Published for SISSA by Springer

Received: November 27, 2017
Accepted: April 19, 2018
Published: May 7, 2018

Wess-Zumino and super Yang-Mills theories in D=4 integral superspace

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Abstract: We reconstruct the action of $N = 1, D = 4$ Wess-Zumino and $N = 1, 2, D = 4$ super-Yang-Mills theories, using integral top forms on the supermanifold $\mathcal{M}^{(4|4)}$. Choosing different Picture Changing Operators, we show the equivalence of their rheonomic and superspace actions. The corresponding supergeometry and integration theory are discussed in detail. This formalism is an efficient tool for building supersymmetric models in a geometrical framework.

Keywords: Differential and Algebraic Geometry, Superspaces, Supersymmetric Gauge Theory

ArXiv ePrint: 1711.07194

https://doi.org/10.1007/JHEP05(2018)040

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1 Introduction

In some recent papers [1–3], we explored the rôle of the supermanifolds and their integration theory for applications to gauge theories, supergravity and string theories.

The superspace technique has been invented to describe supersymmetric theories with manifestly supersymmetric actions. This is achieved by adding fermionic coordinates to the bosonic manifold and using Berezin integration. Nonetheless the geometric point of view needs further clarification. During the recent years, due to progress in fundamental string theory [4, 5] and due to progress in the understanding of integration theory on

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supermanifolds (see for ex. [5, 6]), a more solid and fruitful framework for superspace actions has been built.

A convenient way to write a supersymmetric action in superspace, as an integral of integral forms on supermanifolds $\mathcal{M}^{(n|m)}$, is the following

$$S = \int_{\mathcal{M}^{(n|m)}} \mathcal{L}^{(n|0)} \wedge \mathbb{Y}^{(0|m)} \tag{1.1}$$

where the Lagrangian $\mathcal{L}^{(n|0)}$ is a superfield $(n|0)$ superform and $\mathbb{Y}^{(0|m)}$ is a *Picture Changing Operator* (PCO), or using a proper mathematical identification (see for ex. [7]), is the Poincaré dual form of the embedding of a n-dimensional bosonic submanifold into the supermanifold $\mathcal{M}^{(n|m)}$. $\mathbb{Y}^{(0|m)}$ belongs to the super de Rham cohomology $H^{(0|m)}(\mathcal{M}^{(n|m)})$.

The choice of the PCO determines the representation for the supersymmetric theory: the simplest PCO constructed in terms of the fermionic coordinates θ^α (with $\alpha = 1, \dots m$), and their corresponding one-forms $\psi^\alpha = d\theta^\alpha$, is given by $\theta^m \delta^m(\psi)$. When inserted into (1.1) it reproduces the component action. When instead a supersymmetric PCO is used, it yields a superfield action with manifest supersymmetry. Different choices of the PCO's produce different representations of the same theory with different amounts of manifest supersymmetry and passing from one to another leads to equivalent theories when the PCO's differ by exact terms and $\mathcal{L}^{(n|0)}$ is closed.

The purpose of the present paper is to study the four dimensional case, with different amounts of supersymmetry. In particular, we will study the case $N = 1$ and $N = 2$. The cases $D = 1$, $D = 2$ and $D = 3$ are treated in [7–9].

The paper is organized as follows:

1. In section 2, we review the Wess-Zumino model for a chiral field from the superspace point of view. This is the usual construction of the textbooks and we use it to set the stage. Then, we consider the geometric formulation of the *rheonomic* formalism. That framework uses only geometric ingredients: superforms, exterior differential and wedge product. Finally, we rewrite the action using the integral form formulation which projects the geometric action to the superspace action. We perform the computation explicitly to illustrate all steps and we postpone the mathematical construction of the PCO in later sections.

2. In section 3, we review the $N = 1$ super Yang-Mills theory in the superspace framework. Differently from the usual prepotential construction (see for example the textbooks [10, 11]), suitable only for $D = 4$, we use the form language (see [12–14]) and we discuss the solution of the constraints. This allows us to write both the superspace action and the geometric action in terms of the gaugino field strength $W^\alpha, \bar{W}^{\dot{\alpha}}$. The dependence of the geometric action upon the rigid gravitinos $\psi^\alpha, \bar{\psi}^{\dot{\alpha}}$ admits a straightforward generalization to supergravity couplings and it encodes all possible information. The geometric action is built and the equations of motion are given. Finally, we explore two possible choices of the PCO's leading either to the component action or to the well-kwown superspace action.

3. As a further example, in section 4 we consider the case of $N = 2$ super-Yang-Mills. We briefly review the $N = 2$ superspace action (which consists of only one term integrated over the full superspace) and we discuss the rheonomic action. In the long and complicated rheonomic action displayed in the textbook [12–14], only one term is relevant in order to reproduce the superspace action. The relation with the $N = 2, D = 4$ action is achieved by changing the PCO, using the closure of the rheonomic action.

4. In section 5, we summarize the mathematical aspects of the derivation. We review the structure of the integral superspace, considering the full complex of integral forms and of superforms. We review the action of different operators and the notion of picture number. An important issue is the Lorentz symmetry for integral forms, discussed in section 5.2. The volume forms and the PCO's are built in the subsequent sections with detailed derivations. The final two theorems are needed for the supergravity extension of the present framework.

2 D=4 N=1 integral Wess-Zumino model

It is important to clarify the integral form formulation of the most well-known example of supersymmetric model, namely the Wess-Zumino model. It describes a chiral multiplet and the field content is given by a complex scalar ϕ, two fermions $\lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}$ and a complex auxiliary field f. The auxiliary field f guarantees the closure of the off-shell supersymmetry. On shell, f is set to zero and the degrees of freedom of the fermions are halved by the equations of motion, so that they match the bosonic degrees of freedom.

In section 2.1 we review the superspace action in the conventional Weyl/anti-Weyl notation. We give the action in component fields. In section 2.2 we review the geometric (rheonomic) action described in the book [12–14], rewriting it into chiral notation. In section 2.3 we construct the action on the supermanifold $\mathcal{M}^{(4|4)}$ and show how to reproduce the superspace action and the component action. For that we need suitable PCO's to project the geometric action along different supersymmetry realizations. The relevant PCO's will be described later in section 5.

2.1 WZ superspace action

The spinors are taken in the Weyl/anti-Weyl representation in order to compare our formulas with the usual D=4 N=1 superspace [10, 11]. In that framework the supermultiplet is described by a single complex superfield $\Phi(x, \theta, \bar{\theta})$ satisfying

$$\bar{D}_{\dot{\alpha}} \Phi = 0 \,. \tag{2.1}$$

where $\bar{D}_{\dot{\alpha}} = \partial_{\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \partial_{x^{\alpha\dot{\alpha}}}$ (see also section 5.1 for notations, differential operators and their algebra). Equations (2.1) are easily solved by introducing the chiral coordinates $(y^{\alpha\dot{\alpha}} \equiv x^{\alpha\dot{\alpha}} - i\theta^\alpha\bar{\theta}^{\dot{\alpha}}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. The chiral superfield Φ is independent of $\bar{\theta}$ and can be decomposed as follows

$$\Phi(y,\theta) = \phi(y) + \lambda_\alpha(y)\theta^\alpha + f(y)\frac{\theta^2}{2} \tag{2.2}$$
$$= \phi + \lambda_\alpha\theta^\alpha + \left(\frac{1}{2}f\theta^2 - i\theta^\alpha\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\phi\right) - \frac{i}{2}\theta^2\bar{\theta}^{\dot{\alpha}}\partial_{\dot{\alpha}\alpha}\lambda^\alpha + \frac{1}{8}\theta^2\bar{\theta}^2\partial^2\phi\,. \tag{2.3}$$

where $\theta^2 = \theta^\alpha\epsilon_{\alpha\beta}\theta^\beta$, $\bar{\theta}^2 = \bar{\theta}^{\dot{\alpha}}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}$, and the components ϕ, λ^α and f in the last line depend on x. The free equations of motion (we comment later on the introduction of a superpotential) are

$$\bar{D}^2 D^2 \Phi = 0\,. \tag{2.4}$$

In components they read

$$\partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\phi = 0\,, \qquad i\partial_{\alpha\dot{\alpha}}\lambda^\alpha = 0\,, \qquad f = 0\,, \tag{2.5}$$

with analogous equations for the conjugated fields. They derive from the superspace action

$$S = \int [d^4x d^2\theta d^2\bar{\theta}]\Phi\bar{\Phi}\,. \tag{2.6}$$

As explained in [2, 5], the symbol $[d^4xd^2\theta d^2\bar{\theta}]$ is not a measure in the usual sense. The form of the integral, where both θ and $\bar{\theta}$ are present, is known as non-chiral superspace integral.

There are two ways to derive the equations of motion (2.5) from (2.6):

1) compute the Berezin integral over θ's and $\bar{\theta}$'s to obtain the component action:

$$S = \int d^4x \left(\frac{1}{2}\partial^{\alpha\dot{\alpha}}\bar{\phi}\partial_{\alpha\dot{\alpha}}\phi + i\bar{\lambda}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\lambda^\alpha + f\bar{f}\right), \tag{2.7}$$

Then, derive eqs. (2.4) by considering the variations with respect to $\bar{\phi}, \bar{\lambda}$ and $\bar{f}$.

2) vary the action with respect to the superfield Φ or $\bar{\Phi}$. This must be done with care since they are constrained fields. First one performs a Berezin integration over $\bar{\theta}$ leading to

$$S = \int [d^4xd^2\theta]\ (\bar{D}^2\bar{\Phi})\Phi\big|_{\bar{\theta}=0} \tag{2.8}$$

where both Φ and $\bar{D}^2\bar{\Phi}$ are computed at $\bar{\theta} = 0$. Notice that due to $D^3 = 0$ and $\bar{D}^3 = 0$ (valid in the case $D = 4$), the superfield $\bar{D}^2\bar{\Phi}$ is also a chiral field. The variation with respect to Φ gives the equations of motion (2.4).

Likewise, one could also integrate with respect to θ to get another version of the action

$$S = \int [d^4xd^2\bar{\theta}]\ \bar{\Phi}(D^2\Phi)\big|_{\theta=0} \tag{2.9}$$

which is the anti-chiral version. Again, the equations of motion are given by (2.4). In (2.6), (2.8) or (2.9) the supersymmetry is manifest since they are written in terms of superfields. Any variation of the Lagrangian under supersymmetry is a total derivative and then the variation of the action vanishes.

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In superspace, the supersymmetric transformations are implemented by the supersymmetry generators $Q_\alpha = \partial_{\theta^\alpha} + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}$ and $\bar Q_{\dot\alpha} = \partial_{\bar\theta^{\dot\alpha}} + i\theta^\alpha\partial_{\alpha\dot\alpha}$ (which commute with the superderivatives D_α and $\bar D_{\dot\alpha}$) as follows

$$\delta_\epsilon\Phi = (\epsilon^\alpha Q_\alpha + \bar\epsilon^{\dot\alpha}\bar Q_{\dot\alpha})\Phi\,, \qquad \delta_\epsilon D_\alpha\Phi = D_\alpha(\delta_\epsilon\Phi)\,. \tag{2.10}$$

In order to add interactions, we need to introduce the superpotential. The superfield $\mathcal{W}(\Phi)$ is an holomorphic function of Φ (for a renormalizable theory a polynomial of maximum degree $= 3$) and the full action is written as (see [11] and [10])

$$S = \int [d^4x d^2\theta d^2\bar\theta]\Phi\bar\Phi + \int [d^4x d^2\theta]\mathcal{W}(\Phi) + \int [d^4x d^2\bar\theta]\overline{\mathcal{W}}(\bar\Phi)\,. \tag{2.11}$$

The contribution of the superpotential is automatically supersymmetric invariant and its holomorphicity w.r.t. Φ implies the non-renormalization properties of the WZ action. The equations of motion are computed as above, by converting the first integral into a chiral or antichiral integral (see eqs. (2.8) or (2.9)) and then varying with respect to Φ (or w.r.t. $\bar\Phi$) to get

$$D^2\Phi + \overline{\mathcal{W}}'(\bar\Phi) = 0\,, \qquad \bar D^2\bar\Phi + \mathcal{W}'(\Phi) = 0\,. \tag{2.12}$$

As a consistency check observe that acting with D_α on the l.h.s. of the first equation, both terms vanish and, similarly acting with $\bar D_{\dot\alpha}$ on the second equation. Acting with $\bar D^2$ on the l.h.s. of the first equation we get

$$\bar D^2 D^2\Phi = \mathcal{W}'(\Phi)\overline{\mathcal{W}}''(\bar\Phi) + \bar D_{\dot\alpha}\bar\Phi\bar D^{\dot\alpha}\bar\Phi\overline{\mathcal{W}}'''(\bar\Phi)\,, \tag{2.13}$$

which reduces to (2.4) in absence of $\mathcal{W}$ and its conjugate.

The generalization to multiple superfields Φ^I with $I = 1,\ldots,N$ is straightforward. The superpotential $\mathcal{W}$ becomes a generic polynomial in the superfields Φ^I, and the kinetic term becomes a quadratic form $\bar\Phi\Phi \to g_{\bar I J}\bar\Phi^{\bar I}\Phi^J$.

To couple the superfields to abelian gauge fields by minimal coupling, one promotes to local superfield the chiral parameter Λ of the rigid symmetry

$$\Phi^I \to e^{ie_I\Lambda}\Phi^I\,, \qquad \bar\Phi^{\bar I} \to e^{-ie_I\overline{\Lambda}}\bar\Phi^{\bar I}\,, \tag{2.14}$$

of the action. The gauge fields are introduced by modifying the action as follows

$$S = \sum_I \int [d^4x d^2\theta d^2\bar\theta] g_{\bar I J}\bar\Phi^{\bar I} e^V \Phi^J + \int [d^4x d^2\theta]\mathcal{W}(\Phi^I) + \int [d^4x d^2\bar\theta]\overline{\mathcal{W}}(\bar\Phi^{\bar I})\,. \tag{2.15}$$

Here V is the prepotential of the gauge fields (see [10] for more details) which transforms as $V \to V + i(\Lambda - \overline{\Lambda})$.

As a final remark, one can convert the action (2.11) into an integral on the complete superspace:

$$S = \int [d^4x d^2\theta d^2\bar\theta]\Big(\bar\Phi\Phi + \mathcal{W}(\Phi)\bar\theta^2 + \overline{\mathcal{W}}(\Phi)\theta^2\Big)\,. \tag{2.16}$$

where we have inserted the θ-terms. Integrating the second term with respect to $\bar\theta$ we obtain again the chiral integral, and likewise for the third term.

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In the following, we need some algebraic relations between superderivatives. In particular, given a superfield $\mathcal{F}_{\alpha\dot\alpha}(x,\theta,\bar\theta)$, we need the relation

$$D^2\bar D^2\Big(\mathcal{F}_{\alpha\dot\alpha}\theta^\alpha\bar\theta^{\dot\alpha}\Big)\Big|_{\theta=\bar\theta=0} = D^\alpha\bar D^{\dot\alpha}\mathcal{F}_{\alpha\dot\alpha}\big|_{\theta=\bar\theta=0} + \text{total deriv.} \tag{2.17}$$

This implies

$$\int [d^4xd^2\theta d^2\bar\theta]\Big(\mathcal{F}_{\alpha\dot\alpha}\theta^\alpha\bar\theta^{\dot\alpha}\Big) = \int [d^4x]\; D^\alpha\bar D^{\dot\alpha}\mathcal{F}_{\alpha\dot\alpha}\big|_{\theta=\bar\theta=0}\,. \tag{2.18}$$

2.2 Geometric WZ action

In the geometrical formulation, we start again from the complex scalar superfield Φ and we impose the following condition

$$\begin{aligned} d\Phi &= V^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\Phi + \psi^\alpha D_\alpha\Phi + \bar\psi^{\dot\alpha}\bar D_{\dot\alpha}\Phi \\ &= V^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\Phi + \psi^\alpha W_\alpha\,, \end{aligned} \tag{2.19}$$

where $(V^{\alpha\dot\alpha},\psi^\alpha,\bar\psi^{\dot\alpha})$ is the supervielbein (see also section 5.1). The differential d is the usual super-differential (it is an anticommuting operator and therefore we assume it anticommutes with θ and $\bar\theta$ as well). Comparing the two lines, we get

$$\bar D_{\dot\alpha}\Phi = 0\,, \qquad D_\alpha\Phi = W_\alpha\,. \tag{2.20}$$

The new superfield W_α of (2.19) has as first component the fermion of the supermultiplet λ_α. Applying d on the left hand side, we have a consistency condition on W_α leading to

$$dW_\alpha = V^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}W_\alpha - 2i\bar\psi^{\dot\alpha}\partial_{\alpha\dot\alpha}\Phi + \psi_\alpha F\,, \tag{2.21}$$

where the new superfield F has as first component the auxiliary field f and $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$. On W_α, we have the conditions

$$D_\alpha W_\beta = -\epsilon_{\alpha\beta}F\,, \qquad \bar D_{\dot\alpha}W_\alpha = -2i\partial_{\alpha\dot\alpha}\Phi\,. \tag{2.22}$$

Again, applying the differential d, we find the differential of F

$$dF = V^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}F + 2i\bar\psi^{\dot\alpha}\partial_{\dot\alpha\alpha}W^\alpha\,, \tag{2.23}$$

and the constraints

$$D_\alpha F = 0\,, \qquad \bar D_{\dot\alpha}F = 2i\partial_{\alpha\dot\alpha}W^\alpha\,, \qquad F = \frac{1}{2}\epsilon^{\alpha\beta}D_\alpha W_\beta = \frac{1}{2}\epsilon^{\alpha\beta}D_\alpha D_\beta\Phi\,, \tag{2.24}$$

where ϕ,λ^α,f are the fields of the Wess-Zumino multiplet. It can be checked that no additional superfields are needed. The first components of the superfields Φ, W^α, F are

$$\Phi = \phi + O(\theta)\,, \qquad W_\alpha = \lambda_\alpha + O(\theta)\,, \qquad F = f + O(\theta)\,. \tag{2.25}$$

In terms of these superfields, the equations of motion are

$$\partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\Phi = 0\,, \qquad \partial_{\alpha\dot\alpha}W^\alpha = 0\,, \qquad F = 0\,, \tag{2.26}$$

and their conjugates. These equations reduce to the spacetime equations, by setting $\theta = \bar\theta = 0$. Note that all components in the $\theta, \bar\theta$ expansion satisfy the same equations, for example, by expanding the superfield at second order $\Phi = \phi + \theta^\alpha \lambda_\alpha + \bar\theta^{\dot\alpha} \bar\lambda_{\dot\alpha} + \mathcal{O}(\theta^2)$ we find

$$\partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\Phi = \partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\phi + \theta^\beta\left(\partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\lambda_\beta\right) + \bar\theta^{\dot\beta}\left(\partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\bar\lambda_{\dot\beta}\right) + \mathcal{O}(\theta^2) \tag{2.27}$$

and $\partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\lambda_\beta = 0$ and $\partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\bar\lambda_{\dot\beta} = 0$ which follow from the Dirac equations (the second eq. in (2.26) and its conjugate) by acting with $\partial^{\dot\beta\alpha}$ on $\partial_{\alpha\dot\alpha}W^\alpha = 0$.

