}$$

reflecting that, as opposed to the gauge field A, it transforms covariantly.

It was proposed that for writing down an action for these equations of motion one needs an inner product

$$\langle \ , \ \rangle : X_{-1} \otimes X_{-2} \to \mathbb{R} \tag{2.32}$$

satisfying the cyclicity property

$$\langle A_0, \ell_n(A_1,\dots,A_n)\rangle = \langle A_1, \ell_n(A_0,\dots,A_n)\rangle \tag{2.33}$$

for all $A_i \in X_{-1}$. Then, the equations of motion follow from varying the action

$$\begin{aligned} S &= \sum_{n\geq 1} \frac{1}{(n+1)!} (-1)^{\frac{n(n-1)}{2}} \, \langle A, \ell_n(A^n)\rangle \\ &= \frac{1}{2}\langle A, \ell_1(A)\rangle - \frac{1}{3!}\langle A, \ell_2(A^2)\rangle - \frac{1}{4!}\langle A, \ell_3(A^3)\rangle + \dots \,. \end{aligned} \tag{2.34}$$

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2.3 L_∞ algebras for NC-CS and NC-YM gauge theories

Now, as two examples, we analyze how U(1) NC Chern-Simons and Yang-Mills theories fit into the scheme of L_∞ algebras. In this section, we consider the case of the Moyal-Weyl star-product, i.e. the NC-parameter Θ is constant. In this case, the computation is very similar to the analysis of ordinary (non-abelian) CS and YM theories discussed in [18].

L_∞ structure of non-commutative CS. The vector spaces are still as in eq. (2.24) or as in eq. (2.29). Some of the relevant relations have already been given in section 2.1. From the gauge variation (2.6), we can read off[3]

$$\ell_1(f) = \partial_a f\,, \qquad \ell_2(f,A) = i[f,A_a]_\star \tag{2.35}$$

and from the off-shell closure condition (2.7)

$$\ell_2(f,g) = i[f,g]_\star\,, \qquad \ell_3(f,g,E) = 0 \tag{2.36}$$

with all higher products vanishing, e.g. $\ell_{n+1}(f,A^n) = 0$ for $n \geq 2$. The equation of motion (2.11) motivates the choice for the non-vanishing products

$$\ell_1(A) = \epsilon_c{}^{ab}\partial_a A_b\,, \qquad \ell_2(A,B) = i\epsilon_c{}^{ab}[A_a,B_b]_\star\,. \tag{2.37}$$

Therefore, only ℓ_1 and ℓ_2 products are non-vanishing and one only has to check the finite number of L_∞ relations listed below

$$\begin{aligned}
&\mathcal{J}_1(f) \in X_{-2}\\
&\mathcal{J}_2(f,g) \in X_{-1}\,, \qquad \mathcal{J}_2(f,A) \in X_{-2},\\
&\mathcal{J}_3(f,g,h) \in X_0\,, \quad \mathcal{J}_3(f,g,A) \in X_{-1}\,, \quad \mathcal{J}_3(f,A,B) \in X_{-2}\\
&\mathcal{J}_3(f,g,E) \in X_{-2}\,.
\end{aligned} \tag{2.38}$$

The first relation $\mathcal{J}_1(f) = \ell_1(\ell_1(f)) = \epsilon_c{}^{ab}\partial_a\partial_b f = 0$ can be readily checked. The relation $\mathcal{J}_2(f,g) = 0$ is nothing else than the Leibniz-rule for the star commutator. The full third relation reads

$$\ell_1(\ell_2(f,A)) = \ell_2(\ell_1(f),A) + \ell_2(f,\ell_1(A)) \tag{2.39}$$

which fixes the last term to be

$$\ell_2(f,E) = i[f,E_a]_\star\,. \tag{2.40}$$

Since all ℓ_3 are vanishing, the remaining four $\mathcal{J}_3$ relations do only contain $\ell_2\ell_2$-terms. As the star commutator satisfies the Jacobi identity, these are all satisfied. Let us also mention that the field strength can be expressed as

$$\ell_1(A) - \frac{1}{2}\ell_2(A,A) = \frac{1}{2}\epsilon_c{}^{ab}\Big(\partial_a A_b - \partial_b A_a - i[A_a,A_b]_\star\Big) = \frac{1}{2}\epsilon_c{}^{ab}F_{ab}\,. \tag{2.41}$$

[3]Note that by writing $\ell_2(f,A)$ it is understood that the object also carries an index a like $\ell_2(f,A)_a$. In order not to clutter the notation, in the following we leave this index out, as it is usually clear from the free index on the r.h.s.

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Clearly, by setting all elements in X_{-2} to zero, one gets the sub-algebra $\mathrm{L}_\infty^{\text{gauge}}$. The latter is the same for NC-CS and NC-YM. Defining the inner product as

$$\langle A, E \rangle = \int d^3x \; \eta^{ab} A_a E_b \tag{2.42}$$

one can integrate this to an action (2.34).

Thus, we have seen that the U(1) NC-CS theory fits nicely into the L_∞ framework, where the highest appearing products are ℓ_2.

L_∞ structure of non-commutative YM. A similar computation can also be done for NC-YM theory. The case of usual non-abelian YM-theory was first formulated in [29, 30]. Here we follow the same path as in the more recent paper [18].

Since the action of a gauge transformation on the fields and its closure are the same as for NC-CS theory, the products $\ell_1(f)$ and $\ell_2(f, A)$ from (2.35) and $\ell_2(f, g)$ from (2.36) are still valid. The equations of motion (2.10) allow one to read-off the higher products

$$\begin{aligned}
\ell_1(A) &= \Box A_a - \partial_a(\partial \cdot A) \\
\ell_2(A, B) &= i\,\partial^b [A_b, B_a]_\star + i[A^b, \partial_b B_a - \partial_a B_b]_\star + (A \leftrightarrow B) \\
\ell_3(A, B, C) &= [A^b, [B_b, C_a]_\star]_\star + [B^b, [C_b, A_a]_\star]_\star + [C^b, [A_b, B_a]_\star]_\star \\
&\quad + [A^b, [C_b, B_a]_\star]_\star + [C^b, [B_b, A_a]_\star]_\star + [B^b, [A_b, C_a]_\star]_\star \,.
\end{aligned} \tag{2.43}$$

Note that $\ell_1(A)$ has changed from the NC-CS case and that for NC-YM there also exist a non-vanishing ℓ_3. Therefore, besides (2.38) one also has to check the L_∞ relations

$$\begin{aligned}
&\mathcal{J}_4(f, g, h, A) \in X_0 \,, \qquad \mathcal{J}_4(f, g, A, B) \in X_{-1} \,, \qquad \mathcal{J}_4(f, g, h, E) \in X_{-1} \\
&\mathcal{J}_4(f, A, B, C) \in X_{-2} \,, \qquad \mathcal{J}_4(f, g, A, E) \in X_{-2} \,.
\end{aligned} \tag{2.44}$$

The nil-potency condition $\ell_1(\ell_1(f)) = 0$ can readily be checked. Similarly to the NC-CS theory, the Leibniz-rule $\mathcal{J}_2(f, A)$ fixes $\ell_2(f, E) = i[f, E_a]_\star$. Setting now all other higher products to zero, one realizes that the three relations $\mathcal{J}_3(f, g, h) = \mathcal{J}_3(f, g, A) = \mathcal{J}_3(f, g, E) = 0$ involve only star-commutators and are satisfied by their Jacobi identity. The identity $\mathcal{J}_3(f, A, B) = 0$ is more non-trivial and also involves the three-product $\ell_3(A, B, C)$. However, by spelling out all terms in the relation, one realizes that they indeed all cancel. From the next order relations in (2.44) only $\mathcal{J}_4(f, A, B, C) = 0$ is non-trivial, but can be checked by applying the Jacobi identity for the star-commutator. In principle also $\mathcal{J}_5$ could be relevant, but due to $\ell_3(\ell_3(A^3), A^2) \in X_{-3}$ these relations are satisfied trivially.

3 NC gauge theories arising in string theory

We just showed that both NC-CS and NC-YM on flat Minkowski space with constant NC-structure Θ fit into the scheme of L_∞ algebras. However, not all consistent D-branes (boundary states) in string theory are of this simple type, as there do also exist D-branes wrapping curved submanifolds and carrying a non-constant gauge flux on the brane world-volume. Therefore, the question arises whether also the expected NC gauge theory on such

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branes fit into the scheme of L_∞ algebras. In this case, the Kontsevich star product (2.3) indicates that one gets extra derivative terms $\partial\Theta$.

Before we continue along these lines, in this section we want to remind the reader of a few stringy circumstances where non-commutativity with non-constant Θ does appear. These will be branes in exactly solvable WZW models for compact groups and recent advances related to integrable deformations of AdS_5 sigma models.

3.1 D-Branes in WZW models

In this section we review some of the features of D-branes[4] in WZW models relevant for us. WZW models are exactly solvable sigma models whose target spaces are group manifolds equipped with non-trivial NS-NS three-form fluxes. Their distinctive feature is that the corresponding two-dimensional conformal field theories are explicitly known and given by the unitary series of Kac-Moody algebras. As a consequence it was possible to construct boundary states in the CFT that turned out to correspond to certain branes wrapping conjugacy classes of the group manifold [3, 31] carrying non-constant two-form flux. In this section, we will review the semi-classical description of these consistent branes.

Preliminaries. The starting point is the two-dimensional world-sheet action of a WZW model

$$\begin{aligned} S_{\mathrm{WZW}} = &\frac{k}{16\pi}\int_{\partial\Sigma} d^2\sigma\, \mathrm{Tr}\left(\partial_i h^{-1}\partial^i h\right) \\ &+\frac{k}{24\pi}\int_{\Sigma} d^3\tilde\sigma\, \epsilon^{\tilde i\tilde j\tilde k}\,\mathrm{Tr}\Big((h^{-1}\partial_{\tilde i}h)(h^{-1}\partial_{\tilde j}h)(h^{-1}\partial_{\tilde k}h)\Big) \end{aligned} \tag{3.1}$$

where h denotes the general element of a (simple) Lie-group $\mathcal{G}$ and Σ a three-manifold whose boundary is the closed string world-sheet. From the WZW sigma model action one can directly read off the metric

$$g = \frac{k}{2}\,\mathrm{Tr}\left(dh^{-1}\otimes dh\right) \tag{3.2}$$

and the NS-NS three-form flux

$$H = \frac{k}{6}\,\mathrm{Tr}\Big((h^{-1}dh)\wedge(h^{-1}dh)\wedge(h^{-1}dh)\Big)\,. \tag{3.3}$$

Here the total derivative is with respect to the target space coordinates. Since this gives a CFT, the metric and the H-flux satisfy the string equations of motion for the metric and the B-field at any power in α', it only needs some additional input to also satisfy the dilaton equation of motion. This can be a linear dilaton $\varphi(z)$ depending on an orthogonal direction (like it appears for the deep throat limit of the NS5-brane solution).

The question which D-branes can be consistently introduced into these closed string backgrounds has been under intensive investigation. Here we just focus on the most simple

[4]Even though we are working with the bosonic string and there are no R-R fields, we call these branes D-branes, as their tension $T\sim g_s^{-1}$ scales with string coupling in the same manner as for D-branes in type II string theory.

set of such branes. Since the WZW model describes a background with a non-trivial B-field, three issues arise.

First, one expects that the effective theory for the gauge field on the brane becomes non-commutative with the non-commutativity being controlled by an antisymmetric bi-vector $\Theta = \Theta^{ij}\partial_i \wedge \partial_j$, which is part of the so-called open string fields,

$$\begin{aligned} &G = g - \mathcal{F}\, g^{-1}\, \mathcal{F}\,, \qquad \Theta = (\mathcal{F} - g\, \mathcal{F}^{-1}\, g)^{-1}\,, \\ &e^{-2\phi}\sqrt{G} = e^{-2\varphi}\sqrt{g}\,. \end{aligned} \tag{3.4}$$

Here $\mathcal{F} = B + 2\pi F$ (with $\alpha' = 1$ and $F = dA$) denotes the gauge invariant open string two-form and G and ϕ are the open string metric and dilaton.

Second, the gauge field A on the brane provides new degrees of freedom that are also governed by equations of motion. Varying the Dirac-Born-Infeld action with respect to A, one arrives at

$$0 = \partial_i\Big(e^{-\varphi}\sqrt{g+\mathcal{F}}\,\Theta^{ij}\Big) = \partial_i\Big(e^{-\phi}\sqrt{G}\,\Theta^{ij}\Big)\,, \tag{3.5}$$

where the indices i, j are along the brane world-volume. Since the DBI action is established only for adiabatic field configurations, there will presumably be higher derivative corrections to this field equation. However, for constant $\mathcal{F}$ it includes all higher order α' corrections.

Third, due to the non-trivial H-flux in the bulk, its restriction on the brane has to satisfy $d\mathcal{F} = H$, i.e. it is a total derivative of a globally defined two-form. Therefore, H must be trivial in the cohomology on the brane $[H|_{\mathrm{D}}] = 0$. This is also called the Freed-Witten anomaly cancellation condition [32]. Note that this does not mean that $H|_{\mathrm{D}}$ has to vanish identically on the brane.