We can write the free Lagrangian $\mathcal{L}^{(4|0)}_{\rm kin}$ for the kinetic terms as follows

$$\begin{aligned}\mathcal{L}^{(4|0)}_{\rm kin} = {} & (V^4)\,(\bar\xi^{\alpha\dot\alpha}\xi_{\alpha\dot\alpha} + \bar F F) \\ & + (V^3)^{\alpha\dot\alpha}\Big[(d\Phi - \psi^\beta W_\beta)\bar\xi_{\alpha\dot\alpha} + (d\bar\Phi - \bar\psi^{\dot\beta}\bar W_{\dot\beta})\xi_{\alpha\dot\alpha} + a_1(\bar W_{\dot\alpha} dW_\alpha + d\bar W_{\dot\alpha} W_\alpha)\Big] \\ & + (V^2_+)^{\alpha\beta}\left[a_2(W_\alpha\psi_\beta d\bar\Phi) + a_3(W_\alpha\psi_\beta \bar W^{\dot\gamma}\bar\psi_{\dot\gamma})\right] \\ & + (V^2_-)^{\dot\alpha\dot\beta}\left[a_2(\bar W_{\dot\alpha}\bar\psi_{\dot\beta} d\Phi) + a_3(\bar W_{\dot\alpha}\bar\psi_{\dot\beta} W^\gamma\psi_\gamma)\right] \\ & + V^{\alpha\dot\alpha}\left[a_4(\bar\Phi d\Phi - d\bar\Phi\Phi)\psi_\alpha\bar\psi_{\dot\alpha}\right]\,.\end{aligned} \tag{2.28}$$

where we have adopted the following definitions (see also appendix B)

$$\begin{aligned} V^4 &= \frac{1}{4!}V_{\alpha\dot\alpha}\wedge V^{\dot\alpha\beta}\wedge V_{\beta\dot\gamma}\wedge V^{\dot\gamma\alpha}\,, & (V^3)^{\alpha\dot\alpha} &= \frac{1}{3!}V^{\alpha\dot\beta}\wedge V^{\dot\gamma\beta}\wedge V^{\rho\dot\alpha}\epsilon_{\dot\beta\dot\gamma}\epsilon_{\beta\rho} \\ (V^2_+)^{\alpha\beta} &= \frac{1}{2!}V^{\alpha\dot\beta}\wedge V^{\dot\beta\beta}\epsilon_{\dot\alpha\dot\beta}\,, & (V^2_-)^{\dot\alpha\dot\beta} &= \frac{1}{2!}V^{\dot\alpha\alpha}\wedge V^{\beta\dot\beta}\epsilon_{\alpha\beta}\,, \end{aligned} \tag{2.29}$$

for the wedge products of the vielbeins $V^{\alpha\dot\alpha}$.

The Lagrangian is organized in powers of V's. The first line, proportional to the volume form V^4, contains two terms: one with the auxiliary fields F and $\bar F$ and the other with the first-order-formalism field $\xi^{\alpha\dot\alpha}$ and its conjugate. The latter are needed in order to write the action without using the Hodge dual operator. This is required for the Lagrangian to be a pure 4-form built exclusively with fields, their differentials and the supervielbeins. We have written all possible terms compatible with the scaling dimensions and with the form degree. The constants a_1, a_2, a_3, a_4 are fixed by requiring the closure of the Lagrangian and the correct equations of motion.

We have four fields $F, \xi_{\alpha\dot\alpha}, \Phi, W_\alpha$ and their conjugates. Therefore, we need four equations of motion.

The equation of F is obtained by varying $\mathcal{L}^{(4|0)}_{\rm kin}$ with respect to $\bar F$. This simply gives

$$(V^4)F = 0 \tag{2.30}$$

which is the free equation of the auxiliary field. The equation for the auxiliary field $\xi_{\alpha\dot\alpha}$ is

$$(V^4)\xi^{\alpha\dot\alpha} + (V^3)^{\alpha\dot\alpha}\left(\psi^\beta D_\beta\Phi + V^{\beta\dot\beta}\partial_{\beta\dot\beta}\Phi - \psi^\beta W_\beta\right) = 0 \tag{2.31}$$

which implies

$$W_\beta = D_\beta\Phi\,,\qquad \bar D_{\dot\beta}\Phi = 0\,,\qquad \xi^{\alpha\dot\alpha} = \partial^{\alpha\dot\alpha}\Phi\,. \tag{2.32}$$

These relations identify the superfield W_α and the auxiliary field $\xi^{\alpha\dot\alpha}$ with derivatives of Φ . In addition, the second equation establishes the chirality of the superfield Φ. The equation of motion for Φ is obtained by taking the functional derivative of the action with respect to the superfield $\bar\Phi$. After integration by parts it becomes

$$\begin{aligned}
&i\left(\psi^\alpha(V_-^2\bar\psi)^{\dot\alpha}-\bar\psi^{\dot\alpha}(V_+^2\psi)^\alpha\right)\xi_{\alpha\dot\alpha}-(V^3)^{\alpha\dot\alpha}d\xi_{\alpha\dot\alpha}\\
&+a_2\left(-2i\psi^{(\alpha}(V\bar\psi)^{\beta)}\right)W_\alpha\psi_\beta+a_2(V_+^2)^{\alpha\beta}dW_\alpha\psi_\beta+2a_4(\psi V\bar\psi)d\Phi=0
\end{aligned}\tag{2.33}$$

where $(\psi V\bar\psi)=\psi^\alpha V^{\beta\dot\beta}\bar\psi^{\dot\alpha}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}$. This equation implies

$$\partial^{\alpha\dot\alpha}\xi_{\alpha\dot\alpha}=0\Longrightarrow\partial^2\Phi=0\,,\qquad a_4=-\frac{i}{2}a_2\,,\qquad a_2=1\Longrightarrow a_4=-\frac{i}{2}\,.\tag{2.34}$$

Finally, the equation for W_α is given by

$$(V^3)^{\alpha\dot\alpha}\bar\psi^{\dot\beta}\xi_{\alpha\dot\alpha}-2a_1(V^3)^{\alpha\dot\beta}dW_\alpha+a_1\,d(V^3)^{\alpha\dot\beta}W_\alpha+(V_-^2)^{\dot\alpha\dot\beta}\left(\bar\psi_{\dot\alpha}d\Phi+a_3\,\bar\psi_{\dot\alpha}W\cdot\psi\right)=0\tag{2.35}$$

(where $W\cdot\psi=W^\alpha\epsilon_{\alpha\beta}\psi^\beta$) yielding the equations of motion for the spinor superfield W_α. We fix the remaining coefficients a_1 and a_3

$$\partial^{\alpha\dot\beta}W_\alpha=0\,,\qquad a_1=\frac{1}{2}\,,\qquad a_3=1\,.\tag{2.36}$$

One can check the consistency among the four equations (2.30), (2.31), (2.33), and (2.35).

To complete the Lagrangian we need the interaction and the superpotential terms. These are written as follows

$$\begin{aligned}
\mathcal{L}^{(4|0)}_{\rm sup}=&\left(\mathcal{W}'(\Phi)F-\frac{1}{2}\mathcal{W}''(\Phi)W_\alpha W^\alpha\right)(V^4)+\mathcal{W}'(\Phi)W^\alpha\bar\psi^{\dot\alpha}(V^3)_{\alpha\dot\alpha}\\
&+\mathcal{W}(\Phi)\bar\psi^{\dot\alpha}\bar\psi^{\dot\beta}(V_-^2)_{\dot\alpha\dot\beta}+\text{h.c.}
\end{aligned}\tag{2.37}$$

where $\mathcal{W}(\Phi)$ is the superpotential introduced in the previous section and $\mathcal{W}'(\Phi),\mathcal{W}''(\Phi)$ are the first and the second derivative of $\mathcal{W}(\Phi)$ with respect to Φ.

The Lagrangian $\mathcal{L}^{(4|0)}=\mathcal{L}^{(4|0)}_{\rm kin}+\mathcal{L}^{(4|0)}_{\rm sup}$ is closed as can be verified by using the definitions of the curvatures $d\Phi, dW_\alpha, dF$ as in (2.19), (2.21), (2.23) and the algebraic equations (2.31).

2.3 WZ action on the supermanifold $\mathcal{M}^{(4|4)}$

Now we show that the action (2.16) can be obtained from the supermanifold integral

$$S=\int_{\mathcal{SM}^{(4|4)}}\mathcal{L}^{(4|0)}(\Phi,W,F)\wedge\mathbb{Y}^{(0|4)}\,,\tag{2.38}$$

where the Lagrangian $\mathcal{L}^{(4|0)}$ is given in the previous section.

The PCO $\mathbb{Y}^{(0|4)}$ is a $(0|4)$-form which depends upon the superspace data. As the Lagrangian is d-closed, we can shift $\mathbb{Y}^{(0|4)}\to\mathbb{Y}^{(0|4)}+d\Lambda^{(-1|4)}$ by an exact term *without changing the action*. The PCO's are discussed in section 5.1 (see also [1–3, 7]).

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The first PCO we consider is given by

$$\mathbb{Y}^{(0|4)}_{s.t.} = \theta^2 \delta^2(\psi) \wedge \bar{\theta}^2 \delta(\bar{\psi}) \,, \tag{2.39}$$

which is closed, not exact and Lorentz invariant. It is not supersymmetric, but its variation under supersymmetry is d-exact. The Dirac delta functions $\delta(\psi)$ and $\delta(\bar{\psi})$ are needed to set ψ and $\bar{\psi}$ in $\mathcal{L}^{(4|0)}$ to zero and the factor $\theta^2\bar{\theta}^2$ sets $\theta = \bar{\theta} = 0$. Thus the integrand (2.38) takes the form

$$\begin{aligned} \mathcal{L}^{(4|0)} \wedge \mathbb{Y}^{(0|4)}_{s.t.} = \Bigg[& (\bar{\xi}^{\alpha\dot{\alpha}} \xi_{\alpha\dot{\alpha}} + \bar{f} f) d^4x \\ & + \left(d\phi \bar{\xi}^{\alpha\dot{\alpha}} + d\bar{\phi} \xi^{\alpha\dot{\alpha}} + \frac{i}{2} (\bar{\lambda}^{\dot{\alpha}} d\lambda^\alpha + d\bar{\lambda}^{\dot{\alpha}} \lambda^\alpha) \right) (d^3x)_{\alpha\dot{\alpha}} \\ & + \left(\mathcal{W}'(\phi) f - \frac{1}{2} \mathcal{W}''(\phi) \lambda^\alpha \epsilon_{\alpha\beta} \lambda^\beta \right) d^4x + \text{h.c.} \Bigg] \theta^2 \bar{\theta}^2 \delta^2(\psi) \delta^2(\bar{\psi}) \,, \end{aligned} \tag{2.40}$$

where $(d^3x)_{\alpha\dot{\alpha}} = dx_{\alpha\dot{\beta}} dx^{\dot{\beta}\gamma} dx_{\gamma\dot{\alpha}}$ By solving the algebraic equations of motion for $\xi^{\alpha\dot{\alpha}}$ and its conjugate, and using

$$d\phi \wedge \delta^2(\psi)\delta^2(\bar{\psi}) = dx^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \phi \wedge \delta^2(\psi)\delta^2(\bar{\psi}) \,, \tag{2.41}$$

one ends up with the component Lagrangian given in (2.11). The choice of the PCO (2.39) represents the trivial embedding of the bosonic submanifold $\mathcal{M}^4$ into the supermanifold $\mathcal{M}^{(4|4)}$.

To derive an action with manifest supersymmetry, we need a different PCO. That will be discussed in the forthcoming section 5.1, and here we report the main result:

$$\mathbb{Y}^{(0|4)}_{s.s.} = \Big(-4(\theta V \bar{\iota}) \wedge (\bar{\theta} V \iota) + \theta^2 (\iota V \wedge V \iota) + \bar{\theta}^2 (\bar{\iota} V \wedge V \bar{\iota}) \Big) \delta^4(\psi) \tag{2.42}$$

where $\iota = \partial_\psi$ (and similar for $\bar{\iota}$). Notice that it still depends upon θ and $\bar{\theta}$. This is needed to produce the superspace action in the usual form. In addition, we notice that the first term is non-chiral and the other two are chiral and anti-chiral, respectively.

With this PCO, the action becomes

$$\begin{aligned} S &= \int_{\mathcal{M}^{(4|4)}} \mathcal{L}^{(4|0)} \wedge \mathbb{Y}^{(0|4)}_{s.s.} \\ &= \int_{\mathcal{M}^{(4|4)}} \Big((\overline{W} V \psi)(\bar{\psi} V W) + \mathcal{W}(\Phi)(\bar{\psi} V \wedge V \bar{\psi}) + \overline{\mathcal{W}}(\bar{\Phi})(\psi V \wedge V \psi) \Big) \wedge \mathbb{Y}^{(0|4)}_{s.s.} \\ &= \int_{\mathcal{M}^{(4|4)}} \Big(\overline{W}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} W_\alpha \theta^\alpha + \mathcal{W}(\Phi) \bar{\theta}^2 + \overline{\mathcal{W}}(\bar{\Phi}) \theta^2 \Big) V^4 \delta^4(\psi) \\ &= \int [d^4x d^2\theta d^2\bar{\theta}] \Big(\overline{W}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} W_\alpha \theta^\alpha + \mathcal{W}(\Phi) \bar{\theta}^2 + \overline{\mathcal{W}}(\bar{\Phi}) \theta^2 \Big) . \end{aligned} \tag{2.43}$$

and, using the algebraic relations among superderivatives given in (2.17) and (2.18), recalling $W_\alpha = D_\alpha \Phi, \bar{W}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} \bar{\Phi}$, and

$$(D_\alpha \Phi)\theta^\alpha = D_\alpha(\Phi \theta^\alpha) + 2\Phi \,, \tag{2.44}$$

and integrating by parts, one arrives at the usual superspace action (2.16). Notice that the three pieces of the PCO $\mathbb{Y}^{(0|4)}_{s.s.}$ in (2.42) are essential to get the complete action since the terms for the kinetic part and for the superpotential have completely different algebraic structures. Notice also the unusual form of the kinetic term which has a non-chiral structure as said above.

3 D=4 N=1 integral super Yang-Mills

Using the same strategy, we now study the SYM action in this framework. In section 3.1, we review SYM in the superspace formulation (see [10] for further details). In section 3.2 we review the geometric (rheonomic) formulation of SYM and we discuss the equations of motion. In section 3.3, we prove that both the component action and the superspace action can be retrieved from the same supermanifold action by changing the PCO; the same PCO given in (2.42) produces the superspace action.

3.1 SYM superspace action

It is convenient to adopt again a Weyl/anti-Weyl notation in order to describe the superspace action in its most common formulation [10]. The gauge field is identified with the (1|0)-superconnection

$$A^{(1|0)} = A_{\alpha\dot\alpha}V^{\alpha\dot\alpha} + A_\alpha\psi^\alpha + A_{\dot\alpha}\bar\psi^{\dot\alpha}\,, \tag{3.1}$$

and the field strength is

$$\begin{aligned}\mathcal{F} &= dA^{(1|0)} + A^{(1|0)}\wedge A^{(1|0)} \\ &= F_{\alpha\dot\alpha\beta\dot\beta}V^{\alpha\dot\alpha}\wedge V^{\beta\dot\beta} + F_{\alpha\dot\alpha\beta}V^{\alpha\dot\alpha}\wedge\psi^\beta + F_{\alpha\dot\alpha\dot\beta}V^{\alpha\dot\alpha}\wedge\bar\psi^{\dot\beta} \\ &\quad + F_{\alpha\beta}\psi^\alpha\wedge\psi^\beta + F_{\alpha\dot\beta}\psi^\alpha\wedge\bar\psi^{\dot\beta} + F_{\dot\alpha\dot\beta}\bar\psi^{\dot\alpha}\wedge\bar\psi^{\dot\beta}\,.\end{aligned} \tag{3.2}$$

The superfield $A^{(1|0)}$ contains several independent components exceeding the physical ones. Therefore, to reduce that number one needs additional constraints. It is customary to set all spinorial field strengths to zero

$$F_{\alpha\beta} = 0\,, \qquad F_{\alpha\dot\beta} = 0\,, \qquad F_{\dot\alpha\dot\beta} = 0\,. \tag{3.3}$$

Consequently, the Bianchi identities $d\mathcal{F} + A^{(1|0)}\wedge\mathcal{F} = 0$ imply some constraints on the remaining field strengths which can be easily solved.

The parametrizations of the curvatures are

$$\begin{aligned}\mathcal{F} &= F^+_{\alpha\beta}(V^2_+)^{\alpha\beta} + F^-_{\dot\alpha\dot\beta}(V^2_-)^{\dot\alpha\dot\beta} + 2i\,\overline{W}_{\dot\alpha}(V\psi)^{\dot\alpha} + 2i\,W_\alpha(V\bar\psi)^\alpha \\ \nabla W_\alpha &= V^{\beta\dot\beta}\nabla_{\beta\dot\beta}W_\alpha - (F^+\psi)_\alpha + \mathcal{D}\,\epsilon_{\alpha\beta}\psi^\beta\,, \\ \nabla\overline{W}_{\dot\alpha} &= V^{\beta\dot\beta}\nabla_{\beta\dot\beta}\overline{W}_{\dot\alpha} - (F^-\bar\psi)_{\dot\alpha} - \mathcal{D}\,\epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}\,, \\ \nabla\mathcal{D} &= V^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}\mathcal{D} - \bar\psi^{\dot\alpha}\nabla_{\alpha\dot\alpha}W^\alpha - \psi^\alpha\nabla_{\alpha\dot\alpha}\overline{W}^{\dot\alpha}\end{aligned} \tag{3.4}$$

where $(F^+\psi)_\alpha = F^+_{\alpha\beta}\psi^\beta$, $(F^-\bar\psi)_{\dot\alpha} = F^-_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}$, $W^\alpha = \epsilon^{\alpha\beta}W_\beta$ and $\bar W^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar W_{\dot\beta}$. The real scalar field $\mathcal{D}$ is an auxiliary field needed to close the algebra off-shell. Notice that

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setting $\mathcal{D}=0$, the last line implies the Dirac equations for W_α and $\bar{W}_{\dot\alpha}$: $\partial_{\alpha\dot\alpha}\bar{W}^{\dot\alpha}=0$ and $\partial_{\alpha\dot\alpha}W^\alpha=0$.

The Bianchi identities for the curvatures $\mathcal{F}, dW_\alpha, d\bar{W}_{\dot\alpha}, d\mathcal{D}$ together with their parametrization (3.4) yield the constraints

$$\begin{aligned} &F^+_{\alpha\beta}=D_{(\alpha}W_{\beta)}\,, && \overline{D}_{\dot\alpha}W^\beta=0\,, && \mathcal{D}=D_\alpha W^\alpha\,,\\ &F^-_{\dot\alpha\dot\beta}=\overline{D}_{(\dot\alpha}\overline{W}_{\dot\beta)}\,, && D_\alpha\overline{W}^{\dot\beta}=0\,, && \mathcal{D}=\overline{D}_{\dot\alpha}\overline{W}^{\dot\alpha}\,, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &D_\rho F^-_{\dot\alpha\dot\beta}=-2i\,\partial_{\rho(\dot\alpha}\overline{W}_{\dot\beta)}\,, && \overline{D}_{\dot\rho}F^-_{\dot\alpha\dot\beta}=-2i\,\epsilon_{\dot\rho(\dot\alpha}\partial_{\dot\beta)\rho}W^\rho\,,\\ &\overline{D}_{\dot\rho}F^+_{\alpha\beta}=-2i\,\partial_{\dot\rho(\alpha}W_{\beta)}\,, && D_\rho F^+_{\alpha\beta}=-2i\,\epsilon_{\rho(\alpha}\partial_{\beta)\dot\rho}\overline{W}^{\dot\rho}\,, \end{aligned} \tag{3.6}$$

The latter can be verified by using (3.5) together with the algebra of superderivatives and with the Schouten identities $\epsilon_{\rho\alpha}\epsilon^{\tau\sigma}=(\delta^\tau_\alpha\delta^\sigma_\rho-\delta^\tau_\rho\delta^\sigma_\alpha)$ and $\epsilon_{\dot\rho\dot\alpha}\epsilon^{\dot\tau\dot\sigma}=(\delta^{\dot\tau}_{\dot\alpha}\delta^{\dot\sigma}_{\dot\rho}-\delta^{\dot\tau}_{\dot\rho}\delta^{\dot\sigma}_{\dot\alpha})$.

The second equation of the first line of (3.5) implies that the superfield W^α is chiral and therefore can be decomposed as follows

$$W_\alpha=\lambda_\alpha+(f_{\alpha\beta}+\epsilon_{\alpha\beta}\hat{\mathcal{D}})\theta^\beta+\frac{1}{2}\partial_{\alpha\dot\alpha}\overline{\lambda}^{\dot\alpha}\theta^2 \tag{3.7}$$

where $\lambda_\alpha(x), \overline{\lambda}_{\dot\alpha}(x)$ are the Weyl/anti-Weyl components of the gaugino, $f_{\alpha\beta}(x), f_{\dot\alpha\dot\beta}(x)$ are the self-dual and anti-self dual part of the Maxwell tensor and $\hat{\mathcal{D}}$ is the real auxiliary field (the first component of $\mathcal{D}=\hat{\mathcal{D}}(x)+O(\theta)$).

In terms of these fields the superspace action can be written as

$$S=\int[d^4xd^2\theta]W^\alpha W_\alpha+\int[d^4xd^2\bar\theta]\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha} \tag{3.8}$$

separating the chiral and the antichiral part. Again, as in the WZ case, we can rewrite the action as an integral on the full superspace (non-chiral integral) as follows

$$S=\int[d^4xd^2\theta d^2\bar\theta]\Big(W^\alpha W_\alpha\bar\theta^2+\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}\theta^2\Big) \tag{3.9}$$

where the powers of θ and $\bar\theta$ are needed to reproduce the correct action. The last equation (3.9) will be useful for the comparison with the supermanifold approach.

3.2 Geometric SYM action

Following the method described in the book [12–14], based on scaling dimensions of the fields, form degree, Lorentz invariance and gauge invariance, the geometric (rheonomic)

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Lagrangian for N=1 super Yang-Mills is found to be

$$
\begin{aligned}
\mathcal{L}^{(4|0)} = & \operatorname{Tr}(\mathcal{F}F^{+}_{\alpha\beta})_{\wedge}(V_{+}^{2})^{\alpha\beta} + \operatorname{Tr}(\mathcal{F}F^{-}_{\dot\alpha\dot\beta})_{\wedge}(V_{-}^{2})^{\dot\alpha\dot\beta} \\
& - \operatorname{Tr}\left(F^{+}_{\alpha\beta}F^{+\alpha\beta} + F^{-}_{\dot\alpha\dot\beta}F^{-\dot\alpha\dot\beta} + \frac{1}{2}\mathcal{D}^{2}\right)(V^{4}) \\
& - \frac{1}{2}\operatorname{Tr}(\bar W_{\dot\alpha}\nabla W_{\alpha} + \nabla \bar W_{\dot\alpha} W_{\alpha})_{\wedge}(V^{3})^{\dot\alpha\alpha} \\
& - 4i\Big(\operatorname{Tr}(F^{+}_{\alpha\beta}\bar W_{\dot\sigma})(V^{3})^{\alpha\dot\sigma}{}_{\wedge}\psi^{\beta} - \operatorname{Tr}(F^{-}_{\dot\alpha\dot\beta}\bar W^{\dot\beta})(V^{3}\psi)^{\dot\alpha} \\
& \quad - \operatorname{Tr}(F^{+}_{\alpha\beta}W^{\beta})(V^{3}\bar\psi)^{\alpha} + \operatorname{Tr}(F^{-}_{\dot\alpha\dot\beta}W_{\sigma})(V^{3})^{\dot\alpha\sigma}{}_{\wedge}\bar\psi^{\dot\beta}\Big) + \\
& + 2i\left(\operatorname{Tr}(\mathcal{F}\bar W_{\dot\alpha})_{\wedge}(V\psi)^{\dot\alpha} + \operatorname{Tr}(\mathcal{F}W_{\alpha})_{\wedge}(V\bar\psi)^{\alpha}\right) \\
& + 2\Big(\operatorname{Tr}(W_{\alpha}W_{\beta})\epsilon^{\alpha\beta}(\bar\psi V_{-}^{2}\bar\psi) + \operatorname{Tr}(\bar W_{\dot\alpha}\bar W_{\dot\beta})\epsilon^{\dot\alpha\dot\beta}(\psi V_{+}^{2}\psi)\Big)
\end{aligned}
\tag{3.10}
$$

The Lagrangian is closed, by using the parametrization of curvatures (3.4) and the algebraic equation for $F^{+}_{\alpha\beta}$ and for $F^{-}_{\dot\alpha\dot\beta}$. The closure of $\mathcal{L}^{(4|0)}$ implies also the supersymmetry invariance of the action since $\ell_{\epsilon}\mathcal{L}^{(4|0)} = d\iota_{\epsilon}\mathcal{L}^{(4|0)}$.

The first three lines contain those terms which reduce to the component action by using the simplest PCO

$$
\mathbb{Y}^{(0|4)}_{s.t.} = \theta^{2}\bar\theta^{2}\delta^{2}(\psi)\delta^{2}(\bar\psi)\,. \tag{3.11}
$$

The action is

$$
S = \int_{\mathcal{M}^{(4|4)}} \mathcal{L}^{(4|0)}(A, F^{\pm}, W, \bar W) \wedge \mathbb{Y}^{(0|4)}_{s.t.}\,. \tag{3.12}
$$

The Dirac delta's for ψ and $\bar\psi$ set the last four lines to zero, whereas the factor $\theta^{2}\bar\theta^{2}$ extracts the lowest components of the superfields $\mathcal{F}, W_{\alpha}$ and $\bar W_{\dot\alpha}$. These coincide with the curvature of the gauge field (after using the algebraic equations of motion for $F^{+}_{\alpha\beta}, F^{-}_{\dot\alpha\dot\beta}$) and with the gauginos, respectively.

3.3 SYM action on the supermanifold $\mathcal{M}^{(4|4)}$

The way to get the superspace action is to consider the following supermanifold integral

$$
S_{SYM} = \int_{\mathcal{M}^{(4|4)}} \mathcal{L}^{(4|0)} \wedge \mathbb{Y}^{(0|4)}_{s.s.} \tag{3.13}
$$

where the integral is extended to the full supermanifold $\mathcal{M}^{(4|4)}$. Now, in order to reproduce the superspace action, we use the real PCO discussed in section 2.3 (see also section 5.1 for the computational details).

$$
\mathbb{Y}^{(0|4)} = \Big(-4(\theta V\bar\iota) \wedge (\bar\theta V\iota) + \theta^{2}(\iota V \wedge V\iota) + \bar\theta^{2}(\bar\iota V \wedge V\bar\iota)\Big)\delta^{4}(\psi) \tag{3.14}
$$

The last two terms in the Lagrangian (3.10) can be rewritten as follows

$$
\begin{aligned}
& 2\int \Big(\operatorname{Tr}(W_{\alpha}W_{\beta})\epsilon^{\alpha\beta}(\bar\psi V_{-}^{2}\bar\psi) + \operatorname{Tr}(\bar W_{\dot\alpha}\bar W_{\dot\beta})\epsilon^{\dot\alpha\dot\beta}(\psi V_{+}^{2}\psi)\Big) \wedge \mathbb{Y}^{(0|4)} = \\
& = \int \Big(W_{\rho}W^{\rho}\omega^{(4|2)}\bar\theta^{2}\delta^{2}(\bar\psi) + \text{h.c.}\Big) = \int [d^{4}xd^{2}\theta]W^{\alpha}W_{\alpha} + \int [d^{4}xd^{2}\bar\theta]\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}
\end{aligned}
\tag{3.15}
$$

where $\omega^{(4|2)} = V^4\delta^2(\psi)$ is the chiral volume form discussed in section 5.3. The last two integrals are computed with respect to the chiral superspaces (x,θ) and $(x,\bar{\theta})$. The final answer coincides with the usual superspace Lagrangian. Notice that there is no other contribution from the complicated action (3.10) because of the power of V's and the derivatives of delta functions.

4 D=4 N=2 integral SYM

4.1 N=2 vector superfields

To discuss the N=2 case, we consider the simplest case, namely the N=2 vector multiplet. This contains $4 \oplus 4$ on-shell degrees of freedom. The superspace is described by the coordinates $(x^a, \theta^\alpha_A, \bar{\theta}^{\dot{\alpha}}_A)$ with $A = 1, 2$.

These degrees of freedom are easily understood in terms of N=1 superfields: one chiral superfield Φ and one real superfield V (better expressed in terms of the chiral superfield W^α). The off-shell degrees of freedom are 3 bosonic d.o.f. for the gauge field (with one gauge degree of freedom), 1 d.o.f. for the auxiliary field $\mathcal{D}$, a complex scalar ϕ and the complex auxiliary field F; on the other side, there are 8 fermions for the $N = 2$ gaugino.

We define a N=2 chiral superfield as a complex scalar superfield Ψ constrained by the conditions

$$\bar{D}_{\dot{\alpha},A}\Psi = 0\,, \qquad A = 1, 2 \tag{4.1}$$

where $D_{\dot{\alpha}A}$ is the superderivative with the algebra $\{D^A_\alpha, D^B_\beta\} = 0$ and $\{D^A_\alpha, \bar{D}_{B\dot{\beta}}\} = 2i\delta^A_{\ B}\gamma^a_{\alpha\dot{\beta}}\partial_a$. Solving the constraints, we get the expression

$$\Psi(x,\theta_A) = \Phi(x,\theta_1) + W_\alpha(x,\theta_1)\theta^\alpha_2 + F(x,\theta_1)\,(\theta_2)^2 \tag{4.2}$$

where F is related to the complex conjugate of Φ and of W^α (see [15, 16]).

In (4.2), we expanded the superfield Ψ in terms of θ^α_2. The components Φ, W^α, F are superfields depending on (x^a, θ^α_1). The action for the vector superfield Ψ reads

$$S = \mathrm{Im}\frac{1}{2}\int [d^4x d^2\theta_1 d^2\theta_2]\,\Psi^2 \tag{4.3}$$

Performing the Berezin integral over θ^a_2 produces the action of N=1 superfield W^α coupled to a chiral superfield Φ.

Let us move to the rheonomic action. We should consider the rheonomic parametrization. The first equation is

$$d\Psi = \partial_a\Psi V^a + \lambda^A_\alpha\psi^\alpha_A\,, \tag{4.4}$$

where we have denoted by $\lambda^A_\alpha = D^A_\alpha\Psi$ the gauginos. In the same way we define the Maxwell tensor F^+, F^- and the scalar $\Phi = A + iB$

$$\begin{aligned} F^+_{\alpha\beta} &= \epsilon_{AB}D^A_{(\alpha}\lambda^B_{\beta)}\,, & \overline{F}^-_{\dot{\alpha}\dot{\beta}} &= \epsilon^{AB}\overline{D}^{(\dot{\alpha}}_A\overline{\lambda}^{\dot{\beta})}_B\,, \\ \Phi &= \epsilon_{AB}\epsilon^{\alpha\beta}D^A_\alpha\lambda^B_\beta\,, & \overline{\Phi} &= \epsilon^{AB}\epsilon_{\dot{\alpha}\dot{\beta}}\overline{D}^{\dot{\alpha}}_A\overline{\lambda}^{\dot{\beta}}_B \end{aligned} \tag{4.5}$$

In terms of those fields, the rheonomic action is given in the book [12–14] (in eq. II.9.34). Here we reproduce only the relevant terms

$$\mathcal{L}^{(4|0)}_{\rm rheo} = \cdots + \frac{i}{4}(A^2 - B^2)\epsilon^{AB}\epsilon^{CD}\overline{\psi}_A\psi_B\overline{\psi}_C\gamma_5\psi_D + \frac{1}{4}A\,B\epsilon^{AB}\epsilon^{CD}\overline{\psi}_A\psi_B\overline{\psi}_C\psi_D + \dots \tag{4.6}$$

where the ellipsis stand for other terms of the action which do not contribute. The superfields A and B are the real and imaginary part of the chiral superfield Φ. We selected those terms of the action which contain four gravitinos ψ_A. All other terms contain at least one power of V^a.

Now we study the PCO. As discussed in the above sections, we have the simplest PCO

$$\mathbb{Y}^{(0|8)} = \theta^8\delta^8(\psi) \tag{4.7}$$

where

$$\theta^8 = (\epsilon^{AC}\epsilon^{BD}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\theta^\alpha_A\theta^\beta_B\theta^\gamma_C\theta^\delta_D)(\epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\delta}\epsilon_{AC}\epsilon_{BD}\bar\theta^A_{\dot\alpha}\bar\theta^B_{\dot\beta}\bar\theta^C_{\dot\gamma}\bar\theta^D_{\dot\delta}) \tag{4.8}$$

and equivalently for $\delta^8(\psi)$. The PCO is closed and not exact. Computing the action

$$S = \int_{\mathcal{M}^{(4|8)}} \mathcal{L}^{(4|0)}_{\rm rheo} \wedge \mathbb{Y}^{(0|8)} \tag{4.9}$$

we get the component action for $N = 2$ SYM in $d = 4$. Since the Lagrangian $\mathcal{L}^{(4|0)}_{\rm rheo}$ is closed we can change the PCO at will (in the same cohomology class). In particular, we can choose a supersymmetric PCO. For this we notice that we can construct such an operator by multiplying two PCO's of the $N = 1$ type given in section 5.3:

$$\mathbb{Y}^{(0|4)}_A = V^a \wedge V^b(\bar\theta^2_A\iota_A\gamma_{ab}\iota_A + \text{h.c.})\delta^4(\psi_A)\,, \qquad A = 1, 2 \tag{4.10}$$

where $\iota_{A\alpha} = \partial/\partial\psi^{\alpha A}$ and we obtain

$$\begin{aligned}\mathbb{Y}^{(0|8)} &= V^a \wedge V^b(\bar\theta^2_1\iota_1\gamma_{ab}\iota_1 + \text{h.c.}) \wedge V^c \wedge V^d(\bar\theta^2_2\iota_2\gamma_{cd}\iota_2 + \text{h.c.})\delta^8(\psi) \\ &= V^4\epsilon^{abcd}\left(\bar\theta^4\iota_1\gamma_{ab}\iota_1\iota_2\gamma_{cd}\iota_2 + \text{h.c.}\right)\delta^8(\psi)\end{aligned} \tag{4.11}$$

which is closed and not exact. Notice that closure is easily verified by using the MC equations $dV^a = \bar\psi^A\gamma^a\psi_A$. The presence of the factor $\bar\theta^4$ is essential for the non-exactness. The other terms are needed to have a real PCO.