Geometry of D-branes in WZW models. In this section we provide a set of branes for which the geometric semi-classical identification is known [33–36]. Our presentation follows the appendix of [36]. Indeed in the CFT there exist boundary states corresponding to branes wrapping conjugacy classes

$$\mathcal{O}(h) := \{k^{-1}hk, \text{for all } k \in \mathcal{G}\}\,. \tag{3.6}$$

Here, for the element h one can always choose a representative from the Cartan torus $M(\chi) = \exp(i\chi \cdot H)$. Then the position of the brane is labelled by χ and the coordinates along the D-brane can be parametrized by angular variables ψ according to

$$g = N(\psi)^{-1}\, M(\chi)\, N(\psi)\,. \tag{3.7}$$

This provides suitable coordinates for this brane configuration that admits a very explicit description of the geometry and the fluxes. Note that the (generic) dimension of these branes is $d = \dim\mathcal{G} - \mathrm{rk}\mathcal{G}$. Since these configurations do correspond to boundary states in the CFT, one expects that the open string equation of motion and the Freed-Witten anomaly condition are satisfied. Now one defines one-forms $\theta^\alpha = \theta^\alpha{}_i\, d\psi^i$ on the D-brane via

$$dN N^{-1} = \theta^\alpha\, E_\alpha - \theta^{\overline{\alpha}}\, E_{\overline{\alpha}} + i\rho^i H_i \tag{3.8}$$

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where the generators in the Cartan-Weyl basis are normalized as $\mathrm{Tr}(H_i\, H_j) = \delta_{ij}$ and $\mathrm{Tr}(E_\alpha\, E_{\overline{\beta}}) = \delta_{\alpha\beta}$. As usual, the dual vector-fields are given by $\hat{\theta}_\alpha = \hat{\theta}_\alpha{}^i\, \partial_i$ with $\hat{\theta} = (\theta^{-1})^T$. Then one can show that the metric (3.2) restricted to the brane world-volume can be expressed in the nice way

$$g|_{\mathrm{D}} = 2k \sum_{\alpha>0} \sin^2\left(\frac{\alpha\cdot\chi}{2}\right)\left(\theta^\alpha\otimes\theta^{\overline{\alpha}} + \theta^{\overline{\alpha}}\otimes\theta^\alpha\right) \tag{3.9}$$

where the sum is over all positive roots. This form neatly shows the separation of the dependence on the brane positions χ and the angular coordinates ψ along the brane. Moreover, one can choose a gauge so that the NS-NS two-form has legs only along the brane. Indeed, Choosing

$$B = -ik\sum_{\alpha>0}\Big(\alpha\cdot\chi - \sin(\alpha\cdot\chi)\Big)\Big(\theta^\alpha\otimes\theta^{\overline{\alpha}} - \theta^{\overline{\alpha}}\otimes\theta^\alpha\Big) \tag{3.10}$$

gives $H = dB$. Therefore, the restriction of the B-field onto the brane is also given by this expression, i.e. $B|_{\mathrm{D}} = B$. This by itself does not satisfy the open string equation motion (3.5), but has to be supplemented by a non-trivial gauge flux on the D-brane. This is also known quite explicitly as

$$F = \frac{ik}{2\pi}\sum_{\alpha>0}\Big(\alpha\cdot\chi\Big)\Big(\theta^\alpha\otimes\theta^{\overline{\alpha}} - \theta^{\overline{\alpha}}\otimes\theta^\alpha\Big)\,. \tag{3.11}$$

The quantization of the gauge flux fixes $\chi = 2\pi(\lambda+\rho)/k$, where $\rho = \sum_{\alpha>0}\alpha/2$ denotes the Weyl-vector and ρ an element from the weight-lattice. Thus, the total two-form flux on the brane is given by

$$\mathcal{F} = B|_{\mathrm{D}} + 2\pi F = ik\sum_{\alpha>0}\sin(\alpha\cdot\chi)\Big(\theta^\alpha\otimes\theta^{\overline{\alpha}} - \theta^{\overline{\alpha}}\otimes\theta^\alpha\Big)\,. \tag{3.12}$$

Now, one can explicitly compute similar expressions for the fields in the open string frame (3.4). For the metric we find the simple result

$$G = 2k\sum_{\alpha>0}\Big(\theta^\alpha\otimes\theta^{\overline{\alpha}} + \theta^{\overline{\alpha}}\otimes\theta^\alpha\Big) \tag{3.13}$$

and for the anti-symmetric bi-vector

$$\Theta = \frac{ik}{2}\sum_{\alpha>0}\cot\left(\frac{\alpha\cdot\chi}{2}\right)\Big(\hat{\theta}_\alpha\otimes\hat{\theta}_{\overline{\alpha}} - \hat{\theta}_{\overline{\alpha}}\otimes\hat{\theta}_\alpha\Big)\,. \tag{3.14}$$

Note that in the last expression the dual one-vectors $\hat{\theta}_\alpha$ appear. For the dilaton in the open string frame one gets

$$e^{-2\phi} = e^{-2\varphi(z)}\prod_{\alpha>0}\sin^2\left(\frac{\alpha\cdot\chi}{2}\right), \tag{3.15}$$

which does not depend on the coordinates along the brane. As a consequence, the open string equation of motion (3.5) is equivalent to

$$\nabla_i\Theta^{ij} = 0 \tag{3.16}$$

which involves the Levi-Civita connection with respect to the open string metric G.

Example: SU(2) WZW. Let us discuss the most familiar case of the SU(2) WZW model. In this case the target space is an S^3 with a non-constant H-flux through it. Due to the Freed-Witten anomaly condition, it is clear that there does not exist a D-brane wrapping the entire S^3. However, a class of D-branes is given by the orbit $\mathcal{O}(D) := \{k^{-1}Dk\}$ with D denoting an element from the one-dimensional Cartan torus. Therefore, generically this describes a brane wrapping a two-dimensional submanifold of S^3.

To apply the construction from the last section, we introduce a basis of correctly normalized generators of SU(2): $H = \frac{1}{\sqrt{2}}\sigma_3$ and $E_{\alpha(\overline{\alpha})} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ with the positive root $\alpha = \sqrt{2}$. Here σ_i denote the Pauli matrices. The Cartan torus $D(\chi) = \exp(i\chi H)$ is

$$D(\chi) = \begin{pmatrix} e^{i\frac{\chi}{\sqrt{2}}} & 0 \\ 0 & e^{-i\frac{\chi}{\sqrt{2}}} \end{pmatrix} \tag{3.17}$$

and the orthogonal directions to the D-brane can be parametrized by

$$N(\varphi,\psi) = \begin{pmatrix} \cos\varphi & \sin\varphi\, e^{i\psi} \\ -\sin\varphi\, e^{-i\psi} & \cos\varphi \end{pmatrix} \tag{3.18}$$

so that we write an element of SU(2) as $M = N^{-1}D(\chi)N$. Evaluating (3.2), the metric on the SU(2) group manifold reads

$$k^{-1}\, ds^2 = \frac{1}{2}d\chi^2 + 4\sin^2\left(\frac{\chi}{\sqrt{2}}\right)d\varphi^2 + \sin^2\left(\frac{\chi}{\sqrt{2}}\right)\sin^2(2\varphi)\, d\psi^2 \tag{3.19}$$

with $\sqrt{g} = \sqrt{2}\sin^2(\frac{\chi}{\sqrt{2}})\sin(2\varphi)\, R^3$ with the radius $R = \sqrt{k}$. Thus, the semi-classical large radius limit corresponds to $k \to \infty$.

Computing the total volume of the S^3 and comparing to other parametrizations of the SU(2) we can fix the ranges of the variables as $0 \le \chi \le \sqrt{2}\,\pi$, $0 \le \varphi \le \frac{\pi}{2}$ and $0 \le \psi \le 2\pi$. Indeed we get $\int \sqrt{G}\, d\chi d\varphi d\psi = 2\pi^2 R^3$. Next we evaluate (3.3) and obtain

$$H = -k\sqrt{8}\sin^2\left(\frac{\chi}{\sqrt{2}}\right)\sin(2\varphi)\, d\chi\wedge d\varphi\wedge d\psi = -\frac{2}{\sqrt{k}}\mathrm{vol}(S^3) \tag{3.20}$$

so that the flux integral $\frac{1}{(2\pi)^2}\int_{S^3} H = -k$ is indeed quantized and the H-flux goes to zero for large k.

For the holomorphic one-forms (3.8) on the D-brane one obtains

$$\theta^\alpha = e^{i\psi}d\varphi + \frac{i}{2}\sin(2\varphi)\, e^{i\psi}d\psi\,, \qquad \theta^{\overline{\alpha}} = e^{-i\psi}d\varphi - \frac{i}{2}\sin(2\varphi)\, e^{-i\psi}d\psi\,. \tag{3.21}$$

Inserting this into (3.9), we indeed find the metric (3.19) on S^3 restricted to the D-brane world-volume

$$ds^2|_{\text{brane}} = k\,\sin^2\left(\frac{\chi}{\sqrt{2}}\right)\left(4d\varphi^2 + \sin^2(2\varphi)\, d\psi^2\right). \tag{3.22}$$

This metric describes an S^2 of radius $r = R\,\sin(\frac{\chi}{\sqrt{2}})$. As described, for the B-field one can choose a gauge so that it has only legs on the D-brane. For SU(2) this simply reads

$$B = k\left(-\sqrt{2}\chi + \sin\left(\sqrt{2}\chi\right)\right)\sin(2\varphi)\, d\varphi\wedge d\psi \tag{3.23}$$

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and via $H = dB$ indeed gives the H-flux from (3.20). Clearly, the restriction of the H-flux to the brane is vanishing and one can also show that the restriction of the B-field (3.23) to the D-branes does not satisfy the open string equation of motion. However, this latter point can be reconciled by also turning on a non-trivial gauge flux on the brane

$$F = dA = \frac{\sqrt{2}k}{2\pi}\,\chi\,\sin(2\varphi)\,d\varphi\wedge d\psi\,. \tag{3.24}$$

The gauge flux quantization condition $\frac{1}{2\pi}\int F \in \mathbb{Z}$ leads to $\chi = \frac{2\pi}{k}\frac{m}{\sqrt{2}}$ with $0 \le m \le k$, which agrees with the formula below eq. (3.11) by observing that the weight lattice of SU(2) is $\lambda = \mathbb{Z}/\sqrt{2}$ and that its Weyl-vector reads $\rho = 1/\sqrt{2}$. For the two choices $m = 0, k$, the co-dimension one D-brane degenerates to a point-like D-brane sitting at the north- or south-pole of the S^3, respectively.

Now, one can simply proceed by computing the globally defined two-form flux $\mathcal{F} = B + 2\pi F$ on the brane and the open string measure as

$$\mathcal{F} = k\,\sin(\sqrt{2}\chi)\,\sin(2\varphi)\,d\varphi\wedge d\psi\,,\qquad \sqrt{g+\mathcal{F}} = 2k\sin(2\varphi)\sin\left(\frac{\chi}{\sqrt{2}}\right). \tag{3.25}$$

Using the dual holomorphic one-vectors

$$\hat{\theta}_\alpha = \frac{1}{2}e^{-i\psi}\partial_\varphi - \frac{i}{\sin(2\varphi)}\,e^{-i\psi}\,\partial_\psi\,,\qquad \hat{\theta}_{\overline{\alpha}} = \frac{1}{2}e^{i\psi}\partial_\varphi + \frac{i}{\sin(2\varphi)}\,e^{i\psi}\,\partial_\psi \tag{3.26}$$

and evaluating (3.14), we get for the antisymmetric bi-vector Θ^{ij} on the brane

$$\Theta = -\frac{k}{2\sin(2\varphi)}\cot\left(\frac{\chi}{\sqrt{2}}\right)\,\partial_\varphi\wedge\partial_\psi\,. \tag{3.27}$$

Multiplying this with the measure (3.25) one realizes that the φ dependence drops out so that the open string equation of motion (3.5) is trivially satisfied on the D-brane.

Since the brane is two-dimensional, one trivially has $\Pi^{ijk} = 0$, as well as $\nabla_k\Theta^{ij} = 0$. In appendix B we also work out the SU(3) WZW case and show that there one has a co-dimension two D-brane supporting a non-vanishing Π^{ijk}. Therefore, not all brane solutions of the leading order string equations of motion necessarily have $\Pi^{ijk} = 0$ so that in our later analysis we will also admit a non-vanishing Π^{ijk}.

In the semi-classical limit, the two-dimensional world-volume of the D-brane is expected to support a non-commutative (but still associative) gauge theory. Since the world-volume is compact, for fixed but large k, there can only be a finite number of quantum cells so that the non-commutative gauge theory turned out not to be a field theory but rather a matrix theory. Using the operator product expansion of the corresponding vertex operators, this theory has been derived in [22] and, as we discuss next, provides the first non-trivial application of our L_∞ bootstrap program.

3.2 L_∞ algebra for the fuzzy 2-sphere

Let us first briefly review some relevant features of this construction of the NC gauge theory in the fuzzy sphere limit (for a little review see [37])

$$\alpha' \to 0,\quad \alpha' k \to \infty\,. \tag{3.28}$$

This means that one takes the zero-slope and the large radius limit.

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The rational boundary states are known explicitly and the open string excitations at lowest energy are given in terms of the ground states in the open string sector. As discussed above, branes wrapping the conjugacy classes $S^2 \subset S^3$ are labelled by an integer $0 \leq m \leq k$. This integer determines a representation of the SU(2) current algebra. In this section, we use instead the half-integer representation labels j. The open string spectrum can be organized into the $\mathrm{SU}(2)_k$ representations that appear in the fusion product of m with itself

$$(j) \otimes (j) = \oplus_{j'=0}^{2j} (j') \,. \tag{3.29}$$

Here, one was working in the large k limit, such that no truncation appears in the fusion rules. It can now be observed that the representation (j') contains $2j'+1$ ground states $|Y_n^{j'}\rangle$ labeled by j', n with $n \in -j', \ldots, j'$. Geometrically, one can think about them in terms of spherical harmonics on S^2. These finitely many states are analogous to the infinitely many states $\exp(ikX)$ in the flat Moyal-Weyl case.

Note that the space of ground states is $(2j+1)^2$ dimensional. As proposed in [31], these ground states can be identified with square matrices $\mathrm{Mat}(2j+1)$. SU(2) acts on these matrices in the adjoint representation. This representation is not irreducible; decomposing it into irreducible representations reproduces precisely the decomposition (3.29).

Furthermore, there is a product structure on the space of ground states arising from the (truncated) operator product expansion (OPE) between the corresponding vertex operators. Since their conformal weight $h_j = j(j+1)/(k+2)$ goes to zero in the $k \to \infty$ limit, the boundary OPE becomes regular in this limit. As it turns out [22], the information about these OPEs is precisely encoded in the non-commutative matrix product $f \cdot g$ in $\mathrm{Mat}(2j+1)$. This in particular allows to compute the correlation functions of arbitrary vertex operators in terms of traces over products of matrices. The upshot is that one deals with an associative matrix algebra. The matrix product plays the role of the Moyal-Weyl star-product, so that we define

$$:= f \cdot g - g \cdot f \,. \tag{3.30}$$

The above structure includes an action of angular momentum on the spherical harmonics. It is obtained from the OPE between the WZW currents and the vertex operators corresponding to the ground states. From this one obtains for any matrix $A \in \mathrm{Mat}(2j+1)$

$$L_a A = \frac{1}{\sqrt{2}}[Y_a^1, A] \quad , a \in \{1,2,3\} \,. \tag{3.31}$$

In the flat space limit, the operators L_a can be thought of as derivatives $L^a \to -i\partial^a$. However, these operators do not commute but satisfy

$$[L_a, L_b] = i f_{ab}{}^c L_c \tag{3.32}$$

where f_{abc} are the totally antisymmetric SU(2) structure constants.

The effective theory on N branes of type j wrapping this fuzzy 2-sphere was described as a gauge theory with the gauge potential $A_a \in \mathrm{Mat}(N) \otimes \mathrm{Mat}(2j+1)$. Here $\mathrm{Mat}(N)$ labels the Chan-Paton factors. This gauge field has to satisfy the physical state condition $L^a A_a = 0$.