The main issue is the overall factor V^4. This is due to the two factors V^a in the factorized PCO's $\mathbb{Y}^{(0|4)}_A$ and to their anti-symmetrization. That factor is essential to provide the bosonic part of the volume integral form. On the other side, the four derivatives ι^A_α must act on four gravitino terms in the action. Thus the four-gravitino terms of the action (4.9) are selected, giving a term proportional to the scalar $(A+iB)^2 = (A^2-B^2)+2iAB$. In addition, the PCO selects the chiral part of the superfields leading to the correct action (4.3).

5 The geometry of D=4 N=1 supermanifolds

The integral forms are the crucial ingredients to define a geometric integration theory for supermanifolds inheriting all the good properties of integration theory in conventional (purely bosonic) geometry. In this section we briefly describe the notations and the most relevant definitions (see [5] and also [1–3, 7]). We introduce the complexes of superforms, of integral forms and of pseudo-forms. These complexes are represented in the figure 1 below. Horizontally the operator is the usual odd differential, vertically (up and down) the *picture changing operators* (PCO's) map cohomology classes into cohomology classes. The PCO's are not coboundary operators, so figure 1 does not represent a double complex. The complexes are filtered by two numbers (the form number and the picture number) as described below.

The present section is organized as follows: 1) we first review some of the properties of the complex of superforms and the differential operators acting on it, 2) we discuss the properties of forms under Lorentz and linear transformations, 3) we discuss the space of superfields and of the volume forms, 4) we construct a few new cohomology classes needed for applications, 5) we build the PCO of type $\mathbb{Y}$ (raising the picture number) with manifest supersymmetry, 6) we check that by consistency the action of the PCO of type $\mathbb{Z}$ (lowering the picture number) indeed maps cohomology into cohomology. In section 5.7 we rederive, in the integral form framework, two well-known theorems for superspace field theories.

5.1 Flat D=4 N=1 integral superspace

Let us first discuss the $D = 4$ $N = 1$ supermanifold $\mathcal{M}^{(4|4)}$. Locally it is described in terms of the coordinates ($x^{\alpha\dot{\alpha}}, \theta^a, \bar{\theta}^{\dot{\alpha}}$ with $\alpha, \dot{\alpha} = 1, 2$) of the superspace $\mathbb{R}^{(4|4)}$. We recall that $x^a = x^{\alpha\dot{\alpha}}\gamma^a_{\alpha\dot{\alpha}}$ ($a = 1 \dots 4$, see appendix A for details on the relations between vectorial and chiral notations). We will use the notation $(4|4)$ to denote quantities in the real representation, and the notations $(4|2, 0)$ and $(4|0, 2)$ for chiral (or anti-chiral) quantities.

Let us fix our conventions. We define the flat supervielbeins

$$V^{\alpha\dot{\alpha}} = dx^{\alpha\dot{\alpha}} + i(\theta^\alpha d\bar{\theta}^{\dot{\alpha}} + d\theta^\alpha\bar{\theta}^{\dot{\alpha}}) , \qquad \psi^\alpha = d\theta^\alpha , \qquad \bar{\psi}^{\dot{\alpha}} = d\bar{\theta}^{\dot{\alpha}} , \tag{5.1}$$

which satisfy

$$dV^{\alpha\dot{\alpha}} = 2\, i\psi^\alpha \wedge \bar{\psi}^{\dot{\alpha}} , \qquad d\psi^\alpha = 0 , \qquad d\bar{\psi}^{\dot{\alpha}} = 0 . \tag{5.2}$$

We also denote the derivatives as follows

$$\partial_{\alpha\dot{\alpha}} , \qquad D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}} , \qquad \bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\partial_{\alpha\dot{\alpha}} , \tag{5.3}$$

with the commutation relations

$$\{D_\alpha, D_\beta\} = 0 , \qquad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 , \qquad \{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\partial_{\alpha\dot{\alpha}} , \tag{5.4}$$

while $\partial_{\alpha\dot{\alpha}}$ commutes with the other differential operators. We introduce the contraction operators

$$\iota_{\alpha\dot{\alpha}} = \iota_{\partial_{\alpha\dot{\alpha}}} , \qquad \iota_\alpha \equiv \iota_{D_\alpha} = \iota_{\partial_\alpha} - i\bar{\theta}^{\dot{\alpha}}\iota_{\alpha\dot{\alpha}} , \qquad \bar{\iota}_{\dot{\alpha}} = \iota_{\bar{D}_{\dot{\alpha}}} = \bar{\iota}_{\partial_{\dot{\alpha}}} - i\theta^\alpha\iota_{\alpha\dot{\alpha}} , \tag{5.5}$$

where $\iota_{\partial_\alpha} \equiv \frac{\partial}{\partial\psi^\alpha}$ and $\iota_{\partial_{\dot{\alpha}}} = \frac{\partial}{\partial\bar{\psi}^{\dot{\alpha}}}$. The following relations hold:

$$\begin{aligned} &\iota_{\alpha\dot{\alpha}}V^{\beta\dot{\beta}} = \delta_\alpha^{\ \beta}\delta_{\dot{\alpha}}^{\ \dot{\beta}} , && \iota_\alpha\psi^\beta = \delta^\beta_\alpha , && \iota_{\dot{\alpha}}\bar{\psi}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\ \dot{\beta}} , \\ &\iota_{\alpha\dot{\alpha}}\psi^\beta = \iota_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\beta}} = 0 , && \iota_\alpha V^{\beta\dot{\beta}} = \iota_\alpha\bar{\psi}^{\dot{\beta}} = 0 , && \bar{\iota}_{\dot{\alpha}}V^{\beta\dot{\beta}} = \bar{\iota}_{\dot{\alpha}}\psi^\beta = 0 , \end{aligned} \tag{5.6}$$

$$
\begin{array}{ccccccccc}
\Omega^{(0|0)} & \xrightarrow{d} & \Omega^{(1|0)} & \xrightarrow{d} & \Omega^{(2|0)} & \xrightarrow{d} & \Omega^{(3|0)} & \xrightarrow{d} & \Omega^{(4|0)} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\Omega^{(0|1)} & \xrightarrow{d} & \Omega^{(1|1)} & \xrightarrow{d} & \Omega^{(2|1)} & \xrightarrow{d} & \Omega^{(3|0)} & \xrightarrow{d} & \Omega^{(4|0)} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\Omega^{(0|2)} & \xrightarrow{d} & \Omega^{(1|2)} & \xrightarrow{d} & \Omega^{(2|2)} & \xrightarrow{d} & \Omega^{(3|2)} & \xrightarrow{d} & \Omega^{(4|2)} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\Omega^{(0|3)} & \xrightarrow{d} & \Omega^{(1|3)} & \xrightarrow{d} & \Omega^{(2|3)} & \xrightarrow{d} & \Omega^{(3|3)} & \xrightarrow{d} & \Omega^{(4|3)} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\Omega^{(0|4)} & \xrightarrow{d} & \Omega^{(1|4)} & \xrightarrow{d} & \Omega^{(2|4)} & \xrightarrow{d} & \Omega^{(3|4)} & \xrightarrow{d} & \Omega^{(4|4)}
\end{array}
$$

Figure 1. The complex of pseudoforms for the supermanifold $\mathcal{M}^{(4|4)}$.

The contraction operator $\iota_{\alpha\dot{\alpha}}$ is an odd differential operator, while ι_α and $\bar{\iota}_{\dot{\alpha}}$ are even. Their (anti)commutation relations are all vanishing.

The first row in figure 1 is the complex of superforms and the last one is the complex of integral forms (the pseudoforms of maximal picture). The differential d is the usual odd differential. Along the vertical line (up and down), the picture changing operators (PCO's) act by increasing or decreasing the picture (i.e.the number of delta forms).

We denote by Ω the space of all pseudoforms. It is filtered by two integers numbers p and q:

$$\Omega = \bigoplus_{p,q} \Omega^{(p|q)} \left(\mathcal{M}^{(4|4)}\right) \tag{5.7}$$

where q denotes the picture number and p is the form number. The picture q ranges between $0 \leq q \leq 4$. The range of values for p depends on q. At picture zero ($q = 0$), we have the space of superforms $\Omega^{(p|0)}$. A generic element $\omega^{(p|0)}$ is given by:

$$\omega^{(p|0)} = \sum_{r,s,t,r+s+t=p} \omega_{[a_1 \dots a_r](\alpha_1 \dots \alpha_s)(\dot{\alpha}_1 \dots \dot{\alpha}_t)} V^{a_1} \dots V^{a_r} \psi^{\alpha_1} \dots \psi^{\alpha_s} \bar{\psi}^{\dot{\alpha}_1} \dots \bar{\psi}^{\dot{\alpha}_t} \tag{5.8}$$

where the coefficients $\omega_{[a_1 \dots a_r](\alpha_1 \dots \alpha_s)(\dot{\alpha}_1 \dots \dot{\alpha}_t)}(x, \theta, \bar{\theta})$ are superfields. There is no upper bound in the number of ψ's and $\bar{\psi}$'s, therefore $p \geq 0$ for $q = 0$. However, it will be seen that there are no nontrivial cohomology classes for $p > 4$. The total form number is

$$p = r + s + t\,. \tag{5.9}$$

At maximal picture we have the space of the integral forms $\Omega^{(p|4)}$. A generic element $\omega^{(p|4)}$ is given by:

$$\omega^{(p|4)} = \sum_r \sum_{\beta_1 \beta_2} \sum_{\gamma_1 \gamma_2} \omega_{[a_1 \dots a_r]} V^{a_1} \dots V^{a_r} \delta^{(\beta_1)}(\psi^1) \delta^{(\beta_2)}(\psi^2) \delta^{(\gamma_1)}(\bar{\psi}^{\dot{1}}) \delta^{(\gamma_2)}(\bar{\psi}^{\dot{2}}) \tag{5.10}$$

where $\delta^{(\beta_1)}(\psi^1) = (\iota_1)^{\beta_1} \delta(\psi^1) = \frac{\partial^{\beta_1}}{\partial(\psi^1)^{\beta_1}} \delta(\psi^1)$ denotes the β_1- th derivative of $\delta(\psi^1)$ with respect to its argument (and analogously for the other terms in the monomial). The derivatives of the delta's carry negative form degree and therefore the total form number of $\omega^{(p|4)}$ is

$$p = r - (\beta_1 + \beta_2 + \gamma_1 + \gamma_2)\,. \tag{5.11}$$

Thus the complex of integral forms is bounded from above, but it is unbounded from below. Notice that $\delta(\psi)$ and $\delta(\bar{\psi})$ carry zero form degree and that $\psi^1\iota_1\delta(\psi^1) = -\delta(\psi^1)$ (and analogously for the other terms).

For $p > 4$, $\Omega^{(p|4)} = 0$, but we can have any negative-degree integral form in the spaces $\Omega^{(-p|4)}$ with $p > 0$. It is important to notice that each space $\Omega^{(p|4)}$ for any p is finitely generated and that its dimension increases when the form degree decreases. This parallels the case of superforms whose complex is also finitely generated with a dimension that increases with higher ψ and $\bar{\psi}$ powers. That is the basis for establishing the Hodge dual correspondence between the two complexes

$$\star : \Omega^{(p|0)}(\mathcal{M}) \longrightarrow \Omega^{(4-p|4)}(\mathcal{M}) \tag{5.12}$$

as discussed in [3] and [17].

Finally, we have the spaces of pseudoforms with $0 < q < 4$. Each space $\Omega^{(p|q)}$ is not finitely generated and these complexes are unbounded from above and from below. Since there are no nontrivial cohomology classes in $\Omega^{(p|q)}$ with $p > 4$ and $p < 0$ (as discussed for example in [18]), we restrict our analysis to the square box formed by the complexes $\Omega^{(p|q)}$ with $0 \leq q \leq 4$ and $0 \leq p \leq 4$. Note that even for pseudoforms there is a Hodge duality operator

$$\star : \Omega^{(p|q)} \longrightarrow \Omega^{(4-p|4-q)} \tag{5.13}$$

We consider now some operators. The odd differential d acts horizontally

$$d : \Omega^{(p|q)} \longrightarrow \Omega^{(p+1|q)} \tag{5.14}$$

increasing the form degree and leaving unmodified the picture number. We have already introduced the contraction operators $\iota_a, \iota_\alpha, \iota_{\dot\alpha}$ and consequently the Lie derivatives $\mathcal{L}_a = i_a d + d i_a$ etc. The $d-$cohomology is well-defined in the present framework and we denote by $H_d(\Omega^{(p|q)})$ de Rham cohomology classes of $(p|q)$ pseudoforms.

Following the discussion in [2, 7], we need also the *Picture Changing Operators* $\mathbb{Y}_k^{(0|1)}$. They act multiplicatively (using the graded wedge product of pseudoforms) on the spaces $\Omega^{(p|q)}$:

$$\mathbb{Y}_k^{(0|1)} : \Omega^{(p|q)} \longrightarrow \Omega^{(p|q+1)}\,, \tag{5.15}$$

with $\omega^{(p|q+1)} = \omega^{(p|q)} \wedge \mathbb{Y}_k^{(0|1)}$. There are here four possible independent directions along which $\mathbb{Y}_k^{(0|1)}$ can act, labelled by the index k. This means, for example, that $\mathbb{Y}_\alpha^{(0|1)}$ is proportional to $\delta(\psi^\alpha)$, and $\mathbb{Y}_{\dot\alpha}^{(0|1)}$ is proportional to $\delta(\bar{\psi}^{\dot\alpha})$. We denote by $\mathbb{Y}^{(0|4)}$ the product of four PCO's along the four possibile independent directions. As discussed for example in [5] and [18], the product of two delta's is anticommuting (e.g. for $\delta(\psi^1) \wedge \delta(\bar{\psi}^{\dot 2}) = -\delta(\bar{\psi}^{\dot 2}) \wedge \delta(\psi^1)$), guaranteeing that no singularity arises when multiplying two or more PCO's. Thus $\mathbb{Y}_1^{(0|1)} \wedge \mathbb{Y}_1^{(0|1)} = 0$, etc.

As discussed in [2, 7], the PCO's of type $\mathbb{Y}$ represent the Poincaré form dual to the embedding of the reduced bosonic submanifold $\mathcal{M}^{(4|0)}$ into the supermanifold $\mathcal{M}^{(4|4)}$. They are elements of the de Rham cohomology with the properties

$$d\mathbb{Y}_k^{(0|1)} = 0\,, \qquad \mathbb{Y}_k^{(0|1)} \neq d\eta_k^{(-1|1)}\,, \qquad \delta\mathbb{Y}_k^{(0|1)} = d\Lambda_k^{(-1|1)} \tag{5.16}$$

The last equation means that any variation (under a diff.) of the PCO is d-exact. This gives

$$d\omega^{(p|q+1)} = d\left[\omega^{(p|q)} \wedge \mathbb{Y}_k^{(0|1)}\right] = d\omega^{(p|q)} \wedge \mathbb{Y}_k^{(0|1)} \tag{5.17}$$

which implies also that $\mathbb{Y}_k^{(0|1)}$ maps cohomology classes into cohomology classes:

$$\mathbb{Y}_k^{(0|1)} : H_d(\Omega^{(p|q)}) \longrightarrow H_d(\Omega^{(p|q+1)}) \tag{5.18}$$

The explicit form of $\mathbb{Y}_k^{(0|1)}$ is important in the applications and we will elaborate on it in the forthcoming sections. In particular there are choices with manifest symmetries, playing a crucial rôle in building manifestly supersymmetric actions.

To decrease the picture, we use a different PCO operator denoted by $\mathbb{Z}_k^{(0|-1)}$, acting as a double differential operator on the space of pseudoforms

$$\mathbb{Z}_k^{(0|-1)} : \Omega^{(p|q)} \longrightarrow \Omega^{(p|q-1)} \tag{5.19}$$

These operators act along different directions k by removing the corresponding delta forms of type $\delta(\psi^\alpha)$ or $\delta(\bar\psi^{\dot\alpha})$. A convenient way to represent $\mathbb{Z}_k^{(0|-1)}$ is given by

$$\mathbb{Z}_k^{(0|-1)} = [d, \Theta(\iota_k)] = \delta(\iota_k)\ell_k \tag{5.20}$$

(see for examples again [7]) where $\Theta(\iota_k)$ is the Heaviside step function and ι_k is the contraction along the ψ^α or $\bar\psi^{\dot\alpha}$. Notice that $\Theta(\iota_k)$ is not a compact-support distribution and therefore it has to be treated carefully. Nonetheless the explicit form of (5.20) shows that $\mathbb{Z}_k^{(0|-1)}$ is expressed only in terms of compact-support distributions. ℓ_k is the Lie derivative along one of the vector fields D_α or $\bar D_{\dot\alpha}$. The form (5.20) is computationally convenient when it acts on closed forms as will be seen later. In addition, we also notice that the formula (5.20) shows that the operator $\mathbb{Z}_k^{(0|-1)}$ is "closed" but it fails to be "exact" since $\Theta(\iota_k)$ is not a compact-support distribution.