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Using CFT techniques, the effective action was shown to be a sum of a Yang-Mills term and a Chern-Simons term. The two terms are separately invariant under the following gauge transformation

$$\delta_f A_a = iL_a f + i[A_a, f]\,, \tag{3.33}$$

where f is an arbitrary matrix in $\mathrm{Mat}(N)\otimes\mathrm{Mat}(2j+1)$. Note that the derivative operator L_a only acts non-trivially on the degrees of freedom in $\mathrm{Mat}(2j+1)$. The closure of two such gauge variations gives

$$[\delta_f, \delta_g] = \delta_{i[f,g]}\,. \tag{3.34}$$

Introducing a field strength

$$F_{ab} = iL_a A_b - iL_b A_a + i[A_a, A_b] + f_{ab}{}^{c} A_c\,, \tag{3.35}$$

the action can be expressed as

$$S = \frac{1}{4}\mathrm{tr}\Big(F_{ab}\, F^{ab}\Big) - \frac{i}{2}\mathrm{tr}\Big(f^{abc}\,\mathrm{CS}_{abc}\Big) \tag{3.36}$$

with

$$\mathrm{CS}_{abc} = L_a A_b\, A_c + \frac{1}{3}A_a[A_b, A_c] - \frac{i}{2} f_{ab}{}^{d} A_d A_c\,. \tag{3.37}$$

The resulting equation of motion can be written as

$$\begin{aligned} 0 &= L^b F_{ba} + [A^b, F_{ba}] \\ &= iL^b L_b A_a - iL^b(L_a A_b) - f_a{}^{bc} L_b A_c \\ &\quad + iL^b[A_b, A_a] + [A^b, iL_b A_a - iL_a A_b] - f_a{}^{bc}[A_b, A_c] \\ &\quad + i[A^b, [A_b, A_a]]\,. \end{aligned} \tag{3.38}$$

As a first application of our approach, we now show that the form of this NC gauge theory on the fuzzy sphere can be bootstrapped by invoking an L_∞ structure. The computation turns out to be similar to the Moyal-Weyl case but includes some corrections terms that can be traced back to the non-trivial commutator (3.32) of the derivatives. Let emphasize that we proceed not just by simply checking the L_∞ algebra but by bootstrapping the higher products via the L_∞ relations. Of course, one needs some initial information to get started.

As usual, we consider the graded vector space $X_0\oplus X_{-1}\oplus X_{-2}$ with now matrix valued gauge parameters in X_0, gauge fields in X_{-1} and equations of motion in X_{-2}. From the gauge variation (3.33) and the closure condition (3.34) we read-off

$$\ell_1(f) = iL_a f, \qquad \ell_2(f,g) = -i[f,g]\,, \tag{3.39}$$

Then, imposing the L_∞ relation $\mathcal{J}_2(f,g)=0$ fixes

$$\ell_2(f,A) = i[A_a, f]\,. \tag{3.40}$$

From the linear term in the equation of motion (3.38) we read-off

$$\begin{aligned} \ell_1(A) &= \ell_1^{\mathrm{YM}}(A) + \ell_1^{\mathrm{CS}}(A) \\ &= iL^b L_b A_a - iL^b(L_a A_b) - f_a{}^{bc} L_b A_c \end{aligned} \tag{3.41}$$

where, as indicated, the first two terms come from the variation of the YM action and the last term from the variation of the CS action. First, after using (3.32) we realize that

$$\ell_1^{\rm YM}(\ell_1(f)) = -L_a f\,, \qquad \ell_1^{\rm CS}(\ell_1(f)) = L_a f \tag{3.42}$$

so that only the combination of the two kinetic terms satisfies the relation $\mathcal{J}_1(f) = \ell_1(\ell_1(f)) = 0$. Therefore, if one were missing the contribution to the kinetic energy from the CS-term, one would be forced to introduce it by the nilpotency condition in the L_∞ algebra. Moreover, one can further simplify

$$\ell_1(A) = iL^b L_b A_a - iL_a(L^b A_b) \tag{3.43}$$

where the second term actually vanishes by the physical state condition $L^b A_b = 0$. Next, we consider the L_∞ relation $\mathcal{J}_2(f,A) = 0$. A straightforward computation reveals that this can be satisfied by defining

$$\ell_2(f,E) = i[E_a, f] \tag{3.44}$$

and

$$\begin{aligned}\ell_2(A,B) = &-iL^b[A_b, B_a] - i[A^b, L_b B_a - L_a B_b]\\ &+ f_a{}^{bc}[A_b, B_c] + (A \leftrightarrow B)\,.\end{aligned} \tag{3.45}$$

This looks very similar to the Moyal-Weyl case, except for the term in the second line. Note that $-\frac{1}{2}\ell_2(A,A)$ gives precisely the order $O(A^2)$ terms in the equation of motion (3.38), that we bootstrapped from an L_∞ relation.

Next, we observe that the $\ell_2\ell_2$-terms in relations $\mathcal{J}_3(f,g,h) = \mathcal{J}_3(f,g,A) = \mathcal{J}_3(f,g,E) = 0$ involve only matrix commutators so that they can directly be satisfied by setting

$$\ell_3(f,g,A) = \ell_3(f,g,E) = \ell_3(f,A,B) = \ell_3(f,A,E) = 0\,. \tag{3.46}$$

The only non-trivial relation is $\mathcal{J}_3(f,A,B) = 0$. However, one can check that the extra terms coming from the second line in (3.45) cancel against each other so that the computation is analogous to the Moyal-Weyl case presented in section 2.3. Thus, this relation fixes

$$\begin{aligned}\ell_3(A,B,C) = &- i[A^b,[B_b,C_a]] - i[B^b,[C_b,A_a]] - i[C^b,[A_b,B_a]]\\ &- i[A^b,[C_b,B_a]] - i[C^b,[B_b,A_a]] - i[B^b,[A_b,C_a]]\end{aligned} \tag{3.47}$$

which is again consistent with the order $O(A^3)$ term in the equation of motion. From the higher order relations only $\mathcal{J}_4(f,A,B,C) = 0$ is not trivially satisfied, but eventually vanishes by the Jacobi identity of the matrix commutator.

Thus, after taking the initial data $\ell_1(f)$, $\ell_1(A)$ and $\ell_2(f,g)$ we have bootstrapped the remaining terms appearing in the gauge variations and the equations of motion by imposing the relations of an L_∞ algebra. In particular, we found the extra correction terms ($\sim f_{abc}$) in the equations of motion. We consider this as first compelling evidence that the form of NC gauge theories (arising from string theory) is governed by an L_∞ structure. We will continue to elaborate on this idea in section 4.

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3.3 Non-constant Θ via integrable deformations

As we have seen, branes in WZW models can lead to a non-constant Θ on a curved space. However, for compact group manifolds the effective NC-gauge theory in the large volume limit is rather a matrix model than a NC field theory based on a star-product with non-constant Θ.

In this section, we recall that recently string theory examples have been identified that are supposed to give rise to NC-field theories on flat Minkowski space with non-constant Θ. These appeared in the context of integrable deformations of the AdS_5 sigma model [4]. Here, we do not want to review the whole construction and its refinements, as we are only interested in one of its aspects.

It was shown that from certain solutions to the classical Yang-Baxter equation one can extract a closed string metric g_{ij}, Kalb-Ramond field B_{ij} and dilaton ϕ (and R-R fields) that satisfies the string equations of motion and, for the deformation going to zero, gives back the AdS_5 geometry. This construction can be seen as a generalization of the early analysis [38, 39] of the supergravity background dual to the Moyal-Weyl NC-gauge theory.

Expressing the deformed solution in the open string frame [40–43] revealed that the open string metric is still the one on AdS_5, the open string dilaton is constant and the only change is the anti-symmetric bi-vector Θ. There are cases, where the latter restricted to the boundary $\mathbb{R}_{1,3} \subset \mathrm{AdS}_5$ satisfies indeed the open string equation of motion (3.5). Therefore, one expects that the gravity theory in the bulk is dual to a NC-gauge theory on the flat boundary with non-constant Θ^{ij}.

Here, we just present one example that we took from [42, 43]. Taking one specific solution to the classical Yang-Baxter equation, in the open string frame gives a flat metric with

$$\Theta(x) = \begin{pmatrix} 0 & 0 & -\eta x_1 x_3 & \eta x_1 x_2 \\ 0 & 0 & -\eta x_0 x_3 & \eta x_0 x_2 \\ \eta x_1 x_3 & \eta x_0 x_3 & 0 & 0 \\ -\eta x_1 x_2 & -\eta x_0 x_2 & 0 & 0 \end{pmatrix} . \tag{3.48}$$

Here η is the deformation parameter. One can readily check that $\partial_i \Theta^{ij} = 0$ is satisfied for all $j = 0, \ldots, 3$ and that $\Pi^{ijk} = 0$. Therefore, string theory admits solutions that are expected to give rise to NC gauge theories on flat (or curved) spaces. Here we just cite one comment from [40]: 'While likely technically involved, we believe it should in principle be possible to construct (supersymmetric) gauge theories on such non-commutative spaces using the methods developed in [1, 6, 8]'. Our proposal rather is that such theories can be bootstrapped via L_∞ algebras.

4 L_∞ structures for non-constant flux

As we have just seen, the question now arises whether one can deform the Moyal-Weyl case and formulate a consistent NC-gauge theory for non-constant Θ. Motivated also by the previous example of branes in the SU(2) WZW model, we investigate whether the L_∞ bootstrapping approach works and gives reasonable results. In this main section of the paper, we investigate this novel approach for flat/curved space with a completely generic non-constant $\Theta^{ij}(x)$ with even non-vanishing Π^{ijk}.

First we consider just the action of the gauge symmetry and its closure, i.e. we construct the corresponding $\mathrm{L}_\infty^{\text{gauge}}$ algebra. In appendix B we explicitly show that this can be refined to an $\mathrm{A}_\infty^{\text{gauge}}$ algebra. First, let us explain that for non-constant flux a serious problem appears, for which L_∞ offers a new solution.

4.1 An issue for non-constant Θ

In the previous computations in section 2.2 it was essential that the non-commutativity structure Θ^{ij} was constant. For non-constant Θ^{ij}, even at linear order one runs into the following issue concerning the Leibniz-rule. Consider the generic star-product between functions

$$f \bullet g = f \cdot g + \frac{i}{2}\Theta^{ij}(x)\,\partial_i f\,\partial_j g + O(\Theta^2) \tag{4.1}$$

and apply ∂_a. One finds

$$\partial_a(f \bullet g) = \partial_a f \bullet g + f \bullet \partial_a g + \frac{i}{2}(\partial_a \Theta^{ij})\,\partial_i f \partial_j g + O(\Theta^2)\,, \tag{4.2}$$

i.e. the derivative does not satisfy the usual Leibniz-rule with respect to the star-product. One way to resolve this issue is to generalize the co-product, leading to the general structure of a Hopf-algebra. For this purpose one recalls that for the usual product of functions $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ with

$$\mu(f \otimes g) = f \cdot g \tag{4.3}$$

the enveloping algebra $\mathcal{H}$ of the variations $\delta_a = \partial_a$ defines a Hopf-algebra with product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$

$$m(\delta_a \otimes \delta_b) = \delta_a\,\delta_b \tag{4.4}$$

and co-product $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

$$\Delta(\delta_a) = \delta_a \otimes 1 + 1 \otimes \delta_a \tag{4.5}$$

where we have used the action $\mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$ with $\delta_a(f) = \partial_a f$. For consistency, the co-product should be co-associative,

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta\,, \tag{4.6}$$

as well as admit a co-unit and an antipode, see e.g. [13] for details. Then, the Leibniz-rule can be abstractly written as

$$\delta_a(\mu(f \otimes g)) = \mu \circ \Delta(\delta_a)(f \otimes g)\,. \tag{4.7}$$

The relation (4.2) can be written in an analogous manner by defining the new product

$$\mu^\star(f \otimes g) = f \bullet g \tag{4.8}$$

between functions and the adjusted or deformed co-product $\Delta^\star(\delta_a)$ for elements in $\mathcal{H}$. Then (4.2) can still be written as

$$\delta_a(\mu^\star(f\otimes g)) = \mu^\star \circ \Delta^\star(\delta_a)(f\otimes g)\,. \tag{4.9}$$

Note, that the consistent definition of the deformed co-product satisfying the co-associativity condition (4.6) and the relation (4.9) is a highly nontrivial problem. The known solution is to use an invertible element $\mathscr{F}\in\mathcal{H}\otimes\mathcal{H}$, called a twist, with the help of which the original star product can be represented as

$$f\bullet g = \mu^\star(f\otimes g) = \mu\circ\mathscr{F}^{-1}(f\otimes g)\ . \tag{4.10}$$

Then the deformed co-product is given by

$$\Delta^\star(\delta_a) = \mathscr{F}\Delta(\delta_a)\mathscr{F}^{-1}\ . \tag{4.11}$$

However, only very few examples of star products originating from a twist are known.

Another point which should be mentioned here is that the deformed co-product is still co-associative and that is why in the Hopf-algebra approach no higher products or brackets are needed to compensate the violation of the original Leibniz rule. This is the key difference with our proposal in this paper. Nevertheless, we leave for the future a better understanding of the relation between our approach and other previous approaches to the construction of non-commutative gauge theories.

In view of the proposal that generic (gauge) symmetries in string theory are related to L_∞ structures, let us have a second look at the violation of the naive Leibniz-rule (4.2). Recall that for NC-Yang-Mills theory we found $\ell_1(f) = \partial_a f$. If we define $\ell_2(f,g) = i[f,g]_\bullet = i(f\bullet g - g\bullet f)\in X_0$ then by anti-symmetrization the relation (4.2) is closely related to

$$\begin{aligned}\ell_1(\ell_2(f,g)) &= i[\overbrace{\ell_1(f)}^{\in X_{-1}},g]_\bullet + i[f,\overbrace{\ell_1(g)}^{\in X_{-1}}]_\bullet - (\partial_a\Theta^{ij})\,\partial_i f\partial_j g + O(\Theta^2)\,,\\ &= \ell_2(\ell_1(f),g) + \ell_2(f,\ell_1(g))\,.\end{aligned} \tag{4.12}$$

From this point of view, the correction term $\partial\Theta$ only indicates that we should better not define $\ell_2(f,A) = i[f,A]_\bullet\in X_{-1}$ but instead

$$\ell_2(f,A) = i[f,A_a]_\bullet - \frac{1}{2}(\partial_a\Theta^{ij})\,\partial_i f A_j + O(\Theta^2)\,. \tag{4.13}$$

Note that in the L_∞ algebra, $\ell_2(f,g)\in X_0$ and $\ell_2(f,A)\in X_{-1}$ are a priori *different* products. Thus, we can still satisfy the usual L_∞ Leibniz-rule by changing the action of the NC-gauge symmetry on the gauge fields

$$\begin{aligned}\delta_f A &= \ell_1(f) + \ell_2(f,A) + \dots\\ &= \partial_a f + i[f,A_a]_\bullet - \frac{1}{2}(\partial_a\Theta^{ij})\,\partial_i f A_j + \dots\,.\end{aligned} \tag{4.14}$$

By construction, this guarantees that the closure condition

$$A = \delta_{-i[f,g]_\bullet} A \tag{4.15}$$

is indeed satisfied up to linear order in Θ.

Proceeding in this way, the higher products of the L_∞ algebra will receive higher derivative corrections, leading to corrections to the action of the symmetries on the gauge fields and eventually to the equations of motion. This latter approach seems to be completely different from the resolution of the problem via twisted symmetries and Hopf-algebras and much closer to the structure of symmetries in string theory. In the remainder of this section, we will work this out in more detail and compute the corresponding derivative corrections to $\mathrm{L}_\infty^{\mathrm{gauge}}$ up to second order in Θ. At this order also the non-associativity enters.

4.2 $\mathrm{L}_\infty^{\mathrm{gauge}}$ algebra at order $O(\Theta^2)$

The resolution of the Leibniz-rule was done at linear order in Θ and it is of course not clear whether the procedure can be consistently continued to higher orders in Θ. To get some confidence, starting with the Kontsevich star-product (2.3) at second order in Θ, in this section we construct the corresponding $\mathrm{L}_\infty^{\mathrm{gauge}}$ algebra.

Thus, we have the vector space $X_0 \oplus X_{-1}$ containing gauge parameters and gauge fields still choose

$$\ell_1(f) = \partial_a f\,. \tag{4.16}$$

Moreover, for $\ell_2(f,g)$ we want to have the Kontsevich star-commutator

$$\ell_2(f,g) = i(f\bullet g - g\bullet f) = -\Theta^{ij}\partial_i f\partial_j g + O(\Theta^3)\,. \tag{4.17}$$

Note that the even order terms in Θ drop out in the star-commutator. Therefore, the analysis of the Leibniz rule $\mathcal{J}_2(f,g)=0$ from the previous section is still valid and we get

$$\ell_2(f,A) = i[f,A_a]_\bullet - \frac{1}{2}(\partial_a\Theta^{ij})\,\partial_i f A_j + O(\Theta^3)\,. \tag{4.18}$$

Next, we have to impose the L_∞ relations $\mathcal{J}_3(f,g,h)=0$ and $\mathcal{J}_3(f,g,A)=0$. The first relation explicitly reads

$$\begin{aligned} 0 = {} & \ell_2(\ell_2(f,g),h) + \ell_2(\ell_2(g,h),f) + \ell_2(\ell_2(h,f),g) \\ & + \ell_3(\ell_1(f),g,h) + \ell_3(f,\ell_1(g),h) + \ell_3(f,g,\ell_1(h))\,. \end{aligned} \tag{4.19}$$

The first line is just the Jacobiator for the star-commutator and yields

$$-\Pi^{ijk}\partial_i f\partial_j g\partial_k h \tag{4.20}$$

that we do not require to be vanishing. Taking into account the graded symmetry of the brackets we find

$$\ell_3(A,f,g) = \frac{1}{3}\Pi^{ijk}A_i\partial_j f\partial_k g\,, \tag{4.21}$$

Next, we have to analyze the relation $\mathcal{J}_3(f,g,A)=0$, which is explicitly given by

$$\begin{aligned}0 &= \ell_2(\ell_2(A,f),g)+\ell_2(\ell_2(f,g),A)+\ell_2(\ell_2(g,A),f)\\ &\quad+\ell_1(\ell_3(A,f,g))-\ell_3(A,\ell_1(f),g)-\ell_3(A,f,\ell_1(g))\end{aligned} \tag{4.22}$$

where we used $\ell_1(A)=0$. The last two terms involve the three-product $\ell_3(A,B,f)$ that needs to be determined from this relation. For this purpose, we calculate

$$\begin{aligned}&\ell_2(\ell_2(A,f),g)+\ell_2(\ell_2(f,g),A)+\ell_2(\ell_2(g,A),f)+\ell_1(\ell_3(A,f,g))\\ &\qquad=-\frac{1}{2}G_a{}^{ijk}\,A_i\,\partial_j f\,\partial_k g-\Pi^{ijk}\,\partial_i A_a\,\partial_j f\,\partial_k g+\frac{1}{3}\Pi^{ijk}\,\partial_a A_i\,\partial_j f\,\partial_k g\\ &\qquad\quad+\frac{1}{3}\Pi^{ijk}\,A_i\,\partial_a\partial_j f\,\partial_k g+\frac{1}{3}\Pi^{ijk}A_i\,\partial_j f\,\partial_a\partial_k g\end{aligned} \tag{4.23}$$

with

$$G_a{}^{ijk}=\frac{1}{3}\partial_a\Pi^{ijk}-\Theta^{im}\partial_m\partial_a\Theta^{jk}-\frac{1}{2}\partial_a\Theta^{jm}\partial_m\Theta^{ki}-\frac{1}{2}\partial_a\Theta^{km}\partial_m\Theta^{ij} \tag{4.24}$$

that apparently satisfies the relation

$$G_a{}^{ijk}+G_a{}^{kij}+G_a{}^{jki}=0\,. \tag{4.25}$$

Taking into account the symmetry $\ell_3(A,B,f)=\ell_3(B,A,f)$, (4.23) motivates to make the ansatz

$$\begin{aligned}\ell_3(A,B,f)&=\alpha(G_a{}^{ijk}+G_a{}^{jik})A_iB_j\partial_k f\\ &\quad+\beta\Pi^{ijk}(\partial_a A_iB_j\partial_k f-A_i\partial_a B_j\partial_k f)\\ &\quad+\gamma\Pi^{ijk}(\partial_i A_aB_j\partial_k f-A_i\partial_j B_a\partial_k f)\,.\end{aligned} \tag{4.26}$$

Inserting this into the L_∞ relation (4.19), using (4.25) and comparing coefficients we find

$$\alpha=-\frac{1}{6}\,,\qquad \beta=\frac{1}{6}\,,\qquad \gamma=-\frac{1}{2}\,. \tag{4.27}$$

Note that for β and γ we have 3 different equations for the two unknowns. Moreover, the relation (4.25) was used to solve for α, and is in fact the consistency condition for the existence of the solution of the equation (4.19), see [25] for more details. Thus, it is highly non-trivial that indeed the L_∞ relation $\mathcal{J}_3(f,g,A)=0$ can be satisfied via

$$\begin{aligned}\ell_3(A,B,f)=&-\frac{1}{6}\Big(G_a{}^{ijk}+G_a{}^{jik}\Big)A_iB_j\partial_k f\\ &+\frac{1}{6}\Pi^{ijk}(\partial_a A_iB_j\partial_k f-A_i\partial_a B_j\partial_k f)\\ &-\frac{1}{2}\Pi^{ijk}(\partial_i A_aB_j\partial_k f-A_i\partial_j B_a\partial_k f)\,.\end{aligned} \tag{4.28}$$

Note that, as opposed to $\ell_3(A,f,g)$, the three-product $\ell_3(A,B,f)$ is non-vanishing in the associative case, either. Recall that our computation was exact only up to second order in Θ. Setting now all higher products to zero up to this order, all higher relations $\mathcal{J}_n=0$ for

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$n \geq 4$ are automatically satisfied, as well. This is because $\ell_2\ell_3$ is already third order in Θ. Therefore, for non-constant Θ we have constructed a consistent $\mathrm{L}_\infty^{\mathrm{gauge}}$ algebra, for which derivative $\partial\Theta$ corrections induce non-vanishing higher products (even in the associative case). This is very compelling, as in the course of this computation there arose non-trivial consistency conditions that just happened to be satisfied.

We also analyzed whether the gauge structure features an underlying A_∞ algebra. Since the higher products are not any longer graded symmetric, for that purpose one has to determine many more individual higher products that are also constrained by more A_∞ relations. In this sense, an A_∞ structure can be considered as a refinement of an L_∞ structure. Since the computations turned out to be quite lengthy and involved, we delegated the presentation of the positive results into appendix B.

Let us proceed and extend the former $\mathrm{L}_\infty^{\mathrm{gauge}}$ algebra for the action of a gauge symmetry on gauge fields by a vector space X_{-2} that contains the equation of motion of a star-deformed 3D Chern-Simons theory and a star-deformed U(1) Yang-Mills theory.

4.3 NC Chern-Simons theory

Let us first consider a NC-CS theory. Since a metric does not appear neither in the Kontsevich star-product nor in the topological CS-theory, we suspect that the following considerations are valid irrespective of a metric. Let us first consider the appearing structure up to linear order in Θ. From the former discussion, we expect to find also here $\partial\Theta$ corrections.

As said, we have three vector spaces like

$$\begin{array}{ccc} X_0 & X_{-1} & X_{-2} \\ f & A_a & E_a \end{array} \tag{4.29}$$

with the following derivatives

$$\ell_1(f) = \partial_a f\,, \qquad \ell_1(A) = \epsilon_c{}^{ab}\,\partial_a A_b \tag{4.30}$$

that clearly satisfy $\ell_1(\ell_1(f)) = 0$. With

$$\ell_2(f,g) = i[f,g]_\bullet = -\Theta^{ij}\partial_i f \partial_j g + O(\Theta^2) \tag{4.31}$$

we have already seen that the Leibniz-rule $\ell_1(\ell_2(f,g)) = \ldots$ fixes

$$\ell_2(f,A) = i[f,A_a]_\bullet - \frac{1}{2}\partial_a\Theta^{ij}\partial_i f A_j + O(\Theta^2)\,. \tag{4.32}$$

Next, one has to check the Leibniz rule

$$\ell_1(\ell_2(f,A)) = \ell_2(\ell_1(f),A) + \ell_2(f,\ell_1(A))\,. \tag{4.33}$$

By making an ansatz for $\ell_2(A,B)$ and $\ell_2(f,E)$, using that the former is symmetric and fixing the coefficients one finds

$$\ell_2(f,E) = i[f,E_a]_\bullet + O(\Theta^2) \tag{4.34}$$

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and

$$\ell_2(A,B) = \epsilon_c{}^{ab}\, i[A_a, B_b]_\bullet - \epsilon_c{}^{ab}\partial_a\Theta^{ij}\Big(A_i\partial_j B_b + B_i\partial_j A_b\Big) + \frac{1}{2}\epsilon_c{}^{ab}\partial_a\Theta^{ij}\Big(A_i\partial_b B_j + B_i\partial_b A_j\Big) + O(\Theta^2)\,. \tag{4.35}$$

Note that indeed $\ell_2(A,B)$ is symmetric under exchange of the two arguments and that $\ell_2(f,E)$ does not receive any $\partial\Theta$ correction. The Leibniz-rules for $(AA),(fE),(AE)$ are trivially satisfied, as they lie in trivial vector spaces X_{-3} and X_{-4}. Since all ℓ_2 products are linear in Θ, all $\ell_2 \circ \ell_2$ relations are trivially satisfied up to linear order in Θ.

Therefore, by requiring the consistency of an underlying L_∞ algebra for the case of a Θ deformed 3D Chern-Simons theory, we have extracted a $\partial\Theta$ correction to the equation of motion

$$\begin{aligned}\mathcal{F} &= \ell_1(A) - \frac{1}{2}\ell_2(A,A) + O(\Theta^2)\\ &= \epsilon_c{}^{ab}\bigg(\partial_a A_b - \frac{i}{2}[A_a, A_b]_\bullet + \partial_a\Theta^{ij}\Big(A_i\partial_j A_b - \frac{1}{2}A_i\partial_b A_j\Big)\bigg) + O(\Theta^2)\,.\end{aligned} \tag{4.36}$$

We find it tantalizing that the algebraic structure of an L_∞ algebra allowed one to fix derivative corrections to the equations of motion of a non-commutative CS theory. Of course, so far this computation is only up to linear order in Θ, but we conjecture that it can be extended in a consistent way to higher orders. As a non-trivial check let us now consider the corrections at second order in Θ. This is the first instance where also the associators appear.

NC-CS at order $O(\Theta^2)$. The star commutator to this order remains unchanged, so do the previously defined structures for ℓ_2. However, at this order there appear higher brackets ℓ_3. The expressions for $\ell_3(A,f,g)$ and $\ell_3(A,B,f)$ were found in section 3.1. Taking into account that now X_{-2} is non trivial, one may also have non-vanishing $\ell_3(E,f,g) \in X_{-1}$, $\ell_3(E,A,f) \in X_{-2}$ and $\ell_3(A,B,C) \in X_{-2}$.

Let us start with $\ell_3(E,f,g)$. Such a term contributes to the closure condition $\mathcal{J}_3(f,g,A) = 0$, which are however satisfied without it. Therefore, we can set $\ell_3(E,f,g) = 0$. Next we consider $\mathcal{J}_3(E,f,g) = 0$

$$\begin{aligned}0 &= \ell_2(\ell_2(E,f),g) + \ell_2(\ell_2(g,E),f) + \ell_2(\ell_2(f,g),E)\\ &\quad + \ell_3(E,\ell_1(f),g) + \ell_3(E,f,\ell_1(g))\end{aligned} \tag{4.37}$$

from which we derive

$$\ell_3(E,A,f) = \frac{1}{2}\,\Pi^{ijk}\partial_i E_a A_j \partial_k f\,. \tag{4.38}$$

It is understood, that all expressions are (only) correct up to order $O(\Theta^2)$. Finally, to determine $\ell_3(A,B,C)$, we consider $\mathcal{J}(A,B,f)$ and write is as

$$\begin{aligned}\ell_3(A,B,\ell_1(f)) = &-\ell_1(\ell_3(A,B,f)) - \ell_3(\ell_1(A),B,f) + \ell_3(A,\ell_1(B),f)\\ &-\ell_2(\ell_2(A,B),f) - \ell_2(\ell_2(f,A),B) + \ell_2(\ell_2(B,f),A)\,.\end{aligned} \tag{4.39}$$

The right hand side of this relation is quite involved, so we follow the same strategy as in the section 4.2 and collect structures with the same number of derivatives acting on the arguments A, B and f. In principle we can get terms from one up to four derivates so that we write

$$\ell_3(A,B,\ell_1(f)) = \sum_{N=1}^{4} \ell_3^{(N)}(A,B,\ell_1(f))\,. \tag{4.40}$$

After a tedious computation we find a couple of cancellations and simplifications. First we obtain that two of the four terms do vanish

$$\ell_3^{(1)}(A,B,\ell_1(f)) = \ell_3^{(4)}(A,B,\ell_1(f)) = 0\,. \tag{4.41}$$

The term with two derivatives using the relation (4.25) can be written in the convenient way

$$\begin{aligned}
\ell_3^{(2)}(A,B,\ell_1(f)) = &-\epsilon_c{}^{ab}\Big(\frac{1}{3}\Theta^{km}\partial_b\partial_m\Theta^{ij} - \frac{1}{6}\partial_b\Theta^{km}\partial_m\Theta^{ij} + (j\leftrightarrow k)\Big)\\
&\Big(\partial_a A_i B_j \partial_k f + A_j \partial_a B_i \partial_k f\Big)\\
&-\epsilon_c{}^{ab}\Big(-\frac{1}{2}\Theta^{km}\partial_b\partial_m\Theta^{ij} + \frac{1}{2}\partial_b\Theta^{km}\partial_m\Theta^{ij} + (j\leftrightarrow\ \Big)\\
&\Big(\partial_i A_a B_j \partial_k f + A_j \partial_i B_a \partial_k f\Big)\\
&-\epsilon_c{}^{ab}\Big(-\frac{1}{6}\Theta^{km}\partial_b\partial_m\Theta^{ij} + \frac{1}{3}\partial_b\Theta^{km}\partial_m\Theta^{ij} + (j\leftrightarrow k)\Big)\\
&\Big(A_j B_k \partial_a \partial_i f\Big)\\
&-\epsilon_c{}^{ab}\Big(\frac{1}{2}\partial_a\Theta^{ij}\partial_b\Theta^{kl}\Big)\\
&\Big((\partial_i A_k - \partial_k A_i)B_j\partial_l f + A_j(\partial_i B_k - \partial_k B_i)\partial_l f\Big)\,.
\end{aligned} \tag{4.42}$$