5.2 Lorentz transformations on $\Omega^{(p|q)}$

Before discussing in detail some of the relevant spaces $\Omega^{(p|q)}$, we would like to clarify how the Lorentz symmetry is implemented in the complex of pseudoforms. This is a crucial point in order to understand how the covariance is recovered at any picture number.

Let us consider an infinitesimal Lorentz transformation $\Lambda^a{}_b$ of SO(3,1). It acts on the coordinates $x^a, \theta^\alpha, \bar\theta^{\dot\alpha}$ linearly according to vector and spinor representations

$$\delta x^a = \Lambda^a{}_b x^b\,, \qquad \delta\theta^\alpha = \frac{1}{4}\Lambda_{ab}(\gamma^{ab})^\alpha{}_\beta\theta^\beta\,, \qquad \delta\bar\theta^{\dot\alpha} = \frac{1}{4}\Lambda_{ab}(\gamma^{ab})^{\dot\alpha}{}_{\dot\beta}\bar\theta^{\dot\beta}\,. \tag{5.21}$$

In the same way, the (1|0)-superforms $(V^a, \psi^\alpha, \bar\psi^{\dot\alpha})$ transform, respectively, in the vector and in the spinor representations. Thus, all forms belonging to the complex with zero picture, namely $\Omega^{(p|0)}$, transform in the tensorial representations of each single monomial. For example, given $\omega_{[ab](\alpha_1\dots\alpha_n)}V^aV^b\psi^{\alpha_1}\dots\psi^{\alpha_n}$, the components $\omega_{[ab](\alpha_1\dots\alpha_n)}(x,\theta)$ transform in the anti-symmetrized product of the vector representation tensored with n-symmetrized spinor representation.

If we consider the complex of integral forms $\Omega^{(p|4)}$, and we perform an infinitesimal Lorentz transformation, we have to use distributional relations as for example

$$\delta(a\psi^1 + b\psi^2)\delta(c\psi^1 + d\psi^2) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \delta(\psi^1)\delta(\psi^2)\,, \tag{5.22}$$

implying that the product of $\delta(\psi^1)\delta(\psi^2)$ transforms as the inverse of a density. Therefore each monomial of the complex $\Omega^{(p|4)}$ transforms according to a tensorial representation of the Lorentz group. For example, a finite variation of an integral top form $\omega^{(4|4)} = f(x,\theta)V^4\delta^2(\psi)\delta^2(\bar\psi)$ gives

$$\omega^{(4|4)} \longrightarrow \frac{\det(\Lambda^a{}_b)}{\det(\Lambda^\alpha{}_\beta)\det(\bar\Lambda^{\dot\alpha}{}_{\dot\beta})} f\left(\Lambda^a_b x^a, \Lambda^\alpha{}_\beta\theta^\beta, \bar\Lambda^{\dot\alpha}_{\dot\beta}\bar\theta^{\dot\beta}\right) V^4\delta^2(\psi)\delta^2(\bar\psi) \tag{5.23}$$

where $\Lambda^\alpha{}_\beta = \frac{1}{4}\Lambda_{ab}(\gamma^{ab})^\alpha{}_\beta$ and $\bar\Lambda^{\dot\alpha}{}_{\dot\beta} = \frac{1}{4}\Lambda_{ab}(\gamma^{ab})^{\dot\alpha}{}_{\dot\beta}$. Since Λ is a Lorentz transformation, i.e. $\Lambda \in \mathrm{SO}(3,1)$, all determinants appearing in the front factor are equal to one and the top form is invariant if

$$f\left(\Lambda^a_b x^a, \Lambda^\alpha{}_\beta\theta^\beta, \bar\Lambda^{\dot\alpha}_{\dot\beta}\bar\theta^{\dot\beta}\right) = f(x,\theta,\bar\theta) \tag{5.24}$$

Let us now consider the complexes of pseudoforms, for example at picture one: $\Omega^{(p|1)}$ for any $p \in \mathbb{Z}$. As seen above, it is unbounded from above and from below and each space is infinite dimensional. For a single Dirac delta function $\delta(\psi^1)$, we cannot use the distributional identity (5.22), but we observe that

$$\begin{aligned}\delta(\psi^1) &\longrightarrow \delta\Big(\psi^1 + \frac{1}{4}\Lambda_{ab}(\gamma^{ab})^1{}_\beta\psi^\beta\Big) \\ &= \left(1 - \frac{1}{4}\Lambda_{ab}(\gamma^{ab})^1{}_1\right)\delta(\psi^1) + \frac{1}{4}\Lambda_{ab}(\gamma^{ab})^1{}_2\psi^2\delta^{(1)}(\psi^1) + \mathcal{O}(\Lambda^2)\end{aligned} \tag{5.25}$$

where $\delta^{(1)}(\psi_1)$ is the first derivative of $\delta(\psi^1)$ and we have neglected the infinitesimal terms. The first term is obtained by using the rule $\psi^1\delta^{(1)}(\psi^1) = -\delta(\psi^1)$ and the second term comes from the Taylor expansion of the delta function. Then, in order to implement the Lorentz symmetry in the space of pseudoforms $\Omega^{(p|1)}$, all the components in the expansion of a generic superform in $\Omega^{(p|1)}$ are needed, and span an infinite dimensional space.

5.3 Superfields, volume forms and chiral volume forms

A superfield Φ is a (0|0)-superform and it has the conventional superfield properties. Its supersymmetry transformations are deduced from its differential

$$\delta\Phi = \ell_\epsilon\Phi = \iota_\epsilon d\Phi\,. \tag{5.26}$$

where ϵ is the constant supersymmetry parameter.

An important ingredient for the subsequent sections are the volume forms, necessary to build integral forms and therefore integrable quantities on the entire supermanifold without referring to a specific coordinate system. As in general relativity, where the use of differential forms is a powerful tool to construct diff. invariant objects, here the construction of integral forms is needed to have superdiff. invariant objects, that are in turn also invariant under rigid supersymmetry. For this reason, we provide here some remarks concerning the real and the chiral volume forms.

The top integral forms of $\Omega^{(4|4)}$ are represented by

$$\omega^{(4|4)} = \Phi(x,\theta,\bar{\theta})\epsilon_{abcd}V^a \wedge V^b \wedge V^c \wedge V^d \epsilon_{\alpha\beta}\delta(\psi^a)\delta(\psi^\beta)\epsilon_{\dot{\alpha}\dot{\beta}}\delta(\psi^{\dot{\alpha}})\delta(\psi^{\dot{\beta}}) \tag{5.27}$$

Rewriting the supervielbein $E^A = (V^a, \psi^\alpha, \bar{\psi}^{\dot{\alpha}})$ on a curved basis:

$$\begin{aligned} V^a &= E^a_m dx^m + E^a_\mu d\theta^\mu + E^a_{\dot{\mu}} d\bar{\theta}^{\dot{\mu}} \\ \psi^a &= E^a_m dx^m + E^a_\mu d\theta^\mu + E^a_{\dot{\mu}} d\bar{\theta}^{\dot{\mu}} \\ \bar{\psi}^{\dot{\alpha}} &= E^{\dot{\alpha}}_m dx^m + E^{\dot{\alpha}}_\mu d\theta^\mu + E^{\dot{\alpha}}_{\dot{\mu}} d\bar{\theta}^{\dot{\mu}} \end{aligned} \tag{5.28}$$

we find also:

$$\begin{aligned} \omega^{(4|4)} &= \Phi(x,\theta,\bar{\theta})V^4\delta^4(\psi) \equiv \epsilon_{abcd}V^a \wedge V^b \wedge V^c \wedge V^d \epsilon_{\alpha\beta}\delta(\psi^a)\delta(\psi)\epsilon_{\dot{\alpha}\dot{\beta}}\delta(\psi^{\dot{\alpha}})\delta(\psi^{\dot{\beta}}) \\ &= \Phi(x,\theta,\bar{\theta})\mathrm{Sdet}(E)d^4x\delta^4(d\theta) \end{aligned} \tag{5.29}$$

This (4|4) form is trivially closed (being a top integral form), and not exact if $\Phi(x,\theta,\bar{\theta})\mathrm{Sdet}(E) \neq$ constant. Its supersymmetry variation is

$$\delta\omega^{(4|4)} = \ell_\epsilon \omega^{(4|4)} = d\left(\iota_\epsilon \omega^{(4|4)}\right) \tag{5.30}$$

Notice that if $\Phi\mathrm{Sdet(E)} = 1$, the top form $\omega^{(4|4)}$ cannot be regarded as the true volume form. Indeed

$$\tilde{\omega}^{(4|4)} \equiv V^4\delta^4(\psi) = \epsilon_{abcd}V^a \wedge V^b \wedge V^c \wedge V^d \epsilon_{\alpha\beta}\delta(\psi^\alpha)\delta(\psi^\beta)\epsilon_{\dot{\alpha}\dot{\beta}}\delta(\bar{\psi}^{\dot{\alpha}})\delta(\bar{\psi}^{\dot{\beta}})\,. \tag{5.31}$$

is closed, but it is also exact as can be shown using the relation

$$\epsilon_{\dot{\alpha}\dot{\beta}}\delta(\bar{\psi}^{\dot{\alpha}})\delta(\bar{\psi}^{\dot{\beta}}) = d\Big[\bar{\theta}^{\dot{\alpha}}\bar{\iota}_{\dot{\alpha}}\delta^2(\bar{\psi})\Big] \tag{5.32}$$

to write $\tilde{\omega}^{(4|4)}$ as

$$\tilde{\omega}^{(4|4)} = d\Big[V^4\delta^2(\psi) \wedge \bar{\theta}^{\dot{\alpha}}\bar{\iota}_{\dot{\alpha}}\delta^2(\bar{\psi})\Big]\,. \tag{5.33}$$

so that

$$\int_{\mathcal{M}^{(4|4)}} \tilde{\omega}^{(4|4)} = 0 \tag{5.34}$$

by Stokes theorem.[1] Nevertheless the form $\tilde{\omega}^{(4|4)}$ can be used to construct integral forms that can be integrated on the entire supermanifold. Given a superfield $\Phi(x,\theta,\bar{\theta})$ we have:

$$\int_{\mathcal{M}^{(4|4)}} \Phi(x,\theta,\bar{\theta})\tilde{\omega}^{(4|4)} = \int_{\mathcal{M}^{(4)}} d^4x\, D^2\bar{D}^2\Phi\big|_{\theta=\bar{\theta}=0} \tag{5.35}$$

[1] For notations and the integration theory of superfields and integral forms we refer mainly to [5, 7]. Stokes theorem for integral forms integration is discussed in reference [5].

which in general does not vanish if Φ is not constant.

We can construct the chiral volume forms as follows. Given $\omega^{(4|2)} = V^4 \delta^2(\psi)$, $\overline{\omega}^{(4|\bar{2})} = V^4 \delta^2(\bar{\psi})$ and the PCO's $\mathbb{Y}^{(0|2)} = \theta^2 \delta^2(\psi)$ and $\overline{\mathbb{Y}}^{(0|2)} = \bar{\theta}^2 \delta^2(\bar{\psi})$, we have

$$\omega_C^{(4|4)} = V^4 \delta^2(\psi) \wedge \overline{\mathbb{Y}}^{(0|2)} \,, \qquad \overline{\omega}_C^{(4|4)} = \mathbb{Y}^{(0|2)} \wedge V^4 \delta^2(\bar{\psi}) \,, \tag{5.36}$$

They are conjugated to each other. They are closed, and in fact are exact. This can be easily seen by using again the equation (5.32). The differential of V^a produces one ψ^α — annihilated by the contraction ι_α — and one $\bar{\psi}^{\dot{\alpha}}$ which however is not cancelled by the Dirac deltas $\delta^2(\bar{\psi})$ which are present in $\omega^{(4|4)}$, but not in $\omega^{(4|2)}$. Then, we have

$$\begin{aligned}
&\int_{\mathcal{M}^{(4|4)}} \Phi(x,\theta,\bar{\theta}) \left(V^4 \delta^2(\psi) \wedge \overline{\mathbb{Y}}^{(0|2)} + \mathbb{Y}^{(0|2)} \wedge V^4 \delta^2(\bar{\psi}) \right) \left[d^4 x d^2\theta d^2\bar{\theta} d^2\psi d^2\bar{\psi} \right] = \\
&= \int_{\mathcal{M}^{(4|2,0)}} \Phi(x,\theta,0) V^4 \delta^2(\psi) \left[d^4 x d^2\theta d^2\psi \right] + \int_{\mathcal{M}^{(4|0,2)}} \Phi(x,0,\bar{\theta}) V^4 \delta^2(\bar{\psi}) \left[d^4 x d^2\bar{\theta} d^2\bar{\psi} \right] = \\
&= \int_{\mathcal{M}^{(4)}} d^4 x \; D^2 \Phi\big|_{\theta=0} + \int_{\mathcal{M}^{(4)}} d^4 x \; \bar{D}^2 \Phi\big|_{\bar{\theta}=0}
\end{aligned} \tag{5.37}$$

The result is a sum of a chiral and an anti-chiral term integrated over the reduced bosonic submanifold of the supermanifold.

5.4 Chevalley-Eilenberg cohomology

The next step is to analyze some other interesting sectors of the cohomology. In particular those which are relevant for Wess-Zumino and super-Yang-Mills actions. It turns out that the crucial ingredients for the forthcoming sections are elements of the cohomology $H_d(\Omega^{(4|0)})$ with two vectorial vielbeins and two spinorial vielbeins, i.e. with the generic form:

$$\omega^{(4|0)} \sim \bar{\theta}\theta \, \bar{\psi} \wedge \psi \wedge V \wedge V + \theta^2 \bar{\psi}^2 \wedge V \wedge V + \text{h.c.} \tag{5.38}$$

These differential forms are dual to the PCO's listed in (5.52), in the sense that:

$$\omega^{(4|0)} \wedge \mathbb{Y}^{(0|4)} \sim \bar{\theta}^2 \theta^2 V^4 \delta^4(\psi) \,. \tag{5.39}$$

The factor $\bar{\theta}^2\theta^2$ appearing in the r.h.s. is crucial in order to have a closed, but not exact, integral form.

Now, in order to find the appropriate expression for $\omega^{(4|0)}$ we list the possible Lorentz invariant forms with two θ's and two ψ's:

$$\begin{aligned}
\omega_1 &= \frac{1}{2} \left(\theta_\alpha \bar{\theta}_{\dot{\alpha}} \psi_\beta \bar{\psi}_{\dot{\beta}} - \theta_\beta \bar{\theta}_{\dot{\beta}} \psi_\alpha \bar{\psi}_{\dot{\alpha}} \right) V^{\alpha\dot{\alpha}} \wedge V^{\beta\dot{\beta}} = (\theta V \bar{\theta})(\psi V \bar{\psi}) \,, \\
\omega_2 &= \frac{1}{2} \left(\theta_\alpha \bar{\theta}_{\dot{\beta}} \psi_\beta \bar{\psi}_{\dot{\alpha}} - \theta_\beta \bar{\theta}_{\dot{\alpha}} \psi_\alpha \bar{\psi}_{\dot{\beta}} \right) V^{\alpha\dot{\alpha}} \wedge V^{\beta\dot{\beta}} = (\theta V \bar{\psi})(\psi V \bar{\theta}) \,, \\
\omega_3 &= \frac{1}{2} (\theta_\gamma \psi^\gamma) \left(\bar{\theta}_{\dot{\alpha}} \bar{\psi}_{\dot{\beta}} + \bar{\theta}_{\dot{\beta}} \bar{\psi}_{\dot{\alpha}} \right) \epsilon_{\alpha\beta} V^{\alpha\dot{\alpha}} \wedge V^{\beta\dot{\beta}} = (\theta \cdot \psi)(\bar{\theta} V_-^2 \bar{\psi}) \,, \\
\omega_4 &= \frac{1}{2} (\bar{\theta}_{\dot{\gamma}} \bar{\psi}^{\dot{\gamma}}) \left(\theta_\alpha \psi_\beta + \theta_\beta \psi_\alpha \right) \epsilon_{\dot{\alpha}\dot{\beta}} V^{\alpha\dot{\alpha}} \wedge V^{\beta\dot{\beta}} = (\bar{\theta} \cdot \bar{\psi})(\theta V_+^2 \psi) \,, \\
\omega_5 &= \theta^\gamma \epsilon_{\gamma\rho} \theta^\rho (\bar{\psi}_{\dot{\alpha}} \bar{\psi}_{\dot{\beta}}) \epsilon_{\alpha\beta} V^{\alpha\dot{\alpha}} \wedge V^{\beta\dot{\beta}} = \theta^2 (\bar{\psi} V_-^2 \bar{\psi}) \\
\omega_6 &= \bar{\theta}^{\dot{\gamma}} \epsilon_{\dot{\gamma}\dot{\rho}} \bar{\theta}^{\dot{\rho}} (\psi_\alpha \psi_\beta) \epsilon_{\dot{\alpha}\dot{\beta}} V^{\alpha\dot{\alpha}} \wedge V^{\beta\dot{\beta}} = \bar{\theta}^2 (\psi V_+^2 \psi) \,,
\end{aligned} \tag{5.40}$$