Note that this is explicitly symmetric under the exchange of the gauge fields A and B. Moreover, the numerical prefactors are just right to directly read off a totally symmetric three product

$$\boxed{\begin{aligned}
\ell_3^{(2)}(A,B,C) = &-\epsilon_c{}^{ab}\Big(\frac{1}{3}\Theta^{km}\partial_b\partial_m\Theta^{ij} - \frac{1}{6}\partial_b\Theta^{km}\partial_m\Theta^{ij} + (j\leftrightarrow k)\Big)\\
&\Big(\partial_a A_i B_j C_k + A_j \partial_a B_i C_k + A_k B_j \partial_a C_i\Big)\\
&-\epsilon_c{}^{ab}\Big(-\frac{1}{2}\Theta^{km}\partial_b\partial_m\Theta^{ij} + \frac{1}{2}\partial_b\Theta^{km}\partial_m\Theta^{ij} + (j\leftrightarrow k)\Big)\\
&\Big(\partial_i A_a B_j C_k + A_j \partial_i B_a C_k + A_k B_j \partial_i C_a\Big)\\
&-\epsilon_c{}^{ab}\Big(\frac{1}{2}\partial_a\Theta^{ij}\partial_b\Theta^{kl}\Big)\\
&\Big((\partial_i A_k - \partial_k A_i)B_j C_l + A_j(\partial_i B_k - \partial_k B_i)C_l\\
&\quad + A_l B_j(\partial_i C_k - \partial_k C_i)\Big)\,.
\end{aligned}} \tag{4.43}$$

Next we come to the contribution with three derivatives. Here one has essentially two terms, one proportional to Π^{ijk} and one that does not vanish in the associative case

$$
\begin{aligned}
\ell_3^{(3)}(A,B,\ell_1(f)) = \epsilon_c{}^{ab}\Pi^{ijk}\Big(& \frac{1}{3}\partial_a A_i \partial_b B_j \partial_k f + \partial_i A_a \partial_j B_b \partial_k f \\
& + \frac{1}{6}\big(\partial_a A_i B_j \partial_b \partial_k f + A_j \partial_a B_i \partial_b \partial_k f\big) \\
& - \frac{1}{2}\big(\partial_i A_a B_j \partial_b \partial_k f + A_j \partial_i B_a \partial_b \partial_k f\big) \\
& - \frac{1}{2}\big(\partial_i A_a \partial_b B_j \partial_k f + \partial_b A_j \partial_i B_a \partial_b \partial_k f\big)\Big) \\
-\epsilon_c{}^{ab}\Theta^{kl}\partial_b\Theta^{ij}\Big(& \frac{1}{2}\big(\partial_k A_a B_i \partial_j \partial_l f + A_i \partial_k B_a \partial_j \partial_l f\big) \\
& - \frac{1}{2}\big(\partial_k A_a \partial_l B_i \partial_j f + \partial_l A_i \partial_k B_a \partial_j f\big) \\
& - \frac{1}{2}\big(\partial_l A_i B_j \partial_k \partial_a f + A_j \partial_l B_i \partial_k \partial_a f\big)\Big)\,.
\end{aligned}
\tag{4.44}
$$

Again the relative coefficients are just right to define a totally symmetric three-product

$$
\boxed{
\begin{aligned}
\ell_3^{(3)}(A,B,C) = {} & \frac{1}{3}\epsilon_c{}^{ab}\Pi^{ijk}\Big(\partial_a A_i \partial_b B_j C_k + A_i \partial_a B_j \partial_b C_k + \partial_b A_i B_j \partial_a C_k\Big) \\
& + \epsilon_c{}^{ab}\Pi^{ijk}\Big(\partial_i A_a \partial_j B_b C_k + A_i \partial_j B_a \partial_k C_b + \partial_i A_b B_j \partial_k C_a\Big) \\
& - \frac{1}{2}\epsilon_c{}^{ab}\Pi^{ijk}\Big(\partial_i A_a \partial_b B_j C_k + A_i \partial_j B_a \partial_b C_k + \partial_b A_i B_j \partial_k C_a \\
& + \partial_a A_i \partial_j B_b C_k + A_i \partial_a B_j \partial_k C_b + \partial_i A_b B_j \partial_a C_k\Big) \\
& + \frac{1}{2}\epsilon_c{}^{ab}\Theta^{kl}\partial_b\Theta^{ij}\Big(\partial_k A_a(\partial_l B_i C_j - B_i \partial_l C_j) + \partial_k B_a(\partial_l A_i C_j - A_i \partial_l C_j) \\
& + \partial_k C_a(\partial_l A_i B_j - A_i \partial_l B_j)\Big)\,.
\end{aligned}
}
\tag{4.45}
$$

Therefore invoking the L_∞ relations, in (4.43) and (4.45) we have determined a totally symmetric three-product $\ell_3(A,B,C) = \ell_3^{(2)}(A,B,C) + \ell_3^{(3)}(A,B,C)$. Note that all the formerly determined higher products went into this computation and that all the prefactors just came out right to fit into the L_∞ structure. This procedure works both for the associative and the non-associative cases. Even in the associative case the three-product is non-vanishing and receives derivative corrections.

Moreover, all higher L_∞ relations are automatically satisfied at $O(\Theta^2)$. This is all very compelling and makes us believe that this procedure can be continued to higher orders in Θ. As a consequence, up to quadratic order, the equation of motion of the NC-CS gauge theory reads

$$
\begin{aligned}
\mathcal{F} &= \ell_1(A) - \frac{1}{2}\ell_2(A^2) - \frac{1}{3!}\ell_3(A^3) + O(\Theta^3) \\
&= \epsilon_c{}^{ab}\Big[\partial_a A_b - \frac{i}{2}[A_a,A_b]_\bullet + \partial_a\Theta^{ij}\Big(A_i\partial_j A_b - \frac{1}{2}A_i \partial_b A_j\Big) \\
&\quad + \frac{1}{6}\Big(\Theta^{km}\partial_b\partial_m\Theta^{ij} - \frac{1}{2}\partial_b\Theta^{km}\partial_m\Theta^{ij} + (j \leftrightarrow k)\Big)\Big(\partial_a A_i A_j A_k\Big)
\end{aligned}
$$

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$$
\begin{aligned}
&-\frac{1}{4}\Big(\Theta^{km}\partial_b\partial_m\Theta^{ij}-\partial_b\Theta^{km}\partial_m\Theta^{ij}+(j\leftrightarrow k)\Big)\Big(\partial_iA_aA_jA_k\Big) \qquad (4.46)\\
&+\frac{1}{2}\Big(\partial_a\Theta^{ij}\partial_b\Theta^{kl}\Big)\Big(\partial_iA_kA_jA_l\Big)\\
&-\frac{1}{2}\Pi^{ijk}\Big(\frac{1}{3}\partial_aA_i\partial_bA_jA_k+\partial_iA_a(\partial_jA_b-\partial_bA_j)A_k\Big)\\
&-\frac{1}{2}\Big(\Theta^{kl}\partial_b\Theta^{ij}\Big)\Big(\partial_kA_a\partial_lA_iA_j\Big)\Big]+O(\Theta^3)\,.
\end{aligned}
$$

Let us emphasize that this is designed such that it transforms covariantly under an $(\partial\Theta)$ corrected NC gauge transformations

$$
\begin{aligned}
\delta_fA_a&=\ell_1(f)+\ell_2(f,A)-\frac{1}{2}\ell_3(f,A,A)+O(\Theta^3)\\
&=\partial_af+i[f,A_a]_\bullet-\frac{1}{2}\partial_a\Theta^{ij}\partial_ifA_j\\
&\quad-\frac{1}{6}\Big(\Theta^{im}\partial_a\partial_m\Theta^{jk}-\frac{1}{2}\partial_a\Theta^{im}\partial_m\Theta^{jk}\Big)A_iA_j\partial_kf\\
&\quad-\frac{1}{2}\Pi^{ijk}\Big(\frac{1}{3}\partial_aA_iA_j\partial_kf-\partial_iA_aA_j\partial_kf\Big)+O(\Theta^3)\,.
\end{aligned}
\tag{4.47}
$$

Comments on an action. After we have successfully derived the equations of motion for the NC-CY theory up to order Θ^2, one can ask whether these can be integrated to an action. Just to show what kind of issues appear, here we consider the simple case of the equations of motion up to linear order in Θ. Recall that the equation of motion is

$$
\begin{aligned}
\mathcal{F}&=\ell_1(A)-\frac{1}{2}\ell_2(A,A)+O(\Theta^2) \qquad (4.48)\\
&={\epsilon_c}^{ab}\Big(\partial_aA_b+\frac{1}{2}\Theta^{ij}\partial_iA_a\partial_jA_b+\partial_a\Theta^{ij}\Big(A_i\partial_jA_b-\frac{1}{2}A_i\partial_bA_j\Big)\Big)+O(\Theta^2)\,.
\end{aligned}
$$

As mentioned in section 2.2, for defining an action one needs a inner product, which we choose to be the same one as for the Moyal-Weyl case

$$
\langle A,E\rangle=\int d^3x\,\eta^{ab}A_a\,E_b\,. \tag{4.49}
$$

For the action we also use the same form

$$
S=\frac{1}{2}\langle A,\ell_1(A)\rangle-\frac{1}{3!}\langle A,\ell_2(A,A)\rangle+O(\Theta^2)\,. \tag{4.50}
$$

Varying now this action with respect to the gauge field one has to do a partial integration. In doing this, we require

$$
\partial_i\Theta^{ij}=0 \tag{4.51}
$$

which can be considered as the open string equation of motion for a flat brane or a natural topological generalization of the latter. Thus, after variation we can express the result as

$$\begin{aligned}\delta S = \int d^3x \, \Big[& \delta A_c \, \epsilon^{abc} \Big(\partial_a A_b + \frac{1}{2} \Theta^{ij} \partial_i A_a \partial_j A_b \Big) \\ & + \delta A_c \, \epsilon^{abc} \partial_a \Theta^{ij} \Big(A_i \partial_j A_b - \frac{1}{2} A_i \partial_b A_j \Big) \\ & - \delta A_c \, \epsilon^{abc} \partial_a \Theta^{ij} \Big(\frac{1}{3} A_i (\partial_j A_b - \partial_b A_j) - \frac{1}{6} A_b (\partial_j A_i - \partial_i A_j) \Big) \\ & + \delta A_i \, \epsilon^{abc} \partial_a \Theta^{ij} \Big(\frac{1}{3} A_c (\partial_j A_b - \partial_b A_j) - \frac{1}{12} A_j (\partial_b A_c - \partial_c A_b) \Big) \Big] \, .\end{aligned} \tag{4.52}$$

Let us make two remarks: first, the first four terms are already the ones appearing in the equation of motion. Second, from the variation one gets extra terms, e.g. those multiplying δA_i. However, all these terms explicitly contain a factor of $\partial\Theta$ and are proportional to $\partial_k A_l - \partial_l A_k$. Since the leading order equation of motion tells us that this combination vanishes, all the terms like $\partial\Theta(\partial_k A_l - \partial_l A_k)$ are already of order Θ^2 and can be safely dropped from the equation of motion at first order in Θ. Therefore, only after carefully working at linear order in Θ one gets the equation of motion (4.48) from the action (4.50). This means that the inner product itself does not satisfy the cyclicity property (2.33).

It would be interesting to see whether such a reasoning also works up to second order in Θ or whether also the inner product receives some derivative $\partial\Theta$ corrections. We leave this involved study for future research.

4.4 NC Yang-Mills theory

So far we have considered NC deformations of the topological CS-theory. To really get into contact with the D-branes appearing as solutions of string theory, we finally consider non-commutative U(1) YM theory. Since in this case, the computations turn out to be more involved than in the CS case, we restrict ourselves in this section only to linear order in Θ and leave the generalization to higher orders for the future.

Since the equation of motion involves the metric, we first discuss the simplest case of a flat background with a non-constant NC-structure Θ and then generalize this to a curved background of metric g_{ij} and non-constant Θ.

NC-YM on flat background. Recall from section 2.3 that for the Moyal-Weyl star-product we have three vector spaces like

$$\begin{array}{ccc} X_0 & X_{-1} & X_{-2} \\ f & A_a & E_a \end{array} \tag{4.53}$$

with the following ℓ-products

$$\begin{aligned} \ell_1(A) &= \Box A_a - \partial_a(\partial \cdot A) \\ \ell_2^{(0)}(A,B) &= i[\partial \cdot A, B_a]_\star + i[A_k, \partial^k B_a]_\star + i[A^k, \partial_k B_a - \partial_a B_k]_\star + (A \leftrightarrow B) \\ \ell_3(A,B,C) &= [A^k, [B_k, C_a]_\star]_\star + 5 \text{ terms} \, . \end{aligned} \tag{4.54}$$

Note that at linear order in Θ the three-product $\ell_3(A,B,C)$ is vanishing. As for the NC-CS theory, imposing the Leibniz-rule we expect to get a derivative correction $\ell_2^{(1)}(A,B)$. Indeed, going through the computation we arrive at the familiar expression

$$\ell_2(f,E) = i[f,E_a]_\star \tag{4.55}$$

and the remaining term

$$\begin{aligned}
\ell_2^{(1)}(\ell_1(f),A) = &-\Box\Theta^{ij}\Big(\partial_i f\partial_j A_a - \frac{1}{2}\partial_a\partial_i f A_j - \frac{1}{2}\partial_i f\partial_a A_j\Big) \\
&+ \big(\partial^k\partial_a\Theta^{ij}\big)\Big(\partial_i f\partial_j A_k - \frac{1}{2}\partial_i\partial_k f A_j - \frac{1}{2}\partial_i f\partial_k A_j\Big) \\
&- \big(\partial_a\Theta^{ij}\big)\Big(\partial^k\partial_i f(\partial_j A_k - \partial_k A_j) - \partial_i f\partial_j(\partial\cdot A) \\
&\qquad + \frac{1}{2}\partial_i f\Box A_j + \frac{1}{2}\Box\partial_i f A_j\Big) \\
&- \big(\partial^k\Theta^{ij}\big)\Big(2\partial_k\partial_i f\partial_j A_a + 2\partial_i f\partial_k\partial_j A_a - \partial_a\partial_i f\partial_j A_k - \partial_i f\partial_a\partial_j A_k \\
&\qquad - \frac{1}{2}\partial_a\partial_k\partial_i f A_j - \frac{1}{2}\partial_k\partial_i f\partial_a A_j \\
&\qquad - \frac{1}{2}\partial_a\partial_i f\partial_k A_j - \frac{1}{2}\partial_i f\partial_a\partial_k A_j\Big)\,.
\end{aligned} \tag{4.56}$$