We have defined:

$$
\begin{aligned}
(\theta V\bar{\theta}) &= \theta^{\alpha}V^{\beta\dot{\beta}}\bar{\theta}^{\dot{\alpha}}\epsilon_{\alpha\beta}\epsilon_{\dot{\beta}\dot{\alpha}}\,, & (\psi V\bar{\psi}) &= \psi^{\alpha}V^{\beta\dot{\beta}}\bar{\psi}^{\dot{\alpha}}\epsilon_{\alpha\beta}\epsilon_{\dot{\beta}\dot{\alpha}}\,,\\
(\theta V\bar{\psi}) &= \theta^{\alpha}V^{\beta\dot{\beta}}\bar{\psi}^{\dot{\alpha}}\epsilon_{\alpha\beta}\epsilon_{\dot{\beta}\dot{\alpha}}\,, & (\psi V\bar{\theta}) &= \psi^{\alpha}V^{\beta\dot{\beta}}\bar{\theta}^{\dot{\alpha}}\epsilon_{\alpha\beta}\epsilon_{\dot{\beta}\dot{\alpha}}\,,\\
(\theta\cdot\psi) &= \theta^{\alpha}\psi^{\beta}\epsilon_{\alpha\beta}\,, & (\bar{\theta}V_-^2\bar{\psi}) &= \bar{\theta}^{\dot{\alpha}}(V_-^2)^{\dot{\beta}\dot{\gamma}}\bar{\psi}^{\dot{\delta}}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}}\,,\\
(\bar{\theta}\cdot\bar{\psi}) &= \bar{\theta}^{\dot{\alpha}}\bar{\psi}^{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}}\,, & (\theta V_+^2\psi) &= \theta^{\alpha}(V_+^2)^{\beta\gamma}\psi^{\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\,,\\
\theta^2 &= \theta^{\alpha}\theta^{\beta}\epsilon_{\alpha\beta}\,, & \bar{\theta}^2 &= \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}}\,,\\
(\bar{\psi}V_-^2\bar{\psi}) &= \bar{\psi}^{\dot{\alpha}}(V_-^2)^{\dot{\beta}\dot{\gamma}}\bar{\psi}^{\dot{\delta}}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}}\,, & (\psi V_+^2\psi) &= \psi^{\alpha}(V_+^2)^{\beta\gamma}\psi^{\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}
\end{aligned}
\tag{5.41}
$$

whose differentials are

$$
\begin{aligned}
&d(\theta V\bar{\theta}) = (\psi V\bar{\theta}) + (\theta V\bar{\psi}) - 2i(\theta\cdot\psi)(\bar{\psi}\cdot\bar{\theta})\\
&d(\theta V\bar{\psi}) = (\psi V\bar{\psi})\,, && d(\psi V\bar{\theta}) = -(\psi V\bar{\psi})\\
&d(\psi V\bar{\psi}) = 0\,, && d(\psi V_+^2\psi) = 0\,,\\
&d(\bar{\psi}V_-^2\bar{\psi}) = 0\,, && d(\theta\cdot\psi) = 0\,, \qquad d(\bar{\theta}\cdot\bar{\psi}) = 0\,,
\end{aligned}
\tag{5.42}
$$

The linear combination

$$\omega^{(4|0)} = a\,\omega_1 + b\,\omega_2 + c\,\omega_3 + d\,\omega_4 + e\,\omega_5 + f\,\omega_6 \tag{5.43}$$

is closed if $a = c-d, e = f$ and $b = \frac{1}{2}(c+d)+2e$. If, in addition, we require the hermiticity of ω, one finds $c = d$ and therefore $a = 0$. Then, we find that the combination

$$\omega^{(4|0)} = c(\omega_2 + \omega_3 + \omega_4) + e(2\omega_2 + \omega_5 + \omega_6)\,, \tag{5.44}$$

is closed, real, and depends upon the two parameters c and e. Furthermore, we have to check whether this expression is exact. We observe that there is only one real candidate (with r a real parameter):

$$\gamma^{(3|0)} = r\Big(\theta^2(\bar{\psi}V_-^2\bar{\theta}) + \bar{\theta}^2(\psi V_+^2\theta)\Big)\,. \tag{5.45}$$

such that $d\gamma^{(3|0)}$ has a structure similar to the ones listed in (5.40). Computing $d\gamma^{(3|0)}$ and adding it to $\omega^{(4|0)}$, we finally end up with the expression

$$\omega^{(4|0)} = (c+2e)\omega_2 + (c-2r)(\omega_3+\omega_4) + (e+r)(\omega_5+\omega_6) \tag{5.46}$$

and we can use the parameter r to set one of the two combinations to zero. If we choose $c = 2r$, we see that the full expression is proportional to $(c+2e)$. In the same way by choosing $r = -e$, we obtain again an expression which is proportional to the combination $(c+2e)$. Therefore, after subtracting the exact piece, we get a single representative in the cohomology class.

Notice that $\omega^{(4|0)}$ is not manifestly supersymmetric since it depends upon θ and $\bar{\theta}$. This is the reason why this cohomology was never used. However, its supersymmetry variation is d-exact.

5.5 The PCO's $\mathbb{Y}^{(0|1)}$

The easiest example of PCO that we can build is the one that projects the theory on the bosonic submanifold by switching off the θ coordinates and their differentials. For each coordinate we have the following four PCO's acting along the θ-directions

$$\mathbb{Y}^{(0|1)}_1 = \theta^1\delta(\psi^1)\,, \quad \mathbb{Y}^{(0|1)}_2 = \theta^2\delta(\psi^2)\,, \quad \mathbb{Y}^{(0|1)}_{\dot 1} = \bar\theta^{\dot 1}\delta(\bar\psi^{\dot 1})\,, \quad \mathbb{Y}^{(0|1)}_{\dot 2} = \bar\theta^{\dot 2}\delta(\bar\psi^{\dot 2})\,. \tag{5.47}$$

Each of them increases by one the picture of the form and projects to zero the corresponding coordinate. Notice that they have a non-trivial kernel, for example the kernel $\mathbb{Y}^{(0|1)}_1$ consists of linear functions of θ^1, ψ^1 and $\delta(\psi^1)$ (due to the anticommutation properties of the deltas). All PCO's in (5.47) are closed and not exact. They are invariant under partial supersymmetry (for example $\mathbb{Y}^{(0|1)}_1$ is invariant under the supersymmetries along $\theta^2, \bar\theta^{\dot 1}$ and $\bar\theta^{\dot 2}$). As already noticed, its supersymmetry variation is exact. The wedge product of all four PCO's produces a single operator (up to an overall sign) which we denote by

$$\mathbb{Y}^{(0|4)} = \theta^2\delta^2(\psi)\bar\theta^2\delta^2(\bar\psi) \tag{5.48}$$

This PCO is trivially closed, it is not exact and it is not manifestly supersymmetric. Nonetheless, its supersymmetry transformation is d-exact. Therefore, given a closed superform $\mathcal{L}^{(4|0)}$, we can write an action

$$S = \int_{\mathcal{M}^{(4|4)}} \mathcal{L}^{(4|0)} \wedge \mathbb{Y}^{(0|4)} \tag{5.49}$$

which reduces to the component action (which means the integral of $\mathcal{L}^{(4|0)}$ computed at $\theta = \bar\theta = 0$ and $\psi = \bar\psi = 0$ over $\mathcal{M}^{(4)}$).

The closure of $\mathcal{L}^{(4|0)}$ guarantees the supersymmetry invariance of the action up to boundary terms. A milder condition can be imposed on $\mathcal{L}^{(4|0)}$ in order for S to be supersymmetric invariant:

$$\iota_\epsilon d\mathcal{L}^{(4|0)} = d\xi \tag{5.50}$$

i.e. the differential along the supersymmetry directions must be exact. The computation of the integral in (5.49) along the θ's and the ψ's leads to

$$S = \int_{\mathcal{M}^{(4)}} \mathcal{L}^{(4|0)}\Big|_{\theta=0,\psi=0}\,, \tag{5.51}$$

which is the component action and it is supersymmetric invariant if the supersymmetry variation of the Lagrangian $\mathcal{L}^{(4|0)}|_{\theta=0,\psi=0}$ is an exact differential.

To rewrite the action in a manifestly supersymmetric way, we need another PCO which is manifestly supersymmetric. It should have picture number equal to 4 and zero form degree. To get from the Lagrangian $\mathcal{L}^{(4|0)}$ a top integral form, the PCO should be closed, not exact, and possibly invariant under supersymmetry. For that purpose, we consider the following six combinations

$$
\begin{aligned}
Y_1 &= V^{\alpha\dot\alpha} \wedge V^{\beta\dot\beta} \left(\theta_\alpha \bar\theta_{\dot\alpha} \iota_\beta \bar\iota_{\dot\beta} - \theta_\beta \bar\theta_{\dot\beta} \iota_\alpha \bar\iota_{\dot\alpha}\right) \delta^4(\psi)\,,\\
Y_2 &= V^{\alpha\dot\alpha} \wedge V^{\beta\dot\beta} \left(\theta_\alpha \bar\theta_{\dot\beta} \iota_\beta \bar\iota_{\dot\alpha} - \theta_\beta \bar\theta_{\dot\alpha} \iota_\alpha \bar\iota_{\dot\beta}\right) \delta^4(\psi)\,,\\
Y_3 &= V^{\alpha\dot\alpha} \wedge V^{\beta\dot\beta} \epsilon_{\alpha\beta} \left(\bar\theta_{\dot\alpha} \bar\iota_{\dot\beta} + \bar\theta_{\dot\beta} \bar\iota_{\dot\alpha}\right) \theta^\gamma \iota_\gamma \delta^4(\psi)\,,\\
Y_4 &= V^{\alpha\dot\alpha} \wedge V^{\beta\dot\beta} \epsilon_{\dot\alpha\dot\beta} \left(\theta_\alpha \iota_\beta + \theta_\beta \iota_\alpha\right) \bar\theta^{\dot\gamma} \iota_{\dot\gamma} \delta^4(\psi)\,,\\
Y_5 &= V^{\alpha\dot\alpha} \wedge V^{\beta\dot\beta} \epsilon_{\alpha\beta} (\bar\theta_{\dot\gamma} \bar\theta^{\dot\gamma})\, \bar\iota_{\dot\alpha} \bar\iota_{\dot\beta} \delta^4(\psi)\,,\\
Y_6 &= V^{\alpha\dot\alpha} \wedge V^{\beta\dot\beta} \epsilon_{\dot\alpha\dot\beta} (\theta_\gamma \theta^\gamma)\, \iota_\alpha \iota_\beta \delta^4(\psi)\,.
\end{aligned} \tag{5.52}
$$

The six possible forms reproduce the terms appearing in the Lagrangian (2.28) (see the $\psi\bar\psi W\bar W V V$ terms in section 2).

They are indeed the terms needed to reproduce the full superspace action. The two contraction operators $\iota\bar\iota$ appearing in the operators act on the Lagrangian by selecting the terms proportional to the combination $\bar\psi\psi$. In addition, the factors $\theta\bar\theta$ are needed to prevent the PCO being exact.

By adjusting the six constants a_i we can make the combination

$$
\mathbb{Y}^{(0|4)} = \sum_{i=1}^{6} a_i Y_i\,, \tag{5.53}
$$

closed. Let us first impose the hermiticity by setting $a_3 = a_4$ and $a_5 = a_6$. This reduces the structures to the four combinations $Y_1, Y_2, Y_3 + Y_4, Y_5 + Y_6$. Imposing the closure, we get $a_1 = 0$, $a_2 = -2(a_3 + a_5)$. Therefore, there are two independent structures which are closed. However, there is a combination which is also exact.

This can be easily derived by computing the variation of

$$
\eta^{(-1|4)} = \left(\theta^2\bar\theta \cdot \bar\iota(\iota V_+^2 \iota) + \bar\theta^2\theta \cdot \iota(\bar\iota V_-^2 \bar\iota)\right)\delta^4(\psi)\,. \tag{5.54}
$$

Therefore, to select a representative of the cohomology class we fix one of the coefficients, avoiding the exact combination. For example we can set $a_3 = 0$ to simplify the structure as much as possible:

$$
\mathbb{Y}^{(0|4)} = \Big(-4(\theta V\bar\iota) \wedge (\bar\theta V\iota) + \theta^2(\iota V \wedge V\iota) + \bar\theta^2(\bar\iota V \wedge V\bar\iota)\Big)\delta^4(\psi) \tag{5.55}
$$

Notice that there is a single non-chiral and two chiral and anti-chiral terms. This already suggests how the three terms of the action in superspace emerge from the geometrical action.

5.6 The PCO's $\mathbb{Z}^{(0|-1)}$

We have seen that the complexes of pseudoforms are connected by the picture changing operators. In the previous section we also observed that there are some non-trivial cohomology classes needed for physics applications. We check here that these cohomology classes are related by the PCO's.

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Let us first analyze the action of $\mathbb{Z}^{(0|-1)}$ on the chiral forms.

They are supersymmetric invariant and we can apply the PCO's $Z_\alpha = [d, \Theta(\iota_\alpha)]$ to get the image in $\Omega^{(4|0)}$. Since this computation is very instructive we report it here in some detail. We have to act with the PCO's as follows:

$$Z_1\Big(V^4\delta^2(\psi)\Big) = [d,\Theta(\iota_1)]\, V^4\delta^2(\psi) = d\Big[\Theta(\iota_1)V^4\delta^2(\psi)\Big] \tag{5.56}$$

$$= d\left[\frac{V^4}{\psi^1}\delta(\psi^2)\right] = \Big(\bar\psi^{\dot 1}V^{1\dot 2}V^{2\dot 1}V^{2\dot 2} - \bar\psi^{\dot 2}V^{1\dot 1}V^{2\dot 1}V^{2\dot 2}\Big)\delta(\psi^2)\,. \tag{5.57}$$

Notice that the result does not contain inverse powers[2] of ψ's. In the same way, we have $Z_2 = [d, \Theta(\iota_2)]$ and

$$Z_2\Big(\bar\psi^{\dot 1}V^{1\dot 2}V^{2\dot 1}V^{2\dot 2} - \bar\psi^{\dot 2}V^{1\dot 1}V^{2\dot 1}V^{2\dot 2}\Big)\delta(\psi^2) \tag{5.58}$$

$$= d\left[\frac{1}{\psi^2}\Big(\bar\psi^{\dot 1}V^{1\dot 2}V^{2\dot 1}V^{2\dot 2} - \bar\psi^{\dot 2}V^{1\dot 1}V^{2\dot 1}V^{2\dot 2}\Big)\right] \tag{5.59}$$

$$= \Big[\bar\psi^{\dot 1}\bar\psi^{\dot 2}(V^{1\dot 2}V^{2\dot 1} + V^{1\dot 1}V^{2\dot 2}) - (\bar\psi^{\dot 1})^2V^{1\dot 2}V^{2\dot 2} - (\bar\psi^{\dot 2})^2V^{1\dot 1}V^{2\dot 1}\Big] \tag{5.60}$$

$$= V^{\alpha\dot\alpha}\wedge V^{\beta\dot\beta}\epsilon_{\alpha\beta}\bar\psi_{\dot\alpha}\bar\psi_{\dot\beta}\,. \tag{5.61}$$

(with $\bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}$). This form is closed, supersymmetric invariant and polynomial in V^a, ψ^a and $\bar\psi^{\dot\alpha}$ (this means that it is indeed a superform). Notice that we get only the chiral part of the cohomology of $\Omega^{(4|0)}$. Starting from the antichiral integral form $V^4\delta^2(\bar\psi)$, we would get the other class in $H_d^{(4|0)}$.

We consider now the following volume form where we have chosen $\Phi(x,\theta,\bar\theta)$ in (5.27) to be equal to the product of the θ's and $\bar\theta$'s,

$$\mathrm{Vol}^{(4|4)} = V^4\theta^1\theta^2\bar\theta^{\dot 1}\bar\theta^{\dot 2}\,\delta^2(\psi)\delta^2(\bar\psi) \tag{5.62}$$

and where we have written the spinorial indices explicitly to simplify the derivation. We use the notations in appendix B for the product of the vielbeins.

We act with the PCO $Z_1 = [d, \Theta(\iota_{D_1})]$ on the volume form:

$$Z_1\mathrm{Vol}^{(4|4)} = [d,\Theta(\iota_{D_1})]\mathrm{Vol}^{(4|4)} = d\Big[\Theta(\iota_{D_1})\mathrm{Vol}^{(4|4)}\Big] \tag{5.63}$$

$$= d\left[V^4\theta^1\theta^2\bar\theta^{\dot 1}\bar\theta^{\dot 2}\,\frac{1}{\psi_1}\delta(\psi_2)\delta^2(\bar\psi)\right] = V^4\theta^2\bar\theta^{\dot 1}\bar\theta^{\dot 2}\,\delta(\psi_2)\delta^2(\bar\psi)\,. \tag{5.64}$$

Acting with $Z_2 = [d, \Theta(\iota_{D_2})]$, we find

$$Z_2Z_1\mathrm{Vol}^{(4|4)} = [d,\Theta(\iota_{D_2})]V^4\theta^2\bar\theta^{\dot 1}\bar\theta^{\dot 2}\,\delta(\psi_2)\delta^2(\bar\psi) = V^4\bar\theta^{\dot 1}\bar\theta^{\dot 2}\,\delta^2(\bar\psi)$$

[2]Negative powers of the forms ψ exist and are well defined only in picture 0. In this case the inverses of the $\psi's$ are closed and exact and behave as negative degree superforms. The enlarged modules that contain also these inverses extend to the left the complex of superforms (the first line in figure 1). In picture $\neq 0$ negative powers are not defined because of the distributional relation $\psi\delta(\psi) = 0$.

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This form is the chiral volume form which is closed and not exact. To proceed, we can act with the PCO removing the δ's depending on $\bar{\psi}$'s:

$$\begin{aligned}\bar{Z}_1 Z_2 Z_1 \mathrm{Vol}^{(4|4)} &= d\left[V^4 \bar{\theta}^{\dot{1}} \bar{\theta}^{\dot{2}} \frac{1}{\bar{\psi}_{\dot{1}}} \delta(\bar{\psi}_{\dot{2}})\right] \\ &= \left[\psi_\alpha (V_3)^{\alpha\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}\, \bar{\theta}^{\dot{2}} \frac{1}{\bar{\psi}_{\dot{1}}} \delta(\bar{\psi}_{\dot{2}}) + V^4 \bar{\theta} \bar{\theta}^{\dot{2}} \delta(\bar{\psi}_{\dot{2}})\right] \\ &= \left[\psi_\alpha (V_3)^{\alpha\dot{1}} \bar{\theta}^{\dot{2}} \delta(\bar{\psi}_{\dot{2}}) + V^4 \bar{\theta}^{\dot{2}} \delta(\bar{\psi}_{\dot{2}})\right]\end{aligned} \tag{5.65}$$

where all the inverse powers of $\bar{\psi}$'s disappeared. For the last step, we act with $\bar{Z}_2$, and we have

$$\begin{aligned}\bar{Z}_2 \bar{Z}_1 Z_2 Z_1 \mathrm{Vol}^{(4|4)} &= d\left[\psi_\alpha (V_3)^{\alpha\dot{1}} \bar{\theta}^{\dot{2}} \frac{1}{\bar{\psi}_{\dot{2}}} + V^4 \bar{\theta}^{\dot{2}} \frac{1}{\bar{\psi}_{\dot{2}}}\right] \\ &= -\frac{i}{2} (\psi V_+^2 \psi) \bar{\theta}^2 + \psi_\alpha (V_3)^{\alpha\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} + V_4\end{aligned} \tag{5.66}$$

where $\bar{\theta}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta}^{\dot{\beta}}$.

One obtains a covariant expression since all indices are suitably contracted. In the same way, one could act first with the $\bar{Z}$'s and then with the Z's to find

$$Z_2 Z_1 \bar{Z}_2 \bar{Z}_1 \mathrm{Vol}^{(4|4)} = \frac{i}{2} (\psi V_-^2 \psi) \theta^2 + \bar{\psi}_{\dot{\alpha}} (V_3)^{\alpha\dot{\alpha}} \theta_\alpha + V_4 \tag{5.67}$$

where $\theta_\alpha = \epsilon_{\alpha\beta} \theta^\beta$.

Note that we can relate the two formulae above by observing that:

$$d[\theta_\alpha (V_3)^{\alpha\dot{\alpha}} \bar{\theta}_{\dot{\alpha}}] = i\theta_\alpha (V_3)^{\alpha\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} - i\psi_\alpha (V_3)^{\alpha\dot{\alpha}} \bar{\psi}_{\dot{\alpha}} + i(\theta \cdot \psi)(\bar{\theta} V_-^2 \bar{\psi}) - i(\bar{\theta} \cdot \bar{\psi})(\theta V_+^2 \psi) \tag{5.68}$$

which allows us to rewrite the second term in (5.66) as the second term in (5.67). Combining the two expressions we end up with the final result

$$\bar{Z}_2 \bar{Z}_1 Z_2 Z_1 \mathrm{Vol}^{(4|4)} + Z_2 Z_1 \bar{Z}_2 \bar{Z}_1 \mathrm{Vol}^{(4|4)} = \omega^{(4|0)} + d\eta \tag{5.69}$$

where $\omega^{(4|0)}$ is in the Chevalley-Eilenberg cohomology class discussed above. Thus, we have shown that acting with the PCO's Z on the volume form (5.62) reproduces the Chevalley-Eilenberg cohomology discussed in the previous sections. Notice that the presence of θ's and $\bar{\theta}$'s is essential to reconstruct the cohomology by acting with PCO's.

5.7 Two useful theorems

As an application of the previous discussions, we illustrate in this section two theorems playing an important rôle in the superspace analysis of physical theories (see also [19]).

The first is an application of Stokes theorem to supermanifolds with torsion, and is very useful in manipulating the superspace Lagrangians since it simplifies many computations. The use of integral forms is very well adapted to such manipulations since Stokes' theorem is valid for integral forms (and is a strong motivation for their integration theory) and well-known techniques can be employed here.

The second theorem is very useful for treating supergravity theories. In that framework some important quantities, such as the Ricci scalar or the Riemann tensor, appear in the superspace expansion of some superfields. Therefore, disentangling those physical components from a given superfield is crucial for building actions. One important example is the relation between curved chiral and anti-chiral volume forms with the Ricci scalar of the manifold and the non-chiral volume form. We show that this is very natural in the context of integral forms where the volume form plays an essential rôle.

In studying the relation between the chiral volume forms and the non-chiral one, we face the problem of computing the variation of the superdeterminant of the supervielbein. For that purpose, we use the integral forms for a straight derivation.

We recall that, if we denote by ∇_A the supercovariant derivative (w.r.t. the spin connection ω^{ab}), we have the equations

$$\begin{aligned} \nabla_A V^a &= T^a{}_{Ab} V^b + T^a{}_{A\beta} \psi^\beta \,, \\ \nabla_A \psi^\alpha &= T^\alpha_{Ab} V^b + T^\alpha_{A\beta} \psi^\beta \,, \\ \nabla_A \omega^{ab} + \omega^a_{A,c} \omega^{cb} &= R^{ab}{}_{Ac} V^c + R^{ab}{}_{A\beta} \psi^\beta \,, \end{aligned} \tag{5.70}$$

where $T^A{}_{BC}$ are the components of the torsion $T^a = \frac{1}{2} T^a{}_{AB} E^A \wedge E^B$ and where $E^A = (V^a, \psi^\alpha)$ (we do not impose any constraints and we use the greek indices to denote the 4 spinors components in the Majorana representation).

We act with ∇_A on $\omega^{(4|4)}$ as follows

$$\begin{aligned} \nabla_A \omega^{(4|4)} &= \nabla_A \Big(\epsilon_{abcd} V^a \dots V^b \delta^4(\psi) \Big) \\ &= 4 \epsilon_{abcd} (\nabla_A V^a) \dots V^d \delta^4(\psi) + \epsilon_{abcd} V^a \dots V^d (\nabla_A \psi^\alpha) \iota_\alpha \delta^4(\psi) \\ &= 4 \epsilon_{abcd} \Big(T^a_{Ae} V^e + T^a_{A\beta} \psi^\beta \Big) \dots V^d \delta^4(\psi) + \epsilon_{abcd} V^a \dots V^d \Big(T^\alpha_{Ae} V^e + T^\alpha_{A\beta} \psi^\beta \Big) \iota_\alpha \delta^4(\psi) \\ &= \epsilon_{abcd} \Big(T^a_{Ae} V^e \Big) \dots V^d \delta^4(\psi) + \epsilon_{abcd} V^a \dots V^d \Big(T^\alpha_{A\beta} \psi^\beta \Big) \iota_\alpha \delta^4(\psi) \end{aligned} \tag{5.71}$$

where we have used $\psi^\alpha \delta^4(\psi) = 0$ and $V^1 \wedge \dots \wedge V^5 = 0$. In addition, using $\psi^\alpha \iota_\beta \delta^4(\psi) = -\delta^\alpha_\beta \delta^4(\psi)$ and $V^a \wedge \dots \wedge V^d = \epsilon^{abcd} (V)^4$, we finally find

$$\nabla_A \omega^{(4|4)} = (-1)^B T^B{}_{BA} \omega^{(4|4)} \tag{5.72}$$

This guarantees, for $T^B{}_{BA} = 0$ the integration by parts formula

$$\int_{\mathcal{M}^{(4|4)}} \omega^{(4|4)} \, \nabla_A \Phi^{(0|0)} = - \int_{\mathcal{M}^{(4|4)}} (\nabla_A \omega^{(4|4)}) \Phi^{(0|0)} = 0 \tag{5.73}$$

for a superfield $\Phi^{(0|0)}$.

Now we consider again the top integral form $\omega^{(4|4)}$ and we express it in terms of curved coordinates as

$$\begin{aligned} \omega^{(4|4)} &= \Big(\epsilon_{abcd} V^a \dots V^b \delta^4(\psi) \Big) \\ &= (\epsilon_{abcd} E^a_m \dots E^b_p)(\epsilon_{\alpha\beta\gamma\delta} E^\alpha_\mu \dots E^\delta_\sigma) dx^m \dots dx^p \, \delta(d\theta^\mu) \dots \delta(\delta\theta^\sigma) \\ &= E \, d^4 x \delta^4(d\theta) \,. \end{aligned} \tag{5.74}$$

where $E = \text{Sdet}(E_M^A)$ is the superdeterminant of the supervielbein. E is a function of $(x, \theta, \bar\theta)$ using the chiral/anti-chiral decomposition. Then, we can expand it according to θ or $\bar\theta$ as follows

$$\omega^{(4|4)} = E\, d^4x\delta^4(d\theta) = \Big(E|_{\bar\theta=0} + \bar\theta^{\dot\alpha}\, \bar D_{\dot\alpha} E\big|_{\bar\theta=0} + \bar\theta^2\, \bar D^2 E\big|_{\bar\theta=0}\Big) d^4x\delta^4(d\theta) + \text{h.c.} \tag{5.75}$$

Using (5.74), we can set

$$\omega^{(4|4)} = E\, d^4x\delta^4(d\theta) = d\Xi + \bar D^2 E\big|_{\bar\theta=0} d^4x\delta^2(d\theta)\bar\theta^2\delta^2(d\bar\theta) + \text{h.c.} \tag{5.76}$$

where the first and the second terms in the expansion in eq. (5.75) are cohomologically trivial, while the third term provides the factor $\bar\theta^2$ needed to construct the PCO. In eq. (5.76) we have collected the exact terms into $d\Xi$.

Looking at the superdeterminant $\text{Sdet}(E)$, by choosing a gauge such that $E_{\dot\mu}^a = 0$ (no mixing between the chiral and the anti-chiral representation), we have:

$$\text{Sdet}(E) = \frac{\det\Big(E_m^a - E_\mu^a(E^{-1})^\mu_\beta E_m^\beta - E_{\dot\mu}^a(\bar E^{-1})^{\dot\mu}_{\dot\beta} E_m^{\dot\beta}\Big)}{\det(E_\mu^a)\det(\bar E_{\dot\mu}^{\dot\alpha})} = \frac{\text{Sdet}_C(\hat E)}{\det(\bar E_{\dot\mu}^{\dot\alpha})} \tag{5.77}$$

where $\text{Sdet}_C(\hat E)$ is the chiral super determinant written in terms of a redefined vielbein $\hat E_m^a = E_m^a - E_{\dot\mu}^a(\bar E^{-1})^{\dot\mu}_{\dot\beta} E_m^{\dot\beta}$. It can be proved that, by a suitable gauge fixing (chiral representation) $\text{Sdet}_C(\hat E)$ is chiral, namely $\bar D_{\dot\alpha}\text{Sdet}_C(\hat E) = 0$. We can than rewrite the above expression as follows:

$$\begin{aligned}\omega^{(4|4)} &= E\, d^4x\delta^4(d\theta) = d\Omega + \text{Sdet}_C(\hat E)\, \bar D^2 \left(\frac{1}{\det(\bar E_{\dot\mu}^{\dot\alpha})}\right)\Bigg|_{\bar\theta=0} d^4x\delta^2(d\theta)\bar\theta^2\delta^2(d\bar\theta) + \text{h.c.} \\ &= \omega^{(4|2)}\, \bar D^2 \left(\frac{1}{\det(\bar E_{\dot\mu}^{\dot\alpha})}\right)\Bigg|_{\bar\theta=0} \bar\theta^2\delta^2(d\bar\theta)\end{aligned} \tag{5.78}$$

and using the notations of [11] we set $\mathcal{R} = \bar D^2 \left(\det(\bar E_{\dot\mu}^{\dot\alpha})\right)^{-1}\Big|_{\bar\theta=0}$. The superfield $\mathcal{R}$ contains the auxiliary fields and the Ricci scalar and it appears in the commutation relation $\{\nabla_\alpha, \nabla_\beta\} = -\bar{\mathcal{R}}\mathcal{M}_{\alpha\beta}$, namely it is one of the components of the torsion T^A.

Finally, recalling that $\omega^{(4|2)} = \text{Sdet}_C(\hat E)d^4x\delta^2(d\theta)$ we have:

$$\omega^{(4|4)} = \frac{1}{\mathcal{R}}\omega^{(4|2)} \wedge \mathbb{Y}^{(0|\bar 2)} + \text{h.c.} \tag{5.79}$$

which reproduces Siegel chiral integration formula in terms of integral forms [11].

Acknowledgments

We thank P. Fré and C. Maccaferri for fruitful discussions.

A Gamma matrix conventions and two-component formalism

Clifford algebra.

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}, \qquad \eta_{ab} = (1, -1, -1, -1) \tag{A.1}$$

Matrix representation.

$$\gamma_0 = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix}, \quad \gamma_{i=1,2,3} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix} \tag{A.2}$$

where σ_i are the Pauli matrices. The Weyl projectors $P_\pm = (1 \pm \gamma_5)/2$ are therefore given by

$$P_+ = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1_{2\times 2} \end{pmatrix} \tag{A.3}$$

Two-component formalism. The four dimensional spinor index is decomposed into α=1,2, $\dot\alpha$=1,2. Thus

$$\gamma_0 = \begin{pmatrix} 0 & \delta^\alpha_{\dot\beta} \\ \delta^{\dot\alpha}_\beta & 0 \end{pmatrix}, \quad \gamma_{i=1,2,3} = \begin{pmatrix} 0 & \sigma_i{}^\alpha{}_{\dot\beta} \\ -\sigma_i{}^{\dot\alpha}{}_\beta & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} \delta^\alpha_\beta & 0 \\ 0 & -\delta^{\dot\alpha}_{\dot\beta} \end{pmatrix} \tag{A.4}$$

A four-component spinor gets decomposed into two two-component spinors $\psi = (\psi^\alpha_+, \psi^{\dot\alpha}_-)$, where the $\pm$ subscripts remind us that they are the $P_\pm$ projected parts of ψ. These subscripts may be omitted when the α or $\dot\alpha$ indices suffice to identify ψ_+ or ψ_-.

A compact way to express $\gamma_{a=0,1,2,3}$ is

$$\gamma_a = \begin{pmatrix} 0 & \sigma_a{}^\alpha{}_{\dot\beta} \\ -\sigma_a{}^{\dot\alpha}{}_\beta & 0 \end{pmatrix}, \quad \text{with } \sigma_a{}^\alpha{}_{\dot\beta} = (1, \sigma_i)^\alpha{}_{\dot\beta}, \quad \sigma_a{}^{\dot\alpha}{}_\beta = (-1, \sigma_i)^{\dot\alpha}{}_\beta \tag{A.5}$$

The matrices σ_a satisfy the completeness and the trace relations

$$\eta^{ab}\sigma_a{}^\alpha{}_{\dot\beta}\,\sigma_b{}^{\dot\gamma}{}_\delta = 2\,\delta^\alpha_\delta\,\delta^{\dot\gamma}_{\dot\beta}, \qquad Tr(\sigma_a\sigma_b) = 2\eta_{ab} \tag{A.6}$$

Charge conjugation. In the above matrix representation, the charge conjugation takes the form

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & -\epsilon_{\dot\alpha\dot\beta} \end{pmatrix} \tag{A.7}$$

where ϵ is the usual Levi-Civita symbol in two dimensions. One can check that

$$\gamma_a^T = -C\gamma_a C^{-1} \tag{A.8}$$

so that $C\gamma_a$, $C\gamma_{ab}$ are symmetric, while C, $C\gamma_5$, $C\gamma_{ab}\gamma_5$ are antisymmetric.

Majorana condition. We can impose the Majorana condition on the spinor ψ:

$$\psi^\dagger\gamma_0 = \psi^T C \tag{A.9}$$

relating $\psi^\alpha_+, \psi^{\dot\alpha}_-$ to the components of the conjugated spinor $(\psi^*_+)_\alpha, (\psi^*_-)_{\dot\alpha}$ as follows:

$$\psi^\alpha_+\epsilon_{\alpha\beta} = (\psi^*_-)_\beta, \quad \psi^{\dot\alpha}_-\epsilon_{\dot\alpha\dot\beta} = -(\psi^*_+)_{\dot\beta} \tag{A.10}$$

Note that a spinor cannot be both Majorana and Weyl in 4 dimensions, since the Majorana condition mixes the ψ_+ and ψ_- components.

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Raising and lowering spinor indices. The charge conjugation matrix C and its inverse C^{-1} can be used to lower and raise spinor indices. Correspondingly $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot\alpha\dot\beta}$, and their inverses, are used to lower and raise two-component spinor indices with the "upper left to lower right" convention. Thus for example

$$A_\alpha = A^\beta \epsilon_{\beta\alpha}, \quad A^\alpha = \epsilon^{\alpha\beta} A_\beta \tag{A.11}$$

Note that $A^\alpha B_\alpha = -A_\alpha B^\alpha$ and similar for dotted indices. We can also define σ_a matrices with both indices up or down:

$$\sigma_a{}^{\alpha\dot\beta} \equiv \epsilon^{\dot\beta\dot\gamma}\sigma_a{}^\alpha{}_{\dot\gamma}, \quad \sigma_{a\ \alpha\dot\beta} \equiv \sigma_a{}^\gamma{}_{\dot\beta}\,\epsilon_{\gamma\alpha}, \quad \sigma_a{}^{\dot\alpha\beta} \equiv \epsilon^{\beta\gamma}\sigma_a{}^{\dot\alpha}{}_\gamma, \quad \sigma_{a\ \dot\alpha\beta} \equiv \sigma_a{}^{\dot\gamma}{}_\beta\,\epsilon_{\dot\gamma\dot\alpha} \tag{A.12}$$

With these definitions one finds

$$\sigma_a{}^{\alpha\dot\beta} = \sigma_a{}^{\dot\beta\alpha}, \quad \sigma_{a\ \alpha\dot\beta} = \sigma_{a\ \dot\beta\alpha} \tag{A.13}$$

i.e. the σ_a matrices with both indices up or down are symmetric.