Again, the relative coefficients are just right to be able to read off a symmetric $\ell_2(A,B)$

$$\boxed{\begin{aligned}
\ell_2(A,B) = &\; i[\partial\cdot A, B_a]_\star + i[A_k,\partial^k B_a]_\star + i[A^k,\partial_k B_a - \partial_a B_k]_\star \\
&- \Box\Theta^{ij}\Big(A_i\partial_j B_a - \frac{1}{2}A_i\partial_a B_j\Big) \\
&+ \big(\partial^k\partial_a\Theta^{ij}\big)\Big(A_i\partial_j B_k - \frac{1}{2}A_i\partial_k B_j\Big) \\
&- \big(\partial_a\Theta^{ij}\big)\Big(\partial^k A_i\partial_j B_k - A_i\partial_j(\partial\cdot B) + \frac{1}{2}A_i\Box B_j\Big) \\
&- \big(\partial^k\Theta^{ij}\big)\Big(\frac{3}{2}\partial_i A_k\partial_j B_a + \frac{1}{2}\partial_k A_i\partial_j B_a + \frac{1}{2}\partial_a A_i\partial_j B_k \\
&\qquad + 2A_i\partial_k\partial_j B_a - A_i\partial_a\partial_j B_k - \frac{1}{2}A_i\partial_a\partial_k B_j\Big) \\
&+ (A\leftrightarrow B)\,.
\end{aligned}} \tag{4.57}$$

Then, the vacuum equation of motion for the non-commutative U(1) Yang-Mills theory up to linear order in Θ reads

$$\begin{aligned}
0 = &\,\Box A_a - \partial_a(\partial\cdot A) - i[\partial\cdot A, A_a]_\star - i[A_k,\partial^k A_a]_\star - i[A^k,\partial_k A_a - \partial_a A_k]_\star \\
&+ \Box\Theta^{ij}\Big(A_i\partial_j A_a - \frac{1}{2}A_i\partial_a A_j\Big)
\end{aligned}$$

$$
\begin{aligned}
&- (\partial^k \partial_a \Theta^{ij}) \Big(A_i \partial_j A_k - \frac{1}{2} A_i \partial_k A_j \Big) \\
&+ (\partial_a \Theta^{ij}) \Big(\partial^k A_i \partial_j A_k - A_i \partial_j (\partial \cdot A) + \frac{1}{2} A_i \Box A_j \Big) \\
&+ (\partial^k \Theta^{ij}) \Big(\frac{3}{2} \partial_i A_k \partial_j A_a + \frac{1}{2} \partial_k A_i \partial_j A_a + \frac{1}{2} \partial_a A_i \partial_j A_k \\
&\qquad + 2 A_i \partial_k \partial_j A_a - A_i \partial_a \partial_j A_k - \frac{1}{2} A_i \partial_a \partial_k A_j \Big) \,.
\end{aligned}
\tag{4.58}
$$

As we see, the linear order corrections are much more involved than for NC-CS theory so that we stop here and leave higher order computations for future work. Here it is important to note that so far we did not encounter any obstacle for solving the L_∞ relations.

NC-YM on curved background. As we have seen in section 3, the general situation for on-shell D-brane configurations involves a curved manifold equipped with the (open string) metric G and a non-constant bi-vector Θ. On such a Riemannian manifold the easiest case should be one where Θ is covariantly constant with respect to the Levi-Civita connection. This requirement is natural from the open string equation of motion (3.16) for the WZW branes. For instance, the two-dimensional branes for the SU(2) WZW model really feature a covariantly constant Θ.

Clearly, here we are entering new territory, as the usual star-product is constructed with respect to a Poisson structure only, without any mentioning of a metric or a connection. In this section, at least up to linear order in Θ, we investigate whether one can also bootstrap the first terms of a NC-YM theory on a curved manifold following our strategy of imposing the L_∞ algebra.

Looking at the usual abelian Yang-Mills theory on a Riemannian manifold, our starting point is that

$$
\ell_1(f) = \partial_a f \,, \qquad \ell_1(A) = \nabla^b \nabla_b A_a - \nabla^b \nabla_a A_b \tag{4.59}
$$

where the second definition follows from varying the action

$$
S = \int d^n x \sqrt{G}\, F_{ab}\, F^{ab} \tag{4.60}
$$

where indices are pulled up and down with the metric G_{ab}. Moreover, for the star-product between two functions up to linear order we keep

$$
f \star g = f \cdot g + \frac{i}{2} \Theta^{ij} \partial_i f \partial_j g + O(\Theta^2) \tag{4.61}
$$

and define

$$
\ell_2(f, g) = i[f, g]_\star \,. \tag{4.62}
$$

The Leibniz rule $\mathcal{J}_2(f, g) = 0$ can now be used to bootstrap the form of $\ell_2(f, A)$, assuming that $\nabla_k \Theta^{ij} = 0$

$$
\ell_2(f, A) = i[f, A]_\star := -\Theta^{ij}\, \nabla_i f\, \nabla_j A_a + O(\Theta^2) \tag{4.63}
$$

where of course $\nabla_i f = \partial_i f$. Thus, we realize that from this perspective it is more natural that indeed the covariant derivative appears when star-multiplying tensors. Since two covariant derivatives do not commute and give

$$T_c = R_{ab,c}{}^d \, T_d \,, \tag{4.64}$$

we must be prepared that there will arise curvature[5] corrections to the expression we encountered in the previous sections. Next, we impose the Leibniz rule $\mathcal{J}_2(f, A) = 0$ from which we are able to read-off $\ell_2(A, B)$ and $\ell_2(f, E)$. After reordering covariant derivatives and applying Bianchi-identities for the curvature, we finally arrive at

$$\ell_2(f, E) = i[f, E]_\star \tag{4.65}$$

and the more involved expression

$$\begin{aligned} \ell_2(A, B) = \nabla^b \Big(& i[A_b, B_a]_\star + \frac{1}{2}\Theta^{ij} R_{ab,j}{}^c \, A_i \, B_c + (A \leftrightarrow B) \Big) \\ + \Big(& i[A^b, \partial_b B_a - \partial_a B_b]_\star - \Theta^{ij} R_{jc} \, A_i \, (\partial^c B_a - \partial_a B^c) \\ & - \Theta^{ij} R_{jb,a}{}^c \, A_i \, (\partial_c B^b - \partial^b B_c) + (A \leftrightarrow B) \Big) \,. \end{aligned} \tag{4.66}$$

Here, we have used $\partial_{[k} B_{l]} = \nabla_{[k} B_{l]}$. Note that this expression has the correct limit in the flat case and manifestly shows the curvature corrections. Since all ℓ_2-product are at first order in Θ, all higher order relations are satisfied up to linear order and we have succeeded to bootstrap a NC-YM theory on a curved manifold up to linear order in a covariantly constant Θ.

This could be continued to higher orders but we do not pursue this further here and just state that all the discussed examples exemplify that the string theory motivated L_∞ bootstrap program provides a promising novel approach to algebraically construct non-commutative gauge theories in regimes that were not completely accessible yet.

5 Conclusions

Motivated by its appearance in string theory and first successes when applied to the matrix valued NC gauge theory on the fuzzy 2-sphere, in the main part of this paper we have successfully carried out an L_∞ bootstrap program for determining higher derivative corrections to NC gauge theories for non-constant and in general non-associative NC-structure Θ. What is changed is both, the action of the gauge symmetry on the gauge fields and their equations of motion. An interesting open question is whether one can also find an action for them. We leave this non-trivial question for future work.

We believe that this approach is different from former attempts to solve this problem but is based on a string theoretic well motivated guiding principle. By successively applying or solving the L_∞ relations we managed to determine higher ℓ-products. Since we were pursuing a perturbative approach in Θ, the actual computations become more and more

[5]Note that the torsion vanishes for the Levi-Civita connection.

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involved. Note that at higher orders also the Kontsevich star-commutator receives derivative corrections that one needs to take into account. (For the A_∞ algebra from appendix B such corrections already appeared at second order and could consistently be handled.) But it is promising that, up to second order in Θ, we did not encounter any obstacle, even not in the non-associative case. To gain even more evidence for the self-consistency of this approach, one could try to implement this bootstrap algorithm and push the computer to iteratively produce higher and higher orders. Mathematically, one could also ask for a proof that our algorithm always works.

When considering NC-YM theory on a curved space with covariantly constant Θ, the L_∞ structure was telling us that one should better introduce a star-product that uses covariant derivatives when acting on tensors. It would be interesting to investigate whether our first order results can be extended to higher orders.

Of course, generally one could contemplate on other possible problems where such an L_∞ based bootstrap approach might be worthwhile to pursue.

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A Semi classical branes in SU(3) WZW

As we have seen, the SU(2) group manifold is not rich enough to non-trivially check whether on-shell brane configurations exist with non-vanishing Π^{ijk}. The reason was that the boundary states describe at most two-dimensional branes on which H restricts trivially. Thus, we now consider the eight-dimensional SU(3) WZW model which admit six-dimensional branes. The non-vanishing Betti numbers of this group manifold are

$$b^0 = b^3 = b^5 = b^8 = 1\,. \tag{A.1}$$

This manifold can be considered as a S^3 fibration over a five-dimensional base M_5. Clearly, in order to satisfy the Freed-Witten anomaly condition, the D-brane world-volume should better not contain the S^3.

From these topological considerations this example seems to be rich enough to provide a non-trivial example with $\Pi^{ijk} \neq 0$. To explicitly describe the SU(3) group manifold we introduce the matrices

$$D_1(\chi_1) = \begin{pmatrix} e^{\frac{i}{\sqrt{2}}\chi_1} & 0 & 0 \\ 0 & e^{-\frac{i}{\sqrt{2}}\chi_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; D_2(\chi_2) = \begin{pmatrix} e^{\frac{i}{\sqrt{6}}\chi_2} & 0 & 0 \\ 0 & e^{\frac{i}{\sqrt{6}}\chi_2} & 0 \\ 0 & 0 & e^{-i\sqrt{\frac{2}{3}}\chi_2} \end{pmatrix} \tag{A.2}$$

and

$$M_{12}(\varphi,\psi) = \begin{pmatrix} \cos\varphi & \sin\varphi\, e^{i\psi} & 0 \\ -\sin\varphi\, e^{-i\psi} & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.3}$$

and similarly for M_{13} and M_{23}. Now we write an element of SU(3) as

$$M = N^{-1}\, D_1(\chi_1)\, D_2(\chi_2)\, N \tag{A.4}$$

with $N = M_{12}(\varphi_1,\psi_1)\, M_{23}(\varphi_2,\psi_2)\, M_{12}(\varphi_3,\psi_3)$. Thus, the six coordinates ϕ_i along the brane are $\phi \in \{\varphi_1,\psi_1,\varphi_2,\psi_2,\varphi_3,\psi_3\}$ Moreover, in this normalization the positive roots are given by

$$\alpha_1 = (\sqrt{2},0)\,, \quad \alpha_2 = \left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right), \quad \alpha_3 = \left(-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right). \tag{A.5}$$

For the metric on SU(3) one obtain

$$k^{-1}\, ds^2 = \frac{1}{2} d\chi_1^2 + \frac{1}{2} d\chi_2^2 + g_{ij}(\phi)\, d\phi^i \otimes d\phi^j \tag{A.6}$$

where the second term is the metric restricted to the D-brane. The metric components $g_{ij}(\phi)$ are partially long expressions in terms of the coordinates along the brane. Therefore, we just list a few components to convince the reader that the expressions are indeed very explicit

$$\begin{aligned} g_{11} &= 4\sin^2\left(\frac{\chi_1}{\sqrt{2}}\right), \qquad g_{12} = g_{13} = g_{14} = 0 \\ g_{15} &= 4\sin^2\left(\frac{\chi_1}{\sqrt{2}}\right) \cos\varphi_2 \cos(\psi_1 - \psi_3) \\ g_{16} &= 2\sin^2\left(\frac{\chi_1}{\sqrt{2}}\right) \cos\varphi_2 \sin(2\varphi_3) \sin(\psi_1 - \psi_3) \\ g_{22} &= \sin^2\left(\frac{\chi_1}{\sqrt{2}}\right) \sin^2(2\varphi_1)\,, \ldots \end{aligned} \tag{A.7}$$

At generic positions $\chi_{1,2}$ this gives a smooth metric on the $D6$-brane. However, at the three boundaries $\alpha_i \cdot \chi = 0$ the metric degenerates to a four-dimensional metric. Therefore, at these positions one gets $D4$-branes. At the intersection of two such lines the whole metric degenerates thus yielding $D0$-branes. Therefore, the position moduli space has the form displayed in figure 1.

The determinant of the metric has a simple form

$$\sqrt{g} = 4k^4 \prod_{\alpha>0} \sin^2\left(\frac{\alpha\cdot\chi}{2}\right) \sin(2\varphi_1)\sin(2\varphi_2)\sin^2(\varphi_2)\sin(2\varphi_3)\,. \tag{A.8}$$

Integrating this over the domain

$$0 \le \chi_1 \le \sqrt{2}\pi\,, \quad -\frac{\chi_1}{\sqrt{3}} \le \chi_2 \le \frac{\chi_1}{\sqrt{3}}\,, \quad 0 \le \varphi_i \le \frac{\pi}{2}\,, \quad 0 \le \psi_i \le 2\pi \tag{A.9}$$

one finds for the volume of the SU(3) group manifold

$$V(\mathrm{SU}(3)) = \sqrt{3}\pi^5 k^4\,. \tag{A.10}$$

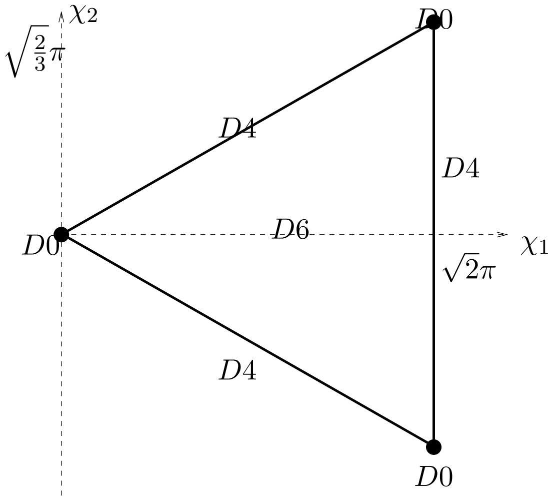

Figure 1. Domain for position of the $D6$-brane.