Converting vector into spinor indices. Finally, the σ_a matrices can be used to convert a 4-dim vector index into a couple of two-component spinor indices, and viceversa:

$$V^{\alpha\dot\alpha} \equiv V^a \sigma_a{}^{\alpha\dot\alpha} \implies V^a = \frac{1}{2}\sigma^a{}_{\alpha\dot\alpha} V^{\alpha\dot\alpha} \tag{A.14}$$

The second formula can be deduced from the first, and from the trace relation

$$\sigma_{a\ \alpha\dot\alpha}\sigma_b{}^{\alpha\dot\alpha} = 2\eta_{ab} \tag{A.15}$$

Examples. i) the current $\bar\psi\gamma^a\psi$ (ψ Majorana spinor 1-form) becomes, in two-component formalism:

$$\bar\psi\gamma_a\psi = \psi^T C\gamma_a\psi = \psi^\alpha\epsilon_{\alpha\beta}\sigma_a{}^\beta{}_{\dot\gamma}\psi^{\dot\gamma} + \psi^{\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\sigma_a{}^{\dot\beta}{}_\gamma\psi^\gamma = -\psi^\alpha\sigma_{a\ \alpha\dot\gamma}\psi^{\dot\gamma} - \psi^{\dot\alpha}\sigma_{a\ \dot\alpha\gamma}\psi^\gamma \tag{A.16}$$

$$= -\psi^\alpha\sigma_{a\ \alpha\dot\gamma}\psi^{\dot\gamma} - \psi^{\dot\gamma}\sigma_{a\ \dot\gamma\alpha}\psi^\alpha = -2\psi^\alpha\psi^{\dot\gamma}\sigma_{a\ \alpha\dot\gamma} \tag{A.17}$$

having used $\sigma_{a\ \dot\gamma\alpha} = \sigma_{a\ \alpha\dot\gamma}$. Converrting the vector index into two-component spinor indices yields:

$$\bar\psi\gamma^a\psi\ \sigma_a{}^{\beta\dot\delta} = -2\psi^\alpha\psi^{\dot\gamma}\sigma^a{}_{\alpha\dot\gamma}\sigma_a{}^{\beta\dot\delta} = 4\psi^\beta\psi^{\dot\delta} \tag{A.18}$$

using the completeness relation. Thus the flat superspace Cartan-Maurer equation $dV^a = \frac{i}{2}\bar\psi\gamma^a\psi$ becomes $dV^{\alpha\dot\alpha} = 2i\psi^\alpha\psi^{\dot\alpha}$.

ii) chiral and antichiral projections of VV:

$$\begin{aligned}(V_+^2)^{\alpha\beta} &\equiv [P_+(VV)]^{\alpha\beta} = V^aV^b[P_+\gamma_{ab}]^{\alpha\beta} = V^aV^b\sigma_a^\alpha{}_{\dot\beta}\,\sigma_b{}^{\dot\beta\beta} \\ &= V^aV^b\sigma_a{}^{\alpha\dot\alpha}\sigma_b{}^{\dot\beta\beta}\epsilon_{\dot\alpha\dot\beta} = V^{\alpha\dot\alpha}V^{\beta\dot\beta}\epsilon_{\dot\alpha\dot\beta}\end{aligned} \tag{A.19}$$

and similarly $(V_-^2)^{\dot\alpha\dot\beta} = V^{\alpha\dot\alpha}V^{\beta\dot\beta}\epsilon_{\alpha\beta}$.

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Pros and cons. Pros: two-component formalism can simplify calculations, since gamma matrices disappear in most cases (see the examples above), and Fierz rearrangements are automatically implemented. Cons: the notation is less compact (two spinor indices replace one vector index), and it is necessary to remember minus signs in relations like $A^\alpha B_\alpha = -A_\alpha B^\alpha$.

B Some useful formulas

We consider the supervielbeins $(V^{\alpha\dot\alpha}, \psi^\alpha, \bar\psi^{\dot\alpha})$ such that

$$dV^{\alpha\dot\alpha} = 2i\psi^\alpha \bar\psi^{\dot\alpha}\,, \qquad d\psi^\alpha = 0\,, \qquad d\bar\psi^{\dot\alpha} = 0\,. \tag{B.1}$$

We define the following combinations

$$\begin{aligned}
(V_+^2)^{\alpha\beta} &= \frac{1}{2!} V^{\alpha\dot\alpha} \wedge V^{\beta\dot\beta} \epsilon_{\dot\alpha\dot\beta}\,, \\
(V_-^2)^{\dot\alpha\dot\beta} &= \frac{1}{2!} V^{\alpha\dot\alpha} \wedge V^{\beta\dot\beta} \epsilon_{\alpha\beta}\,, \\
(V^3)^{\alpha\dot\alpha} &= \frac{1}{3!} V^{\alpha\dot\beta} \wedge V^{\dot\gamma\gamma} \wedge V^{\beta\dot\alpha} \epsilon_{\dot\beta\dot\gamma}\epsilon_{\gamma\beta}\,, \\
(V^4) &= \frac{1}{4!} V^{\alpha\dot\alpha} \wedge V^{\dot\gamma\gamma} \wedge V^{\beta\dot\beta} \wedge V^{\dot\sigma\sigma} \epsilon_{\dot\alpha\dot\gamma}\epsilon_{\gamma\beta}\epsilon_{\dot\beta\dot\sigma}\epsilon_{\sigma\alpha} = \det(V^{\alpha\dot\alpha})\,,
\end{aligned} \tag{B.2}$$

The first two combinations $V_\pm^2$ are the self-dual and anti-self dual part of the wedge product of two vielbeins $V^{\alpha\dot\alpha}$. The last one is the singlet combination (corresponding to the determinant) of the vielbein. By multiplying with $V^{\alpha\dot\alpha}$ we find the following relations (recall that $e_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$ and $e_{\alpha\beta}\epsilon^{\alpha\beta} = -2$)

$$\begin{aligned}
V^{\alpha\dot\alpha} \wedge V^{\beta\dot\beta} &= -\epsilon^{\alpha\beta}(V_-^2)^{\dot\alpha\dot\beta} - \epsilon^{\dot\alpha\dot\beta}(V_+^2)^{\alpha\beta}\,, \\
V^{\alpha\dot\alpha} \wedge V^{\beta\dot\beta} \wedge V^{\gamma\dot\gamma} &= 2\epsilon^{\dot\beta\dot\gamma}\epsilon^{\alpha(\beta}(V^3)^{\gamma)\dot\alpha} + 2\epsilon^{\beta\gamma}\epsilon^{\dot\alpha(\dot\beta}(V^3)^{\dot\gamma)\alpha}\,, \\
V^{\alpha\dot\alpha} \wedge (V_+^2)^{\beta\gamma} &= -2\epsilon^{\alpha(\beta}(V^3)^{\gamma)\dot\alpha}\,, \\
V^{\alpha\dot\alpha} \wedge (V_-^2)^{\dot\beta\dot\gamma} &= -2\epsilon^{\dot\alpha(\dot\beta}(V^3)^{\dot\gamma)\alpha}\,, \\
V^{\alpha\dot\alpha} \wedge (V^3)^{\beta\dot\beta} &= \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}(V^4)\,, \\
V^{\alpha\dot\alpha} \wedge V^{\beta\dot\beta} \wedge V^{\gamma\dot\gamma} \wedge V^{\sigma\dot\sigma} &= t^{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma\sigma\dot\sigma}(V^4)\,, \\
(V_+^2)^{\alpha\beta} \wedge (V_+^2)^{\gamma\delta} &= (\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} + \epsilon^{\alpha\delta}\epsilon^{\beta\gamma})(V^4)\,, \\
(V_-^2)^{\dot\alpha\dot\beta} \wedge (V_-^2)^{\dot\gamma\dot\delta} &= (\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\delta} + \epsilon^{\dot\alpha\dot\delta}\epsilon^{\dot\beta\dot\gamma})(V^4)\,, \\
(V_+^2)^{\alpha\beta} \wedge (V_-^2)^{\dot\gamma\dot\delta} &= 0\,,
\end{aligned} \tag{B.3}$$

where $A^{(\alpha\beta)} = \frac{1}{2}(A^{\alpha\beta} + A^{\beta\alpha})$ and the tensor

$$\begin{aligned}
t^{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma\sigma\dot\sigma} &= -\epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\gamma}\epsilon^{\gamma\sigma}\epsilon^{\dot\sigma\dot\alpha} + \epsilon^{\alpha\gamma}\epsilon^{\dot\gamma\dot\beta}\epsilon^{\beta\sigma}\epsilon^{\dot\sigma\dot\alpha} - \epsilon^{\alpha\sigma}\epsilon^{\dot\sigma\dot\gamma}\epsilon^{\gamma\beta}\epsilon^{\dot\beta\dot\alpha} + \epsilon^{\alpha\sigma}\epsilon^{\dot\sigma\dot\beta}\epsilon^{\beta\gamma}\epsilon^{\dot\gamma\dot\alpha} \\
&= \epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\sigma}\epsilon^{\alpha\gamma}\epsilon^{\beta\sigma} + \epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\sigma}\epsilon^{\alpha\sigma}\epsilon^{\beta\gamma} + \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\sigma}\epsilon^{\alpha\beta}\epsilon^{\gamma\sigma} + \epsilon^{\dot\alpha\dot\sigma}\epsilon^{\dot\beta\dot\gamma}\epsilon^{\alpha\beta}\epsilon^{\gamma\sigma}
\end{aligned} \tag{B.4}$$

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respects all properties of form multiplication. The second line is obtained by using relations like

$$\epsilon^{\alpha\beta}\epsilon^{\gamma\delta} + \epsilon^{\alpha\gamma}\epsilon^{\delta\beta} + \epsilon^{\alpha\delta}\epsilon^{\beta\gamma} = 0\,.$$

This invariant tensor is obtained by contracting with the Dirac gamma matrices the Levi-Civita tensor

$$t^{\alpha\dot\alpha\beta\dot\beta\gamma\dot\gamma\sigma\dot\sigma} = \frac{1}{4!}\epsilon_{abcd}(\gamma^a)^{\alpha\dot\alpha}(\gamma^b)^{\beta\dot\beta}(\gamma^c)^{\gamma\dot\gamma}(\gamma^d)^{\sigma\dot\sigma}\,. \tag{B.5}$$

The differentials are

$$\begin{aligned} dV^{\alpha\dot\alpha} &= 2i\psi^\alpha\bar\psi^{\dot\alpha}\,, \\ d(V_+^2)^{\alpha\beta} &= -2i\psi^{(\alpha}(V\bar\psi)^{\beta)}\,, \\ d(V_-^2)^{\dot\alpha\dot\beta} &= -2i\bar\psi^{(\dot\alpha}(V\psi)^{\dot\beta)}\,, \\ d(V^3)^{\alpha\dot\alpha} &= i\Big(\psi^\alpha(V_-^2\bar\psi)^{\dot\alpha} - \bar\psi^{\dot\alpha}(V_+^2\psi)^\alpha\Big)\,, \\ d(V^4) &= 2i\psi_\alpha(V^3)^{\alpha\dot\alpha}\bar\psi_{\dot\alpha}\,, \end{aligned} \tag{B.6}$$

where

$$\begin{aligned} (V\bar\psi)^\alpha &= V^{\alpha\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\,\bar\psi^{\dot\beta}\,, \\ (V\psi)^{\dot\alpha} &= V^{\dot\alpha\alpha}\epsilon_{\alpha\beta}\,\psi^\beta\,, \\ (V_-^2\bar\psi)^{\dot\alpha} &= (V_-^2)^{\dot\alpha\dot\beta}\epsilon_{\dot\beta\dot\gamma}\,\bar\psi^{\dot\gamma}\,, \\ (V_+^2\psi)^\alpha &= (V_+^2)^{\alpha\beta}\epsilon_{\beta\gamma}\,\psi^\gamma\,, \end{aligned} \tag{B.7}$$

with

$$\begin{aligned} d(V\bar\psi)^\alpha &= 2i\psi^\alpha\bar\psi^{\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\,\bar\psi^{\dot\beta} = 0\,, \\ d(V\psi)^{\dot\alpha} &= 2i\psi^\alpha\bar\psi^{\dot\alpha}\epsilon_{\alpha\beta}\,\psi^\beta = 0\,, \\ d(V_-^2\bar\psi)^{\dot\alpha} &= i\bar\psi^{\dot\alpha}(\psi V\bar\psi)\,, \\ d(V_+^2\psi)^\alpha &= i\psi^\alpha(\psi V\bar\psi)\,. \\ d(\bar\psi V_-^2\bar\psi) &= 0\,, \\ d(\psi V_+^2\psi) &= 0\,, \\ d(\psi V\bar\psi) &= 0\,. \end{aligned} \tag{B.8}$$

C The curved supermanifold $\boldsymbol{Osp}(1|4)$

Let us consider the case of curved supermanifolds, for example the supercoset manifold

$$\mathrm{Osp}(1|4)/\mathrm{SO}(1,3) \sim (\mathrm{AdS}_4|4)$$

which is a supermanifold whose bosonic submanifold is $4d$ anti-de Sitter and with 4 fermionic coordinates. We have

$$\nabla V^{\alpha\dot\alpha} = 2i\psi^\alpha\wedge\bar\psi^{\dot\alpha}\,, \tag{C.1}$$

$$\nabla\psi^\alpha = i\Lambda\,V^{\alpha\dot\alpha}\bar\psi_{\dot\alpha}\,, \tag{C.2}$$

$$\nabla\bar\psi^{\dot\alpha} = -i\Lambda\, V^{\alpha\dot\alpha}\psi_\alpha\,. \tag{C.3}$$
$$R_+^{\alpha\beta} = 4\Lambda^2(V_+^2)^{\alpha\beta} + 2\Lambda\psi^\alpha\psi^\beta\,, \tag{C.4}$$
$$R_-^{\dot\alpha\dot\beta} = -4\Lambda^2(V_-^2)^{\dot\alpha\dot\beta} - 2\Lambda\bar\psi^{\dot\alpha}\bar\psi^{\dot\beta}\,, \tag{C.5}$$

where R^{ab}, decomposed into the self-dual and the anti-self dual parts ($R_+^{\alpha\beta}$ and $R_-^{\dot\alpha\dot\beta}$), is the curvature of the supermanifold and ∇ is the covariant derivative w.r.t. to the connection of SO(1,3). It is easy to check the Bianchi identities using these definitions.

In addition, by using (C.3), we can verify that

$$\begin{aligned} d\left((V^4)\delta^4(\psi)\right) &= \nabla\left((V^4)\delta^4(\psi)\right) \\ &= 2i\psi_\alpha(V^3)^{\alpha\dot\alpha}\bar\psi_{\dot\alpha}\delta^4(\psi) + (V^4)i\Lambda V^{\alpha\dot\alpha}\left(\bar\psi_{\dot\alpha}\iota_\alpha - \psi_\alpha\bar\iota_{\dot\alpha}\right)\delta^4(\psi)\,. \end{aligned} \tag{C.6}$$

where the first equality follows from the Lorentz invariance of the volume form $V^4\delta^4(\psi)$ and the last follows from the distributional law $\psi\delta(\psi) = 0$ and the properties of top forms.

For the curved case, we have that

$$\nabla\Big(V^a\bar\psi\gamma_a\psi\Big) = i\Lambda V^a\wedge V^b\Big(\psi\gamma_{ab}\psi - \bar\psi\gamma_{ab}\bar\psi\Big) \tag{C.7}$$

and since $\Big(V^a\bar\psi\gamma_a\psi\Big)$ is a scalar, we find $\nabla^2\Big(V^a\bar\psi\gamma_a\psi\Big) = 0$. This means that only the class $iV^a\wedge V^b\big(\psi\gamma_{ab}\psi - \bar\psi\gamma_{ab}\bar\psi\big)$ is closed. In the limit $\Lambda\to 0$, one recovers the flat case.

Let us now consider the same problem in the curved space. We start with Osp(1|4) case. We use the relations given in (C.3) and the volume form has the expression

$$\omega^{(4|4)} = \epsilon_{abcd}V^a\wedge\cdots\wedge V^d\wedge\delta^4(\psi)\,, \tag{C.8}$$

which is closed (the variation of V^a is cancelled because of the Dirac delta's, while the variation of ψ's is cancelled by the presence of four V's. Using the definitions

$$V^a = V^a_m dx^m + V^a_\mu d\theta^\mu\,, \qquad \psi^a = \psi^a_m dx^m + \psi^a_\mu\theta^\mu\,, \tag{C.9}$$

we find

$$\omega^{(4|4)} = \mathrm{Sdet}(E)\epsilon_{abcd}dx^a\wedge\cdots\wedge dx^d\delta^4(d\theta) \tag{C.10}$$

with $E = \begin{pmatrix} V^a_m & V^a_\mu \\ \psi^\alpha_m & \psi^\alpha_\mu\end{pmatrix}$. The bosonic space is Sp(4)/SO(1,3), namely the curved space AdS_4, and therefore we have

$$\mathrm{Vol}_{\mathrm{Osp}(1|4)/\mathrm{SO}(1,3)} = \int_{AdS_4} d^4x\; D^4\mathrm{Sdet}(E)\big|_{\theta=0} \tag{C.11}$$

where $D^4 = \epsilon_{\alpha_1\ldots\alpha_4}D^{\alpha_1}\ldots D^{\alpha_4}$. In the present case the (4|4)-integral form $\omega^{(4|4)}$ is closed, but it is not exact since Sdet(E) has a non-trival θ-dependence.

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