Next, utilizing (3.10) one computes the B-field and its total derivative to get the H-flux. To find a compact expression for the sechs-bein along the D-brane world-volume it is useful to define the structure "constants"

$$F_{AB}{}^{C}(\phi) := 2\,\hat{\theta}_{[A}{}^{i}\,\partial_i\hat{\theta}_{B]}{}^{j}\,\theta^{C}{}_{j} \tag{A.11}$$

where $A, B, C \in \{\alpha_i, \overline{\alpha}_i\}$. There are non-zero and non-constant elements $F_{AB}{}^{A}(\phi)$ (no sum over A), but they do not contribute to H. The really relevant constant non-zero elements turn out to be

$$F_{31}{}^{2} = F_{\overline{3}\overline{1}}{}^{\overline{2}} = F_{\overline{1}2}{}^{3} = F_{\overline{2}3}{}^{\overline{1}} = F_{1\overline{2}}{}^{\overline{3}} = F_{2\overline{3}}{}^{1} = 1\,. \tag{A.12}$$

Now, using

$$d\theta^{A} = -\frac{1}{2}F_{BC}{}^{A}\,\theta^{B}\wedge\theta^{C} \tag{A.13}$$

one can express the H-flux in a very compelling way as

$$H|_{\mathrm{D}} = h(\chi)\left[\theta^{\alpha_1}\wedge\theta^{\overline{\alpha}_2}\wedge\theta^{\alpha_3} - \theta^{\overline{\alpha}_1}\wedge\theta^{\alpha_2}\wedge\theta^{\overline{\alpha}_3}\right] \tag{A.14}$$

with

$$\begin{aligned} h(\chi) &= ik\sum_{i=1}^{3}(-1)^{i}\Big(\alpha_i\cdot\chi - \sin(\alpha_i\cdot\chi)\Big) \\ &= 2ik\sin\left(\frac{\chi_1}{\sqrt{2}}\right)\left[\cos\left(\frac{\chi_1}{\sqrt{2}}\right) - \cos\left(\sqrt{\frac{3}{2}}\chi_2\right)\right]\,. \end{aligned} \tag{A.15}$$

Thus, in contrast to the former SU(2) case, the restriction of H onto the $D6$-brane is not vanishing. However, the restriction onto the $D4$ and $D0$ branes at boundary of the moduli space vanishes. Due to $\sum_i(-1)^i\alpha_i\cdot\chi = 0$ these linear terms in χ do not contribute to the three-form flux $H|_{\mathrm{D}}$ on the brane. Taking into account (3.11), this is nothing else than the manifestation of the fact that a pure gauge flux $F = dA$ satisfies $dF = 0$. Therefore, in accord with the Freed-Witten anomaly, the restriction of the bulk H-flux onto the brane can be expressed as $H|_{\mathrm{D}} = d\mathcal{F}$. Being non-vanishing, there is a good chance to finally also get a non-vanishing Π^{ijk}.

The quantization of the gauge flux F only admits a finite number of allowed D-branes. These are parametrized by

$$\chi_1 = \sqrt{2}\pi\,\frac{m_1}{k}\,, \qquad \chi_2 = \sqrt{\frac{2}{3}}\pi\,\frac{(2m_2 - m_1)}{k} \tag{A.16}$$

with $m_1, m_2 \in \mathbb{Z}$ such that they lie inside the domain in figure 1. For $k = 1$ one only gets the three $D0$-branes but for $k = 3$, for the first time, also a $D6$ is allowed.

Next one can derive an explicit expression of the flux $\mathcal{F}$ and compute

$$\sqrt{g+\mathcal{F}} = \frac{k^3}{2}\prod_{\alpha>0}\sin\left(\frac{\alpha\cdot\chi}{2}\right)\,\sin(2\varphi_1)\sin(2\varphi_2)\sin^2(\varphi_2)\sin(2\varphi_3)\,. \tag{A.17}$$

Similarly, from (3.14) one can get an expression for the anti-symmetric bi-vector Θ and explicitly check that the equation of motion (3.5) is indeed satisfied. Starting with the general expression (3.14), one can check in more detail that the underlying reason for this result are the relations

$$\begin{aligned}
\partial_i\left(\sqrt{g+\mathcal{F}}\,\hat\theta_\alpha{}^i\right) &= -\sqrt{g+\mathcal{F}}\,F_{\alpha\overline{\alpha}}{}^{\overline{\alpha}}(\phi)\\
\partial_i\left(\sqrt{g+\mathcal{F}}\,\hat\theta_{\overline{\alpha}}{}^i\right) &= -\sqrt{g+\mathcal{F}}\,F_{\overline{\alpha}\alpha}{}^{\alpha}(\phi)
\end{aligned} \tag{A.18}$$

(no sum over α in $F_{\overline{\alpha}\alpha}{}^{\alpha}$) leading to a cancellation already for each term in the sum over the positive roots in (3.14). Thus, as expected, the highly curved, fluxed $D6$-brane is a consistent solution of the string equations of motion. Similarly to the H-flux one can write the non-vanishing antisymmetric three-vector $\Pi = [\Theta,\Theta]_S$ in the very compelling form

$$\boxed{\Pi = \frac{k^2}{4}\pi(\chi)\left[\hat\theta_{\alpha_1}\wedge\hat\theta_{\overline{\alpha}_2}\wedge\hat\theta_{\alpha_3} + \hat\theta_{\overline{\alpha}_1}\wedge\hat\theta_{\alpha_2}\wedge\hat\theta_{\overline{\alpha}_3}\right]} \tag{A.19}$$

with

$$\begin{aligned}
\pi(\chi) &= \cot\left(\frac{\alpha_1\cdot\chi}{2}\right)\cot\left(\frac{\alpha_2\cdot\chi}{2}\right) - \cot\left(\frac{\alpha_1\cdot\chi}{2}\right)\cot\left(\frac{\alpha_3\cdot\chi}{2}\right)\\
&\quad + \cot\left(\frac{\alpha_2\cdot\chi}{2}\right)\cot\left(\frac{\alpha_3\cdot\chi}{2}\right)\\
&= -1\,.
\end{aligned} \tag{A.20}$$

This explicitly shows that the SU(3) WZW model admits a six-dimensional brane that carries a non-trivial Π. That means that Θ is not a Poisson structure and the related star-product becomes non-associative. For concreteness, we display the components of Π

$$
\begin{aligned}
\Pi^{123} &= \frac{\sin(\psi_1-\psi_3)\ \sin\phi_3}{8\sin\phi_2\cos\phi_3}\\
\Pi^{124} &= \frac{\cos(2\phi_1)}{4\sin(2\phi_1)\sin^2(\phi_2)} - \frac{\cos(\psi_1-\psi_3)(5+3\cos(2\phi_2))\sin\phi_3}{32\sin^2(\phi_2)\cos\phi_2\cos\phi_3}\\
\Pi^{134} &= -\frac{\sin(\psi_1-\psi_3)\ \sin\phi_3}{16\sin\phi_2\cos\phi_3}\\
\Pi^{234} &= -\frac{1}{4\sin(2\phi_2)} - \frac{\cos(\psi_1-\psi_3)\cos(2\phi_1)\sin\phi_3}{8\sin(2\phi_1)\sin\phi_2\cos\phi_3}\\
\Pi^{125} &= \frac{\sin(\psi_1-\psi_3)(1+3\cos(2\phi_2))}{32\sin^2(\phi_2)\cos\phi_2}\\
\Pi^{135} &= -\frac{\cos(\psi_1-\psi_3)}{16\sin\phi_2}\\
\Pi^{235} &= \frac{\sin(\psi_1-\psi_3)\cos(2\phi_1)}{16\sin\phi_1\cos\phi_1\sin\phi_2}\\
\Pi^{145} &= -\frac{\sin(\psi_1-\psi_3)}{8\sin\phi_2\sin(2\phi_2)}\\
\Pi^{245} &= \frac{\sin(\phi_3)}{8\sin^2(\phi_2)\cos\phi_3} - \frac{\cos(\psi_1-\psi_3)\cos(2\phi_1)}{8\sin(2\phi_1)\sin^2(\phi_2)\cos\phi_2}\\
\Pi^{345} &= 0\\
\Pi^{126} &= -\frac{\cos(\psi_1-\psi_3)(1+3\cos(2\phi_2))}{16\sin^2(\phi_2)\cos\phi_2\sin(2\phi_3)}\\
\Pi^{136} &= -\frac{\sin(\psi_1-\psi_3)}{8\sin\phi_2\sin(2\phi_3)}\\
\Pi^{236} &= -\frac{\cos(\psi_1-\psi_3)\cos(2\phi_1)}{4\sin(2\phi_1)\sin\phi_2\sin(2\phi_3)}\\
\Pi^{146} &= \frac{\cos(\psi_1-\psi_3)}{16\cos\phi_2\sin^2(\phi_2)\sin\phi_3\cos\phi_3}\\
\Pi^{246} &= -\frac{\sin(\psi_1-\psi_3)\cos(2\phi_1)}{2\sin(2\phi_1)\sin\phi_2\sin(2\phi_2)\sin(2\phi_3)}\\
\Pi^{346} &= \Pi^{156} = 0\\
\Pi^{256} &= -\frac{1}{8\sin^2(\phi_2)\sin\phi_3\cos\phi_3}\\
\Pi^{356} &= \Pi^{456} = 0
\end{aligned}
\tag{A.21}
$$

B Refinement to an $\mathrm{A}_\infty^{\mathrm{gauge}}$ algebra

Since in string theory, open-closed string field theory has an underlying A_∞ algebra, one might expect that NC-gauge theory admits an A_∞ algebra, as well. Thus, in this appendix we refine the $\mathrm{L}_\infty^{\mathrm{gauge}}$ structure from section 4 and construct an $\mathrm{A}_\infty^{\mathrm{gauge}}$ algebra that underlies the, in general non-associative, gauge theory up to order $O(\Theta^2)$.

Recall that we consider two non-trivial vector spaces, X_0 and X_{-1} where X_0 contains functions (gauge parameters) and X_{-1} vectors (gauge fields). Moreover, to simplify the notation, in this section we use

$$\hat{\Theta}^{ij} = \frac{i}{2}\Theta^{ij}\,, \qquad \hat{\Pi}^{ijk} = -\frac{1}{4}\Pi^{ijk}. \tag{B.1}$$

Similar to the L_∞ case, the first order products are defined as[6]

$$\boxed{\begin{aligned} m_1(f) &= \partial_a f \in X_{-1} \\ m_2(f,g) &= f \star g + \frac{1}{3}(\hat{\Theta}^{im}\partial_m\hat{\Theta}^{jk})(\partial_i\partial_j f\,\partial_k g + \partial_i\partial_j g\,\partial_k f) \in X_0 \end{aligned}} \tag{B.2}$$

where the second line is just the full Kontsevich star-product and $f \star g$ denotes the Moyal-Weyl part of it

$$f \star g = fg + \hat{\Theta}^{ij}\partial_i f \partial_g + \frac{1}{2}\hat{\Theta}^{im}\hat{\Theta}^{jn}\,\partial_i\partial_j f\,\partial_m\partial_n g + O(\hat{\Theta}^3)\,. \tag{B.3}$$

By this split, we explicitly take care of all appearing derivative $\partial\Theta$ terms. Note that $m_2(f,g)$ is neither symmetric nor anti-symmetric under exchanging the arguments f and g. The relation to the corresponding higher product in the L_∞ algebra is given by graded symmetrization

$$-i\ell_2(f,g) = m_2(f,g) - m_2(g,f)\,. \tag{B.4}$$

The A_∞ relation

$$\mathcal{A}_2(f,g) = m_1(m_2(f,g)) - m_2(m_1(f),g) - m_2(f,m_1(g)) = 0 \tag{B.5}$$

is nothing else than the Leibniz-rule for the two-product and can be satisfied, up to order $O(\hat{\Theta}^2)$, by defining

$$\boxed{\begin{aligned} m_2(A,g) &= A_a \star g + \frac{1}{3}(\hat{\Theta}^{im}\partial_m\hat{\Theta}^{jk})(\partial_i\partial_j A_a\,\partial_k g + \partial_i\partial_j g\,\partial_k A_a) \\ &\qquad + \frac{1}{2}\partial_a\hat{\Theta}^{ij}\,A_i \star \partial_j g + \frac{1}{3}\partial_a(\hat{\Theta}^{im}\partial_m\hat{\Theta}^{jk})\,\partial_j A_i\,\partial_k g \\ m_2(f,A) &= f \star A_a + \frac{1}{3}(\hat{\Theta}^{im}\partial_m\hat{\Theta}^{jk})(\partial_i\partial_j f\,\partial_k A_a + \partial_i\partial_j A_a\,\partial_k f) \\ &\qquad + \frac{1}{2}\partial_a\hat{\Theta}^{ij}\,\partial_i f \star A_j + \frac{1}{3}\partial_a(\hat{\Theta}^{im}\partial_m\hat{\Theta}^{jk})\,\partial_j A_i\,\partial_k f\,. \end{aligned}} \tag{B.6}$$

Note that the two terms in the second line are correction terms that arise for non-constant $\hat{\Theta}$. These terms make the construction of the A_∞ algebra much more complicated than in the Moyal-Weyl case of constant $\hat{\Theta}$. Again, by anti-symmetrization we can confirm

$$-i\ell_2(f,A) = m_2(f,A) - m_2(A,f)\,. \tag{B.7}$$

[6] Like in (B.2), in the following the definitions of the higher products are put in boxes.

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Next we analyze the A_∞ relation

$$\begin{aligned}\mathcal{A}_3(f,g,h) = m_2(m_2(f,g),h) - m_2(f,m_2(g,h)) + m_1(m_3(f,g,h))\\ + m_3(m_1(f),g,h) + m_3(f,m_1(g),h) + m_3(f,g,m_1(h)) = 0\,.\end{aligned} \tag{B.8}$$

For the associator one finds

$$m_2(m_2(f,g),h) - m_2(f,m_2(g,h)) = -\frac{2}{3}\hat{\Pi}^{ijk}\partial_i f\partial_j g\partial_k h\,. \tag{B.9}$$

Since $m_1(m_3(f,g,h)) \in X_1$ vanishes one can solve (B.8) by

$$\boxed{\begin{aligned} m_3(A,f,g) &= \alpha\,\hat{\Pi}^{ijk}A_i\partial_j f\partial_k g\\ m_3(f,A,g) &= \beta\,\hat{\Pi}^{ijk}A_i\partial_j f\partial_k g\\ m_3(f,g,A) &= \gamma\,\hat{\Pi}^{ijk}A_i\partial_j f\partial_k g\end{aligned}} \tag{B.10}$$

with $\alpha-\beta+\gamma = 2/3$. This guarantees that in the associative case, all these three-products vanish. It is straightforward to confirm the relation to the corresponding L_∞ three-product

$$\begin{aligned}(-i)^2\ell_3(A,f,g) = m_3(A,f,g) - m_3(A,g,f) - m_3(f,A,g) + m_3(g,A,f)\\ + m_3(f,g,A) - m_3(g,f,A)\,.\end{aligned} \tag{B.11}$$

Now we proceed invoking the remaining $\mathcal{A}_3$ relations

$$\begin{aligned}\mathcal{A}_3(A,f,g) &= m_2(m_2(A,f),g) - m_2(A,m_2(f,g)) + m_1(m_3(A,f,g))\\ &\quad + m_3(m_1(A),f,g) - m_3(A,m_1(f),g) - m_3(A,f,m_1(g)) = 0\\ \mathcal{A}_3(f,g,A) &= m_2(m_2(f,g),A) - m_2(f,m_2(g,A)) + m_1(m_3(f,g,A))\\ &\quad + m_3(m_1(f),g,A) + m_3(f,m_1(g),A) + m_3(f,g,m_1(A)) = 0\\ \mathcal{A}_3(f,A,g) &= m_2(m_2(g,A),f) - m_2(g,m_2(A,f)) + m_1(m_3(g,A,f))\\ &\quad + m_3(m_1(g),A,f) + m_3(g,m_1(A),f) - m_3(g,A,m_1(f)) = 0\,.\end{aligned} \tag{B.12}$$

Note the signs appearing in front of the $m_3\,m_1$-terms, which involve extra signs relative to (2.22) whenever m_1 is permuted through an odd element $A \in X_{-1}$. Let us first compute the associators

$$\begin{aligned}&m_2(m_2(A,f),g) - m_2(A,m_2(f,g))\\ &\quad= \left(\frac{1}{3}\hat{\Theta}^{im}\partial_a\partial_m\hat{\Theta}^{jk} - \frac{1}{2}\hat{\Theta}^{km}\partial_a\partial_m\hat{\Theta}^{ij} - \frac{1}{6}\partial_a\hat{\Theta}^{im}\partial_m\hat{\Theta}^{jk} - \frac{1}{4}\partial_a\hat{\Theta}^{km}\partial_m\hat{\Theta}^{ij}\right)A_a\partial_j f\partial_k g\\ &\qquad - \frac{2}{3}\hat{\Pi}^{ijk}\partial_i A_a\partial_j f\partial_k g\\ &m_2(m_2(f,g),A) - m_2(f,m_2(g,A))\\ &\quad= \left(\frac{1}{3}\hat{\Theta}^{im}\partial_a\partial_m\hat{\Theta}^{jk} - \frac{1}{2}\hat{\Theta}^{jm}\partial_a\partial_m\hat{\Theta}^{ki} - \frac{1}{6}\partial_a\hat{\Theta}^{im}\partial_m\hat{\Theta}^{jk} - \frac{1}{4}\partial_a\hat{\Theta}^{jm}\partial_m\hat{\Theta}^{ki}\right)A_a\partial_j f\partial_k g\\ &\qquad - \frac{2}{3}\hat{\Pi}^{ijk}\partial_i A_a\partial_j f\partial_k g\end{aligned} \tag{B.13}$$

and

$$
\begin{aligned}
&m_2(m_2(g,A),f)-m_2(g,m_2(A,f))\\
&\quad=\Big(-\frac{2}{3}\hat{\Theta}^{im}\partial_a\partial_m\hat{\Theta}^{jk}-\frac{1}{2}\hat{\Theta}^{km}\partial_a\partial_m\hat{\Theta}^{ij}-\frac{1}{2}\hat{\Theta}^{jm}\partial_a\partial_m\hat{\Theta}^{ki}\\
&\qquad-\frac{2}{3}\partial_a\hat{\Theta}^{im}\partial_m\hat{\Theta}^{jk}-\frac{1}{4}\partial_a\hat{\Theta}^{km}\partial_m\hat{\Theta}^{ij}-\frac{1}{4}\partial_a\hat{\Theta}^{jm}\partial_m\hat{\Theta}^{ki}\Big)A_a\partial_jf\partial_kg\\
&\qquad-\frac{2}{3}\hat{\Pi}^{ijk}\partial_iA_a\partial_jf\partial_kg\,.
\end{aligned}
\tag{B.14}
$$

To solve (B.12), we first observe that $m_1(A)=0$ and make the general ansatz

$$
\begin{aligned}
m_3(A,B,f)=\Sigma^{ijk}A_iB_j\partial_kf+\hat{\Pi}^{ijk}\Big(&x_1\,\partial_iA_aB_j\partial_kf+x_2\,A_i\partial_jB_a\partial_kf\\
&+x_3\,\partial_aA_iB_j\partial_kf+x_4\,A_i\partial_aB_j\partial_kf\Big)
\end{aligned}
\tag{B.15}
$$

and similarly for $m_3(A,f,B)$ and $m_3(f,A,B)$. This gives a set of conditions that only admit a solution iff $\alpha-\beta+\gamma=2/3$, i.e. precisely the condition that followed from the relation $\mathcal{A}_3(f,g,h)=0$ shown in (B.8). Eliminating β in favor of α and γ, the general set of solutions is given as

$$
\boxed{
\begin{aligned}
m_3(A,B,f)=\Big[&\kappa_1\,\hat{\Theta}^{im}\partial_a\partial_m\hat{\Theta}^{jk}+\kappa_2\,\hat{\Theta}^{km}\partial_a\partial_m\hat{\Theta}^{ij}+\kappa_3\,\hat{\Theta}^{jm}\partial_a\partial_m\hat{\Theta}^{ki}\\
&+\lambda_1\,\partial_a\hat{\Theta}^{im}\partial_m\hat{\Theta}^{jk}+\lambda_2\,\partial_a\hat{\Theta}^{km}\partial_m\hat{\Theta}^{ij}+\lambda_3\,\partial_a\hat{\Theta}^{jm}\partial_m\hat{\Theta}^{ki}\Big]\,A_iB_j\partial_kf\\
+\hat{\Pi}^{ijk}\Big[&x_1\,\partial_iA_aB_j\partial_kf+x_2\,A_i\partial_jB_a\partial_kf\\
&+\Big(-\frac{2}{3}+\alpha+\gamma-x_1\Big)\,\partial_aA_iB_j\partial_kf+(\alpha-x_2)\,A_i\partial_aB_j\partial_kf\Big]
\end{aligned}
}
$$

and

$$
\boxed{
\begin{aligned}
m_3(A,f,B)=\Big[&\Big(-\frac{1}{3}-\alpha+\kappa_1\Big)\,\hat{\Theta}^{im}\partial_a\partial_m\hat{\Theta}^{jk}+(-\alpha+\kappa_3)\,\hat{\Theta}^{km}\partial_a\partial_m\hat{\Theta}^{ij}\\
&+\Big(\frac{1}{2}-\alpha+\kappa_2\Big)\,\hat{\Theta}^{jm}\partial_a\partial_m\hat{\Theta}^{ki}\Big]\,A_iB_j\partial_kf\\
+\Big[&\Big(\frac{1}{6}-\alpha+\lambda_1\Big)\,\partial_a\hat{\Theta}^{im}\partial_m\hat{\Theta}^{jk}+(-\alpha+\lambda_3)\,\partial_a\hat{\Theta}^{km}\partial_m\hat{\Theta}^{ij}\\
&+\Big(\frac{1}{4}-\alpha+\lambda_2\Big)\,\partial_a\hat{\Theta}^{jm}\partial_m\hat{\Theta}^{ki}\Big]\,A_iB_j\partial_kf\\
+\hat{\Pi}^{ijk}\Big[&\Big(\frac{2}{3}+x_1\Big)\,\partial_iA_aB_j\partial_kf+y_2\,A_i\partial_jB_a\partial_kf\\
&+\Big(-\frac{2}{3}+\gamma-x_1\Big)\,\partial_aA_iB_j\partial_kf+(-\alpha-y_2)\,A_i\partial_aB_j\partial_kf\Big]
\end{aligned}
}
\tag{B.16}
$$

and

$$\boxed{\begin{aligned} m_3(f,A,B) = \Big[&(-\alpha-\gamma+\kappa_3)\,\hat\Theta^{im}\partial_a\partial_m\hat\Theta^{jk} + \Big(\frac{1}{6}-\alpha-\gamma+\kappa_1\Big)\,\hat\Theta^{km}\partial_a\partial_m\hat\Theta^{ij} \\ &+\Big(\frac{1}{6}-\alpha-\gamma+\kappa_2\Big)\,\hat\Theta^{jm}\partial_a\partial_m\hat\Theta^{ki}\Big]\,A_iB_j\partial_k f \\ &+\Big[(-\alpha-\gamma+\lambda_3)\,\partial_a\hat\Theta^{im}\partial_m\hat\Theta^{jk} + \Big(\frac{5}{12}-\alpha-\gamma+\lambda_1\Big)\,\partial_a\hat\Theta^{km}\partial_m\hat\Theta^{ij} \\ &+\Big(\frac{5}{12}-\alpha-\gamma+\lambda_2\Big)\,\partial_a\hat\Theta^{jm}\partial_m\hat\Theta^{ki}\Big]\,A_iB_j\partial_k f \\ &+\hat\Pi^{ijk}\Big[\Big(-\frac{2}{3}+x_2\Big)\,\partial_iA_aB_j\partial_k f + \Big(\frac{2}{3}+y_2\Big)\,A_i\partial_jB_a\partial_k f \\ &+\Big(\frac{2}{3}-\gamma-x_2\Big)\,\partial_aA_iB_j\partial_k f + (-\alpha-\gamma-y_2)\,A_i\partial_aB_j\partial_k f\Big] \end{aligned}}$$

where besides α,γ the κ_i, λ_i and x_1,x_2,y_2 are still free parameters. However, when computing the graded symmetrization of these m-products, all these parameters precisely cancel and one gets the corresponding ℓ_3-product

$$\begin{aligned}(-i)^2\ell_3(A,B,f) = &\, m_3(A,B,f)+m_3(B,A,f)-m_3(A,f,B) \\ &- m_3(B,f,A)+m_3(f,A,B)+m_3(B,A,f)\,.\end{aligned} \tag{B.17}$$

Finally, one has to check the A_∞ relation $\mathcal{A}_4$ (2.23). There are only two possible sets of a priori non-trivial relations with entries $\mathcal{A}_4(f,g,h,A)$ and $\mathcal{A}_4(f,g,A,B)$ and permutations thereof. The $\mathcal{A}_4(f,g,h,A)$ relations are all satisfied up to order $O(\Theta^2)$ so that we choose a vanishing four-product $m_4(f,g,A,B)\in X_0$. The $\mathcal{A}_4(f,g,A,B)$ relations are also all satisfied in the associative case, but in the non-associative case, one needs to introduce non-trivial four-products $m_4(f,A,B,C)\in X_{-1}$ that are proportional to $\hat\Pi^{ijk}$. As before, we make a general ansatz

$$\boxed{\begin{aligned} m_4(f,A,B,C) = &\,\hat\Pi^{ijk}\Big(\mu_1A_aB_iC_j+\mu_2A_jB_aC_i+\mu_3A_iB_jC_a\Big)\partial_k f \\ &+\hat\Pi^{ijk}\mu_4\partial_a f A_iB_jC_k \end{aligned}} \tag{B.18}$$

and similarly for $m_4(A,f,B,C)$, $m_4(A,B,f,C)$ and $m_4(A,B,C,f)$. One realizes that there appear consistency conditions for the existence of a solution, that are however satisfied once the relations that we encountered before are satisfied. After all, the four parameters in $m_4(f,A,B,C)$ remain as free parameters with the other three four-products given as

$$\boxed{\begin{aligned} m_4(A,f,B,C) = &\,\hat\Pi^{ijk}\Big(-\mu_4A_aB_iC_j+\Big(\frac{2}{3}+\mu_2-x_2\Big)A_jB_aC_i \\ &+(\gamma+\mu_3)A_iB_jC_a\Big)\partial_k f \\ &+\hat\Pi^{ijk}\Big(-\frac{2}{3}+\gamma-\mu_1+x_2\Big)\partial_a f A_iB_jC_k \end{aligned}} \tag{B.19}$$

and

$$\boxed{\begin{aligned} m_4(A,B,f,C) &= \hat{\Pi}^{ijk}\bigg(\Big(-\frac{2}{3}+\gamma-\mu_4-x_1\Big)A_aB_iC_j+(\mu_1-x_1-x_2)A_jB_aC_i \\ &\qquad +(\alpha+\gamma+\mu_3+y_2)A_iB_jC_a\bigg)\partial_k f \\ &\quad +\hat{\Pi}^{ijk}\Big(-\frac{2}{3}-\alpha-\mu_2+x_2-y_2\Big)\partial_a f A_iB_jC_k \\ m_4(A,B,C,f) &= \hat{\Pi}^{ijk}\bigg(\Big(-\frac{2}{3}+\alpha+\gamma-\mu_4-x_1\Big)A_aB_iC_j+(\mu_1-x_1)A_jB_aC_i \\ &\qquad +\Big(\frac{2}{3}+\mu_2+y_2\Big)A_iB_jC_a\bigg)\partial_k f \\ &\quad +\hat{\Pi}^{ijk}(-\alpha-\gamma-\mu_3-y_2)\partial_a f A_iB_jC_k\,. \end{aligned}}$$

We note that via graded symmetrization the corresponding ℓ_4-product is vanishing, being consistent with our findings in section 4.

Having now a non-trivial m_4-product one also has to worry about the relation $\mathcal{A}_5$ that for $m_5 = 0$ contains the order $O(\hat{\Theta}^2)$ term

$$\begin{aligned} \mathcal{A}_5 = m_4(m_2\otimes 1^3 - 1\otimes m_2\otimes 1^2 + 1^2\otimes m_2\otimes 1 - 1^3\otimes m_2) \\ + m_2(m_4\otimes 1 - 1\otimes m_4) + O(\hat{\Theta}^3)\,. \end{aligned} \tag{B.20}$$

We have checked that all ten relations of the type $\mathcal{A}_5(f,g,A,B,C)$ are satisfied. All higher relations are trivially satisfied up to order $\hat{\Theta}^2$.

Let us summarize our findings: we have explicitly constructed the A_∞ algebra up to order $O(\hat{\Theta}^2)$ that underlies the non-commutative gauge theory governed by a non-constant and in general even non-associative star-product. By constructing the higher products in a step-by-step procedure, we encountered many derivative $\partial\Theta$-corrections that make the whole algebra and relations highly non-trivial. At each step, we observed that the A_∞ relation under question led to some consistency conditions that were automatically satisfied once the lower A_∞ relations were already satisfied. This is very encouraging and makes us believe that the whole procedure continues also to higher orders in Θ. Up to the level that we were considering, we found that in the associative case, a non-constant Θ induces non-vanishing higher products of type

$$m_2(f,g)\,,\quad m_2(A,g)\quad m_3(A,B,g)\,. \tag{B.21}$$

In the non-associative case a further three and a four-product had to be introduced

$$m_2(f,g)\,,\quad m_2(A,g)\,,\quad m_3(A,f,g)\,,\quad m_3(A,B,g)\,,\quad m_4(f,A,B,C)\,. \tag{B.22}$$

In the conclusion of this section we stress that both, for the consistency of the proposed construction of A_∞ and for the correct relation to the L_∞, the product $m_2(f,g)$ was taken to be the Kontsevich star product $f\bullet g$. Up to second order in Θ, $\ell_2(f,g) = i[f,g]_\bullet$, coincides with the "classical" (quasi)-Poisson bracket, $-\{f,g\}$, which is not the case of the product m_2. It contains "quantum" information in the sense of deformation quantization corrections. A separate question is whether there exists a "classical" L_∞ algebra, where the two-product is always simply $\ell_2(f,g) = \{f,g\}$ or whether for consistency one should necessarily take $\ell_2(f,g)$ as a star commutator, i.e. construct "quantum" L_∞.

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